



**HAL**  
open science

## Statistical test for an urn model with random multidrawing and random addition

Irene Crimaldi, Pierre-Yves Louis, Ida Germana Minelli

► **To cite this version:**

Irene Crimaldi, Pierre-Yves Louis, Ida Germana Minelli. Statistical test for an urn model with random multidrawing and random addition. *Stochastic Processes and their Applications*, 2023, 158, pp.342-360. 10.1016/j.spa.2022.12.012 . hal-03697574v1

**HAL Id: hal-03697574**

**<https://hal.science/hal-03697574v1>**

Submitted on 20 Oct 2023 (v1), last revised 18 Dec 2023 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STATISTICAL TEST FOR AN URN MODEL WITH RANDOM MULTIDRAWING AND RANDOM ADDITION

ABSTRACT. We complete the study of the model introduced in [11]. It is a two-color urn model with multiple drawing and random (non-balanced) time-dependent reinforcement matrix. The number of sampled balls at each time-step is random. We identify the exact rates at which the number of balls of each color grows to  $+\infty$  and define two strongly consistent estimators for the limiting reinforcement averages. Then we prove a Central Limit Theorem, which allows to design a statistical test for such averages.

Irene Crimaldi<sup>1</sup>, Pierre-Yves Louis<sup>2,3</sup>, Ida G. Minelli<sup>4</sup>

**Keywords.** Hypergeometric Urn; Multiple drawing urn; Pólya urn; Randomly reinforced urn; Central limit theorem; Stable convergence; hypothesis testing; Population dynamics; Opinion dynamics; Response-adaptive design.

## 1. INTRODUCTION

An urn model can be described as follows: an urn contains balls of different colors. At each time step a number  $k$  of balls is drawn and replaced into the urn together with other balls, whose color depends on the composition of the sample. It applies typically whenever one wants to describe a (*self-*)*reinforcement* phenomenon in a stochastic evolution, which is the tendency to increase the probability of an event in relation with the number of times this event occurred in the past.

We focus on urns with balls of two colors, say  $A$  and  $B$ . The first urn model was from Pólya-Eggenberger [13] and it consists in drawing *one* ball at each time and returning it back into the urn together with  $a$  balls of the same color. A variant of this model is the Friedman urn [14], where, at each time step,  $a$  balls of the color drawn and  $b$  balls of the color not drawn are added into the urn. A natural generalization consists in drawing more than one ball and changing the replacement scheme at each time step. More precisely, assume that the urn contains balls of color  $A$  and  $B$ . If  $X_n$  (resp.,  $k - X_n$ ) denotes the number of balls of color  $A$  (resp., color  $B$ ) in the sample at time  $n$ , the number of balls of color  $A$  (resp.,  $B$ ) added into the urn is  $A_n X_n$  (resp.,  $B_n(k - X_n)$ ), where the *reinforcement factors*  $A_n$  and  $B_n$  may change over time.

This class of models has been studied by many authors in almost two decades and the usual assumptions are that the number  $k$  of sampled balls and the number  $(A_n + B_n)k$  of balls added into the urn at each step are constant (e.g. [5, 6, 16, 18–23]).

In more recent models, the replacement scheme is *not balanced*, i.e.,  $A_n + B_n$  change in time and the reinforcement factors  $A_n$ , and  $B_n$  are *random*. For example, in [1, 2],  $(A_n)$  and  $(B_n)$  are two independent sequences of i.i.d. random variables. Other authors have studied the case when the sample size, denoted by  $N_n$ , is *time-dependent*, *random* and *possibly dependent of the past* and

<sup>1</sup>IMT School for Advanced Studies Lucca, Piazza San Ponziano 6, 55100 Lucca, Italy, [irene.crimaldi@imtlucca.it](mailto:irene.crimaldi@imtlucca.it)

<sup>2</sup>PAM UMR 02.102, Université Bourgogne Franche-Comté, AgroSup Dijon, 1 esplanade Erasme, F-21000, Dijon, France, [pierre-yves.louis@agrosupdijon.fr](mailto:pierre-yves.louis@agrosupdijon.fr)

<sup>3</sup>Institut de Mathématiques de Bourgogne, UMR 5584 CNRS, Université Bourgogne Franche-Comté, F-21000, Dijon, France, [pierre-yves.louis@math.cnrs.fr](mailto:pierre-yves.louis@math.cnrs.fr)

<sup>4</sup>Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, Via Vetoio (Coppito 1), 67100 L'Aquila, Italy, [idagermana.minelli@univaq.it](mailto:idagermana.minelli@univaq.it)

both colors are reinforced at each step with the same i.i.d. reinforcement factors, i.e.  $A_n = B_n$  for all  $n$  (see [3, 7], that include as a particular case the model recently given in [4]).

In [11], the above models have been generalized to the case when  $A_n, B_n$  may be different and eventually *correlated*. The purpose of this article is to complete the study of this last model by proving additional theoretical results and providing a statistical tool for inference on the reinforcement factors.

Going into more detail, in [11], the number of balls added into the urn at each time-step is  $(A_n + B_n)N_n$ , of which  $A_n X_n$  are colored  $A$  and  $B_n(N_n - X_n)$  are colored  $B$ . The pair of reinforcement factors  $[A_n, B_n]$  is assumed to be independent from all the past until  $n - 1$  and from the composition of the sample at time-step  $n$ . Thinking about possible applications of the model, this assumption of independence means that the reinforcement deals with some exogenous random factors, not related to those in the past and external to the urn mechanism. On the other hand, the joint distribution of the pair  $[A_n, B_n]$  may be whatever: the two random factors may be correlated and their distribution may depend on  $n$ . The main object of study is the asymptotic behavior of the urn composition. For the proportion of color  $A$  balls in the urn, almost sure convergence results, as well as fluctuation results (through Central Limit Theorems in the sense of stable convergence and of almost sure conditional convergence), have been proven in [11]. Specifically, two settings have been considered. If the factors  $A_n$  and  $B_n$  have the same mean (equal reinforcement averages case), the limit proportion  $Z \in [0, 1]$  is random without atoms and the proven CLTs provides asymptotic confidence intervals for it. In the case of unequal limit reinforcement averages, i.e., when  $E[A_n] \rightarrow m_A, E[B_n] \rightarrow m_B$ , with  $m_A > m_B$ , the proportion converges almost surely to 1: in the present paper we identify the exact rate of convergence of such proportion towards its limit and we show that the total number of color  $A$  and color  $B$  balls observed in the samples up to time  $n$  are quantities of order  $n$  and  $n^{m_B/m_A}$  respectively. Note that, in the applications, when convergence of the proportion of color  $A$  balls is slow (as can be seen in some of the examples illustrated in [11]), it could be difficult to understand whether we are in the first or in the second setting. Using the above mentioned result we are able to define two strongly consistent estimators for  $m_A$  and  $m_B$  and we prove a CLT for these estimators. We then use this result to design a statistical test for the hypothesis  $m_A = m_B$  versus  $m_A > m_B$ .

Below we discuss some possible applications of the model to *population/social dynamics* and *response-adaptive designs*.

Generally speaking, assume to have a population, where the units can be of two types, say  $A$  and  $B$ . The units already present in the population are represented by the colored balls inside the urn. At each time-step, a group (with random size  $N_n$ , possibly dependent of the past) of units generates other units: each unit in the group generates a certain number of new units of the same type. At time-step  $n$ , the reinforcement factors  $A_n$  and  $B_n$  represent the ability to generate new units for the two different sub-populations (specific example are given below). Indeed, it is reasonable to assume the ability to give rise to new units to be specific for each of the two sub-populations, depend on time and on exogenous random factors. Moreover, the possible correlation between  $A_n$  and  $B_n$  models the competition or the cooperation between the two sub-populations. If one of the reinforcement factors is eventually larger in mean, then the associated sub-population will eventually dominate. If both reinforcement factors are equal in mean then in the long-run some random equilibrium takes place. As specific examples, one may think to bacterial populations or to the diffusion of genetic variants of a virus, but also to examples in opinion dynamics: for instance,

- (Elections) There are two candidates,  $A$  and  $B$ , and people who have already (irreversibly) decided who they are going to vote for. They are color  $A$  and color  $B$  balls into the urn, that is the two sub-populations. At each time-step, a random group  $N_n$  of voters becomes “active” and try to persuade others to adopt their same choice. The ability to persuade

may depend on several factors (intrinsic of the voters, good campaign or reputation of the candidate, etc.) and may change in time. Therefore, it is reasonable to assume that  $A_n$  and  $B_n$  are random and eventually correlated. Here we assume that the “observable” quantities are  $N_n, X_n, A_n, B_n$ , where  $A_n$  and  $B_n$  may be interpreted as trends associated with one or the other choice, which can be obtained through polls, study of the specific context, analysis of voting flows, etc.

- (Diffusion of a binary opinion through social networks) Imagine a connected population on the internet, where communities of agents share the same binary opinion on a given subject and grow in time through the addition of new “followers”, who support one of these two opinions. In such communities some misperceptions may spread faster than others, regardless of their rationality. This can generate what is called “behavioral epidemics”, i.e., there is a group reinforcement in adopting a given opinion (or choice)  $A$  due to a behavioral pattern perceived by the agents. In this case, the two different opinions ( $A$  and its opposite  $B$ ) identify the two sub-populations and we expect a strong correlation between  $A_n$  and  $B_n$ , where we interpret  $B_n$  as the rate at which misperception  $A$  is debunked with success. On the other hand, choices have a cost (psychological, material): this implies that the number  $N_n$  of “influencers” who are imitated by others is random and may depend on the past, and the reinforcement factors  $A_n, B_n$  may change along time. Again, in this case we assume that  $N_n, A_n$  and  $B_n$  are observable quantities: for example, on *Twitter*,  $A_n$  and  $B_n$  may be desumed from the current number of “retweets” of comments or hashtags supporting the two opinions.

