



HAL
open science

Pressure jump and radial stationary solutions of the degenerate Cahn-Hilliard equation

Charles Elbar, Benoît Perthame, Jakub Skrzeczkowski

► **To cite this version:**

Charles Elbar, Benoît Perthame, Jakub Skrzeczkowski. Pressure jump and radial stationary solutions of the degenerate Cahn-Hilliard equation. 2022. hal-03696545v1

HAL Id: hal-03696545

<https://hal.science/hal-03696545v1>

Preprint submitted on 16 Jun 2022 (v1), last revised 13 Jan 2023 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Pressure jump and radial stationary solutions of the degenerate Cahn-Hilliard equation

Charles Elbar^{*†} Benoît Perthame^{*‡} Jakub Skrzeczkowski^{§¶}

June 16, 2022

Abstract

The Cahn-Hilliard equation with degenerate mobility is used in several areas including the modeling of living tissues. We are interested in quantifying the pressure jump at the interface in the case of incompressible flows. To do so, we include an external force and consider stationary radial solutions. This allows us to compute the pressure jump in the small dispersion regime. We also characterize compactly supported stationary solutions in the incompressible case, prove the incompressible limit and prove convergence of the parabolic problems to stationary states.

2010 *Mathematics Subject Classification.* 35B40; 35B45; 35G20 ; 35Q92

Keywords and phrases. Degenerate Cahn-Hilliard equation; Asymptotic Analysis; Incompressible limit; Hele-Shaw equations; Surface tension; Pressure jump.

1 Introduction

The degenerate Cahn-Hilliard equation is now commonly used in tumor growth modeling and takes into account surface tensions between different cells, leading to a jump condition on pressure. In order to compute this jump, we propose to consider a spherically symmetric domain and include an external force, and therefore we write the equation

$$\partial_t n - \operatorname{div}(n \nabla \mu) = \operatorname{div}(n \nabla V(|x|)), \quad \text{in } (0, +\infty) \times B_{R_b}, \quad (1.1)$$

$$\mu = n^\gamma - \delta \Delta n, \quad \text{in } (0, +\infty) \times B_{R_b}, \quad (1.2)$$

equipped with an initial data n_0 and Neumann boundary conditions

$$\frac{\partial n}{\partial \nu} = n \frac{\partial(\mu + V)}{\partial \nu} = 0 \text{ on } \partial B_{R_b}, \quad 0 \leq n^0 \leq 1 \text{ a.e.}$$

Here, B_R denotes a ball of radius R , ν the outward normal at its boundary ∂B_{R_b} , while $V(|x|) \in C^1(\mathbb{R})$ is a radially symmetric confining potential, for instance $V(|x|) = |x|^2$. In this work, n^γ actually stands for $\max(0, n)^\gamma$ but since the solutions are nonnegative, we keep the notation n^γ for

^{*}Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions (LJLL), F-75005 Paris, France

[†]email: charles.elbar@sorbonne-universite.fr

[‡]email: benoit.perthame@sorbonne-universite.fr

[§]Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Stefana Banacha 2, 02-097 Warsaw, Poland

[¶]email: jakub.skrzeczkowski@student.uw.edu.pl

sake of clarity.

We also consider radial symmetric solutions and dimension $d = 2$ for simplicity, which means we focus on the system

$$\frac{\partial(rn)}{\partial t} - \frac{\partial}{\partial r} \left(rn \frac{\partial(\mu + V)}{\partial r} \right) = 0, \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (1.3)$$

$$\mu = n^\gamma - \frac{\delta}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right), \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (1.4)$$

where $I_{R_b} = (0, R_b)$ is the line segment of length R_b . Equations (1.3)–(1.4) are equipped with an initial condition $n_0 \in H^1(I_{R_b})$, $0 \leq n_0 \leq 1$ and Neumann boundary conditions

$$\frac{\partial n}{\partial r} \Big|_{r=0} = \frac{\partial n}{\partial r} \Big|_{r=R_b} = n \frac{\partial(\mu + V)}{\partial r} \Big|_{r=0} = n \frac{\partial(\mu + V)}{\partial r} \Big|_{r=R_b} = 0. \quad (1.5)$$

Our first result concerns the existence of solutions to (1.3)–(1.4), their regularity, and asymptotic behaviour.

Theorem 1 (Existence of solutions and asymptotic behaviour). *There exists a global weak solution to (1.3)–(1.4) with initial data n_0 in the sense of Definition 2.1. Moreover, it satisfies estimates as in Remark 2.2. Finally, up to a subsequence, $\{rn(t+k, r)\}_k$ converges to the stationary solution $rn_\infty(r)$ where $n_\infty \in C^1(\overline{I_{R_b}})$ satisfies*

$$rn_\infty \frac{\partial(\mu_\infty + V)}{\partial r} = 0, \quad \mu_\infty = n_\infty^\gamma - \frac{\delta}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n_\infty}{\partial r} \right) \quad n'_\infty(0) = n'_\infty(R_b) = 0. \quad (1.6)$$

Our second result characterizes possible stationary states. From now on we consider the confining potential $V(r) = r^2$ for simplicity. The proof may be adapted to any increasing potential.

Theorem 2 (Characterization of the stationary states). **(A)** *Let $n_\infty \in C^1([0, R_b])$ be a nonnegative solution of (1.6). Then n_∞ is a nonincreasing. Furthermore, $n_\infty(R_b) = 0$ or $0 < n_\infty(R_b) < C/R_b^2$ where C depends on the initial mass.*

(B) *Let $n_\infty(R_b) = 0$ and let $R > 0$ be the smallest argument such that $n_\infty(R) = 0$. Then, there exists a unique value of $\lambda \in (0, R^2)$ such that n_∞ is the unique radial solution of*

$$\begin{cases} n^\gamma - \delta \Delta n = R^2 - r^2 - \lambda & \text{in } B_R \\ n(x) = \frac{\partial n}{\partial \mathbf{n}} = 0 & \text{on } \partial B_R \\ n(x) > 0 & \text{on } \partial B_R \end{cases} \quad (1.7)$$

Furthermore, there is at most one couple (n, λ) solving (1.7).

(C) *Let $\delta \in (0, 1)$ and $m = \int_0^{R_b} r n_\infty(r) dr$. There exists $R(m)$ such that for all $R_b > R(m)$, it is not possible that $n_\infty(R_b) > 0$.*

We focus on the incompressible limit of the solutions of (1.7).

Theorem 3 (Incompressible limit of the stationary states). *Let $\{\gamma_k\}_{k \in \mathbb{N}}$ be any sequence such that $\gamma_k \rightarrow \infty$. Let $\{n_k\}_{k \in \mathbb{N}}$ be a sequence of stationary states with the same mass*

$$\int_\Omega r n_k(r) dr = \int_\Omega r n_0(r) dr$$

and with radius R_k being the first argument such that $n_k(r) = 0$. Then, $n_k \rightarrow \tilde{u}_c$ strongly and $R_k \rightarrow R$, where \tilde{u}_c and R are defined in Proposition 3.8. Moreover the sequence of pressures $\{n_k^{\gamma_k}\}_{k \in \mathbb{N}}$ converges weakly to some pressure p such that $p(\tilde{u}_c - 1) = 0$ and p has a jump at $\partial\{\tilde{u}_c = 1\}$

$$\llbracket p \rrbracket \approx \sqrt[3]{6} R^{2/3} \delta^{1/3}, \quad \text{as } \delta \rightarrow 0.$$

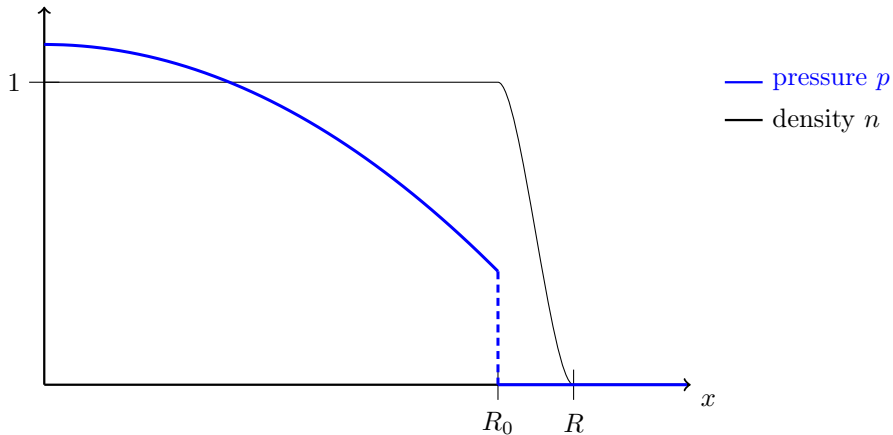


Figure 1: Plot of the limiting profile for the potential $V(r) = r^2$. The tumor is situated in the zone $\{n = 1\} = \{p > 0\}$.

The profile \tilde{u}_c obtained for the incompressible limit of stationary states is as follows; see Figure 1. The density is equal to 1 on a certain interval where the pressure is strictly positive. Then, this density decreases on a small interval towards 0. At the boundary, i.e. when the density starts to decrease, the pressure undergoes a jump, which depends on the surface tension coefficient δ and on the shape of the confinement potential. More precisely, for a general potential $V(r)$, this jump is

$$\llbracket p \rrbracket \approx \frac{\sqrt[3]{12}}{2} \delta^{1/3} (V'(R))^{2/3} \quad \text{as } \delta \rightarrow 0, \quad (1.8)$$

where R is the first time when $n(R) = 0$. We point out that the limiting profile (including parameters R_0 and R) is uniquely determined in terms of mass $\int_{\Omega} r n_0(r) dr$ and δ cf. Proposition 3.8.

The main novelty of our work concerns the incompressible limit $\gamma \rightarrow \infty$ for the stationary states. Previous work in this direction [13] made use of viscosity relaxation, which provided additional estimates implying compactness. In our case, assuming the radial symmetry of the problem, we are able to characterize the incompressible limit of the sequence of compactly supported stationary solutions. While our setting is restrictive, it allows performing many computations explicitly. In particular, we find how the pressure jump depends on V and δ , cf. (1.8).

Open question. In this paper, we prove that the stationary states are compactly supported or at least zero on the boundary if the domain is large enough. It is logical to ask whether the solutions of the parabolic equation are compactly supported for a large domain and a strong confining potential. This question is still open. However, work in this direction [9] has proved that in dimension 1, one could expect the solutions of the Cahn-Hilliard equation without confining potential to propagate with finite speed. By adding this potential, we can expect to have a better result, and compactly supported solutions, with time-independent support.

Contents of the paper. In Section 2, we prove Theorem 1. In Section 3 we prove Theorems 2 and 3. Numerical simulations of the model with a source term and no confining potential are presented in Section 4. The appendix contains the computation of the pressure jump for a general confining potential.

Notations. For a function $n(x, t)$ we associate a function in radial coordinates that is still denoted by $n(r, t)$. For $1 \leq p, s \leq +\infty$ or $s = -1$ and Ω a domain, $L^p(\Omega), H^s(\Omega)$ denote the usual Lebesgue and Sobolev spaces. When $s = -1$, $H^{-1}(\Omega)$ is the topological dual of $H^1(\Omega)$. Here $H^s(\Omega) = W^{s,2}(\Omega)$ in the usual notation. We also consider the Bochner spaces $L^p(0, T; H^s(\Omega))$ associated with the norm

$$\|f\|_{L^p(0, T; H^s(\Omega))} = \left(\int_0^T \|f\|_{H^s(\Omega)}^p \right)^{1/p}$$

∂_r denotes the partial derivative with respect to the radial variable, $\partial_r u(r) = \frac{\partial u}{\partial r}(r) = u'(r)$. Finally, C denotes a generic constant which appears in inequalities and whose value can change from one line to another. This constant can depend on various parameters unless when it is specified otherwise.

1.1 Literature review and biological relevancy of the system

Tissue growth models and Hele-Shaw limits. Development of tissue growth models is a major line of research in mathematical biology since it was initiated by the influential work of Greenspan [22]. A number of models are now available [6, 17, 24] with the common feature that they use the tissue internal pressure as the main driver of both the cell movement and proliferation. The simplest example of a mechanical model of living tissue is the *compressible* equation

$$\partial_t n = \operatorname{div}(n \nabla p) + nG(p), \quad p = P_\gamma(n) := n^\gamma, \quad (1.9)$$

in which $p(t, x) = P(n(t, x))$, with P a law of state, is the pressure and n the density of cell number. Here, the cell velocity is given via Darcy's law which captures the effect of cells moving away from regions of high compression. Dependence on growth function pressure has also been used to model the sensitivity of tissue proliferation to compression (contact inhibition, [5]).

An important problem is to understand the so-called incompressible limit (*i.e.* $\gamma \rightarrow \infty$) of this model. Perthame *et al.* [27] have shown that in this limit, solutions of (1.9) converge to a limit solution (n_∞, p_∞) of a Hele-Shaw-type free boundary limit problem for which the speed of the free boundary is given by the normal component of ∇p_∞ . In this limit, the solution of (1.9) is organized into 2 regions: $\Omega(t)$ in which the pressure is positive (corresponding to the tissue) and outside of this zone where $p = 0$. Furthermore, the free boundary problem is supplemented by a complementary equation that indicates that the pressure satisfies

$$-\Delta p_\infty = G(p_\infty), \quad \text{in } \Omega(t), \quad \text{or similarly } p_\infty(\Delta p_\infty + G(p_\infty)) = 0 \quad \text{a.e. in } \Omega. \quad (1.10)$$

In this model, the pressure stays continuous and is equal to 0 at the interface. This is because only repulsive forces were taken into account. Hence, the crucial role of the cell-cell adhesion and thus the pressure jump at the surface of the tissue is not retrieved at the limit. On the other hand, as pointed out by Lowengrub *et al.* [24], the velocity of the free surface should depend on its geometry and more precisely on the local curvature denoted by κ .

This motivated considering variants of the general model (1.9), where other physical effects of mechanical models of tissue growth are introduced. One of them is the addition of the effect of viscosity in the model, which has been made to represent the friction between cells [3, 4] through the use of Stokes' or Brinkman's law (see [2] for a rigorous derivation of Brinkman's law in inhomogeneous materials). Moreover, as pointed out by Perthame and Vauchelet [28], Brinkman's law leads to a simpler version of the model and, therefore, is a preferential choice for its mathematical analysis. Adding viscosity through the use of Brinkman's law leads to the model

$$\begin{cases} \partial_t n = \operatorname{div}(n \nabla \mu) + nG(p), & \text{in } (0, +\infty) \times \Omega, \\ -\sigma \Delta \mu + \mu = p, & \text{in } (0, +\infty) \times \Omega. \end{cases} \quad (1.11)$$

The incompressible limit of this system also yields the complementary relation (see [28])

$$p_\infty(p_\infty - \mu_\infty - \sigma G(p_\infty)) = 0, \quad \text{a.e. in } \Omega.$$

In the incompressible limit, notable changes compared to the system with Darcy's law are found. First, the previous complementary relation is different compared to Equation (1.10), and the pressure p_∞ in the limit is discontinuous, *i.e.* there is a jump of the pressure located at the surface of $\Omega(t)$. However, the pressure jump is related to the potential μ and not to the local curvature of the free boundary $\partial\Omega(t)$. The authors already indicated that a possible explanation for this is that the previous model does not include the effect of surface tension.

Surface tension and pressure jump. Surface tension is a concept associated with the internal cohesive forces between the molecules of a fluid: hydrogen bonds, van der Waals forces, metallic bonds, etc. Inside the fluid, molecules are attracted equally in all directions leading to a net force of zero; however molecules on the surface experience an attractive force that tends to pull them to the interior of the fluid: this is the origin of the surface energy. This energy is equivalent to the work or energy required to remove the surface layer of molecules in a unit area. The value of the surface tension will vary greatly depending on the nature of the forces exerted between the atoms or molecules. In the case of solid tumor cells in a tissue, it reflects the cell-cell adhesion tendency between the cells and depends on the parameter δ and the geometry of the tumor.

In the previous definition, the surface tension is associated with a single body that has an interface with the vacuum. When one considers two bodies, the surface energy of each body is modified by the presence of the other and we speak of interfacial tension. The latter depends on the surface tension of each of the two compounds, as well as the interaction energy between the two compounds. In the system considered above, it is then possible to imagine that the vacuum in which the tumor grows is in fact another body that has an internal pressure of the form $V(r)$ which increases with respect to r so that the tumor is stopped at some point and we can consider the stationary states.

