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POSITIVITY OF ADELIC LINE BUNDLES OVER AN ADELIC CURVE

HUAYI CHEN AND ATSUSHI MORIWAKI

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Abstract. The notions of "positivity" and their properties are the most important subjects in algebraic geometry. In this article, we consider an arithmetic analogue over a general adelic curve as a generalization of the classical Arakelov geomerty over a number field.

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1. Introduction

The positivity of line bundles is one of the most fundamental and important notions in algebraic geometry. In Arakelov geometry, the analogue of ampleness and Nakai-Moishezon criterion have been studied by Zhang [21, 22]. The arithmetic bigness has been introduced in the works [17, 19, 18] of Moriwaki and Yuan. These positivity conditions and their properties have various applications in Diophantine geometry.

The purpose of this article is to revisit the arithmetic positivity notions in the context of Arakelov geometry over adelic curves. Recall that an adelic curve consists of a field K equipped with a family of a family of absolute values on K (with possible repetitions), which is parametrized by a measure space. This notion is a very natural generalization to any countable field of Weil's adelic approche of number theory. The fundament of height theory and Arakelov geometry for projective varieties over an adelic curve have been established in the works of Gubler [14] (in a slightly different setting of M-fields) and Chen-Moriwaki [10].

Let S be an adelic curve with a countable and perfect underlying field K. We assume that S is proper, namely the family of absolute values in S satisfies an analogue of product formula in the number field case. Let X be a projective scheme over $\operatorname{Spec} K$. Given an adelic line bundle $\overline{L}$ on X, namely an invertible $\mathcal{O}_X$-module equipped with a measurable and dominated family of continuous metrics on the analytification of X (in the sense of Berkovich [2]) according to absolute values of K in the adelic curve structure of S, we are interested in various positivity conditions of the adelic line bundle $\overline{L}$. We say that the adelic line bundle $\overline{L}$ is *relatively ample* if the invertible $\mathcal{O}_X$-module is ample and if the metrics of $\overline{L}$ are all semi-positive. The relative nefness can then been defined in a limit form of relative ampleness, similarly to the classic case in algebraic geometry. Recall that the global intersection number of relatively ample adelic line bundles (or more generally, integral adelic line bundles) can be defined as the integral of local heights along the measure space in the adelic structure (cf. [11]). This construction is fundamental in the Arakelov height theory of projective varieties.

The first contribution of the current article is to introduce a numerical invariant — asymptotic minimal slope — to describe the global positivity of an adelic line bundle $\overline{L}$ such that L is ample. This invariant, which is denoted by $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$, describes the asymptotic behaviour (when $n \to +\infty$) of the minimal slopes of the sectional spaces $H^0(X, L^{\otimes n})$ equipped with sup norms (which are adelic vector bundles on S). It turns out that this invariant is super-additive with respect to $\overline{L}$. This convexity property allows to extend the construction of the asymptotic minimal slope to the cone of adelic line bundles with nef underlying invertible $\mathcal{O}_X$-module (see §3.2 for the construction of the asymptotic minimal slope and its properties). The importance of this invariant can be shown by the following height estimate (see Theorem 3.10 for the proof and Proposition 3.18 for its generalization to the relatively nef case).

Theorem 1.1. *Let X be a reduced projective k-scheme of dimension $d \geqslant 0$ over $\operatorname{Spec} K$, and $\overline{L}_0, \dots, \overline{L}_d$ be a family of relatively ample adelic line bundles on X. For any $i \in \{0, \dots, d\}$, let δ_i be the geometric intersection number*

$$(L_0 \cdots L_{i-1} L_{i+1} \cdots L_d).$$

Then the following inequality holds:

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \sum_{i=1}^{d} \delta_i \, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i),$$

where $(\overline{L}_0 \cdots \overline{L}_d)_S$ *denotes the arithmetic intersection number of* $\overline{L}_0, \ldots, \overline{L}_d$ *(cf.* [11]).

The asymptotic minimal slope always increases if one replaces the adelic line bundle by its pull-back by a projective morphism (see Theorem 3.25): if $g : X \to P$ is a projective morphism of reduced K-schemes of dimension $\geqslant 0$, then for any adelic line bundle $\overline{M}$ on P such that M is nef, one has $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{M})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{M})$. Typical situations includes a closed embedding of X into a projective space, or a finite covering over a projective space, which allows to obtain lower bounds of $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i)$ in the application of the above theorem. Note that the particular case where $\overline{L}_0, \ldots, \overline{L}_d$ are all equal to the same adelic line bundle $\overline{L}$ gives the following inequality

$$\frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)} \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}), \tag{1.1}$$

which relates the normalized height of X with respect to $\overline{L}$ and the asymptotic minimal slope of the latter. This inequality is similar to the first part of [22, Theorem 5.2]. However, the imitation of the devissage argument using the intersection of hypersurfaces defined by small sections would not work in the setting of general adelic curves. This is mainly due to the fact that the analogue of Minkowski's first theorem fails for adelic vector bundles on a general adelic curve. Although the inequality (1.1) could be obtained in an alternative way by using the arithmetic Hilbert-Samuel formula of $\overline{L}$ (cf. [12]), together with the fact that the minimal slope of an adelic vector bundle on S is always bounded from the above by its slope (see Proposition 3.26), the proof of Theorem 1.1 needs a new idea. Our approach consists in combining an analogue of the slope theory of Bost [4, 5] with the height of multi-resultant to achieve the proof.

Bigness is another type of positivity conditions which describes the growth of the total graded linear series of a line bundle. In Arakelov geometry of number fields, the arithmetic bigness describes the asymptotic behaviour of the number of small sections in the graded sectional algebra of adelic vector bundles. This notion can be generalized to the setting of Arakelov geometry of adelic curve in replace the logarithm of the number of small sections by the positive degree of an adelic vector bundle (namely the supremum of the Arakelov degrees of adelic vector subbundles). In [10, Proposition 6.4.18], the arithmetic bigness has been related to an arithmetic sectional invariant — asymptotic maximal slope, which is quite similar to asymptotic minimal slope: for any integral projective K-scheme and any adelic line bundle $\overline{L}$ on X such that L is big, we introduce a numerical invariant $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ which describes the asymptotic behaviour (when $n \to +\infty$) of the maximal slopes of $H^0(X, L^{\otimes n})$ equipped with sup norms (see §4.2 for its construction and properties). It turns out that this invariant is also super-additive with respect to $\overline{L}$, which allows to extend the function $\widehat{\mu}^{\mathrm{asy}}_{\max}(\cdot)$ to the cone of adelic line bundles $\overline{L}$ such that L is pseudo-effective. Moreover, in the case where L is nef, the inequality $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ holds.

Recall that Fujita's approximation theorem asserts that a big line bundle can be decomposed in a birational modification into the tensor product of two $\mathbb{Q}$-line bundles which are respectively ample and effective, with a good approximation of the volume function. In this article, establish the following relative version of Fujita's approximation theorem for the asymptotic maximal slope (see Theorem 4.16 and Remark 4.17).

Theorem 1.2. *Let $\overline{L}$ be an adelic line bundle on X such that L is big. For any real number $t < \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$, there exist a positive integer p, a birational projective K-morphism $g : X' \to X$, a relatively ample adelic line bundle $\overline{A}$ and an effective adelic line bundle $\overline{M}$ on X' such that A is big, $g^*(\overline{L}^{\otimes p})$ is isomorphic to $\overline{A} \otimes \overline{M}$, and $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) \geqslant pt$.*

As an application, in the case where X is an integral scheme, we can improve the height inequality in Theorem 1.1 in relaxing the positivity condition of one of the adelic line bundles and in replacing the asymptotic minimal slope of this adelic line bundle by the asymptotic maximal slope (see Theorem 4.18).

Theorem 1.3. *Let X be an integral projective scheme of degree d over $\operatorname{Spec} K$, and $\overline{L}_0, \dots, \overline{L}_d$ be adelic line bundles on X such that $\overline{L}_1, \dots, \overline{L}_d$ are relatively ample and L_0 is big. For any $i \in \{0, \dots, d\}$, let $\delta_i = (L_0 \cdots L_{i-1} L_{i+1} \cdots L_d)$. Then the following inequality holds:*

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_0) + \sum_{i=1}^{d} \delta_i\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i).$$

In the case where $\overline{L}_0, \dots, \overline{L}_d$ are all equal to the same relatively adelic line bundle $\overline{L}$, the above inequality leads to

$$\frac{(\overline{L}^{d+1})_S}{(L^d)} \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) + d\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

In the case where the adelic curve S comes from the canonical adelic structure of a number field, if $\overline{L}$ is a relatively ample adelic line bundle, then $\widehat{\mu}_{\min}(\overline{L})$ is equal to the absolue minimum of the Arakelov (absolute) height function $h_{\overline{L}}$ on the set of closed points of X. This is essentially a consequence of [22, Corollary 5.7]. Similarly, the asymptotic maximal slope $\widehat{\mu}^{\mathrm{sym}}_{\max}(\overline{L})$ is equal to the essential minimum of the height function $h_{\overline{L}}$. This is a result of Ballaÿ [1, Theorem 1.1]. In this article, we show that these results can essentially be extended to the case of general adelic curves, if we consider all integral closed subschemes of X. More precisely, we obtain the following result (see Theorem 4.23 and Proposition 4.32).

Theorem 1.4. *Let X be a non-empty reduced projective scheme over $\operatorname{Spec} K$ and Θ_X be the set of integral closed subschemes of X. For any relatively ample adelic line bundle $\overline{L}$ on X, the following equality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) = \inf_{Y \in \Theta_X} \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)(L|_Y^{\dim(Y)})} = \inf_{Y \in \Theta_X} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y).$$

Moreover, if X is an integral scheme, the following equality holds:

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) = \sup_{\substack{Y \in \Theta_X \\ Y \neq X}} \inf_{\substack{Z \in \Theta_X \\ Z \not\subseteq Y}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y).$$

We also show that a property similar to Minkowski's first theorem permits to recover the link between the asymptotic maximal/minimal slopes and the Arakelov height of closed points in the number field case. More precisely, we say that a relatively ample adelic line bundle $\overline{L}$ is *strongly Minkowskian* if for any $Y \in \Theta_X$ one has

$$\lim_{n\to+\infty} \frac{1}{n} \sup_{\substack{s\in H^0(Y, L|_Y^{\otimes n})\\ s\neq 0}} \widehat{\deg}(s) \geqslant \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)(L|_Y^{\dim(Y)})}.$$

This condition is automatically satisfied always satisfied notably when the adelic curve S comes from a number field (consequence of Minkowski's first theorem) or a function field of a projective curve (consequence of Riemann-Roch theorem). We then establish the following result (see Corollary 4.28).

Theorem 1.5. *Let X be an integral projective scheme over* $\operatorname{Spec} K$ *and $\overline{L}$ be a relatively ample adelic line bundle on X which is strongly Minkowskian. Denote by $X^{(0)}$ the set of closed points of X. Then the equality $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) = \inf_{x\in X^{(0)}} h_{\overline{L}}(x)$ holds.*

Motivated by Theorem 1.4, we propose the following analogue of successive minima for relatively ample adelic line bundles. Let X be an integral projective scheme of dimension d over $\operatorname{Spec} K$ and $\overline{L}$ be a relatively ample adelic line bundle on X. For $i \in \{1, \dots, d+1\}$, let

$$e_i(\overline{L}) = \sup_{\substack{Y\subseteq X \text{ closed}\\ \operatorname{codim}(Y)\geqslant i}} \inf_{\substack{Z\in\Theta_X\\ Z\not\subseteq Y}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Z).$$

With this notation, one can rewrite the assertion of Theorem 1.4 as

$$e_1(\overline{L}) = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}), \quad e_{d+1}(\overline{L}) = \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

We show in Remark 4.33 that, in the number field case, one has

$$\forall\, i \in \{1, \dots, d+1\}, \quad e_i(\overline{L}) = \sup_{\substack{Y\subseteq X \text{ closed}\\ \operatorname{codim}(Y)\geqslant i}} \inf_{x\in(X\setminus Y)^{(0)}} h_{\overline{L}}(x). \tag{1.2}$$

Thus we recover the definition of successive minima in the sens of [21, §5]. We propose several fundamental questions about these invariants:

(1) Do the equalities holds in the case of a general adelic curve, under the assumption that the $\overline{L}$ is strongly Minkowskian?
(2) What is the relation between the invariants $e_2(\overline{L}), \dots, e_d(\overline{L})$ and the sectional algebra $\bigoplus_{n\in\mathbb{N}} H^0(X, L^{\otimes n})$ in adelic vector bundles?
(3) Does the analogue of some classic results in Diophantine geometry concerning the successive minima, such as the inequality

$$\frac{(\overline{L}^{d+1})_S}{(L^d)} \geqslant \sum_{i=1}^{d+1} e_i(\overline{L}),$$

still holds for general adelic curve?
(4) In the case where (X, L) is a polarized toric variety and the metrics in φ are toric metrics, is it possible to describe in a combinatoric way the positivity conditions of $\overline{L}$, and express the the invariants $e_i(\overline{L})$ in terms of the combinatoric data of $(X, \overline{L})$, generalizing some results of [6, 7] for example?

The last part of the article is devoted to the study of global positivity of adelic line bundles. Motivated by Nakai-Moishezon criterion of ampleness, we say that an adelic line bundle $\overline{L}$ on X is *ample* if it is relatively ample and if the normalized height with respect to $\overline{L}$ of integral closed subschemes of X has a positive lower bound. We show that this condition is equivalent to the relative ampleness together with the positivity of the invariant $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$. Therefore, we deduced from Theorem 1.1 that, if $\overline{L}_0, \ldots, \overline{L}_d$ are ample adelic line bundles on X, where d is the dimension of X, then one has (see Proposition 5.3)

$$(\overline{L}_0 \cdots \overline{L}_d)_S > 0.$$

In the case where $\overline{L}$ is strongly Minkowskian, $\overline{L}$ is ample if and only if it is relatively ample and the height function $h_{\overline{L}}$ on the set of closed points of X has a positive lower bound (see Proposition 5.4). Once the ample cone is specified, one can naturally define the nef cone as its closure. It turns out that the nefness can also be described in a numerical way: an adelic line bundle $\overline{L}$ is nef if and only if it is relatively nef and $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \geqslant 0$ (see Proposition 5.6).

Bigness and pseudo-effectivity are also described in a numerical way by the invariant $\widehat{\mu}^{\mathrm{asy}}_{\max}(\cdot)$: an adelic line bundle $\overline{L}$ is big if and only if L is big and $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) > 0$ (which coincides with the bigness in [10]); it is pseudo-effective if and only if L is pseudo-effective and $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \geqslant 0$ (see [10, Proposition 6.4.18] and Proposition 5.13). We deduce from Theorem 1.3 that, if $\overline{L}_0, \ldots, \overline{L}_d$ are adelic line bundles on X such that $\overline{L}_0$ is pseudo-effective and that $\overline{L}_1, \ldots, \overline{L}_d$ are nef, then the inequality $(\overline{L}_0, \ldots, \overline{L}_d)_S \geqslant 0$ holds (see Proposition 5.14).

The article is organized as follows. In the second section, we recall some terminologies and facts of Arakelov geometry over adelic curves. In the third section, we study relative ampleness and nefness. In the fourth section, we focus on the asymptotic maximal slope and its relation with positivity of adelic line bundles. In the fifth and last section, we discuss global positivity conditions.

2. Notation and preliminaries

2.1. By *proper adelic curve*, we mean the data $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ consisting of a field K, a measure space $(\Omega, \mathcal{A}, \nu)$ and a family $\phi = (|\cdot|_\omega)_{\omega\in\Omega}$ of absolute values on K, such that, for any $a \in K \setminus \{0\}$, the function $(\omega \in \Omega) \mapsto \ln |a|_\omega$ is integrable on $(\Omega, \mathcal{A}, \nu)$ and of integral 0. In this article, we consider an adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ which satisfies the following conditions:

(a) K is a perfect field, that is, either K is of characteristic 0, or K is of characteristic $p > 0$ and the Frobenius map $(a \in K) \mapsto a^p$ is surjective;
(b) either the field K is countable, or the σ-algebra $\mathcal{A}$ is discrete.

For any $\omega \in \Omega$, we denote by K_ω the completion of K with respect to the absolute value $|\cdot|_\omega$.

2.2. We call *adelic vector bundle* on S the data $\overline{E} = (E, (\|\cdot\|_\omega)_{\omega\in\Omega})$ consisting of a finite-dimensional vector space E over K and family of norms, where each $\|\cdot\|_\omega$ is a norm on $E_\omega = E \otimes_K K_\omega$, which satisfy the following conditions:

(1) (measurability) for any $s \in E$, the function
$$(\omega \in \Omega) \longmapsto \|s\|_\omega$$
is $\mathcal{A}$-measurable,
(2) (dominancy) there exists a basis $\boldsymbol{e} = (e_i)_{i=1}^r$ of E over K and an integrable function $A(\cdot)$ on $(\Omega, \mathcal{A}, \nu)$ such that, for any $\omega \in \Omega$ and any $(\lambda_1, \dots, \lambda_r) \in K_\omega^r \setminus \{(0, \dots, 0)\}$,
$$\Big|\ln \|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_\omega - \ln \max\{|\lambda_1|_\omega, \dots, |\lambda_r|_\omega\}\Big| \leqslant A(\omega).$$

In this article, all adelic vector bundles are assumed to be ultrametric on non-Archimedean places, namely we assume that $\|\cdot\|_\omega$ is ultrametric when $|\cdot|_\omega$ is non-Archimedean.

