Continuous boundary condition at the interface for two coupled fluids

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Abstract

We consider two laminar incompressible flows coupled by the continuous law at a fixed interface Γ_I. We approach the system by one that satisfies a friction Navier law at Γ_I, and we show that when the friction coefficient goes to ∞, the solutions converges to a solution of the initial system. We then write a numerical Schwarz-like coupling algorithm and run 2D-simulations, that yields same convergence result.

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1 Introduction

We consider the two coupled fluids problem with a rigid lid assumption, given by two 3D stokes equations,

$$-\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i, \quad \nabla \cdot \mathbf{u}_i = 0, \tag{1.1}$$

$$\mathbf{u}_{1,h|\Gamma_I} = \mathbf{u}_{2,h|\Gamma_I}, \quad w_{i|\Gamma_I} = \mathbf{u}_i \cdot \mathbf{n}_{i|\Gamma_I} = 0, \tag{1.2}$$

$$\mathbf{u}_{1|\Gamma_1} = \mathbf{u}_{2|\Gamma_2} = 0, \tag{1.3}$$

for $i = 1, 2$, where the velocities $(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1(\mathbf{x}_h, z_1), \mathbf{u}_2(\mathbf{x}_h, z_2))$ are decomposed as $\mathbf{u}_i = (\mathbf{u}_{i,h}, w)$, $\mathbf{u}_{i,h} = (u_{i,x}, u_{i,y})$. Moreover, $\mathbf{x}_h \in \mathbb{T}_2$, where $\mathbb{T}_2$ is the two dimensional torus, which means that we consider horizontal periodic boundary conditions. The interface Γ_I is given by $\Gamma_I = \{(\mathbf{x}_h, 0), \mathbf{x}_h \in \mathbb{T}_2\}$, the boundaries Γ_i are given by $\Gamma_1 = \{(\mathbf{x}_h, z_1^+), \mathbf{x}_h \in \mathbb{T}_2\}$, $\Gamma_2 = \{(\mathbf{x}_h, z_2^-), \mathbf{x}_h \in \mathbb{T}_2\}$, $z_1 \in J_1 = [0, z_1^+]$, $z_2 \in J_2 = [z_2^-, 0]$, where $z_1^+ > 0$ and $z_2^- < 0$. The coefficient $\nu_i > 0$ is the viscosity of the fluid i, p_i its pressure.
The main characteristic of this problem is the continuity boundary condition (1.2), which is natural and physical [2], and usually considered for free interfaces [6]. Notice that the rigid lid assumption we consider is reasonable for laminar coupled flows, as well as for large scales. In this paper we adress the question of the existence and uniqueness of a weak solution to Problem (1.1)-(1.2)-(1.3), given as the limit of "frictional solutions", for

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which we can write a numerical Schwarz-like algorithm. More specifically, we approach this problem by the following problem

$$-\nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i, \quad \nabla \cdot \mathbf{u}_i = 0, \tag{1.4}$$

$$\nu_i \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i}|_{\Gamma_I} = -\alpha(\mathbf{u}_{i,h} - \mathbf{u}_{j,h}), \quad w_{i|\Gamma_I} = \mathbf{u}_i \cdot \mathbf{n}_{i|\Gamma_I} = 0, \tag{1.5}$$

$$\mathbf{u}_{i|\Gamma_i} = 0, \tag{1.6}$$

$i, j = 1, 2$, in which the continuity condition (1.2) is replaced by the Navier law (1.5) where $i \neq j$. Similar problems have been already studied before, see [3, 4, 5, 8], and the existence and uniqueness of a weak solution is guaranteed. We aim to investigate how Problem (1.4)-(1.5)-(1.6) approaches Problem (1.1)-(1.2)-(1.3) when the friction coefficient α goes to infinity. Such question has already been adressed in [1] for a single fluid, where it is proved that the corresponding solution strongly converges to a solution to the corresponding Stokes (Navier-Stokes) equations with a no slip boundary condition when $\alpha \to \infty$. We show in this paper the convergence in H^1 space type of the solution of (1.4)-(1.5)-(1.6) to a solution of (1.1)-(1.2)-(1.3) (see Theorem 2.1).
As we shall see, numerical simulations are easily carried out by (1.4)-(1.5)-(1.6) thanks to a Schwarz-like coupling algorithm, that does not work for (1.1)-(1.2)-(1.3). This method has already been successfully implemented for coupled problems, see for example [9].
The note is organized as follows. In the first part we set the functional framework and then we prove the convergence result, namely Theorem 2.1. In the second part, we describe our algorithm and show some numerical results in the 2D case. In particular we check the numerical convergence of the algorithm.

