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Bipartite instances of INFLUENCE

Eric Duchêne, Nacim Oijid and Aline Parreau

October 2020

Abstract

The game INFLUENCE is a scoring combinatorial game that has been introduced in 2020 by Duchene et al [DGP⁺21]. It is a good representative of Milnor’s universe of scoring games, i.e. games where it is never interesting for a player to miss his turn. New general results are first given for this universe, by transposing the notions of mean and temperature derived from non-scoring combinatorial games. Such results are then applied to INFLUENCE to refine the case of unions of segments started in [DGP⁺21]. The computational complexity of the score of the game is also solved and proved to be PSPACE-complete. We finally focus on some specific cases of INFLUENCE when the graph is bipartite, by giving explicit strategies and bounds on the optimal score on structures like grids, hypercubes or torus.

1 Introduction

INFLUENCE is a two player scoring combinatorial game introduced by Duchene et al in [DGP⁺21]. It is played on a directed graph where each vertex is colored either black or white. Alternately, both players (called Left and Right) take a vertex of their color: if it is Left’s turn, she takes a black vertex, remove it and all its successors. If it is Right’s turn, he takes a white vertex, remove it and all its predecessors. The game ends when the graph is empty. Each player scores the total number of vertices that he/she removed. Naturally, the objective of each player is to maximize his/her score.

The work presented in [DGP⁺21] is the first study of a particular combinatorial game within the recent scoring framework of Larsson et al. [LNdS17]. Indeed, if scoring combinatorial game theory has been introduced by Milnor [Mil53] and Hanner [Han59] in the 50’s, their work was not followed up, partly because of the hardness of solving such games. This is only since the last decade that new results appeared on the topic, in particular with the introduction of general frameworks of resolution, for particular families (also called *universes*) of scoring games.

In [DGP⁺21], the authors show that INFLUENCE belongs to the so-called *Milnor’s universe* of scoring games. Roughly speaking, the major property that derived from this universe is that a player has never interest to miss his/her turn. They also partially solve the game on particular classes of digraphs, called *segments*, that correspond to alternated black and white oriented paths, with arcs from black to white vertices.

In this paper, we extend the case of segments to more general families of bipartite graphs having the same properties. For that purpose, we consider a particular version of INFLUENCE that is played on bipartite digraphs, where each set of the bipartition corresponds to vertices of the same color. In addition, all the arcs are oriented from black to white vertices. According to these constraints, this version amounts to playing the game on an undirected bipartite graph (where black and white vertices form the two sets of the bipartition), and where each player takes a vertex of his/her color, removes it and all its neighbours. To simplify the notation, this particular version of INFLUENCE will be called BIPARTITE INFLUENCE.

BIPARTITE INFLUENCE is somehow very natural to play with, as bipartite bicolored graphs are a model for many 2-player abstract games. This is for example the case for checkerboard grids. Figure 1 illustrates the first two moves of a BIPARTITE INFLUENCE game played on a 5×5 checkerboard grid. After these first two moves, Left has scored 5 points (first move in the center), and Right 4 points (second move with the white bottom left vertex). As discussed in [DGP⁺21], since isolated vertices can only be taken by the player who owns them, they are generally

immediately removed from the graph and added to the score of the corresponding player.

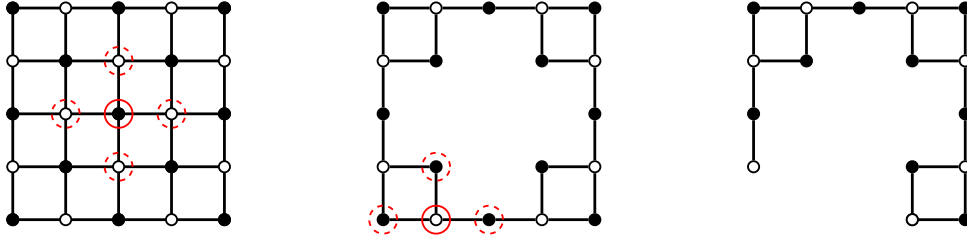


Figure 1: First moves of BIPARTITE INFLUENCE on a 5×5 grid

In addition, this version of the game can be considered as a scoring variation of the well-known combinatorial game NODE-KAYLES (in its partisan version). Indeed, in NODE-KAYLES, each player removes a vertex and its neighborhood. The goal of each player is to be the last player to move. The correlation between NODE-KAYLES and BIPARTITE INFLUENCE is thus immediate.

For the rest of the paper, each instance of BIPARTITE INFLUENCE will be denoted by a bipartite graph $G = (B \cup W, E)$, where B and W correspond respectively to the sets of black and white vertices, and every edge of E is between a vertex of B and W .

The main issue when studying a combinatorial game is the computation of the highest score of each player, by assuming both are playing optimally. This score is well defined as the game is finite and with no chance nor hidden information. The main results of this paper are described below.

In Section 2, we provide the necessary material from scoring combinatorial theory that applies for the game. In particular and for the first time, we provide a formal framework inspired from Milnor and Hanner, and also from combinatorial (non-scoring) game theory [Sie13] for the concept of temperature and mean of a scoring game. In Section 3, we give general basic or known results on BIPARTITE INFLUENCE. In Section 4, we prove that INFLUENCE and even BIPARTITE INFLUENCE are PSPACE-complete. The next section deals with BIPARTITE INFLUENCE on graphs having some symmetry properties that induce a draw strategy. Section 6 is about a refinement of the results of [DGP⁺21] on segments, in particular by using the concepts of mean and temperature. Finally, the game is considered on grids with two and three rows, with an exact value or a bound for the score in almost all cases. A conjecture is given for general grids.

2 Background for scoring combinatorial game theory

If scoring combinatorial game theory has been introduced in the 50's, the results of Milnor and Hanner have mainly been a bootstrap for combinatorial and economic game theory. A couple of recent studies (see [LNdS17] for a survey) have settled general results for scoring game theory. For the reasons detailed in [LNdS17], the one of Larsson et al. in [LNdS15] unifies the different previous frameworks. It has also more advantages, in particular because of its proximity with classical Combinatorial Game Theory. Our results will be presented according to this framework.

2.1 Main definitions and notations

The definitions below are derived from [LNdS15].

A scoring game G is recursively defined as a couple $\langle G^L | G^R \rangle$ where G^L (resp. G^R) are two sets of scoring games. G^L (resp. G^R) is called the set of Left (resp. Right) *options*. In terms of play, the Left (resp. Right) options of G correspond to the games that can be reached by Left (resp. Right) in one move.

If G is a game with no Left option, then we write G^L as \emptyset^s (with s being a real number) to indicate that if it is Left's turn, the game is over and the score of the game is s . The same notation applies for Right. Final positions

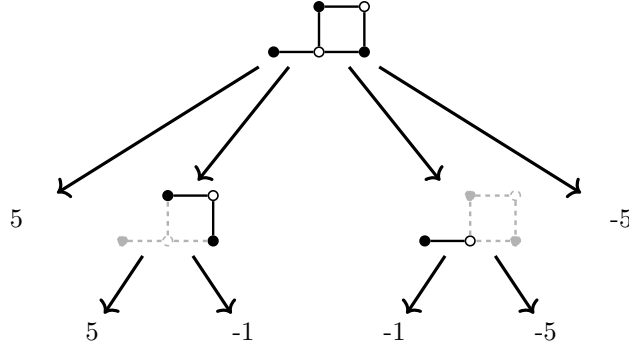


Figure 2: Example of a game tree from a position of BIPARTITE INFLUENCE

correspond to games of the form $\langle \emptyset^s | \emptyset^t \rangle$.

Note that in the rest of the paper, we will consider only *short* games, i.e. games $\langle G^L | G^R \rangle$ where the sets G^L and G^R are finite sets of short games.

In many games having symmetric rules, there exist no position of the form $\langle \emptyset^s | \emptyset^t \rangle$ with $s \neq t$. In particular, this is the case of INFLUENCE. As they correspond to final positions, games of the form $\langle \emptyset^s | \emptyset^s \rangle$ are called *numbers* and will be simply denoted by s .

In scoring combinatorial game theory, the so-called *game tree* is a nice way to represent a game and all the possible moves. Given a game G , the set of nodes of its game tree are all the positions that can be reached when playing from G . The root of the tree is the starting position of the game. Given a node, its left (resp. right) sons are the Left (resp. Right) options. An example of game tree for BIPARTITE INFLUENCE is given in Figure 2. The scores on the leaves correspond to the difference between the number of vertices removed by Left and Right during the play.

If the notion of score is well defined when G^L or G^R is empty, Larsson et al. [LNdS15] extended it recursively to any game G . Inspired by the notions of Left and Right stops in combinatorial game theory [Sie13], they defined in a similar way the notions of Left and Right scores corresponding to the score of the game depending on whether Left or Right starts the game.

Definition 1 (Larsson et al [LNdS15]). *Let $G = \langle G^L | G^R \rangle$ be a scoring game. We call Left score and Right score of G , respectively denoted by $Ls(G)$ and $Rs(G)$, the following functions :*

$$\begin{aligned}
 Ls(\langle \emptyset^s | G^R \rangle) &= Rs(\langle G^L | \emptyset^s \rangle) = s \text{ if } s \text{ is a real number} \\
 Ls(\langle G^L | G^R \rangle) &= \max_{G^l \in G^L} Rs(G^l) \text{ if } G^L \neq \emptyset^s \\
 Rs(\langle G^L | G^R \rangle) &= \min_{G^r \in G^R} Ls(G^r) \text{ if } G^R \neq \emptyset^s
 \end{aligned}$$

These definitions yield a partial order on the games. The convention is that positive scores are in favor of Left, whereas negative scores are for Right. Games with a score equal to zero are draws. On the example of Figure 2, if G is the root, we have $Ls(G) = 5$ and $Rs(G) = -5$. This means that the player who starts wins the game.

The following definition concerns the length of a game.