2.3. Let $\overline{E} = (E, (\|\cdot\|_\omega)_{\omega\in\Omega})$ be an adelic vector bundle on S and r be the dimension of E over $\operatorname{Spec} K$. The *Arakelov degree* of $\overline{E}$ is defined as
$$\widehat{\deg}(\overline{E}) := -\int_\Omega \ln \|e_1 \wedge \cdots \wedge e_r\|_{\omega,\det} \, \nu(\mathrm{d}\omega),$$
where the determinant norm $\|\cdot\|_{\omega,\det}$ on $\det(E_\omega)$ is define as
$$\forall\, \eta \in \det(E_\omega), \quad \|\eta\|_{\omega,\det} = \inf_{\substack{(x_i)_{i=1}^r \in E_\omega^r \\ \eta = x_1 \wedge \cdots \wedge x_r}} \|x_1\|_\omega \cdots \|x_r\|_\omega.$$

If E is non-zero, we define the *slope* of $\overline{E}$ as
$$\widehat{\mu}(\overline{E}) := \frac{\widehat{\deg}(\overline{E})}{\dim_K(E)},$$
the *maximal slope* of $\overline{E}$ as
$$\widehat{\mu}_{\max}(\overline{E}) := \sup_{0\neq F\subseteq E} \widehat{\mu}(\overline{F}),$$
where $\overline{F}$ runs over the set of non-zero vector subspaces of E equipped with restricted norms, and the *minimal slope* of $\overline{E}$ as
$$\widehat{\mu}_{\min}(\overline{E}) := \inf_{E\twoheadrightarrow G\neq 0} \widehat{\mu}(\overline{G}),$$

where $\overline{G}$ runs over the set of non-zero quotient vector spaces of E equipped with quotient norms.

2.4. Let $\overline{E}$ and $\overline{F}$ be two adelic vector bundles on S and $\varphi : E \to F$ be a K-linear map. We define the height of φ as

$$h(\varphi) := \int_\Omega \ln \|\varphi\|_\omega \, \nu(\mathrm{d}\omega),$$

where $\|\varphi\|_\omega$ denotes the operator norm of the K_ω-linear map $E_\omega \to F_\omega$ induced by φ. Moreover, if E is non-zero and if φ is injective, then the following slope inequality holds (see [10, Proposition 4.3.31]):

$$\widehat{\mu}_{\max}(\overline{E}) \leqslant \widehat{\mu}_{\max}(\overline{F}) + h(\varphi).$$

2.5. Let $\overline{E} = (E, (\|\cdot\|_{E,\omega})_{\omega\in\Omega})$ and $\overline{F} = (F, (\|\cdot\|_{F,\omega})_{\omega\in\Omega})$ be two adelic vector bundles on S. We denote by $\overline{E} \otimes_{\varepsilon,\pi} \overline{F}$ the adelic vector bundle $(E \otimes F, (\|\cdot\|_\omega)_{\omega\in\Omega})$ defined as follows: when $|\cdot|_\omega$ is non-Archimedean (resp. Archimedean), the norm $\|\cdot\|_\omega$ is the ε-tensor product (resp. π-tensor product) of $\|\cdot\|_{E,\omega}$ and $\|\cdot\|_{F,\omega}$, which is the greatest ultrametric norm (resp. greatest norm) on $E_\omega \otimes F_\omega$ such that

$$\forall\, (s,t) \in E_\omega \times F_\omega, \quad \|s \otimes t\|_\omega = \|s\|_{E,\omega} \|t\|_{F,\omega}.$$

2.6. Let $f : X \to \operatorname{Spec} K$ be a reduced projective scheme over $\operatorname{Spec} K$ (which is geometrically reduced since K is assumed to be perfect). For any $\omega \in \Omega$, let X_ω be $X \times_{\operatorname{Spec} K} \operatorname{Spec} K_\omega$ and let X_ω^{an} be the analytic space associated with X_ω (in the sense of Berkovich if $|\cdot|_\omega$ is non-Archimedean). If L is an invertible $\mathcal{O}_X$-module, we call *metric family* on L any family $\varphi = (\varphi_\omega)_{\omega\in\Omega}$, where φ_ω is a continuous metric on $L_\omega = L|_{X_\omega}$. We say that $\overline{L} = (L, \varphi)$ is an *adelic line bundle* on X if the metric family φ is dominated and measurable (see [10, §6.1]). For example, if $\overline{E}$ is an adelic vector bundle on S, $g : X \to \mathbb{P}(E)$ is a projective K-morphism and $L = g^*(\mathcal{O}_E(1))$, then the invertible $\mathcal{O}_X$-module L, equipped with the pullbacks of Fubini-Study metrics (called a *quotient metric family*), forms and adelic line bundle on X.

Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X. We denote by $f_*(\overline{L})$ the couple $(H^0(X, L), (\|\cdot\|_{\varphi_\omega})_{\omega\in\Omega})$, where for $s \in H^0(X_\omega, L_\omega)$,

$$\|s\|_{\varphi_\omega} = \sup_{x \in X_\omega^{\mathrm{an}}} |s|_{\varphi_\omega}(x).$$

It turns out that $f_*(\overline{L})$ is an adelic line bundle on S (see [10, Theorems 6.1.13 and 6.1.32]).

2.7. Let X be a reduced projective scheme over $\operatorname{Spec} K$. Let L be an invertible $\mathcal{O}_X$-module, $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ and $\psi = (\psi_\omega)_{\omega\in\Omega}$ be metric families on L such that (L, φ) and (L, ψ) are both adelic line bundles. Then we define the *distance* between φ and ψ as

$$d(\varphi, \psi) := \int_\Omega \sup_{x \in X_\omega} \left| \ln \frac{|\cdot|_{\varphi_\omega}(x)}{|\cdot|_{\psi_\omega}(x)} \right| \nu(\mathrm{d}\omega).$$

If L is semiample and if there exists a positive integer m and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of quotient metric families (where φ_n is a metric family of $L^{\otimes nm}$), such that

$$\lim_{n\to+\infty} d(nm\varphi, \varphi_n) = 0,$$

we say that the metric family φ is *semi-positive*.

2.8. Let X be a non-empty reduced projective scheme over $\operatorname{Spec} K$ and d be the dimension of X. We denote by $\widehat{\operatorname{Int}}(X)$ the set of adelic line bundles $\overline{L}$ on X which can be written in the form $\overline{A}_1 \otimes \overline{A}_2^{\vee}$, where each $\overline{A}_i$ is an ample invertible $\mathcal{O}_X$-module equipped with a semi-positive metric family. In [11], we have constructed an *arithmetic intersection product*

$$\big((\overline{L}_0, \ldots, \overline{L}_d) \in \widehat{\operatorname{Int}}(X)^{d+1}\big) \longmapsto (\overline{L}_0 \cdots \overline{L}_d)_S \in \mathbb{R},$$

which is multi-linear with respect to tensor product. We have also related the arithmetic intersection number $(\overline{L}_0 \cdots \overline{L}_d)_S$ to the height of the multi-resultant of $L_0, \ldots, L_d$.

2.9. Let $f : X \to \operatorname{Spec} K$ be an integral projective scheme over $\operatorname{Spec} K$ and $\overline{L}$ be an adelic line bundle on X such that L is ample. Then the sequence

$$\frac{\widehat{\deg}(f_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N}_{\geqslant 1}$$

converges to a real number which is denoted by $\widehat{\operatorname{vol}}_\chi(\overline{L})$. This result has been proved in [12, Theorem-Definition 5.5] under the assumption that X is geometrically integral. This hypothesis has been used there to ensure that the K-scheme X remains integral by a base change to the perfect closure of K. Since here we assume the field K to be perfect, this result also applies to integral K-schemes. Moreover, in the case where the metrics in the adelic line bundle structure of $\overline{L}$ are all semi-positive, the following Hilbert-Samuel formula holds:

$$\widehat{\operatorname{vol}}_\chi(\overline{L}) = (\overline{L}^{d+1})_S,$$

where d is the dimension of X.

3. Relative ampleness and nefness

Throughout the section, we fix a proper adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ such that, either K is countable, or $(\Omega, \mathcal{A})$ is discrete. We assume in addition that K is perfect.

3.1. Convergence of minimal slopes.

Lemma 3.1. *Let k be a field, X and Y be projective k-schemes and $g : Y \to X$ be a projective k-morphism such that $g_*(\mathcal{O}_Y) = \mathcal{O}_X$. Let L be an ample line bundle on Y and M be an ample line bundle on X. Then there exists $N \in \mathbb{N}_{\geqslant 1}$ such that, for any $(n, m) \in \mathbb{N}^2$ satisfying $\min\{n, m\} \geqslant N$, the k-linear map*

$$H^0(Y, L^{\otimes n}) \otimes_K H^0(X, M^{\otimes m}) = H^0(Y, L^{\otimes n}) \otimes_K H^0(Y, g^*(M^{\otimes m})) \longrightarrow H^0(Y, L^{\otimes n} \otimes g^*(M)^{\otimes m})$$

defined by multiplication of sections is surjective.

Proof. Consider the graphe

$$\Gamma_g : Y \longrightarrow Y \times_k X$$

of the morphism $g : Y \to X$. It is a closed immersion since g is separated. Denote by I the ideal sheaf of the image of Γ_g. Let $p : Y \times_k X \to Y$ and $q : Y \times_k X \to X$ be the two projections, and $A = p^*(L) \otimes q^*(M)$. Since M and L are both ample, the line bundle A on $Y \times_k X$ is ample. Moreover, one has $\Gamma_g^*(A) = L \otimes g^*(M)$. The short exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_{Y \times_K X} \longrightarrow \mathcal{O}_{Y \times_k X}/I \longrightarrow 0$$

induces, by tensor product with the invertible sheaf $p^*(L^{\otimes n}) \otimes q^*(M^{\otimes m})$ and then by taking cohomology groups on $Y \times_k X$, an exact sequence of K-vector spaces

$$H^0(Y, L^{\otimes n}) \otimes_k H^0(Y, g^*(M)^{\otimes m}) \longrightarrow H^0(Y, L^{\otimes n} \otimes g^*(M)^{\otimes m}) \longrightarrow H^1(Y \times_k X, I \otimes p^*(L^{\otimes n}) \otimes q^*(M^{\otimes m})).$$

By [15, Example 1.4.4], the line bundles $p^*(L)$ and $q^*(M)$ are nef. By Fujita's vanishing theorem (cf. [13, Theorem 5.1]), there exists $N \in \mathbb{N}_{\geqslant 1}$ such that, for any $(n, m) \in \mathbb{N}^2$ such that $\min\{n, m\} > N$, one has

$$\begin{aligned} &H^1(Y \times_k X, I \otimes p^*(L^{\otimes n}) \otimes q^*(M^{\otimes m})) \\ &= H^1(Y \times_k X, I \otimes A^{\otimes N} \otimes p^*(L^{\otimes (n-N)}) \otimes q^*(M^{\otimes (m-N)})) = \mathbf{0}. \end{aligned}$$

Therefore the assertion follows. □

Lemma 3.2. *Let $(k, |\cdot|)$ be a filed equipped with a complete absolute value. Let X be a projective scheme over k, L be a semi-ample line bundle on X and φ be a semi-positive metric of L. Then, for any projective K-morphism $g : Y \to X$, $g^*(\varphi)$ is also semi-positive.*

Proof. Replacing L by a tensor power, we may assume that L is generated by global sections, and that there exists a sequence of quotient metric families $(\varphi_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to +\infty} \frac{1}{n} d(n\varphi, \varphi_n) = 0.$$

Note that for each $n \in \mathbb{N}$, the pull-back $g^*(\varphi_n)$ is still a quotient metric, and one has

$$d(ng^*(\varphi), g^*(\varphi_n)) \leqslant d(n\varphi, \varphi_n).$$

Therefore we obtain that $g^*(\varphi)$ is semi-positive. □

In the remaining of the section, we let $f : X \to \operatorname{Spec} K$ be a *non-empty and reduced* projective scheme over $\operatorname{Spec} K$. Since the base field K is supposed to be perfect, the K-scheme X is geometrically reduced.

Proposition 3.3. *Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X such that L is ample. Then the sequence*

$$\frac{\widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n}))}{n}, \quad n \in \mathbb{N},\ n \geqslant 1 \tag{3.1}$$

converges in $\mathbb{R}$.

Proof. For any $n \in \mathbb{N}_{\geqslant 1}$ let $\overline{E}_n = (E_n, \xi_n)$ be the adelic vector bundle $f_*(\overline{L}^{\otimes n})$. Since L is ample, by Lemma 3.1 there exists $N \in \mathbb{N}_{\geqslant 1}$ such that, for any $(n, m) \in \mathbb{N}^2_{\geqslant N}$, the map

$$E_n \otimes_K E_m \longrightarrow E_{n+m}, \quad s \otimes t \longmapsto st$$

is surjective. Moreover, if we equip $E_n \otimes E_m$ with the ε, π-tensor product of the norm families ξ_n and ξ_m, the above map has height $\leqslant 0$. By [10, Proposition 4.3.31], one has

$$\widehat{\mu}_{\min}(\overline{E}_{n+m}) \geqslant \widehat{\mu}_{\min}(\overline{E}_n \otimes_{\varepsilon,\pi} \overline{E}_m).$$

Moreover, since the field K is assumed to be perfect, by [10, Corollary 5.6.2] (see also [12, Remark C.3]), one has

$$\begin{aligned}\widehat{\mu}_{\min}(\overline{E}_n \otimes_{\varepsilon,\pi} \overline{E}_m) \geqslant \widehat{\mu}_{\min}(\overline{E}_n) + \widehat{\mu}_{\min}(\overline{E}_m) \\ - \frac{3}{2}\nu(\Omega_\infty)(\ln(\dim_K(E_n)) + \ln(\dim_K(E_m))).\end{aligned}$$

Note that

$$\ln(\dim_K(E_n)) = O(\ln(n)),$$

and, by [10, Propositions 6.4.4 and 6.2.7], there exists a constant $C > 0$ such that

$$\widehat{\mu}_{\min}(\overline{E}_n) \leqslant \widehat{\mu}_{\max}(\overline{E}_n) \leqslant Cn.$$

Therefore, by [9, Corollary 3.6], we obtain the convergence of the sequence (3.1). □

3.2. **Relative ampleness and asymptotic minimal slope.**

Definition 3.4. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X. If L is ample, we define the *asymptotic minimal slope of* $\overline{L}$ as

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) := \lim_{n \to +\infty} \frac{\widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n}))}{n}.$$

By definition, for any $m \in \mathbb{N}$ such that $m \geqslant 1$, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes m}) = m\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}). \tag{3.2}$$

Proposition 3.5. *Let $\overline{L} = (L, \varphi)$ and $\overline{M} = (M, \psi)$ be adelic line bundles on X such that L and M are ample. Then one has*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L} \otimes \overline{M}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) + \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{M}). \tag{3.3}$$

Proof. By Lemma 3.1, for sufficiently large natural number n, the K-linear map

$$H^0(X, L^{\otimes n}) \otimes_K H^0(X, M^{\otimes n}) \longrightarrow H^0(X, (L \otimes M)^{\otimes n}), \quad s \otimes t \longmapsto st$$

is surjective. Moreover, for any $\omega \in \Omega$, the following inequality holds:

$$\forall\, (s,t) \in H^0(X_\omega, L_\omega^{\otimes n}) \times H^0(X_\omega, M_\omega^{\otimes n}), \quad \|st\|_{n(\varphi_\omega+\psi_\omega)} \leqslant \|s\|_{n\varphi_\omega} \cdot \|t\|_{n\psi_\omega}.$$

Therefore, if we equip $H^0(X, L^{\otimes n}) \otimes_K H^0(X, M^{\otimes n})$ with the ε, π-tensor product norm family, then the above K-linear map has height $\leqslant 0$. Hence, by [10, Proposition 4.3.31 and Corollary 5.6.2] (see also [12, Remark C.3]), we obtain

$$\begin{aligned}\widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n} \otimes \overline{M}^{\otimes n})) &\geqslant \widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n})) + \widehat{\mu}_{\min}(f_*(\overline{M}^{\otimes n})) \\ &\quad - \frac{3}{2}\nu(\Omega_\infty)\Big(\ln(\dim_K(H^0(X, L^{\otimes n}))) + \ln(\dim_K(H^0(X, M^{\otimes n})))\Big).\end{aligned}$$

We divide the two sides of the inequality by n and then take the limit when $n \to +\infty$, using

$$\lim_{n\to+\infty} \frac{1}{n} \ln(\dim_K(H^0(X, L^{\otimes n}))) = \lim_{n\to+\infty} \frac{1}{n} \ln(\dim_K(H^0(X, M^{\otimes n}))) = 0.$$

we obtain the inequality (3.3). □

Proposition 3.6. *Let L be an ample line bundle on X and φ_1 and φ_2 be metric families on L such that (L, φ_1) and (L, φ_2) are both adelic line bundles. Then the following inequality holds:*

$$\Big|\, \widehat{\mu}^{\mathrm{asy}}_{\min}(L, \varphi_1) - \widehat{\mu}^{\mathrm{asy}}_{\min}(L, \varphi_2) \Big| \leqslant d(\varphi_1, \varphi_2). \tag{3.4}$$

Proof. For any $n \in \mathbb{N}$, the identity maps

$$f_*(L^{\otimes n}, n\varphi_1) \longrightarrow f_*(L^{\otimes n}, n\varphi_2)$$

and

$$f_*(L^{\otimes n}, n\varphi_2) \longrightarrow f_*(L^{\otimes n}, n\varphi_1)$$

have heights $\leqslant d(n\varphi_1, n\varphi_2) = n\, d(\varphi_1, \varphi_2)$. By [10, Proposition 4.3.31], we obtain that

$$\Big|\, \widehat{\mu}_{\min}(f_*(L^{\otimes n}, n\varphi_1)) - \widehat{\mu}_{\min}(f_*(L^{\otimes n}, n\varphi_2)) \Big| \leqslant n\, d(\varphi_1, \varphi_2).$$

Dividing the two sides of the inequality by n and then taking the limit when $n \to +\infty$, we obtain (3.4). □

3.3. **Asymptotic slope and intersection number.** In this subsection, we assume that X is integral. Let $\overline{L}$ be an adelic line bundle on X such that L is ample. Note that

$$\dim_K(H^0(X, L^{\otimes n})) = \frac{(L^d)}{d!} n^d + o(n^d), \quad n \to +\infty.$$

Therefore, one has

$$\lim_{n\to+\infty} \frac{\widehat{\mu}(f_*(\overline{L}^{\otimes n}))}{n} = \frac{\widehat{\mathrm{vol}}_\chi(\overline{L})}{(d+1)(L^d)}. \tag{3.5}$$

We denote by $\widehat{\mu}^{\mathrm{asy}}(\overline{L})$ the value $\dfrac{\widehat{\mathrm{vol}}_\chi(\overline{L})}{(d+1)(L^d)}$ and call it the *asymptotic slope* of $\overline{L}$.