For such a tumor to be in equilibrium, it is necessary that the interior is overpressured relative to the exterior by an amount. This amount is called the pressure jump and is computed explicitly in our case.

Surface tension effects can be introduced in the Hele-Shaw model as follows (see *e.g.* [15])

$$\begin{cases} -\Delta\mu = 0 & \text{in } \Omega \setminus \partial\Omega(t), \\ \mu = \sigma\kappa & \text{on } \partial\Omega(t). \end{cases} \quad (1.12)$$

where σ is a positive constant, called a surface tension and κ is a mean curvature of $\partial\Omega(t)$. This correct Hele-Shaw limit has been formally obtained as the sharp-interface asymptotic model of the Cahn-Hilliard equation [1]; see also [10] for a convergence result in a weak varifold formulation. This suggests that the Cahn-Hilliard equation is an appropriate model to capture surface tension effects.

The Cahn Hilliard equation. System (1.1)–(1.2) is an example of a Cahn-Hilliard type equation that is widely used nowadays to represent living tissues and in particular tumors [29, 16, 20, 19, 18, 11, 12, 26]. Originally introduced in the context of materials sciences [8, 7], it is currently applied in numerous fields, including complex fluids, polymer science, and mathematical biology. For the overview of mathematical theory, we refer to [25].

Cahn-Hilliard equation takes the form of

$$\partial_t n = \operatorname{div}(b(n)\nabla(\psi'(n) - \delta\Delta n)) \rightarrow \begin{cases} \partial_t n & = \operatorname{div}(b(n)\nabla\mu), \\ \mu & = -\delta\Delta n + \psi'(n), \end{cases} \quad (1.13)$$

where n represents the relative density of cells $n = n_1/(n_1 + n_2)$, b is the mobility, ψ is the potential while μ is the quantity of chemical potential, which is a quantity related to the effective pressure.

From the point of view of mathematical biology, the most relevant case is $b(n) = n$, which is referred to as degenerate mobility. Then, (1.1) models the motion of a population of cells constituting a biological tissue in the form of a continuity equation. It takes into account pressure, the surface tension occurring at the surface of the tissue and its viscosity. More precisely, the equation for μ (*i.e.* equation (1.2)) includes the effects of both the pressure, through the term n^γ with $\gamma > 1$ that controls the stiffness of the pressure law, and surface tension by $-\delta\Delta n$, where $\sqrt{\delta}$ is the width of the interface in which partial mixing of the two components n_1, n_2 occurs.

System (1.1)–(1.2) has previously been considered in [13] with a viscosity term and a proliferation source term in place of the confinement potential. In the incompressible limit, the authors obtain a jump in pressure at the interface at all times for the relaxed system. The aim here is to make a rigorous limit without the viscosity relaxation and to compute the pressure jump by analyzing the stationary states of a system with confining potential.

2 Existence, regularity, and asymptotic behavior for the parabolic problem

The existence of weak solutions for the Cahn-Hilliard equation with degenerate mobility usually follows the method from [14]. The idea is to apply a Galerkin scheme with a non-degenerate regularized mobility, *i.e.*, calling $b(n)$ the mobility, then one considers an approximation $b_\varepsilon(n) \geq \varepsilon$. Then, using standard compactness methods one can prove the existence of weak solutions for the initial system. However, the uniqueness of the weak solutions is still an open question.

In the case of a radially symmetric solution, the resulting system has only one dimension in space, and it is possible to apply a fixed point theorem, see [30], to obtain better regularity results. Since the solutions have radial symmetry, the equation is singular at $r = 0$. Therefore, the first step is to consider the system with $r + \varepsilon$ instead of r and a regularized mobility. The existence of solutions for a similar regularized system has been achieved in [30] based on a result of [23] and we do not repeat the arguments here. We can then pass to the limit $\varepsilon \rightarrow 0$. Finally, the nonnegativity of the limiting solution is achieved with the bounds provided by the entropy.

We finally point out that since we are also interested in the convergence to the stationary states, one needs to carefully verify that the bounds do not depend on time.

Definition 2.1 (Weak solutions). We say that $n(t, r)$ is a global weak solution of the equation (1.3)–(1.4) provided that:

- n is nonnegative,
- rn is continuous in $[0, \infty) \times \overline{I_{R_b}}$, $\sqrt{r}n \in L^\infty((0, \infty) \times I_{R_b})$ and $r \partial_t n \in L^2((0, \infty); H^{-1}(I_{R_b}))$,
- $\sqrt{rn} \partial_r \mu \in L^2((0, \infty) \times \overline{I_{R_b}} \setminus \{rn = 0\})$ and μ is defined in (1.4),
- for every test function $\varphi \in L^2((0, \infty); H^1(I_{R_b})) \cap C_c^1([0, \infty) \times I_{R_b})$

$$\int_0^T r \langle \partial_t n, \varphi \rangle_{H^{-1}, H^1} dt + \int_0^T \int_0^{R_b} \mathbf{1}_{rn > 0} r n \partial_r (\mu + V) \partial_r \varphi dr dt = 0,$$

and

$$\int_0^T r \langle \partial_t n, \varphi \rangle_{H^{-1}, H^1} dt = - \int_0^T \int_0^{R_b} rn \partial_t \varphi dr dt - \int_0^{R_b} \varphi(0, r) n_0(r) dr,$$

- $n'(t, R_b) = 0$ for a.e. $t \in (0, T)$.

Remark 2.2 (Better properties of weak solutions). In fact, we will construct solutions satisfying additionally mass, energy, and entropy inequalities as follows: for a.e. $\tau \in [0, T]$

$$\int_0^{R_b} r n(\tau, r) dr = \int_0^{R_b} r n_0(r) dr, \quad (2.1)$$

$$\mathcal{E}[n(\tau, \cdot)] + \int_0^\tau \int_0^{R_b} \mathbb{1}_{rn>0} rn |\partial_r(\mu + V)|^2 dr dt \leq \mathcal{E}[n_0], \quad (2.2)$$

$$\Phi[n(\tau, \cdot)] + \int_0^\tau \int_0^{R_b} \left(\gamma r n^{\gamma-1} |\partial_r n|^2 + \delta r |\partial_{rr} n|^2 + \delta \frac{|\partial_r n|^2}{r} + r \partial_r n \partial_r V \right) dr dt \leq \Phi[n_0]. \quad (2.3)$$

where energy and entropy are defined as follows:

$$\mathcal{E}[n] = \int_0^{R_b} r \left(\frac{n^{\gamma+1}}{\gamma+1} + \frac{\delta}{2} |\partial_r n|^2 + n V \right) dr, \quad \Phi[n] = \int_0^{R_b} r \phi(n) dr,$$

and $\phi(n) = n(\log(n) - 1) + 1$. Equations (2.1)–(2.3) provide additional a priori estimates. Moreover, the constructed solution will be Hölder continuous in time and space: there is a constant C , such that for all $r, r_1, r_2 \in [0, R_b]$, $t, t_1, t_2 \in [0, \infty)$

$$|r_1 n(t, r_1) - r_2 n(t, r_2)| \leq C |r_1 - r_2|^{1/2}, \quad (2.4)$$

$$|r(n(t_2, r) - n(t_1, r))| \leq C |t_2 - t_1|^{1/8}. \quad (2.5)$$

2.1 Regularized system

We consider the existence of a regularized system, which reads:

$$\partial_t(r + \varepsilon)n_\varepsilon - \partial_r((r + \varepsilon)B_\varepsilon(n_\varepsilon) \partial_r(\mu_\varepsilon + V)) = 0, \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (2.6)$$

$$\mu_\varepsilon = n_\varepsilon^\gamma - \frac{\delta}{r + \varepsilon} \partial_r((r + \varepsilon)\partial_r n_\varepsilon), \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (2.7)$$

where

$$B_\varepsilon(n) = \begin{cases} \varepsilon & \text{for } n \leq \varepsilon, \\ n & \text{otherwise.} \end{cases} \quad (2.8)$$

We impose Neumann boundary conditions

$$\frac{\partial n_\varepsilon}{\partial r} \Big|_{r=0} = \frac{\partial n_\varepsilon}{\partial r} \Big|_{r=R_b} = B_\varepsilon(n_\varepsilon) \frac{\partial(\mu_\varepsilon + V)}{\partial r} \Big|_{r=0} = B_\varepsilon(n_\varepsilon) \frac{\partial(\mu_\varepsilon + V)}{\partial r} \Big|_{r=R_b} = 0. \quad (2.9)$$

From [30] we have the

Theorem 2.3. *For $\varepsilon > 0$ and $T > 0$, Problem (2.6)–(2.7) with boundary conditions (2.9) admits a unique classical solution n_ε .*

Next, we prove certain conservation properties for the system (2.6)–(2.7), see for instance [13, 26].

Lemma 2.4 (Conservation of mass, energy and entropy). *Let us define*

$$\begin{aligned} \mathcal{E}_\varepsilon[n] &:= \int_0^{R_b} (r + \varepsilon) \left(\frac{n^{\gamma+1}}{\gamma+1} + \frac{\delta}{2} |\partial_r n|^2 + n V \right) dr, \\ \Phi_\varepsilon[n] &:= \int_0^{R_b} (r + \varepsilon) \phi_\varepsilon(n) dr, \end{aligned}$$

where ϕ_ε is such that $\phi_\varepsilon''(n) = \frac{1}{B_\varepsilon(n)}$ and $\phi_\varepsilon(1) = \phi_\varepsilon'(1) = 0$. Then,

$$\frac{d}{dt} \int_0^{R_b} (r + \varepsilon) n_\varepsilon(t, r) dr = 0, \quad (2.10)$$

$$\frac{d}{dt} \mathcal{E}_\varepsilon[n_\varepsilon] + \int_0^{R_b} (r + \varepsilon) B_\varepsilon(n_\varepsilon) |\partial_r(\mu_\varepsilon + V)|^2 dr = 0, \quad (2.11)$$

$$\frac{d}{dt} \Phi_\varepsilon[n_\varepsilon] + \int_0^{R_b} \left(\gamma(r + \varepsilon) n_\varepsilon^{\gamma-1} |\partial_r n_\varepsilon|^2 + \delta(r + \varepsilon) |\partial_{rr} n_\varepsilon|^2 + \delta \frac{|\partial_r n_\varepsilon|^2}{r + \varepsilon} + (r + \varepsilon) \partial_r n_\varepsilon \partial_r V \right) dr = 0. \quad (2.12)$$

Remark 2.5. The function ϕ_ε is given by an explicit formula

$$\phi_\varepsilon(x) = \begin{cases} x(\log(\varepsilon) - 1) + 1 + x^2/(2\varepsilon) - \varepsilon/2 & \text{if } x \leq \varepsilon \\ x(\log(x) - 1) + 1 & \text{if } \varepsilon < x. \end{cases} \quad (2.13)$$

With $\varepsilon < 1$, it enjoys three properties:

1. $\phi_\varepsilon(x) \rightarrow \phi(x) := x(\log(x) - 1) + 1$ for $x \geq 0$ as $\varepsilon \rightarrow 0$,
2. $\phi_\varepsilon(x) \geq 0$ for all $x \in \mathbb{R}$,
3. $\phi_\varepsilon(x) \leq \phi(x) + 1 - \varepsilon/2$ for $x \geq 0$.

The first one is trivial. To see the second one, we observe that the function $x \mapsto x(\log(x) - 1) + 1$ is nonnegative, which implies $\phi_\varepsilon(x) \geq 0$ for $x \geq \varepsilon$. Then, for $x \leq \varepsilon$ we discover

$$\phi_\varepsilon'(x) = \log(\varepsilon) - 1 + \frac{x}{\varepsilon} \leq 0.$$

As $\phi_\varepsilon(\varepsilon) \geq 0$, this implies $\phi_\varepsilon(x) \geq 0$ for all $x \in \mathbb{R}$. The third property follows by estimating $\phi_\varepsilon(x) \leq \phi_\varepsilon(0)$ for $x \leq \varepsilon$.

Proof of Lemma 2.4. Mass conservation (2.10) follows from integrating (2.6) in space. To see (2.11), we multiply (2.6) by $\mu_\varepsilon + V$, integrate in space and use boundary conditions to obtain:

$$\int_0^{R_b} (r + \varepsilon) \partial_t n_\varepsilon (\mu_\varepsilon + V) dr + \int_0^{R_b} (r + \varepsilon) B_\varepsilon(n_\varepsilon) |\partial_r(\mu_\varepsilon + V)|^2 dr = 0.$$

Using (2.7) and integrating by parts, we obtain

$$\int_0^{R_b} (r + \varepsilon) \partial_t n_\varepsilon (\mu_\varepsilon + V) dr = \frac{d}{dt} \mathcal{E}_\varepsilon[n_\varepsilon],$$

concluding the proof of (2.11). To see (2.12), we multiply (2.6) by $\phi_\varepsilon'(n_\varepsilon)$ and integrate in space to obtain

$$\frac{d}{dt} \Phi_\varepsilon[n_\varepsilon] + \int_0^{R_b} (r + \varepsilon) n_\varepsilon'(r) \partial_r(\mu_\varepsilon + V) dr = 0.$$

In view of (2.12), it is sufficient to prove

$$\int_0^{R_b} (r + \varepsilon) n_\varepsilon'(r) \partial_r \mu_\varepsilon dr = \gamma \int_0^{R_b} (r + \varepsilon) |n_\varepsilon'(r)|^2 n_\varepsilon^{\gamma-1} dr + \delta \int_0^{R_b} \frac{|n_\varepsilon'|^2}{r + \varepsilon} + (r + \varepsilon) |n_\varepsilon''|^2 dr.$$

We have

$$\int_0^{R_b} (r+\varepsilon) n'_\varepsilon(r) \partial_r \mu_\varepsilon \, dr = \gamma \int_0^{R_b} (r+\varepsilon) |n'_\varepsilon(r)|^2 n_\varepsilon^{\gamma-1} \, dr - \delta \int_0^{R_b} (r+\varepsilon) n'_\varepsilon(r) \partial_r \left(\frac{1}{r+\varepsilon} \partial_r ((r+\varepsilon) \partial_r n_\varepsilon) \right) \, dr.$$

In the second part, we can integrate by parts (using Neumann's boundary conditions)

$$\begin{aligned} & - \int_0^{R_b} (r+\varepsilon) n'_\varepsilon(r) \partial_r \left(\frac{1}{r+\varepsilon} \partial_r ((r+\varepsilon) \partial_r n_\varepsilon) \right) \, dr \\ &= \int_0^{R_b} n'_\varepsilon(r) \frac{1}{r+\varepsilon} \partial_r ((r+\varepsilon) \partial_r n_\varepsilon) \, dr + \int_0^{R_b} (r+\varepsilon) n''_\varepsilon(r) \frac{1}{r+\varepsilon} \partial_r ((r+\varepsilon) \partial_r n_\varepsilon) \, dr \\ &= \int_0^{R_b} \frac{|n'_\varepsilon|^2}{r+\varepsilon} + (r+\varepsilon) |n''_\varepsilon|^2 \, dr + 2 \int_0^{R_b} n'_\varepsilon(r) n''_\varepsilon(r) \, dr. \end{aligned}$$

The last term vanishes thanks to boundary conditions:

$$2 \int_0^{R_b} n'_\varepsilon(r) n''_\varepsilon(r) \, dr = \int_0^{R_b} \partial_r |n'_\varepsilon(r)|^2 \, dr = 0$$

and this concludes the proof. \square

From Lemma 2.4, we deduce the following uniform bounds (in ε) of the solutions n_ε in the following

Proposition 2.6. *Let $T > 0$. The following sequences are uniformly bounded with respect to $\varepsilon > 0$:*

- (A1) $\{\sqrt{r+\varepsilon} \partial_r n_\varepsilon\}$ in $L^\infty((0, \infty); L^2(I_{R_b}))$,
- (A2) $\{\sqrt{r+\varepsilon} n_\varepsilon\}$ in $L^\infty((0, \infty) \times I_{R_b})$,
- (A3) $\{\sqrt{(r+\varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r (\mu_\varepsilon + V)\}$ in $L^2((0, \infty) \times I_{R_b})$,
- (A4) $\{\sqrt{r+\varepsilon} \partial_{rr} n_\varepsilon\}$ and $\left\{ \frac{\partial_r n_\varepsilon}{\sqrt{r+\varepsilon}} \right\}$ in $L^2((0, T) \times I_{R_b})$,
- (A5) $\{\Phi_\varepsilon(n_\varepsilon)\}$ in $L^\infty(0, T)$,
- (A6) $\{(r+\varepsilon) \partial_t n_\varepsilon\}$ in $L^2((0, \infty); H^{-1}(I_{R_b}))$,

where the estimates (A1)–(A3) and (A6) depend only on the initial energy $\mathcal{E}(n_0)$. Moreover, there is a constant C , independent of ε , such that for all $r, r_1, r_2 \in [0, R_b]$, $t, t_1, t_2 \in [0, \infty)$

$$|(r_1 + \varepsilon) n_\varepsilon(r_1, t) - (r_2 + \varepsilon) n_\varepsilon(r_2, t)| \leq C |r_1 - r_2|^{1/2}, \quad (2.14)$$

$$|(r + \varepsilon) (n_\varepsilon(t_2, r) - n_\varepsilon(t_1, r))| \leq C |t_2 - t_1|^{1/8}. \quad (2.15)$$

In fact, the constant C depends only on initial energy $\mathcal{E}(n_0)$.