This model can be also applied in the context of response-adaptive designs (see [17, 26, 27] for exhaustive reviews and see [24] for an urn model without multidrawing, applied to clinical trials, which is included in the model considered in the present paper as the very special case when the sample-size  $N_n$  is equal to 1,  $E[A_n] = m_A$  and  $E[B_n] = m_B$  for each  $n$ ). In such a design, units enter the experiment sequentially and are allocated randomly to one out of two “treatments”,  $A$  or  $B$ , according to a rule that depends on the previous allocations and the previous observed responses. The experimenter has two simultaneous goals: 1) collecting evidence to determine the superior treatment and 2) biasing the allocations towards the better treatment, in order to reduce the proportion of units in the experiment that receive the inferior treatment. The design driven by the model considered in this paper satisfies this twice requirement when we take the reinforcement factors  $A_n$  and  $B_n$  of the model as representative of the goodness of the two treatments  $A$  and  $B$ . Indeed, if one treatment is better than the other in average, the design will allocate the units to the best treatment with a proportion that converges almost surely to 1. When the two treatments are equivalent, the design allocates the proportion of units with a random limit (see Remark 3.3).

As an example of application in the industry sector, think to a firm that has to select the size of its production for two different kinds of products, say product  $A$  and product  $B$ . At the same time, the firm wants to get information on which of the two products is more in demand by potential buyers. The firm may perform the following design associated to our model. At each time-step  $n$ , the total of the production of the firm is  $N_n$ . The firm decides to produce  $X_n$  products of type- $A$  and  $N_n - X_n$  products of type- $B$ , according to a sampling without replacement of  $N_n$  balls from an urn following the dynamics here described. The factors  $A_n$  and  $B_n$ , which will affect the decision about the production at time-step  $n + 1$ , are related to a market survey on the two products, conducted at time-step  $n$ . The market survey can be assumed to be independent of the production of the firm and of the previous surveys, and this justifies the hypothesis of independence for  $[A_n, B_n]$  required in the model.

The paper is organized as follows. In Section 2 we recall the model and set the assumptions. In Section 3 we state and prove the main theoretical results. Section 4 provides the test on the reinforcement averages. Finally, in Appendix A we prove some technical results, and, in Appendix B and in Appendix C, we recall some auxiliary results and the definition of stable convergence.

## 2. MODEL SPECIFICATION

We consider the model introduced in [11]. An urn initially contains  $a \in \mathbb{N} \setminus \{0\}$  balls of color A and  $b \in \mathbb{N} \setminus \{0\}$  balls of color B. At each discrete time  $n \geq 1$ , we simultaneously (*i.e.* without replacement) draw a random number  $N_n$  of balls. Let  $X_n$  be the number of extracted balls of color A. Then we return the extracted balls into the urn together with  $A_n X_n$  balls of color A and  $B_n(N_n - X_n)$  balls of color B. More precisely, we take a probability space  $(\Omega, \mathcal{A}, P)$  and some random variables  $N_n, X_n, A_n, B_n$  defined on it and such that, for each  $n \geq 1$ , we have:

(A1) the conditional distribution of the random variable  $N_n$  given

$$[N_1, X_1, A_1, B_1, \dots, N_{n-1}, X_{n-1}, A_{n-1}, B_{n-1}]$$

is concentrated on  $\{1, \dots, S_{n-1}\}$  where  $S_{n-1}$  is the total number of balls in the urn at time  $n - 1$ , that is

$$S_{n-1} = a + b + \sum_{j=1}^{n-1} A_j X_j + \sum_{j=1}^{n-1} B_j (N_j - X_j); \quad (1)$$

(A2) the conditional distribution of the random variable  $X_n$  given

$$[N_1, X_1, A_1, B_1, \dots, N_{n-1}, X_{n-1}, A_{n-1}, B_{n-1}, N_n]$$

is hypergeometric with parameters  $N_n, S_{n-1}$  and  $H_{n-1}$ , where  $H_{n-1}$  is the total number of balls of color A at time  $n - 1$ , that is

$$H_{n-1} = a + \sum_{j=1}^{n-1} A_j X_j; \quad (2)$$

(A3) the random vector  $[A_n, B_n]$  takes values in  $\mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\}$  and it is independent of

$$[N_1, X_1, A_1, B_1, \dots, N_{n-1}, X_{n-1}, A_{n-1}, B_{n-1}, N_n, X_n].$$

According to the above notation, the random variable  $X_n$  corresponds to the number of balls having the color A in a random sample without replacement of size  $N_n$  from an urn with  $H_{n-1}$  balls of color A and  $K_{n-1} = (S_{n-1} - H_{n-1})$  balls of color B. The reinforcement rule is of the ‘‘multiplicative’’ type: indeed, at  $n \geq 1$ , we add to the urn  $A_n X_n$  balls of color A and  $B_n(N_n - X_n)$  balls of color B. Therefore, the total number of balls added to the urn, that is  $A_n X_n + B_n(N_n - X_n)$ , is random and depends on  $n$ .

Note that we do not specify the conditional distribution of the random variable  $N_n$  (the sample size) given the past

$$[N_1, X_1, A_1, B_1, \dots, N_{n-1}, X_{n-1}, A_{n-1}, B_{n-1}]$$

nor the distribution of  $[A_n, B_n]$  (the random reinforcement factors  $A_n$  and  $B_n$  may have different distributions, they may be correlated and their joint and marginal distributions may vary with  $n$ ). Several examples can be found in [11].

It is worthwhile to remark that this model includes the Hypergeometric Randomly Reinforced Urn studied in [3, 8] (take  $A_n = B_n$  for all  $n$ ), which in turn includes the model recently given in [4]. In particular, two special cases are the classical Pólya urn (the case with  $N_n = 1$  and  $A_n = B_n = k \in \mathbb{N} \setminus \{0\}$  for each  $n$ ) and the 2-colors randomly reinforced urn with the reinforcements for the two colors equal or different in mean (the case with  $N_n = 1$  for each  $n$  and  $[A_n, B_n]$

arbitrarily random in  $\mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\}$ ). Moreover, as told in Section 1, previous literature (we refer to the quoted papers in Sec. 1) deals with the case when the sample size  $N_n$  is a fixed constant, not depending on  $n$ , and/or the balanced case (constant number of balls added to the urn each time).

We set  $Z_n$  equal to the proportion of balls of color A in the urn immediately after the  $n$ th update, that is  $Z_0 = a/(a+b)$  and

$$Z_n = \frac{H_n}{S_n} \quad \text{for } n \geq 1.$$

Moreover we set

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma(N_1, X_1, A_1, B_1, \dots, N_n, X_n, A_n, B_n) \quad \text{for } n \geq 1,$$

and

$$\mathcal{G}_n = \mathcal{F}_n \vee \sigma(N_{n+1}), \quad \mathcal{H}_n = \mathcal{G}_n \vee \sigma(A_{n+1}, B_{n+1}) \quad \text{for } n \geq 0.$$

By the above assumptions and notation, we have

$$E[A_{n+1} | \mathcal{G}_n] = E[A_{n+1}], \quad E[B_{n+1} | \mathcal{G}_n] = E[B_{n+1}] \quad (3)$$

and

$$\begin{aligned} E[X_{n+1} | \mathcal{H}_n] &= E[X_{n+1} | \mathcal{G}_n] = N_{n+1} Z_n, \\ E[N_{n+1} - X_{n+1} | \mathcal{H}_n] &= E[N_{n+1} - X_{n+1} | \mathcal{G}_n] = N_{n+1}(1 - Z_n). \end{aligned} \quad (4)$$

Finally, we set  $\mathcal{X}_n = \{0 \vee [N_n - (S_{n-1} - H_{n-1})], \dots, N_n \wedge H_{n-1}\}$  and, for each  $k \in \mathcal{X}_n$ ,

$$p_{n,k} = p_k(N_n, S_{n-1}, H_{n-1}) = \frac{\binom{H_{n-1}}{k} \binom{S_{n-1} - H_{n-1}}{N_n - k}}{\binom{S_{n-1}}{N_n}}. \quad (5)$$

For all  $n$ , set the reinforcement averages/means  $m_{A,n} = E[A_n] \geq 1$  and  $m_{B,n} = E[B_n] \geq 1$ . We will assume that the two sequences  $(m_{A,n})_n$  and  $(m_{B,n})_n$  respectively converge to  $m_A < +\infty$  and  $m_B < +\infty$ . In the present paper, we will consider the following cases:

CASE  $m_A > m_B$ : We have  $m_A > m_B$  with

$$\sum_n \frac{|m_{A,n+1} m_B - m_A m_{B,n+1}|}{n} < +\infty; \quad (6)$$

CASE  $m_A = m_B$ : We have  $m_{A,n} = m_{B,n} = m_n$  for each  $n$  and so  $m_A = m_B = m \in [1, +\infty)$ .

Note that assumption (6) is trivially satisfied when  $|m_{A,n} - m_A| = O(n^{-\epsilon_A})$  and  $|m_{B,n} - m_B| = O(n^{-\epsilon_B})$ , with  $\epsilon_A > 0$  and  $\epsilon_B > 0$ . In particular, it is verified when  $m_{A,n} = m_A$  and  $m_{B,n} = m_B$  for each  $n$  (constant average values).

In the sequel, we will refer to the above two cases as “case  $m_A > m_B$ ” and “case  $m_A = m_B$ ”, respectively. Therefore, *condition (6) will be tacitly assumed* when we are in case  $m_A > m_B$ . Moreover, for simplicity, throughout the paper we will assume

$$A_n \vee B_n \vee N_n \leq C \quad \text{for some (integer) constant } C.$$

As mentioned in [11], sometimes this assumption can be removed or replaced by an assumption of uniform integrability, but we will not focus in the present work on this possibility.