We recall below the Hilbert-Samuel formula for adelic line bundles in the framework of Arakelov geometry over an adelic curve.

Theorem 3.7 (Hilbert-Samuel formula)**.** *If $\overline{L}=(L,\varphi)$ is an adelic line bundle on X such that L is ample and φ is semi-positive, that is, $\overline{L}$ is relatively ample in the sense of Definition 3.9 below, then $\widehat{\mathrm{vol}}_\chi(\overline{L})=(\overline{L}^{d+1})_S$ and hence*

$$\widehat{\mu}^{\mathrm{asy}}(\overline{L})=\frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)}. \tag{3.6}$$

Proof. See [12, Theorem 1.2] or §2.9. □

Remark 3.8. Let $\overline{L}$ be an adelic line bundle on X such that L is ample. By definition the following inequality holds:

$$\widehat{\mu}^{\mathrm{asy}}(\overline{L})\geqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}). \tag{3.7}$$

3.4. **Lower bound of intersection number for relatively ample adelic line bundles.**

Definition 3.9. Let (L,φ) be an adelic line bundle on X. We say (L,φ) is *relatively ample* if L is ample and φ is semi-positive. By [10, Proposition 2.3.5], if $\overline{L}$ and $\overline{M}$ are relatively ample adelic line bundle, then the tensor product $\overline{L}\otimes\overline{M}$ is relatively ample.

Theorem 3.10. *Let $\overline{L}_i=(L_i,\varphi_i)$ be a family of relatively ample adelic line bundles on X, where $i\in\{0,\dots,d\}$. For any $i\in\{0,\dots,d\}$, let*

$$\delta_i=(L_0\cdots L_{i-1}L_{i+1}\cdots L_d).$$

Then the following inequality holds:

$$(\overline{L}_0\cdots\overline{L}_d)_S\geqslant\sum_{i=0}^{d}\delta_i\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i). \tag{3.8}$$

Proof. Without loss of generality, we may assume that $L_0,\dots,L_d$ are very ample. For any $n\in\mathbb{N}_{\geqslant1}$ and any $i\in\{0,\dots,d\}$, we denote by $E_{i,n}$ the K-vector space $H^0(X,L_i^{\otimes n})$, and set $r_{i,n}=\dim_K(E_{i,n})-1$. We denote by $\xi_{n\varphi_i}$ the norm family $(\|\cdot\|_{n\varphi_{i,\omega}})_{\omega\in\Omega}$ on $E_{i,n}$, and let $\xi_{i,n}$ be a Hermitian norm family on $E_{i,n}$ such that $(E_{i,n},\xi_{i,n})$ forms an adelic vector bundle and that

$$d_\omega(\xi_{i,n},\xi_{n\varphi_i})\leqslant\frac{1}{2}\mathbb{1}_{\Omega_\infty}(\omega)\ln(r_{i,n}+2).$$

The existence of such a Hermitian norm family is ensured by [10, Theorem 4.1.26]. Let $\varphi_i^{(n)}$ be the metric family on L_i such that $n\varphi_i^{(n)}$ identifies with the quotient metric family induced by the closed embedding $X\to\mathbb{P}(E_{i,n})$ and the norm family $\xi_{i,n}$. Since

$$\lim_{n\to+\infty}\frac{1}{n}\ln(r_{i,n}+2)=0$$

and the metric families φ_i are semi-positive, by [11, Proposition 3.3.12], we obtain that

$$\lim_{n\to+\infty}d(\varphi_i^{(n)},\varphi_i)=\lim_{n\to+\infty}\int_\Omega d_\omega(\varphi_i^{(n)},\varphi_i)\,\nu(\mathrm{d}\omega)=0.$$

For any $n\in\mathbb{N}_{\geqslant1}$, let R_n be the one-dimensional vector space of

$$S^{n^d\delta_0}(E_{0,n}^\vee)\otimes_k\cdots\otimes_k S^{n^d\delta_d}(E_{d,n}^\vee) \tag{3.9}$$

spanned by any resultant of the closed embeddings $X \to \mathbb{P}(E_{i,n})$. We equip each $S^{n^d\delta_0}(E_{i,n}^\vee)$ with the orthogonal symmetric power norm family of $\xi_{i,n}^\vee$, and the tensor product space (3.9) with the orthogonal tensor product norm family. By [11, Remark 4.2.14] and [3, Corollary 1.4.3 and Lemma 4.3.8], we obtain that

$$\begin{aligned}((L_0,\varphi_0^{(n)})\cdots(L_d,\varphi_d^{(n)}))_S &\geqslant -\frac{1}{n^{d+1}}\biggl(\widehat{\deg}(\overline{R}_n)+\nu(\Omega_\infty)\sum_{i=0}^d \ln\biggl(\begin{pmatrix} r_{i,n}+n^d\delta_i \\ n^d\delta_i\end{pmatrix}\biggr)\biggr)\\ &\geqslant -\frac{1}{n^{d+1}}\bigl(\widehat{\deg}(\overline{R}_n)+\nu(\Omega_\infty)\sum_{i=0}^d n^d\delta_i\ln(r_{i,n}+1)\bigr),\end{aligned} \tag{3.10}$$

where the second inequality comes from

$$\forall\,(a,b)\in\mathbb{N}^2_{\geqslant 1},\quad \begin{pmatrix} a+b\\ b\end{pmatrix}\leqslant (a+1)^b.$$

Note that

$$\widehat{\deg}(\overline{R}_n)\leqslant \widehat{\mu}_{\max}(S^{n^d\delta_0}(E_{0,n}^\vee,\xi_{0,n}^\vee)\otimes\cdots\otimes S^{n^d\delta_d}(E_{d,n}^\vee,\xi_{d,n}^\vee)). \tag{3.11}$$

In the case where K is of characteristic 0, by [12, Remark B.6] and [10, Proposition 4.3.31], we obtain

$$\begin{aligned}&\widehat{\mu}_{\max}\bigl(S^{n^d\delta_0}(E_{0,n}^\vee,\xi_{0,n}^\vee)\otimes\cdots\otimes S^{n^d\delta_d}(E_{d,n}^\vee,\xi_{d,n}^\vee)\bigr)\\ &\leqslant \widehat{\mu}_{\max}\bigl((E_{0,n}^\vee,\xi_{0,n}^\vee)^{\otimes n^d\delta_0}\otimes\cdots\otimes(E_{d,n}^\vee,\xi_{d,n}^\vee)^{\otimes n^d\delta_d}\bigr)+\nu(\Omega_\infty)\sum_{i=0}^d n^d\delta_i\ln(n^d\delta_i).\end{aligned} \tag{3.12}$$

By [10, Corollaries 4.3.27 and 5.6.2], we have

$$\begin{aligned}&\widehat{\mu}_{\max}\bigl((E_{0,n}^\vee,\xi_{0,n}^\vee)^{\otimes n^d\delta_0}\otimes\cdots\otimes(E_{d,n}^\vee,\xi_{d,n}^\vee)^{\otimes n^d\delta_d}\bigr)\\ &\leqslant \sum_{i=0}^d n^d\delta_i\Bigl(\widehat{\mu}_{\max}(E_{i,n}^\vee,\xi_{i,n}^\vee)+\frac{1}{2}\nu(\Omega_\infty)\ln(r_{i,n}+1)\Bigr)\\ &= \sum_{i=0}^d n^d\delta_i\Bigl(-\widehat{\mu}_{\min}(E_{i,n},\xi_{i,n})+\frac{1}{2}\nu(\Omega_\infty)\ln(r_{i,n}+1)\Bigr)\end{aligned} \tag{3.13}$$

Combining (3.10), (3.11), (3.12) and (3.13), we obtain

$$\begin{aligned}((L_0,\varphi_0^{(n)})\cdots(L_d,\varphi_d^{(n)}))_S &\geqslant \sum_{i=0}^d \delta_i\frac{\widehat{\mu}_{\min}(E_{i,n},\xi_{i,n})}{n}\\ &\quad -\frac{3}{2}\nu(\Omega_\infty)\sum_{i=0}^d\frac{\delta_i}{n}\ln(r_{i,n}+1)-\nu(\Omega_\infty)\sum_{i=0}^d\frac{\delta_i}{n}\ln(n^d\delta_i).\end{aligned} \tag{3.14}$$

In the case where K is of positive characteristic, by [12, Corollary C.2 and Theorem C.5], we obtain

$$\widehat{\mu}_{\max}\bigl(S^{n^d\delta_0}(E_{0,n}^\vee,\xi_{0,n}^\vee)\otimes\cdots\otimes S^{n^d\delta_d}(E_{d,n}^\vee,\xi_{d,n}^\vee)\bigr)\leqslant\sum_{i=0}^d n^d\delta_i\,\widehat{\mu}_{\max}(E_{i,n}^\vee,\xi_{i,n}^\vee).$$

Hence the inequality (3.14) still holds in this case. Since $r_{i,n} = O(n^d)$, taking the limit when n goes to the infinity, we obtain the inequality (3.8). □

3.5. Relative nefness and continuous extension of $\widehat{\mu}^{\mathrm{asy}}_{\min}$.

Proposition 3.11. *Let $\overline{L}$ and $\overline{A}$ be adelic line bundle on X. Assume that L is nef and A is ample. Then the sequence*

$$\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{A}),\quad n\in\mathbb{N}_{\geqslant 1} \tag{3.15}$$

converges in $\mathbb{R}\cup\{-\infty\}$, and the limit does not depend on the choice of $\overline{A}$. In particular, in the case where L is ample, the following equality holds:

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}(\overline{L}^{\otimes n}\otimes\overline{A}) = \widehat{\mu}_{\min}(\overline{L}). \tag{3.16}$$

Proof. Let p be a positive integer. By Proposition 3.5, for any $\ell\in\mathbb{N}_{\geqslant 1}$ and any $r\in\{1,\dots,p\}$, one has

$$\begin{aligned}\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p}\otimes\overline{A}) &= \frac{1}{\ell+1}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes(\ell+1)p}\otimes\overline{A}^{\otimes(\ell+1)})\\ &\geqslant \frac{1}{\ell+1}\Big(\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes(\ell p+r)}\otimes\overline{A}) + \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes(p-r)}\otimes\overline{A}) + (\ell-1)\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A})\Big).\end{aligned}$$

Taking the limit superior when $\ell p + r\to+\infty$, we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p}\otimes\overline{A}) \geqslant p\limsup_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{A}) + \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}),$$

which leads to

$$\liminf_{p\to+\infty}\frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p}\otimes\overline{A}) \geqslant \limsup_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{A}).$$

Therefore the sequence (3.15) converges in $[-\infty,+\infty]$. Moreover, still by Proposition 3.5, for any $p\in\mathbb{N}_{\geqslant 1}$, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}\otimes\overline{A}) = \frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p}\otimes\overline{A}^{\otimes p}) \geqslant \frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p}\otimes\overline{A}) + \frac{p-1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}),$$

which shows that

$$\lim_{p\to+\infty}\frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p}\otimes\overline{A}) \leqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}\otimes\overline{A}) - \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) < +\infty.$$

To prove the second assertion, we first show that the limit of the sequence does not depend on the choice of the metric family on $\overline{A}$. For this purpose, we consider two metric families φ_1 et φ_2 on A such that both (A,φ_1) and (A,φ_2) are adelic line bundles on X. By Proposition 3.6, for any $n\in\mathbb{N}$ one has

$$\Big|\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes(A,\varphi_1)) - \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes(A,\varphi_1))\Big| \leqslant d(\varphi_1,\varphi_2),$$

so that

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes(A,\varphi_1)) = \lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes(A,\varphi_2)). \tag{3.17}$$

We then show that, for any $p\in\mathbb{N}_{\geqslant 2}$, the following inequality holds:

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{A}) = \lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{A}^{\otimes p}). \tag{3.18}$$

In fact, by (3.2), for any $n \in \mathbb{N}_{\geqslant 1}$ one has

$$\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}) = \frac{1}{np}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes np} \otimes \overline{A}^{\otimes p}).$$

Taking the limit when $n \to +\infty$, we obtain the equality (3.18).

Note that if $\overline{B}$ is another adelic line bundle such that B is ample, then the following inequality holds:

$$\lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}) \leqslant \lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A} \otimes \overline{B}). \tag{3.19}$$

In fact, by Proposition 3.5, for any $n \in \mathbb{N}_{\geqslant 1}$, one has

$$\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A} \otimes \overline{B}) \geqslant \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}) + \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{B}).$$

Taking the limit when $n \to +\infty$, we obtain (3.19).

Finally, we show that, if $\overline{B}$ is an arbitrary adelic line bundle such that B is ample, then the equality

$$\lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}) = \lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{B}) \tag{3.20}$$

holds. In fact, there exists $p \in \mathbb{N}_{\geqslant 1}$ such that $N = B^{\otimes p} \otimes A^{\vee}$ is ample. We equip it with an arbitrary metric family such that $\overline{N}$ forms an adelic line bundle. By (3.19) we obtain

$$\lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}) \leqslant \lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A} \otimes \overline{N}).$$

Since $A \otimes N$ is isomorphic to $B^{\otimes p}$, by (3.17) and (3.18) we obtain

$$\lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A} \otimes \overline{N}) = \lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{B}).$$

Therefore, we deduce

$$\lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}) \leqslant \lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{B}).$$

Interchanging the roles of $\overline{A}$ and $\overline{B}$ we obtain the converse inequality.

To obtain the equality (3.16), it suffices to apply the equality (3.20) in the particular case where $\overline{A} = \overline{L}$. The proposition is thus proved. □

Definition 3.12. Let $\overline{L}$ be an adelic line bundle on X such that L is nef, we define

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) := \lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}),$$

where $\overline{A}$ is an arbitrary adelic line bundle such that A is ample. The element $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$ of $\mathbb{R} \cup \{-\infty\}$ is called *asymptotic minimal slope* of $\overline{L}$.

Remark 3.13. It is an interesting question to ask when the asymptotic minimal slope is a real number. As we will show in Theorem 3.25, the asymptotic minimal slope does not decrease if we replace the adelic line bundle by its pullback by a projective morphism. In particular, if L is the pullback of an ample line bundle by a projective morphism, then $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \in \mathbb{R}$.

Proposition 3.14. *Let $\overline{L}$ and $\overline{M}$ be adelic line bundles on X such that L and M are nef. One has*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}\otimes\overline{M})\geqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})+\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{M}). \tag{3.21}$$

Moreover, one has

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{M})=\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \tag{3.22}$$

provided that $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{M})>-\infty$.

Proof. Let $\overline{A}$ be an adelic line bundle on X such that A is ample. For any $n\in\mathbb{N}_{\geqslant 1}$, by Proposition 3.5 one has

$$\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{M}^{\otimes n}\otimes\overline{A}^{\otimes 2})\geqslant\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{A})+\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{M}^{\otimes n}\otimes\overline{A}).$$

Taking the limit when $n\to+\infty$, we obtain the inequality (3.21).