2 Convergence analysis

2.1 Energy balance

This section is devoted to the derivation of the main a priori estimate, which is standard. Let $(\mathbf{u}_1, \mathbf{u}_2)$ be any enough smooth solution to Problem (1.4)-(1.5)-(1.6). Taking the scalar product of equation $(1.4)_i$ by $\mathbf{u}_i$ in integrating over $\mathbb{T}_2 \times J_i$ over yields by $(1.6)_i$, because of the periodic boundary conditions in the $x - y$ axes, the incompressibility condition and $\mathbf{u}_i \cdot \mathbf{n}_i = 0$ at Γ_I,

$$\nu_i \int_{\mathbb{T}_2 \times J_i} |\nabla \mathbf{u}_i|^2 - \nu_i \int_{\Gamma_I} \frac{\partial \mathbf{u}_{i,h}}{\partial \mathbf{n}_i} = \int_{\mathbb{T}_2 \times J_i} \mathbf{f}_i \cdot \mathbf{u}_i,$$

giving by (1.5), $\nu_i \int_{\mathbb{T}_2 \times J_i} |\nabla \mathbf{u}_i|^2 + \alpha \int_{\Gamma_I} \mathbf{u}_{i,h} \cdot (\mathbf{u}_{i,h} - \mathbf{u}_{j,h}) = \int_{\mathbb{T}_2 \times J_i} \mathbf{f}_i \cdot \mathbf{u}_i$. Summing up the two equalities yields the following energy balance,

$$\nu_1 \int_{\mathbb{T}_2 \times J_1} |\nabla \mathbf{u}_1|^2 + \nu_2 \int_{\mathbb{T}_2 \times J_2} |\nabla \mathbf{u}_2|^2 + \alpha \int_{\Gamma_I} |\mathbf{u}_{1,h} - \mathbf{u}_{2,h}|^2 = \int_{\mathbb{T}_2 \times J_1} \mathbf{f}_1 \cdot \mathbf{u}_1 + \int_{\mathbb{T}_2 \times J_2} \mathbf{f}_2 \cdot \mathbf{u}_2. \tag{2.1}$$

2.2 Functions spaces, variational formulation

Let $\mathcal{W}_i = \{\mathbf{u} \in C^\infty(\mathbb{T}_2 \times J_i), \quad \mathbf{u}_{|\Gamma_i} = 0, \quad \mathbf{u} \cdot \mathbf{n}_i|_{\Gamma_I} = 0, \quad \nabla \cdot \mathbf{u}_i = 0\}$, equipped with $||\mathbf{u}||_{i,1} = ||\nabla \mathbf{u}||_{L^2(\mathbb{T}_2 \times J_i)}$ which is indeed a norm due to the condition $\mathbf{u}_{|\Gamma_i} = 0$. Let W_i denotes the completion of $\mathcal{W}_i$ with respect to this norm,

$$W = W_1 \times W_2, \quad W_0 = \{(\mathbf{u}_1, \mathbf{u}_2) \in W, \ \mathbf{u}_{1,h|\Gamma_I} = \mathbf{u}_{2,h|\Gamma_I} \, a.e. \text{ in } \Gamma_I\}. \tag{2.2}$$

We equip W with the scalar product, for any $\mathbf{U}=(\mathbf{u}_1,\mathbf{u}_2),\mathbf{V}=(\mathbf{v}_1,\mathbf{v}_2)\in W$,

$$\Lambda(\mathbf{U},\mathbf{V})=\nu_1\int_{\mathbb{T}_2\times J_1}\nabla\mathbf{u}_1\cdot\nabla\mathbf{v}_1+\nu_2\int_{\mathbb{T}_2\times J_2}\nabla\mathbf{u}_2\cdot\nabla\mathbf{v}_2. \tag{2.3}$$

