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► **To cite this version:**

Miroslav Ploščica, Friedrich Wehrung. Spectral subspaces of spectra of Abelian lattice-ordered groups in size aleph one. *Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum*, 2023, 89 (3-4), pp.339–356. 10.1007/s44146-023-00080-z . hal-03696927

**HAL Id: hal-03696927**

**<https://hal.science/hal-03696927>**

Submitted on 16 Jun 2022

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# SPECTRAL SUBSPACES OF SPECTRA OF ABELIAN LATTICE-ORDERED GROUPS IN SIZE ALEPH ONE

MIROSLAV PLOŠČICA AND FRIEDRICH WEHRUNG

ABSTRACT. It is well known that the lattice  $\text{Id}_c G$  of all principal  $\ell$ -ideals of any Abelian  $\ell$ -group  $G$  is a completely normal distributive 0-lattice, and that not every completely normal distributive 0-lattice is a homomorphic image of some  $\text{Id}_c G$ , *via* a counterexample of cardinality  $\aleph_2$ . We prove that every completely normal distributive 0-lattice with at most  $\aleph_1$  elements is a homomorphic image of some  $\text{Id}_c G$ . By Stone duality, this means that every completely normal generalized spectral space, with at most  $\aleph_1$  compact open sets, is homeomorphic to a spectral subspace of the  $\ell$ -spectrum of some Abelian  $\ell$ -group.

## 1. INTRODUCTION

A subset  $I$ , in a lattice-ordered group (in short  $\ell$ -group)  $G$ , is an  $\ell$ -ideal if it is an order-convex normal subgroup closed under the lattice operations. If  $I \neq G$ , we say in addition that  $I$  is *prime* if  $x \wedge y \in I$  implies that  $\{x, y\} \cap I \neq \emptyset$ , whenever  $x, y \in G$ . In case  $G$  is *Abelian*, the  $\ell$ -spectrum of  $G$  is defined as the set  $\text{Spec } G$  of all prime  $\ell$ -ideals of  $G$ , endowed with the topology whose closed subsets are the  $\{P \in \text{Spec } G \mid X \subseteq P\}$  for  $X \subseteq G$  (often called the *hull-kernel topology*). Denote by  $\mathcal{G}$  the class of all Abelian  $\ell$ -groups.

The problem of the description of  $\ell$ -spectra of all Abelian  $\ell$ -groups (say the  $\ell$ -spectrum problem) is stated, in the language of MV-algebras, in Mundici [10, Problem 2]. Now under Stone duality (cf. Grätzer [6, § II.5], Johnstone [8, § II.3], Rump and Yang [12] for the case without top element, and Wehrung [17, § 2.2] for a summary), for any  $G \in \mathcal{G}$ ,  $\text{Spec } G$  corresponds to the lattice  $\text{Id}_c G$  of all principal  $\ell$ -ideals of  $G$ ; that is,  $\text{Id}_c G = \{\langle a \rangle \mid a \in G^+\}$  where each  $\langle a \rangle \stackrel{\text{def}}{=} \{x \in G \mid (\exists n \in \mathbb{N})(|x| \leq na)\}$ . This enables us to restate the  $\ell$ -spectrum problem as the description problem of the class  $\text{Id}_c \mathcal{G} \stackrel{\text{def}}{=} \{D \mid (\exists G \in \mathcal{G})(D \cong \text{Id}_c G)\}$ . All such lattices are clearly distributive with smallest element (usually denoted by 0). They are also *completely normal* (cf. Bigard, Keimel, and Wolfenstein [3, Ch. 10]), that is, they satisfy the statement

$$(\forall a, b)(\exists x, y)(a \vee b = a \vee y = x \vee b \text{ and } x \wedge y = 0).$$

Delzell and Madden observed in [4, Theorem 2], *via* a counterexample of cardinality  $\aleph_1$ , that those properties are not sufficient to characterize  $\text{Id}_c \mathcal{G}$ . On the

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*Date:* June 16, 2022.

2010 *Mathematics Subject Classification.* 06D05; 06D20; 06D35; 06D50; 06F20; 46A55; 52A05; 52C35.

*Key words and phrases.* Lattice-ordered; Abelian; group; vector lattice; ideal; homomorphic image; completely normal; distributive; lattice; countable; tree; relatively complete; join-irreducible; Heyting algebra; closed map; consonance; spectrum.

other hand, the second author proved in [14] that every *countable* completely normal distributive 0-lattice belongs to  $\text{Id}_c \mathcal{G}$ . The categorical concept of *condensate*, initiated in the second author's work [5] with Pierre Gillibert, together with the main result of [15], enabled the second author to prove in [16] that  $\text{Id}_c \mathcal{G}$  is not the class of models of any class of  $\mathcal{L}_{\infty\lambda}$  sentences of lattice theory, for any infinite cardinal  $\lambda$ . Using further tools from infinitary logic, the second author extended those results in [19] by proving that  $\text{Id}_c \mathcal{G}$  is not the *complement of a projective class* over  $\mathcal{L}_{\infty\infty}$ , thus verifying in particular that the additional property of all lattices  $\text{Id}_c G$  coined by the first author in his proof of [11, Theorem 2.1] is still not sufficient to characterize  $\text{Id}_c \mathcal{G}$ .

As observed in the above-cited references, all those results extend to the class of all (lattice) *homomorphic images* of lattices  $\text{Id}_c G$ . On the other hand, not every homomorphic image of a lattice of the form  $\text{Id}_c G$  belongs to  $\text{Id}_c \mathcal{G}$  (cf. Wehrung [14, Example 10.6]). Recast in terms of spectra, *via* Stone duality, this means that *not every spectral subspace of an  $\ell$ -spectrum is an  $\ell$ -spectrum*.

Moreover, not every completely normal bounded distributive lattice is a homomorphic image of some  $\text{Id}_c G$ : a counterexample of cardinality  $\aleph_2$  is constructed in Wehrung [15].

In this paper we complete the picture above, by establishing that *every completely normal distributive 0-lattice  $D$ , with at most  $\aleph_1$  elements, is a homomorphic image of  $\text{Id}_c G$  for some Abelian  $\ell$ -group  $G$* . This also strengthens the first author's result, obtained in [11], that  $D$  is *Cevian*. In fact, we verify the slightly more general statement that  $G$  may be taken a vector lattice over any given countable totally ordered division ring  $\mathbb{k}$  (cf. Theorem 7.4), modulo the obvious change in the definition of an  $\ell$ -ideal (i.e.,  $\ell$ -ideals need to be closed under scalar multiplication by elements of  $\mathbb{k}$ ; see Wehrung [17, § 2.3] for more detail). Due to the results of [17, § 9], the countability assumption on  $\mathbb{k}$  cannot be dispensed with.

Our argument will roughly follow the one from Wehrung [14], with the “Main Extension Lemma” [14, Lemma 4.2] strengthened from finite lattices to certain infinite lattices, and streamlined *via* the introduction of *consonance kernels* (cf. Definition 3.1), as Lemma 4.4. The proof of the “closure step” [14, Lemma 7.2] fails in that more general context, so we get only “homomorphic image” as opposed to “isomorphic copy”, of  $\text{Id}_c G$ . This will also require a few known additional properties of finite distributive lattices and their homomorphisms, *via* Birkhoff duality (see in particular Lemma 2.4). Our final argument, given a completely normal distributive 0-lattice  $L$ , will start by expressing  $L$  as a directed union of an ascending  $\omega_1$ -sequence  $\vec{L} = (L_\xi \mid \xi < \omega_1)$  of countable completely normal distributive 0-lattices, and then, with the help of Lemma 4.4, iteratively lift all subdiagrams  $(L_\xi \mid \xi < \alpha)$ , with  $\alpha < \omega_1$ , with respect to the functor  $\text{Id}_c$ . That part of our argument turns out to be valid not only for the chain  $\omega_1$  but for any tree in which every element has countable height (cf. Theorem 7.3).

## 2. BASIC CONCEPTS

**2.1. Sets, posets, lattices.** For any set  $X$ ,  $\text{Pow } X$  denotes the powerset algebra of  $X$ . By “countable” we will mean “at most countable”. For an element  $a$  in a partially ordered set (from now on *poset*)  $P$ , we set  $P \downarrow a \stackrel{\text{def}}{=} \{p \in P \mid p \leq a\}$  (or  $\downarrow a$  if  $P$  is understood). A subset  $A$  of  $P$  is a *lower subset* of  $P$  if  $P \downarrow a \in A$  whenever

$a \in A$ . A poset  $P$  with bottom element is a *tree* if  $P \downarrow a$  is well-ordered under the induced order whenever  $a \in P$ .

For a subset  $P$  in a poset  $Q$  and for  $x \in Q$ ,  $x^P$  (resp.,  $x_P$ ) denotes the least  $y \in P$  such that  $x \leq y$  (resp., the largest  $y \in P$  such that  $y \leq x$ ) if it exists. We say that  $P$  is *relatively complete in  $Q$*  if  $x^P$  and  $x_P$  both exist for all  $x \in P$ . If  $P$  is a subalgebra of a Boolean algebra  $Q$ , it suffices to verify that  $x^P$  exists whenever  $x \in Q$  (resp.,  $x_P$  exists whenever  $x \in Q$ ).