By (3.21), we obtain that, for any positive integer n, the inequality

$$\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{M})\geqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})+\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{M}).$$

Since $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{M})\in\mathbb{R}$, taking the limit inferior when $n\to+\infty$, we obtain

$$\liminf_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{M})\geqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

Pick an adelic line bundle $\overline{A}$ on X such that A is ample. Since $A\otimes M$ is ample, one has

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{M}\otimes\overline{A})=\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}). \tag{3.23}$$

Moreover, by (3.21) one has

$$\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{M}\otimes A)\geqslant\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{M})+\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}).$$

Taking the limit superior, by (3.23) we obtain

$$\limsup_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{M})\leqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

Hence the equality (3.22) holds. □

Definition 3.15. Let $\overline{L}=(L,\varphi)$ be an adelic line bundle on X. We say that $\overline{L}$ is *relatively nef* if there exists a relatively ample adelic line bundle $\overline{A}$ on X and a positive integer N such that, for any $n\in\mathbb{N}_{\geqslant N}$, the tensor product $\overline{L}^{\otimes n}\otimes\overline{A}$ is relatively ample.

Proposition 3.16. *Let $\overline{L}=(L,\varphi)$ be an adelic line bundle on X such that L is semi-ample and φ is semi-positive. Then, for any adelic line bundle $\overline{A}=(A,\psi)$ on X which is relatively ample and any $n\in\mathbb{N}$, the tensor product $\overline{L}^{\otimes n}\otimes\overline{A}$ is relatively ample. In particular, $\overline{L}$ is relatively nef.*

Proof. Since L is semi-ample, we obtain that, for any $n\in\mathbb{N}$, $L^{\otimes n}\otimes A$ is ample. Moreover, by [10, Proposition 2.3.5], $n\varphi+\psi$ is semi-positive. Hence $\overline{L}^{\otimes n}\otimes\overline{A}$ is relatively ample. □

Proposition 3.17. *Let $\overline{L}$ and $\overline{M}$ be adelic line bundles on X which are relatively nef. Then the tensor product $\overline{L}\otimes\overline{M}$ is also relatively nef.*

Proof. Let $\overline{A}$ and $\overline{B}$ be relatively ample adelic line bundles on X, and N be a positive integer such that $\overline{L}^{\otimes n} \otimes \overline{A}$ and $\overline{M}^{\otimes n} \otimes \overline{B}$ are relatively ample for any integer $n \geqslant N$. We then obtain that $(\overline{L} \otimes \overline{M})^{\otimes n} \otimes (\overline{A} \otimes \overline{B})$ is relatively ample. Therefore $\overline{L} \otimes \overline{M}$ is relatively nef. □

Proposition 3.18. *Let $\overline{L}_0, \dots, \overline{L}_d$ be a family of relatively nef adelic line bundles on X. For any $i \in \{0, \dots, d\}$, let*

$$\delta_i = (L_0 \cdots L_{i-1} L_{i+1} \cdots L_d).$$

Assume that $\delta_i > 0$ for those $i \in \{0, \dots, d\}$ such that $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i) = -\infty$. Then the following inequality holds:

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \sum_{i=0}^{d} \delta_i \, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i). \tag{3.24}$$

Proof. If there is $i \in \{0, \dots, d\}$ such that $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i) = -\infty$, then the assertion is obvious, so that we may assume that $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i) > -\infty$ for all $i \in \{0, \dots, d\}$.

Let $\overline{A}_i$ be a relatively ample adelic line bundle on X such that $\overline{L}_i^{\otimes n} \otimes \overline{A}_i$ is relatively ample for sufficiently large positive integer n. For any $i \in \{0, \dots, d\}$ and any positive integer n, let

$$\overline{L}_{i,n} = \overline{L}_i^{\otimes n} \otimes \overline{A}_i,$$
$$\delta_{i,n} = (L_{0,n} \cdots L_{i-1,n} L_{i+1,n} \cdots L_{d,n}).$$

By the multi-linearity of intersection product, we obtain that

$$\lim_{n \to +\infty} \frac{\delta_{i,n}}{n^d} = \delta_i, \quad \lim_{n \to +\infty} \frac{(\overline{L}_{0,n} \cdots \overline{L}_{d,n})_S}{n^{d+1}} = (\overline{L}_0 \cdots \overline{L}_d)_S.$$

Note that Theorem 3.10 leads to

$$(\overline{L}_{0,n} \cdots \overline{L}_{d,n})_S \geqslant \sum_{i=0}^{d} \delta_{i,n} \, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_{i,n})$$

for sufficiently large positive integer n. Dividing the two sides by n^{d+1} and then taking the limit when $n \to +\infty$, we obtain the inequality (3.24). □

3.6. **Pull-back by a projective morphism.**

Lemma 3.19. *If $\overline{L} = (L, \varphi)$ is a relatively nef adelic line bundle on X and if $g : Y \to X$ is projective morphism from a reduced K-scheme Y to X, then the pull-back $g^*(\overline{L})$ is a relatively nef adelic line bundle on Y.*

Proof. Let $\overline{A} = (A, \psi)$ be a relatively ample line bundle on X and N be a positive integer such that $\overline{L}^{\otimes n} \otimes \overline{A} = (L^{\otimes n} \otimes A, n\varphi + \psi)$ is relatively ample for any $n \in \mathbb{N}_{\geqslant N}$. Note that $L^{\otimes n} \otimes A$ is ample and hence $g^*(L)^{\otimes n} \otimes g^*(A)$ is semi-ample. Moreover, by Lemma 3.2, $n\varphi + \psi$ is semi-positive. We choose an arbitrary relatively ample adelic line bundle $\overline{B}$ on Y. By Proposition 3.16, we obtain that $g^*(\overline{L})^{\otimes n} \otimes (g^*(\overline{A}) \otimes \overline{B})$ is relatively ample for any $n \in \mathbb{N}_{\geqslant N}$. Thus the assertion follows. □

Proposition 3.20. *Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X such that L is nef. For any non-empty and reduced closed subscheme Y of X, the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}|_Y) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}). \tag{3.25}$$

Proof. We first consider the case where L is ample. Clearly the restriction of L to Y is ample, and there exists $n_0 \in \mathbb{N}$ such that the restriction map

$$\pi_n : H^0(X, L^{\otimes n}) \longrightarrow H^0(Y, L|_Y^{\otimes n})$$

is surjective for any $n \in \mathbb{N}_{\geqslant n_0}$. Moreover, if we denote by φ_ω^Y the restriction of the metric φ_ω to $L_\omega|_{Y_\omega}$, then, for any $s \in H^0(X_\omega, L_\omega^{\otimes n})$, the inequality

$$\|s\|_{n\varphi_\omega} \geqslant \|\pi_{n,\omega}(s)\|_{n\varphi_\omega^Y}$$

holds, so that, by [10, Proposition 4.3.31], we obtain

$$\widehat{\mu}_{\min}(H^0(Y, L|_Y^{\otimes n}), (\|\cdot\|_{n\varphi_\omega^Y})_{\omega\in\Omega}) \geqslant \widehat{\mu}_{\min}(H^0(X, L^{\otimes n}), (\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega})$$

for any $n \in \mathbb{N}_{\geqslant n_0}$. Dividing the two sides of the inequality by n and taking the limit when $n \to +\infty$, we obtain the inequality (3.25).

In general, let $\overline{A}$ be an adelic line bundle on X such that A is ample. By the above argument, one has $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}|_Y) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) > -\infty$. Since L is nef, $L|_Y$ is also nef (see [15, Example 1.4.4]) and therefore $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_Y)$ is well defined. By (3.22) and the above case, one has

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_Y) = \lim_{n\to+\infty} \frac{1}{n}\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_Y^{\otimes n} \otimes \overline{A}|_Y) \geqslant \lim_{n\to+\infty} \frac{1}{n}\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{A}) = \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}),$$

as required. □

Proposition 3.21. *Let Y be a reduced and non-empty closed subscheme of X and r be the dimension of Y. Let $\overline{L}_0, \ldots, \overline{L}_r$ be a family of relatively nef adelic line bundles on X. For any $i \in \{0, \ldots, r\}$, let*

$$\delta_i = (L_0|_Y \cdots L_{i-1}|_Y L_{i+1}|_Y \cdots L_r|_Y).$$

Assume that, for any $i \in \{0, \ldots, r\}$, $\delta_i > 0$ once $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i|_Y) = -\infty$. Then the following inequality holds:

$$(\overline{L}_0|_Y \cdots \overline{L}_r|_Y)_S \geqslant \sum_{i=0}^{r} \delta_i\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i|_Y). \tag{3.26}$$

Proof. This is a consequence of Proposition 3.18 and Lemma 3.19. □

Proposition 3.22. *Let $\overline{L}_0 = (L_0, \varphi_0), \ldots, \overline{L}_d = (L_d, \varphi_d)$ be a family of adelic line bundles on X. For any $i \in \{0, \ldots, d\}$, let*

$$\delta_i = (L_0 \cdots L_{i-1} L_{i+1} \cdots L_d).$$

Assume that $\overline{L}_1, \ldots, \overline{L}_d$ are relatively nef, L_0 admits a global section s which is a regular meromorphic section, and, for any $i \in \{1, \ldots, d\}$, $\delta_i > 0$ once $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i) = -\infty$. Then the following inequality holds:

$$\begin{aligned}(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \sum_{i=1}^{d} \delta_i\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i) \\ - \int_\Omega \int_{X_\omega^{\mathrm{an}}} \ln \|s\|_{\varphi_{0,\omega}} c_1(L_{1,\omega}, \varphi_{1,\omega}) \cdots c_1(L_{d,\omega}, \varphi_{d,\omega})\, \nu(\mathrm{d}\omega).\end{aligned} \tag{3.27}$$

Proof. If there exists $i \in \{1, \dots, d\}$ such that $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i) = -\infty$, then the inequality (3.27) is trivial. Therefore, we can assume that $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i) \in \mathbb{R}$ for any $i \in \{1, \dots, n\}$. Let $\operatorname{div}(s) = a_1 Z_1 + \cdots + a_n Z_n$ be the decomposition of $\operatorname{div}(s)$ as linear combination of prime divisors, where $a_1, \dots, a_n$ are non-negative integers since s is a global section. By proposition 3.20, for any $i \in \{1, \dots, d\}$ and and any $j \in \{1, \dots, n\}$, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i|_{Z_j}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i). \tag{3.28}$$

By [11, Proposition 4.4.4], one has

$$\begin{aligned}(\overline{L}_0 \cdots \overline{L}_d)_S = &\sum_{j=1}^{n} a_j (\overline{L}_1|_{Z_j} \cdots \overline{L}_d|_{Z_j})_S \\ &- \int_{\Omega} \int_{X^{\mathrm{an}}_{\omega}} \ln |s|_{\varphi_{0,\omega}}(x)\, c_1(L_{1,\omega}, \varphi_{1,\omega}) \cdots c_1(L_{d,\omega}, \varphi_{d,\omega})(\mathrm{d}x)\, \nu(\mathrm{d}\omega).\end{aligned}$$

By Proposition 3.21, one has

$$\sum_{j=1}^{n} a_j (\overline{L}_1|_{Z_j} \cdots \overline{L}_d|_{Z_j})_S \geqslant \sum_{j=1}^{n} a_j \sum_{i=1}^{d} \delta_{i,j}\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i|_{Z_j}) \geqslant \sum_{j=1}^{n} a_j \sum_{i=1}^{d} \delta_{i,j}\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i),$$

where

$$\delta_{i,j} := (L_1|_{Z_j} \cdots L_{i-1}|_{Z_j} L_{i+1}|_{Z_j} \cdots L_d|_{Z_j}),$$

and the second inequality comes from (3.28). Note that, for any $i \in \{1, \dots, d\}$, one has

$$\sum_{j=1}^{n} a_j \delta_{i,j} = \delta_i.$$

Hence we obtain the desired inequality. □

3.7. **Asymptotic minimal slope of a quotient adelic line bundle.**

Proposition 3.23. *Let (E, ξ) be an adelic vector bundle on S, L be a quotient line bundle of $f^*(E)$ and φ be the quotient metric family induced by ξ. Then the adelic line bundle (L, φ) is relatively nef. Moreover, the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \geqslant \widehat{\mu}_{\min}(\overline{E}) - \frac{3}{2}\nu(\Omega_\infty) \ln(\dim_K(E)) \tag{3.29}$$

Proof. By [16, Propositions 6.1.8 and 6.1.2], $f^*(E)$ is a nef vector bundle on X and hence L is a nef line bundle. Moreover, since quotient metrics are semi-positive (see [10, Remark 2.3.1]), the adelic line bundle $\overline{L}$ is relatively nef.

In the following, we prove the inequality (3.29). Let p be an integer and $\overline{A}$ be a relatively ample adelic line bundle on X. Then $\overline{L}^{\otimes p} \otimes \overline{A}$ is relatively ample. Let $Y = \mathbb{P}(f^*(E)^{\otimes p})$ and $g : Y \to X$ be the structural morphism. The quotient homomorphism $f^*(E) \to L$ induces by taking the tensor product a surjective homomorphism $f^*(E)^{\otimes p} \to L^{\otimes p}$, which corresponds to a section $s : X \to Y$ such that $s^*(\mathcal{O}_Y(1)) \cong L^{\otimes p}$. Hence

$$s^*(\mathcal{O}_Y(1) \otimes g^*(A)) \cong L^{\otimes p} \otimes A.$$

By Proposition 3.20, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p} \otimes \overline{A}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{\mathcal{O}_Y(1)} \otimes g^*(\overline{A})), \tag{3.30}$$

where we consider Fubini-Study metric fiber by fiber on $\mathcal{O}_Y(1)$. Note that, for any integer $n \in \mathbb{N}_{\geqslant 1}$, by the adjunction formula one has

$$H^0(Y, \mathcal{O}_Y(n) \otimes g^*(A)^{\otimes n}) = H^0(X, S^n(f^*(E)^{\otimes p}) \otimes A^{\otimes n}) = S^n(E^{\otimes p}) \otimes H^0(X, A^{\otimes n}).$$

Moreover, the projection map

$$E^{\otimes np} \otimes H^0(X, A^{\otimes n}) \longrightarrow S^n(E^{\otimes p}) \otimes H^0(X, A^{\otimes n}),$$

where we consider the ε, π-tensor product norm family on the left hand side of the arrow, and the adelic vector bundle structure of $(fg)_*(\overline{\mathcal{O}_Y(n)} \otimes g^*(\overline{A})^{\otimes n})$ on the right hand side. By [10, Corollary 5.6.2] (see also [12, Remark C.3]), we obtain

$$\begin{aligned}\frac{\widehat{\mu}_{\min}\big(g_*(\overline{\mathcal{O}_Y(n)} \otimes g^*(\overline{A}^{\otimes n})\big)}{n} \geqslant \frac{1}{n}\bigg(& np\,\widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}_{\min}(f_*(\overline{A}^{\otimes n})) \\ & - \frac{3}{2}\nu(\Omega_\infty)\ln\big(\dim_K(E)^{np} \cdot \dim_K(H^0(X, A^{\otimes n}))\big)\bigg).\end{aligned}$$

Taking the limit when $n \to +\infty$, we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{\mathcal{O}}_Y(1) \otimes g^*(\overline{A})) \geqslant p\,\widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) - \frac{3}{2}\nu(\Omega_\infty)p\ln(\dim_K(E)).$$

Combining this inequality with (3.30), we obtain

$$\frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p} \otimes \overline{A}) \geqslant \widehat{\mu}_{\min}(\overline{E}) + \frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E)).$$

Thus, due to Definition 3.12, we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \geqslant \limsup_{p\to\infty} \frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p} \otimes \overline{A}) \geqslant \widehat{\mu}_{\min}(\overline{E}) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E)),$$

as required. □

3.8. **Asymptotic minimal slope of a pull-back.**

Proposition 3.24. *Let $g : Y \to X$ be a projective morphism of K-schemes, which is surjective and such that $g_*(\mathcal{O}_Y) = \mathcal{O}_X$. Let $\overline{L}$ be an adelic line bundle on X such that L is nef. Then the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

Proof. By [15, Example 1.4.4], the line bundle $g^*(L)$ is nef, and hence $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L}))$ is well defined. We first consider the case where L is ample. Let p be a positive integer and $\overline{A}$ be an adelic line bundle on Y such that A is ample. By Lemma 3.1, for sufficiently positive integer n, the K-linear map

$$H^0(Y, A^{\otimes n}) \otimes H^0(X, L^{\otimes pn}) \longrightarrow H^0(Y, A^{\otimes n} \otimes g^*(L)^{\otimes pn})$$

is surjective. Moreover, if we equip the left hand side of the arrow with the ε, π-tensor product norm family of those of $(fg)_*(\overline{A}^{\otimes n})$ and $f_*(\overline{L}^{\otimes pn})$, then the K-linear map has height $\leqslant 0$. Therefore, by [10, Corollary 5.6.2] we obtain

$$\begin{aligned}\widehat{\mu}_{\min}((fg)_*(\overline{A}^{\otimes n} \otimes g^*(\overline{L})^{\otimes pn})) \geqslant\ & \widehat{\mu}_{\min}((fg)_*(\overline{A}^{\otimes n})) + \widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes pn})) \\ & - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(H^0(Y, A^{\otimes n}))\dim_K(H^0(X, L^{\otimes pn}))).\end{aligned}$$

Dividing the two sides of the inequality by pn and then taking the limit when $n \to +\infty$, we obtain

$$\frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A} \otimes g^*(\overline{L})^{\otimes p}) \geqslant \frac{\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A})}{p} + \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}),$$

which leads to

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L})) \geqslant \limsup_{p\to+\infty} \frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A} \otimes g^*(\overline{L})^{\otimes p}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

We now consider the general case. Let $\overline{B}$ be an adelic line bundle on X such that B is ample. By the above argument we obtain that $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{B})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{B}) > -\infty$ and, for any positive integer n,

$$\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L}^{\otimes n}) \otimes g^*(\overline{B})) \geqslant \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{B}).$$

Taking the limit when $n \to +\infty$, by (3.22) we obtain $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$. □

Theorem 3.25. *Let $g : Y \to X$ be a projective morphism of K-schemes. We assume that Y is non-empty and reduced. For any adelic line bundle $\overline{L}$ on X such that L is nef, one has $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$.*

Proof. The projective morphism g can be written as the composition of a closed immersion from Y into a projective bundle on X and the projection from the projective bundle to X. Hence the inequality follows from Propositions 3.24 and 3.20. □

3.9. **Comparison with the normalized height.** The following height estimate can be deduced from Theorem 3.18. Here we provide an alternative proof in the particular case where X is integral by using the arithmetic Hilbert-Samuel formula.