Proof of Proposition 2.6. We divide the reasoning into a few steps.

Step 1: Estimates (A1)–(A2). First, from (2.11) we deduce (A1). For estimate (A2) we adapt the method from [30]. For any $\rho \in (0, R_b)$,

$$\begin{aligned}
& \frac{R_b^2 + 2\varepsilon R_b}{2} n_\varepsilon(t, \rho) - \int_0^{R_b} (z + \varepsilon) n_\varepsilon(t, z) \, dz \\
&= \int_0^{R_b} (z + \varepsilon) [n(t, \rho) - n(t, z)] \, dz \\
&= \int_0^{R_b} \int_z^\rho (z + \varepsilon) \partial_r n_\varepsilon(t, r) \, dr \, dz \\
&= \int_0^\rho \int_z^\rho (z + \varepsilon) \partial_r n_\varepsilon(t, r) \, dr \, dz + \int_\rho^{R_b} \int_z^\rho (z + \varepsilon) \partial_r n_\varepsilon(t, r) \, dr \, dz \\
&= \int_0^\rho \int_0^r (z + \varepsilon) \partial_r n_\varepsilon(t, r) \, dz \, dr + \int_\rho^{R_b} \int_r^{R_b} (z + \varepsilon) \partial_r n_\varepsilon(t, r) \, dz \, dr \\
&= \int_0^\rho \left(\frac{r^2}{2} + \varepsilon r \right) \partial_r n_\varepsilon(t, r) \, dr + \int_\rho^{R_b} \left[\frac{1}{2}(R_b^2 - r^2) + \varepsilon(R_b - r) \right] \partial_r n_\varepsilon(t, r) \, dr \\
&\leq R_b \int_0^\rho (r + \varepsilon) |\partial_r n_\varepsilon(t, r)| \, dr + 2R_b^2 \int_\rho^{R_b} |\partial_r n_\varepsilon(t, r)| \, dr.
\end{aligned}$$

Multiplying the previous inequality by $2(\rho + \varepsilon)^{1/2}$ yields

$$\begin{aligned}
& \left| (R_b^2 + 2\varepsilon R_b)(\rho + \varepsilon)^{1/2} n_\varepsilon(t, \rho) - 2(\rho + \varepsilon)^{1/2} \int_0^{R_b} (z + \varepsilon) n_\varepsilon(t, z) \, dz \right| \leq \\
& \leq 2(\rho + \varepsilon)^{1/2} R_b \int_0^\rho (r + \varepsilon) |\partial_r n_\varepsilon(t, r)| \, dr + 4R_b^2 \int_\rho^{R_b} (r + \varepsilon)^{1/2} |\partial_r n_\varepsilon(t, r)| \, dr \\
& \leq C(R_b) \left(\int_0^{R_b} (r + \varepsilon) |\partial_r n_\varepsilon(t, r)|^2 \, dr \right)^{1/2}.
\end{aligned}$$

Thanks to the conservation of mass (2.10), we obtain (A2).

Step 2: Estimates (A3)–(A5). The bound (A3) follows from the conservation of energy (2.11). To see (A4) and (A5), we want to use the conservation of entropy (2.12), but this has to be done carefully, as the term $(r + \varepsilon) \partial_r n_\varepsilon \partial_r V$ can be negative. Therefore, we fix $T > 0$, consider ϕ_ε as in Remark 2.5 and integrate (2.12) on $(0, T)$ to deduce

$$\Phi_\varepsilon(n_\varepsilon(T, \cdot)) + \delta \int_0^T \int_0^{R_b} (r + \varepsilon) |\partial_{rr} n_\varepsilon|^2 + \frac{|\partial_r n_\varepsilon|^2}{r + \varepsilon} \, dr \, dt \leq \Phi_\varepsilon(n_0) + \int_0^T \int_0^{R_b} (r + \varepsilon) \partial_r n_\varepsilon \partial_r V \, dr \, dt.$$

The last term can be easily bounded (using estimates (A1)–(A2)) by a constant depending on T . The conclusion follows from $\Phi_\varepsilon(n_\varepsilon(T, \cdot)) \geq 0$, cf. Remark 2.5.

Step 3: Estimate (A6). Let $\chi \in L^2(0, T; H^1(I_{R_b}))$. We multiply (2.6) by χ and integrate with respect to r between 0 and R_b . Using an integration by parts and Neumann boundary conditions, we obtain

$$\begin{aligned}
\int_0^\infty \int_0^{R_b} (r + \varepsilon) \partial_t n_\varepsilon \chi \, dr \, dt &= - \int_0^\infty \int_0^{R_b} (r + \varepsilon) B_\varepsilon(n_\varepsilon) \partial_r (\mu_\varepsilon + V) \partial_r \chi \, dr \, dt \\
&= - \int_0^\infty \int_0^{R_b} \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r (\mu_\varepsilon + V) \partial_r \chi \, dr \, dt \\
&\leq \| \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \|_\infty \| \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r (\mu_\varepsilon + V) \|_2 \| \partial_r \chi \|_2.
\end{aligned}$$

where the norms are taken over $(0, \infty) \times I_{R_b}$. The conclusion follows.

Step 4: Hölder estimate in space (2.14). By differentiation

$$(r_2 + \varepsilon) n(t, r_2) - (r_1 + \varepsilon) n(t, r_1) = \int_{r_1}^{r_2} (r + \varepsilon) \partial_r n(t, r) \, dr + \int_{r_1}^{r_2} n(t, r) \, dr.$$

For the first term, we have

$$\int_{r_1}^{r_2} (r + \varepsilon) \partial_r n_\varepsilon(t, r) \, dr \leq \left(\int_{r_1}^{r_2} (r + \varepsilon) \, dr \right)^{1/2} \| \sqrt{r + \varepsilon} \partial_r n_\varepsilon(t, \cdot) \|_2 \leq C |r_1 - r_2|^{1/2}$$

due to (A1). For the second term, we compute, using (A2),

$$\begin{aligned}
\int_{r_1}^{r_2} n_\varepsilon(t, r) \, dr &= \int_{r_1}^{r_2} n_\varepsilon(t, r) \frac{\sqrt{r + \varepsilon}}{\sqrt{r + \varepsilon}} \, dr \leq \| \sqrt{r + \varepsilon} n_\varepsilon \|_\infty \int_{r_1}^{r_2} \frac{1}{\sqrt{r + \varepsilon}} \, dr \leq \\
&\leq C | \sqrt{r_2 + \varepsilon} - \sqrt{r_1 + \varepsilon} | \leq C |r_1 - r_2|^{1/2}.
\end{aligned}$$

Step 5: Hölder estimate in time (2.15). The idea is to deduce the regularity in time from the regularity in space. We extend the function $r \mapsto n_\varepsilon r$ for $r < 0$ with a constant to preserve continuity. We consider η_ν to be a usual one-dimensional mollifier in the spatial variable r where ν will be chosen later in terms of $|t_2 - t_1|$. Mollifying (2.6) with η_ν and integrating in time (from t_1 to t_2) we obtain

$$((r + \varepsilon) n_\varepsilon) * \eta_\nu(t_2, r) - ((r + \varepsilon) n_\varepsilon) * \eta_\nu(t_1, r) = \int_{t_1}^{t_2} \partial_r \eta_\nu * ((r + \varepsilon) B_\varepsilon(n_\varepsilon) \partial_r (\mu_\varepsilon + V)) \, dt. \quad (2.16)$$

First, we estimate (RHS). We notice that Young's convolutional inequality and Hölder's inequality are implying for fixed $t \in [t_1, t_2]$

$$\begin{aligned}
\| \partial_r \eta_\nu * ((r + \varepsilon) B_\varepsilon(n_\varepsilon) \partial_r (\mu_\varepsilon + V)) \|_\infty &\leq \\
&\leq \| \partial_r \eta_\nu \|_2 \| \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \|_\infty \| \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r (\mu_\varepsilon + V) \|_2.
\end{aligned}$$

By the definition of a mollifier,

$$\| \partial_r \eta_\nu \|_2 = \frac{1}{\nu^2} \left| \int_{\mathbb{R}} (\eta')^2 \left(\frac{r}{\nu} \right) \, dr \right|^{1/2} \leq \frac{C}{\nu^{3/2}}.$$

Therefore, applying (A2), (A3) and Hölder's inequality in time, we deduce

$$\int_{t_1}^{t_2} \| \partial_r \eta_\nu * ((r + \varepsilon) B_\varepsilon(n_\varepsilon) \partial_r (\mu_\varepsilon + V)) \|_\infty \, dt \leq C \frac{|t_2 - t_1|^{1/2}}{\nu^{3/2}}. \quad (2.17)$$

To conclude the proof, we need to correct (2.16) using (2.14) from Step 4:

$$\begin{aligned}
|((r + \varepsilon)n_\varepsilon) * \eta_\nu(t_1, r) - ((r + \varepsilon)n_\varepsilon)(t_1, r)| &\leq \\
&\leq \int_{\mathbb{R}} |(r + y + \varepsilon)n_\varepsilon(t_1, r + y) - (r + \varepsilon)n_\varepsilon(t_1, r)| \eta_\nu(y) \, dy \\
&\leq C \int_{\mathbb{R}} |y|^{1/2} \eta_\nu(y) \, dy \leq C \nu^{1/2},
\end{aligned} \tag{2.18}$$

where we used the fact that on the support of η_ν we have $|y| \leq \nu$. Exactly the same estimate holds if we replace t_1 with t_2 . Combining (2.16), (2.17) and (2.18) we obtain

$$|(r + \varepsilon)n_\varepsilon(t_2, r) - (r + \varepsilon)n_\varepsilon(t_1, r)| \leq C \frac{|t_2 - t_1|^{1/2}}{\nu^{3/2}} + C \nu^{1/2}.$$

We choose $\nu = |t_2 - t_1|^{1/4}$ and this concludes the proof. \square

Remark 2.7. In fact, Step 1 of the proof above shows something more general: if $n(t, r) : [0, \infty) \times [0, R_b] \rightarrow \mathbb{R}$ satisfies $\int_0^{R_b} r n(t, r) \, dr = \int_0^{R_b} r n(0, r) \, dr$ and $\sqrt{r} \partial_r n \in L^\infty(0, T; L^2(0, R_b))$ then

$$|\sqrt{r} n(t, r)| \leq C \left(R_b, \int_0^{R_b} r n(0, r) \, dr, \|\sqrt{r} \partial_r n\|_{L^\infty(0, T; L^2(0, R_b))} \right).$$

2.2 Proof of Theorem 1 (existence)

We are concerned with the first part of Theorem 1 *i.e.* the convergence $\varepsilon \rightarrow 0$ of the approximated scheme.

Proof of Theorem 1 (existence). The proof is divided into several steps.

Step 1: compactness. By the estimates in Proposition 2.6, Banach-Alaoglu, and Arzela-Ascoli theorems, we can extract a subsequence such that

$$(C1) \quad (r + \varepsilon) n_\varepsilon \rightarrow r n \text{ uniformly in } C([0, T] \times I_{R_b}),$$

$$(C2) \quad (r + \varepsilon) \partial_t n_\varepsilon \rightharpoonup r \partial_t n \text{ in } L^2(0, T; H^{-1}(I_{R_b})),$$

$$(C3) \quad \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r (\mu_\varepsilon + V) \rightharpoonup \xi \text{ in } L^2((0, T) \times I_{R_b}),$$

$$(C4) \quad \sqrt{r + \varepsilon} \partial_{rr} n_\varepsilon \rightharpoonup \sqrt{r} \partial_{rr} n \text{ and } \frac{\partial_r n_\varepsilon}{\sqrt{r + \varepsilon}} \rightharpoonup \frac{\partial_r n}{\sqrt{r}} \text{ in } L^2((0, T) \times I_{R_b}).$$

for some $\xi \in L^2((0, T) \times I_{R_b})$.

Step 2: nonnegativity of n . The plan is to obtain a contradiction with the uniform estimate of the entropy. For $\alpha > 0$, we define the sets

$$V_{\alpha, \varepsilon} = \{(t, r) \in (0, T) \times I_{R_b} : n_\varepsilon(t, r) \leq -\alpha, r \geq \alpha\},$$

$$V_{\alpha, 0} = \{(t, r) \in (0, T) \times I_{R_b} : n(t, r) \leq -\alpha, r \geq \alpha\}.$$

By Remark 2.5 (nonnegativity of ϕ_ε) and (A5) in Lemma 2.6 there is a constant such that

$$\int_{V_{\alpha, \varepsilon}} (r + \varepsilon) \phi_\varepsilon(n_\varepsilon) \, dr \, dt \leq \int_{(0, T) \times I_{R_b}} (r + \varepsilon) \phi_\varepsilon(n_\varepsilon) \, dr \, dt \leq C(T).$$

For $n_\varepsilon \leq -\alpha$, we have $0 \leq \phi_\varepsilon(-\alpha) \leq \phi_\varepsilon(n_\varepsilon)$ (Remark 2.5) so that

$$(-\alpha(\log(\varepsilon) - 1) + 1 + \alpha^2/(2\varepsilon) - \varepsilon/2) \int_{V_{\alpha,\varepsilon}} (r + \varepsilon) dr dt \leq C(T).$$

Sending $\varepsilon \rightarrow 0$ and using uniform convergence of $n_\varepsilon \rightarrow n$ for $r \geq \alpha > 0$ we discover

$$\int_{V_{\alpha,0}} r dr dt = \lim_{\varepsilon \rightarrow 0} \int_{V_{\alpha,\varepsilon}} (r + \varepsilon) dr dt = 0$$

(we use here the fact from measure theory asserting that on the measure space (X, μ) if $f_n, f : X \rightarrow \mathbb{R}$ and $f_n \rightarrow f$ in $L^1(X, \mu)$ then for a.e. $\alpha \in \mathbb{R}$ we have $\int_{f_n < \alpha} d\mu \rightarrow \int_{f < \alpha} d\mu$ as $n \rightarrow \infty$). This means that $V_{\alpha,0}$ is a null set for each $\alpha > 0$, concluding the proof.

Step 3: Identification of the limit $(r + \varepsilon)B_\varepsilon(n_\varepsilon)\partial_r(\mu_\varepsilon + V)$. The last difficulty is to pass to the limit in $\int_0^T \int_0^{R_b} (r + \varepsilon)B_\varepsilon(n_\varepsilon)\partial_r(\mu_\varepsilon + V)\partial_r\varphi dr dt$. Indeed, since the mobility is degenerate it is not clear that we can identify the derivative of the potential $\partial_r\mu$ in the limit. However, due to the uniform convergence of $(r + \varepsilon)n_\varepsilon$ and the nonnegativity of n we can conclude. By (C3) and the uniform convergence of $\sqrt{(r + \varepsilon)B_\varepsilon(n_\varepsilon)}$, we have

$$(r + \varepsilon)B_\varepsilon(n_\varepsilon)\partial_r(\mu_\varepsilon + V) \rightharpoonup \sqrt{rn}\xi = \begin{cases} \sqrt{rn}\xi & \text{if } rn > 0 \\ 0 & \text{if } rn = 0 \end{cases} \text{ in } L^2((0, T) \times I_{R_b}). \quad (2.19)$$

We first claim that

$$\xi(t, r) = \sqrt{rn}\partial_r(\mu + V) \text{ if } rn > 0. \quad (2.20)$$

We introduce the family of open sets

$$\{(t, r) : rn(t, r) > 0\} = \cup_{\nu > 0} P_\nu, \quad P_\nu = \{(t, r) : rn(r, t) > \nu, r > \nu\},$$

so that it is sufficient to identify the limit ξ in P_ν for fixed $\nu > 0$.