Relative completeness has been used in a description of projective Boolean algebras. For the proof of the following (easy) assertion see Heindorf and Shapiro [7, Lemma 1.2.7].

**Lemma 2.1.** *Let  $A, A'$  be subalgebras of a Boolean algebra  $B$  with  $A'$  finitely generated over  $A$ . If  $A$  is relatively complete in  $B$ , then so is  $A'$ .*

For posets  $P$  and  $Q$  with respective top elements  $\top_P$  and  $\top_Q$ , a map  $f: P \rightarrow Q$  is *top-faithful* if  $f^{-1}\{\top_Q\} = \{\top_P\}$ . For any poset  $P$ ,  $P^{\sqcup\infty}$  denotes the poset obtained by adding an extra element, usually denoted by  $\infty$ , atop of  $P$ . For any map  $f: P \rightarrow Q$ , we denote by  $f^{\sqcup\infty}: P^{\sqcup\infty} \rightarrow Q^{\sqcup\infty}$  the unique extension of  $f$  sending  $\infty$  to  $\infty$ . Such maps are exactly the top-faithful maps from  $P^{\sqcup\infty}$  to  $Q^{\sqcup\infty}$ .

We denote by  $\text{Ji } L$  (resp.,  $\text{Mi } L$ ) the set of all join-irreducible (resp., meet-irreducible) elements in a lattice  $L$ , endowed with the induced ordering. For any join-irreducible element  $p$  in a finite distributive lattice  $D$ , we denote by  $p_*$  the unique lower cover of  $p$  in  $D$ , and by  $p^\dagger$  the largest element of  $D$  not above  $p$ ; so  $p_* = p \wedge p^\dagger$ . The assignment  $p \mapsto p^\dagger$  defines an order-isomorphism from  $\text{Ji } D$  onto  $\text{Mi } D$ .

As in Wehrung [14, 17], two elements  $a$  and  $b$  in a 0-lattice (i.e., lattice with a bottom element)  $D$  are *consonant* if there exist  $u, v \in D$  such that  $a \leq u \vee b$ ,  $b \leq a \vee v$ , and  $u \wedge v = 0$ . A subset  $X$  of  $D$  is consonant if any pair of elements in  $X$  is consonant. The lattice  $D$  is *completely normal* if it is consonant within itself.

We denote by  $\text{Ji } L$  (resp.,  $\text{Mi } L$ ) the set of all join-irreducible (resp., meet-irreducible) elements in a lattice  $L$ , endowed with the induced partial ordering. The assignment  $D \mapsto \text{Ji } D$  is part of *Birkhoff duality* between finite distributive lattices, with 0, 1-lattice homomorphisms, and finite posets, with isotone maps (cf. Grätzer [6, § II.1.3]). The Birkhoff dual of a 0, 1-lattice homomorphism  $\varphi: D \rightarrow E$  is the map  $\text{Ji } E \rightarrow \text{Ji } D$ ,  $q \mapsto q^\varphi \stackrel{\text{def}}{=} \min \{x \in D \mid q \leq \varphi(x)\}$ .

For any distributive 0-lattice  $D$ , we denote by  $\text{BR}(D)$  the generalized Boolean algebra  *$R$ -generated by  $D$*  in the sense of Grätzer [6, § II.4] (aka the *Boolean envelope* of  $D$ ). Equivalently,  $\text{BR}(D)$  is the universal generalized Boolean algebra of  $D$ . Up to isomorphism,  $\text{BR}(D)$  is the unique generalized Boolean algebra generated by  $D$  as a 0-sublattice. The assignment  $D \mapsto \text{BR}(D)$  canonically extends to a functor, which turns 0-lattice embeddings to embeddings of generalized Boolean algebras. For a 0-sublattice  $D$  of a distributive lattice  $E$  with 0, we will thus identify  $\text{BR}(D)$  with its canonical image in  $\text{BR}(E)$ . If  $D$  is a finite distributive lattice and  $P \stackrel{\text{def}}{=} \text{Ji } D$ , then the assignment  $x \mapsto P \downarrow x$  defines an isomorphism from  $D$  onto the lattice  $\text{Down } P$  of all lower subsets of  $P$ . Since the universal Boolean algebra of  $\text{Down } P$  is the powerset lattice of  $P$ , with each  $\{p\} = (\downarrow p) \setminus (\downarrow p)_*$ , it follows that the atoms of  $\text{BR}(D)$  are exactly the  $p \wedge \neg p_*$  for  $p \in \text{Ji } D$ .

**Lemma 2.2.** *The following statements hold, for any distributive 0-lattice  $D$ :*

- (1) For all  $a_1, a_2, b_1, b_2 \in D$ ,  $a_1 \wedge \neg b_1 \leq a_2 \wedge \neg b_2$  within  $\text{BR}(D)$  iff  $a_1 \leq a_2 \vee b_1$  and  $a_1 \wedge b_2 \leq b_1$  within  $D$ .
- (2) If  $D$  is finite, then  $a \wedge \neg b = \bigvee \{p \wedge \neg p_* \mid p \in \text{Ji } D, p \leq a, p \not\leq b\}$  within  $\text{BR}(D)$ , whenever  $a, b \in D$ .

**Lemma 2.3.** *Let  $D$  and  $L$  be distributive 0-lattices with  $D$  finite, let  $\varphi: D \rightarrow L$  be a 0-lattice homomorphism, let  $a, b \in D$ , and let  $c \in L$ . Then  $\varphi(a) \leq \varphi(b) \vee c$  iff  $\varphi(p) \leq \varphi(p_*) \vee c$  whenever  $p \in \text{Ji } D$  with  $p \leq a$  and  $p \not\leq b$ .*

*Proof.*  $\varphi(a) \leq \varphi(b) \vee c$  iff  $\text{BR}(\varphi)(a \wedge \neg b) \leq c$ , iff  $\text{BR}(\varphi)(p \wedge \neg p_*) \leq c$  whenever  $p \in \text{Ji } D$  such that  $p \leq a$  and  $p \not\leq b$  (we apply Lemma 2.2(2)). Now  $\text{BR}(\varphi)(p \wedge \neg p_*) \leq c$  iff  $\varphi(p) \leq \varphi(p_*) \vee c$ .  $\square$

For any elements  $x$  and  $y$  in a lattice  $E$  let  $x \rightarrow_E y$  denote the largest  $z \in E$ , if it exists, such that  $x \wedge z \leq y$  (it is also called the *pseudocomplement of  $x$  relative to  $y$* ); so  $\rightarrow_E$  is the *Heyting implication* on  $E$ . If  $\rightarrow_E$  is defined on every pair of elements then we say that  $E$  is a *generalized Heyting algebra*. If, in addition,  $E$  has a bottom element, then we say that  $E$  is a *Heyting algebra*. Every Heyting algebra is a bounded distributive lattice, and every finite distributive lattice is a Heyting algebra<sup>1</sup>.

Dually, we denote by  $x \searrow_E y$  the least  $z \in E$ , if it exists, such that  $x \leq y \vee z$ . It is the *dual pseudocomplement of  $x$  relative to  $y$* .

A lattice homomorphism  $\varphi: D \rightarrow E$  is *closed* if whenever  $a_0, a_1 \in D$  and  $b \in E$ , if  $\varphi(a_0) \leq \varphi(a_1) \vee b$ , then there exists  $x \in D$  such that  $a_0 \leq a_1 \vee x$  and  $\varphi(x) \leq b$ . If  $\varphi$  is an inclusion map we will say that  $D$  is a *closed sublattice* of  $E$ .

The following folklore lemma, whose easy proof we leave to the reader as an exercise, enables to read, on the Birkhoff dual, whether a given homomorphism, between finite distributive lattices, is a homomorphism of Heyting algebras or a closed homomorphism, respectively.

**Lemma 2.4.** *The following statements hold, for any finite distributive lattices  $D$  and  $E$  and any 0, 1-lattice homomorphism  $\varphi: D \rightarrow E$ :*

- (1)  $\varphi$  is a homomorphism of Heyting algebras iff for all  $p \in \text{Ji } D$  and all  $q \in \text{Ji } E$ , if  $p \leq q^\varphi$ , then there exists  $x \in \text{Ji } E$  such that  $x \leq q$  and  $x^\varphi = p$ .
- (2)  $\varphi$  is closed iff for all  $p \in \text{Ji } D$  and all  $q \in \text{Ji } E$ , if  $q^\varphi \leq p$ , then there exists  $x \in \text{Ji } E$  such that  $q \leq x$  and  $x^\varphi = p$ .

**2.2. The lattices  $\text{Bool}(\mathcal{F}, \Omega)$ ,  $\text{Op}(\mathcal{F}, \Omega)$ , and  $\text{Op}^-(\mathcal{F}, \Omega)$ .** For more detail on this subsection we refer the reader to Wehrung [14, 17]. For a right vector space  $\mathbb{E}$  over a totally ordered division ring  $\mathbb{k}$ , a map  $f: \mathbb{E} \rightarrow \mathbb{k}$  is an *affine functional* if  $f - f(0)$  is a linear functional. Note that the affine functionals on  $\mathbb{E}$  form a *left* vector space over  $\mathbb{k}$ .