The space W_0 is the kernel of the form $L:(\mathbf{u}_1,\mathbf{u}_2)\to\mathbf{u}_{1,h|\Gamma_I}-\mathbf{u}_{2,h|\Gamma_I}$, which is continuous by the trace theorem. Therefore W_0 is a closed hyperplane of W. Let P denotes the orthogonal projection over W_0, and $\mathbf{\Phi}=(\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)$ a unit orthogonal vector to W_0, so that $W_0^\perp=\text{vect}\,\mathbf{\Phi}$.

Definition 2.1. *(weak solution) A couple $\mathbf{U}=(\mathbf{u}_1,\mathbf{u}_2)\in W$ is a weak solution to Problem (1.4)-(1.5)-(1.6) when $\forall\,\mathbf{V}=(\mathbf{v}_1,\mathbf{v}_2)\in W$,*

$$\Lambda(\mathbf{U},\mathbf{V})+\alpha\int_{\Gamma_I}(\mathbf{u}_{1,h}-\mathbf{u}_{2,h})\cdot(\mathbf{v}_{1,h}-\mathbf{v}_{2,h})=\int_{\mathbb{T}_2\times J_1}\mathbf{f}_1\cdot\mathbf{v}_1+\int_{\mathbb{T}_2\times J_2}\mathbf{f}_2\cdot\mathbf{v}_2=(\mathbf{F},\mathbf{V}). \tag{2.4}$$

Throughout the rest of the paper, we assume that $\mathbf{f}_i\in L^2(\mathbb{T}_2\times J_i)$, $i=1,2$. The existence and the uniqueness of a weak solution to Problem (1.4)-(1.5)-(1.6) that satisfies the energy balance (2.1) is straightforward by the Lax-Milgram Theorem for any given $\alpha>0$. Notice that work remains to be done about the pressures, by a suitable adaptation of a De Rham like theorem in this framework, which is an open problem.

2.3 Convergence

Let $\mathbf{U}_\alpha=(\mathbf{u}_1^\alpha,\mathbf{u}_2^\alpha)\in W$ be the solution of (1.4)-(1.5)-(1.6). We study in this section the convergence of the familly $(\mathbf{U}_\alpha)_{\alpha>0}$ when $\alpha\to\infty$, proving the following result.

Theorem 2.1. *The familly $(\mathbf{U}_\alpha)_{\alpha>0}$ strongly converges in W to a weak solution $\mathbf{U}=(\mathbf{u}_1,\mathbf{u}_2)\in W_0$ of Problem (1.1)-(1.2)-(1.3) when $\alpha\to\infty$, in the sense:*

$$\forall\,\mathbf{V}=(\mathbf{v}_1,\mathbf{v}_2)\in W_0,\quad \Lambda(\mathbf{U},\mathbf{V})=(\mathbf{F},\mathbf{V}). \tag{2.5}$$

Moreover, the solution of (2.5) is unique.

Proof. Let $\mathbf{U}^\alpha=(\mathbf{u}_1^\alpha,\mathbf{u}_2^\alpha)\in W=W_1\times W_2$ be the solution of (S_1,S_2). We first show that the familly $(\mathbf{U}^\alpha)_{\alpha>0}$ is bounded in W. We have, by (2.1),

$$\|\mathbf{U}^\alpha\|_W^2+\alpha\int_{\Gamma_I}|\mathbf{u}_{1,h}^\alpha-\mathbf{u}_{2,h}^\alpha|^2=\int_{\mathbb{T}_2\times J_1}\mathbf{f}_1\cdot\mathbf{u}_1^\alpha+\int_{\mathbb{T}_2\times J_2}\mathbf{f}_2\cdot\mathbf{u}_2^\alpha, \tag{2.6}$$

which yields

$$\|\mathbf{U}^\alpha\|_W^2\le\int_{\mathbb{T}_2\times J_1}\mathbf{f}_1\cdot\mathbf{u}_1^\alpha+\int_{\mathbb{T}_2\times J_2}\mathbf{f}_2\cdot\mathbf{u}_2^\alpha. \tag{2.7}$$