Because of the uniform convergence we know that for every $\varepsilon < \varepsilon(\nu)$ for $\varepsilon(\nu)$ small enough,

$$(r + \varepsilon)B_\varepsilon(n_\varepsilon(r, t)) \geq \frac{\nu}{2}, \quad (r, t) \in P_\nu.$$

Therefore, the estimate (A3) implies

$$\|\partial_r(\mu_\varepsilon + V)\|_{L^2(P_\nu)} \leq \frac{C}{\nu^{1/2}}.$$

As $\partial_r V$ is uniformly bounded, we deduce that

$$\|\partial_r\mu_\varepsilon\|_{L^2(P_\nu)} \leq \frac{C}{\nu^{1/2}}.$$

By definition of μ_ε

$$\partial_r\mu_\varepsilon = \gamma n_\varepsilon^{\gamma-1} \partial_r n_\varepsilon - \delta \partial_r \left(\frac{1}{r + \varepsilon} \partial_r((r + \varepsilon)\partial_r n_\varepsilon) \right).$$

For the first term of (RHS), we use the strong convergence (C1) that yields a uniform convergence of n_ε in the zone $\{(r, t) : r > \nu\}$. Then, because $P_\nu \subset \{(r, t) : r > \nu\}$ we obtain

$$\gamma n_\varepsilon^{\gamma-1} \rightarrow \gamma n^{\gamma-1} \quad \text{uniformly in } L^\infty(P_\nu).$$

Combined with the weak convergence provided by estimate (A1) in P_ν we obtain that up to a subsequence,

$$\gamma n_\varepsilon^{\gamma-1} \partial_r n_\varepsilon \rightharpoonup \gamma n^{\gamma-1} \partial_r n \quad \text{weakly in } L^2(P_\nu).$$

Then we combine the $L^2(P_\nu)$ bound on $\gamma n_\varepsilon^{\gamma-1} \partial_r n_\varepsilon$ with the $L^2(P_\nu)$ estimate of $\partial_r \mu_\varepsilon$. We obtain an $L^2(P_\nu)$ bound on the second term on the right-hand side. Together with estimates (A1)-(A2)-(A3)-(A4) we obtain the weak convergence up to a subsequence

$$\partial_r \left(\frac{1}{r+\varepsilon} \partial_r ((r+\varepsilon) \partial_r n_\varepsilon) \right) \rightharpoonup \partial_r \left(\frac{1}{r} \partial_r (r \partial_r n) \right) \quad \text{weakly in } L^2(P_\nu).$$

Finally,

$$\partial_r \mu_\varepsilon \rightharpoonup \partial_r \mu = \gamma n^{\gamma-1} \partial_r n - \delta \partial_r \left(\frac{1}{r} \partial_r (r \partial_r n) \right) \quad \text{weakly in } L^2(P_\nu). \quad (2.21)$$

Using uniform convergence, we conclude the proof of (2.20). Finally, (2.19) and (2.20) implies

$$\int_0^T \int_0^{R_b} (r+\varepsilon) B_\varepsilon(n_\varepsilon) \partial_r (\mu_\varepsilon + V) \partial_r \varphi \, dr \, dt \rightarrow \int_{rn>0} r n \partial_r (\mu + V) \partial_r \varphi \, dr \, dt.$$

Step 4: existence of a weak solution. Steps 1-3 show that n satisfies the condition of Definition 2.1.

Step 5: Properties (2.1)–(2.5) from Remark 2.2. First, properties (2.1), (2.4) and (2.5) follow from uniform convergence (C1) and estimates (2.14)–(2.15) for $r n_\varepsilon$. To see (2.2), we notice that weak lowersemicontinuity of L^2 norm implies

$$\mathcal{E}[n(\tau, \cdot)] + \int_0^\tau \int_0^{R_b} |\xi(t, r)|^2 \, dr \, dt \leq \mathcal{E}[n_0],$$

By (2.20), the integral on the (LHS) can be estimated from below by

$$\mathcal{E}[n(\tau, \cdot)] + \int_0^\tau \int_0^{R_b} \mathbb{1}_{rn>0} r n |\partial_r (\mu + V)|^2 \, dr \, dt \leq \mathcal{E}[n_0],$$

To see (2.3), it is sufficient to prove

$$\Phi(n(t, \cdot)) \leq \liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon(n_\varepsilon(t, \cdot)).$$

Let $\delta > 0$. By nonnegativity of ϕ_ε we estimate

$$\liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon(n_\varepsilon(t, \cdot)) \geq \liminf_{\varepsilon \rightarrow 0} \int_{r \geq \delta} \phi_\varepsilon(n_\varepsilon)(r+\varepsilon) \, dr = \int_{r \geq \delta} \phi(n) r \, dr,$$

because on the set $\{r \geq \delta\}$, we have uniform convergence $n_\varepsilon \rightarrow n$. As $\phi(n) \geq 0$, we can send $\delta \rightarrow 0$ by monotone convergence and conclude the proof.

Step 6: Neumann boundary condition $n'(t, R_b) = 0$. First, if $\varphi, \phi \in C^1[a, b] \cap H^2(a, b)$ we have (via approximation)

$$\int_a^b \varphi'(r) \phi'(r) + \varphi''(r) \phi(r) \, dr = \varphi'(b) \phi(b) - \varphi'(a) \phi(a). \quad (2.22)$$

Let ϕ be a smooth function with $\phi(R_0) = 0$ and $\phi(R_b) = 1$ for some $R_0 \in (0, R_b)$. We know from estimates (C1)–(C4) that $n \in L^2(0, T; H^2(R_0, R_b))$. Let t be such that $r \mapsto n(t, r) \in H^2(R_0, R_b)$. We apply (2.22) with $\varphi(r) = n_\varepsilon(t, r)$ to deduce

$$\int_{R_0}^{R_b} n'_\varepsilon(t, r) \phi'(r) + n''_\varepsilon(t, r) \phi(r) \, dr = 0$$

by the Neumann boundary condition $n'_\varepsilon(t, R_b) = 0$. Multiplying by a smooth test function $\eta(t)$ we have

$$\int_0^T \int_{R_0}^{R_b} \eta(t) (n'_\varepsilon(t, r) \phi'(r) + n''_\varepsilon(t, r) \phi(r)) dr dt = 0.$$

Passing to the weak limit $\varepsilon \rightarrow 0$ and using that η is arbitrary we conclude

$$\int_{R_0}^{R_b} n'(t, r) \phi'(r) + n''(t, r) \phi(r) dr = 0$$

for a.e. t . As $n(t, \cdot) \in H^2(R_0, R_b)$ we can apply (2.22) again and deduce

$$\int_{R_0}^{R_b} n'(t, r) \phi'(r) + n''(t, r) \phi(r) dr = n'(t, R_b).$$

which finally proves $n'(t, R_b) = 0$. □

2.3 Proof of Theorem 1 (asymptotics)

We constructed solutions on $t \in (0, \infty)$. For a fixed k, T , $k \geq T$ we consider the solution n in the interval $(-T + k, T + k)$. This solution satisfies

$$\int_{-T+k}^{T+k} r \langle \partial_t n, \varphi \rangle_{H^{-1}, H^1} dt + \int_{-T+k}^{T+k} \int_0^{R_b} \mathbb{1}_{rn>0} r n \partial_r (\mu + V) \partial_r \varphi dr dt = 0.$$

We define $n_k(t, x) = n(t + k, x)$ and $\mu_k(t, x) = \mu(t + k, x)$. A change of variables yields

$$\int_{-T}^T r \langle \partial_t n_k, \varphi \rangle_{H^{-1}, H^1} dt + \int_{-T}^T \int_0^{R_b} \mathbb{1}_{rn_k>0} r n_k \partial_r (\mu_k + V) \partial_r \varphi dr dt = 0. \quad (2.23)$$

We also recall the Neumann boundary condition $n'_k(t, R_b) = 0$ and the conservation of mass $\int_0^{R_b} r n_k dr = \int_0^{R_b} r n_0 dr$. We want to pass to the limit $k \rightarrow \infty$ in this equation and prove the

Proposition 2.8. *Let (n, μ) be a weak solution of (1.3)-(1.4). Then, we can extract a subsequence, still denoted by the index k , of (n_k, μ_k) such that $\sqrt{r} n_k \rightarrow \sqrt{r} n_\infty$ strongly in $L^\infty((-T, T) \times I_{R_b})$ and $\sqrt{r n_k} \partial_r (\mu_k + V) \rightharpoonup \sqrt{r n} \partial_r (\mu_\infty + V)$ weakly in $L^2((-T, T) \times \overline{I_{R_b}} \setminus \{rn = 0\})$. We have $n_\infty \in C^1(\mathbb{R} \times \overline{B_{R_b}})$ and the relation*

$$r n_\infty \partial_r (\mu_\infty + V) = 0, \quad \mu_\infty = n_\infty^\gamma - \frac{\delta}{r} \partial_r (r \partial_r n_\infty). \quad (2.24)$$

with the boundary conditions

$$\frac{\partial n_\infty}{\partial r} \Big|_{r=0} = \frac{\partial n_\infty}{\partial r} \Big|_{r=R_b} = 0.$$

The mass $\int_0^{R_b} r n_\infty(t) dr$ is constant and equal to the initial mass $\int_0^{R_b} r n_0 dr$.

This proposition implies assertions of Theorem 1.

Proof. Step 1: Bounds coming from the energy. We claim that the following uniform estimates (with respect to k) are true:

(B1) $\{\sqrt{r} \partial_r n_k\}$ in $L^\infty((-T, T); L^2(I_{R_b}))$,

(B2) $\{\sqrt{r} n_k\}$ in $L^\infty((-T, T) \times I_{R_b})$,

(B3) $\{r \partial_t n_k\}$ in $L^2((-T, T); H^{-1}(I_{R_b}))$

$$(B4) \quad |r_2 n_k(t_2, r_2) - r_1 n_k(t_1, r_1)| \leq C(|t_2 - t_1|^{1/8} + |r_2 - r_1|^{1/2}),$$

$$(B5) \quad L_k(T) := \int_{-T}^T \int_0^{R_b} \mathbf{1}_{rn_k > 0} r n_k |\partial_r(\mu_k + V)|^2 \xrightarrow{k \rightarrow +\infty} 0.$$

The energy decay (2.2) and assumption $\mathcal{E}[n_0] < \infty$ implies that $\mathcal{E}[n_k(t)]$ remains bounded with respect to k for all $k > T$. Therefore, (B1) follows directly from (2.2) and then (B2) follows from Remark 2.7. As $r n_k(t, r)$ is obtained as the pointwise limit of $(r + \varepsilon) n_\varepsilon(t + k, r)$, estimates (B3) and (B4) follow directly from passing to the limit $\varepsilon \rightarrow 0$ in (A6) and (2.14)–(2.15) in Proposition 2.6. Finally, to see (B5), we note that

$$\int_0^\infty \int_0^{R_b} \mathbf{1}_{rn > 0} r n |\partial_r(\mu + V)|^2 dr dt \leq \mathcal{E}(n_0)$$

so by change of variables we obtain

$$L_k(T) \leq \int_{k-T}^\infty \int_0^{R_b} \mathbf{1}_{rn > 0} r n |\partial_r(\mu + V)|^2 dr dt \xrightarrow{k \rightarrow +\infty} 0.$$

Step 2: Bounds coming from the entropy. We prove now uniform estimates

$$(C1) \quad \{\sqrt{r} \partial_{rr} n_k\} \text{ in } L^2((-T, T); L^2(I_{R_b})),$$

$$(C2) \quad \left\{ \frac{\partial_r n_k}{\sqrt{r}} \right\} \text{ in } L^2((-T, T); L^2(I_{R_b})).$$

To this end, we integrate the entropy relation (2.3) between $k - T$ and $k + T$ and perform a change of variables to obtain

$$\begin{aligned} \int_{-T}^T \int_0^{R_b} \left(\gamma r n_k^{\gamma-1} |\partial_r n_k|^2 + \delta r |\partial_{rr} n_k|^2 + \delta \frac{|\partial_r n_k|^2}{r} \right) dr dt &\leq \\ &\leq \Phi[n_k(-T, \cdot)] - \Phi[n_k(T, \cdot)] + \int_{-T}^T \int_0^{R_b} r \partial_r n_k \partial_r V dr dt. \end{aligned}$$

We need to bound the right-hand side. Concerning the entropy term, we recall the inequality $\log n \leq n - 1$ valid for $n \geq 0$ so that

$$\begin{aligned} \Phi(n_k(T, \cdot)) &= \int_0^{R_b} r(n_k(T, r)(\log n_k(T, r) - 1) + 1) dr \leq \\ &\leq \int_0^{R_b} r((n_k(T, r))^2 + n_k(T, r)) dr \leq C \|\sqrt{r} n_k\|_\infty \leq C \end{aligned}$$

by bound (B2). The same estimate is satisfied by $\Phi(n_k(-T))$. Concerning $\int_{-T}^T \int_0^{R_b} r \partial_r n_k \partial_r V dr dt$, we estimate it using (B1) and uniform bound on $\partial_r V$. Therefore,

$$\int_{-T}^T \int_0^{R_b} \left(\gamma r n_k^{\gamma-1} |\partial_r n_k|^2 + \delta r |\partial_{rr} n_k|^2 + \delta \frac{|\partial_r n_k|^2}{r} \right) dr dt \leq C(T, \mathcal{E}(n_0)).$$

Step 3: Convergence in equation (2.23). Reasoning as in the proof of Theorem 1 we obtain in the limit $k \rightarrow \infty$

$$\int_{-T}^T r \langle \partial_t n_\infty, \varphi \rangle_{H^{-1}, H^1} dt + \int_{-T}^T \int_0^{R_b} \mathbf{1}_{rn_\infty > 0} r n_\infty \partial_r(\mu_\infty + V) \partial_r \varphi dr dt = 0.$$

We can show even better, namely that $\partial_t n_\infty = 0$. Indeed, from the Cauchy-Schwarz inequality we obtain that for every test function χ compactly supported in $(-T, T) \times (0, R_b)$,

$$\begin{aligned} \left| \int_{-T}^T \int_0^{R_b} r \partial_t n_k \chi \right| &= \left| \int_{-T}^T \int_0^{R_b} \mathbb{1}_{rn_k > 0} rn_k \partial_r (\mu_k + V) \partial_r \chi \right| \\ &\leq C(T, R_b) \|\partial_r \chi\|_{L^\infty} \int_{-T}^T \int_0^{R_b} \mathbb{1}_{rn_k > 0} rn_k |\partial_r (\mu_k + V)|^2 \xrightarrow[k \rightarrow \infty]{} 0. \end{aligned}$$

where we used (B2) and (B5). This means that in the limit, n_∞ does not depend on the time t . Then we obtain that in the limit for every test function χ ,

$$\int_{-T}^T \int_0^{R_b} \mathbb{1}_{rn_\infty > 0} rn_\infty \partial_r (\mu_\infty + V) \partial_r \chi = 0.$$

Step 4: n'_∞ is uniformly continuous and n_∞ satisfies Neumann boundary condition $n'_\infty(0) = 0$. We recall that n_∞ does not depend on time. Moreover, the estimate (C1) implies that n'_∞ is continuous on $(0, R_b]$. Furthermore, from the estimates (C1)-(C2), we obtain the absolute continuity in space of the derivative of n_∞ . Indeed, for every $r_1, r_2 \in (0, R_b)$ we obtain

$$\begin{aligned} (\partial_r n_\infty(r_2))^2 - (\partial_r n_\infty(r_1))^2 &= 2 \int_{r_1}^{r_2} \partial_r n_\infty(r) \partial_{rr} n_\infty(r) dr \\ &= 2 \int_{r_1}^{r_2} \frac{\partial_r n_\infty(r)}{\sqrt{r}} \sqrt{r} \partial_{rr} n_\infty(r) dr \\ &\leq 2 \left(\int_{r_1}^{r_2} \frac{|\partial_r n_\infty(r)|^2}{r} dr \right)^{1/2} \left(\int_{r_1}^{r_2} r |\partial_{rr} n_\infty(r)|^2 dr \right)^{1/2} \end{aligned}$$

From this estimate, we deduce that $\partial_r n_\infty$ is bounded so that by Sobolev embedding n_∞ is continuous and

$$n_\infty(r_2) - n_\infty(r_1) = \int_{r_1}^{r_2} \partial_r n_\infty(r) dr.$$

Second, we discover that $(0, R_b] \ni r \mapsto (\partial_r n_\infty(r))^2$ is uniformly continuous, so that by Lemma 2.9 below, $n'_\infty(r)$ and $|n'_\infty(r)|$ are uniformly continuous on $(0, R_b]$. Therefore, there is the unique extension of $r \mapsto n_\infty(r)$ to $[0, R_b]$ which is uniformly continuous. Furthermore, in view of

$$\int_0^{R_b} \frac{|\partial_r n_\infty|^2}{r} dr \leq C,$$

the extension has to be obtained by setting 0 at $r = 0$.