## 3. MAIN RESULTS

In [11], without assumption (6), we proved that  $H_n$  and  $K_n = S_n - H_n$  go almost surely to  $+\infty$  and that, under the condition

$$E[\widetilde{N}_{n+1}|\mathcal{F}_n] \xrightarrow{a.s.} N, \quad (7)$$

where  $N$  is a (finite, strictly positive) random variable, we have

$$\frac{H_n}{n} \xrightarrow{a.s.} m_A N Z, \quad \text{and} \quad \frac{K_n}{n} \xrightarrow{a.s.} m_B N (1 - Z),$$

where  $Z = 1$  in the case  $m_A > m_B$  and a non-atomic random variable with values in  $[0, 1]$  in the case  $m_A = m_B$ . Therefore,  $H_n$  and  $S_n$  grow like  $n$  in both cases, while  $K_n$  grows as  $n$  only in the case  $m_A = m_B$ . For the case  $m_A > m_B$ , we obtained  $K_n = o(n^\theta)$ , that is  $1 - Z_n = o(n^{-(1-\theta)})$ , for all  $\theta \in (m_B/m_A, 1)$ . With the following result, we prove here that, under the additional assumption (6),  $K_n$  goes to  $+\infty$  exactly as  $n^{m_B/m_A}$ .

**Theorem 3.1.** (Rate of  $K_n$  in case  $m_A > m_B$ )

Suppose to be in case  $m_A > m_B$  and assume (7). Then:

- $(K_n)_n$  increases to  $+\infty$  as  $n^{m_B/m_A}$ ;
- $n^{1-m_B/m_A}(1 - Z_n) \xrightarrow{a.s.} \widetilde{Z}$ , where  $\widetilde{Z}$  is a random variable taking values in  $(0, +\infty)$ ;
- setting  $N_{B,n} = \sum_{j=1}^n (N_j - X_j)$ , we have  $\frac{N_{B,n}}{n^{m_B/m_A}} \xrightarrow{a.s.} \frac{m_A}{m_B} N \widetilde{Z}$ .

*Proof.* The main part of the proof is the proof of Lemma A.2. Indeed, given this result, that is the fact that  $H_n/K_n^{m_A/m_B}$  converges almost surely to a random variable  $Y$  taking values in  $(0, +\infty)$ , we have

$$\frac{K_n}{n^{m_B/m_A}} = \frac{K_n}{H_n^{m_B/m_A}} \left( \frac{H_n}{n} \right)^{m_B/m_A} \xrightarrow{a.s.} \widetilde{Y} = Y^{-m_B/m_A} (m_A N)^{m_B/m_A} \in (0, +\infty).$$

As a consequence, we have

$$n^{1-m_B/m_A}(1 - Z_n) = n^{1-m_B/m_A} \frac{K_n}{S_n} = \frac{K_n}{n^{m_B/m_A}} \frac{n}{S_n} \xrightarrow{a.s.} \widetilde{Z} = \frac{\widetilde{Y}}{m_A N} = \frac{Y^{-m_B/m_A}}{(m_A N)^{1-m_B/m_A}} \in (0, +\infty).$$

Finally, we observe that

$$n^{1-m_B/m_A} E[N_{n+1} - X_{n+1}|\mathcal{F}_n] = E[N_{n+1}|\mathcal{F}_n] n^{1-m_B/m_A} (1 - Z_n) \xrightarrow{a.s.} N \widetilde{Z}$$

and so, by Lemma B.1 (applied with  $Y_j = j^{1-m_B/m_A}(N_j - X_j)$  and  $\alpha_j = j^{1-m_B/m_A}$  and  $\beta_n = n^{m_B/m_A}$ ), we get

$$\frac{N_{B,n}}{n^{m_B/m_A}} = \frac{1}{n^{m_B/m_A}} \sum_{j=1}^n (N_j - X_j) \xrightarrow{a.s.} \frac{m_A}{m_B} N \widetilde{Z}. \quad \blacksquare$$

**Corollary 3.2.** (Strongly consistent estimators)

Assume condition (7) and

$$E[N_{n+1}^2|\mathcal{F}_n] \xrightarrow{a.s.} Q, \quad (8)$$

where  $Q$  is a (strictly positive finite) random variable. In both cases  $m_A = m_B$  and  $m_A > m_B$ ,

$$\widehat{\mu}_n = \frac{\sum_{j=1}^n N_j}{n}, \quad \widehat{q}_{N,n} = \frac{\sum_{j=1}^n N_j^2}{n}, \quad (9)$$

are strongly consistent estimators of the random variables  $N$  and  $Q$ . Moreover, setting  $N_{A,n} = \sum_{j=1}^n X_j$  and  $N_{B,n} = \sum_{j=1}^n (N_j - X_j)$ , the random variables

$$\hat{m}_{A,n} = \frac{\sum_{j=1}^n A_j X_j}{N_{A,n}}, \quad \hat{m}_{B,n} = \frac{\sum_{j=1}^n B_j (N_j - X_j)}{N_{B,n}} \quad (10)$$

are strongly consistent estimators of  $m_A$  and  $m_B$ , respectively. Further, assuming

$$q_{A,n} = E[A_n^2] \longrightarrow q_A, \quad q_{B,n} = E[B_n^2] \longrightarrow q_B, \quad (11)$$

$$\hat{q}_{A,n} = \frac{\sum_{j=1}^n A_j^2 X_j}{N_{A,n}} \quad \text{and} \quad \hat{q}_{B,n} = \frac{\sum_{j=1}^n B_j^2 (N_j - X_j)}{N_{B,n}} \quad (12)$$

are strongly consistent estimators of  $q_A$  and  $q_B$ . Finally, assuming

$$q_{AB,n} = E[A_n B_n] \longrightarrow q_{AB} \quad (13)$$

and  $P(N = 1) < 1$ , we have

$$\hat{q}_{AB,n} = \frac{\sum_{j=1}^n A_j B_j X_j (N_j - X_j)}{\sum_{j=1}^n X_j (N_j - X_j)} \xrightarrow{a.s.} q_{AB} \quad \text{on } \{N > 1\}. \quad (14)$$

*Proof.* For  $\hat{\mu}_n$  and  $\hat{q}_{N,n}$ , it is enough to apply Lemma B.1 using the two assumptions (7) and (8). For the other random variables, we have to employ Lemma B.1, using the suitable rates given in [11] and in the above Theorem 3.1. As an example, we show the proof only of the statement for  $\hat{q}_{AB,n}$ . To this regards, we observe that

$$E[X_{n+1}(N_{n+1} - X_{n+1})|\mathcal{H}_n] = Z_n(1 - Z_n) \frac{S_n}{S_n - 1} (N_{n+1}^2 - N_{n+1})$$

and so

$$n^{1-m_B/m_A} E[X_{n+1}(N_{n+1} - X_{n+1})|\mathcal{F}_n] \xrightarrow{a.s.} \begin{cases} Z(1 - Z)(Q - N) & \text{in the case } m_A = m_B \\ \tilde{Z}(Q - N) & \text{in the case } m_A > m_B. \end{cases}$$

Therefore, Lemma B.1 implies

$$\frac{\sum_{j=1}^n X_j (N_j - X_j)}{n^{m_B/m_A}} \xrightarrow{a.s.} \begin{cases} Z(1 - Z)(Q - N) & \text{in the case } m_A = m_B \\ \frac{m_A}{m_B} \tilde{Z}(Q - N) & \text{in the case } m_A > m_B, \end{cases}$$

Similarly, since  $E[A_{n+1}B_{n+1}X_{n+1}(N_{n+1} - X_{n+1})|\mathcal{F}_n] = q_{AB,n+1}E[X_{n+1}(N_{n+1} - X_{n+1})|\mathcal{F}_n]$ , we get

$$\frac{\sum_{j=1}^n A_j B_j X_j (N_j - X_j)}{n^{m_B/m_A}} \xrightarrow{a.s.} \begin{cases} q_{AB} Z(1 - Z)(Q - N) & \text{in the case } m_A = m_B \\ q_{AB} \frac{m_A}{m_B} \tilde{Z}(Q - N) & \text{in the case } m_A > m_B, \end{cases}$$

and so  $\hat{q}_{AB,n} \xrightarrow{a.s.} q_{AB}$  on  $\{Q - N > 0\} = \{N > 1\}$ .  $\blacksquare$

**Remark 3.3.** The proportion of balls of color  $A$  along time is

$$\frac{N_{A,n}}{\sum_{j=1}^n N_j} = \frac{\sum_{j=1}^n X_j}{n} \frac{n}{\sum_{j=1}^n N_j}.$$

By the above Corollary 3.2,  $\sum_{j=1}^n N_j/n$  converges almost surely to  $N$ , while, with a similar argument,  $\sum_{j=1}^n X_j/n$  converges almost surely to  $Z$ , because  $E[X_{n+1}|\mathcal{H}_n] = N_{n+1}Z_n$  and so, under (7), we have  $E[X_{n+1}|\mathcal{F}_n] \xrightarrow{a.s.} NZ$ . Therefore, the proportion  $N_{A,n}/\sum_{j=1}^n N_j$  converges to 1 in the case  $m_A > m_B$  and to a non-atomic random variable  $Z \in [0, 1]$  in the case  $m_A = m_B$ . This is the formalization of the property required for response-adaptive designs, mentioned in the Introduction.



**Theorem 3.4.** (*Asymptotic normality of the estimators*)

With the same notation as in Corollary 3.2, assume conditions (7), (8), (11) and (13) and set

$$\sigma_A^2 = q_A - m_A^2, \quad \sigma_B^2 = q_B - m_B^2 \quad \text{and} \quad c_{AB} = q_{AB} - m_A m_B.$$

In both cases  $m_A > m_B$  and  $m_A = m_B$ , we have

$$\left[ \sqrt{N_{A,n}} (\hat{m}_{A,n} - m_A), \sqrt{N_{B,n}} (\hat{m}_{B,n} - m_B) \right] \xrightarrow{\text{stably}} \mathcal{N}(0, \Sigma),$$

where

$$\Sigma = \begin{pmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{AB} & \Sigma_{BB} \end{pmatrix}$$

with

$$\Sigma_{AA} = \begin{cases} \sigma_A^2 Q/N & \text{when } m_A > m_B \\ \sigma_A^2 [(1-Z) + ZQ/N] & \text{when } m_A = m_B \end{cases}, \quad \Sigma_{BB} = \begin{cases} \sigma_B^2 & \text{when } m_A > m_B \\ \sigma_B^2 [Z + (1-Z)Q/N] & \text{when } m_A = m_B \end{cases}$$

and

$$\Sigma_{AB} = \begin{cases} 0 & \text{in the case } m_A > m_B \\ c_{AB}(Q/N - 1) & \text{in the case } m_A = m_B. \end{cases}$$

Note that the asymptotic covariance is always zero in the case  $m_A > m_B$  and, in the case  $m_A = m_B$ , it is equal to zero when  $c_{AB} = 0$  or only on the event  $\{Q = N\} = \{N = 1\}$ .