We deduce from Poincaré and Cauchy-Schwarz inequalities that $(\mathbf{U}^\alpha)_{\alpha>0}$ is indeed bounded in W. Therefore, we can extract a subsequence $(\mathbf{U}^{\alpha_n})_{n\in\mathbf{N}}$ ($\alpha_n\to\infty$ as $n\to\infty$) which converges weakly in W to some $U\in W$. Moreover, by the trace theorem and usual Sobolev compactness results, the corresponding traces are strongly convergent in $L^2(\Gamma_I)$. As by (2.6) $\lim_{n\to\infty}tr(\mathbf{u}_{1,h}^{\alpha_n}-\mathbf{u}_{2,h}^{\alpha_n})=0$ in $L^2(\Gamma_I)$, then $U\in W_0$. Finally, take $\mathbf{V}\in W_0$ in (2.4) as test, so that the boundary term vanishes. By passing to the limit in this case when $\alpha\to\infty$, we obtain that U is a weak solution to (1.1)-(1.2)-(1.3). Uniqueness is straightforward, which in addition garanties that the entire familly does converge to $\mathbf{U}$.

It remains to show the strong convergence. Let $\lambda_\alpha \in \mathbb{R}$, be such that $\mathbf{U}^\alpha = P\mathbf{U}^\alpha + \lambda_\alpha \mathbf{\Phi}$ ($\mathbf{\Phi}$ being given in section 2.2). We first show the strong convergence of $(P\mathbf{U}^\alpha)_{\alpha>0}$ to $\mathbf{U}$ by taking $P\mathbf{U}^\alpha$ as a test in (2.4) which gives, by using the orthogonal decomposition of $\mathbf{U}^\alpha$,

$$\Lambda(\mathbf{U}^\alpha, P\mathbf{U}^\alpha) = \|P\mathbf{U}^\alpha\|_W^2 = (\mathbf{F}, P\mathbf{U}^\alpha), \tag{2.8}$$

since the boundary term on Γ_I equals to zero by orthogonality. Therefore $(P\mathbf{U}^\alpha)_{\alpha>0}$ is bounded in W_0, then converges weakly -*up to a subsequence* (keeping the same notation)- to a limit $\mathbf{W}$, strongly in $L^2((\mathbb{T}_2 \times J_1) \times (\mathbb{T}_2 \times J_2))$. Taking $\mathbf{V} \in W_0$ in (2.4) as test, noting that in this case $\Lambda(\mathbf{U}^\alpha, \mathbf{V}) = \Lambda(P\mathbf{U}^\alpha, \mathbf{V})$, and passing to the limit when $\alpha \to \infty$, we see that $\mathbf{W}$ is solution of the problem (1.1)-(1.2)-(1.3), hence $\mathbf{W} = \mathbf{U}$ by uniqueness, and the entire sequence converges. Therefore, passing to the limit in (2.8) yields

$$\lim_{\alpha\to\infty} ||P\mathbf{U}^\alpha||_W^2 = (\mathbf{F}, \mathbf{U}) = \Lambda(\mathbf{U}, \mathbf{U}) = ||\mathbf{U}||_W^2,$$

which, together with the weak convergence, ensures the strong convergence as claimed. To conclude, it remains to prove that $\lim_{\alpha\to\infty} \lambda_\alpha = 0$. By the energy balance (2.6), we have

$$\limsup_{\alpha\to\infty} ||\mathbf{U}^\alpha||_W^2 + \limsup_{\alpha\to\infty} \alpha \int_{\Gamma_I} |\mathbf{u}_{1,h}^\alpha - \mathbf{u}_{2,h}^\alpha|^2 = (\mathbf{F}, \mathbf{U}). \tag{2.9}$$