It remains to be proved that n_∞ is differentiable (in the classical sense) at $r = 0$ and $n'_\infty(0) = 0$. To this end, we write

$$\left| \frac{n_\infty(r) - n_\infty(0)}{r} \right| \leq \frac{1}{r} \int_0^r |\partial_r n_\infty(u)| du \leq \sup_{u \in (0, r]} |\partial_r n_\infty(u)| \rightarrow 0$$

as $r \rightarrow 0$ by uniform continuity which implies that $n'_\infty(0)$ exists and $n'_\infty(0) = 0$.

Step 5: Neumann boundary condition $n'_\infty(R_b) = 0$. The proof is similar to Step 6 in Section 2.2. For fixed $k \in \mathbb{N}$, there is a set of times $\mathcal{N}_k \subset (0, T)$ of full measure such that if $t \in \mathcal{N}_k$ we have

$n'_k(t, R_b) = 0$ and $n_k(t, \cdot) \in H^2(R_0, R_b)$. Let $\mathcal{N} = \cap_{k \in \mathbb{N}} \mathcal{N}_k$, which is again the set of full measure. For $t \in \mathcal{N}$ and ϕ as in Step 6 in Section 2.2, we have

$$\int_{R_0}^{R_b} n'_k(t, r) \phi'(r) + n''_k(t, r) \phi(r) dr = 0.$$

We multiply by a smooth test function $\eta(t)$ and pass to the weak limit $\varepsilon \rightarrow 0$ to deduce

$$\int_0^T \eta(t) dt \int_{R_0}^{R_b} (n'_\infty(r) \phi'(r) + n''_\infty(r) \phi(r)) dr = 0.$$

As $n_\infty \in H^2(R_0, R_b)$ we deduce $n'_\infty(R_b) = 0$. \square

Lemma 2.9. *Let $f : (a, b) \rightarrow \mathbb{R}$ be a continuous function such that f^2 is uniformly continuous. Then $|f|$ and f are also uniformly continuous.*

Proof. First, we observe that $|f|$ is uniformly continuous as a composition of a $\frac{1}{2}$ -Hölder continuous function and a uniformly continuous one. Therefore,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in (a, b) \quad |x - y| \leq \delta \implies ||f(x)| - |f(y)|| \leq \varepsilon. \quad (2.25)$$

Fix $\varepsilon > 0$ and choose $\delta > 0$ such that (2.25) holds with $\varepsilon/2$. Let $x, y \in (a, b)$ be such that $|x - y| \leq \delta$. If $f(x), f(y)$ have the same sign we are done. Otherwise, by continuity, there exists z between x and y such that $f(z) = 0$. As $|x - z|, |y - z| \leq \delta$, we can apply (2.25) again to deduce

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \square$$

3 Study of the stationary states

We now prove Theorems 2 and 3, limiting ourselves to $V(r) = r^2$. We postpone to Appendix A the case of a more general potential $V(r)$.

3.1 Proof of Theorem 2 (A)

Lemma 3.1 (n is nonincreasing). *With the notations of Theorem 2 (A), suppose there exists $R_* > 0$ such that $n(R_*) > 0$, $n'(R_*) = 0$. Then, $n'(r) \leq 0$ for all $r \leq R_*$.*

Proof. We argue by contradiction. Suppose that there is $R \in (0, R_*)$ such that $n'(r) > 0$ for $r \in (R, R_*)$. Let $R_0 = \max\{r \in [0, R_*] : n(r) = 0\}$ (it is possible that $R_0 = 0$). By C^1 regularity of n and its nonnegativity, $n'(R_0) = 0$. Moreover, n is C^2 on $(R_0, R_*]$ because n solves

$$n^\gamma(r) - \frac{\delta}{r} n'(r) - \delta n''(r) = C - r^2 \quad r \in (R_0, R_*] \quad (3.1)$$

for some constant C . By definition, $n''(R_*) = \lim_{h \rightarrow 0^+} \frac{-n'(R_* - h)}{h} \leq 0$. As $n'(R_0) = n'(R_*) = 0$, by Rolle's theorem, there exists at least one $R_c \in (R_0, R_*)$ such that $n''(R_c) = 0$. We distinguish two cases.

- There exists maximal $R_c < R_*$. Then, by maximality, $n''(r)$ is either strictly positive or strictly negative on (R_c, R_*) . We deduce that it is strictly negative because $n'(r) > 0$ for all $r \in (R, R_*)$ and $n'(r) = - \int_r^{R_*} n''(\rho) d\rho$. Then, the same formula actually shows that $n'(r) > 0$ for $r \in [R_c, R_*)$ so in particular $n'(R_c) > 0$. From (3.1) evaluated at R_* and R_c we have

$$n^\gamma(R_*) - n^\gamma(R_c) + \frac{\delta}{r} n'(R_c) - \delta n''(R_*) = R_c^2 - R_*^2.$$

The (LHS) is nonnegative while the (RHS) is strictly negative raising contradiction.

- There is no maximal $R_c < R_*$. In this case, we may choose R_c arbitrarily close to R_* and we choose R_c such that $R_c \in (R, R_*)$. Then $n'(R_c) > 0$ and $n''(R_c) = 0$ so that we can obtain contradiction as above.

It follows that $n'(r) \leq 0$ in some left neighborhood of R_* . To extend it for all $r \leq R^*$, we suppose that $R_p := \sup\{r \in [0, R_*) : n'(r) > 0\}$ exists. We know that $R_p < R_*$, $n(R_p) \geq n(R_*) > 0$ and $n'(r) < 0$ in a left neighbourhood of R_p raising contradiction with the first part of the proof (where we replace R_* with R_p). \square

Proof of Theorem 2 (A). We recall that the stationary states satisfy the Neumann boundary conditions $n'(0) = n'(R_b) = 0$.

Step 1: Stationary solutions such that $n(R_b) = 0$. By nonnegativity and conservation of mass, n cannot be constantly zero so it has to be decreasing on some interval (R_2, R_3) with $n(R_2) > 0$ and $n(R_3) = 0$. Then, if n is strictly increasing on (R_1, R_2) we have $n'(R_2) = 0$ and we apply Lemma 3.1 to obtain contradiction.

Step 2: Stationary solutions such that $n(R_b) > 0$. We apply directly Lemma 3.1 with $R_* = R_b$. \square

3.2 Proof of Theorem 2 (B)

In the sequel, we consider stationary states such that $n(R_b) = 0$. So there is an $R \in [0, R_b]$ such that $n(R) = 0$. Since n is radially decreasing we have $n(r) = 0$ on $[R, R_b]$ and n is positive on $[0, R)$. Hence, on $[0, R]$, the relation (2.24) shows that there exists C that we write $C = R^2 - \lambda$ such that the stationary states are the radial solutions of

$$\begin{cases} n^\gamma - \delta \Delta n = R^2 - r^2 - \lambda & \text{in } B_R \\ n = 0 & \text{on } \partial B_R \\ n > 0 & \text{on } B_R. \end{cases} \quad (3.2)$$

Of course, as we will see in the next lemma, the previous problem does not admit a solution for every λ . Notice that we fixed only Dirichlet's boundary conditions. We want to prove that there exists only one λ such that the solution of the previous equation also satisfies the Neumann boundary condition $\frac{\partial n}{\partial \nu} = 0$ on ∂B_R .

First, we understand which values of λ can be excluded.

Lemma 3.2. *For every λ , let n be the radially symmetric solution of*

$$\begin{cases} n^\gamma - \delta \Delta n = R^2 - r^2 - \lambda & \text{in } B_R \\ n = 0 & \text{on } \partial B_R \end{cases} \quad (3.3)$$

Then,

- if $\lambda \geq R^2$, then $n(r) \leq 0$ for $r \in [0, R]$ and $n'(R) > 0$,
- if $\lambda \leq 0$, then $n(r) \geq 0$ for $r \in [0, R]$ and $n'(R) < 0$.

Proof. For $\lambda \geq R^2$, we multiply the equation by $n_+ = \max(n, 0)$ and integrate by parts to get

$$\int_{B_R} n_+^{\gamma+1} + \delta \int_{B_R} |\nabla n_+|^2 = \int_{B_R} (R^2 - r^2 - \lambda) n_+ \leq 0.$$

so that $n_+ = 0$ and $n \leq 0$. It follows that $n'(R) \geq 0$. If $n'(R) = 0$ we obtain by the equation $n''(R) = \frac{\lambda}{\delta} > 0$ which is in a contradiction with the fact that n is nonpositive in a small neighborhood of R .

For $\lambda \leq 0$, we first multiply the equation by $n_- = \min(n, 0)$ and integrate by parts to get

$$\delta \int_{B_R} |\nabla n_-|^2 = \int_{B_R} (R^2 - r^2 - \lambda) n_- \leq 0.$$

We recall here that the notation n^γ was a notation for $\max(0, n)^\gamma$. Therefore $n_- = 0$ and $n \geq 0$. It follows that $n'(R) \leq 0$. Second, we exclude the possibility that $n'(R) = 0$. Suppose by contradiction that $n'(R) = 0$. Since we also have $n(R) = 0$, we find that $n''(R) = \frac{\lambda}{\delta}$. When $\lambda < 0$, we find contradiction with $n \geq 0$. On the other hand, when $\lambda = 0$, we have $n''(R) = 0$. Differentiating the equation, we find

$$\gamma n^{\gamma-1}(r) n'(r) - \delta n^{(3)}(r) - \delta \frac{n''(r)}{r} + \delta \frac{n'(r)}{r^2} = -2r,$$

and thus $n^{(3)}(R) = 2R/\delta > 0$. As $n(R) = n'(R) = n''(R) = 0$, it follows that in a small neighbourhood of R , n has to be negative raising contradiction. The lemma is proved. \square

From Lemma 3.2, we may conclude that there is at least one $\lambda \in (0, R^2)$ such that the Neumann condition is satisfied

Lemma 3.3. *There exists at most one $\lambda \in (0, R^2)$ such that the radial solution of (3.3) satisfies $n'(R) = 0$.*

Proof. For each $\lambda \in (0, R^2)$ let n be solution of (3.3) (it always exists). We differentiate (3.3) in λ and obtain

$$\gamma n^{\gamma-1} f - \delta \Delta f = -1 \text{ on } B_R, \quad f = 0 \text{ on } \partial B_R, \quad f = \frac{\partial n}{\partial \lambda}.$$

Multiplying by a positive part f_+ , we deduce from this equation that $f \leq 0$. This means that n is decreasing everywhere with λ . However the inequality is not strict everywhere and we cannot conclude directly from the condition $n(R) = 0$.

So suppose there are two radial solutions n_1, n_2 to (3.3) with $0 < \lambda_1 < \lambda_2 < R^2$ such that $n_i(R) = n_i'(R) = 0$ for $i = 1, 2$. From (3.3) in radial coordinates, we find $n_i''(R) = \frac{\lambda_i}{\delta}$. Therefore $0 < n_1''(R) < n_2''(R)$ and we conclude by a Taylor expansion that n_1 is smaller than n_2 in a small neighborhood of R . This is in contradiction with the behaviour of n with respect to λ . \square

Proof of Theorem 2 (B). Clearly, n_∞ is a radial solution to the problem (3.2) with some λ . By Lemma 3.2, we know that $\lambda \in (0, R^2)$ and then Lemma 3.3 yields the unique value of λ .

For the second assertion, if there are two solutions $(n_1, \lambda_1), (n_2, \lambda_2)$ of (1.7), we know from [21, Theorem 1', p. 13] that both n_1 and n_2 are radially symmetric. Then, Lemma 3.3 applies and we obtain that $\lambda_1 = \lambda_2$. The conclusion follows from uniqueness of solutions to elliptic PDE (3.3). \square

3.3 Proof of Theorem 2 (C)

First, to simplify the notation, we write n for the stationary state n_∞ and R for the length of the interval R_b . Second, we observe that if function n with $n(R) > 0$ is a stationary solution, it is a C^1 solution of (1.6). As $n' \leq 0$ (Theorem 2 (A)) and $n(R) > 0$, we have $n > 0$ so that (1.6) boils down to

$$\begin{cases} n^\gamma - \delta \Delta n = R^2 - r^2 - \lambda & \text{in } B_R \\ n(x) = a, \quad \frac{\partial n}{\partial \mathbf{n}} = 0 & \text{on } \partial B_R, \\ m = \int_0^R r n(r) dr \end{cases} \quad (3.4)$$

where $a > 0$ and λ is some constant. The target of this section is to prove that if R is sufficiently large, there is no such solution to (3.4).

First, we provide lower and upper bound on admissible values of constant λ only in terms of R .

Lemma 3.4. *If $2m \leq R^2$ and $1 \leq R$, then $\lambda \geq -R^2$.*

Proof. From (3.4) evaluated at R we have

$$n^\gamma(R) - \delta n''(R) = -\lambda \implies \lambda \geq -n^\gamma(R) \quad (3.5)$$

because $\delta > 0$ and $n''(R) \geq 0$ because n is decreasing and $n(R) = n'(R) = 0$. By a lower bound on mass, $n(R) \leq 2m/R^2$ so that by assumptions $n(R) \leq 1$ and $n^\gamma(R) \leq 1$. As $R \geq 1$, we have $n^\gamma(R) \leq R^2$ so that (3.5) implies $\lambda \geq -R^2$. \square

Lemma 3.5. *Suppose that $\delta J m < R^6$ where $J := \frac{64}{\frac{3}{4} - \log(2)} > 0$. Then $\lambda \leq R^2/4$.*

Proof. We consider function u solving equation

$$\begin{cases} -\delta \Delta u = R^2 - r^2 - \lambda_u & \text{in } B_R \\ u(x) = a, \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial B_R, \end{cases} \quad (3.6)$$

where λ_u will be chosen to satisfy $\lambda_u < \frac{R^2}{2}$. In fact, we have explicit formula for u cf. Proposition 3.7 (A):

$$\begin{aligned} u(r) &= a + \frac{R^2}{4\delta} (R^2 - 2\lambda_u) \ln\left(\frac{r}{R}\right) + \frac{(r^2 - R^2)^2}{16\delta} + \frac{R^2 - 2\lambda_u}{8\delta} (R^2 - r^2), \\ u'(r) &= \frac{(R^2 - r^2)(R^2 - r^2 - 2\lambda_u)}{4\delta r}, \end{aligned}$$

Function u attains its maximum at $r^2 = R^2 - 2\lambda_u$ which equals

$$\begin{aligned} u(\sqrt{R^2 - 2\lambda_u}) &= a + \frac{R^2}{8\delta} (R^2 - 2\lambda_u) \ln\left(1 - \frac{2\lambda_u}{R^2}\right) + \frac{4\lambda_u^2}{16\delta} + \frac{R^2 - 2\lambda_u}{8\delta} 2\lambda_u \\ &= a + \frac{R^2}{8\delta} (R^2 - 2\lambda_u) \ln\left(1 - \frac{2\lambda_u}{R^2}\right) + \frac{R^2 - \lambda_u}{8\delta} 2\lambda_u. \end{aligned}$$

By Proposition 3.7 (C) applied to functions $n - a$ and $u - a$, we see that if $\lambda \geq \lambda_c$ then $n \geq u$.

After this preparatory steps, aiming at contradiction, let us assume that $\lambda > R^2/4$. We choose $\lambda_u = R^2/4$ so that $R^2 - 2\lambda_u = R^2/2$ and

$$n(R/\sqrt{2}) \geq u(R/\sqrt{2}) \geq \frac{R^4}{16\delta} \ln(1/2) + \frac{3}{4} \frac{R^4}{16\delta} = \frac{R^4}{16\delta} \left(\frac{3}{4} - \log(2)\right)$$

where we estimated $a \geq 0$. As $n' \leq 0$, $n(r) \geq n(R/\sqrt{2})$ for $r \in [0, R/\sqrt{2}]$. Therefore, we can estimate the mass from below as follows:

$$m \geq n(R/\sqrt{2}) \int_0^{R/\sqrt{2}} r \, dr = \frac{R^2}{4} \frac{R^4}{16\delta} \left(\frac{3}{4} - \log(2)\right) = \frac{R^6}{64\delta} \left(\frac{3}{4} - \log(2)\right),$$

which is a contradiction. \square

Now, we provide a formula for $\int_0^R r n^\gamma \, dr$ which is the only mass-related information that comes from the equation.