For functions  $f$  and  $g$  with common domain  $\Omega$  and values in a poset  $T$ , we set  $\llbracket f \leq g \rrbracket \stackrel{\text{def}}{=} \{x \in \Omega \mid f(x) \leq g(x)\}$ ; and similarly for  $\llbracket f < g \rrbracket$ ,  $\llbracket f = g \rrbracket$ ,  $\llbracket f \neq g \rrbracket$ , and so on. Throughout this paper,  $f$  and  $g$  will always be restrictions, to a convex set  $\Omega$ , of continuous affine functionals on a topological vector space  $\mathbb{E}$  over a totally ordered division ring  $\mathbb{k}$ . For a set  $\mathcal{F}$  of maps from  $\Omega$  to  $\mathbb{k}$ , we will denote

<sup>1</sup>Strictly speaking we should set the Heyting implication  $\rightarrow$  apart from the lattice signature, thus for example stating that “every finite distributive lattice expands to a unique Heyting algebra”. The shorter formulation, which we shall keep for the sake of simplicity, reflects a standard abuse of terminology that should not create any confusion here.

by  $\text{Bool}(\mathcal{F}, \Omega)$  the Boolean subalgebra of the powerset of  $\Omega$  generated by all subsets  $\llbracket f > 0 \rrbracket$  and  $\llbracket f < 0 \rrbracket$  for  $f \in \mathcal{F}$ . As in [17], we will also denote by  $\text{Op}^-(\mathcal{F}, \Omega)$  the sublattice of  $\text{Bool}(\mathcal{F}, \Omega)$  generated by all  $\llbracket f > 0 \rrbracket$  and  $\llbracket f < 0 \rrbracket$  where  $f \in \mathcal{F}$ , and then set  $\text{Op}(\mathcal{F}, \Omega) \stackrel{\text{def}}{=} \text{Op}^-(\mathcal{F}, \Omega) \cup \{\Omega\}$ . Evidently,  $\text{Bool}(\mathcal{F}, \Omega)$  is generated, as a Boolean algebra, by its 0-sublattice  $\text{Op}(\mathcal{F}, \Omega)$ ; so  $\text{Bool}(\mathcal{F}, \Omega) = \text{BR}(\text{Op}(\mathcal{F}, \Omega))$ .

For any set  $I$  and any totally ordered division ring  $\mathbb{k}$ , we will occasionally identify every element  $a = (a_i \mid i \in I) \in \mathbb{k}^{(I)}$  with the corresponding (continuous) linear functional  $\sum_{i \in I} a_i \delta_i$  (where  $\delta_i$  denotes the  $i$ th projection), thus justifying such notations as  $\text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(I)})$  and  $\text{Op}(\mathbb{k}^{(I)}, \mathbb{k}^{(I)})$ ; observe that in those notations, the first (resp., second) occurrence of  $\mathbb{k}^{(I)}$  is endowed with its structure of *left* (resp., *right*) vector space over  $\mathbb{k}$ . Moreover, in its second occurrence,  $\mathbb{k}^{(I)}$  is endowed with the coarsest topology making all canonical projections  $\delta_i$  continuous.

Denote by  $F_\ell(I, \mathbb{k})$  the free left<sup>2</sup>  $\mathbb{k}$ -vector lattice on a set  $I$ . As observed in Baker [1], Bernau [2], Madden [9, Ch. III] (see also Wehrung [17, page 13] for a summary),  $F_\ell(I, \mathbb{k})$  canonically embeds into  $\mathbb{k}^{\mathbb{k}^{(I)}}$ . We sum up a few related facts.

**Lemma 2.5** (Folklore).

- (1)  $F_\ell(I, \mathbb{k})$  is isomorphic to the sublattice of  $\mathbb{k}^{\mathbb{k}^{(I)}}$  generated by all linear functionals  $\sum_{i \in I} a_i \delta_i$  associated to elements  $a \in \mathbb{k}^{(I)}$ , via the assignment  $i \mapsto \delta_i$ .
- (2) The assignment  $\langle x \rangle \mapsto \llbracket x \neq 0 \rrbracket$  defines an isomorphism from the lattice  $\text{Id}_c F_\ell(I, \mathbb{k})$ , of all principal  $\ell$ -ideals of the left  $\mathbb{k}$ -vector lattice  $F_\ell(I, \mathbb{k})$ , onto  $\text{Op}^-(\mathbb{k}^{(I)}, \mathbb{k}^{(I)})$ .

### 3. CONSONANCE KERNELS

In this section we introduce a tool, the *consonance kernels*, expressing the consonance of the image a lattice homomorphism *via* its behavior on join-irreducible elements.

**Definition 3.1.** Let  $D$  and  $L$  be distributive lattices, with  $D$  finite and  $L$  with a zero element, and let  $f: D \rightarrow L$  be a join-homomorphism. Set  $P \stackrel{\text{def}}{=} \text{Ji } D$ . A *consonance kernel* for  $f$  is a family  $(e_p \mid p \in P)$  of elements of  $L$  such that

$$f(p) = f(p_*) \vee e_p, \quad \text{whenever } p \in P; \quad (3.1)$$

$$e_p \wedge e_q = 0, \quad \text{whenever } p, q \in P \text{ are incomparable.} \quad (3.2)$$

We then set  $x \otimes_{\bar{e}} y \stackrel{\text{def}}{=} \bigvee \{e_p \mid p \in (P \downarrow x) \setminus (P \downarrow y)\}$ , whenever  $x, y \in D$ .

**Lemma 3.2.** *In the context of Definition 3.1,  $f(x) = f(x \wedge y) \vee (x \otimes_{\bar{e}} y)$  whenever  $x, y \in D$ . Moreover,  $f$  is a lattice homomorphism.*

*Proof.* Setting  $c \stackrel{\text{def}}{=} f(x \wedge y) \vee (x \otimes_{\bar{e}} y)$ , it is obvious that  $c \leq f(x)$ . In order to prove that  $f(x) \leq c$ , it suffices to prove that  $f(p) \leq c$  whenever  $p \in P \downarrow x$ . By way of contradiction, let  $p$  be a minimal element of  $P \downarrow x$  with  $f(p) \not\leq c$ . Since  $p \leq y$  implies  $f(p) \leq f(x \wedge y) \leq c$ , we get  $p \in (P \downarrow x) \setminus (P \downarrow y)$ , so  $f(p) = f(p_*) \vee e_p$ . Since  $e_p \leq c$ , we get  $f(p_*) \not\leq c$ . The case  $p_* = 0$  is impossible, because  $f(0) \leq f(x \wedge y) \leq c$ . Since  $f$  is a join-homomorphism, we get  $f(p_*) = \bigvee \{f(q) \mid q \in P \downarrow p_*\}$ . By the minimality assumption on  $p$ , we get  $f(q) \leq c$  for every  $q \in P \downarrow p_*$ , hence  $f(p_*) \leq c$ , a contradiction.

<sup>2</sup>“Right” and “left” appear to have been unfortunately mixed up at various places in [17], particularly on pages 12 and 13. Since this is mostly a matter of choosing sides, that paper’s results are unaffected. We nonetheless attempt to fix this here.

Now let  $x, y \in D$ . By the result of the paragraph above,  $f(x) = f(x \wedge y) \vee (x \odot_{\vec{e}} y)$  and  $f(y) = f(x \wedge y) \vee (y \odot_{\vec{e}} x)$ . Due to (3.2),  $(x \odot_{\vec{e}} y) \wedge (y \odot_{\vec{e}} x) = 0$ ; whence  $f(x) \wedge f(y) = f(x \wedge y)$ .  $\square$

**Lemma 3.3.** *Let  $D$  and  $L$  be distributive lattices, with  $D$  finite and  $L$  with a zero element. Then a lattice homomorphism  $f: D \rightarrow L$  has a consonance kernel iff the range of  $f$  is consonant in  $L$ .*

*Proof.* Suppose first that the range of  $f$  is consonant in  $L$ . Since  $D$  is finite, there exists a finite 0-sublattice  $K$  of  $L$ , containing  $f[D]$ , such that the range of  $f$  is consonant in  $K$ . Setting  $e_p \stackrel{\text{def}}{=} f(p) \searrow_K f(p_*)$  for each  $p \in \text{Ji } D$ , Condition (3.1) is obviously satisfied. Let  $p, q \in \text{Ji } D$  be incomparable. From  $p \wedge q \leq p_*$  we get

$$e_p = f(p) \searrow_K f(p_*) \leq f(p) \searrow_K f(p \wedge q) = f(p) \searrow_K (f(p) \wedge f(q)) = f(p) \searrow_K f(q),$$

and, similarly,  $e_q \leq f(q) \searrow_K f(p)$ . Since  $f(p)$  and  $f(q)$  are consonant within  $K$ , we get  $(f(p) \searrow_K f(q)) \wedge (f(q) \searrow_K f(p)) = 0$ ; whence  $e_p \wedge e_q = 0$ .