Proposition 3.26. *Let $\overline{L}$ be a relatively nef adelic line bundle on X such that $(L^d) > 0$. Then the following inequality holds*

$$\frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)} \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}). \tag{3.31}$$

Proof. We assume that X is integral. In the case where $\overline{L}$ is relatively ample, it is a consequence of Theorem 3.7 and Remark 3.8.

We now consider the general case where $\overline{L}$ is only relatively nef. Let $\overline{A}$ be a relatively ample adelic line bundle and N be a positive integer such that $L^{\otimes n} \otimes \overline{A}$ is relatively ample for any $n \in \mathbb{N}_{\geqslant N}$. For any $n \in \mathbb{N}_{\geqslant N}$, the adelic line bundle $\overline{L}_n = \overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample. Hence the particular case of the proposition proved above shows that

$$\forall\, n \in \mathbb{N}_{\geqslant N}, \quad \frac{(\overline{L}_n^{d+1})_S}{(d+1)(\overline{L}_n^d)} \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_n).$$

Moreover, by the multi-linearity of intersection product, one has

$$\lim_{n\to+\infty} \frac{(\overline{L}_n^{d+1})_S}{n^{d+1}} = (\overline{L}^{d+1})_S, \quad \lim_{n\to+\infty} \frac{(L_n^d)}{n^d} = (L^d).$$

Therefore, one obtains

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) = \lim_{n\to+\infty} \frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_n) \leqslant \lim_{n\to+\infty} \frac{(\overline{L}_n^{d+1})_S}{n(d+1)(\overline{L}_n^d)} = \frac{(\overline{L}^{d+1})}{(d+1)(L^d)}.$$

□

Corollary 3.27. *Let $\overline{L}$ be a relatively nef adelic line bundle on X. For any non-empty and reduced closed subscheme Y of X, the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \leqslant \frac{h_{\overline{L}}(Y)}{(\dim_L(Y)+1)\deg_L(Y)}. \tag{3.32}$$

In particular, for any closed point x, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \leqslant h_{\overline{L}}(x).$$

Proof. By Lemma 3.19, the restriction of $\overline{L}$ to Y is relatively nef. By Proposition 3.26 one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}|_Y) \leqslant \frac{h_{\overline{L}}(Y)}{(\dim_L(Y)+1)\deg_L(Y)}.$$

By Proposition 3.20, we obtain (3.32). □

4. Geometrically big and pseudoeffective adelic line bundles

As in the previous section, we fix a proper adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ such that, either K is countable, or $(\Omega, \mathcal{A})$ is discrete. We assume that K is perfect.

4.1. Convergence of maximal slopes.

Proposition 4.1. *Let X be an integral projective scheme over $\operatorname{Spec} K$, and $\overline{L} = (L, \varphi)$ and $\overline{M} = (M, \psi)$ be adelic line bundles on X such that $H^0(X, L)$ and $H^0(X, M)$ are non-zero. Then the following inequality holds:*

$$\widehat{\mu}_{\max}(f_*(\overline{L} \otimes \overline{M})) \geqslant \widehat{\mu}_{\max}(f_*(\overline{L})) + \widehat{\mu}_{\max}(f_*(\overline{M})) - \frac{3}{2}\nu(\Omega_\infty)(\ln(h^0(L) \cdot h^0(M))),$$

where $h^0(L) = \dim_K(H^0(X, L))$ and $h^0(M) = \dim_K(H^0(X, M))$.

Proof. By [10, Theorem 4.3.58], there exist non-zero vector subspaces E and F of $H^0(X, L)$ and $H^0(X, M)$, respectively, such that

$$\widehat{\mu}_{\min}(\overline{E}) = \widehat{\mu}_{\max}(f_*(\overline{L})), \quad \widehat{\mu}_{\min}(\overline{F}) = \widehat{\mu}_{\max}(f_*(\overline{M})),$$

where we consider restricted norm families on E and F. Since X is integral, the map

$$E \otimes_K F \longrightarrow H^0(X, L \otimes M), \quad s \otimes t \longmapsto st$$

is non-zero. Moreover, for any $\omega \in \Omega$, one has

$$\forall\, (s, t) \in E_\omega \times F_\omega, \quad \|st\|_{\varphi_\omega + \psi_\omega} \leqslant \|s\|_{\varphi_\omega} \cdot \|t\|_{\psi_\omega}.$$

Therefore, the height of the above K-linear map is $\leqslant 0$ if we consider the ε, π-tensor product norm family on $E \otimes_K F$. By [10, Theorem 4.3.31 and Corollary 5.6.2], we obtain

$$\begin{aligned}
&\widehat{\mu}_{\max}(f_*(\overline{L} \otimes \overline{M})) \\
&\geqslant \widehat{\mu}_{\min}(\overline{E} \otimes_{\varepsilon,\pi} \overline{F}) \geqslant \widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}_{\min}(\overline{F}) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E) \cdot \dim_K(F)) \\
&= \widehat{\mu}_{\max}(f_*(\overline{L})) + \widehat{\mu}_{\max}(f_*(\overline{M})) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E) \cdot \dim_K(F)) \\
&\geqslant \widehat{\mu}_{\max}(f_*(\overline{L})) + \widehat{\mu}_{\max}(f_*(\overline{M})) - \frac{3}{2}\nu(\Omega_\infty)(\ln(h^0(L) \cdot h^0(M))),
\end{aligned}$$

as required. □

Corollary 4.2. *Let $\overline{L}$ be an adelic line bundle on X such that $H^0(X, L^{\otimes n})$ is non-zero for sufficiently large natural number n. The sequence*

$$\frac{1}{n}\,\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n})), \quad n \in \mathbb{N}_{\geqslant 1}$$

converges in $\mathbb{R}$.

Proof. The convergence of the sequence follows from Proposition 4.1, using the same argument as in the proof of Proposition 3.3. □

4.2. **Asymptotic maximal slope.** In this subsection, we let $f : X \to \operatorname{Spec} K$ be an integral projective K-scheme.

Definition 4.3. Let $\overline{L}$ be an adelic line bundle on X such that L is big. We define

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) := \lim_{n\to+\infty} \frac{\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n}))}{n}.$$

By definition, for any $p \in \mathbb{N}_{\geqslant 1}$, the following equality holds:

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes p}) = p\,\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}).$$

Proposition 4.4. *Let $\overline{L}$ and $\overline{M}$ be adelic line bundles on X such that L and M are both big. One has*

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}\otimes\overline{M}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) + \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{M}). \tag{4.1}$$

Proof. For any $n \in \mathbb{N}_{\geqslant 1}$, let $a_n = \dim_K(H^0(X, L^{\otimes n}))$ and $b_n = \dim_K(X, M^{\otimes n})$. One has

$$\ln(a_n) = O(\ln(n)), \quad \ln(b_n) = O(\ln(n)), \quad n \to +\infty.$$

By Proposition 4.1, for sufficiently large n, one has

$$\frac{\widehat{\mu}_{\max}(f_*((\overline{L}\otimes\overline{M})^{\otimes n}))}{n} \geqslant \frac{\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n}))}{n} + \frac{\widehat{\mu}_{\max}(f_*(\overline{M}^{\otimes n}))}{n} - \frac{3}{2}\nu(\Omega_\infty)\frac{\ln(a_n b_n)}{n}.$$

Taking the limit when $n \to +\infty$, we obtain the inequality (4.1). □

Proposition 4.5. *Let $\overline{L}$ and $\overline{A}$ be adelic line bundle on X. We assume that L is pseudo-effective and A is big. Then the sequence*

$$\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes n}\otimes\overline{A}), \quad n \in \mathbb{N}_{\geqslant 1}$$

converges in $\mathbb{R}\cup\{-\infty\}$. Moreover, its limit does not depend on the choice of $\overline{A}$. In particular, in the case where L is big, the following equality holds:

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes n}\otimes\overline{A}) = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}). \tag{4.2}$$

Proof. The proof relies on the super-additivity of the function $\widehat{\mu}^{\mathrm{asy}}_{\max}(\cdot)$ (see Proposition 4.4) and follows the same strategy as that of Proposition 3.11. We omit the details. □

Definition 4.6. Let $\overline{L}$ be an adelic line bundle on X such that L is pseudo-effective. We define $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ as the limite

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes n}\otimes\overline{A}),$$

where $\overline{A}$ is an arbitrary adelic line bundle on X such that A is big. The element $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ of $\mathbb{R}\cup\{-\infty\}$ is called the *asymptotic maximal slope* of $\overline{L}$.

Proposition 4.7. *Let $\overline{L}$ and $\overline{M}$ be adelic line bundles on X such that L and M are pseudo-effective. Then the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}\otimes\overline{M}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) + \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{M}).$$

Proof. Let $\overline{A}$ be an adelic line bundle on X such that A is big. For any $n \in \mathbb{N}$,

$$(L \otimes M)^{\otimes n} \otimes A^{\otimes 2} = (L^{\otimes n} \otimes A) \otimes (M^{\otimes n} \otimes A)$$

is big. Moreover, by Proposition 4.4, one has

$$\frac{1}{n}\widehat{\mu}^{\mathrm{asy}}_{\max}\big((\overline{L} \otimes M)^{\otimes n} \otimes \overline{A}^{\otimes 2}\big) \geqslant \frac{1}{n}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes n} \otimes \overline{A}) + \frac{1}{n}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{M}^{\otimes n} \otimes \overline{A}).$$

Taking the limit when $n \to +\infty$, we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L} \otimes \overline{M}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) + \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{M}).$$

□

4.3. **Pullback by a surjective projective morphism.** Let X and Y be integral projective K-schemes and $g : Y \to X$ be a surjective projective morphism.

Lemma 4.8. *Let L be an invertible $\mathcal{O}_X$-module. If L is pseudo-effective, then the pullback $g^*(L)$ is also pseudo-effective.*

Proof. Let A be a big invertible $\mathcal{O}_X$-module and B be a big invertible $\mathcal{O}_Y$-module. For any positive integer p, the invertible $\mathcal{O}_X$-module $L^{\otimes p} \otimes A$ is big and hence $g^*(L^{\otimes p} \otimes A)$ is pseudo-effective since it has a tensor power which is effective. Similarly, $g^*(A)$ is also pseudo-effective. Thus we obtain that $g^*(A) \otimes B$ and $g^*(L)^{\otimes p} \otimes g^*(A) \otimes B$ are big. In particular, $g^*(L)$ is pseudo-effective. □

Proposition 4.9. *Let $\overline{L}$ be an adelic line bundle on X such that L is pseudo-effective. Then the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(g^*(\overline{L})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}).$$

Proof. We have seen in Lemma 4.8 that the invertible $\mathcal{O}_X$-module L is pseudo-effective, so that $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ is well defined. We choose an adelic line bundle $\overline{A}$ on X such that A is big.

We first assume that L is big. Let n and p be positive integers. We consider the K-linear map

$$\begin{aligned} H^0(X, g_*(A^{\otimes n})) \otimes H^0(X, L^{\otimes np}) \longrightarrow H^0(X, g_*(A^{\otimes n}) \otimes L^{\otimes np}) \\ = H^0(Y, A^{\otimes n} \otimes g^*(L^{\otimes np})) \end{aligned}$$

induced by multiplication of sections. Let E be the destabilizing vector subspace of $(fg)_*(\overline{A}^{\otimes n})$ and let F be the destabilizing vector subspace of $f_*(\overline{L}^{\otimes np})$. By [10, Proposition 4.3.31], one has

$$\widehat{\mu}_{\min}(\overline{E} \otimes_{\varepsilon,\pi} \overline{F}) \leqslant \widehat{\mu}_{\max}((fg)_*(\overline{A}^{\otimes n} \otimes g^*(\overline{L}^{\otimes np}))).$$

by [10, Corollary 5.6.2 and Remark 4.3.48] (see also [12, Remark C.3]), one deduces

$$\begin{aligned} &\widehat{\mu}_{\max}((fg)_*(\overline{A}^{\otimes n} \otimes g^*(\overline{L}^{\otimes np}))) \\ &\geqslant \widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}_{\min}(\overline{F}) - \frac{3}{2}\nu(\Omega_\infty)(\ln(\dim_K(E)) + \ln(\dim_K(F))) \\ &\geqslant \widehat{\mu}_{\max}((fg)_*(\overline{A}^{\otimes n})) + \widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes np})) - 2\nu(\Omega_\infty)\ln\dim_K(H^0(Y, A^{\otimes n})) \\ &\quad - 2\nu(\Omega_\infty)\ln\dim_K(H^0(X, L^{\otimes np})). \end{aligned}$$

If we divide the two sides by np, taking the limit when $n \to +\infty$, we obtain

$$\frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{A} \otimes g^*(\overline{L}^{\otimes p})) \geqslant \frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{A}) + \widehat{\mu}_{\max}(\overline{L}).$$

Taking the limit when $p \to +\infty$, we obtain $\widehat{\mu}^{\mathrm{asy}}_{\max}(g^*(\overline{L})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$, as required.

We then consider the general case where L is only assumed to be pseudo-effective. Let $\overline{B}$ be an adelic line bundle on X such that B is big. Note that, for any positive integer p, $L^{\otimes p} \otimes B$ is big. Hence, by the particular case of the proposition shown above, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(g^*(\overline{L})^{\otimes p} \otimes g^*(\overline{B})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes p} \otimes B).$$

Therefore, by Proposition 4.7, we obtain

$$\frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(g^*(\overline{L})^{\otimes p} \otimes g^*(\overline{B}) \otimes \overline{A}) \geqslant \frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes p} \otimes \overline{B}) + \frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{A}).$$

Taking the limit when $p \to +\infty$, we obtain $\widehat{\mu}^{\mathrm{asy}}_{\max}(g^*(\overline{L})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$. □

Remark 4.10. Let $\overline{L}$ be an adelic line bundle on X. Assume that L is the pull-back of a big line bundle by a surjective projective morphism. Then Proposition 4.9 shows that $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \in \mathbb{R}$.

4.4. **Relative Fujita approximation.** Let $f : X \to \operatorname{Spec} K$ be a projective K-scheme, $K(X)$ be the field of rational functions on X, and $\mathscr{M}_X$ be the sheaf of meromorphic function on X.

Definition 4.11. Let L be a big line bundle on X. Note that $\mathscr{M}_X \otimes_{\mathcal{O}_X} L$ is isomorphic to the trivial invertible $\mathscr{M}_X$-module. In particular, if s and t are two global sections of L such that $s \neq 0$, then there existe a unique rational function $\lambda \in K(X)$ such that $t = \lambda s$. We denote by t/s this rational function. If E is a K-vector subspace of $H^0(X, L)$. We denote by $K(E)$ the sub-extension of $K(X)/K$ generated by elements of the form t/s, where t and s are non-zero sections in $K(E)$. We say that E is *birational* if $K(E) = K(X)$. Moreover L is said to be *birational* if $K(H^0(X, L)) = K(X)$.

Remark 4.12. Let L and M be line bundle on X, E be a vector subspace of $H^0(X, L)$, s be a non-zero global section of M and

$$F = \{ts \,|\, t \in E\} \subseteq H^0(X, L \otimes M).$$

Then by definition one has $K(F) = K(E)$. In particular, if E is birational, so is F; if L is birational, so is $L \otimes M$.