Lemma 3.6. *Let n be solution to (3.4). Then*

$$\int_0^R r n^\gamma dr = \frac{R^2(R^2 - 2\lambda)}{4}.$$

Proof. Thanks to the Neumann boundary condition,

$$2\pi \int_0^R n^\gamma r dr = \int_{B_R} n^\gamma dx = \int_{B_R} (R^2 - r^2 - \lambda) dx = \int_0^R 2\pi r (R^2 - r^2 - \lambda) dr.$$

Then we conclude by computing

$$\int_0^R r (R^2 - r^2 - \lambda) dr = \frac{R^2(R^2 - 2\lambda)}{4}.$$

□

Proof of Theorem 2 (C). We find $R(m)$ such that for all $R \geq R(m)$ the following inequalities are satisfied:

$$\frac{R^4}{8} > 2R^2 m + m, \quad R^2 > 2m + 1, \quad R^6 > \delta J m,$$

where $J > 0$ is a numerical constant from Lemma 3.5. Now, suppose that there is a solution for $R \geq R(m)$. We want to estimate $\int_0^R n^\gamma r dr$ by the mass. This estimate is possible only when $n \leq 1$. However, as n is nonincreasing, we can estimate the maximal interval $[0, R_1]$ where $n \geq 1$ by mass:

$$m = \int_0^R r n(r) dr \geq \int_0^{R_1} r dr = \frac{R_1^2}{2}.$$

It follows that $R_1 \leq \sqrt{2m}$. Therefore,

$$\int_0^R n^\gamma r dr \leq \int_0^{R_1} n^\gamma r dr + \int_{R_1}^R n r dr \leq n^\gamma(0) \frac{R_1^2}{2} + m \leq n^\gamma(0) m + m. \quad (3.7)$$

To estimate $n^\gamma(0)$, we use the equation evaluated at 0

$$n^\gamma(0) - \delta n''(0) = R^2 - \lambda.$$

By definition, $n''(0) = \lim_{r \rightarrow 0} \frac{n'(r)}{r} \leq 0$ so that

$$0 \leq n^\gamma(0) \leq R^2 - \lambda \leq 2R^2$$

thanks to Lemma 3.4. Using Lemma 3.6 and (3.7) we conclude

$$\frac{R^2(R^2 - 2\lambda)}{4} \leq 2R^2 m + m.$$

As $\lambda \leq R^2/4$ (Lemma 3.5) we conclude

$$\frac{R^4}{8} \leq 2R^2 m + m$$

which is a contradiction. □

3.4 Preliminary steps for the proof of Theorem 3

We consider a sequence $\gamma_k \rightarrow \infty$ of stationary states, but to reduce notations, we drop the ∞ index in the sequel and keep the full family γ , which is a posteriori possible by uniqueness of the limit.

In the limit $\gamma \rightarrow \infty$ we expect to have $n^\gamma \rightarrow p$ where $p(n-1) = 0$. So, when $n < 1$, we have $p = 0$. Therefore, we study equation (3.8) from the following theorem

Proposition 3.7 (Lower bound profile). *Let n be a solution of (1.7) as in Theorem 2 and let u be a solution of*

$$\begin{cases} -\delta\Delta u = R^2 - r^2 - \lambda_u & \text{in } B_R \setminus \{0\} \\ u(x) = \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial B_R. \end{cases} \quad (3.8)$$

where $\lambda_u \in [0, R^2]$ is arbitrary. Then,

(A) we have explicit formula for u which in radial coordinates reads

$$\begin{aligned} u(r) &= \frac{R^2}{4\delta}(R^2 - 2\lambda_u) \ln\left(\frac{r}{R}\right) + \frac{(r^2 - R^2)^2}{16\delta} + \frac{R^2 - 2\lambda_u}{8\delta}(R^2 - r^2), \\ u'(r) &= \frac{(R^2 - r^2)(R^2 - r^2 - 2\lambda_u)}{4\delta r}, \end{aligned}$$

(B) $u(r)$ is strictly decreasing and nonnegative for r such that $R^2 - r^2 \leq 2\lambda_u$,

(C) if $\lambda \geq \lambda_u$ then $n(r) \geq u(r)$ for $r \in (0, R]$.

Since n is radially decreasing and $p(n-1) = 0$ in the limit we expect to have $n = 1$ in the zone $p > 0$. Therefore, the limiting profile is expected to be the solution of the following theorem

Proposition 3.8 (Existence of unique limiting profile). *Let $m > 0$ and suppose that $72\delta^{1/2} \leq m$. Then there exists uniquely determined $R, \lambda_c \in (0, R^2)$ and $R_0 \in (0, R)$ such that the following problem has solution*

$$\begin{cases} -\delta\Delta u_c = R^2 - r^2 - \lambda_c & \text{in } B_R \setminus B_{R_0} \\ u_c(x) = \frac{\partial u_c}{\partial \mathbf{n}} = 0 & \text{on } \partial B_R, \\ u_c(x) = 1, \frac{\partial u_c}{\partial \mathbf{n}} = 0 & \text{on } \partial B_{R_0}, \\ \int_{\Omega} r \tilde{u}_c(x) dx = m, \end{cases} \quad (3.9)$$

where \tilde{u} is the extension of u_c to B_{R_0} by setting $\tilde{u} = 1$. In fact, $R_0 = \sqrt{R^2 - 2\lambda_c}$ and $\lambda_c \approx \sqrt[3]{6} R^{2/3} \delta^{1/3}$ for small $\delta > 0$.

Then, it remains to prove Theorem 3 i.e. the convergence of the stationary states to this limiting profile.

3.5 Proof of Proposition 3.7.

Proof of (A): We first compute $u'(r)$ and $u''(r)$:

$$\begin{aligned} u'(r) &= \frac{R^2}{4\delta}(R^2 - 2\lambda_u) \frac{1}{r} + \frac{(r^2 - R^2)r}{4\delta} - \frac{R^2 - 2\lambda_u}{4\delta} r = \frac{(R^2 - r^2)(R^2 - r^2 - 2\lambda_u)}{4\delta r}, \\ u''(r) &= \frac{-4r(R^2 - r^2) + 4\lambda_u r}{4\delta r} - \frac{(R^2 - r^2)(R^2 - r^2 - 2\lambda_u)}{4\delta r^2} = -\frac{1}{\delta}(R^2 - r^2 - \lambda_u) - \frac{u'(r)}{r}. \end{aligned}$$

Therefore,

$$-\delta\Delta u = -\delta u''(r) - \delta \frac{u'(r)}{r} = R^2 - r^2 - \lambda_u$$

as desired.

Proof of (B): We use directly the formula for $u'(r)$: indeed, when $R^2 - r^2 - 2\lambda_u < 0$ we have $u'(r) < 0$ so that u is strictly decreasing. As $u(R) = 0$ this proves that $u(r) > 0$ when $R^2 - r^2 - 2\lambda_u < 0$.

Proof of (C): We introduce $h(r) = n(r) - u(r)$ and we have to prove that $h(r) \geq 0$. From the equations we get

$$n^\gamma - \delta \Delta h = \lambda_u - \lambda \text{ in } (0, R]$$

so from assumptions we have

$$\Delta h \geq 0 \text{ in } (0, R].$$

Now, in radial coordinates this reads

$$h''(r) + \frac{h'(r)}{r} \geq 0$$

so if we let $g'(r) = rh'(r)$ we have $g''(r) \geq 0$. Integrating this from r to R and using boundary conditions we obtain

$$g'(r) \leq 0 \implies rh'(r) \leq 0 \implies h'(r) \leq 0.$$

Integrating this once again and using boundary conditions, we discover $h(r) \geq 0$ as desired.

3.6 Proof of Proposition 3.8.

The plan is to characterize λ_c and R_0 using an explicit solution from Proposition 3.7. Indeed, we are looking for λ_c and R_0 such that

$$u'_c(R_0) = \frac{(R^2 - R_0^2)(R^2 - R_0^2 - 2\lambda_c)}{4\delta r} = 0, \quad (3.10)$$

$$u_c(R_0) = \frac{R^2}{4\delta}(R^2 - 2\lambda_c) \ln\left(\frac{R_0}{R}\right) + \frac{(R_0^2 - R^2)^2}{16\delta} + \frac{R^2 - 2\lambda_c}{8\delta}(R^2 - R_0^2) = 1. \quad (3.11)$$

Lemma 3.9 (solving for R_0 and λ_c). *Let $R > 0$. Then (3.10)-(3.11) has a unique solution if and only if $16\delta < R^4$. Moreover, the solution is given by*

$$R_0 = \sqrt{R^2 - 2\lambda_c}, \quad \lambda_c = \frac{R^2 x_c}{2}, \quad (3.12)$$

where x_c is the unique solution of

$$(1 - x_c) \ln(1 - x_c) + \frac{1}{2}x_c^2 + (1 - x_c)x_c = \frac{8\delta}{R^4}. \quad (3.13)$$

Proof. We split the reasoning into a few steps.

Equation for R_0 . From (3.10) we deduce $R_0 = R$ or $R^2 - R_0^2 = 2\lambda_c$. As the first possibility leads to the contradiction, we conclude

$$R_0 = \sqrt{R^2 - 2\lambda_c}. \quad (3.14)$$

Equation for λ_c . We first plug (3.14) into (3.11) to deduce

$$\frac{R^2}{4\delta}(R^2 - 2\lambda_c) \ln\left(\frac{\sqrt{R^2 - 2\lambda_c}}{R}\right) + \frac{4\lambda_c^2}{16\delta} + \frac{R^2 - 2\lambda_c}{8\delta}2\lambda_c = 1.$$

Using properties of logarithm and simple algebra we have

$$\frac{R^2}{8\delta}(R^2 - 2\lambda_c) \ln\left(1 - \frac{2\lambda_c}{R^2}\right) + \frac{\lambda_c^2}{4\delta} + \frac{R^2 - 2\lambda_c}{4\delta}\lambda_c = 1.$$

We introduce an auxiliary variable $x_c = \frac{2\lambda_c}{R^2}$ and multiply equation by $\frac{8\delta}{R^4}$ to get

$$(1 - x_c) \ln(1 - x_c) + \frac{1}{2}x_c^2 + (1 - x_c)x_c = \frac{8\delta}{R^4}. \quad (3.15)$$

It follows that if x_c is a solution of (3.15), then $\lambda_c = \frac{R^2 x_c}{2}$.

Existence and uniqueness of x_c and λ_c . We prove that if $16\delta < R^4$, equation (3.15) has a unique solution. To this end, we define

$$f(x) := (1 - x) \ln(1 - x) + \frac{1}{2}x^2 + (1 - x)x. \quad (3.16)$$

Then,

$$f'(x) = -\ln(1 - x) - 1 + x + 1 - 2x = -\ln(1 - x) - x, \quad f''(x) = \frac{1}{1 - x} - 1. \quad (3.17)$$

It follows that $f''(x) > 0$ for $x \in (0, 1)$ so that $f'(x)$ is increasing. As $f'(0) = 0$ it follows that $f'(x) > 0$ so that $f(x)$ is increasing. It follows that f is a bijection from $(0, 1)$ into $(0, \frac{1}{2})$. Therefore, when $16\delta < R^4$, there exists a unique $x_c \in (0, 1)$ such that $f(x_c) = \frac{8\delta}{R^4}$. \square

Lemma 3.10 (Estimates for x_c). *Let x_c be a solution to (3.13) and $16\delta < R^4$. Then, Taylor expansion implies*

$$x_c \approx 2\sqrt[3]{6}\delta^{1/3}R^{-4/3}, \quad \lambda_c \approx \sqrt[3]{6}\delta^{1/3}R^{2/3} \quad (\text{as } \delta \rightarrow 0).$$

More precisely, we have

$$x_c \leq 2\sqrt[3]{6}\delta^{1/3}R^{-4/3}. \quad (3.18)$$

Moreover, if $6^4 8\delta < R^4$, we have

$$x_c \geq 2\sqrt[3]{5}\delta^{1/3}R^{-4/3}. \quad (3.19)$$

Proof. First, we prove Taylor estimates. As δ is small, we expect the solution of (3.13) to be small too. Using (3.16) and (3.17) we obtain $f(0) = f'(0) = f''(0) = 0$. Moreover, $f^{(3)}(x) = \frac{1}{(1-x)^2}$ so that $f^{(3)}(0) = 1$. Hence, by Taylor expansion, for small x , $f(x) \approx \frac{x^3}{6}$. Plugging this approximation into (3.15) and using $\lambda_c = \frac{R^2 x_c}{2}$ we deduce as desired.

Now, we prove precise lower and upper estimates. First, we observe that $f^{(3)}(x) = \frac{1}{(1-x)^2}$. Therefore, by simple induction,

$$f^{(k)}(x) = \frac{(k-2)!}{(1-x)^{k-1}}, \quad f^{(k)}(0) = (k-2)!$$

In particular, Taylor expansion around $x = 0$ implies

$$f(x) = \sum_{k \geq 3} \frac{(k-2)!}{k!} x^k = \sum_{k \geq 3} \frac{1}{k(k-1)} x^k.$$

Therefore, on the one hand, $f(x) \geq \frac{x^3}{6}$. On the other hand,

$$f(x) \leq \frac{x^3}{6} \sum_{k \geq 0} x^k = \frac{x^3}{6(1-x)}. \quad (3.20)$$

First, we deduce that $x_c \leq 2\sqrt[3]{6}\delta^{1/3}R^{-4/3}$. Using this, we can find δ such that $2\sqrt[3]{6}\delta^{1/3}R^{-4/3} \leq \frac{1}{6}$, namely $6^4 8\delta \leq R^4$. Then, we have $x_c \leq \frac{1}{6}$ so that $1 - x_c \geq \frac{5}{6}$ and then the estimate (3.20) gives us

$$\frac{8\delta}{R^4} = f(x_c) \leq \frac{x_c^3}{6(1-x_c)} \leq \frac{x_c^3}{5}$$

so that

$$\frac{8\delta}{R^4} \leq \frac{x_c^3}{5} \iff \frac{40\delta}{R^4} \leq x_c^3 \iff 2\sqrt[3]{5}\delta^{1/3}R^{-4/3} \leq x_c.$$

□

Lemma 3.11. *If u_c is as in Proposition 3.8 then the total mass of u_c satisfies*

$$\mathcal{M}(u_c) := \int_0^R r u_c(r) dr = \frac{R^6 x_c^3(R)}{96\delta}.$$

Moreover, the map $R \mapsto \frac{R^6 x_c^3(R)}{96\delta}$ is strictly increasing if $R^4 x_c^3(R) > 32\delta$.