Definition 2. *Let $G = \langle G^L | G^R \rangle$ be a scoring game. We define the length of G , denoted by $l(G)$ as the size of the longest sequence of moves in G .*

Recursively, $l(G)$ is thus defined as follows:

$$l(\langle \emptyset^s | \emptyset^t \rangle) = 0 \text{ for any real numbers } s \text{ and } t$$

$$l(\langle G^L | G^R \rangle) = 1 + \max_{G' \in G^L \cup G^R} l(G')$$

As for standard combinatorial games, the disjunctive sum operator $+$ applied to scoring games is defined as follows:

Definition 3 (Larsson et al. [LNdS15]). *Given two games G_1 and G_2 , their disjunctive sum, written $G_1 + G_2$ is defined as the following game:*

$$\begin{aligned} G_1 + G_2 &= \langle \emptyset^{\ell_1 + \ell_2} | \emptyset^{r_1 + r_2} \rangle \text{ if } G_1 = \langle \emptyset^{\ell_1} | \emptyset^{r_1} \rangle \text{ and } G_2 = \langle \emptyset^{\ell_2} | \emptyset^{r_2} \rangle \\ &= \langle \emptyset^{\ell_1 + \ell_2} | G_1^R + G_2, G_1 + G_2^R \rangle \text{ if } G_1 = \langle \emptyset^{\ell_1} | G_1^R \rangle \text{ and } G_2 = \langle \emptyset^{\ell_2} | G_2^R \rangle \text{ and at least } G_1^R \text{ or } G_2^R \text{ is not empty} \\ &= \langle G_1^L + G_2, G_1 + G_2^L | \emptyset^{r_1 + r_2} \rangle \text{ if } G_1 = \langle G_1^L | \emptyset^{r_1} \rangle \text{ and } G_2 = \langle G_2^L | \emptyset^{r_2} \rangle \text{ and at least } G_1^L \text{ or } G_2^L \text{ is not empty} \\ &= \langle G_1^L + G_2, G_1 + G_2^L | G_1^R + G_2, G_1 + G_2^R \rangle \text{ otherwise} \end{aligned}$$

For the accuracy of this definition, note that $G_1^L + G_2$ does not exist if G_1^L is empty.

In addition, the *opposite* (or *negative*) of a game G , denoted by $-G$, is defined as the game where the roles of Left and Right are exchanged. In BIPARTITE INFLUENCE, it consists in exchanging the colors of the vertices. By definition of the scores, we have ([DGP⁺21], after Corollary 15) that for all G ,

$$Ls(-G) = -Rs(G).$$

Finally, the notion of equivalence has been defined in order to replace big games by smaller ones in disjunctive sums, without changing the score:

Definition 4 (Milnor [Mil53]). *Two games G_1 and G_2 are equivalent (write $G_1 = G_2$) if for any game G , we have $Ls(G + G_1) = Ls(G + G_2)$ and $Rs(G + G_1) = Rs(G + G_2)$.*

Note that this definition makes sense when G , G_1 and G_2 have similar properties (same universe - see next paragraph - or same ruleset).

2.2 Milnor's Universe

In scoring combinatorial game theory, a *universe* is generally defined as a set of games closed under disjunctive sum, by taking options and negative [LNdS15]. Several universes of scoring games have been considered in the literature (Stewart [Ste11], Ettinger [Ett96], Larsson [LNdS15]). In his paper, although it is not formulated in this way, Milnor introduced the universe of dicotic and nonzugzwang games [Mil53]. Note that in [LNdS15], this universe is also denoted $\mathbb{P}\mathbb{S}$. It is defined as follows:

Definition 5 (Milnor [Mil53]). *A game G is dicotic if both players can move from every nonempty position in the game tree of G .*

Formally, it is equivalent to say that for all $G = \langle G^L | G^R \rangle$, $G^L = \emptyset \Leftrightarrow G^R = \emptyset$.

Definition 6 (Milnor [Mil53]). *A game is nonzugzwang, or with no zugzwang, if for each position G in its game tree, we have $Ls(G) \geq Rs(G)$.*

Roughly speaking, in a nonzugzwang game, a player never has interest in skipping his turn.

The disjunctive sum of games is natural for games that split into several small components during the play. Although the score of a sum of games is in general not equal to the sum of the scores of each game, it can be bounded according to the following theorem for the games that belong to Milnor's universe:

Theorem 7 (Milnor [Mil53]). *Let G and H be two dicotic nonzugzwang games, we have*

$$Rs(G) + Rs(H) \leq Rs(G + H) \leq Ls(G) + Rs(H) \leq Ls(G + H) \leq Ls(G) + Ls(H).$$

In particular, if H is a number s we have $Ls(G + s) = Ls(G) + s$ and $Rs(G + s) = Rs(G) + s$.

Being in Milnor's universe induces a couple of nice properties concerning the equivalence of games. The first result below means that every game G of Milnor's universe has an inverse (i.e. $-G$), leading to the fact that this universe is an abelian group. This is not the case of the other universes studied in the literature. The second result yields a simple test to show that two games are equivalent.

Lemma 8. *For any games G and H that are dicotic nonzugzwang, we have:*

- $Ls(G - G) = Rs(G - G) = 0$;
- $Ls(G - H) = Rs(G - H) = 0$ if and only if G and H are equivalent.

As for standard combinatorial games, a relation order can be set on general scoring games (not necessary from Milnor's universe) [Ett96]. In Milnor's universe, this relation can be defined as follows:

Definition 9. *Let G and H be two games that are dicotic and nonzugzwang. We says that G dominates H , and denote it by $G \geq H$, if $Rs(G - H) \geq 0$.*

If a game has dominated options, then it can be simplified by simply removing them:

Theorem 10. *Let $G = \langle G^L | G^R \rangle$ be a dicotic nonzugzwang game. Assume that there exist two options G_1 and G_2 in G^L such that $G_2^l \geq G_1^l$. Then $G = \langle G^L \setminus G_1^l | G^R \rangle$.*

Similarly, if there exist $G_1^r \geq G_2^r$ in G^R , then $G = \langle G^L | G^R \setminus G_1^r \rangle$.

Proof. By symmetry, we just prove the first part of the theorem. Let $G' = \langle G^L \setminus G_1^l | G^R \rangle$.

We first prove that G' belongs to Milnor's Universe. Since G is dicotic and G has at least two left options, G' is still dicotic. All the positions of G' are positions of G except G' itself. Thus to prove that G' is nonzugzwang, we still need to prove that $Ls(G') \geq Rs(G')$. Since G_2^l dominates G_1^l , we have in particular $Rs(G_2^l) \geq Rs(G_1^l)$. Indeed, by Theorem 7, $Rs(G_2^l) \geq Rs(G_2^l - G_1^l) + Rs(G_1^l)$ and $Rs(G_2^l - G_1^l) \geq 0$. Thus $\max_{G^l \in G^L} (Rs(G^l)) = \max_{G^l \in G^L \setminus G_1^l} (Rs(G^l))$ and $Ls(G) = Ls(G')$. On the other side, $Rs(G') = Rs(G)$ since the Right options of G and G' are the same. Thus $Ls(G') \geq Rs(G')$ and G' is dicotic nonzugzwang.

Therefore, to prove that $G = G'$, one need to prove that $Ls(G - G') = Rs(G - G') = 0$. Since $G - G'$ is dicotic nonzugzwang, we already have $Ls(G - G') \geq Rs(G - G')$. Thus one just need to prove that $Ls(G - G') \leq 0$ and that $Rs(G - G') \geq 0$.

Left can answer to any Right move in $G - G'$ by a move leading to a game $H - H = 0$ since all the Right options in $G - G'$ have a symmetric Left option. Hence Left has a strategy to insure a score 0 when playing second, which means that $Rs(G - G') \geq 0$.

Assume now that Left starts in the game $G - G'$. If Left does not play to $G_1^l - G'$, then Right can answer to a game $H - H = 0$ and insure a score of 0. If Left play to $G_1^l - G'$, Right can answer to $G_1^l - G_2^l$ and insure a non positive score since $Ls(G_1^l - G_2^l) = -Rs(G_2^l - G_1^l) \leq 0$. Thus Right has a strategy in second to insure non-positive score, which means that $Ls(G - G') \leq 0$. □

Application to BIPARTITE INFLUENCE

In [DGP⁺21], the authors proved that INFLUENCE belongs to Milnor's universe. It also naturally holds for BIPARTITE INFLUENCE:

Theorem 11 ([DGP⁺21]). *For any instance G of BIPARTITE INFLUENCE, we have that G is dicotic and nonzugzwang.*

Consequently, all the properties of Milnor's universe detailed above apply to BIPARTITE INFLUENCE. In addition, Example 12 below illustrates equivalent games in BIPARTITE INFLUENCE, as well as dominated options.

Example 12. Denote by S_5 the path of length 5, with the two extremities and the middle vertex colored black, and the two other vertices colored white. Denote also by S_2 the path of length two with one black and one white vertex. See Figure 3.

We will prove that $2S_5 = 2 + S_2$. According to Lemma 8, it is sufficient to prove $Ls(2S_5 - (2 + S_2)) = Rs(2S_5 - (2 + S_2)) = 0$. As $S_2 = -S_2$ and as numbers can be removed from a sum according to Theorem 7, we will prove $Ls(2S_5 + S_2) = Rs(2S_5 + S_2) = 2$. As a first step, when considering the Left options from $2S_5 + S_2$, playing an extremity of a S_5 (i.e. the vertex u in Figure 3) always leads to a position that is dominated by playing the middle of S_5 (i.e. the vertex v). Therefore, when computing $Ls(2S_5 + S_2)$, one only take into account the option v (that yields 5 points) and the move on S_2 (that yields 2 points):

$$\begin{aligned}
Ls(2S_5 + S_2) &= \max(5 + Rs(S_5 + S_2), 2 + Rs(2S_5)) \\
&= \max(5 + \min(-2 + Ls(S_5), -3 + Ls(S_2 + S_2)), 2 - 3 + Ls(S_5 + S_2)) \\
&= \max(5 + \min(3, -3), -1 + \max(5 + Rs(S_2), 2 + Rs(S_5))) \\
&= \max(2, -1 + \max(3, 2 - 3 + Ls(S_2))) \\
&= \max(2, -1 + \max(3, -1 + 2)) \\
&= 2.
\end{aligned}$$

$$\begin{aligned}
Rs(2S_5 + S_2) &= \min(-3 + Ls(S_5 + S_2 + S_2), -2 + Ls(2S_5)) \\
&= \min(-3 + Ls(S_5), -2 + 5 + Rs(S_5)) \\
&= \min(2, 3 - 3 + Ls(S_2)) \\
&= \min(2, 2) \\
&= 2.
\end{aligned}$$

This proves that $2S_5 = 2 + S_2$. Moreover, as $S_2 = -S_2$, we also have that $4S_5 = 2 + S_2 + 2 + S_2 = 4$.

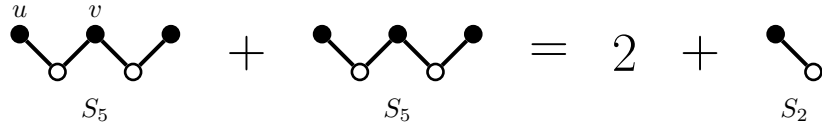


Figure 3: In BIPARTITE INFLUENCE, the game $S_5 + S_5$ is equivalent to the game $2 + S_2$.

2.3 Mean and Temperature

In this section, we develop the so-called concepts of *mean* and *temperature* of a game. These two notions have been introduced by Milnor and Hanner [Mil53, Han59] for scoring games, and then have been formalized in the context of combinatorial game theory by Conway [Con76]. In his book, Siegel [Sie13] gives formal definitions and proofs about them. In this section, we propose to do a similar work in the context of scoring combinatorial game theory. Our results cover the results of Hanner and extend them by following the CGT formalism. In addition, some of them will be used in Section 6 to yield results about BIPARTITE INFLUENCE on segments.