*Proof.* We are going to apply Theorem C.1. To this purpose, we define the 2-dimensional random vector  $\mathbf{T}_{n,k}$  as  $[\mathbf{T}_{n,k}]_1 = \frac{X_k(A_k - m_{A,k})}{\sqrt{n}}$  and  $[\mathbf{T}_{n,k}]_2 = \frac{(N_k - X_k)(B_k - m_{B,k})}{\sqrt{n^{m_B/m_A}}}$ . Moreover, set  $\mathcal{G}_{n,k} = \mathcal{G}_k$ .

We observe that

$$E[X_k(A_k - m_{A,k})|\mathcal{G}_{n,k-1}] = E[X_k(A_k - m_{A,k})|\mathcal{G}_{k-1}] = E[X_k|\mathcal{G}_{k-1}]E[A_k - m_{A,k}] = 0$$

and, similarly,  $E[(N_k - X_k)(B_k - m_{B,k})|\mathcal{G}_{k-1}] = 0$ . Therefore, for any fixed  $n$ ,  $(\mathbf{T}_{n,k})_{1 \leq k \leq n}$  is a martingale difference array with respect to the filtration  $(\mathcal{G}_{n,k})_{k \geq 0}$ , which satisfies condition (c1) of Theorem C.1. Moreover, we have

$$\sum_{k=1}^n \mathbf{T}_{n,k} \mathbf{T}_{n,k}^\top = \begin{pmatrix} \frac{\sum_{k=1}^n X_k^2 (A_k - m_{A,k})^2}{n} & \frac{\sum_{k=1}^n X_k (N_k - X_k) (A_k - m_{A,k}) (B_k - m_{B,k})}{\sum_{k=1}^n \frac{n^{m_B/m_A} n^{1/2(1-m_B/m_A)}}{n^{m_B/m_A}}} \\ \frac{\sum_{k=1}^n X_k (N_k - X_k) (A_k - m_{A,k}) (B_k - m_{B,k})}{\sum_{k=1}^n \frac{n^{m_B/m_A} n^{1/2(1-m_B/m_A)}}{n^{m_B/m_A}}} & \frac{\sum_{k=1}^n (N_k - X_k)^2 (B_k - m_{B,k})^2}{n^{m_B/m_A}} \end{pmatrix}.$$

Now, we set  $\sigma_{A,k}^2 = q_{A,k} - m_{A,k}^2 = E[A_k^2] - E[A_k]^2$  and we observe that

$$\begin{aligned} E[X_k^2 (A_k - m_{A,k})^2 | \mathcal{F}_{k-1}] &= E[X_k^2 | \mathcal{F}_{k-1}] \sigma_{A,k}^2 = \\ \sigma_{A,k}^2 \left\{ E[N_k | \mathcal{F}_{k-1}] Z_{k-1} (1 - Z_{k-1}) \frac{S_{k-1} - E[N_k | \mathcal{F}_{k-1}]}{S_{k-1} - 1} + E[N_k^2 | \mathcal{F}_{k-1}] Z_{k-1}^2 \right\} \\ \xrightarrow{\text{a.s.}} \tilde{\Sigma}_{AA} &= \begin{cases} \sigma_A^2 Q & \text{when } m_A > m_B \\ \sigma_A^2 [Z(1-Z)N + Z^2 Q] & \text{when } m_A = m_B \end{cases} \end{aligned}$$

and so, by Lemma B.1 (applied to  $Y_j = X_j^2 (A_j - m_{A,j})^2$ ,  $\alpha_j = 1$  and  $\beta_n = n$ ), we have

$$\sum_{k=1}^n X_k^2 (A_k - m_{A,k})^2 / n \xrightarrow{\text{a.s.}} \tilde{\Sigma}_{AA}.$$

Similarly, setting  $\sigma_{B,k}^2 = q_{B,k} - m_{B,k}^2 = E[B_k^2] - E[B_k]^2$ , we have

$$\begin{aligned} & k^{1-m_B/m_A} E[(N_k - X_k)^2 (B_k - m_{B,k})^2 | \mathcal{F}_{k-1}] = \\ & \sigma_{B,k}^2 \left\{ E[N_k | \mathcal{F}_{k-1}] k^{1-m_B/m_A} (1 - Z_{k-1}) Z_{k-1} \frac{S_{k-1} - E[N_k | \mathcal{F}_{k-1}]}{S_{k-1} - 1} + E[N_k^2 | \mathcal{F}_{k-1}] k^{1-m_B/m_A} (1 - Z_{k-1})^2 \right\} \\ & \xrightarrow{a.s.} \tilde{\Sigma}_{BB} = \begin{cases} \sigma_B^2 N \tilde{Z} & \text{when } m_A > m_B \\ \sigma_B^2 [Z(1-Z)N + (1-Z)^2 Q] & \text{when } m_A = m_B \end{cases} \end{aligned}$$

and so, by Lemma B.1 (applied to  $Y_j = j^{1-m_B/m_A} (N_j - X_j)^2 (B_j - m_{B,j})^2$ ,  $\alpha_j = j^{1-m_B/m_A}$  and  $\beta_n = n^{m_B/m_A}$ ), we have  $\sum_{k=1}^n (N_k - X_k)^2 (B_k - m_{B,k})^2 / n^{m_B/m_A} \xrightarrow{a.s.} \tilde{\Sigma}_{BB} = \frac{m_A}{m_B} \tilde{\Sigma}_{BB}$ . Finally, looking at the computations reported in the proof of Theorem 3.2, we have

$$\begin{aligned} & k^{1-m_B/m_A} E[X_k (N_k - X_k) (A_k - m_{A,k}) (B_k - m_{B,k}) | \mathcal{F}_{k-1}] = c_{AB,k} k^{1-m_B/m_A} E[X_k (N_k - X_k) | \mathcal{F}_{k-1}] \\ & \xrightarrow{a.s.} \tilde{\Sigma}_{AB} = \begin{cases} c_{AB} Z(1-Z)(Q-N) & \text{in the case } m_A = m_B \\ c_{AB} \tilde{Z}(Q-N) & \text{in the case } m_A > m_B. \end{cases} \end{aligned}$$

Therefore, by Lemma B.1 (applied to  $Y_j = j^{1-m_B/m_A} E[X_j (N_j - X_j) (A_j - m_{A,j}) (B_j - m_{B,j})]$ ,  $\alpha_j = j^{1-m_B/m_A}$  and  $\beta_n = n^{m_B/m_A}$ ), we have

$$\sum_{k=1}^n X_k (N_k - X_k) (A_k - m_{A,k}) (B_k - m_{B,k}) / n^{m_B/m_A} \xrightarrow{a.s.} \frac{m_A}{m_B} \tilde{\Sigma}_{AB}$$

and so, setting  $\tilde{\Sigma}_{AB}$  equal to  $\frac{m_A}{m_B} \tilde{\Sigma}_{AB}$  when  $m_A > m_B$  and zero otherwise, we have

$$\sum_{k=1}^n X_k (N_k - X_k) (A_k - m_{A,k}) (B_k - m_{B,k}) / (n^{m_B/m_A} n^{1/2(1-m_B/m_A)}) \xrightarrow{a.s.} \tilde{\Sigma}_{AB}.$$

Summing up, condition (c2) of Theorem C.1 is satisfied. Regarding the last condition (c3), we observe that  $|\mathbf{T}_{n,k}| = O(1/\sqrt{n}) + O(1/\sqrt{n^{m_B/m_A}}) \xrightarrow{a.s.} 0$ . We are now ready to apply Theorem C.1, that gives the stable convergence of  $\sum_{k=1}^n \mathbf{T}_{n,k}$  toward  $\mathcal{N}(0, \tilde{\Sigma})$ , where

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_{AA} & \tilde{\Sigma}_{AB} \\ \tilde{\Sigma}_{AB} & \tilde{\Sigma}_{BB} \end{pmatrix}.$$

In order to conclude, it is enough to observe that

$$\begin{aligned} \sqrt{N_{A,n}} (\hat{m}_{A,n} - m_A) &= \sqrt{\frac{n}{N_{A,n}}} \frac{\sum_{k=1}^n X_k A_k}{\sqrt{n}} - \sqrt{N_{A,n}} m_A = \\ & \sqrt{\frac{n}{N_{A,n}}} \frac{\sum_{k=1}^n X_k (A_k - m_{A,k})}{\sqrt{n}} + \sqrt{N_{A,n}} \left( \frac{\sum_{k=1}^n X_k m_{A,k}}{N_{A,n}} - m_A \right) = \\ & \sqrt{\frac{n}{N_{A,n}}} \sum_{k=1}^n [\mathbf{T}_{n,k}]_1 + \sqrt{\frac{n}{N_{A,n}}} \frac{\sum_{k=1}^n X_k (m_{A,k} - m_A)}{n}, \end{aligned}$$

and, similarly,

$$\begin{aligned} \sqrt{N_{B,n}}(\widehat{m}_{B,n} - m_B) &= \sqrt{\frac{n^{m_B/m_A} \sum_{k=1}^n (N_k - X_k) B_k}{N_{B,n} \sqrt{n^{m_B/m_A}}} - \sqrt{N_{B,n}} m_B = \\ &= \sqrt{\frac{n^{m_B/m_A} \sum_{k=1}^n (N_k - X_k)(B_k - m_{B,k})}{N_{B,n} \sqrt{n^{m_B/m_A}}} + \sqrt{N_{B,n}} \left( \frac{\sum_{k=1}^n (N_k - X_k) m_{B,k}}{N_{B,n}} - m_B \right) = \\ &= \sqrt{\frac{n^{m_B/m_A} \sum_{k=1}^n [\mathbf{T}_{n,k}]_2}{N_{B,n}}} + \sqrt{\frac{n^{m_B/m_A} \sum_{k=1}^n (N_k - X_k)(m_{B,k} - m_B)}{N_{A,n} n^{m_B/m_A}}}. \end{aligned}$$