Let, conversely,  $(e_p \mid p \in \text{Ji } D)$  be a consonance kernel for  $f$  and set  $P \stackrel{\text{def}}{=} \text{Ji } D$ . Let  $x, y \in D$ , set  $u \stackrel{\text{def}}{=} x \odot_{\vec{e}} y$  and  $v \stackrel{\text{def}}{=} y \odot_{\vec{e}} x$ . It follows from Lemma 3.2 that  $f(x) \leq f(y) \vee u$  and  $f(y) \leq f(x) \vee v$ . Moreover, for all  $p \in (P \downarrow x) \setminus (P \downarrow y)$  and  $q \in (P \downarrow y) \setminus (P \downarrow x)$ ,  $p$  and  $q$  are incomparable, thus  $e_p \wedge e_q = 0$ ; whence  $u \wedge v = 0$ . Therefore, the pair  $(u, v)$  witnesses the consonance of  $f(x)$  and  $f(y)$  in  $L$ .  $\square$

#### 4. AN EXTENSION LEMMA FOR INFINITE DISTRIBUTIVE LATTICES

This section's main result, Lemma 4.4, states conditions under which a homomorphism  $f: D \rightarrow L$  of distributive lattices can be extended to a homomorphism  $f: E \rightarrow L$  in case  $E$  is generated over  $D$  by two disjoint elements  $a$  and  $b$ . One of its main improvements, over the original [14, Lemma 4.2] it stems from, is the possibility of  $D$  be infinite.

**Definition 4.1.** A 0, 1-sublattice  $D$  of a bounded distributive lattice  $E$  is a *semi-Heyting sublattice* if for all  $x, y \in D$ ,  $x \rightarrow_D y$  and  $x \rightarrow_E y$  both exist and are equal.

In particular, every semi-Heyting sublattice of  $E$  is a Heyting algebra ( $E$  itself may not be a Heyting algebra).

**Notation 4.2.** Let  $D$  be a finite 0, 1-sublattice of a bounded distributive lattice  $E$  and let  $f: D \rightarrow L$  be a 0-lattice homomorphism. We set

$$f_{\vec{e}}(a) \stackrel{\text{def}}{=} \bigvee \{e_p \mid p \in \text{Ji } D, p \leq p_* \vee a\}$$

for every consonance kernel  $\vec{e}$  of  $f$  and every  $a \in E$ .

The following lemma arises from Wehrung [17, Remark 4.6]. We include a proof for convenience.

**Lemma 4.3.** *Let  $D$  be a finite semi-Heyting sublattice of a bounded distributive lattice  $E$ , let  $f: D \rightarrow L$  be a 0-lattice homomorphism, and let  $a, b \in E$  such that  $a \wedge b = 0$ . Then any join-irreducible elements  $p$  and  $q$  in  $D$  such that  $p \leq p_* \vee a$  and  $q \leq q_* \vee b$  are incomparable. In particular,  $f_{\vec{e}}(a) \wedge f_{\vec{e}}(b) = 0$  for any consonance kernel  $\vec{e}$  for  $f$ .*

*Proof.* Suppose otherwise, say  $p \leq q$ ; thus  $p^\dagger \leq q^\dagger$ . From  $a \wedge b = 0$  we get  $p \wedge b \leq (p_* \vee a) \wedge b = p_* \wedge b \leq p_*$ , thus, by assumption,  $b \leq p \rightarrow_E p_* = p \rightarrow_D p_* = p^\dagger$ . Since  $p^\dagger \leq q^\dagger$ , we get  $b \leq q^\dagger$ , so  $q \leq q_* \vee b \leq q^\dagger$ , a contradiction.  $\square$

We are now reaching this section's main goal. In the next proof we use the following well known extension criterion. Let  $D$  and  $L$  be distributive lattices and  $X$  a generating subset of  $D$ . Then a map  $f: X \rightarrow L$  can be extended to a (necessarily unique) homomorphism  $g: D \rightarrow L$  if and only if

$$\bigwedge_{i=1}^m x_i \leq \bigvee_{j=1}^n y_j \implies \bigwedge_{i=1}^m f(x_i) \leq \bigvee_{j=1}^n f(y_j) \quad (4.1)$$

for all  $m, n > 0$  and all  $x_1, \dots, x_m, y_1, \dots, y_n \in X$ .

**Lemma 4.4** (Main Extension Lemma). *Let  $D$  be a semi-Heyting sublattice of a bounded distributive lattice  $E$  and let  $a, b \in E$ . Setting  $B \stackrel{\text{def}}{=} \text{BR}(D)$ , we assume the following:*

- (1)  $E$  is generated, as a lattice, by  $D \cup \{a, b\}$ .
- (2)  $a \wedge b = 0$ .
- (3) All elements  $a_B, b_B, (a \vee b)_B, a^B$ , and  $b^B$  are defined.
- (4)  $(a \vee b)_B = a_B \vee b_B$ .

Then

$$c^B \in D \text{ whenever } c \in \{a, b, a \vee b\}. \quad (4.2)$$

Further, for every 0-lattice homomorphism  $f: D \rightarrow L$  and all  $\alpha, \beta \in L$ , the following conditions are equivalent.

- (i)  $(\alpha, \beta) = (g(a), g(b))$  for some lattice homomorphism  $g: E \rightarrow L$  extending  $f$ ;
- (ii)  $\alpha \leq f(a^B)$ ,  $\beta \leq f(b^B)$ ,  $\alpha \wedge \beta = 0$ ,  $\text{BR}(f)(a_B) \leq \alpha$ , and  $\text{BR}(f)(b_B) \leq \beta$ .

Moreover, for any finite semi-Heyting sublattice  $D'$  of  $D$  such that  $\{a_B, b_B\} \subseteq \text{BR}(D')$  and  $\{a^B, b^B\} \subseteq D'$ , and any consonance kernel  $\bar{c}$  of  $f' \stackrel{\text{def}}{=} f \upharpoonright_{D'}$ ,  $(f'_\bar{c}(a), f'_\bar{c}(b))$  is a pair satisfying (ii).

*Note.* By the same token as the one used in the proof of Lemma 2.3, the condition that  $\text{BR}(f)(a_B) \leq \alpha$  is equivalent to saying that for all  $x, y \in D$ ,  $x \leq y \vee a \implies f(x) \leq f(y) \vee \alpha$ . By Lemma 2.3, if  $D$  is finite, then it suffices to restrict ourselves to the case where  $x = p \in \text{Ji } D$  and  $y = p_*$ . Note that  $\text{BR}(f)(a_B)$  is an element of  $\text{BR}(L)$ , usually not in  $L$ , so it cannot be taken as the lowest possible value of  $\alpha$  a priori.

*Proof.* We start by proving (4.2). By (3), there is an expression of the form  $c^B = \bigwedge_{i < n} (\neg u_i \vee v_i)$  (within  $B$ ) where  $n < \omega$  and all  $u_i, v_i \in D$ . For each  $i < n$ ,  $c \leq \neg u_i \vee v_i$  within  $\text{BR}(E)$ , thus  $u_i \wedge c \leq v_i$ , and thus, since  $D$  is a semi-Heyting sublattice of  $E$ ,  $c \leq u_i \rightarrow_E v_i = u_i \rightarrow_D v_i$ ; whence, setting  $w \stackrel{\text{def}}{=} \bigwedge_{i < n} (u_i \rightarrow_D v_i)$ , we get  $c \leq w$ . For each  $i < n$ ,  $w \leq u_i \rightarrow_D v_i$  with  $w \in D$ , thus  $u_i \wedge w \leq v_i$ , so  $w \leq \neg u_i \vee v_i$  within  $B$ , and so  $w \leq c^B$ . Since  $w \in D$ , it follows that  $w = c^B = c^D$ .

Now it is obvious that for every lattice homomorphism  $g: E \rightarrow L$  extending  $f$ , the pair  $(\alpha, \beta) \stackrel{\text{def}}{=} (g(a), g(b))$  satisfies  $\alpha \leq f(a^B)$ ,  $\beta \leq f(b^B)$ ,  $\alpha \wedge \beta = 0$ ,  $\text{BR}(f)(a_B) \leq \alpha$ , and  $\text{BR}(f)(b_B) \leq \beta$ . Let, conversely,  $(\alpha, \beta)$  be such a pair.

We need to show the implication (4.1) for  $x_i, y_j \in D \cup \{a, b\}$ . Since  $f$  is a lattice homomorphism, we can assume that exactly one  $x_i$  and exactly one  $y_j$  belong to  $D$ .