While playing on a (disjunctive) sum of games, general information can be deduced from the information on each term of the sum. As said previously, the Left score of a sum of games is not generally the sum of the Left scores. Nevertheless, Milnor and Hanner have introduced several tools to get some knowledge on sums of games. The most intuitive one is the *mean* of a game, which is very useful if the same game is repeated many times in a sum.

Definition 13 (Hanner [Han59]). *The left (resp. right) mean of a game G , denoted m_L (resp. m_R), is the score obtained in average by playing G a huge number of times if Left (resp. Right) starts, if it exists. More formally:*

$$m_L(G) = \lim_{n \rightarrow \infty} \frac{Ls(nG)}{n}$$

$$m_R(G) = \lim_{n \rightarrow \infty} \frac{Rs(nG)}{n}$$

where nG corresponds to the disjunctive sum $G + \dots + G$ with n terms.

In Milnor's universe, both left and right means always exist:

Theorem 14 (Milnor [Mil53]). *Let G be a dicotic nonzugzwang game. The values $m_L(G)$ and $m_R(G)$ exist and are equal.*

The proof derives from the fact that asymptotically, the first move is made on a game that cannot significantly modify the rest of the sum. According to this result, in Milnor's universe, the mean value is thus unique and will be denoted $m(G)$.

The mean of a game can be interpreted as what a player can expect from a game in a large sum. However, it gives no clue about which game of the sum it is strategic to play in. In order to answer to this question, Hanner [Han59] introduced the so-called parameter $\sigma(G)$. The more $\sigma(G)$ is high, the more the game is said *hot*, meaning a player will want to play in it in priority. This parameter is similar to the *temperature* notion known in standard combinatorial games [Sie13]. Nevertheless, Hanner's definition of σ requires the use of an algorithm and is not very tractable. In what follows, we propose a formalization of this notion of temperature for scoring games, by adapting the formalism used in combinatorial games. Many results from Hanner [Han59] about σ remain true within this formalism.

Definition 15. *Let $G = \langle G^L | G^R \rangle$ be a dicotic nonzugzwang game. Given a real number $t \geq 0$, we define G cooled by t , denoted by G_t , as follows. If G is a real number k , put $G_t = k$ and $t_0 = 0$. Otherwise, let $\tilde{G}_t = \langle G_t^L - t | G_t^R + t \rangle$, $t_0 = \min \{t \geq 0 | Ls(\tilde{G}_t) = Rs(\tilde{G}_t)\}$ and define*

$$G_t = \begin{cases} \tilde{G}_t & \text{if } t \leq t_0 \\ Ls(G_{t_0}) & \text{otherwise} \end{cases}$$

The value t_0 is called the *temperature* of G and is denoted by $\sigma(G)$.

If t is higher or equal to the temperature of G , the game G_t is said to be *frozen* as it behaves as a number. If t is lower than the temperature of G , playing in G_t can be understood as "playing in G but each move costs t to the player".

This definition of temperature matches with the usual definition of temperature in combinatorial games without scores. In both cases, the temperature is defined as a minimum value, which may not exist.

Before proving that the temperature is well-defined, we give an example of the computation of the temperature.

Example 16. *We compute in this example the temperature of the instance S_5 of BIPARTITE INFLUENCE defined in Example 12. The tree of S_5 is given in Figure 4 where the dominated moves corresponding to the extremities have been removed. We have $S_5 = \langle 5 | \langle -1 | 5 \rangle \rangle$.*

Let $t \geq 0$. By definition of a cooled game, we have

$$(\widetilde{S_5})_t = \langle 5_t - t | \langle -1, -5 \rangle_t + t \rangle.$$

Since numbers are already cooled, we have

$$(\widetilde{S_5})_t = \langle 5 - t | \langle -1, -5 \rangle_t + t \rangle.$$

The temperature of $\langle -1 | -5 \rangle$ is 2. Hence, for $t \leq 2$, we have that $\langle -1 | -5 \rangle_t + t = \langle -1 | -5 + 2t \rangle$, and $(\widetilde{S_5})_t = \langle 5 - t | \langle -1 | -5 + 2t \rangle \rangle$ which corresponds to the second case of Figure 4. For $t \geq 2$, the game $\langle -1 | -5 \rangle_t$ is frozen and equals to -3 . Thus $(\widetilde{S_5})_t = \langle 5 - t | -3 + t \rangle$ which corresponds to the rightmost case of Figure 4. When $t = 4$, we have for the first time $Ls((\widetilde{S_5})_t) = Rs((\widetilde{S_5})_t)$ and thus the temperature of (S_5) equals 4. After $t = 4$, $(S_5)_t$ is frozen and equals to 1.

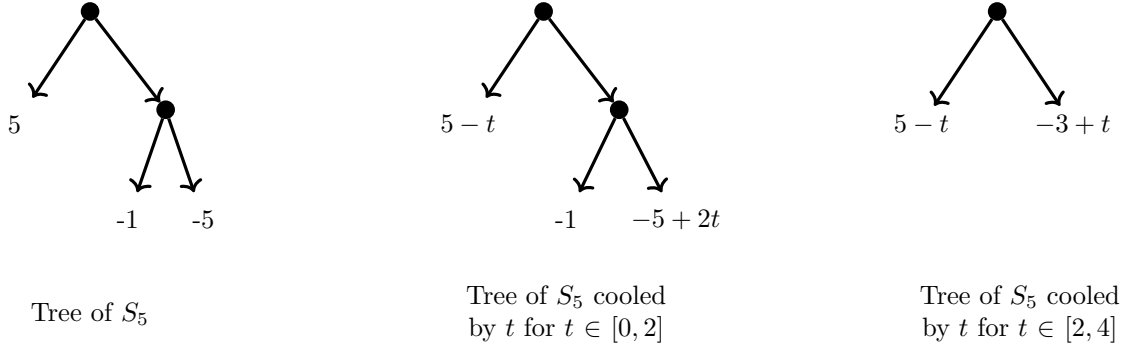


Figure 4: Tree of the game S_5 and the same game cooled by t for different values of t .

Proof of validity of Definition 15. We prove by induction on the length of G that for any $t \geq 0$, G_t exists and the two maps $t \mapsto Rs(G_t)$ and $t \mapsto Ls(G_t)$ are continuous.

If G is a number, its temperature is 0 and is well defined. Hence G_t exists for all $t \geq 0$ and the two score functions are constant.

Otherwise, $G = \langle G^L | G^R \rangle$ with G^L and G^R non empty since the game is dicotic. By induction, the temperature of all the games in G^L and G^R are well defined. Thus the games in G_t^L and G_t^R exist for all $t \geq 0$ and the corresponding score functions are continuous.

Therefore, the functions $t \mapsto Ls(\widetilde{G}_t)$ and $t \mapsto Rs(\widetilde{G}_t)$ exist and are continuous as maximum and minimum of (a finite number of) continuous functions.

Consider now the function $f : t \mapsto Ls(\widetilde{G}_t) - Rs(\widetilde{G}_t)$ defined on \mathbb{R}^+ . The function f is well defined, continuous and

$$f(t) = \max_{G^l \in G^L} (Rs(G_t^l)) - \min_{G^r \in G^R} (Ls(G_t^r)) - 2t.$$

We have $f(0) = Ls(G) - Rs(G) \geq 0$ as G is nonzugzwang. Furthermore, $\lim_{t \rightarrow \infty} f(t) = -\infty$ since the functions $t \mapsto Rs(G_t^l)$ and $t \mapsto Ls(G_t^r)$ are in a finite number and, by definition, constant for t large enough. Since f is continuous, there exists a minimum t_0 such that $f(t_0) = 0$ which guarantees the existence of the temperature and the game G_t .

Moreover, for any $0 \leq t \leq t_0$, $Ls(G_t)$ and $Rs(G_t)$ are continuous as minimum and maximum of continuous functions; for $t \geq t_0$, they are constant so continuous. Finally, $Ls(G_t)$ and $Rs(G_t)$ are continuous in t_0 since $Ls(G_{t_0}) = Rs(G_{t_0})$. \square

With a similar induction, one can prove the following corollary.

Corollary 17. *Let G be dicotic nonzugzwang game. Then for any $t \geq 0$, G_t is a dicotic nonzugzwang game. The function $t \mapsto Ls(G_t)$ and $t \mapsto Ls(\widetilde{G}_t)$ are decreasing functions. Symmetrically, $t \mapsto Rs(G_t)$ and $t \mapsto Rs(\widetilde{G}_t)$ are increasing functions.*

In particular, if $t \geq \sigma(G)$, we have $Ls(\widetilde{G}_t) \leq G_{\sigma(G)} \leq Rs(\widetilde{G}_t)$.

A natural question is about how a cooled game is far from the original game, in terms of Left and Right scores. The following property yields an answer.

Proposition 18. *Let G be a dicotic nonzugzwang game and let $t \geq 0$. We have $0 \leq Ls(G) - Ls(G_t) \leq t$ and $0 \geq Rs(G) - Rs(G_t) \geq -t$*

Proof. By considering $-G$ instead of G , it suffices to prove the result for Ls since $Rs(-G) = -Ls(G)$ and $Rs(-G_t) = -Ls(G_t)$. By Corollary 17, $t \mapsto Ls(G_t)$ is decreasing, which implies that $Ls(G) - Ls(G_t) \geq 0$. It remains to prove that $Ls(G) - Ls(G_t) \leq t$

If G is a number, we have for any $t \geq 0$, $G_t = G$, which implies the result.

Otherwise, consider the case $0 \leq t \leq \sigma(G)$. Let G^l be an optimal move for Left in G , we have $Ls(G) = Rs(G^l)$. Since $Ls(G_t) = \max_{G^l \in G^L} (Rs(G_t^l) - t)$, we have in particular $Ls(G_t) \geq Rs(G_t^l) - t$. Thus, $Ls(G) - Ls(G_t) \leq Rs(G^l) - Rs(G_t^l) + t$. Since $t \mapsto Rs(G_t^l)$ is increasing, $Rs(G^l) - Rs(G_t^l) \leq 0$ leading to $Ls(G) - Ls(G_t) \leq t$.

Consider now the case $t > \sigma(G)$, we have $Ls(G) - Ls(G_t) = Ls(G) - Ls(G_{\sigma(G)})$. By the last inequality above, we have $t > \sigma(G) \geq Ls(G) - Ls(G_t)$. □

As said before, this definition of a cooled game can be understood as a game where moving has a cost. Hence, the temperature is defined as the highest cost that can be set such that the game remains nonzugzwang. Since the temperature has been defined to handle sums of games, we state preliminary results about the sum of two cooled games, starting by the case where one the games is a number.