Therefore, recalling that  $N_{A,n}/n$  converges almost surely toward  $N$  in the case  $m_A > m_B$  and toward  $NZ$  in the case  $m_A = m_B$  and  $N_{B,n}/n^{m_B/m_A}$  converges almost surely toward  $\frac{m_A}{m_B} N \tilde{Z}$  in the case  $m_A > m_B$  and toward  $N(1 - Z)$  in the case  $m_A = m_B$  and using Lemma B.1 in order to prove that  $\sum_{k=1}^n X_k(m_{A,k} - m_A)/n$  and  $\sum_{k=1}^n (N_k - X_k)(m_{B,k} - m_B)/n^{m_B/m_A}$  converges almost surely toward zero, we can conclude.  $\blacksquare$

**Corollary 3.5.** *With the same notation and assumptions as in Theorem 3.4, set  $\sigma_A^2 = q_A - m_A^2$ ,  $\sigma_B^2 = q_B - m_B^2$  and  $\rho_{AB} = c_{AB}/\sqrt{\sigma_A^2 \sigma_B^2} = (q_{AB} - m_A m_B)/\sqrt{\sigma_A^2 \sigma_B^2}$ . Moreover, assume  $\sigma_A^2 > 0$  and  $\sigma_B^2 > 0$  and define*

$$\lambda_n = \frac{\sigma_A^2 N_{B,n}/n}{\sigma_A^2 N_{B,n}/n + \sigma_B^2 N_{A,n}/n}$$

and

$$\Gamma_n = \begin{cases} 1 + \left(\frac{Q}{N} - 1\right) \lambda_n & \text{if } m_A > m_B \\ \frac{Q}{N} + \left(\frac{Q}{N} - 1\right) \left[ (2Z - 1) \lambda_n + 2\rho_{AB} \sqrt{\lambda_n(1 - \lambda_n)} - Z \right] & \text{if } m_A = m_B. \end{cases}$$

Then, in both cases, we have

$$\zeta_n = \frac{1}{\sqrt{\Gamma_n}} \frac{\widehat{m}_{A,n} - \widehat{m}_{B,n} - (m_A - m_B)}{\sqrt{\sigma_{A,n}^2/N_{A,n} + \sigma_{B,n}^2/N_{B,n}}} \xrightarrow{\text{stably}} \mathcal{N}(0, 1),$$

provided  $\Gamma = a.s. - \lim_n \Gamma_n > 0$  almost surely.

*Proof.* In both cases, we have

$$\lambda_n = \frac{\sigma_A^2 N_{B,n}/n}{\sigma_A^2 N_{B,n}/n + \sigma_B^2 N_{A,n}/n} \xrightarrow{a.s.} \lambda = \frac{\sigma_A^2(1 - Z)}{\sigma_A^2(1 - Z) + \sigma_B^2 Z},$$

where  $Z$  is equal to 1 (and so  $\lambda = 0$ ) in the case  $m_A > m_B$  and it is a non-atomic random variable with values in  $[0, 1]$  in the case  $m_A = m_B$ . Moreover, we have

$$\Gamma_n \xrightarrow{a.s.} \Gamma = \begin{cases} 1 + \left(\frac{Q}{N} - 1\right) \lambda = 1 & \text{if } m_A > m_B \\ \frac{Q}{N} + \left(\frac{Q}{N} - 1\right) \left[ (2Z - 1) \lambda + 2\rho_{AB} \sqrt{\lambda(1 - \lambda)} - Z \right] & \text{if } m_A = m_B, \end{cases}$$

that is

$$\Gamma_n \xrightarrow{a.s.} \Gamma = \frac{\lambda}{\sigma_A^2} \Sigma_{AA} + \frac{1 - \lambda}{\sigma_B^2} \Sigma_{BB} + 2 \frac{\sqrt{\lambda(1 - \lambda)}}{\sqrt{\sigma_A^2 \sigma_B^2}} \Sigma_{AB}.$$

Therefore, in order to conclude it is enough to note that

$$\zeta_n = \frac{1}{\sqrt{\Gamma_n}} \left( \sqrt{\lambda_n} \frac{\sqrt{N_{A,n}}}{\sqrt{\sigma_A^2}} (\hat{m}_{A,n} - m_A) + \sqrt{1 - \lambda_n} \frac{\sqrt{N_{B,n}}}{\sqrt{\sigma_B^2}} (\hat{m}_{B,n} - m_B) \right).$$

Indeed, by the above Theorem 3.4 and the above almost sure convergences, we get that

$$\zeta_n \xrightarrow{\text{stably}} \mathcal{N}(0, 1).$$

■

**Remark 3.6.** Note that, in the case  $m_A = m_B$ , since  $Z$  is a non-atomic random variable with values in  $[0, 1]$ , the limit random variable  $\Gamma$  is equal to 0 with a strictly positive probability if and only if  $P(Q > N) = P(N > 1) > 0$  and, on the event  $\{N > 1\}$ , the random variable  $Q/N$  is a function of  $Z$  such that

$$-\frac{Q}{N} / \left( \frac{Q}{N} - 1 \right) = \left[ (2Z - 1)\lambda + 2\rho_{AB}\sqrt{\lambda(1 - \lambda)} - Z \right],$$

hence, in a very special case.

#### 4. TEST

Set  $M_n = \frac{1}{n} \sum_{j=1}^n X_j/N_j$ , which is the empirical mean of the proportions of balls of color  $A$  in the samples until time-step  $n$ . By Remark 3.6 in [11], we know that  $M_n \xrightarrow{a.s.} Z$ . Moreover, set  $\hat{\sigma}_{A,n}^2 = \hat{q}_{A,n} - \hat{m}_{A,n}^2$ ,  $\hat{\sigma}_{B,n}^2 = \hat{q}_{B,n} - \hat{m}_{B,n}^2$  and  $\hat{\rho}_{AB,n} = (\hat{q}_{AB,n} - \hat{m}_{A,n}\hat{m}_{B,n})/\sqrt{\hat{\sigma}_{A,n}^2\hat{\sigma}_{B,n}^2}$ . The stable convergence stated in Corollary 3.5 still holds true even if we replace all the quantities with their strongly consistent estimators. More precisely, in both cases, assuming  $\sigma_A^2 > 0$ ,  $\sigma_B^2 > 0$  and  $\Gamma = a.s. - \lim_n \Gamma_n > 0$  almost surely, we have

$$\hat{\zeta}_n = \frac{1}{\sqrt{\hat{\Gamma}_n}} \frac{\hat{m}_{A,n} - \hat{m}_{B,n} - (m_A - m_B)}{\sqrt{\hat{\sigma}_{A,n}^2/N_{A,n} + \hat{\sigma}_{B,n}^2/N_{B,n}}} \xrightarrow{\text{stably}} \mathcal{N}(0, 1),$$

where  $\hat{\Gamma}_n$  is defined as  $\Gamma_n$ , but replacing the the random variables  $Z$ ,  $N$  and  $Q$  by  $M_n$ ,  $\hat{\mu}_n$  and  $\hat{q}_{N,n}$ , respectively, and replacing the quantities  $\sigma_A^2$ ,  $\sigma_B^2$  and  $\rho_{AB}$  by their estimators  $\hat{\sigma}_{A,n}^2$ ,  $\hat{\sigma}_{B,n}^2$  and  $\hat{\rho}_{AB,n}$ . Given this fact, a critical region (with asymptotic level  $\theta$ ) for the hypothesis test on the reinforcement means

$$H_0 : m_A = m_B \quad \text{versus} \quad H_1 : m_A > m_B$$

is given by

$$C_\theta = \{\hat{\zeta}_n^0 > q_{1-\theta}\},$$

where  $\hat{\zeta}_n^0$  is defined as the random variable  $\hat{\zeta}_n$  in the case  $m_A = m_B$  and  $q_{1-\theta}$  is the quantile of the standard normal distribution of order  $1 - \theta$ , that is  $\mathcal{N}(0, 1)(q_{1-\theta}, +\infty) = \theta$ .