Proposition 4.13. *Let L be a big line bundle on X. For sufficiently positive integer p, the line bundle $L^{\otimes p}$ is birational.*

Proof. Since L is big, there exists a positive integer q, an ample line bundle A and an effective line bundle M on X such that $L^{\otimes q} \cong A \otimes M$. By replacing q by a multiple, we may assume that the graded K-algebra

$$\bigoplus_{n \in \mathbb{N}} H^0(X, A^{\otimes n})$$

is generated by $H^0(X, A)$ and that $L^{\otimes (q+1)}$ is effective. For any $a \in \mathbb{N}_{\geqslant 1}$, one has

$$X = \operatorname{Proj}\Big(\bigoplus_{n \in \mathbb{N}} H^0(X, A^{\otimes an})\Big),$$

which implies that $A^{\otimes a}$ is birational and hence $L^{\otimes aq}$ is birational. Moreover, since $L^{\otimes (q+1)}$ is effective, for any $b \in \mathbb{N}_{\geqslant 1}$, the line bundle $L^{\otimes b(q+1)}$ is also effective. Therefore, for any $(a, b) \in \mathbb{N}^2_{\geqslant 1}$, the line bundle $L^{\otimes aq + b(q+1)}$ is birational. Since

q and $q+1$ are coprime, we obtain that $L^{\otimes p}$ is birational for sufficiently large $p \in \mathbb{N}$. □

Definition 4.14. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X. If $s \in H^0(X, L)$ is a non-zero global section such that $\|s\|_{\varphi_\omega} \leqslant 1$ for any $\omega \in \Omega$, we say that the global section s is *effective*. We say that $\overline{L}$ is *effective* if it admits at least an effective global section.

Lemma 4.15. *Let $\overline{L}$ be an adelic line bundle such that L is big. For any $t < \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ and any $N \in \mathbb{N}_{\geqslant 1}$, there exists an integer $p \geqslant N$ and a vector subspace E of $H^0(X, L^{\otimes p})$ such that $K(E) = K(X)$ and $\widehat{\mu}_{\min}(\overline{E}) > pt$.*

Proof. By replacing $\overline{L}$ by one of its tensor powers, we may assume without loss of generality that L is birational. For any $n \in \mathbb{N}$, let $r_n = \dim_K(H^0(X, L^{\otimes n}))$. Since $t < \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$, for sufficiently large $n \in \mathbb{N}$, one has

$$\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n})) > (n+1)t - \widehat{\mu}_{\min}(f_*(\overline{L})) + \frac{3}{2}\nu(\Omega_\infty)\ln\big(r_n \cdot r_1\big).$$

Let F be a vector subspace of $H^0(X, L^{\otimes n})$ such that

$$\widehat{\mu}_{\min}(\overline{F}) = \widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n})).$$

The existence of F is ensured by [10, Theorem 4.3.58]. Let E be the image of $F \otimes_K H^0(X, L)$ by the K-linear map

$$H^0(X, L^{\otimes n}) \otimes H^0(X, L) \longrightarrow H^0(X, L^{\otimes n+1}), \quad s \otimes t \longmapsto st.$$

Since L is birational and F is non-zero, we obtain that E is birational. By [10, Corollary 5.6.2], one has

$$\begin{aligned}
&\widehat{\mu}_{\min}(\overline{E}) \geqslant \widehat{\mu}_{\min}(\overline{F} \otimes_{\varepsilon,\pi} f_*(\overline{L})) \\
&\geqslant \widehat{\mu}_{\min}(\overline{F}) + \widehat{\mu}_{\min}(f_*(\overline{L})) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(F)\cdot r_1) \\
&= \widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n})) + \widehat{\mu}_{\min}(f_*(\overline{L})) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(F)\cdot r_1) > (n+1)t.
\end{aligned}$$

□

Theorem 4.16 (Relative Fujita approximation)**.** *Let $\overline{L}$ be an adelic line bundle on X such that L is big. For any real number $t < \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$, there exist a positive integer p, a birational projective K-morphism $g : X' \to X$, a relatively nef adelic line bundle $\overline{A}$ and an effective adelic line bundle $\overline{M}$ on X' such that A is big, $g^*(\overline{L}^{\otimes p})$ is isomorphic to $\overline{A} \otimes \overline{M}$ and $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) \geqslant pt$.*

Proof. We pick a positive integer p and a birational vector subspace V of $H^0(X, L^{\otimes p})$ such that

$$\widehat{\mu}_{\min}(\overline{V}) = \widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes p})) \geqslant pt + \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(H^0(X, L^{\otimes p}))).$$

Let $g : X' \to X$ be the blow-up of L along the base locus of V, namely

$$X' = \operatorname{Proj}\left(\operatorname{Im}\left(\bigoplus_{n\in\mathbb{N}} S^n(f^*(V)) \longrightarrow \bigoplus_{n\in\mathbb{N}} L^{\otimes np}\right)\right).$$

Denote by E the exceptional divisor and by s_E the global section of $\mathcal{O}_X(E)$ which trivializes $\mathcal{O}_X(E)$ outside of the exceptional divisor. One has

$$\mathcal{O}_{X'}(1) \cong g^*(L^{\otimes p}) \otimes \mathcal{O}_X(-E).$$

Moreover, the canonical surjective homomorphism

$$g^*(f^*(V)) \longrightarrow \mathcal{O}_{X'}(1) \tag{4.3}$$

induces a K-morphism $i : X' \to \mathbb{P}(V)$ such that $i^*(\mathcal{O}_V(1)) = \mathcal{O}_{X'}(1)$, where $\mathcal{O}_V(1)$ denotes the universal invertible sheaf on $\mathbb{P}(V)$. Since V is birational, the line bundle $\mathcal{O}_{X'}(1)$ is big.

We equip V with the induced norm family of $(\|\cdot\|_{p\varphi_\omega})_{\omega\in\Omega}$ and $\mathcal{O}_{X'}(1)$ with the quotient metric family $\varphi' = (\varphi'_\omega)_{\omega\in\Omega}$ induced by $(\|\cdot\|_{p\varphi_\omega})_{\omega\in\Omega}$ and the surjective homomorphism (4.3). We identify $\mathcal{O}_X(E)$ with $g^*(L^{\otimes p}) \otimes \mathcal{O}_{X'}(1)^\vee$ and equip it with the tensor product metric family. Then the section s_E is effective. Moreover, by Proposition 3.23, the adelic line bundle $\overline{\mathcal{O}_{X'}(1)}$ is relatively nef, and the following inequality holds

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{\mathcal{O}_{X'}(1)}) \geqslant \widehat{\mu}_{\min}(\overline{V}) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(V)) \geqslant pt,$$

as required.

□

Remark 4.17. Let $\overline{L}$ be an adelic line bundle on X such that L is big. Let $\overline{B}$ be a relatively ample adelic line bundle. There exists a positive integer N such that $L^{\otimes m} \otimes B^\vee$ is big for any $m \in \mathbb{N}_{\geqslant N}$. Let t be a real number such that $t < \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$. There exists $m \in \mathbb{N}_{\geqslant N}$ such that

$$mt - \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{B}) < (m-N)\,\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) + \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes N} \otimes \overline{B}^\vee) \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes m} \otimes \overline{B}^\vee),$$

where the second inequality comes from If we apply Theorem 4.16 to $\overline{L}^{\otimes m} \otimes \overline{B}^\vee$, we obtain the existence of a positive integer p, a birational projective K-morphism $g : X' \to X$, a relatively nef adelic line bundle $\overline{A}$ and an effective adelic line bundle $\overline{M}$ on X' such that A is big, $g^*(\overline{L}^{\otimes mp} \otimes \overline{B}^{\vee\otimes p})$ is isomorphic to $\overline{A} \otimes \overline{M}$ and

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) \geqslant p(mt - \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{B})). \tag{4.4}$$

Let $\overline{N} = \overline{A} \otimes g^*(\overline{B})^{\otimes p}$. This is a relatively ample line bundle, and one has

$$\overline{N} \otimes \overline{M} \cong \overline{A} \otimes \overline{M} \otimes g^*(\overline{B})^{\otimes p} \cong g^*(\overline{L}^{\otimes mp}).$$

Moreover, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{N}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) + p\,\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{B})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) + p\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{B}),$$

where the first inequality comes from Proposition 3.14, and the second comes from Theorem 3.25. By (4.4), we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{N}) \geqslant pmt.$$

Therefore, in Theorem 4.16, the adelic line bundle $\overline{A}$ can be taken to be relatively ample.

4.5. **Lower bound of intersection product.**

Theorem 4.18. *Let X be an integral projective K-scheme, and $\overline{L}_0, \dots, \overline{L}_d$ be adelic line bundles on X. For any $i \in \{0, \dots, d\}$, let*

$$\delta_i = (L_0 \cdots L_{i-1} L_{i+1} \cdots L_d).$$

Suppose that

(1) *$\overline{L}_1, \dots, \overline{L}_d$ are relatively nef and L_0 is pseudo-effective.*
(2) *if $\delta_0 = 0$, then $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_0) > -\infty$,*

(3) *for any* $i \in \{1, \dots, d\}$, *if* $\delta_i = 0$, *then* $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i) > -\infty$.

Then the following inequality holds:

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_0) + \sum_{i=1}^{d} \delta_i\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i). \tag{4.5}$$

Proof. If the set

$$\{\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_0), \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_1), \dots, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_d)\}$$

contains $-\infty$, then the inequality (4.5) is trivial. So we may assume without loss of generality that

$$\{\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_0), \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_1), \dots, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_d)\} \subseteq \mathbb{R}.$$

Let $\overline{M}$ be an adelic line bundle on X such that M is big. For any $n \in \mathbb{N}_{\geqslant 1}$, let

$$\overline{L}_{0,n} = \overline{L}_0^{\otimes n} \otimes \overline{M}.$$

For any $i \in \{1, \dots, n\}$, let

$$\delta'_i = (M L_1 \cdots L_{i-1} L_{i+1} \cdots L_d)$$
$$\delta_{i,n} = (L_{0,n} L_1 \cdots L_{i-1} L_{i+1} \cdots L_d) = n\delta_i + \delta'_{i,n}.$$

By Theorem 4.16 (see also Remark 4.17), for any real number $t < \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_{0,n})$, there exists a positive integer p, a birational projective morphism $g : X' \to X$, a relatively ample adelic line bundle $\overline{A}$ and an effective adelic line bundle E on X' such that

$$g^*(\overline{L}_{0,n}^{\otimes p}) = \overline{A} \otimes \overline{E}, \quad \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) \geqslant pt.$$

By Theorem 3.25 for any $i \in \{1, \dots, d\}$, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L}_i)) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i).$$

Therefore, by Proposition 3.22 and Proposition 3.18, we obtain

$$(\overline{E} \cdot g^*(\overline{L}_1) \cdots g^*(\overline{L}_d))_S \geqslant \sum_{i=1}^{d} (E \cdot L_1 \cdots L_{i-1} \cdot L_i \cdots L_d)\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i),$$

$$(\overline{A} \cdot g^*(\overline{L}_1) \cdots g^*(\overline{L}_d))_S \geqslant \delta_0\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) + \sum_{i=1}^{d} (A \cdot L_1 \cdots L_{i-1} \cdot L_i \cdots L_d)\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i).$$

Taking the sum, we obtain

$$(\overline{L}_{0,n}^{\otimes p} \cdot \overline{L}_1 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) + \sum_{i=1}^{d} p\delta_{i,n}\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i) \geqslant \delta_0 pt + \sum_{i=1}^{d} p\delta_{i,n}\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i).$$

Since t is arbitrary, we deduce

$$(\overline{L}_{0,n}^{\otimes p} \cdot \overline{L}_1 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_{0,n}) + \sum_{i=1}^{d} \delta_{i,n}\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i).$$

Dividing the two sides by n and then taking the limit when $n \to +\infty$, we obtain

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_0) + \sum_{i=1}^{d} \delta_i\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i).$$

□

4.6. **Convergence of the first minimum.** In this section, we let $f : X \to \operatorname{Spec} K$ be an integral projective scheme over $\operatorname{Spec} K$.

Definition 4.19. Let $\overline{E} = (E, (\|\cdot\|_\omega)_{\omega\in\Omega})$ be an adelic vector bundle on S. For any non-zero element s in E, let

$$\widehat{\deg}(s) := -\int_\Omega \ln \|s\|_\omega \, \nu(\mathrm{d}\omega).$$

If E is non zero, we define

$$\lambda_{\max}(\overline{E}) := \sup_{s\in E\setminus\{0\}} \widehat{\deg}(s).$$

Clearly one has

$$\lambda_{\max}(\overline{E}) \leqslant \widehat{\mu}_{\max}(\overline{E}). \tag{4.6}$$

Proposition 4.20. *Let $\overline{L} = (L, \varphi)$ and $\overline{M} = (M, \psi)$ be adelic line bundles on X such that both $H^0(X, L)$ and $H^0(X, M)$ are non-zero. Then the following inequality holds:*

$$\lambda_{\max}(f_*(\overline{L} \otimes \overline{M})) \geqslant \lambda_{\max}(f_*(\overline{L})) + \lambda_{\max}(f_*(\overline{M})).$$

Proof. Let s and t be respectively non-zero elements of $H^0(X, L)$ and $H^0(X, M)$. For any $\omega \in \Omega$, one has

$$\|st\|_{\varphi_\omega+\psi_\omega} \leqslant \|s\|_{\varphi_\omega} \cdot \|t\|_{\psi_\omega},$$

which leads to

$$\lambda_{\max}(f_*(\overline{L} \otimes \overline{M})) \geqslant \widehat{\deg}(st) \geqslant \widehat{\deg}(s) + \widehat{\deg}(t).$$

Taking the supremum with respect to s and t, we obtain the required inequality. □

Let $\overline{L}$ be an adelic line bundle on X such that L is big. Similarly to Corollary 4.2, the sequence

$$\frac{1}{n}\lambda_{\max}(f_*(\overline{L}^{\otimes n})), \quad n \in \mathbb{N}_{\geqslant 1}$$

converges to a real number, which we denote by $\lambda^{\mathrm{asy}}_{\max}(\overline{L})$ and called the *asymptotic first minimum* of $\overline{L}$. By definition, for any $p \in \mathbb{N}_{\geqslant 1}$ one has

$$\lambda^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes p}) = p\lambda^{\mathrm{asy}}_{\max}(\overline{L}).$$

Proposition 4.20 also implies that, if $\overline{L}$ and $\overline{M}$ are adelic line bundles on X such that both L and M are big, one has

$$\lambda^{\mathrm{asy}}_{\max}(\overline{L} \otimes \overline{M}) \geqslant \lambda^{\mathrm{asy}}_{\max}(\overline{L}) + \lambda^{\mathrm{asy}}_{\max}(\overline{M}). \tag{4.7}$$

Similarly to Proposition 4.5, this inequality allows to extend continuously the function $\lambda^{\mathrm{asy}}_{\max}(\cdot)$ to the cone of adelic line bundles $\overline{L}$ such that L is pseudo-effective: if $\overline{L}$ is an adelic line bundle on X such that L is pseudo-effective, then, for any adelic line bundle $\overline{A}$ on X, the sequence

$$\frac{1}{n}\lambda^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes n} \otimes \overline{A}), \quad n \in \mathbb{N}_{\geqslant 1} \tag{4.8}$$

converges in $\mathbb{R} \cup \{-\infty\}$ and its limit does not depend on the choice of $\overline{A}$. For the proof of this statement one can following the strategy of the proof of Proposition 3.11 in using the inequality 4.7 and the fact that, if A is a big line bundle and B

is a line bundle on X, then there exists a positive integer p such that $B^\vee \otimes A^{\otimes p}$ is big. We denote the limit of the sequence (4.8) by $\lambda^{\mathrm{asy}}_{\max}(\overline{L})$. By (4.6) we obtain that

$$\lambda^{\mathrm{asy}}_{\max}(\overline{L}) \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \tag{4.9}$$

for any adelic line bundle $\overline{L}$ such that L is pseudo-effective.

4.7. **Height inequalities.**

Proposition 4.21. *Let $f : X \to \operatorname{Spec} K$ be an integral projective scheme over $\operatorname{Spec} K$ and $\overline{L}$ be an adelic line bundle on X which is relatively nef and such that $(L^d) > 0$. Then the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}(\overline{L}) = \frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)} \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}). \tag{4.10}$$

Proof. We first consider the case where L is relatively ample. As in the proof of Proposition 3.26, one has

$$\frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)} = \lim_{n\to+\infty} \frac{\widehat{\mu}(f_*(\overline{L}^{\otimes n}))}{n} \leqslant \lim_{n\to+\infty} \frac{\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n}))}{n} = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}).$$

We now consider the general case. Let $\overline{A}$ be a relatively ample adelic line bundle on X such that $\overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample for sufficiently large positive integer n. For any $n \in \mathbb{N}_{\geqslant 1}$, let $\overline{L}_n = \overline{L}^{\otimes n} \otimes \overline{A}$. The particular case of the proposition proved above shows that

$$\frac{(\overline{L}_n^{d+1})_S}{(d+1)(L_n^d)} \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}_n)$$

if n is sufficiently large. Taking the limit when $n \to +\infty$, by the relations

$$\lim_{n\to+\infty} \frac{(\overline{L}_n^{d+1})_S}{n^{d+1}} = (\overline{L}^{d+1})_S, \quad \lim_{n\to+\infty} \frac{(L_n^d)}{n^d} = (L^d)$$

and Proposition 4.5 we obtain the desired result. □

Remark 4.22. Combining Propositions 4.21 and 3.26, we obtain that, if $\overline{L}$ is relatively nef and if $(L^d) > 0$, then the inequality $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$. This inequality also holds for relatively nef adelic line bundle $\overline{L}$ with $(L^d) = 0$. It suffices to choose an auxiliary relatively ample adelic line bundle $\overline{M}$ and deduce the inequality from

$$\frac{1}{n}\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{M}) \leqslant \frac{1}{n}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes n} \otimes \overline{M})$$

by taking the limit when $n \to +\infty$.