However, we have $\int_{\Gamma_I} |\mathbf{u}_{1,h}^\alpha - \mathbf{u}_{2,h}^\alpha|^2 = \lambda_\alpha^2 \int_{\Gamma_I} |\boldsymbol{\phi}_{1,h} - \boldsymbol{\phi}_{1,h}|^2$. Therefore, since $(||\mathbf{U}_\alpha||_W^2)_{\alpha>0}$ is bounded, by (2.9) we have $(\alpha\lambda_\alpha^2)_{\alpha>0}$ is bounded, which can happen only if $\lambda_\alpha \to 0$ as $\alpha \to \infty$, concluding the proof. □

3 Numerical simulations

3.1 Algorithm

We solve the problems (1.4)-(1.5)-(1.6) for large values of α, with a coupling Schwarz like algorithm, using the software Freefem++, for solving 2D Stokes problems by the finite element method. Our algorithm is set as follows.
Step 1: We solve the problem on the upper part which gives a first value $\mathbf{u}_1^{\alpha,0}$.

$$-\nu_1 \Delta \mathbf{u}_1^{\alpha,0} + \nabla P_1^{(0)} = \mathbf{f}_1, \quad \nabla \cdot \mathbf{u}_1^{\alpha,0} = 0, \tag{3.1}$$

$$\nu_1 \frac{\partial \mathbf{u}_{1,h}^{\alpha,0}}{\partial \mathbf{n}_1}|_{\Gamma_I} = -\alpha \mathbf{u}_{1,h}^{\alpha,0}, \tag{3.2}$$

$$\mathbf{u}_{1,h}^{\alpha,0}|_{\Gamma_1} = 0, \quad \mathbf{u}_{1,h}^{\alpha,0} \cdot \mathbf{n}_1 = 0. \tag{3.3}$$

This velocity allows us to solve the problem on the lower part, and to calculate the velocities step by step, up and down.
Step 2: We calculate $\mathbf{u}_2^{\alpha,n}$ and $\mathbf{u}_1^{\alpha,n+1}$ by solving

$$-\nu_2 \Delta \mathbf{u}_2^{\alpha,n} + \nabla P_2^{(n)} = \mathbf{f}_2, \quad \nabla \cdot \mathbf{u}_2^{\alpha,n} = 0, \tag{3.4}$$

$$\nu_2 \frac{\partial \mathbf{u}_{2,h}^{\alpha,n}}{\partial \mathbf{n}_2}|_{\Gamma_I} = -\alpha(\mathbf{u}_{2,h}^{\alpha,n} - \mathbf{u}_{1,h}^{\alpha,n}), \tag{3.5}$$

$$\mathbf{u}_{2,h}^{\alpha,n}|_{\Gamma_2} = 0, \quad \mathbf{u}_{2,h}^{\alpha,n} \cdot \mathbf{n}_2 = 0. \tag{3.6}$$

and

$$-\nu_1 \Delta \mathbf{u}_1^{\alpha,n+1} + \nabla P_1^{(n+1)} = \mathbf{f}_1, \quad \nabla \cdot \mathbf{u}_1^{\alpha,n+1} = 0, \tag{3.7}$$

$$\nu_1 \frac{\partial \mathbf{u}_{1,h}^{\alpha,n+1}}{\partial \mathbf{n}_1}|_{\Gamma_I} = -\alpha(\mathbf{u}_{1,h}^{\alpha,n+1} - \mathbf{u}_{2,h}^{\alpha,n}), \tag{3.8}$$

$$\mathbf{u}_{1,h}^{\alpha,n+1}|_{\Gamma_1} = 0, \quad \mathbf{u}_{1,h}^{\alpha,n+1} \cdot \mathbf{n}_1 = 0. \tag{3.9}$$

Note that we are able to prove the stability of this algorithm, and numerical simulation confirm the convergence (see table below). Problem (1.1)-(1.2)-(1.3) cannot be solved in a similar way. Indeed, the interface conditions

$$\mathbf{u}_{2,h}^{(n)}|_{\Gamma_I} = \mathbf{u}_{1,h}^{(n)}|_{\Gamma_I}, \quad \mathbf{u}_{1,h}^{(n+1)}|_{\Gamma_I} = \mathbf{u}_{2,h}^{(n)}|_{\Gamma_I}, \tag{3.10}$$

imply that the sequences $(\mathbf{u}_{1,h}^{(n)})_n$ and $(\mathbf{u}_{2,h}^{(n)})_n$ are constant on the interface Γ_I which doesn't allow any iterations on the coupling algorithm.