Lemma 19. *Let G be a dicotic nonzugzwang game. Let s be a number. Then $(G + s)_t = G_t + s$. In particular, $\sigma(G + s) = \sigma(G)$.*

Proof. The result is prove by induction on the length of G . If G is a number, so is $G + s$ and we have $(G + s)_t = G + s = G_t + s$.

Otherwise:

$$\begin{aligned} \widetilde{(G + s)}_t &= \langle (G^L + s)_t - t | (G^R + s)_t + t \rangle \\ &= \langle G_t^L + s - t | G_t^R + s + t \rangle \text{ by induction} \\ &= \langle G_t^L - t | G_t^R + t \rangle + s \\ &= \widetilde{G}_t + s. \end{aligned}$$

In particular, $Ls(\widetilde{(G + s)}_t) = Ls(\widetilde{G}_t) + s$ and $Rs(\widetilde{(G + s)}_t) = Rs(\widetilde{G}_t) + s$. This implies that $\sigma(G + s) = \sigma(G)$ and that $(G + s)_t = G_t + s$. □

Despite the above result, it is not clear in our opinion that the cooling function is a homomorphism for any two scoring games. If this result is true in the CGT context, a direct adaptation in Milnor's universe is not guaranteed, as \widetilde{G}_t can be outside this universe. However, if the cooled games are not equivalent, the result below show that it is true for the scores. In addition, as for CGT, we also show that the temperature is a submaximal function.

Theorem 20. *Let G and H be two dicotic nonzugzwang games. We have $Ls((G + H)_t) = Ls(G_t + H_t)$ and $Rs((G + H)_t) = Rs(G_t + H_t)$. Moreover, $\sigma(G + H) \leq \max(\sigma(G), \sigma(H))$ and if $\sigma(H) < \sigma(G)$, then $\sigma(G + H) = \sigma(G)$. In particular, we have: $(G + H)_{\sigma(G+H)} = G_{\sigma(G)} + H_{\sigma(H)}$*

Proof. We prove the proposition by induction on the length of $G + H$.

If G or H is a number, the proposition is a direct consequence of Lemma 19.

From now on, we assume that both G and H are not numbers, and, without loss of generality, we assume that $\sigma(G) \geq \sigma(H)$. We first prove that $Ls(\widetilde{(G + H)}_t) = Ls(G_t + H_t)$ for any $t \leq \sigma(G)$.

Let $t \geq 0$.

$$\begin{aligned} Ls(\widetilde{(G + H)}_t) &= \max_{G^l \in G^L, H^l \in H^L} \{Rs((G^l + H)_t - t), Rs((G + H^l)_t - t)\} \\ &= \max_{G^l \in G^L, H^l \in H^L} \{Rs((G^l + H)_t), Rs((G + H^l)_t)\} - t \\ &= \max_{G^l \in G^L, H^l \in H^L} \{Rs(G_t^l + H_t), Rs(G_t + H_t^l)\} - t \text{ by induction} \\ &= \max_{G^l \in G^L, H^l \in H^L} \{Rs(G_t^l + H_t - t), Rs(G_t + H_t^l - t)\}. \end{aligned}$$

Assume first that $t \leq \sigma(H)$, then

$$\begin{aligned} G_t + H_t &= \langle G_t^L - t | G_t^R + t \rangle + \langle H_t^L - t | H_t^R + t \rangle \\ Ls(G_t + H_t) &= \max_{G^l \in G^L, H^l \in H^L} \{Rs(G_t^l + H_t - t), Rs(G_t + H_t^l - t)\} \end{aligned}$$

and $Ls(\widetilde{(G+H)}_t) = Ls(G_t + H_t)$.

Assume now that $\sigma(H) \leq t \leq \sigma(G)$. In particular, H_t is a number equal to $H_{\sigma(H)}$. We prove that there is an option in $G_t^L + H_t - t$ that has a Right Score greater all the options in $G_t + H_t^L - t$. Let $G^l \in G^L$ such that $Ls(G_t) = Rs(G_t^l - t)$ (G^l exists since G_t is not a number). Let any $H^l \in H^L$.

Since $t \geq \sigma(H)$ and since $t \mapsto Ls(\widetilde{H}_t)$ is, by Corollary 17, a decreasing function, we have $Rs(H_t^l - t) \leq Ls(\widetilde{H}_t) \leq Ls(\widetilde{H_{\sigma(H)}}) = H_{\sigma(H)}$.

Therefore, we have the following inequalities:

$$\begin{aligned} Rs(G_t + H_t^l - t) &\leq Ls(G_t) + Rs(H_t^l - t) \text{ since all the components are in Milnor's universe (see 7)} \\ &\leq Rs(G_t^l - t) + H_{\sigma(H)} \\ &\leq Rs(G_t^l - t + H_t) \text{ since } H_t = H_{\sigma(H)} \end{aligned}$$

Thus, the Left score of $(\widetilde{G+H})_t$ is computed only on the options $G^L + H_t - t$ and thus:

$$\begin{aligned} Ls(\widetilde{(G+H)}_t) &= \max_{G^l \in G^L} \{Rs(G_t^l + H_t - t)\} \\ &= \max_{G^l \in G^L} \{Rs(G_t^l - t)\} + H_{\sigma(H)} \\ &= Ls(G_t) + H_{\sigma(H)} \text{ since } t \leq \sigma(G) \\ &= Ls(G_t + H_t). \end{aligned}$$

We have proved that if $t \leq \sigma(G)$, then $Ls(\widetilde{(G+H)}_t) = Ls(G_t + H_t)$. In a similar way, one can prove that $Rs(\widetilde{(G+H)}_t) = Rs(G_t + H_t)$.

The two functions $t \mapsto Ls(\widetilde{(G+H)}_t)$ and $t \mapsto Rs(\widetilde{(G+H)}_t)$ are respectively decreasing and increasing functions. They are both equal to $G_{\sigma(G)} + H_{\sigma(H)}$ in $t = \sigma(G)$. Thus the first time they are equal is before (or in) $\sigma(G)$ which means precisely that $\sigma(G+H) \leq \sigma(G)$. Furthermore, we have:

- for $0 \leq t < \sigma(G+H)$, $Ls((G+H)_t) = Ls(\widetilde{(G+H)}_t) = Ls(G_t + H_t)$ and $Rs((G+H)_t) = Rs(\widetilde{(G+H)}_t) = Rs(G_t + H_t)$;
- for $\sigma(G+H) \leq t \leq \sigma(G)$, $t \mapsto Ls(\widetilde{(G+H)}_t)$ and $t \mapsto Rs(\widetilde{(G+H)}_t)$ are constant and both equal to the value in $\sigma(G+H)$ and $\sigma(G)$. In particular,

$$\begin{aligned} Ls((G+H)_t) &= Ls(\widetilde{(G+H)}_t) = Ls(G_t + H_t) = (G+H)_{\sigma(G)+\sigma(H)} = G_{\sigma(G)} + H_{\sigma(H)}, \\ Rs((G+H)_t) &= Rs(\widetilde{(G+H)}_t) = Rs(G_t + H_t) = (G+H)_{\sigma(G)+\sigma(H)} = G_{\sigma(G)} + H_{\sigma(H)}. \end{aligned}$$

If $\sigma(H) < \sigma(G)$, then for any $\sigma(H) < t < \sigma(G)$,

$$Ls(\widetilde{(G+H)}_t) = Ls(G_t) + H_{\sigma(H)} > Rs(G_t) + H_{\sigma(H)} = Rs(\widetilde{(G+H)}_t),$$

and therefore $\sigma(G+H) = \sigma(G)$.

Finally, for $t \leq \sigma(G+H)$, $Ls((G+H)_t) = (G+H)_{\sigma(G)+\sigma(H)}$ and $Ls(G_t + H_t) = G_{\sigma(G)} + H_{\sigma(H)}$. From what precedes, the two values are equal, which concludes the proof. \square

The score after having cooled a game is defined as the score obtained after having increased the cost of the plays. At the temperature, one can also consider it as the score on which Left and Right agree to play the game regardless who starts. In this sense, this definition is close to the mean of the game. As first announced in [Han59] in another format, the following result formalizes the connection between the two notions.

Proposition 21. *Let G be a nonzugzwang dicotic game. We have $G_{\sigma(G)} = m(G)$.*

Proof. We denote by x the number $G_{\sigma(G)}$. We prove that $m(G) = x$ By Theorem 20, $\sigma(nG) \leq \sigma(G)$ and $(nG)_{\sigma(nG)} = nG_{\sigma(G)} = nx$. In particular, for t large enough, $(nG)_t = nx$.

Therefore, from Proposition 18, we have $Ls(nG) - t \leq Ls((nG)_t) = nx = Rs((nG)_t) \leq Rs(nG) + t$

By dividing by n , we have

$$\frac{Ls(nG)}{n} - \frac{t}{n} \leq x \leq \frac{Rs(nG)}{n} + \frac{t}{n}$$

Finally, as $\lim_{n \rightarrow \infty} \frac{Ls(nG)}{n} = \lim_{n \rightarrow \infty} \frac{Rs(nG)}{n} = m(G)$, and $\frac{t}{n} \rightarrow 0$, we have $x = m(G)$. \square

Next theorem expresses in which sense a small temperature means that playing first is not very important.

Theorem 22. *Let G be a nonzugzwang dicotic game, we have $m(G) - \sigma(G) \leq Rs(G) \leq m(G) \leq Ls(G) \leq m(G) + \sigma(G)$.*

Proof. As G is supposed to be dicotic and nonzugzwang, by symmetry, it suffices to prove $m(G) \leq Ls(G) \leq m(G) + \sigma(G)$. Indeed, by considering $-G$ instead of G , we have:

$$\begin{aligned} & m(-G) \leq Ls(-G) \leq m(-G) + \sigma(-G) \\ \Leftrightarrow & -m(G) \leq -Rs(G) \leq -m(G) + \sigma(G) \text{ as } \sigma(G) = \sigma(-G) \text{ and } m(G) = -m(-G) \\ \Leftrightarrow & m(G) \geq Rs(G) \geq m(G) - \sigma(G) \end{aligned}$$

By Corollary 17, the function $t \mapsto Ls(G_t)$ is decreasing. Since $G_0 = G$ and $Ls(G_t) = m(G)$ for t large enough, we have $Ls(G) \geq m(G)$.

By Proposition 18, $Ls(G) - Ls(G_{\sigma(G)}) \leq \sigma(G)$ which is equivalent to $Ls(G) \leq m(G) + \sigma(G)$. \square

Theorem 22 can be applied to sum of games to bound the overall score according to the mean and the temperature of each term of the sum. The following corollary confirms Theorem 1 of [Han59] with our definition of the temperature. We will use this result for BIPARTITE INFLUENCE on sum of segments in Section 6.