Since  $a \wedge b = 0$ , the inequality  $x \wedge a \leq y \vee b$  is equivalent to  $x \wedge a \leq y$ . So, (4.1) boils down to the equality  $\alpha \wedge \beta = 0$  (which is assumed) and the following implications:

$$x \leq y \vee a \Rightarrow f(x) \leq f(y) \vee \alpha; \quad (4.3)$$

$$x \leq y \vee b \Rightarrow f(x) \leq f(y) \vee \beta; \quad (4.4)$$

$$x \leq y \vee a \vee b \Rightarrow f(x) \leq f(y) \vee \alpha \vee \beta; \quad (4.5)$$

$$x \wedge a \leq y \Rightarrow f(x) \wedge \alpha \leq f(y); \quad (4.6)$$

$$x \wedge b \leq y \Rightarrow f(x) \wedge \beta \leq f(y). \quad (4.7)$$

The implications (4.3) and (4.4) follow from  $\text{BR}(f)(a_B) \leq \alpha$  and  $\text{BR}(f)(b_B) \leq \beta$ . Owing to Condition (4), the implication (4.5) follows from the inequalities

$$\text{BR}(f)((a \vee b)_B) = \text{BR}(f)(a_B \vee b_B) = \text{BR}(f)(a_B) \vee \text{BR}(f)(b_B) \leq \alpha \vee \beta.$$

Suppose that  $x \wedge a \leq y$ . Since  $D$  is a semi-Heyting sublattice of  $E$ , it follows that  $a \leq x \rightarrow_E y = x \rightarrow_D y$ , thus, using (4.2),  $a^D = a^B \leq x \rightarrow_D y$ . It follows that  $\alpha \leq f(a^B) \leq f(x \rightarrow_D y)$ , thus  $f(x) \wedge \alpha \leq f(x) \wedge f(x \rightarrow_D y) \leq f(y)$ . The implication (4.6) follows. The proof of (4.7) is similar.

For the remainder of the proof, let  $D'$  be a finite semi-Heyting sublattice of  $D$  such that  $\{a_B, b_B\} \subseteq \text{BR}(D')$  and  $\{a^B, b^B\} \subseteq D'$  (cf. Figure 4.1), and let  $\vec{e}$  be a consonance kernel of  $f' \stackrel{\text{def}}{=} f \upharpoonright_{D'}$ . Set  $(\alpha, \beta) \stackrel{\text{def}}{=} (f'_{\vec{e}}(a), f'_{\vec{e}}(b))$ .

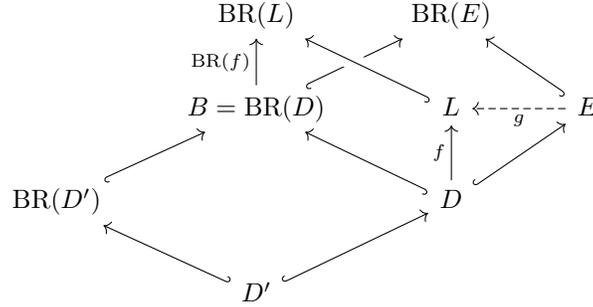


FIGURE 4.1. Illustrating the proof of Lemma 4.4

For every  $p \in \text{Ji } D'$ ,  $p \leq p_* \vee a$  (within  $E$ ) implies that  $p \leq p_* \vee a^B$  (within  $D'$ ), thus, since  $p \in \text{Ji } D'$ , we get  $p \leq a^B$ , whence  $e_p \leq f(p) \leq f(a^B)$ . This proves that  $\alpha \leq f(a^B)$ . Similarly,  $\beta \leq f(b^B)$ . Further, the equation  $\alpha \wedge \beta = 0$  follows from Lemma 4.3.

Let  $c \in \{a, b\}$  and let  $x, y \in D$  such that  $x \leq y \vee c$ , we need to prove that  $f(x) \leq f(y) \vee f'_{\vec{e}}(c)$ . From  $x \wedge \neg y \leq c$  (within  $\text{BR}(E)$ ) it follows that  $x \wedge \neg y \leq c_B$  (within  $\text{BR}(D)$ ). Set  $X \stackrel{\text{def}}{=} \{p \in \text{Ji } D' \mid p \wedge \neg p_* \leq c_B\} = \{p \in \text{Ji } D' \mid p \leq p_* \vee c\}$ . By (3) and since  $\text{BR}(D')$  is a finite Boolean algebra with atoms  $p \wedge \neg p_*$  for  $p \in \text{Ji } D'$ ,

$$c_B = \bigvee \{p \wedge \neg p_* \mid p \in X\} \quad \text{within } B. \quad (4.8)$$

By the definition of  $X$ ,

$$f(p) = f(p_*) \vee e_p \leq f(p_*) \vee f'_{\vec{e}}(c) \quad \text{whenever } p \in X,$$

so  $f(p) \wedge \neg f(p_*) \leq f'_e(c)$  within  $\text{BR}(L)$ , whenever  $p \in X$ ; whence, using (4.8),

$$\text{BR}(f)(c_B) = \bigvee \{f(p) \wedge \neg f(p_*) \mid p \in X\} \leq f'_e(c) \quad \text{within } \text{BR}(L). \quad (4.9)$$

Using (4.9), we get

$$f(x) \wedge \neg f(y) = \text{BR}(f)(x \wedge \neg y) \leq \text{BR}(f)(c_B) \leq f'_e(c),$$

so  $f(x) \leq f(y) \vee f'_e(c)$ .  $\square$

## 5. ADJUNCTIONS BETWEEN LATTICES $\text{Bool}(\mathcal{F}, \mathbb{k}^{(I)})$

Throughout this section  $\mathbb{k}$  will be a totally ordered division ring. In this section we shall state a few properties of Boolean algebras of the form  $\text{Bool}(\mathcal{F}, \Omega)$ , mostly related to relative completeness between such algebras.

The following observation is contained in the proof of Wehrung [14, Lemma 6.6].

**Lemma 5.1.** *Let  $\Omega$  be a convex subset in a right vector space  $\mathbb{E}$  over  $\mathbb{k}$  and let  $\mathcal{F} \cup \{a\}$  be a set of affine functionals on  $\mathbb{E}$ . Set  $A^+ \stackrel{\text{def}}{=} \llbracket a > 0 \rrbracket$  and  $A^- \stackrel{\text{def}}{=} \llbracket a < 0 \rrbracket$ . Then for every  $U \in \text{Bool}(\mathcal{F}, \Omega)$ , if  $U \subseteq A^+ \cup A^-$ , then there are  $U^+, U^- \in \text{Bool}(\mathcal{F}, \Omega)$  such that  $U = U^+ \cup U^-$  whereas  $U^+ \subseteq A^+$  and  $U^- \subseteq A^-$ .*

*Proof.* Since  $U$  is the union of finitely many cells, each of which being the intersection of finitely many sets of the form either  $\llbracket \pm f > 0 \rrbracket$  or  $\llbracket \pm f \geq 0 \rrbracket$  where  $f \in \mathcal{F}$ , it suffices to consider the case where  $U$  is such a cell. If  $U$  meets both  $A^+$  and  $A^-$ , pick  $x \in U \cap A^+$  and  $y \in U \cap A^-$ ; so  $a(x) > 0$  and  $a(y) < 0$ . Then  $\lambda \stackrel{\text{def}}{=} (a(y) - a(x))^{-1} a(y)$  belongs to the open interval  $]0, 1[$  and  $a(x\lambda + y(1 - \lambda)) = 0$ , that is,  $x\lambda + y(1 - \lambda) \notin A^+ \cup A^-$ . On the other hand, since  $U$  is convex,  $x\lambda + y(1 - \lambda) \in U$ ; a contradiction since  $U \subseteq A^+ \cup A^-$ . Therefore,  $U$  is disjoint either from  $A^+$  or from  $A^-$ , thus it is contained either in  $A^+$  or in  $A^-$ .  $\square$

**Corollary 5.2.** *In the context of Lemma 5.2,  $(A^+ \cup A^-)_{\text{Bool}(\mathcal{F}, \Omega)}$  exists iff both  $(A^+)_{\text{Bool}(\mathcal{F}, \Omega)}$  and  $(A^-)_{\text{Bool}(\mathcal{F}, \Omega)}$  exist, and then*

$$(A^+ \cup A^-)_{\text{Bool}(\mathcal{F}, \Omega)} = (A^+)_{\text{Bool}(\mathcal{F}, \Omega)} \cup (A^-)_{\text{Bool}(\mathcal{F}, \Omega)}.$$

In what follows we will identify every element  $f \in \mathbb{k}^{(I)}$  with the associated linear functional on  $\mathbb{k}^{(I)}$ , that is,  $x \mapsto \sum_{i \in I} f_i x_i$ . Moreover, whenever  $I \subseteq J$  we will identify  $\mathbb{k}^{(I)}$  with the subset of  $\mathbb{k}^{(J)}$  consisting of all vectors with support contained in  $I$ .

**Notation 5.3.** For  $I \subseteq J$ , we define mappings

$$\begin{aligned} \varepsilon_{I,J} &: \text{Pow } \mathbb{k}^{(I)} \rightarrow \text{Pow } \mathbb{k}^{(J)}, \\ \rho_{J,I}^\vee, \rho_{J,I}^\wedge &: \text{Pow } \mathbb{k}^{(J)} \rightarrow \text{Pow } \mathbb{k}^{(I)}, \end{aligned}$$

by

$$\begin{aligned} \varepsilon_{I,J}(X) &= \left\{ y \in \mathbb{k}^{(J)} \mid y \upharpoonright_I \in X \right\}, \\ \rho_{J,I}^\wedge(Y) &= \left\{ x \in \mathbb{k}^{(I)} \mid (\forall y \in \mathbb{k}^{(J)})(y \upharpoonright_I = x \Rightarrow y \in Y) \right\}, \\ \rho_{J,I}^\vee(Y) &= \{ y \upharpoonright_I \mid y \in Y \}. \end{aligned}$$

The following statements are immediate consequences of the definitions:

- $\varepsilon_{I,J}$  is an embedding of Boolean algebras,  $\rho_{J,I}^\wedge$  is a meet-homomorphism, and  $\rho_{J,I}^\vee$  is a join-homomorphism. Moreover,  $\rho_{J,I}^\wedge$  and  $\rho_{J,I}^\vee$  are right and left adjoint to  $\varepsilon_{I,J}$ , respectively.
- $\rho_{J,I}^\wedge$  and  $\rho_{J,I}^\vee$  are conjugate, that is,  $\mathbb{k}^{(I)} \setminus \rho_{J,I}^\wedge(Y) = \rho_{J,I}^\vee(\mathbb{k}^{(J)} \setminus Y)$  whenever  $Y \subseteq \mathbb{k}^{(J)}$ .