**Remark 4.1.** (Power of the test)

We have

$$\begin{aligned} \hat{\zeta}_n^0 &= \frac{1}{\sqrt{\hat{\Gamma}_n^0}} \frac{\hat{m}_{A,n} - \hat{m}_{B,n}}{\sqrt{\hat{\sigma}_{A,n}^2/N_{A,n} + \hat{\sigma}_{B,n}^2/N_{B,n}}} \\ &= \sqrt{\frac{\hat{\Gamma}_n^{\neq 0}}{\hat{\Gamma}_n^0}} \frac{1}{\sqrt{\hat{\Gamma}_n^{\neq 0}}} \frac{\hat{m}_{A,n} - \hat{m}_{B,n} - (m_A - m_B)}{\sqrt{\hat{\sigma}_{A,n}^2/N_{A,n} + \hat{\sigma}_{B,n}^2/N_{B,n}}} + \frac{1}{\sqrt{\hat{\Gamma}_n^0}} \frac{m_A - m_B}{\sqrt{\hat{\sigma}_{A,n}^2/N_{A,n} + \hat{\sigma}_{B,n}^2/N_{B,n}}}, \end{aligned}$$

where  $\widehat{\Gamma}_n^0$  (resp.  $\widehat{\Gamma}_n^{\neq 0}$ ) is the random variable defined as  $\Gamma_n$  in the case  $m_A = m_B$  (resp.  $m_A > m_B$ ), but replacing the random variables  $Z$ ,  $N$  and  $Q$  by  $M_n$ ,  $\widehat{\mu}_n$  and  $\widehat{q}_{N,n}$ , respectively, and the quantities  $\sigma_A^2$ ,  $\sigma_B^2$  and  $\rho_{AB}$  by their estimators  $\widehat{\sigma}_{A,n}^2$ ,  $\widehat{\sigma}_{B,n}^2$  and  $\widehat{\rho}_{AB,n}$ . Under the alternative hypothesis (i.e.  $m_A > m_B$ ), we have  $Z = 1$  and so  $\lambda = 0$  and, consequently,  $\widehat{\Gamma}_n^0 \xrightarrow{a.s.} 1$  and  $\widehat{\Gamma}_n^{\neq 0} \xrightarrow{a.s.} 1$  and, hence, by Corollary 3.5, we get

$$\sqrt{\frac{\widehat{\Gamma}_n^{\neq 0}}{\widehat{\Gamma}_n^0}} \frac{1}{\sqrt{\widehat{\Gamma}_n^{\neq 0}}} \frac{\widehat{m}_{A,n} - \widehat{m}_{B,n} - (m_A - m_B)}{\sqrt{\widehat{\sigma}_{A,n}^2/N_{A,n} + \widehat{\sigma}_{B,n}^2/N_{B,n}}} \xrightarrow{\text{stably}} \mathcal{N}(0, 1).$$

Therefore, we can say that, under the alternative hypothesis, we have

$$\widehat{\zeta}_n \stackrel{d}{\approx} \mathcal{N}\left(\frac{1}{\sqrt{\widehat{\Gamma}_n^0}} \frac{m_A - m_B}{\sqrt{\widehat{\sigma}_{A,n}^2/N_{A,n} + \widehat{\sigma}_{B,n}^2/N_{B,n}}}, 1\right).$$

Hence, the power of the test can be approximated by

$$\mathcal{N}\left(\frac{1}{\sqrt{\widehat{\Gamma}_n^0}} \frac{\widehat{m}_{A,n} - \widehat{m}_{B,n}}{\sqrt{\widehat{\sigma}_{A,n}^2/N_{A,n} + \widehat{\sigma}_{B,n}^2/N_{B,n}}}, 1\right) (q_{1-\theta}, +\infty). \quad (15)$$

Note that, under the alternative hypothesis, by Theorem 3.1, we have

$$n^{m_B/m_A} (\widehat{\sigma}_{A,n}^2/N_{A,n} + \widehat{\sigma}_{B,n}^2/N_{B,n}) \xrightarrow{a.s.} \sigma_B^2 \frac{m_B}{m_A} (N\widetilde{Z})^{-1}$$

and so the average value in the above normal distribution goes to  $+\infty$  as  $\sqrt{n^{m_B/m_A}}$ .

**Remark 4.2.** (A very special case)

Note that the very special case when  $N_n = 1$  and  $m_{A,n} = m_A$  and  $m_{B,n} = m_B$  for each  $n$  corresponds to the setting studied in [24].

**Remark 4.3.** (Some possible simplifications)

When  $N$  and  $Q$  are known, we do not need to use their estimators (9). For example, a case is when all the sample sizes  $N_n$  are equal to the same known constant  $\kappa$ . More generally, it might be the case that the generating mechanism of the sample sizes  $N_n$  is decided by the experimenter (for example, this is the case of a response-adaptive design) and so it might be that the random variables  $N$  and  $Q$  can be written explicitly.

**Example**

Take each  $N_n$  independent of  $\mathcal{F}_{n-1}$  and uniformly distributed on  $\{1, \dots, 5\}$ . Moreover, take  $[A_n, B_n]$  such that

$$A_n \stackrel{d}{=} 1 + Y_1 \quad \text{and} \quad B_n \stackrel{d}{=} 1 + Y_2,$$

where  $Y_1$  and  $Y_2$  are, respectively, the first and the second component of a multinomial distribution associated to the parameters: size= 12, probabilities=  $(p_A, p_B, p_3)$  with  $p_A + p_B + p_3 = 1$ . Thus the random variables  $A_n$  and  $B_n$  are negatively correlated. We set  $a = b = 5$ .

Choosing  $p_A = p_B = 4/15$ , the null hypothesis  $m_A = m_B$  holds. Fig. 1 shows a sample path of  $(Z_n)_n$  and of  $(\zeta_n^0)_n$ . The number of iterations is  $n = 20\,000$ .

In order to obtain an empirical power, we make  $\delta = m_A - m_B > 0$  vary (from 0 to 0.1 by 0.005). We keep  $p_B = 4/15$  and the size= 12 and we vary  $p_A$  and  $p_3$  for getting the different values of  $\delta$ . We run the dynamics until time-step  $n = 10\,000$  and we use the approximated power given by (15). Results are sum up in Fig. 2.

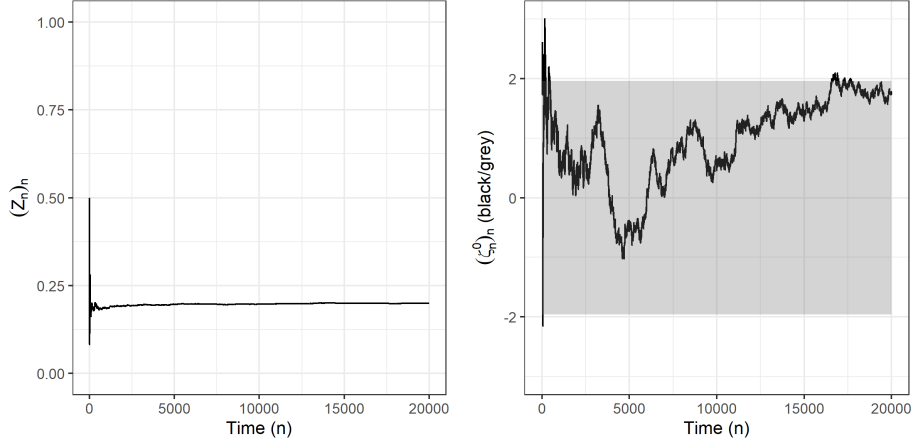


FIGURE 1. Multinomial case under  $H_0$  ( $m_A = m_B$ ), with size= 12,  $p_A = 4/15$ ,  $p_B = 4/15$ . One sample path for  $(Z_n)_n$  and  $(\zeta_n^0)_n$ . The gray region corresponds to a confidence interval of level 5% given by the standard normal distribution.

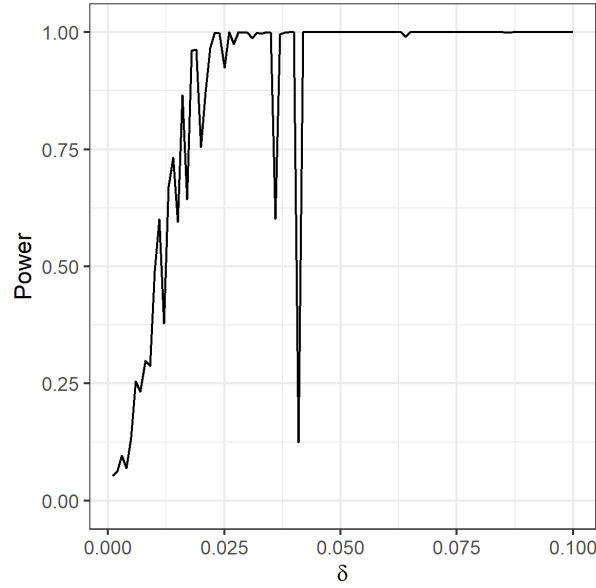


FIGURE 2. Approximated power of the test in the Multinomial case, with size = 12 and  $p_B = 4/15$ . The parameters  $p_A$  and  $p_3$  vary and  $\delta = m_A - m_B > 0$ .

#### APPENDIX A. TECHNICAL RESULTS

**Lemma A.1.** *If  $m_A > m_B$  and condition (7) is satisfied, then, we have  $1/K_n = o(n^{-1/\theta})$  for each  $\theta > m_A/m_B$ .*

*Proof.* Take  $\theta > m_A/m_B$ . We observe that

$$\begin{aligned} E \left[ \frac{H_{n+1}}{K_{n+1}^\theta} - \frac{H_n}{K_n^\theta} \mid \mathcal{H}_n \right] &= E \left[ \frac{H_{n+1}}{K_n^\theta} - \frac{H_n}{K_n^\theta} + \frac{H_{n+1}}{K_{n+1}^\theta} - \frac{H_{n+1}}{K_n^\theta} \mid \mathcal{H}_n \right] = \\ &= \sum_{k \in \mathcal{X}_{n+1}} p_{n+1,k} \left( \frac{H_n + A_{n+1}k}{K_n^\theta} - \frac{H_n}{K_n^\theta} \right) + \sum_{k \in \mathcal{X}_{n+1}} p_{n+1,k} (H_n + A_{n+1}k) \left( \frac{1}{(K_n + B_{n+1}(N_{n+1} - k))^\theta} - \frac{1}{K_n^\theta} \right) = \\ &= \sum_{k \in \mathcal{X}_{n+1}} p_{n+1,k} \frac{A_{n+1}k}{K_n^\theta} + \sum_{k \in \mathcal{X}_{n+1} \setminus \{N_{n+1}\}} p_{n+1,k} (H_n + A_{n+1}k) \left( \frac{1}{(K_n + B_{n+1}(N_{n+1} - k))^\theta} - \frac{1}{K_n^\theta} \right). \end{aligned}$$

Using the Taylor expansion of the function  $f(x) = 1/(c+x)^\theta$  with  $c = K_n$  and  $x = B_{n+1}(N_{n+1} - k)$ , we can choose a constant  $\xi$  such that eventually

$$\left( \frac{1}{(K_n + B_{n+1}(N_{n+1} - k))^\theta} - \frac{1}{K_n^\theta} \right) \leq -\frac{\theta}{K_n^{\theta+1}} \left( B_{n+1}(N_{n+1} - k) - \frac{\xi}{K_n} \right).$$