Theorem 4.23. *Let X be a non-empty and reduced projective K-scheme and Θ_X be the set of all integral closed subschemes of X. Let $\overline{L} = (L, \varphi)$ be a relatively ample adelic line bundle on X. Then the following equality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) = \inf_{Y\in\Theta_X} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y) = \inf_{Y\in\Theta_X} \frac{h_{\overline{L}}(Y)}{(\dim(Y)+1)\deg_L(Y)}. \tag{4.11}$$

Proof. For any $Y \in \Theta_X$ and any $n \in \mathbb{N}$, let $V_{Y,n}(L)$ be the image of the restriction map

$$H^0(X, L^{\otimes n}) \longrightarrow H^0(Y, L|_Y^{\otimes n}).$$

We equip $V_{Y,n}(L)$ with the quotient norm family ξ_n^Y induced by $\xi_{n\varphi} = (\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega}$ to obtain adelic vector bundle on S.

Claim 4.24. *For any $Y \in \Theta_X$, the following equality holds*

$$\lim_{n\to+\infty} \frac{\widehat{\mu}_{\max}(V_{Y,n}(L), \xi_n^Y)}{n} = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y).$$

Proof. Since L is ample, there exists $N \in \mathbb{N}_{\geqslant 1}$ such that, for any $n \in \mathbb{N}_{\geqslant N}$, the following conditions hold:

(a) $L^{\otimes n}$ is generated by global sections, and the canonical K-morphism

$$j_n : X \longrightarrow \mathbb{P}(H^0(X, L^{\otimes n}))$$

is a closed embedding,

(b) one has $V_{Y,n}(L) = H^0(Y, L|_Y^{\otimes n})$.

For any $n \in \mathbb{N}_{\geqslant N}$, we denote by ψ_n the metric family on L such that $n\psi_n$ coincides with the quotient metric family induced by $\xi_{n\varphi}$. Similarly, we denote by ψ_n^Y the metric family on $L|_Y$ such that $n\psi_n^Y$ coincides with the quotient metric family induced by ξ_n^Y. By definition, ψ_n^Y identifies with the restriction of ψ_n to $L|_Y$. Moreover, we denote by φ^Y the restriction of the metric family φ to $L|_Y$. By [10, Proposition 2.2.22], for any $\omega \in \Omega$, one has $\psi_{n,\omega}^Y \geqslant \varphi_\omega^Y$, and

$$\xi_n^Y = (\|\cdot\|_{n\psi_{n,\omega}^Y})_{\omega\in\Omega}.$$

Since φ is semi-positive, for any $\omega \in \Omega$, one has

$$\lim_{n\to+\infty} d_\omega(\psi_n^Y, \varphi^Y) = 0.$$

Denote by $\xi_{n\varphi^Y}$ the norm family $(\|\cdot\|_{n\varphi_\omega^Y})_{\omega\in\Omega}$. Since

$$d_\omega(\xi_n^Y, \xi_{n\varphi^Y}) \leqslant d_\omega(n\psi_n^Y, n\varphi^Y) = nd_\omega(\psi_n^Y, \varphi^Y),$$

we obtain

$$\lim_{n\to+\infty} \frac{1}{n} d_\omega(\xi_n^Y, \xi_{n\varphi^Y}) = 0. \tag{4.12}$$

By [10, Proposition 2.2.22 (5)], the function

$$(\omega \in \Omega) \longmapsto d_\omega(\xi_n^Y, \xi_{n\varphi^Y})$$

is dominated. Therefore, by Lebesgue's dominated convergence theorem we obtain

$$\lim_{n\to+\infty} \frac{1}{n} d(\xi_n^Y, \xi_{n\varphi^Y}) = \lim_{n\to+\infty} \frac{1}{n} \int_{\omega\in\Omega} d_\omega(\xi_n^Y, \xi_{n\varphi^Y})\, \nu(\mathrm{d}\omega). \tag{4.13}$$

Finally, by [10, Proposition 4.3.31], one has

$$\left| \frac{\widehat{\mu}_{\max}(V_{Y,n}(L), \xi_n^Y)}{n} - \frac{\widehat{\mu}_{\max}(V_{Y,n}(L), \xi_{n\varphi^Y})}{n} \right| \leqslant \frac{1}{n} d(\xi_n^Y, \xi_{n\varphi^Y}).$$

Passing to limit when $n \to +\infty$, we obtain the desired equality. □

Combining Claim 4.24 with [10, Theorem 7.2.4], we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \geqslant \inf_{Y\in\Theta_X} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y).$$

By Proposition 4.21, for any $Y \in \Theta_X$, one has

$$\frac{h_{\overline{L}}(Y)}{(\dim(Y)+1)\deg_L(Y)} \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y).$$

Finally, by Corollary 3.27, for any $Y \in \Theta_X$, one has

$$\frac{h_{\overline{L}}(Y)}{(\dim(Y)+1)\deg_L(Y)} \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

Thus (4.11) is proved.

□

Corollary 4.25. *Let X be an integral projective K-scheme and Θ_X be the set of all integral closed subschemes of X. Let $\overline{L} = (L, \varphi)$ be a relatively ample adelic line bundle on X. Then the following inequality holds:*

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{1}{d+1}\lambda_{\max}^{\mathrm{asy}}(\overline{L}) + \frac{d}{d+1}\inf_{Y\in\Theta_X\setminus\{X\}}\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y). \tag{4.14}$$

In particular, if

$$\lambda_{\max}^{\mathrm{asy}}(\overline{L}) \geqslant \frac{(\overline{L}^{(d+1)})}{(d+1)(L^d)},$$

then the inequality

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \inf_{Y\in\Theta_X\setminus\{X\}}\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y) \tag{4.15}$$

holds.

Proof. The case where $d = 0$ is trivial. In the following, we suppose that $d \geqslant 1$. By replacing $\overline{L}$ by a tensor power, we may assume that

$$V_\bullet(L) = \bigoplus_{n\in\mathbb{N}} H^0(X, L^{\otimes n})$$

is generated as K-algebra by $V_1(L)$. For any $n \in \mathbb{N}$, we let $h^0(L^{\otimes n})$ be the dimension of $H^0(X, L^{\otimes n})$ over K. Let s be a non-zero global section of L and $I_\bullet$ be the homogeneous ideal of $V_\bullet(L)$ generated by s. Then one can find a sequence

$$I_\bullet = I_{0,\bullet} \subsetneq I_{1,\bullet} \subsetneq \ldots \subsetneq I_{r,\bullet} = V_\bullet(L)$$

of homogeneous ideals of $R_\bullet$ and non-zero homogeneous prime ideals $P_{i,\bullet}$, $i \in \{1, \ldots, r\}$, of $V_\bullet(L)$ such that

$$\forall\, i \in \{1, \ldots, r\}, \quad P_{i,\bullet} \cdot I_{i,\bullet} \subset I_{i-1,\bullet}.$$

We denote by Y_i the integral closed subscheme of X defined by the homogeneous ideal $P_{i,\bullet}$.

Consider the following sequence

$$\begin{array}{ccccccc}
V_0(L) & \xrightarrow{\cdot s} & I_{0,1} & \hookrightarrow\cdots\hookrightarrow & I_{i,1} & \hookrightarrow\cdots\hookrightarrow & I_{r,1} = V_1(L) \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \xrightarrow{\cdot s} & I_{0,j} & \hookrightarrow\cdots\hookrightarrow & I_{i,j} & \hookrightarrow\cdots\hookrightarrow & I_{r,j} = V_j(\overline{L}) \\
 & \xrightarrow{\cdot s} & I_{0,j+1} & \hookrightarrow\cdots\hookrightarrow & I_{i,j+1} & \hookrightarrow\cdots\hookrightarrow & I_{r,j+1} = V_{j+1}(L) \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \xrightarrow{\cdot s} & I_{0,n} & \hookrightarrow\cdots\hookrightarrow & I_{i,n} & \hookrightarrow\cdots\hookrightarrow & I_{r,n} = V_n(L)
\end{array}$$

By [10, Proposition 4.3.13], one has

$$\widehat{\deg}(f_*(\overline{L}^{\otimes n})) \geqslant \sum_{j=1}^{n}\sum_{i=1}^{r}\widehat{\deg}(\overline{I_{i,j}/I_{i-1,j}}) + \widehat{\deg}(s)\sum_{k=0}^{n-1} h^0(L^{\otimes k}). \tag{4.16}$$

By [10, Proposition 7.1.6] and (4.13), one has

$$\liminf_{m\to+\infty}\frac{\widehat{\mu}_{\min}(\overline{I_{i,m}/I_{i-1,m}})}{m} \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_{Y_i}). \tag{4.17}$$

Moreover, by the asymptotic Riemann-Roch formula, one has

$$h^0(L^{\otimes k}) = \frac{(L^d)}{d!}k^d + O(k^{d-1}),$$

which leads to

$$\lim_{n\to+\infty} \frac{1}{nh^0(L^{\otimes n})} \sum_{j=0}^{n-1} h^0(L^{\otimes j}) = \frac{1}{d+1}.$$

For any integers n and m such that $1 \leqslant m \leqslant n$, we deduce from (4.16) that

$$\begin{aligned}\widehat{\deg}(f_*(\overline{L}^{\otimes n})) \geqslant & \sum_{j=1}^{m}\sum_{i=1}^{r} \widehat{\deg}(\overline{I_{i,j}/I_{i-1,j}}) \\ & + \min_{i\in\{1,\dots,r\}} \inf_{\ell\in\mathbb{N}_{\geqslant m}} \frac{\widehat{\mu}_{\min}(\overline{I_{i,\ell}/I_{i-1,\ell}})}{\ell} \sum_{j=m+1}^{n} j(h^0(L^{\otimes j}) - h^0(L^{\otimes(j-1)})) \\ & + \widehat{\deg}(s)\sum_{k=0}^{n-1} h^0(L^{\otimes k}).\end{aligned}$$

Dividing the two sides by $nh^0(L^{\otimes n})$ and taking the limit when $n \to +\infty$, we obtain

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{d}{d+1} \min_{i\in\{1,\dots,r\}} \inf_{\ell\in\mathbb{N}_{\geqslant m}} \frac{\widehat{\mu}_{\min}(\overline{I_{i,\ell}/I_{i-1,\ell}})}{\ell} + \frac{1}{d+1}\widehat{\deg}(s).$$

Since m is arbitrary, taking the limit when $m \to +\infty$, by (4.17) we obtain

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{d}{d+1} \min_{i\in\{1,\dots,r\}} \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}|_{Y_i}) + \frac{1}{d+1}\widehat{\deg}(s).$$

By Theorem 4.23, for any $i \in \{1,\dots,r\}$, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}|_{Y_i}) = \inf_{Z\in\Theta_{Y_i}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Z) \geqslant \inf_{Z\in\Theta_X\setminus\{X\}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Z).$$

Since s is arbitrary, we obtain

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{1}{d+1}\widehat{\lambda}_{\max}(f_*(\overline{L})) + \frac{d}{d+1} \inf_{\substack{Y\in\Theta_X\\ Y\neq X}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y).$$

Finally, replacing $\overline{L}$ by $\overline{L}^{\otimes p}$ for $p \in \mathbb{N}_{\geqslant 1}$, we obtain

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{1}{p(d+1)}\widehat{\lambda}_{\max}(f_*(\overline{L}^{\otimes p})) + \frac{d}{d+1} \inf_{\substack{Y\in\Theta_X\\ Y\neq X}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y).$$

Taking the limite when $p \to +\infty$, we obtain the inequality (4.14). □

Proposition 4.26. *Let X be an integral projective scheme over* $\operatorname{Spec} K$ *and $\overline{L}$ be an adelic line bundle on X such that L is big. Then the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \leqslant \sup_{\substack{Y\in\Theta_X\\ Y\neq X}} \inf_{x\in(X\setminus Y)^{(0)}} \frac{h_{\overline{L}}(x)}{[K(x):K]},$$

where $(X\setminus Y)^{(0)}$ denotes the set of closed points of $X\setminus Y$.

Proof. See [10, Proposition 6.4.4]. □

4.8. **Minkowskian adelic line bundles.**

Definition 4.27. Let X be a reduced projective K-scheme and $\overline{L}$ be an adelic line bundle on X. We say that $\overline{L}$ is *Minkowskian* if the inequality below holds:

$$\lambda_{\max}^{\mathrm{asy}}(\overline{L}) \geqslant \widehat{\mu}^{\mathrm{asy}}(\overline{L}) = \frac{h_{\overline{L}}(X)}{(\dim(X)+1)\deg_L(X)}.$$

Moreover, $\overline{L}$ is said to be *strongly Minkowskian* if for any integral closed sub-scheme Y of X, one has $\overline{L}|_Y$ is Minkowskian. Note that the strongly Minkowskian condition is satisfied in the following cases:

(1) S is the adelic curve associated with a number field, and the metrics of $\overline{L}$ over non-Archimedean places are almost everywhere induced by a common integral model defined over the ring of algebraic integers in the number field;
(2) S is the adelic curve associated with a regular projective curve over a field, and the metrics of $\overline{L}$ are induced by an integral model of L over the base curve;
(3) S is the adelic curve of a single copy of the trivial absolute value.

The case (1) comes from the classic Minkowski theory of Euclidean lattices. The case (2) is a consequence of Riemann-Roch theorem on curves. The case (3) follows from [10, Remark 4.3.63].

Corollary 4.28. *Let X be an integral projective K-scheme and $\overline{L}$ be a relatively ample adelic line bundle on X. Assume that $\overline{L}$ is strongly Minkowskian. Then the following inequality holds:*

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \inf_{x\in X^{(0)}} h_{\overline{L}}(x), \tag{4.18}$$

where $X^{(0)}$ denotes the set of closed points of X. Moreover, one has

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) = \inf_{x\in X^{(0)}} \frac{h_{\overline{L}}(x)}{[K(x):K]}. \tag{4.19}$$

Proof. We reason by induction on the dimension d of X. The case where $d=0$ is trivial. Assume that $d \geqslant 1$ and that the result is true for any integral projective K-scheme of dimension $< d$. By Corollary 4.25 one has

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \inf_{Y\in\Theta_X\setminus\{X\}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y) \geqslant \inf_{Y\in\Theta_X\setminus\{X\}} \frac{h_{\overline{L}}(Y)}{(\dim(Y)+1)\deg_L(Y)},$$

where the second inequality comes from Proposition 4.21. For any $Y\in\Theta_X$ such that $Y\neq X$, one has $\dim(Y)\leqslant 1$. Hence the induction hypothesis leads to

$$\frac{h_{\overline{L}}(Y)}{(\dim(Y)+1)\deg_L(Y)} \geqslant \inf_{x\in Y^{(0)}} \frac{h_{\overline{L}}(x)}{[K(x):K]} \geqslant \inf_{x\in X^{(0)}} \frac{h_{\overline{L}}(x)}{[K(x):K]}. \tag{4.20}$$

The inequality (4.18) is thus proved.

By Corollary 3.27, the inequality

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \leqslant \inf_{x\in X^{(0)}} \frac{h_{\overline{L}}(x)}{[K(x):K]}$$

holds. Conversely, by Theorem 4.23 and the inequality (4.20), one has

$$\begin{aligned}\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) &= \inf_{Y\in\Theta_X} \frac{h_{\overline{L}}(Y)}{(\dim(Y)+1)\deg_L(Y)} \\ &\geqslant \inf_{Y\in\Theta_X}\inf_{x\in Y^{(0)}} \frac{h_{\overline{L}}(x)}{[K(x):K]} = \inf_{x\in X^{(0)}} \frac{h_{\overline{L}}(x)}{[K(x):K]}.\end{aligned}$$

□

Lemma 4.29. *Let $\pi : X \to Y$ be a generically finite and surjective morphism of d-dimensional projective integral schemes over K. Let $\overline{M}$ be a relatively ample adelic line bundle on Y. Then we have the following:*

(1) $\widehat{\mu}^{\mathrm{asy}}(\pi^*(\overline{M})) = \widehat{\mu}^{\mathrm{asy}}(\overline{M})$.
(2) $\lambda^{\mathrm{asy}}_{\max}(\pi^*(\overline{M})) \geqslant \lambda^{\mathrm{asy}}_{\max}(\overline{M})$.
(3) *If $\overline{M}$ is Minkowskian, then $\pi^*(\overline{M})$ is also Minkowskian.*

Proof. (1) By the Hilbert-Samual formula,

$$\widehat{\mu}^{\mathrm{asy}}(\overline{M}) = \frac{(\overline{M}^{d+1})_S}{(M^d)(\dim Y+1)} \quad \text{and} \quad \widehat{\mu}^{\mathrm{asy}}(\pi^*(\overline{M})) = \frac{(\pi^*(\overline{M})^{d+1})_S}{(\pi^*(M)^d)(\dim X+1)},$$

and hence the assertion follows because

$$(\pi^*(\overline{M})^{d+1})_S = (\deg\pi)(\overline{M}^{d+1})_S \quad \text{and} \quad (\pi^*(M)^d) = (\deg\pi)(M^d).$$

(2) is obvious because $\widehat{\deg}(s) = \widehat{\deg}(\pi^*(s))$ for $s \in H^0(Y, M) \setminus \{0\}$. Moreover, (3) is a consequence (1) and (2). □

Proposition 4.30. *Let $\pi : X \to Y$ be a finite morphism of projective integral schemes over K. Let $\overline{M}$ be an adelic line bundle on Y such that M is ample and $\overline{M}$ is semi-positive. If $\overline{M}$ is strongly Minkowskian, then $\pi^*(\overline{M})$ is also strongly Minkowskian.*

Proof. Let Z be a subvariety of X. Then $\pi|_Z : Z \to \pi(Z)$ is a finite and surjective morphism, and hence, by Lemma 4.29, $\pi^*(\overline{M})\big|_Z = (\pi|_Z)^*(\overline{M}\big|_{\pi(Z)})$ is Minkowskian, as required. □

Remark 4.31. Let L be a very ample line bundle on X. Then there exist a finite and surjective morphism $\pi : X \to \mathbb{P}^d_K$ such that $\pi^*(\mathcal{O}_{\mathbb{P}^d_K}(1)) \simeq L$. It is not difficult to give a semipositive metric family ψ on $\mathcal{O}_{\mathbb{P}^d_K}(1)$ such that $(\mathcal{O}_{\mathbb{P}^d_K}(1), \psi)$ is Minkowskian, so that, by Lemma 4.29, $(L, \pi^*(\psi))$ is Minkowskian.