Proof. For solution u_c satisfying $u_c(R_0) = 1$, $u'_c(R_0) = 0$, its extension by 1 is a C^1 function. Therefore, integrating by parts,

$$\mathcal{M}(u_c) = \int_0^R r u_c(r) dr = -\frac{1}{2} \int_0^R r^2 u'_c(r) dr = -\frac{1}{2} \int_{R_0}^R r^2 u'_c(r) dr.$$

Plugging formula for $u'_c(r)$ we deduce

$$\mathcal{M}(u_c) = -\frac{1}{8\delta} \int_{R_0}^R r (R^2 - r^2)(R^2 - r^2 - 2\lambda_c) dr.$$

Now, we observe that $\lambda_c := R^2 x_c/2$ and $R_0 = R\sqrt{1-x_c}$ so that

$$\mathcal{M}(u_c) = -\frac{1}{8\delta} \int_{R\sqrt{1-x_c}}^R r (R^2 - r^2)(R^2 - r^2 - R^2 x_c) dr.$$

We change variables $\tau = R^2 - r^2$ to get

$$\mathcal{M}(u_c) = -\frac{1}{16\delta} \int_0^{R^2 x_c} \tau(\tau - R^2 x_c) d\tau = \frac{R^6 x_c^3}{96\delta}.$$

For the second assertion, it is sufficient to prove that the map $R \mapsto R^6 x_c^3(R)$ is strictly increasing. Note that $x_c(R)$ is given implicitly via equation (3.15). Differentiating it with respect to R , we discover

$$\frac{dx_c}{dR}(x_c + \log(1-x_c)) = \frac{32\delta}{R^5} \implies \frac{dx_c}{dR} = \frac{32\delta}{R^5(x_c + \log(1-x_c))}.$$

Then, we study the derivative of $R^6 x_c^3(R)$:

$$\frac{d(R^6 x_c^3(R))}{dR} = 6R^5 x_c^3 + 3R^6 x_c^2 \frac{dx_c}{dR} = 6R^5 x_c^3 + \frac{96\delta R x_c^2}{(x_c + \log(1-x_c))}$$

Using a simple Taylor estimate

$$\frac{1}{x + \log(1-x)} \geq \frac{-2}{x^2}$$

we estimate

$$\frac{d(R^6 x_c^3(R))}{dR} \geq 6R^5 x_c^3 - 192\delta R = 6R(R^4 x_c^3 - 32\delta).$$

□

Proof of Proposition 3.8. First, we notice that if (R_0, R, λ) satisfy conditions of the Proposition 3.8, then R_0, λ_c are given by (3.12) (Lemma 3.9) and $\frac{R^6 x_c^3}{96\delta} = m$ (Lemma 3.11). Then, by $6^2 2 \delta^{1/2} \leq m$ as well as an upper bound on x_c , cf. (3.18), we deduce

$$6^2 2 \delta^{1/2} \leq m = \frac{R^6 x_c^3}{96\delta} \leq \frac{R^2}{2} \implies 6^4 4^2 \delta \leq R^4.$$

This means that we can apply the lower bound (3.19) to deduce

$$R^4 x_c^3 \geq 40 \delta^{1/3}.$$

It follows that the necessary condition for existence of (R_0, R, λ) is $6^4 4^2 \delta \leq R^4$ which implies $R^4 x_c^3 \geq 40 \delta^{1/3}$. Therefore, by Lemma 3.11, the map $R \mapsto R^6 x_c^3(R)$ is invertible and we can find uniquely determined R such that

$$m = \frac{R^6 x_c^3(R)}{96\delta}.$$

With such R (because $16\delta < R^4$), we can find unique R_0 and λ_c solving (3.10)-(3.11) so that the formula for the mass is satisfied and the conclusion follows. \square

3.7 Proof of Theorem 3

Since the solutions of Theorem 2 are radially symmetric they satisfy

$$n_k^{\gamma_k} - \delta n_k'' - \frac{\delta}{r} n_k' = R_k^2 - r^2 - \lambda_k \quad \text{on } (0, R_k), \quad n_k = 0 \quad \text{on } (R_k, R_b)$$

with boundary conditions $n_k'(R_k) = n_k'(0) = n_k(R_k) = 0$. Using similar computations as in the proof of Lemma 3.3 we obtain that the sequence $\{n_k^{\gamma_k}\}_k$ is bounded in $L^\infty(I_{R_b})$. Moreover, multiplying this equation by n_k'' and using integration by parts we obtain

$$\int_0^{R_k} \left(\delta (n_k'')^2 + \gamma n_k^{\gamma_k-1} (n_k')^2 + \delta \frac{(n_k')^2}{2r^2} \right) dr = \int_0^{R_k} n_k'' (R_k^2 - r^2 - \lambda_k) dr.$$

Since $0 \leq R_k, \lambda_k \leq R_b$ we see that the right-hand side is bounded by $\frac{\delta}{2} \int_0^{R_k} (n_k'')^2 + C(\delta, R_b)$. Thus, $\{n_k''\}_k$ is uniformly bounded in $L^2(0, R_b)$. Therefore, up to a subsequence, as $k \rightarrow \infty$

$$\begin{aligned} n_k^{\gamma_k} &\rightharpoonup p \quad \text{weakly}^* \text{ in } L^\infty(I_{R_b}), \\ n_k &\rightarrow n \quad \text{in } C^1(\overline{I_{R_b}}). \end{aligned}$$

We have that $n \leq 1$ which follows from the uniform estimate $\|n_k^{\gamma_k}\|_{L^\infty} \leq C$ and uniform convergence $n_k \rightarrow n$.

We also have the graph relation $p(n-1) = 0$. In fact, the inequality $p(n-1) \leq 0$ is found in a straightforward manner using $p \geq 0$ and $n \leq 1$. It remains to show that $p(n-1) \geq 0$. For $\nu > 0$, there exists γ_0 such that for $\gamma_k \geq \gamma_0$

$$n_k^{\gamma_k+1} \geq n_k^{\gamma_k} - \nu$$

(this is because the function $x \mapsto x^\gamma(x-1)$ is nonpositive on $[0, 1]$ and attains its minimum $-\left(\frac{\gamma}{\gamma+1}\right)^\gamma \frac{1}{\gamma+1} \rightarrow 0$ as $\gamma \rightarrow \infty$). Then, from the strong convergence of n_k and the weak convergence of $n_k^{\gamma_k}$ we know that $n_k^{\gamma_k} n_k$ converges weakly to pn . Passing to the limit, we obtain

$$pn \geq p - \nu,$$

for every $\nu > 0$. Letting $\nu \rightarrow 0$ yields the result.

Since $\{\lambda_k\}_k$ and $\{R_k\}_k$ are also bounded subsequences we can extract converging subsequences to λ, R respectively. Thanks to the C^1 convergence we know that n satisfies the boundary condition $n(x) = \frac{\partial n}{\partial \mathbf{n}} = 0$ on ∂B_R and n is radially decreasing as the uniform limit of radially decreasing functions. Finally, we can pass to the limit in the equation of conservation of the mass and obtain $\int_0^R r n dr = m$. To sum up, in the limit we obtain a C^1 , nonincreasing function n satisfying

$$\begin{cases} p - \frac{\delta}{r} n' - \delta n'' = R^2 - r^2 - \lambda & \text{in } (0, \tilde{R}) \\ n(\tilde{R}) = n'(R) = n'(0) = 0, \\ \int_0^{\tilde{R}} r n(r) dr = m, \\ p(n-1) = 0. \end{cases}$$

The limiting ODE is satisfied on $(0, R)$ because the ODE for n_k is satisfied on $(0, \inf_{l \geq k} R_k)$. Passing to the limit, we obtain the ODE on $(0, \lim_{k \rightarrow \infty} \inf_{l \geq k} R_k) = (0, R)$ because $R = \lim_{k \rightarrow \infty} R_k$.

We claim that n reaches value 1. To see this, we note that if $n < 1$ on $[0, R]$, then $p = 0$ so that n is a C^1 solution to the following ODE on $[0, R]$:

$$-\frac{\delta}{r} n' - \delta n'' = R^2 - r^2 - \lambda, \quad n(R) = n'(R) = n'(0) = 0.$$

By Proposition 3.7 (A) such a solution does not exist (it has to blow up at $r = 0$).

By monotonicity and the fact that n reaches value 1 we deduce that there are two zones. In the zone $\{p > 0\}$ we have $n = 1$, and thus $p = R^2 - r^2 - \lambda$. Then, when $n < 1$ (n is decreasing), let us say at $r = R_0$ we have $p = 0$. The pressure jump is equal to $\llbracket p \rrbracket = R^2 - R_0^2 - \lambda$.

Finally, the convergence of the whole sequence follows from uniqueness of the limiting profile as stated in Proposition 3.8.

4 Conclusion and numerical simulations

We established proved that the stationary solutions of the radial Cahn-Hilliard system with a confining potential present a jump of pressure at the boundary $\partial\{n = 1\}$ that we computed explicitly, and is proportional to $\delta^{1/3}$. It is an open question to prove a similar result when the system admits a source term as in [13] for instance. There is a zone $\{n = 0\}$ that induces a degeneracy which is the main difficulty when establishing the a priori estimates. However we can provide numerical simulations to such a system in radial coordinates. To be more precise we focus on the system

$$\frac{\partial(rn)}{\partial t} - \frac{\partial}{\partial r} \left(rn \frac{\partial \mu}{\partial r} \right) = nG(p), \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (4.1)$$

$$\mu = p - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right), \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (4.2)$$

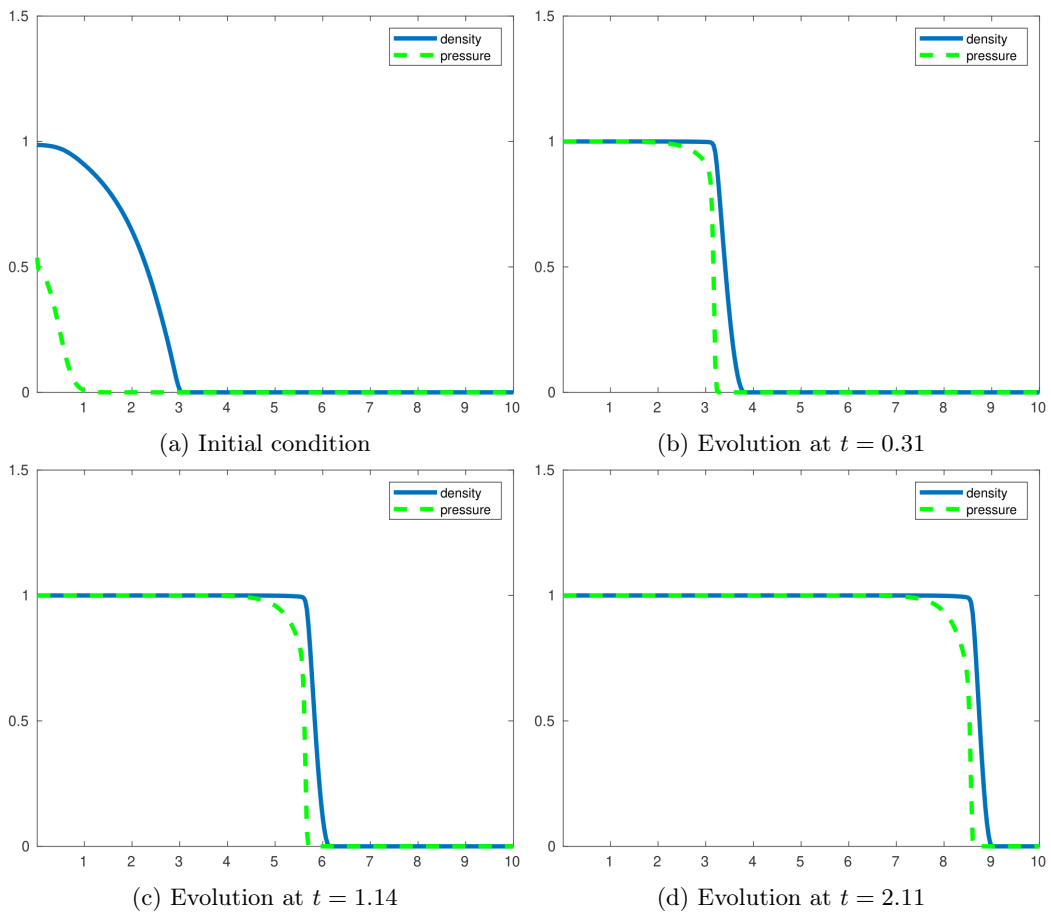
$$p = n^\gamma \quad (4.3)$$

When $\gamma \rightarrow \infty$ we expect to find

$$\begin{cases} -\Delta p = G(p) & \text{in } \{n = 1\}, \\ \llbracket p \rrbracket = -\delta \llbracket \Delta n \rrbracket & \text{on } \partial\{n = 1\}. \end{cases}$$

Then it remains to link the value of the pressure jump to the other parameters of the model, *i.e.* the source term G , the parameter δ and the boundary's curvature. In radial settings the curvature is $\frac{1}{R}$ where R is the radius of the tumor. We present below some numerical simulations for the evolution of the density and the pressure of the tumor. If the pressure jump looks like it decreases to the naked eye as the tumor grows, it is not numerically clear which criterion to choose to characterize the point where the jump begins.

Numerical settings: To remove the degeneracy $r = 0$ in the numerical scheme, we consider $r + \varepsilon$ instead of r for some small $\varepsilon > 0$. For the source term we take $G(p) = 10(1 - p)$. We use an explicit scheme, with time step $dt = 1e-7$, final time $t = 2.11$ and the interval is $I_{R_b} = [0, 10]$ with 300 points. The initial condition is a truncated arctangent. The density is n and the pressure is n^γ .



Note that the pressure p reaches the value 1 as the density. This is due to the homeostatic pressure $p_h = 1$ in the source term $G(p) = 10(p_h - p)$. The homeostatic pressure is the lowest level of pressure that prevents cell multiplication due to contact-inhibition. The pressure cannot exceed this threshold. Of course, different values of p_h are possible.

Acknowledgements

B.P. has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No 740623). J.S. was supported

by the National Science Center grant 2017/26/M/ST1/00783.

A Limit profile for general force in dimension 2

We generalize the pressure jump formula obtained when $V(r) = r^2$ and, for a general strictly increasing force V , we establish that

$$\lambda \approx \frac{\sqrt[3]{12}}{2} \delta^{1/3} (V'(R))^{2/3}.$$

An expression of the solution. We consider the solution of

$$-\delta \Delta n = V(R) - V(r) - \lambda, \quad n'(R) = n(R) = 0 \quad (\text{A.1})$$

and claim that its solution is given by

$$n(r) = \frac{R^2}{2\delta} (V(R) - \lambda) \log\left(\frac{r}{R}\right) + \frac{R^2 - r^2}{4\delta} (V(R) - \lambda) + \frac{1}{\delta} \int_r^R \frac{\mathcal{H}(z)}{z} dz \quad (\text{A.2})$$

with

$$\mathcal{H}(z) := \int_z^R u V(u) du.$$

Indeed, we immediately verify that $n(R) = 0$. Moreover, we have

$$n'(r) = \frac{R^2}{2\delta} (V(R) - \lambda) \frac{1}{r} - \frac{r}{2\delta} (V(R) - \lambda) - \frac{\mathcal{H}(r)}{\delta r}. \quad (\text{A.3})$$

As $\mathcal{H}(R) = 0$, we have $n'(R) = 0$. Finally, we compute n'' using (A.3)

$$n''(r) = -\frac{R^2}{2\delta} (V(R) - \lambda) \frac{1}{r^2} - \frac{1}{2\delta} (V(R) - \lambda) + \frac{1}{\delta} V(r) + \frac{\mathcal{H}(r)}{\delta r^2}. \quad (\text{A.4})$$

Therefore, using (A.3) and (A.4) we compute

$$\delta n''(r) + \delta \frac{n'(r)}{r} = -(V(R) - \lambda) + V(r) = -(V(R) - V(r) - \lambda)$$

so that (A.1) is indeed satisfied.

Limit profile. We are looking for the solution n to (A.1) such that $n(R_0) = 1$, $n'(R_0) = 0$ for some $R_0 < R$ and some λ , i.e. we have two parameters R_0 and λ to be found.