Therefore the last term of the above equalities is eventually smaller than or equal to

$$\frac{H_n}{K_n^\theta} \left\{ \sum_{k \in \mathcal{X}_{n+1}} \left( \frac{A_{n+1}k}{H_n} - \theta \frac{B_{n+1}(N_{n+1} - k)}{K_n} \right) p_{n+1,k} + \theta \xi \sum_{k \in \mathcal{X}_{n+1} \setminus \{N_{n+1}\}} \frac{(1 + A_{n+1}k/H_n)}{K_n^\theta} p_{n+1,k} \right\}.$$

Now, we observe that we have

$$E \left[ \sum_{k \in \mathcal{X}_{n+1}} \left( \frac{A_{n+1}k}{H_n} - \theta \frac{B_{n+1}(N_{n+1} - k)}{K_n} \right) p_{n+1,k} \mid \mathcal{G}_n \right] = \frac{N_{n+1}}{S_n} (m_{n+1,A} - m_{n+1,B}\theta)$$

and, by Lemma A.1 in [11],

$$E \left[ \sum_{k \in \mathcal{X}_{n+1} \setminus \{N_{n+1}\}} \frac{(1 + A_{n+1}k/H_n)}{K_n^\theta} p_{n+1,k} \mid \mathcal{G}_n \right] \leq \frac{(1 - p_{n+1,N_{n+1}})}{K_n^\theta} + \frac{m_{n+1,A}N_{n+1}}{S_n K_n^\theta} = O(1/(S_n K_n)).$$

Therefore, since  $N_{n+1} \geq 1$ , we have

$$E \left[ \frac{H_{n+1}}{K_{n+1}^\theta} - \frac{H_n}{K_n^\theta} \mid \mathcal{G}_n \right] \leq \frac{H_n}{K_n^\theta} \frac{N_{n+1}}{S_n} [-(m_{n+1,B}\theta - m_{n+1,A}) + O(1/K_n)]$$

and so, for  $\theta > m_A/m_B$ , since  $K_n \uparrow +\infty$  (by Lemma 3.1 in [11]), we can conclude that the above conditional expectation is eventually negative. This fact means that  $H_n/K_n^\theta$  is eventually a positive supermartingale and so it converges almost surely toward a finite random variable. Since  $H_n/S_n$  converges toward 1 almost surely and  $S_n$  converges toward  $m_A N > 0$  almost surely, we have that  $H_n/n$  converges almost surely toward a finite random variable and so also  $n/K_n^\theta$  converges almost surely toward a finite random variable for each  $\theta > m_A/m_B$ . This means that  $n/K_n^\theta$  converges almost surely toward 0 for each  $\theta > m_A/m_B$ , that is  $1/K_n = o(n^{1/\theta})$  for each  $\theta > m_A/m_B$ . ■

**Lemma A.2.** *If we are in case  $m_A > m_B$  and condition (7) is satisfied, then  $(H_n/K_n^{m_A/m_B})_n$  converges almost surely toward a random variable taking values in  $(0, +\infty)$ .*

*Proof.* Set  $L_n = \ln(H_n/K_n^{m_A/m_B})$ . If we prove that  $L_n$  converges almost surely to a finite random variable (see Lemma B.2), then  $Y_n = H_n/K_n^{m_A/m_B}$  converges to a random variable  $Y$  with values in  $(0, +\infty)$ . In order to prove the almost sure convergence of  $(L_n)$ , we are going to apply Lemma B.2. Therefore, we set  $\Delta_n = E[L_{n+1} - L_n \mid \mathcal{G}_n]$  and  $Q_n = E[(L_{n+1} - L_n)^2 \mid \mathcal{G}_n]$ . We recall that,

from Lemma A.1 we know that  $1/K_n = O(1/n^\gamma)$  for some  $\gamma > 0$  and, moreover, we know that  $H_n/n \stackrel{a.s.}{\rightarrow} 1/(m_A N) > 0$  and so  $1/H_n = O(1/n)$ . Using the notation (5), we have

$$\begin{aligned} \Delta_n &= E[\ln(H_{n+1}) - \ln(H_n)|\mathcal{H}_n] - \frac{m_A}{m_B} E[\ln(K_{n+1}) - \ln(K_n)|\mathcal{H}_n] = \\ &= \sum_{k \in \mathcal{X}_{n+1}} \left\{ (\ln(H_n + A_{n+1}k) - \ln(H_n)) - \frac{m_A}{m_B} (\ln(K_n + B_{n+1}(N_{n+1} - k)) - \ln(K_n)) \right\} p_{n+1,k} = \\ &= \sum_{k \in \mathcal{X}_{n+1}} \left\{ \int_0^{A_{n+1}k} \frac{1}{H_n + t} dt - \frac{m_A}{m_B} \int_0^{B_{n+1}(N_{n+1} - k)} \frac{1}{K_n + t} dt \right\} p_{n+1,k}. \end{aligned}$$

Since  $1/(H_n + t) \leq 1/H_n$  and  $1/(K_n + t) \geq 1/K_n - t/K_n^2$  for each  $t \geq 0$  and each  $n$ , the last term of the above equalities is eventually smaller than or equal to

$$\sum_{k \in \mathcal{X}_{n+1}} \left\{ \frac{A_{n+1}k}{H_n} - \frac{m_A}{m_B} \frac{B_{n+1}(N_{n+1} - k)}{K_n} + \frac{m_A}{m_B} \frac{B_{n+1}^2(N_{n+1} - k)^2}{2K_n^2} \right\} p_{n+1,k}.$$

Now, we observe that

$$\begin{aligned} E\left[ \sum_{k \in \mathcal{X}_{n+1}} \left( \frac{A_{n+1}k}{H_n} - \frac{m_A}{m_B} \frac{B_{n+1}(N_{n+1} - k)}{K_n} \right) p_{n+1,k} \mid \mathcal{G}_n \right] &= \frac{m_{n+1,A} N_{n+1} H_n}{H_n S_n} - \frac{m_A m_{B,n+1}}{m_B} \frac{N_{n+1} K_n}{K_n S_n} = \\ &= \frac{N_{n+1}}{S_n} \left( m_{A,n+1} - \frac{m_A}{m_B} m_{B,n+1} \right) = O(|m_{A,n+1} m_B - m_A m_{B,n+1}|/n). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \Delta_n &\leq O(|m_{A,n+1} m_B - m_A m_{B,n+1}|/n) + \frac{C^3}{2K_n^2} \sum_{k \in \mathcal{X}_{n+1}} (N_{n+1} - k) p_{n+1,k} \\ &= O(|m_{A,n+1} m_B - m_A m_{B,n+1}|/n) + \frac{C^3}{2K_n^2} N_{n+1} \frac{K_n}{S_n} = O(|m_{A,n+1} m_B - m_A m_{B,n+1}|/n) + O(1/(K_n S_n)) \\ &= O(|m_{A,n+1} m_B - m_A m_{B,n+1}|/n) + O(1/n^{1+\gamma}). \end{aligned}$$

Finally, we note that  $-\Delta_n = \ln(K_{n+1}^{m_B/m_A}/H_{n+1}) - \ln(K_n^{m_B/m_A}/H_n)$  and so, with the same arguments as before, we get  $-\Delta_n \leq O(|m_{A,n+1} m_B - m_A m_{B,n+1}|/n) + O(1/n^{1+\gamma})$ . Thus,  $\sum_n |\Delta_n| < +\infty$  almost surely. Similarly, we have

$$\begin{aligned} E[(\ln(H_{n+1}) - \ln(H_n)) - \frac{m_A}{m_B} \ln(K_{n+1}) + \ln(K_n))^2 | \mathcal{H}_n] &\leq \\ &= 2 \left\{ E[(\ln(H_{n+1}) - \ln(H_n))^2 | \mathcal{H}_n] + \frac{m_A^2}{m_B^2} E[(\ln(K_{n+1}) - \ln(K_n))^2 | \mathcal{H}_n] \right\} \leq \\ &= 2 \sum_{k \in \mathcal{X}_{n+1}} \left( \frac{A_{n+1}^2 k^2}{H_n^2} + \frac{m_A^2}{m_B^2} \frac{B_{n+1}^2 (N_{n+1} - k)^2}{K_n^2} \right) p_{n+1,k} \leq \\ &= 2C^3 \sum_{k \in \mathcal{X}_{n+1}} \left( \frac{k}{H_n^2} + \frac{m_A^2}{m_B^2} \frac{(N_{n+1} - k)}{K_n^2} \right) p_{n+1,k} \leq 2C^4 \left( \frac{1}{H_n S_n} + \frac{m_A^2}{m_B^2} \frac{1}{K_n S_n} \right). \end{aligned}$$

Therefore, we get  $Q_n = O(1/n^{1+\gamma})$  and so  $\sum_n Q_n < +\infty$  almost surely. ■



## APPENDIX B. SOME AUXILIARY RESULTS

We recall two known technical results:

**Lemma B.1.** (Lemma 4.1 in [10])

Let  $(\alpha_j)$  and  $(\beta_j)_j$  be two sequences of strictly positive numbers with

$$\beta_n \uparrow +\infty \quad \text{and} \quad \frac{1}{\beta_n} \sum_{j=1}^n \frac{1}{\alpha_j} \longrightarrow \gamma.$$

Let  $(Y_n)$  be a sequence of real random variables, adapted to a filtration  $\mathcal{F}$ . If  $\sum_{j \geq 1} E[Y_j^2]/(\alpha_j \beta_j)^2 < +\infty$  and  $E[Y_j | \mathcal{F}_{j-1}] \xrightarrow{a.s.} Y$  for some real random variable  $Y$ , then

$$\frac{1}{\beta_n} \sum_{j=1}^n \frac{Y_j}{\alpha_j} \xrightarrow{a.s.} \gamma Y.$$

**Lemma B.2.** (Lemma 3.2 in [25])

Let  $(L_n)$  be a sequence of random variables, adapted to a filtration  $\mathcal{G}_n$ . Set  $\Delta_n = E[L_{n+1} - L_n | \mathcal{G}_n]$  and  $Q_n = E[(L_{n+1} - L_n)^2 | \mathcal{G}_n]$ . If  $\sum_n \Delta_n$  and  $\sum_n Q_n$  are almost surely convergent, then  $(L_n)$  converges almost surely to a finite random variable.