Corollary 23. *Let $G = G_1 + \dots + G_n$ be a sum of dicotic nonzugzwang games. Let m_i be the mean of G_i and $\sigma = \max_{1 \leq i \leq n} \sigma(G_i)$. We have*

$$\sum_{i=0}^n m_i - \sigma \leq Rs(G) \leq \sum_{i=0}^n m_i \leq Ls(G) \leq \sum_{i=0}^n m_i + \sigma.$$

Proof. This result is straightforward as we know that $m(G) = \sum_{i=1}^n m(G_i)$ (as a sum of limits) and $\sigma(G) \leq \max_{1 \leq i \leq n} \sigma(G_i)$ (from Theorem 20). \square

3 General results on BIPARTITE INFLUENCE

The current section gives general results about BIPARTITE INFLUENCE that will be useful in the rest of the paper. We first recall a property introduced in [DGP⁺21] about the so-called relevant graphs. In the context of BIPARTITE INFLUENCE, the vertices from a graph G that can be directly removed by playing a vertex v are v itself, its neighbors and the isolated vertices that possibly appeared (with the same color as v). We denote by $\text{Rmv}(G, v)$ this set of removed vertices. Consequently, the score functions can also be expressed as follows:

$$Ls(G) = \max_{x \in B} \{ |\text{Rmv}(G, x)| + Rs(G \setminus \text{Rmv}(G, x)) \} \quad \text{and} \quad Rs(G) = \min_{y \in W} \{ -|\text{Rmv}(G, y)| + Ls(G \setminus \text{Rmv}(G, y)) \}.$$

3.1 Included moves

In INFLUENCE, it has been proved that a move included in another is dominated by it [DGP⁺21]. More formally, we have the following theorem that is a restatement of Corollary 13 in [DGP⁺21].

Theorem 24 ([DGP⁺21]). *Let $G = (B \cup W, E)$ be a bipartite graph and u, v two vertices of the same colour such that $Rmv(G, u) \subseteq Rmv(G, v)$. Then, the option of playing v dominates the option of playing u . Otherwise said, playing v is always at least as good as playing u .*

3.2 Removing and adding vertices

A way to obtain bounds on the score is to consider that some vertices are automatically given to the opponent, in order to simplify the structure of the graph without those vertices. The following result explains how the score may vary when sets of vertices of the same color are removed.

Theorem 25. *Let $G = (B \cup W, E)$ be a bipartite graph. Let $B_0 \subseteq B$ and $W_0 \subseteq W$.*

We have:

$$Ls(G) \leq Ls(G \setminus W_0) + |W_0| \quad (1)$$

$$Rs(G) \geq Rs(G \setminus B_0) - |B_0| \quad (2)$$

$$Ls(G) \geq Ls(G \setminus B_0) - |B_0| \quad (3)$$

$$Rs(G) \leq Rs(G \setminus W_0) + |W_0|. \quad (4)$$

Roughly speaking, it can be explained by the fact that removing a set of white vertices and giving them to Left may prevent Right from doing some good moves but does not prevent Left to apply the same strategy.

Proof. By symmetry, it suffices to prove (1) and (4). We do this by induction on $|V|$. The result is trivially true if W_0 is empty, if $|V| = 1$, or if the first player has no move. Assume it is true for any graph with at most $n \geq 1$ vertices and where the first player has a move.

Let $G = (B \cup W, E)$ be a graph with $n + 1$ vertices and let $W_0 \subset W$ be a set of white vertices. The main idea of the proof is that by applying the optimal strategy existing in G on the graph with the removed vertices, the player scores at least the same number of points.

We first prove (1) : let $v \in B$ be a vertex played by Left in an optimal move in G . Denote by W_1 the white vertices of W_0 removed by playing v . Namely, we have $W_1 = W_0 \cap Rmv(G, v)$, and define W_2 as the other white vertices of W_0 , i.e. $W_2 = W_0 \setminus W_1$.

Consider now a strategy for Left that starts by playing v in $G \setminus W_0$. By definition of Ls , we have :

$$Ls(G \setminus W_0) \geq Rs((G \setminus W_0) \setminus Rmv(G \setminus W_0, v)) + |Rmv(G \setminus W_0, v)|$$

Since $W_1 = W_0 \cap Rmv(G, v) \subset Rmv(G, v)$, we have $W_0 \cup Rmv(G, v) = W_1 \cup W_2 \cup Rmv(G, v) = W_2 \cup Rmv(G, v)$. Therefore, we have:

$$Ls(G \setminus W_0) + |W_0| \geq |Rmv(G \setminus W_0, v)| + Rs((G \setminus W_2) \setminus Rmv(G, v)) + |W_1| + |W_2|$$

Now, by induction we know that $Rs(G \setminus Rmv(G, v)) \leq Rs((G \setminus Rmv(G, v)) \setminus W_2) + |W_2|$. Thus, by applying this equality to the previous one, we get $Ls(G \setminus W_0) + |W_0| \geq |Rmv(G \setminus W_0, v)| + |W_1| + Rs(G \setminus Rmv(G, v))$.

Finally, by the triangular inequality $|Rmv(G \setminus W_0, v)| + |W_1| \leq |Rmv(G \setminus W_0, v) \cup W_1|$, and as $Rmv(G \setminus W_0, v) \cup W_1 = Rmv(G, v)$, we obtain $Ls(G \setminus W_0) + |W_0| \geq |Rmv(G, v)| + Rs(G \setminus Rmv(G, v))$. Finally, as the move v was supposed optimal in G , recall that $Ls(G) = |Rmv(G, v)| + Rs(G \setminus Rmv(G, v))$, yielding to the desired result.

Now, we prove (4) : Let $v \in W$ be a vertex played by Right in an optimal move in $G \setminus W_0$. By definition, we have

$$Rs(G \setminus W_0) = -|Rmv(G \setminus W_0, v)| + Ls(G \setminus W_0 \setminus Rmv(G \setminus W_0, v))$$

Denote by W_1 the white vertices of W_0 taken by playing v , i.e. $W_1 = W_0 \cap Rmv(G, v)$, and by W_2 the other white vertices of W_0 , i.e. $W_2 = W_0 \setminus W_1$.

We have $Ls(G \setminus W_0 \setminus \text{Rmv}(G \setminus W_0, v)) = Ls(G \setminus W_2 \setminus \text{Rmv}(G, v))$.

Now, by induction hypothesis and by applying (1) to $Ls(G \setminus W_2 \setminus \text{Rmv}(G, v))$, we have

$$Ls(G \setminus W_2 \setminus \text{Rmv}(G, v)) \geq Ls(G \setminus \text{Rmv}(G, v)) - |W_2|$$

As $\text{Rmv}(G \setminus W_0, v) \subset \text{Rmv}(G, v)$, we have $-|\text{Rmv}(G \setminus W_0, v)| \geq -|\text{Rmv}(G, v)|$.

Putting together the previous inequalities we get $Rs(G \setminus W_0) \geq -|\text{Rmv}(G, v)| + Ls(G \setminus \text{Rmv}(G, v)) - |W_2|$.

Since $-|\text{Rmv}(G, v)| + Ls(G \setminus \text{Rmv}(G, v))$ is the score obtained by playing v in G , we have $Rs(G \setminus W_0) \geq Rs(G) - |W_2|$.

Finally, as $W_2 \subset W_0$, we obtain $Rs(G \setminus W_0) \geq Rs(G) - |W_0|$. \square

Example 26. The graph G of Figure 5 is the sum of a graph and its negative with only two additional black vertices (v'_7 and v'_8). To evaluate the score of this sum, Left can give these two vertices to Right, and know that on the remaining graph, the score will be 0 (as being the sum of a graph with its negative). Therefore, by applying (2) and (3) of the above theorem, we deduce that $Ls(G) \geq -2$ and $Rs(G) \geq -2$.

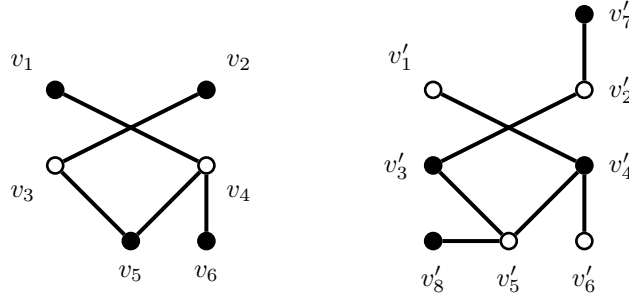


Figure 5: Left, by "giving" the two black vertices v'_7 and v'_8 , can ensure a score of -2 in this graph since the rest of the graph has a (Left and Right) score of 0.

3.3 Twins

Two vertices are called *twins* if they have the same color and the same neighborhood. In BIPARTITE INFLUENCE, twin vertices are necessarily removed together.

Lemma 27. In BIPARTITE INFLUENCE, if a move removes a vertex v of the graph, then this move also removes all the vertices that are twins of v .

Proof. Let $G = (B \cup W, E)$ be a bipartite graph and let a and a' be two twin vertices of G . Without loss of generality, suppose that a and a' are white. Consider a move that removes the vertex a .

If the move is played from a white vertex w , it takes a if and only if it takes all its neighborhood. Thus $N(a) \subseteq N(w)$. But, as $N(a) = N(a')$, we also have $N(a') \subseteq N(w)$. Therefore, a' and a have an empty neighborhood after this move and are won by white.

If the move that removes a is played on a black vertex b , then a must be in the neighborhood of b . Therefore, since a and a' have the same neighborhood, a' is also in the neighborhood of b , which concludes the proof. \square

4 PSPACE-completeness

In this section, we prove that computing the Left score of an instance of BIPARTITE INFLUENCE is PSPACE-complete.

BIPARTITE INFLUENCE

Instance: A bipartite graph $G = (B \cup W, E)$, an integer $k \in \mathbb{N}$

Question: Is it true that $Ls(G) \geq k$?

Theorem 28. BIPARTITE INFLUENCE is PSPACE-complete.

Proof. First, BIPARTITE INFLUENCE is in PSPACE since a game ends after at most $|B \cup W|$ moves. To prove the completeness, we do a reduction from the game POS CNF. An instance of POS CNF is a CNF formula φ over a set of variables $X = \{X_1, \dots, X_n\}$ where all the variables are positive. Alice and Bob alternately choose a variable that has not been chosen yet. Variables chosen by Alice are set to true, those chosen by Bob are set to false. Alice wins if and only if φ is true at the end. The related decision problem is the following one and has been proved to be PSPACE-complete by Schaefer in 1978 [Sch78].

POS CNF

Instance: A set of variable X and a positive CNF formula φ

Question: Does Alice win the game POS CNF played on (X, φ) playing first?