3.2 Simulation results

We take $z_1^+ = 50$, $z_2^- = -5$, $L = 100$, $\nu_1 = \nu_2 = 1$ and the source $\mathbf{F} = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ constant, for the simplicity.

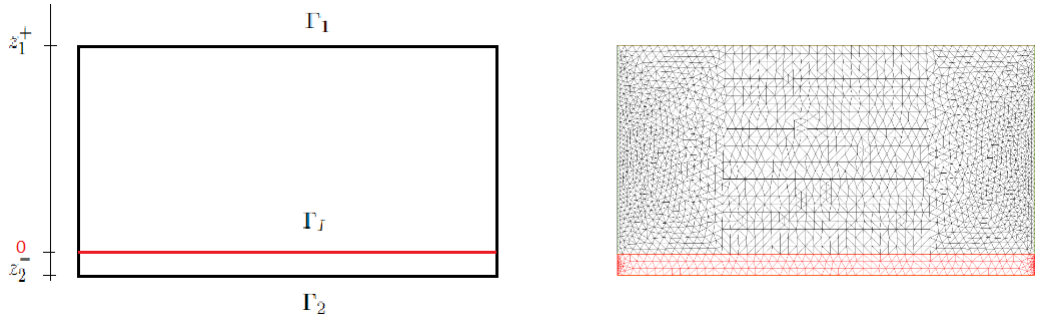


Figure 1: Domain and mesh

α	$\|\mathbf{U}^{\alpha,n+1} - \mathbf{U}^{\alpha,n}\|_{L^2}$
10	15.5
10^2	11.0
10^3	1.33
10^4	$1.32.10^{-1}$
10^5	$1.32.10^{-2}$
10^6	$1.32.10^{-3}$
10^9	$1.32.10^{-6}$

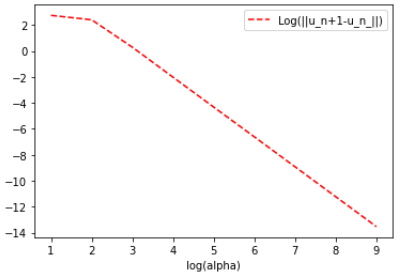


Figure 2: L2 norm of the errors and rate of convergence for n=100

α	n
10	4026
10^2	984
10^3	408
10^4	221
10^5	137
10^6	54
10^9	9

To check the numerical convergence of the method, we study the error term $\|\mathbf{U}^{\alpha,n+1}-\mathbf{U}^{\alpha,n}\|_{L^2}$. On the left, for a given α, we have the first value of n for which $\|\mathbf{U}^{\alpha,n+1}-\mathbf{U}^{\alpha,n}\|_{L^2} < 10^{-3}$. The method is always converging, and the convergence is almost instantaneous for large α ($> 10^6$).

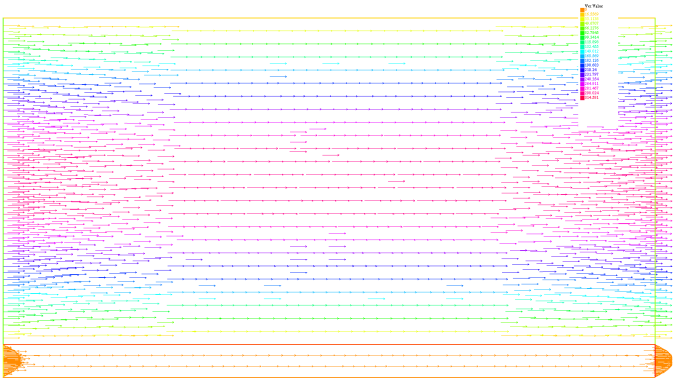


Figure 3: Velocities, $f_1 = (1,-1) = f_2$, $n = 9$, $\alpha = 10^9$

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