**Lemma 5.4.** *Let  $I$  and  $J$  be sets with  $I \subseteq J$ . The following statements hold:*

- (1)  $\varepsilon_{I,J} \rho_{J,I}^\wedge(Z) = \varepsilon_{I,J} \rho_{J,I}^\vee(Z) = Z$  for every  $Z \in \text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(J)})$ .
- (2)  $\varepsilon_{I,J}[\text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(I)})] = \text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(J)}) \subseteq \text{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})$ .
- (3)  $\rho_{J,I}^\wedge[\text{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})] = \rho_{J,I}^\vee[\text{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})] = \text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(I)})$ .

*Proof.* Ad (1) and (2) are both trivial. In order to prove (3), it suffices, since  $\rho_{J,I}^\wedge$  and  $\rho_{J,I}^\vee$  are conjugate, to establish the result for  $\rho_{J,I}^\vee$ . For every  $X \in \text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(I)})$ ,  $X = \rho_{J,I}^\vee \varepsilon_{I,J}(X)$  with  $\varepsilon_{I,J}(X) \in \text{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})$ , thus  $X \in \rho_{J,I}^\vee[\text{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})]$ ; whence  $\rho_{J,I}^\vee[\text{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})]$  contains  $\text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(I)})$ .

Let us establish the converse containment. Since  $\rho_{J,I}^\vee$  is a  $(\vee, 0)$ -homomorphism, it suffices to prove that  $\rho_{J,I}^\vee(Y) \in \text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(I)})$  whenever  $Y$  is a set of the form  $\bigcap_{i < m} \llbracket a_i \geq 0 \rrbracket \cap \bigcap_{j < n} \llbracket b_j > 0 \rrbracket$  where  $m, n < \omega$  and all  $a_i, b_j \in \mathbb{k}^{(J)}$ .

Set  $a'_i \stackrel{\text{def}}{=} a_i \upharpoonright_I$  and  $a''_i \stackrel{\text{def}}{=} a_i \upharpoonright_{J \setminus I}$ , for all  $i < m$ , and define similarly  $b'_j$  and  $b''_j$  for  $j < n$ . An element  $x \in \mathbb{k}^{(I)}$  belongs to  $\rho_{J,I}^\vee(Y)$  iff there exists  $z \in \mathbb{k}^{(J \setminus I)}$  such that each  $a'_i(x) + a''_i(z) \geq 0$  and each  $b'_j(x) + b''_j(z) > 0$ . The set  $V$  of all  $(m+n)$ -tuples of elements of  $\mathbb{k}$  of the form  $(a''_0(z), \dots, a''_{m-1}(z), b''_0(z), \dots, b''_{n-1}(z))$  is a vector subspace of  $\mathbb{k}^{m+n}$ . Hence, an element  $x \in \mathbb{k}^{(I)}$  belongs to  $\rho_{J,I}^\vee(Y)$  iff there exists  $u \in V$  such that  $a'_i(x) + u_i \geq 0$  whenever  $i < m$  and  $b'_j(x) + u_{m+j} > 0$  whenever  $j < n$ . Since membership in  $V$ , of any  $(m+n)$ -tuple of elements of  $\mathbb{k}$ , can be expressed by a finite set of linear equations, the statement that a given  $x \in \mathbb{k}^{(I)}$  belongs to  $\rho_{J,I}^\vee(Y)$  can be expressed by a sentence, over the first-order language  $\mathcal{L} \stackrel{\text{def}}{=} \{<, 0, -, +\} \cup \{\cdot \lambda \mid \lambda \in \mathbb{k}\}$  of ordered Abelian groups augmented with right scalar multiplications by elements of  $\mathbb{k}$ , in  $(a'_0(x), \dots, a'_{m-1}(x), b'_0(x), \dots, b'_{n-1}(x))$ . Now every  $\mathcal{L}$ -sentence is equivalent, over all nonzero totally ordered right  $\mathbb{k}$ -vector spaces, to a quantifier-free  $\mathcal{L}$ -sentence (cf. van den Dries [13, Corollary I.7.8]). Therefore,  $\rho_{J,I}^\vee(Y)$  belongs to  $\text{Bool}(\mathcal{F}, \mathbb{k}^{(I)})$  for a finite set  $\mathcal{F}$  of linear combinations of the  $a'_i$  and the  $b'_j$ .  $\square$

**Proposition 5.5.** *Let  $I$  and  $J$  be sets with  $I \subseteq J$  and let  $\mathcal{D}$  be a finite subset of  $\mathbb{k}^{(J)}$ . Then  $\text{Bool}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)})$  is relatively complete in  $\text{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})$ .*

*Proof.* We first prove that  $\text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(J)})$  is relatively complete in  $\text{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})$ . Let  $Y \in \text{Bool}(\mathbb{k}^{(J)}, \mathbb{k}^{(J)})$ . By Lemma 5.4,  $\varepsilon_{I,J} \rho_{J,I}^\vee(Y), \varepsilon_{I,J} \rho_{J,I}^\wedge(Y) \in \text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(J)})$ . Further,  $Y \subseteq Z \in \text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(J)})$  implies  $\varepsilon_{I,J} \rho_{J,I}^\vee(Y) \subseteq \varepsilon_{I,J} \rho_{J,I}^\vee(Z) = Z$ . Thus,  $Y^{\text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(J)})} = \varepsilon_{I,J} \rho_{J,I}^\vee(Y)$  and similarly,  $Y^{\text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(J)})} = \varepsilon_{I,J} \rho_{J,I}^\wedge(Y)$ .

Since  $\text{Bool}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)})$  is finitely generated over  $\text{Bool}(\mathbb{k}^{(I)}, \mathbb{k}^{(J)})$  (via the additional generators  $\llbracket d > 0 \rrbracket$  and  $\llbracket d < 0 \rrbracket$  for  $d \in \mathcal{D}$ ), the desired conclusion follows from Lemma 2.1.  $\square$

## 6. EXTENDING A TOP-FAITHFUL MAP

In Lemmas 6.1 and 6.2 we fix a totally ordered division ring  $\mathbb{k}$ . The following lemma takes care of the “domain step” required in the proof of Theorem 7.4.

**Lemma 6.1.** *Let  $I$  and  $J$  be sets, let  $L$  be a completely normal distributive 0-lattice, let  $\mathcal{D}$  be a finite subset of  $\mathbb{k}^{(J)}$ , and let  $e \in \mathbb{k}^{(J)}$ . Then every top-faithful 0-lattice homomorphism  $f: \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)}) \rightarrow L^{\sqcup\infty}$  extends to a top-faithful lattice homomorphism  $g: \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{e\}, \mathbb{k}^{(J)}) \rightarrow L^{\sqcup\infty}$  (cf. Figure 6.1).*

$$\begin{array}{ccc}
 & L^{\sqcup\infty} & \\
 & \uparrow f & \swarrow \text{---} g \text{---} \\
 \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)}) & \xrightarrow{\subseteq} & \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{e\}, \mathbb{k}^{(J)})
 \end{array}$$

FIGURE 6.1. A commutative triangle for Lemma 6.1

*Proof.* Set  $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{D} \cup \{e\}$ ,  $D \stackrel{\text{def}}{=} \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)})$ ,  $E \stackrel{\text{def}}{=} \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{E}, \mathbb{k}^{(J)})$ ,  $B \stackrel{\text{def}}{=} \text{BR}(D) = \text{Bool}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)})$ , and  $C \stackrel{\text{def}}{=} \text{Bool}(\mathbb{k}^{(I)} \cup \mathcal{E}, \mathbb{k}^{(J)})$ . By Proposition 5.5,  $B$  is relatively complete in  $C$ . In particular, setting  $a \stackrel{\text{def}}{=} \llbracket e > 0 \rrbracket$  and  $b \stackrel{\text{def}}{=} \llbracket e < 0 \rrbracket$ , the elements  $a^B$ ,  $b^B$ ,  $a_B$ ,  $b_B$ , and  $(a \vee b)_B$  are all defined. By Corollary 5.2,  $(a \vee b)_B = a_B \vee b_B$ . Let  $\mathcal{D}'$  be a finite subset of  $\mathbb{k}^{(I)} \cup \mathcal{D}$  such that  $a^B$ ,  $b^B$ ,  $a_B$ , and  $b_B$  all belong to  $B' \stackrel{\text{def}}{=} \text{Bool}(\mathcal{D}', \mathbb{k}^{(J)})$ . By Wehrung [14, Lemma 5.4] (see also Wehrung [17, Lemma 4.1] for the more general form of that statement),  $D$  is a Heyting subalgebra of  $E$  and  $D' \stackrel{\text{def}}{=} \text{Op}(\mathcal{D}', \mathbb{k}^{(J)})$  is a Heyting subalgebra of  $D$ . Since  $L$  is completely normal and  $f[D']$  is finite, it follows from Lemma 3.3 that  $f' \stackrel{\text{def}}{=} f|_{D'}$  has a consonance kernel  $(e_P \mid P \in \text{Ji } D')$ . By Lemma 4.4,  $f$  extends to a unique lattice homomorphism  $g: D \rightarrow L$  such that  $g(x) = f'_e(x)$  whenever  $x \in \{a, b\}$ . For any  $P \in \text{Ji } D'$  such that  $P \subseteq P_* \cup x$ ,  $0 \notin P_* \cup x$ , thus  $0 \notin P$ , that is,  $P$  is not the top element of  $\text{Op}(\mathbb{k}^{(I)}, \mathbb{k}^{(J)})$ . Since  $f$  is top-faithful, it follows that  $e_P \leq f(P) < \infty$ ; whence  $f'_e(x) < \infty$ . It follows that  $g$  is top-faithful.  $\square$