Let (X, φ) be an instance of POS CNF. Without loss of generality, by adding a useless variable, one can assume that $n = |X|$ is even. Denote $\varphi = \bigwedge_{j=1}^m C_j$ and, for $1 \leq j \leq m$, denote $C_j = \bigvee_{i=1}^{n_j} X_i^j$ where n_j is the number of variables in C_j . We construct an instance of BIPARTITE INFLUENCE $G = (B \cup W, E)$ as follows:

- For each clause C_j , $1 \leq j \leq m$, we add a white vertex c_j .
- For each variable X_i , $1 \leq i \leq n$, we add a white and a black vertex x_i^w, x_i^b , and $m + 2n - 1$ black vertices $v_1^i, \dots, v_{m+2n-1}^i$.
- We finally add all the edges (x_i^w, x_i^b) and (x_i^w, v_k^i) for $1 \leq i \leq n$ and $1 \leq k \leq m + 2n$; and the edges (x_i^b, c_j) whenever $X_i \in C_j$.

See Figure 6 for an illustration of the construction for the formula $\varphi = (X_1 \vee X_2) \wedge (X_2 \vee X_3) \wedge (X_3 \vee X_4) \wedge (X_4 \vee X_1)$. The big black vertices represent the $m + 2n - 1 = 11$ black leaves attached to each vertex x_i^w .

Before proving the reduction, we prove that there are optimal strategies for Left or Right in G consisting in playing only vertices x_i^w or x_i^b (for $1 \leq i \leq n$). For $1 \leq i \leq n$, let B_i be the set of $m + 2n - 1$ black vertices v_k^i . We call such set a *bag*. All these vertices are twins and by Lemma 27 will be removed at the same time, together with the vertex x_i^w , scoring at least $2n + m$ points. However, a move can remove at most one bag at a time, and actually any move except the moves on vertices c_j removes a bag. As a consequence, if at some point Right chooses to play on a vertex c_j whereas a bag is still present, Left will manage to take $n/2 + 1$ bags and will remove at least $(\frac{n}{2} + 1)(2n + m) = n^2 + \frac{nm}{2} + 2n + m$ vertices. On the contrary, if Right always takes a bag when one is left, Left can take at most $n/2$ bags, plus maybe the $n + m$ remaining vertices, removing at most $(\frac{n}{2})(2n + m) + n + m = n^2 + \frac{nm}{2} + n + m$ vertices.

Thus, until all the bags are removed, Right must always choose to play on a vertex x_i^w . When the bag B_i is removed (by Left or Right), the only other vertices that can be removed in the same move are x_i^w, x_i^b and maybe some clause vertices, but no vertices of other variables can be removed. Thus, if Left would have play on a vertex v_k^i in a bag, the vertex x_i^b is still present in the graph. Since $\text{Rmv}(G, v_k^i, \subset) \text{Rmv}(G, x_i^b, \cdot)$, by Theorem 24, playing on x_i^b is always better. Finally, we can suppose that both Left and Right plays on vertices x_i^b and x_i^w until all the bags are removed. But then the graph is empty since only clauses variables could remain and they form an independent set.

Note that the final Left score is exactly the difference between the number of vertices c_j removed by Left (by taking one adjacent vertex x_i^b of c_j) and thus isolated by Right (by removing all the black vertices x_i^b adjacent to c_j).

Assume Alice has a winning strategy in POS CNF on (X, φ) . Then Left follows the same strategy than Alice: when Alice chooses variable X_i , Left removes vertex x_i^b . If Right answers vertex x_i^w , we assume that Bob plays variable $X_{i'}$ and so on. At the end, Left will have take one neighbour of each clause vertex and thus will have removed the m clause vertices. At the end, $Ls(G) \geq m$.

Similarly, if Left has a strategy to have $Ls(G) \geq m$, it means that she can choose vertices x_i^b in such a way that she will touch all the clause variables. If Alice follows the same strategy, φ will be true at the end of the game. \square

Remark 29. In BIPARTITE INFLUENCE, the question was to know whether $Ls(G) \geq k$. Since free points can be given to a player by adding isolated vertices in the graph, the problem is still PSPACE-complete with $k = 0$ by adding k white vertices if $k > 0$ or $-k$ black vertices if $k < 0$. Thus, determining which player has a winning strategy is also PSPACE-complete.

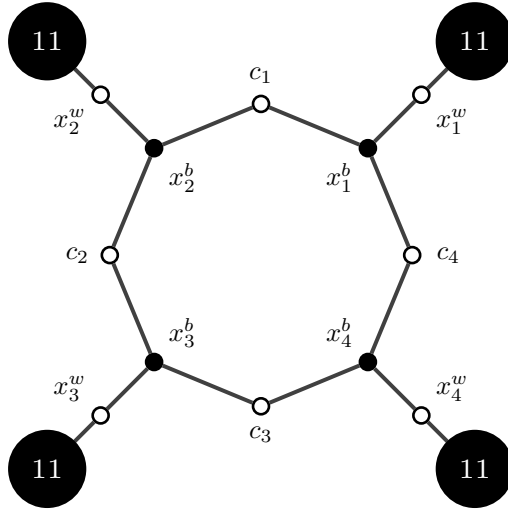


Figure 6: Reduction of $\phi = (X_1 \vee X_2) \wedge (X_2 \vee X_3) \wedge (X_3 \vee X_4) \wedge (X_4 \vee X_1)$ to BIPARTITE INFLUENCE. Big black vertices represent 11 pendant vertices.

Corollary 30. INFLUENCE is PSPACE-complete, even for bipartite graphs.

5 A sufficient condition to have a draw

Identifying *draw* games, i.e. games that are equivalent to 0, is essential in the study of a game in Milnor's universe since they can be removed from any sum. However, as seen in the previous section, determining in general if a game is a draw is hard (since it is already hard to determine if $LS(G) = 0$). In this section, we give a sufficient condition to have a draw when the graph has some symmetries and we apply it to hypercubes, cylinders and torus.

5.1 Sufficient condition

An *automorphism* φ of a graph G is a function from $V(G)$ to $V(G)$ that preserves edges, i.e. if $(u, v) \in E(G)$ if and only if $(\varphi(u), \varphi(v)) \in E(G)$. It is said to be *involutive* if $\varphi(\varphi(u)) = u$ for all $u \in V(G)$. We denote the usual distance (i.e the length of the shortest path) between two vertices u and v by $d(u, v)$. If u and v are not in the same connected component, we let $d(u, v) = \infty$.

Definition 31. Let $G = (B \cup W, E)$ be a bipartite graph. An automorphism φ of G is a *BW-automorphism* if it is involutive, exchanging the colours of the vertices, and if for any $v \in V(G)$, $d(v, \varphi(v)) \geq 3$.

Theorem 32. Let $G = (B \cup W, E)$ be a bipartite graph that has a BW-automorphism, then $G = 0$.

Proof. Let $G = (B \cup W, E)$ a bipartite graph, and let φ be a BW-automorphism of G . To prove that $G = 0$, we just need to prove that $LS(G) = RS(G) = 0$. By symmetry of G and $\varphi(G)$, we actually just need to prove that $LS(G) \leq 0$, that is to prove that the second player can achieve to take as many vertices as the second player. The strategy of the second player will consist in always playing the image by φ of the vertex played by the first player.

To prove that this strategy works, it is sufficient to prove that $\text{Rmv}(G, \varphi(u)) = \varphi(\text{Rmv}(G, u))$ for all $u \in G$ and that $\text{Rmv}(G, \varphi(u)) \cap \text{Rmv}(G, u) = \emptyset$. Indeed, if these two conditions are satisfied, then the second player can always remove the image by φ of the moved played by the first player and will take the same number of vertices. Then after two moves, the graph will still have a BW-automorphism and the second player can repeat his strategy.

Let $u \in V$ and $v \in \text{Rmv}(G, \varphi(u))$. If v is a neighbour of $\varphi(u)$, we have :

$$\begin{aligned} (v, \varphi(u)) \in E &\Leftrightarrow (\varphi(v), \varphi(\varphi(u))) \in E \text{ as } \varphi \text{ respects edges} \\ &\Leftrightarrow (\varphi(v), u) \in E \text{ as } \varphi \text{ is involutive.} \end{aligned}$$

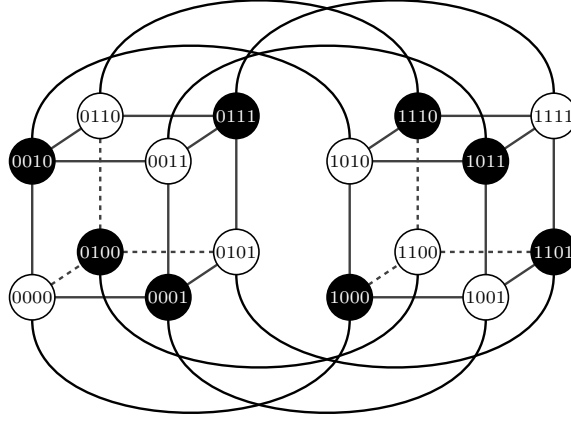


Figure 7: Hypercube H_4

If v is not a neighbour of $\varphi(u)$, v is isolated by playing $\varphi(u)$, i.e. each neighbour w of v is also a neighbour of $\varphi(u)$. As φ preserves edges, the neighbours of $\varphi(v)$ are the images by φ of the neighbours of v . Therefore, they are all removed by playing $\varphi(\varphi(u)) = u$. This proves that $\varphi(v)$ is isolated by playing u , and thus is in $\text{Rmv}(G, u)$.

In both cases, $\varphi(v) \in \text{Rmv}(G, u)$ and thus $v = \varphi(\varphi(v))$ is in $\varphi(\text{Rmv}(G, u))$. This proves $\text{Rmv}(G, \varphi(u)) \subseteq \varphi(\text{Rmv}(G, u))$ for all $u \in V$.

To prove the other direction, we have the following equivalences:

$$\begin{aligned}
& \forall u \in V, \varphi(\text{Rmv}(G, \varphi(u))) \subseteq \varphi(\varphi(\text{Rmv}(G, u))) \\
& \Leftrightarrow \forall u \in V, \varphi(\text{Rmv}(G, \varphi(u))) \subseteq \text{Rmv}(G, u) \text{ as } \varphi \text{ is involutive} \\
& \Leftrightarrow \forall v \in V, \varphi(\text{Rmv}(G, \varphi(\varphi(v)))) \subseteq \text{Rmv}(G, \varphi(v)) \text{ as } \varphi \text{ is a bijection, by taking } v = \varphi(u) \\
& \Leftrightarrow \forall v \in V, \varphi(\text{Rmv}(G, v)) \subseteq \text{Rmv}(G, \varphi(v)) \text{ as } \varphi \text{ is involutive.}
\end{aligned}$$

Finally, we have $\text{Rmv}(G, \varphi(u)) = \varphi(\text{Rmv}(G, u))$ for all $u \in V$.