Condition $n'(R_0) = 0$. This condition, using (A.3), is reduced to

$$\frac{R^2}{2\delta} (V(R) - \lambda) \frac{1}{R_0} - \frac{R_0}{2\delta} (V(R) - \lambda) - \frac{\mathcal{H}(R_0)}{\delta R_0} = 0.$$

Multiplying this equation by $2\delta R_0$ we obtain

$$(R^2 - R_0^2) (V(R) - \lambda) - 2\mathcal{H}(R_0) = 0. \quad (\text{A.5})$$

Condition $n(R_0) = 1$. From (A.2) we obtain

$$\frac{R^2}{2\delta} (V(R) - \lambda) \log\left(\frac{R_0}{R}\right) + \frac{R^2 - R_0^2}{4\delta} (V(R) - \lambda) + \frac{1}{\delta} \int_{R_0}^R \frac{\mathcal{H}(z)}{z} dz = 1.$$

Then, we use (A.5) to remove the term $(V(R) - \lambda)$, and multiply by 2δ to get an equation for R_0

$$R^2 \frac{\mathcal{H}(R_0)}{R^2 - R_0^2} \log\left(\frac{R_0^2}{R^2}\right) + \mathcal{H}(R_0) + 2 \int_{R_0}^R \frac{\mathcal{H}(z)}{z} dz = 2\delta.$$

We introduce the variable $\tau_0 := \frac{R^2 - R_0^2}{R^2}$ so that the equation reads

$$\mathcal{H}(\sqrt{1 - \tau}R) \left(\frac{\log(1 - \tau)}{\tau} + 1 \right) + 2 \int_{\sqrt{1 - \tau}R}^R \frac{\mathcal{H}(z)}{z} dz = 2\delta, \quad \tau_0 := \frac{R^2 - R_0^2}{R^2}. \quad (\text{A.6})$$

Existence and uniqueness of τ_0 and R_0 . We define function

$$\mathcal{F}(\tau) := \mathcal{H}(\sqrt{1 - \tau}R) \left(\frac{\log(1 - \tau)}{\tau} + 1 \right) + 2 \int_{\sqrt{1 - \tau}R}^R \frac{\mathcal{H}(z)}{z} dz.$$

As $\mathcal{H}(R) = 0$ and $\frac{\log(1 - \tau)}{\tau}$ is bounded near $\tau = 0$, we have $\mathcal{F}(\tau) = 0$. Now, we want to compute $\mathcal{F}'(\tau)$. First,

$$\begin{aligned} \frac{d}{d\tau} \mathcal{H}(\sqrt{1 - \tau}R) &= \sqrt{1 - \tau}R V(\sqrt{1 - \tau}R) \frac{R}{2\sqrt{1 - \tau}} = \frac{R^2 V(\sqrt{1 - \tau}R)}{2}, \\ \frac{d}{d\tau} \left(2 \int_{\sqrt{1 - \tau}R}^R \frac{\mathcal{H}(z)}{z} dz \right) &= 2 \left(-\frac{\mathcal{H}(\sqrt{1 - \tau}R)}{\sqrt{1 - \tau}R} \right) \frac{R}{2\sqrt{1 - \tau}} = \frac{\mathcal{H}(\sqrt{1 - \tau}R)}{1 - \tau}, \\ \frac{d}{d\tau} \left(\frac{\log(1 - \tau)}{\tau} + 1 \right) &= \frac{1}{\tau(\tau - 1)} - \frac{\log(1 - \tau)}{\tau^2}. \end{aligned} \quad (\text{A.7})$$

Therefore, by product rule,

$$\begin{aligned} \mathcal{F}'(\tau) &= \\ &= \frac{R^2 V(\sqrt{1 - \tau}R)}{2} \left(\frac{\log(1 - \tau)}{\tau} + 1 \right) + \mathcal{H}(\sqrt{1 - \tau}R) \left(\frac{1}{1 - \tau} + \frac{1}{\tau(\tau - 1)} - \frac{\log(1 - \tau)}{\tau^2} \right) \\ &= \frac{1}{\tau} \left(\frac{\log(1 - \tau)}{\tau} + 1 \right) \left(\frac{R^2 \tau V(\sqrt{1 - \tau}R)}{2} - \mathcal{H}(\sqrt{1 - \tau}R) \right) \end{aligned} \quad (\text{A.8})$$

By a standard calculus $\frac{\log(1 - \tau)}{\tau} + 1 < 0$ for $\tau \in (0, 1)$. Therefore, to prove $\mathcal{F}'(\tau) > 0$, it is sufficient that for $\tau \in (0, 1)$

$$\frac{R^2 \tau V(\sqrt{1 - \tau}R)}{2} - \mathcal{H}(\sqrt{1 - \tau}R) < 0. \quad (\text{A.9})$$

This function vanishes in $\tau = 0$. Moreover, its derivative with respect to τ is equal to

$$\frac{R^2 V(\sqrt{1 - \tau}R)}{2} + \frac{R^2 \tau V'(\sqrt{1 - \tau}R)}{2} \frac{-R}{2\sqrt{1 - \tau}} - \frac{R^2 V(\sqrt{1 - \tau}R)}{2} = -\frac{R^3 \tau V'(\sqrt{1 - \tau}R)}{4\sqrt{1 - \tau}} \quad (\text{A.10})$$

where we used (A.7). As $V' > 0$, we conclude (A.9) which implies $\mathcal{F}'(\tau) > 0$. Hence, in some neighborhood of 0 we can find exactly one τ that solves the equation. Moreover, it is unique as \mathcal{F} is strictly increasing. Then the uniqueness of R_0 and λ follows.

Taylor approximation of τ_0 . As \mathcal{F} is strictly increasing, we expect the solution τ to be small (if δ is small). This justifies to use Taylor expansion around $\tau = 0$. We already know that $\mathcal{F}(0) = \mathcal{F}'(0) = 0$. Now, we claim $\mathcal{F}''(0) = 0$. Indeed,

$$\frac{d}{d\tau} \left(\frac{\log(1 - \tau)}{\tau^2} + \frac{1}{\tau} \right) = -\frac{2 \log(1 - \tau)}{\tau^3} + \frac{\tau - 2}{(1 - \tau)\tau^2} = -\frac{2 \log(1 - \tau)(1 - \tau) + (2 - \tau)\tau}{(1 - \tau)\tau^3}$$

so that using (A.8) and (A.10) we compute

$$\begin{aligned} \mathcal{F}''(\tau) = & -\frac{1}{\tau} \left(\frac{\log(1-\tau)}{\tau} + 1 \right) \frac{R^3 \tau V'(\sqrt{1-\tau}R)}{4\sqrt{1-\tau}} \\ & - \frac{2\log(1-\tau)(1-\tau) + (2-\tau)\tau}{(1-\tau)\tau^3} \left(\frac{R^2 \tau V(\sqrt{1-\tau}R)}{2} - \mathcal{H}(\sqrt{1-\tau}R) \right) \end{aligned} \quad (\text{A.11})$$

A simple Taylor argument shows that $2\log(1-\tau)(1-\tau) + (2-\tau)\tau \approx \tau^3$ for small τ . Therefore, the expressions above are bounded in the neighborhood of 0. Evaluating them at $\tau = 0$, we obtain $\mathcal{F}''(0) = 0$.

Now, we claim that $\mathcal{F}^{(3)}(0) \neq 0$. To see this, we write expression (A.11) in the form

$$\mathcal{F}''(\tau) = A(\tau) B(\tau) + C(\tau) D(\tau)$$

so that

$$\mathcal{F}^{(3)}(\tau) = A'(\tau) B(\tau) + A(\tau) B'(\tau) + C'(\tau) D(\tau) + C(\tau) D'(\tau).$$

We study the four terms above separately.

- $A'(\tau) B(\tau)|_{\tau=0} = 0$. Indeed, $A'(\tau) = -C(\tau)$ and the latter is bounded in the neighbourhood of 0 (see Taylor's expansion above). Moreover, $B(0) = 0$.
- $C(\tau) D'(\tau)|_{\tau=0} = 0$. Indeed, $C(\tau)$ is bounded around $\tau = 0$, while $D'(0) = B(0) = 0$.
- $C'(\tau) D(\tau)|_{\tau=0} = 0$. In fact, $D(0) = 0$, so it is sufficient to prove that $C'(\tau)$ is bounded near $\tau = 0$. We have

$$C'(\tau) = \frac{\tau(2\tau^2 - 9\tau + 6) + 6\log(1-\tau)(1-\tau)^2}{\tau^4(1-\tau^2)}.$$

Using expansion $\log(1-\tau) \approx -\tau - \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3$ we have around $\tau = 0$:

$$\begin{aligned} \tau(2\tau^2 - 9\tau + 6) + 6\log(1-\tau)(1-\tau)^2 & \approx \\ & \approx 2\tau^3 - 9\tau^2 + 6\tau - (6\tau + 3\tau^2 + 2\tau^3)(1 + \tau^2 - 2\tau) \\ & = 2\tau^3 - 9\tau^2 + 6\tau - 6\tau - 3\tau^2 - 2\tau^3 - 6\tau^3 - 3\tau^4 - 2\tau^5 + 12\tau^2 + 6\tau^3 + 4\tau^4 \\ & = \tau^4 - 2\tau^5. \end{aligned}$$

- $A(\tau)B'(\tau)|_{\tau=0} = \frac{R^3 V'(R)}{8}$. Indeed, $A(0) = \frac{1}{2}$. Moreover,

$$B'(\tau) = \frac{R^3 V'(\sqrt{1-\tau}R)}{4\sqrt{1-\tau}} + \frac{R^3 \tau V''(\sqrt{1-\tau}R)}{4\sqrt{1-\tau}} \frac{(-R)}{2\sqrt{1-\tau}} + \frac{R^3 \tau V'''(\sqrt{1-\tau}R)}{8(1-\tau)^{3/2}}$$

which implies $B'(0) = \frac{R^3 V'(R)}{8}$.

Therefore, equation (A.6) can be approximated with

$$\frac{1}{6} \frac{R^3 V'(R)}{8} \tau^3 = 2\delta \iff \tau^3 = \frac{96\delta}{R^3 V'(R)} \iff \tau = \frac{2\sqrt[3]{12}\delta^{1/3}}{R\sqrt[3]{V'(R)}}.$$

Taylor approximation for λ_0 . Small value of τ means that R_0 is close to R . Therefore, we expand in Taylor series equation (A.5) around $R_0 = R$. Writing

$$\mathcal{G}(R_0) := (R^2 - R_0^2)(V(R) - \lambda) - 2\mathcal{H}(R_0) = 0,$$

we have

$$\begin{aligned}\mathcal{G}'(R_0) &= -2R_0(V(R) - \lambda) + 2R_0 V(R_0) = 2R_0(V(R_0) - V(R) + \lambda), \\ \mathcal{G}''(R_0) &= 2(V(R_0) - V(R) + \lambda) + 2R_0 V'(R_0).\end{aligned}$$

It follows that $\mathcal{G}(R) = 0$, $\mathcal{G}'(R) = 2R_0\lambda$ and $\mathcal{G}''(R) = 2\lambda + 2RV'(R)$. Therefore, (A.5) can be approximated with

$$2R_0\lambda(R_0 - R) + \frac{1}{2}(2\lambda + 2RV'(R))(R_0 - R)^2 = 0$$

which can be rewritten as

$$\lambda = \frac{R(R - R_0)V'(R)}{R_0 + R}.$$

Using $\tau_0 = \frac{R^2 - R_0^2}{R^2}$ we have $R - R_0 = \frac{\tau R^2}{R + R_0}$ so that

$$\lambda = \frac{R^3 \tau V'(R)}{(R_0 + R)^2} \approx \frac{R \tau V'(R)}{4} = \frac{\sqrt[3]{12}}{2} \delta^{1/3} (V'(R))^{2/3}.$$

Case $V(r) = r^2$. In this case we have $V'(R) = 2R$. Therefore, $\lambda \approx \sqrt[3]{6} \delta^{1/3} R^{2/3}$ as in the previous approximation.

References

- [1] N. D. ALIKAKOS, P. W. BATES, AND X. CHEN, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model*, Arch. Rational Mech. Anal., 128 (1994), pp. 165–205.
- [2] G. ALLAIRE, *Homogenization of the Navier-Stokes equations and derivation of Brinkman's law*, in Mathématiques appliquées aux sciences de l'ingénieur (Santiago, 1989), Cépaduès, Toulouse, 1991, pp. 7–20.
- [3] M. BASAN, T. RISLER, J. JOANNY, X. SASTRE-GARAU, AND J. PROST, *Homeostatic competition drives tumor growth and metastasis nucleation*, HFSP Journal, 3 (2009), pp. 265–272. PMID: 20119483.
- [4] T. BITTIG, O. WARTLICK, A. KICHEVA, M. GONZÁLEZ-GAITÁN, AND F. JÜLICHER, *Dynamics of anisotropic tissue growth*, New J. Phys., 10 (2008), p. 063001.
- [5] H. BYRNE AND D. DRASDO, *Individual-based and continuum models of growing cell populations: a comparison*, J. Math. Biol., 58 (2009), pp. 657–687.
- [6] H. BYRNE AND L. PREZIOSI, *Modelling solid tumour growth using the theory of mixtures*, Math. Med. Biol., 20 (2003), pp. 341–366.
- [7] J. W. CAHN, *On spinodal decomposition*, Acta Metallurgica, 9 (1961), pp. 795–801.
- [8] J. W. CAHN AND J. E. HILLIARD, *Free energy of a nonuniform system. i. interfacial free energy*, The Journal of Chemical Physics, 28 (1958), pp. 258–267.
- [9] B. CHEN AND C. LIU, *Finite speed of propagation for the cahn-hilliard equation with degenerate mobility*, Applicable Analysis, 100 (2021), pp. 1693–1726.
- [10] X. CHEN, *Global asymptotic limit of solutions of the Cahn-Hilliard equation*, Journal of Differential Geometry, 44 (1996), pp. 262 – 311.
- [11] M. EBENBECK AND H. GARCKE, *On a Cahn-Hilliard-Brinkman model for tumor growth and its singular limits*, SIAM J. Math. Anal., 51 (2019), pp. 1868–1912.

- [12] M. EBENBECK, H. GARCKE, AND R. NÜRNBERG, *Cahn-Hilliard-Brinkman systems for tumour growth*, Discrete Contin. Dyn. Syst. Ser. S, 14 (2021), pp. 3989–4033.
- [13] C. ELBAR, B. PERTHAME, AND A. POULAIN, *Degenerate Cahn-Hilliard and incompressible limit of a Keller-Segel model*, Accepted in Communications in Mathematical Sciences, (2021).
- [14] C. M. ELLIOTT AND H. GARCKE, *On the Cahn-Hilliard equation with degenerate mobility*, SIAM J. Math. Anal., 27 (1996), pp. 404–423.
- [15] J. ESCHER AND G. SIMONETT, *Classical solutions for Hele-Shaw models with surface tension*, Adv. Differential Equations, 2 (1997), pp. 619–642.
- [16] H. B. FRIEBOES, F. JIN, Y.-L. CHUANG, S. M. WISE, J. S. LOWENGRUB, AND V. CRISTINI, *Three-dimensional multispecies nonlinear tumor growth—II: Tumor invasion and angiogenesis*, J. Theor. Biol., 264 (2010), pp. 1254–1278.
- [17] A. FRIEDMAN, *Mathematical analysis and challenges arising from models of tumor growth*, Math. Models Methods Appl. Sci., 17 (2007), pp. 1751–1772.
- [18] S. FRIGERI, K. F. LAM, E. ROCCA, AND G. SCHIMPERNA, *On a multi-species Cahn-Hilliard-Darcy tumor growth model with singular potentials*, Commun. Math. Sci., 16 (2018), pp. 821–856.
- [19] H. GARCKE, K. F. LAM, R. NÜRNBERG, AND E. SITKA, *A multiphase Cahn-Hilliard-Darcy model for tumour growth with necrosis*, Math. Models Methods Appl. Sci., 28 (2018), pp. 525–577.
- [20] H. GARCKE, K. F. LAM, E. SITKA, AND V. STYLES, *A Cahn-Hilliard-Darcy model for tumour growth with chemotaxis and active transport*, Math. Models Methods Appl. Sci., 26 (2016), pp. 1095–1148.
- [21] B. GIDAS, W.-M. NI, AND L. NIRENBERG, *Symmetry and related properties via the maximum principle*, Communications in Mathematical Physics, 68 (1979), pp. 209–243.
- [22] H. P. GREENSPAN, *On the growth and stability of cell cultures and solid tumors*, J. Theoret. Biol., 56 (1976), pp. 229–242.
- [23] Y. JINGXUE, *On the Cahn-Hilliard equation with nonlinear principal part*, Journal of Partial Differential Equations, 7 (1994), pp. 77–96.
- [24] J. LOWENGRUB, H. FRIEBOES, F. JIN, Y. CHUANG, X. LI, P. MACKLIN, S. WISE, AND V. CRISTINI, *Nonlinear modelling of cancer: bridging the gap between cells and tumours.*, Nonlinearity, 23 1 (2010), pp. R1–R9.
- [25] A. MIRANVILLE, *The Cahn-Hilliard equation. Recent advances and applications*, vol. 95 of CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2019. Recent advances and applications.
- [26] B. PERTHAME AND A. POULAIN, *Relaxation of the Cahn-Hilliard equation with singular single-well potential and degenerate mobility*, European J. Appl. Math., 32 (2021), pp. 89–112.
- [27] B. PERTHAME, F. QUIRÓS, AND J. L. VÁZQUEZ, *The Hele-Shaw asymptotics for mechanical models of tumor growth*, Arch. Ration. Mech. Anal., 212 (2014), pp. 93–127.
- [28] B. PERTHAME AND N. VAUCHELET, *Incompressible limit of a mechanical model of tumour growth with viscosity*, Philos. Trans. Roy. Soc. A, 373 (2015), pp. 20140283, 16.

- [29] S. M. WISE, J. S. LOWENGRUB, H. B. FRIEBOES, AND V. CRISTINI, *Three-dimensional multispecies nonlinear tumor growth—I: Model and numerical method*, J. Theor. Biol., 253 (2008), pp. 524–543.
- [30] J. YIN AND C. LIU, *Radial symmetric solutions of the Cahn-Hilliard equation with degenerate mobility.*, Electronic Journal of Qualitative Theory of Differential Equations [electronic only], 2001 (2001), pp. Paper No. 2, 14 p., electronic only—Paper No. 2, 14 p., electronic only.