The “surjectivity step” is much easily taken care of:

**Lemma 6.2.** *Let  $I$  and  $J$  be sets with  $I \subset J$  and  $J \setminus I$  infinite, let  $L$  be a distributive 0-lattice, let  $\mathcal{D}$  be a finite subset of  $\mathbb{k}^{(J)}$ , and let  $c \in L$ . Then every top-faithful 0-lattice homomorphism  $f: \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)}) \rightarrow L^{\sqcup\infty}$ , there are  $e \in \mathbb{k}^{(J)}$  and a top-faithful lattice homomorphism  $g: \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{e\}, \mathbb{k}^{(J)}) \rightarrow L^{\sqcup\infty}$  such that  $g(e) = c$ .*

*Proof.* Since  $\mathcal{D}$  is finite and  $J \setminus I$  is infinite, there exists  $j \in J \setminus I$  not in the support of any element of  $\mathcal{D}$ . Take  $e \stackrel{\text{def}}{=} \delta_j$ , the  $j$ th canonical projection  $\mathbb{k}^{(J)} \rightarrow \mathbb{k}$ . By the argument of Wehrung [14, Lemma 8.3],  $\text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{\delta_j\}, \mathbb{k}^{(J)})$  is the (internal) free amalgamated sum of  $\text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D}, \mathbb{k}^{(J)})$  and  $\{\emptyset, \llbracket \delta_j > 0 \rrbracket, \llbracket \delta_j < 0 \rrbracket, \llbracket \delta_j \neq 0 \rrbracket, \mathbb{k}^{(J)}\}$  within the category of bounded distributive lattices. Hence  $f$  extends to a unique lattice homomorphism  $g: \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{\delta_j\}, \mathbb{k}^{(J)}) \rightarrow L$  such that  $g(\llbracket \delta_j > 0 \rrbracket) = c$  and  $g(\llbracket \delta_j < 0 \rrbracket) = 0$ . Since  $c < \infty$  and  $f$  is top-faithful, it follows that  $g$  is also top-faithful.  $\square$

## 7. REPRESENTING TREES OF COUNTABLE LATTICES

In this section we will reach the paper’s main goal, Theorem 7.4, which states that every completely normal distributive 0-lattice is a homomorphic image of some  $\text{Id}_c F$  for some  $\mathbb{k}$ -vector lattice  $F$ . In order to reach that result we will in fact prove (cf. Theorem 7.3) the apparently stronger statement that every *diagram* of countable completely normal distributive 0-lattices, indexed by a tree in which every element has countable height, can be represented in that fashion.

Towards that goal, our main technical tool is the following “one-step extension” theorem, which relies on the results of Section 6, together with the observation that for  $\mathcal{F} \subseteq \mathbb{k}^{(I)}$ ,  $\text{Op}(\mathcal{F}, \mathbb{k}^{(I)}) = \text{Op}^-(\mathcal{F}, \mathbb{k}^{(I)}) \sqcup \{\infty\}$  (where  $\infty$  denotes here the full space  $\mathbb{k}^{(I)}$ ; so the top-faithful maps  $\text{Op}(\mathcal{F}, \mathbb{k}^{(I)}) \rightarrow L^{\sqcup\infty}$  are exactly the  $g^{\sqcup\infty}$  where  $g: \text{Op}^-(\mathcal{F}, \mathbb{k}^{(I)}) \rightarrow L$ ).

**Theorem 7.1.** *Let  $\mathbb{k}$  be a countable totally ordered division ring, let  $I$  and  $J$  be countable sets with  $I \subset J$  and  $J \setminus I$  infinite, let  $K$  and  $L$  be distributive 0-lattices with  $L$  countable completely normal, let  $\varphi: K \rightarrow L$  be a 0-lattice homomorphism, and let  $f: \text{Op}^-(\mathbb{k}^{(I)}, \mathbb{k}^{(I)}) \rightarrow K$  be a 0-lattice homomorphism. Then there exists a surjective lattice homomorphism  $g: \text{Op}^-(\mathbb{k}^{(J)}, \mathbb{k}^{(J)}) \rightarrow L$  such that  $g \circ \varepsilon_{I,J} = \varphi \circ f$ .*

The settings for Theorem 7.1 can be read on Figure 7.1. Its proof can be followed on Figure 7.2.

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & L \\ f \uparrow & & \uparrow g \\ \text{Op}^-(\mathbb{k}^{(I)}, \mathbb{k}^{(I)}) & \xrightarrow{\varepsilon_{I,J}} & \text{Op}^-(\mathbb{k}^{(J)}, \mathbb{k}^{(J)}) \end{array}$$

FIGURE 7.1. A commutative diagram for Theorem 7.1

$$\begin{array}{ccccc} K & \xrightarrow{\varphi} & L & & \\ f \uparrow & & g_n \uparrow & \swarrow g_{n+1} & \\ \text{Op}^-(\mathbb{k}^{(I)}, \mathbb{k}^{(I)}) & \xrightarrow{\varepsilon_{I,J}} & \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D}_n, \mathbb{k}^{(J)}) & \xrightarrow{\subseteq} & \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D}_{n+1}, \mathbb{k}^{(J)}) \end{array}$$

FIGURE 7.2. Illustrating the proof of Theorem 7.1

*Proof.* An iterative application of Lemmas 6.1 and 6.2, similar to the proof of Wehrung [14, Theorem 9.1] but easier since we do not need any analogue of the “closure step” [14, Lemma 7.1]. Let  $\mathbb{k}^{(J)} = \{v_n \mid n < \omega\}$  and  $L = \{c_n \mid n < \omega\}$ . Given an extension  $g_n: \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D}_n, \mathbb{k}^{(J)}) \rightarrow L$  of  $g_0 \stackrel{\text{def}}{=} \varphi \circ f$ , where  $\mathcal{D}_n \subset \mathbb{k}^{(J)}$  is finite, we extend the top-faithful extension  $g_n^{\sqcup\infty}: \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D}_n, \mathbb{k}^{(J)}) \rightarrow L^{\sqcup\infty}$  of  $g_n$  to a top-faithful lattice homomorphism  $g_{n+1}^{\sqcup\infty}: \text{Op}(\mathbb{k}^{(I)} \cup \mathcal{D}_{n+1}, \mathbb{k}^{(J)}) \rightarrow L^{\sqcup\infty}$ , with  $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ ,  $v_{\lfloor n/2 \rfloor} \in \mathcal{D}_{n+1}$  if  $n$  is even (via Lemma 6.1), and  $c_{\lfloor n/2 \rfloor} \in \text{rng } g_{n+1}$  if  $n$  is odd (via Lemma 6.2). The common extension  $g$  of all  $g_n$  is as required.  $\square$

By virtue of Lemma 2.5, Theorem 7.1 can be recast in terms of  $\ell$ -ideal lattices of free vector lattices over  $\mathbb{k}$ , as follows.

**Theorem 7.2.** *Let  $\mathbb{k}$  be a countable totally ordered division ring, let  $I$  and  $J$  be countable sets with  $I \subset J$  and  $J \setminus I$  infinite, let  $K$  and  $L$  be distributive 0-lattices with  $L$  countable completely normal, let  $\varphi: K \rightarrow L$  be a 0-lattice homomorphism, and let  $f: \text{Id}_c \text{F}_\ell(I, \mathbb{k}) \rightarrow K$  be a 0-lattice homomorphism. Denote by  $\eta_{I,J}: \text{Id}_c \text{F}_\ell(I, \mathbb{k}) \hookrightarrow \text{Id}_c \text{F}_\ell(J, \mathbb{k})$  the canonical embedding. Then there exists a surjective lattice homomorphism  $g: \text{Id}_c \text{F}_\ell(J, \mathbb{k}) \twoheadrightarrow L$  such that  $g \circ \eta_{I,J} = \varphi \circ f$ .*

By using the functoriality of the assignment  $I \mapsto \text{Id}_c \text{F}_\ell(I, \mathbb{k})$ , Theorem 7.2 can further be extended to diagrams indexed by trees, as follows.