A move in BIPARTITE INFLUENCE removes vertices at distance at most 2, and the vertices at distance 2 are removed only if they are made isolated. Assume by contradiction there exists $v \in \text{Rmv}(G, \varphi(u)) \cap \text{Rmv}(G, u)$. Then $d(v, u) \leq 2$ and $d(v, \varphi(u)) \leq 2$. Since $d(u, \varphi(u)) \geq 3$, we actually have $d(u, \varphi(u)) = 3$ (since u and $\varphi(u)$ does not have the same colour, and without loss of generality, we can assume that $d(v, u) = 2$ and $d(v, \varphi(u)) = 1$. But then v cannot be isolated when playing u since it will still be connected to $\varphi(u)$. Thus $\text{Rmv}(G, \varphi(u)) \cap \text{Rmv}(G, u) = \emptyset$. \square

5.2 Applications

In this subsection, we apply Theorem 32 to hypercubes, torus and cylinders.

Definition 33 (Hypercube). *We denote by H_n be the n -dimensional hypercube. The set of vertices of H_n is $\{0, 1\}^n$ and two vertices are adjacent if and only if they differ on exactly one digit. We set black vertices to be the vertices with an odd number of nonzero digits. See Figure 7 for a representation of H_4 .*

Proposition 34. *For $n \geq 3$, $H_n = 0$.*

Proof. By Theorem 32, we just need to find a BW-automorphism. Let $n \geq 3$ and

$$\varphi: \begin{array}{ccc} \{0, 1\}^n & \rightarrow & \{0, 1\}^n \\ (\varepsilon_1, \dots, \varepsilon_n) & \mapsto & (1 - \varepsilon_1, 1 - \varepsilon_2, 1 - \varepsilon_3, \varepsilon_4, \dots, \varepsilon_n) \end{array}$$

The automorphism φ is involutive and $d(u, \varphi(u)) = 3$ for any vertex u , which implies that φ exchanges black and white vertices and. Thus φ is a BW-automorphism and $H_n = 0$. \square

Definition 35 (Cylinder). Let n, m be two positive integers with n even. We denote $C_{n,m}$ the cylinder defined as follows. The $n \times m$ vertices of $C_{n,m}$ are denoted by $v_{i,j}$ with $1 \leq i \leq n$ and $1 \leq j \leq m$. Vertices $v_{i,j}$ and $v_{i',j'}$ are adjacent if and only if $i = i'$ and $j = j' \pm 1$ or $j = j'$ and $i = i' \pm 1 \pmod n$. We set black vertices to be the vertices with $i + j = 0 \pmod 2$.

Definition 36 (Torus). Let n, m two even positive integers. We denote by $T_{n,m}$ the torus defined as follows. The $n \times m$ vertices of $T_{n,m}$ are denoted by $v_{i,j}$ with $1 \leq i \leq n$ and $1 \leq j \leq m$. Vertices $v_{i,j}$ and $v_{i',j'}$ are adjacent if and only if $i = i'$ and $j = j' \pm 1 \pmod m$ or $j = j'$ and $i = i' \pm 1 \pmod n$. We set black vertices to be the vertices with $i + j = 0 \pmod 2$.

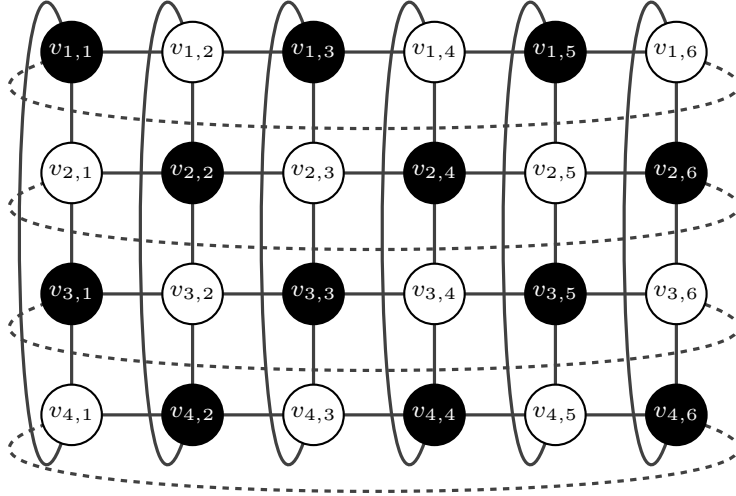


Figure 8: Cylinder $C_{4,6}$ (Torus $T_{4,6}$ with the dashed edges)

See Figure 8 for a representation of $C_{4,6}$ and $T_{4,6}$. Note that a torus is a cylinder where the two extremal columns are connected.

Proposition 37. If $n \geq 4$ or $m \geq 4$, then $T_{n,m} = 0$.

If $n \geq 4$ and m is even or if $n = 4k + 2$ for a positive integer k , we have $C_{n,m} = 0$.

Proof. By symmetry of the torus, we can assume that $n \geq 4$.

Assume first that $n = 2 \pmod 4$. Consider the following morphism on the vertices of $C_{n,m}$ or $T_{n,m}$:

$$\varphi : v_{i,j} \mapsto v_{((i+\frac{n}{2}) \pmod n),j}.$$

It is sending vertices to the symmetric vertex of that is in the same cycle. Thus φ is involutive. Since $n = 2 \pmod 4$, the colour of u and $\varphi(u)$ are opposite. Since $n \geq 6$, they are distance at least 3. Thus φ is a BW-automorphism and $C_{n,m} = T_{n,m} = 0$.

Assume now that $n = 0 \pmod 4$ and that both n and m are even. Consider now the morphism

$$\psi : v_{i,j} \mapsto v_{((i+\frac{n}{2}) \pmod n), (m+1-j)}.$$

One can check that ψ is an automorphism. Furthermore,

$$\psi(\psi(v_{i,j})) = \psi(v_{((i+\frac{n}{2}) \pmod n), (m+1-j)}) = v_{((i+\frac{n}{2}+\frac{n}{2}) \pmod n), (m+1-(m+1-j))} = v_{i,j}$$

and thus ψ is involutive.

Since $i + j$ and $i + \frac{n}{2} + m + 1 - j$ have not the same parity, ψ is exchanging black and white vertices.

Finally, $d(v_{i,j}, \psi(v_{i,j})) \geq \frac{n}{2} + 1 \geq 3$. Indeed, to go from $v_{i,j}$ to $\psi(v_{i,j})$ one needs already $n/2$ steps to go to the same line, and at least one more step to go to the same column since $m + 1 - j \neq j$ (m is even).

Thus ψ is a BW-automorphism and by Theorem 32 we can conclude that $T_{n,m} = C_{n,m} = 0$. \square

5.3 Limits of Theorem 32

Despite Theorem 32 gives a good condition to find some draw graphs, we prove now that deciding if a BW-automorphism exists in a graph is not solvable in polynomial time, unless the graph isomorphism problem is.

Definition 38 (Graph Isomorphism problem). *We recall here that given two graphs G_1 and G_2 , determining if G_1 and G_2 are isomorphic is not known to be solvable in polynomial time. Problems that are known to be at least as difficult as this problems are called GI-Hard problems.*

Note that the graph isomorphism problem is known to be computable in quasi-polynomial time [Bab15].

Theorem 39. *Let $G = (B \cup W, E)$ be a bipartite graph, deciding if G admits a BW-automorphism is GI-Hard.*

Proof. We give a reduction from the graph isomorphism problem. Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two graphs.

We construct an instance $H = (B \cup W, E)$ of BIPARTITE INFLUENCE as follows. The graph H will be divided into two bipartite graphs, H_1 and H_2 . Let $i \in \{1, 2\}$. The graph H_i is the incidence graph of G_i where a leaf is added to each vertex of H_i that corresponds to a vertex of G_i . More precisely, the vertex set of H_i is $V_i \cup E_i \cup V'_i$ where V'_i is a copy of V_i . Two vertices u and v are adjacent in H_i if and only if one of the following items is satisfied

- $u \in V_i$ and v is the copy of u in V'_i , or
- $u \in V_i, v \in E_i$ and v is an edge incident to u in G_i .

The graph H_i is bipartite with parts V_i and $E_i \cup V'_i$. Note that the only vertices of degree 1 are the vertices in V'_i . Then $H = H_1 \cup H_2$. We choose as set of black vertices the set $B = V_1 \cup E_2 \cup V'_2$ and as set of white vertices the set $W = V_2 \cup E_1 \cup V'_1$.

We now prove that G_1 and G_2 are isomorphic if and only if there exists a BW-automorphism of H .

First suppose that G_1 and G_2 are isomorphic. Let ϕ be an isomorphism from G_1 to G_2 . We construct a BW-automorphism φ of H as follows. Vertices of V_1 and V_2 are exchanged as in ϕ . Since ϕ preserves edges, a vertex of H_1 corresponding to an edge (u, v) of G_1 is sent to the vertex of H_2 corresponding to the edge $(\phi(u), \phi(v))$ of G_2 , and conversely for the vertices of H_2 corresponding to the edges of G_2 . Finally, a vertex of V'_1 (respectively V'_2) that is a copy of a vertex u of V_1 (resp. V_2) is sent to the copy of $\phi(u)$ in V'_2 (resp. to the copy of $\phi^{-1}(u)$ in V'_1). Clearly, φ is an automorphism that is involutive, is inverting the colour of the vertices, and since it is sending a vertex to a vertex not in the same component, the distance between u and $\varphi(u)$ is infinite. Thus, φ is a BW-automorphism of H .

Assume now that a BW-automorphism φ of H exists. Since V'_1 and V'_2 are the only vertices of degree 1 of H_1 and H_2 respectively, they must be stable by φ . To respect the exchange of colours, φ must induced a bijection between V'_1 and V'_2 . Indeed, the set V_1 corresponds to the unique neighbours of the vertices of V'_1 and thus must be sent to the unique neighbours of the vertices of V'_2 that correspond to the set V_2 . Finally, the vertices of E_1 can only be sent to the vertices of E'_2 .

Let $\phi : V_1 \rightarrow V_2$ be the restriction of φ to V_1 . Then ϕ is an isomorphism of G_1 . Indeed, φ is involutive so ϕ is a bijection. Consider u and v to neighbours in G_1 and $e = (u, v)$. Then $\varphi(e)$ must be incident to $\phi(u)$ and $\phi(v)$ and thus $\phi(u)$ and $\phi(v)$ are adjacent in G_2 . The reverse can be proved in the same way.

To conclude, G admits a BW-automorphism if and only if G_1 and G_2 are isomorphic, which proves that determining whether a bipartite graph admits a BW-automorphism or not is GI-Hard. \square

6 Segments

6.1 State of the art

Recall that segments, as defined in [DGP⁺21], correspond to instances of BIPARTITE INFLUENCE played on paths. Despite their simple graph structure, their resolution is not obvious.

Definition 40 (Duchêne et al [DGP⁺21]). *The class $C^{segment}$ represents the class of all BIPARTITE INFLUENCE graphs that are paths.*

When playing on a segment, the graph may be split into two smaller segments. This led the study to consider the class of disjunctive sum of segments (also corresponding to unions of paths), in order to play on a family that is closed by a move operation.