## APPENDIX C. STABLE CONVERGENCE

This brief appendix contains some basic definitions and results concerning stable convergence. For more details, we refer the reader to [9, 15] and the references therein.

Let  $(\Omega, \mathcal{A}, P)$  be a probability space, and let  $S$  be a Polish space, endowed with its Borel  $\sigma$ -field. A *kernel* on  $S$ , or a random probability measure on  $S$ , is a collection  $K = \{K(\omega) : \omega \in \Omega\}$  of probability measures on the Borel  $\sigma$ -field of  $S$  such that, for each bounded Borel real function  $f$  on  $S$ , the map

$$\omega \mapsto Kf(\omega) = \int f(x) K(\omega)(dx)$$

is  $\mathcal{A}$ -measurable. Given a sub- $\sigma$ -field  $\mathcal{H}$  of  $\mathcal{A}$ , a kernel  $K$  is said  $\mathcal{H}$ -measurable if all the above random variables  $Kf$  are  $\mathcal{H}$ -measurable.

On  $(\Omega, \mathcal{A}, P)$ , let  $(Y_n)_n$  be a sequence of  $S$ -valued random variables, let  $\mathcal{H}$  be a sub- $\sigma$ -field of  $\mathcal{A}$ , and let  $K$  be a  $\mathcal{H}$ -measurable kernel on  $S$ . Then we say that  $Y_n$  converges  $\mathcal{H}$ -stably to  $K$ , and we write  $Y_n \longrightarrow K$   $\mathcal{H}$ -stably, if

$$P(Y_n \in \cdot | H) \xrightarrow{weakly} E[K(\cdot) | H] \quad \text{for all } H \in \mathcal{H} \text{ with } P(H) > 0,$$

where  $K(\cdot)$  denotes the random variable defined, for each Borel set  $B$  of  $S$ , as  $\omega \mapsto KI_B(\omega) = K(\omega)(B)$ . In the case when  $\mathcal{H} = \mathcal{A}$ , we simply say that  $Y_n$  converges *stably* to  $K$  and we write  $Y_n \longrightarrow K$  stably. Clearly, if  $Y_n \longrightarrow K$   $\mathcal{H}$ -stably, then  $Y_n$  converges in distribution to the probability distribution  $E[K(\cdot)]$ . Moreover, the  $\mathcal{H}$ -stable convergence of  $Y_n$  to  $K$  can be stated in terms of the following convergence of conditional expectations:

$$E[f(Y_n) | \mathcal{H}] \xrightarrow{\sigma(L^1, L^\infty)} Kf \tag{16}$$

for each bounded continuous real function  $f$  on  $S$ .

We denote by  $\mathcal{N}(\mu, C)$  the multivariate Gaussian probability distribution with vector of mean values  $\mu$  and covariance matrix  $C$ . Therefore, when  $\Sigma$  is a random positive semidefinite matrix,

the symbol  $\mathcal{N}(0, \Sigma)$  denotes the Gaussian kernel  $\{\mathcal{N}(0, \Sigma(\omega)) : \omega \in \Omega\}$ .

From Proposition 3.1 in [12], we can get the following result.

**Theorem C.1.** *Let  $(\mathbf{T}_{n,k})_{n \geq 1, 1 \leq k \leq k_n}$  be a triangular array of  $d$ -dimensional real random vectors, such that, for each fixed  $n$ , the finite sequence  $(\mathbf{T}_{n,k})_{1 \leq k \leq k_n}$  is a martingale difference array with respect to a given filtration  $(\mathcal{G}_{n,k})_{k \geq 0}$ . Moreover, let  $(t_n)_n$  be a sequence of real numbers and assume that the following conditions hold:*

- (c1)  $\mathcal{G}_{n,k} \subseteq \mathcal{G}_{n+1,k}$  for each  $n$  and  $1 \leq k \leq k_n$ ;
- (c2)  $\sum_{k=1}^{k_n} (t_n \mathbf{T}_{n,k})(t_n \mathbf{T}_{n,k})^\top = t_n^2 \sum_{k=1}^{k_n} \mathbf{T}_{n,k} \mathbf{T}_{n,k}^\top \xrightarrow{P} \Sigma$ , where  $\Sigma$  is a random positive semidefinite matrix;
- (c3)  $\sup_{1 \leq k \leq k_n} |t_n \mathbf{T}_{n,k}| \xrightarrow{L^1} 0$ .

Then  $t_n \sum_{k=1}^{k_n} \mathbf{T}_{n,k}$  converges stably to the Gaussian kernel  $\mathcal{N}(\mathbf{0}, \Sigma)$ .

### Acknowledgments

Irene Crimaldi and Ida Minelli are members of the Italian Group “Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni” of the Italian Institute “Istituto Nazionale di Alta Matematica”.

### Funding Sources

Irene Crimaldi is partially supported by the Italian “Programma di Attività Integrata” (PAI), project “TOol for Fighting FakeS” (TOFFE) funded by IMT School for Advanced Studies Lucca.

### Declaration

All the authors equally contributed to this work.

### REFERENCES

- [1] R. Aguech, N. Lasmar, and O. Selmi. A generalized urn with multiple drawing and random addition. *Annals of the Institute of Statistical Mathematics*, 71(2):389–408, Apr. 2019.
- [2] R. Aguech and O. Selmi. Unbalanced multi-drawing urn with random addition matrix. *Arab Journal of Mathematical Sciences*, 2019.
- [3] D. A. Aoudia and F. Perron. A new randomized Pólya urn model. *Appl. Math.*, 3:2118–2122, 2012.
- [4] M.-R. Chen. A time dependent Pólya urn with multiple drawings. *Probab. Eng. Informational Sci.*, 34(4):469–483, 2020.
- [5] M.-R. Chen and M. Kuba. On generalized Pólya urn models. *J. Appl. Prob.*, 50:1169–1186, 2013.
- [6] M.-R. Chen and C.-Z. Wei. A new urn model. *Journal of Applied Probability*, 42(4):964–976, Dec. 2005.
- [7] I. Crimaldi. Central limit theorems for a hypergeometric randomly reinforced urn. *J. Appl. Prob.*, 53(3):899–913, 2016.
- [8] I. Crimaldi. Central limit theorems for a hypergeometric randomly reinforced urn. *Journal of Applied Probability*, 53(3):899–913, 2016.
- [9] I. Crimaldi. *Introduzione alla nozione di convergenza stabile e sue varianti (Introduction to the notion of stable convergence and its variants)*, volume 57. Unione Matematica Italiana, Monograf s.r.l., Bologna, Italy., 2016. Book written in Italian.
- [10] I. Crimaldi, P. Dai Pra, and I. G. Minelli. Fluctuation theorems for synchronization of interacting Pólya’s urns. *Stochastic Process. Appl.*, 126(3):930–947, 2016.
- [11] I. Crimaldi, P.-Y. Louis, and I. G. Minelli. An urn model with random multiple drawing and random addition. *Stochastic Processes and their Applications*, 147(C):270–299, 2022.
- [12] I. Crimaldi and L. Pratelli. Convergence results for multivariate martingales. *Stochastic Processes and their Applications*, 115(4):571–577, 2005.
- [13] F. Eggenberger and G. Pólya. Über die Statistik verketteter Vorgänge. *Z. Angewandte Math. Mech.*, 3:279–289, 1923.
- [14] B. Friedman. A simple urn model. *Communications on Pure and Applied Mathematics*, 2(1):59–70, 1949.

- [15] P. Hall and C. C. Heyde. *Martingale limit theory and its application*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. Probability and Mathematical Statistics.
- [16] I. Higuera, J. Moler, F. Plo, and M. San Miguel. Central limit theorems for generalized Pólya urn models. *Journal of Applied Probability*, 43(4):938–951, 2006.
- [17] F. Hu and W. F. Rosenberger. *The Theory of Response-Adaptive Randomization in Clinical Trials*. Wiley, Hoboken, 2006.
- [18] S. Idriss and N. Lasmar. Limit Theorems for Stochastic Approximations Algorithms With Application to General Urn Models. Hal-01726014, 2018.
- [19] N. Johnson, S. Kotz, and H. Mahmoud. Pólya-Type Urn Models with Multiple Drawings. *J. Iran. Stat. Soc.*, 3(2):165–173, 2004.
- [20] M. Kuba. Classification of urn models with multiple drawings. Preprint Arxiv 1612.04354, 2016.
- [21] M. Kuba and H. M. Mahmoud. Two-color balanced affine urn models with multiple drawings. *Adv. in Appl. Math.*, 90:1–26, Sept. 2017.
- [22] N. Lasmar, C. Mailler, and O. Selmi. Multiple drawing multi-colour urns by stochastic approximation. *J. Appl. Probab.*, 55(1):254–281, 2018.
- [23] H. M. Mahmoud. Drawing multisets of balls from tenable balanced linear urns. *Probability in the Engineering and Informational Sciences*, 27(2):147–162, 2013.
- [24] C. May and N. Flournoy. Asymptotics in response-adaptive designs generated by a two-color, randomly reinforced urn. *Ann. Statist.*, 37(2):1058–1078, 04 2009.
- [25] R. Pemantle and S. Volkov. Vertex-reinforced random walk on  $\mathbf{Z}$  has finite range. *Ann. Probab.*, 27(3):1368–1388, July 1999.
- [26] W. F. Rosenberger. Randomized urn models and sequential design. *Sequential Anal.*, 21:1–28, 2002.
- [27] W. F. Rosenberger and J. M. Lachin. *Randomization in Clinical Trials*. Wiley, New York, 2002.