**Theorem 7.3.** *Let  $\mathbb{k}$  be a countable totally ordered division ring, let  $T$  be a tree in which every element has countable height, and let  $\vec{L} \stackrel{\text{def}}{=} (L_s, \varphi_{s,t} \mid s \leq t \text{ in } T)$  be a commutative  $T$ -indexed diagram of distributive 0-lattices such that  $L_t$  is countable completely normal whenever  $t \in T \setminus \{\perp\}$ . Let  $I_\perp \subseteq \{\perp\} \times \omega$  and set  $I_t \stackrel{\text{def}}{=} (T \downarrow t) \times \omega$  whenever  $t \in T \setminus \{\perp\}$ . Set  $\vec{I} \stackrel{\text{def}}{=} (I_s, \eta_{I_s, I_t} \mid s \leq t \text{ in } T)$ . Then every 0-lattice homomorphism  $\chi_\perp: \text{Id}_c \text{F}_\ell(I_\perp, \mathbb{k}) \rightarrow L_\perp$  extends to a natural transformation  $\vec{\chi}: \text{Id}_c \text{F}_\ell(\vec{I}, \mathbb{k}) \rightarrow \vec{L}$  such that  $\chi_t$  is a surjective lattice homomorphism whenever  $t \in T \setminus \{\perp\}$ .*

*Proof.* The proof can be partly followed on Figure 7.3. By Zorn's Lemma, there

$$\begin{array}{ccccc}
 & & \varphi_{s,t} & & \\
 & \nearrow & & \searrow & \\
 L_s & \xrightarrow{\varphi_{s,<t}} & L_{<t} & \xrightarrow{\varphi_{<t,t}} & L_t \\
 \chi_s \uparrow & \xrightarrow{\lim} & \chi_{<t} \uparrow & & \chi_t \uparrow \\
 \text{Id}_c \text{F}_\ell(I_s, \mathbb{k}) & \xrightarrow[\eta_{I_s, I_{<t}}]{} & \text{Id}_c \text{F}_\ell(I_{<t}, \mathbb{k}) & \xrightarrow[\eta_{I_{<t}, I_t}]{} & \text{Id}_c \text{F}_\ell(I_t, \mathbb{k}) \\
 & \searrow & & \nearrow & \\
 & & \eta_{I_s, I_t} & & 
 \end{array}$$

FIGURE 7.3. Illustrating the proof of Theorem 7.3

exists a maximal lower subset  $T'$  of  $T$ , containing  $\{\perp\}$ , on which the conclusion of Theorem 7.3 holds. Suppose, by way of contradiction, that  $T' \neq T$  and let  $t$  be a minimal element of  $T \setminus T'$ ; so  $T' \cup \{t\}$  is also a lower subset of  $T$ . Since the height of  $t$  is countable, so are the lattice  $L_{<t} \stackrel{\text{def}}{=} \varinjlim_{s < t} L_s$  (with transition maps  $\varphi_{s,s'}$  where  $s \leq s' < t$  and limiting maps  $\varphi_{s,<t}: L_s \rightarrow L_{<t}$  for  $s < t$ ) and the set  $I_{<t} \stackrel{\text{def}}{=} \bigcup \{I_s \mid s < t\}$ . The universal property of the colimit ensures the existence of unique 0-lattice homomorphisms

$$\eta_{I_{<t}, I_t}: \text{Id}_c \text{F}_\ell(I_{<t}, \mathbb{k}) = \varinjlim_{s < t} \text{Id}_c \text{F}_\ell(I_s, \mathbb{k}) \rightarrow \text{Id}_c \text{F}_\ell(I_t, \mathbb{k})$$

and  $\varphi_{<t,t}: L_{<t} \rightarrow L_t$ , such that  $\eta_{I_{<t}, I_t} \circ \eta_{I_s, I_{<t}} = \eta_{I_s, I_t}$  and  $\varphi_{<t,t} \circ \varphi_{s,<t} = \varphi_{s,t}$  whenever  $s < t$ . Further, the natural transformation  $(\chi_s \mid s < t)$  induces a unique 0-lattice homomorphism

$$\chi_{<t}: \text{Id}_c \text{F}_\ell(I_{<t}, \mathbb{k}) \rightarrow L_{<t}$$

such that  $\chi_{<t} \circ \eta_{I_s, I_{<t}} = \varphi_{s,<t} \circ \chi_s$  whenever  $s < t$ . By Theorem 7.2, there exists a surjective lattice homomorphism  $\chi_t: \text{Id}_c \text{F}_\ell(I_t, \mathbb{k}) \twoheadrightarrow L_t$  such that  $\chi_t \circ \eta_{I_{<t}, I_t} =$

$\varphi_{<t,t} \circ \chi_{<t}$ . Therefore, for each  $s < t$ ,

$$\chi_t \circ \eta_{I_s, I_t} = \chi_t \circ \eta_{I_{<t}, I_t} \circ \eta_{I_s, I_{<t}} = \varphi_{<t,t} \circ \chi_{<t} \circ \eta_{I_s, I_{<t}} = \varphi_{<t,t} \circ \varphi_{s, <t} \circ \chi_s = \varphi_{s,t} \circ \chi_s.$$

This shows that our conclusion holds at  $T' \cup \{t\}$ , in contradiction with the maximality assumption on  $T'$ .  $\square$

This leads us to the following positive solution of the problem stated at the end of Wehrung [18].

**Theorem 7.4.** *Let  $\mathbb{k}$  be a countable totally ordered division ring. Then every completely normal distributive 0-lattice  $L$  with at most  $\aleph_1$  elements is a surjective homomorphic image of  $\text{Id}_c F$  for some vector lattice  $F$  over  $\mathbb{k}$ .*

*Proof.* Write  $L$  as the directed union of an ascending  $\omega_1$ -sequence  $\vec{L} = (L_\xi \mid \xi < \omega_1)$  of countable completely normal distributive 0-lattices, with  $L_0 = \{0\}$ . Theorem 7.3, applied to the well-ordered chain  $\omega_1$ , yields an  $\omega_1$ -indexed commutative diagram  $\vec{F} = (F_\xi, f_{\xi, \eta} \mid \xi \leq \eta < \omega_1)$  of  $\mathbb{k}$ -vector lattices together with a natural transformation  $\vec{\chi}: \text{Id}_c \vec{F} \rightarrow \vec{L}$  all of whose components are surjective lattice homomorphisms. Letting  $F \stackrel{\text{def}}{=} \varinjlim \vec{F}$ , the universal property of the colimit yields a surjective homomorphism from  $\text{Id}_c F$  onto  $L$ .  $\square$

Due to Wehrung [17, Corollary 9.5], Theorem 7.4 cannot be generalized to *uncountable* totally ordered division rings  $\mathbb{k}$ . On the other hand, setting  $\mathbb{k}$  as any countable Archimedean totally ordered field (for example the rationals),  $\text{Id}_c F$  is identical to the  $\ell$ -ideal lattice of the underlying  $\ell$ -group of  $F$ . Hence,

**Corollary 7.5.** *Every completely normal distributive 0-lattice  $L$  with at most  $\aleph_1$  elements is a surjective homomorphic image of  $\text{Id}_c F$  for some Abelian  $\ell$ -group  $F$ .*

By applying Stone duality for distributive 0-lattices, we obtain the following formulation in terms of spectra.

**Corollary 7.6.** *Every completely normal generalized spectral space with at most  $\aleph_1$  compact open sets embeds, as a spectral subspace, into the  $\ell$ -spectrum of an Abelian  $\ell$ -group.*

Corollary 7.5 also strengthens Ploščica [11, Theorem 3.2], which states that every completely normal distributive 0-lattice of cardinality at most  $\aleph_1$  is Cevian; that is, it carries a binary operation  $(x, y) \mapsto x \setminus y$  such that  $x \leq y \vee (x \setminus y)$ ,  $(x \setminus y) \wedge (y \setminus x) = 0$ , and  $x \setminus z \leq (x \setminus y) \vee (y \setminus z)$  for all  $x, y, z$ . Indeed,  $\text{Id}_c G$  is Cevian for any Abelian  $\ell$ -group  $G$ , and any homomorphic image of a Cevian lattice is Cevian (cf. Wehrung [15, § 5]).

**Problem.** Let  $D$  be a completely normal distributive 0-lattice such that for all  $a, b \in D$  there exists a sequence  $(c_n \mid n < \omega)$  from  $D$  such that for all  $x \in D$ ,  $a \leq b \vee x$  iff there exists  $n < \omega$  such that  $c_n \leq x$  (in [14] we say that  $D$  has *countably based differences*). If  $\text{card } D = \aleph_1$ , does  $D \cong \text{Id}_c G$  for some Abelian  $\ell$ -group  $G$ ?

The cases where  $\text{card } D \leq \aleph_0$  and  $\text{card } D \geq \aleph_2$  are settled in Wehrung [14, 15], in the positive and the negative, respectively (the counterexample constructed in [15] is not even Cevian, thus it is not a homomorphic image of any  $\text{Id}_c G$ ). A Cevian counterexample (of size continuum plus) is constructed in Ploščica [11].

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