4.9. **Successive minima.** Let X be a reduce projective scheme over $\operatorname{Spec} K$ and $\overline{L}$ be a relatively ample adelic line bundle on X. For any $i \in \{1, \dots, d+1\}$, let

$$e_i(\overline{L}) = \sup_{\substack{Y\subseteq X \text{ closed}\\ \operatorname{codim}(Y)\geqslant i}} \inf_{\substack{Z\in\Theta_X\\ Z\not\subseteq Y}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Z).$$

By definition, the following inequalities hold:

$$e_1(\overline{L}) \geqslant \ldots \geqslant e_{d+1}(\overline{L}).$$

Moreover, by Theorem 4.23, one has

$$e_{d+1}(\overline{L}) = \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

Proposition 4.32. *Assume that the scheme X is integral. For any relatively ample adelic line bundle $\overline{L}$ on X, the equality $e_1(\overline{L}) = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ holds.*

Proof. If Y is a closed subscheme of codimension 1 of X, then $X \not\subseteq Y$. Therefore, the inequality $e_1(\overline{L}) \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ holds. In the following, we show the converse inequality. Let t be a real number such that $t > e_1(\overline{L})$. By definition, there exists a family $(Z_i)_{i\in I}$ of integral closed subschemes of Y such that $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_{Z_i}) \leqslant t$ for any $i \in I$ and that the generic points of Z_i form a Zariski dense family in X.

Let m be a positive integer and E_m be a vector subspace of $H^0(X, L^{\otimes m})$ such that

$$\widehat{\mu}_{\min}(\overline{E}_m) = \widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes m})). \tag{4.21}$$

For any positive integer n, let $F_{m,n}$ be the image of $E_m^{\otimes n}$ by the multiplication map

$$H^0(X, L^{\otimes m})^{\otimes n} \longrightarrow H^0(X, L^{\otimes mn}).$$

By [10, Proposition 4.3.31 and Corollary 5.6.2] (see also [12, Remark C.3]), one has

$$\widehat{\mu}_{\min}(\overline{F}_{m,n}) \geqslant n\Big(\widehat{\mu}_{\min}(\overline{E}_m) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E_m))\Big). \tag{4.22}$$

Moreover, there exists $i \in I$ such that the generic point of Z_i does not belong to the base locus of E_m (namely the closed subscheme of X defined by the ideal sheaf $\mathrm{Im}(E_m \otimes L^{\vee\otimes m} \to \mathcal{O}_X)$). Therefore the image of $F_{m,n}$ by the restriction map

$$H^0(X, L^{\otimes mn}) \longrightarrow H^0(Z_i, L^{\otimes mn}|_{Z_i})$$

is non-zero. By [10, Proposition 4.3.31], one has

$$\widehat{\mu}_{\min}(\overline{F}_{m,n}) \leqslant \widehat{\mu}_{\max}((f|_{Z_i})_*(\overline{L}|_{Z_i}^{\otimes mn})).$$

Combining this inequality with (4.21) and (4.22), we obtain

$$\frac{1}{m}\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes m})) \leqslant \frac{1}{mn}\widehat{\mu}_{\max}((f|_{Z_i})_*(\overline{L}|_{Z_i}^{\otimes mn})) + \frac{3}{2m}\nu(\Omega_\infty)\ln(\dim_K(E_m)).$$

Taking the limit when $n \to +\infty$, we obtain

$$\frac{1}{m}\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes m})) \leqslant t + \frac{3}{2m}\nu(\Omega_\infty)\ln(\dim_K(E_m)).$$

Taking the limit when $m \to +\infty$, we obtain $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \leqslant t$. Since $t > e_1(\overline{L})$ is arbitrary, we get $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \leqslant e_1(\overline{L})$, as required. □

Remark 4.33. Let $\overline{L}$ be a relatively ample adelic line bundle on X. For any $t \in \mathbb{R}$ and any positive integer n, we let $V_n^t(\overline{L})$ be the vector subspace of $H^0(X, L^{\otimes n})$ generated by non-zero vector subspaces of minimal slope $\geqslant nt$ and $r_n(t)$ be the dimension of the base locus of $V_n^t(\overline{L})$. For $t \in \mathbb{R}$, let

$$r(t) = \liminf_{n\to+\infty} r_n(t).$$

By using the method used in the proof of Proposition 4.32, we can show that, for any $i \in \{1, \dots, d+1\}$

$$\sup\{t \in \mathbb{R} \mid r(t) \leqslant i\} \leqslant e_i(\overline{L}).$$

It is a natural question to ask if the equality holds. Moreover, we expect that the following inequality is true:

$$(d+1)\widehat{\mu}^{\mathrm{asy}}(\overline{L}) \geqslant \sum_{i=1}^{d+1} e_i(\overline{L}). \tag{4.23}$$

For any $i \in \{1, \dots, d+1\}$, one has

$$e_i(\overline{L}) = \sup_{\substack{Y \subseteq X \text{ closed} \\ \operatorname{codim}(Y) \geqslant i}} \inf_{\substack{Z \in \Theta_X \\ Z \not\subseteq Y}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Z) \leqslant \sup_{\substack{Y \subseteq X \text{ closed} \\ \operatorname{codim}(Y) \geqslant i}} \inf_{x \in (X \setminus Y)^{(0)}} h_{\overline{L}}(x),$$

where $(X \setminus Y)^{(0)}$ denotes the set of closed points of X outside of Y. In the case where S is the adelic curve consisting of places of a number field, by [1, Theorem 1.5], for any integral closed subscheme Z of X, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Z) = \sup_{\substack{W \in \Theta_Z \\ W \neq Z}} \inf_{x \in (Z \setminus W)^{(0)}} h_{\overline{L}}(x).$$

If Z is not contained in Y, then

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Z) \geqslant \inf_{x \in (Z \setminus Y)^{(0)}} h_{\overline{L}}(x) \geqslant \inf_{x \in (X \setminus Y)^{(0)}} h_{\overline{L}}(x).$$

Therefore, in this case $e_i(\overline{L})$ identifies with the i-th minimum of the height function $h_{\overline{L}}$ in the sense of Zhang. In particular, the inequality follows from [22, Theorem 5.2].

5. Positivity conditions for adelic line bundles

Let K be a perfect field and $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve whose underlying field is K. We assume that, either the field K is countable, or the σ-algebra $\mathcal{A}$ is discrete. We also assume that $\nu(\mathcal{A}) \not\subseteq \{0, \infty\}$, and hence there exists an integrable function ϑ on Ω such that

$$\int_\Omega \vartheta(\omega)\, \nu(\mathrm{d}\omega) > 0.$$

5.1. **Ampleness and nefness.** In this subsection, we let X be a non-empty and reduced projective scheme over $\operatorname{Spec} K$, and let d be the dimension of X.

Definition 5.1. We say that an adelic line bundle $\overline{L}$ on X is *ample* if it is relatively ample and if there exists $\varepsilon > 0$ such that the inequality

$$h_{\overline{L}}(Y) \geqslant \varepsilon \deg_L(Y)(\dim(Y) + 1)$$

holds for any integral closed subscheme Y of X.

Proposition 5.2. *Let $\overline{L}$ be an adelic line bundle which is relatively ample. Then the following statements are equivalent:*

(1) *$\overline{L}$ is ample,*
(2) *$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) > 0$,*
(3) *there exists $\varepsilon > 0$ such that, for any integral closed subscheme Y of X, one has $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y) > \varepsilon$.*

Proof. This is a consequence of Theorem 4.23. □

Proposition 5.3. *If $\overline{L}_0, \dots, \overline{L}_d$ are ample adelic line bundles on X, then the inequality*

$$(\overline{L}_0 \cdots \overline{L}_d)_S > 0$$

holds.

Proof. This is a consequence of Theorem 3.10 and Proposition 5.2. □

Proposition 5.4. *Let $\overline{L}$ be an adelic line bundle which is relatively ample and strongly Minkowskian. Then the following conditions are equivalent:*

(1) *$\overline{L}$ is ample,*
(2) *there exists $\varepsilon > 0$ such that, for any closed point x of X, one has $h_{\overline{L}}(x) > \varepsilon$.*

Proof. This is a consequence of Corollary 4.28. □

Definition 5.5. We say that an adelic line bundle $\overline{L}$ on X is *nef* if there exists an ample adelic line bundle $\overline{A}$ and a positive integer N such that $\overline{L}^{\otimes n} \otimes \overline{A}$ is ample for any $n \in \mathbb{N}_{\geqslant N}$.

Proposition 5.6. *Let $\overline{L}$ be an adelic line bundle on X. The following conditions are equivalent:*

(1) *$\overline{L}$ is nef,*
(2) *$\overline{L}$ is relatively nef and $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \geqslant 0$.*

Proof. Assume that $\overline{L}$ is nef. By definition, it is relatively nef. Let $\overline{A}$ be an ample adelic line bundle and N be a positive integer such that $\overline{L}^{\otimes n} \otimes \overline{A}$ is ample for any $n \in \mathbb{N}_{\geqslant N}$. Then one has $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}) > 0$, which leads to

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) = \lim_{n\to+\infty} \frac{1}{n}\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}) \geqslant 0.$$

Conversely, we assume that $\overline{L}$ is relatively nef and $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \geqslant 0$. Since $\overline{L}$ is relatively nef, there exists a relatively ample line bundle $\overline{A}$ and a positive integer N such that $\overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample for any $n \in \mathbb{N}_{\geqslant N}$. By dilating the metrics of $\overline{A}$, we may assume that $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) > 0$. Then, by Proposition 3.14 we obtain that

$$\forall\, n \in \mathbb{N}_{\geqslant N}, \quad \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n} \otimes \overline{A}) \geqslant n\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) + \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) > 0.$$

Therefore $\overline{L}^{\otimes n} \otimes \overline{A}$ is ample. □

Proposition 5.7. (1) *If $\overline{L}_0, \dots, \overline{L}_d$ are nef adelic line bundles on X, then the inequality $(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant 0$ holds.*

(2) *If $\overline{L}$ is a nef adelic line bundle on X and if $g : Y \to X$ is a projective K-morphism, then the pullback $g^*(\overline{L})$ is nef.*

(3) *If $\overline{L}$ is a nef adelic line bundle on X, for any integral closed subscheme Y of X, one has $h_{\overline{L}}(Y) \geqslant 0$.*

(4) *If $\overline{L}$ is a relatively ample adelic line bundle on X such that $h_{\overline{L}}(Y) \geqslant 0$ for any integral closed subscheme Y of X, then $\overline{L}$ is nef.*

(5) *If $\overline{L}$ is a relatively ample adelic line bundle on X such that $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y) \geqslant 0$ for any integral closed subscheme Y of X, then $\overline{L}$ is nef.*

Proof. The first statement is a consequence of Proposition 3.18 and Proposition 5.6. The second statement follows from Lemma 3.19, Theorem 3.25 and Proposition 5.6. The third statement is a consequence of the first and the second ones. The last two statements are consequences of Theorem 4.23 and Proposition 5.6. □

5.2. **Bigness and pseudo-effectivity.** In this subsection, we let X be an integral projective K-scheme $f : X \to \operatorname{Spec} K$ and let d be its dimension.

Definition 5.8. Let $\overline{L}$ be an adelic line bundle on X. We define the *arithmetic volume* of $\overline{L}$ as

$$\widehat{\mathrm{vol}}(\overline{L}) := \limsup_{n\to+\infty} \frac{\widehat{\deg}_+(f_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}.$$

If $\widehat{\mathrm{vol}}(\overline{L}) > 0$, we say that $\overline{L}$ is *big*. It has been shown in [10, Proposition 6.4.18] that $\overline{L}$ is big if and only if L is big and $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) > 0$.

Proposition 5.9. *An ample adelic line bundle is big.*

Proof. Let $\overline{L}$ be an ample adelic line bundle on X. Then one has $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) > 0$, namely for sufficiently large positive integer n one has $\widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n})) > 0$. By [10, Proposition 4.3.13], for such n one has

$$\widehat{\deg}(f_*(\overline{L}^{\otimes n})) = \widehat{\deg}_+(f_*(\overline{L}^{\otimes n})),$$

which leads to, by Theorem 3.7,

$$\widehat{\mathrm{vol}}(\overline{L}) = (\overline{L}^{d+1})_S > 0,$$

where the inequality comes from Proposition 5.3. Hence $\overline{L}$ is big. □

Remark 5.10. We expect that a variant of the method in the proof of Theorem 4.16 leads to an arithmetic version of Fujita's approximation theorem for big adelic line bundles, which generalizes the results of [8, 20].

Proposition 5.11. *Let $\overline{L}_0, \ldots, \overline{L}_d$ be adelic line bundles on X. Assume that $\overline{L}_0$ is big and $\overline{L}_1, \ldots, \overline{L}_d$ are ample, then*

$$(\overline{L}_0 \cdots \overline{L}_d)_S > 0.$$

Proof. This is a consequence of Theorem 4.18. □

Definition 5.12. Let $\overline{L}$ be an adelic line bundle on X. We say $\overline{L}$ is *pseudo-effective* if there exist a big adelic line bundle $\overline{M}$ on X and a positive integer n_0 such that $\overline{L}^{\otimes n} \otimes \overline{M}$ is big for any $n \in \mathbb{N}_{\geqslant n_0}$.

Proposition 5.13. *Let $\overline{L}$ be an adelic line bundle on X. The following assertions are equivalent:*

(1) *$\overline{L}$ is pseudo-effective,*
(2) *L is pseudo-effective and $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \geqslant 0$.*

Proof. Assume that $\overline{L}$ is pseudo-effective. Let $\overline{M}$ be a big adelic line bundle and n_0 be a positive integer such that $\overline{L}^{\otimes n} \otimes \overline{M}$ is big for any integer $n \geqslant n_0$. In particular, $L^{\otimes n} \otimes M$ is big for any integer $n \geqslant n_0$. Hence L is pseudo-effective. Moreover, for $n \geqslant n_0$, one has $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes n} \otimes \overline{M}) > 0$, which implies that

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) = \lim_{n \to +\infty} \frac{1}{n} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes n} \otimes \overline{M}) \geqslant 0.$$

Conversely, assume that L is pseudo-effective and $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \geqslant 0$. Let $\overline{M}$ be a big adelic line bundle on X. Since L is pseudo-effective, for any positive integer n, $L^{\otimes n} \otimes M$ is big. Moreover, by Proposition 4.7 one has

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes n} \otimes \overline{M}) \geqslant n\, \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \otimes \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{M}) > 0.$$

Hence $\overline{L}^{\otimes n} \otimes \overline{M}$ is big for any $n \in \mathbb{N}$, which shows that $\overline{L}$ is pseudo-effective. □

Proposition 5.14. (1) *Let $\overline{L}_0, \ldots, \overline{L}_d$ be adelic line bundles on X. Assume that $\overline{L}_0$ is pseudo-effective and that $\overline{L}_1, \ldots, \overline{L}_d$ are nef, then the inequality $(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant 0$ holds.*

(2) *If $\overline{L}$ is a pseudo-effective adelic line bundle on X and if $g : Y \to X$ is a surjective and projective morphism, then the pullback $g^*(\overline{L})$ is also pseudo-effective.*
(3) *If $\overline{L}$ is nef, then it is pseudo-effective.*

Proof. The first statement is a consequence of Theorem 4.18; the second one is a consequence of Proposition 4.9.

(3) Since $\overline{L}$ is nef, we obtain that L is nef, and hence is pseudo-effective. Let $\overline{A}$ be an ample adelic line bundle. For any positive integer p, by Proposition 4.7 one has

$$\frac{1}{p} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes p} \otimes \overline{A}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) + \frac{1}{p} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{A}).$$

Taking the limit when $p \to +\infty$, we obtain $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \geqslant 0$. By Proposition 5.13, $\overline{L}$ is pseudo-effective. □

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