Definition 41. *The class C represents the class of all bipartite graphs that are finite disjunctive sums of elements in $C^{segment}$.*

A segment of n vertices starting with a black (resp. white) vertex will be denoted by S_n (resp. S_{-n}). See Figure 9 for an example.



Figure 9: Segment S_{-7}

On the class C without isolated vertex (as such vertices can be immediately removed from any instance), remark that a move removes between 2 and 5 vertices. A move that removes k vertices will be called a k -move. 4-moves are the moves made by playing the vertex at distance 2 from the extremity of a segment. 5-moves can only be made by Right on S_{-5} and by Left on S_5 , by playing the middle vertex.

By definition of the negative of a game, remark also that $S_{-n} = -S_n$ for all $n \in \mathbb{N}$. Moreover, by symmetry, we have $S_{2n} = S_{-2n}$.

On the class $C^{segment}$, Duchêne et al. [DGP⁺21] showed that the first player always wins. More precisely, the scores are bounded according to the following result:

Theorem 42 (Duchêne et al [DGP⁺21]). *Let $S_n \in C^{segment}$. We have:*

$$-5 \leq Rs(S_n) < 0 < Ls(S_n) \leq 5$$

The main idea of the proof of this theorem is that, by playing in the middle vertex of a segment, the first player can create two segments having a number of vertices that is very close. Consequently, this player can ensure to lose at most two vertices in the remaining game. As his first move scores three vertices, it ensures a win for the first player.

In addition to this result, there are certain values of n for which the exact value of $Ls(S_n)$ is known (e.g. when $n \equiv 1 \pmod 4$). Yet, for the majority of them, the exact value of the score is an open problem. In the study [DGP⁺21], the first 80 values of Ls and Rs are computed on segments, yielding Figure 10 (depicting the first 38 values). The structure of these results led to the following conjecture:

Conjecture 43 (Duchêne et al [DGP⁺21]). *The sequences $Ls(S_n)$ and $Rs(S_n)$ are ultimately periodic.*

Such a periodicity of the score would directly induce a polynomial time algorithm to solve the game on segments.

Concerning sums of segments, Theorem 42 does not hold when extended to the class C . The best bounds that are currently known are given by the result below.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$Ls(S_n)$	1	2	3	4	5	2	1	2	3	2	1	2	3	4	3	2	3	2	3
$Rs(S_n)$	1	-2	-3	-4	-1	-2	-3	-2	-1	-2	-3	-2	-1	-4	-3	-2	-1	-2	-3
n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
$Ls(S_n)$	2	5	4	3	2	3	2	3	2	5	4	3	2	3	2	3	2	5	2
$Rs(S_n)$	-2	-1	-4	-3	-2	-1	-2	-3	-2	-1	-4	-3	-2	-1	-2	-3	-2	-1	-2

Figure 10: Left and Right score for segments in [DGP⁺21]

Proposition 44 (Duchêne et al [DGP⁺21]). *Let $G \in C$. We have:*

- $-4 \leq Rs(G) \leq 0 \leq Ls(G) \leq 4$ if all the segments of G have even size,
- $-5 \leq Rs(G) \leq 1$ and $-1 \leq Ls(G) \leq 5$ if all the segments of G have even size except one.

In addition, the bounds in this result are tight. For example, consider the sum $G = S_9 + S_2$ for which $Rs(G) = 1$. This case is the smallest non-trivial graph in C where the first player does not win.

6.2 Mean and temperature of segments

As mentioned above, sums of segments have not been considered when there are at least two odd segments in the sum. A way to address this issue is to consider the notion of mean and temperature of segments, by applying Corollary23.

We first deal with the computation of the mean of segments. Recall first that $m_L(S_n) = m_R(S_n)$ by Theorem 14. Consequently, we will denote $m_n = m_L(S_n) = m_R(S_n)$.

Theorem 45. *Let $n \in \mathbb{Z}$. We have*

$$\begin{cases} m_n = 0 & \text{if } n \equiv 0 \pmod{2} \\ 0 \leq m_n \leq 1 & \text{if } n \equiv 1 \pmod{2} \text{ and } n \geq 0 \\ -1 \leq m_n \leq 0 & \text{if } n \equiv 1 \pmod{2} \text{ and } n \leq 0 \end{cases}$$

Proof. First, suppose that n is even. We know that $S_n = S_{-n}$. Therefore, if p is even, $\frac{Ls(pS_n)}{p} = \frac{Ls(0)}{p} = 0$ and if p is odd, we have $\frac{Ls(pS_n)}{p} = \frac{Ls(S_n)}{p}$. As $-\frac{5}{p} \leq \frac{Ls(S_n)}{p} \leq \frac{5}{p}$, we have $\frac{Ls(S_n)}{p} \xrightarrow{p \rightarrow \infty} 0$.

Finally, $\lim_{p \rightarrow \infty} \frac{Ls(pS_n)}{p} = 0$ and $m_n = 0$.

Now suppose that n is odd. Since $S_{-n} = -S_n$, and as the mean is linear, we can assume, without loss of generality, that $n \geq 0$. We will show that $0 \leq m_n \leq 1$.

If $n = 1$, we have $Ls(pS_1) = Rs(pS_1) = p$ as no move is available. Thus $m_1 = 1$.

If $n = 3$, any move of a player removes all the three vertices of S_3 . Ls and Rs can thus be computed as follows:

$$\begin{aligned} Ls(2pS_3) &= 3 + Rs((2p-1)S_3) = 3 - 3 + Ls((2p-2)S_3) = \dots = 0 \\ Ls((2p+1)S_3) &= 3 + Rs(2pS_3) = 3 - 3 + Ls((2p-1)S_3) = \dots = 3 \end{aligned}$$

This proves that $0 \leq Ls(pS_3) \leq 3$ for all p . Consequently, $m_3 = \lim_{p \rightarrow \infty} \frac{Ls(pS_3)}{p} = 0$.

If $n = 5$, by Example 12, we have $4S_5 = 4$. By denoting $p = 4q + r$ with $0 \leq r < 4$, we have $Ls(pS_5) = Ls(rS_5) + 4q$. Thus, since $0 \leq r < 4$, $Ls(rS_5)$ is bounded, $\lim_{p \rightarrow \infty} \frac{Ls(pS_5)}{p} = 1$.

Suppose $n \geq 7$. We first prove that $Ls(pS_n) \leq p + 8$. Assuming that Left starts, we denote by p_k the number of segments of odd length in the resulting graph after the k -th move, and by s_k the relative score, i.e. the difference between the number of vertices taken by Left and the number of vertices taken by Right after the k -th move. We will define a strategy for Right that has the following invariant property: for any $k \geq 0$, if there exists at least one segment of length four or greater, we have $p_{2k} + s_{2k} \leq p$. Since we have $p_0 = p$ and $s_0 = 0$, the invariant is true before the first move. Note that we will only consider p_{2k} and s_{2k} to focus on the state of the game after each move of Right.

The strategy of Right is the following: while there exists one segment of size at least four, Right considers a segment of highest length (if there are several, consider any of them). If it has odd length, Right can, by playing a 3-move on it on an extremity, transform it into a segment of even length; if it has even length, Right can, by playing a 4-move on it (which is always available on an even length segment of size greater than four) take four vertices and leave a segment of the same parity. Therefore, after the k -th move of Right, we have $s_{2k} + p_{2k} = s_{2k-1} + p_{2k-1} - 4$. Note that with such a strategy, Right does not create new odd segment.

Now, we consider the variation of the sequences s_k and p_k after the $(k+1)$ -th move of Left.

- If the move of Left is a 5-move, necessarily, it is played on a S_5 that is removed. We have $p_{2k+1} = p_{2k} - 1$ and $s_{2k+1} = s_{2k} + 5$, thus $p_{2k+1} + s_{2k+1} = p_{2k} + s_{2k} + 4$.
- If the move of Left is a 4-move, it does not change the number of segments of odd length. We have $p_{2k+1} = p_{2k}$ and $s_{2k+1} = s_{2k} + 4$, and thus $p_{2k+1} + s_{2k+1} = p_{2k} + s_{2k} + 4$.
- If the move of Left is a 3-move, and if it is in an even segment, it creates one odd segment and one even segment. If it is in an odd segment, it creates two odd segments, unless it is on a S_3 . In both cases, the number of odd segments is increased by at most one. We have $p_{2k+1} \leq p_{2k} + 1$ and $s_{2k+1} = s_{2k} + 3$, thus $p_{2k+1} + s_{2k+1} \leq p_{2k} + s_{2k} + 4$. Note that since the odd segments are positive (i.e. $n \geq 0$) at the beginning of the game, and since Right's strategy does not create new odd segment, Left cannot create negative odd segment when playing.
- If the last move of Left is a 2-move, it does not change the parity of the number of odd segments. We have $p_{2k+1} = p_{2k}$ and $s_{2k+1} = s_{2k} + 2$, thus $p_{2k+1} + s_{2k+1} = p_{2k} + s_{2k} + 2$.

Finally, we always have $p_{2k+1} + s_{2k+1} \leq p_{2k} + s_{2k} + 4$. Thus, as $p_{2k} + s_{2k} = p_{2k-1} + s_{2k-1} - 4$, the invariant property is satisfied until all segments have length three or less. In the final part of the game when all segments have length three or less, as any move in a segment removes it fully, a greedy strategy ensures that Left takes at most three vertices more than Right. As the largest available value for s_k is $p + 5$ (i.e., after a 5-move of Left, according to the invariant), we obtain $Ls(pS_n) \leq p + 5 + 3 = p + 8$.

It remains to prove that $Ls(pS_n) \geq 0$. We consider a strategy for Left that preserves the following invariant: at any moment of the game, Left has taken at least the same number of vertices than Right. Moreover, if it is Right's turn, there is no segment S_k with $k < 0$, i.e. all the remaining segments have at least one extremity black, and Left has taken at least four vertices more than Right.

We define here the first part of the strategy for Left. Until all segments have size 3 or less, Left considers the following strategy:

- First, Left takes four vertices on a segment. This move is available as we assumed $n \geq 7$.
- If the last move of Right does not create a segment with both extremities white, if at least one segment has size four or more, Left plays a 4-move on it (it might be a 5-move if the segment has length 5). In this way, Left has taken at least four vertices and Right at most four, as 5-moves are only available for Right on S_{-5} and the invariant ensures that there is no segment of this form when it is Right's turn. Therefore, the invariant is satisfied as Left does not create a S_k with $k < 0$.
- If the last move of Right creates a segment with both extremities white, by induction, such a segment did not exist before he made this move. Thus, it has been created by a 3-move on a segment with extremities of different colors. Therefore, this move has created a segment with two white extremities and an even segment (eventually of size zero). Left answers by playing a 3-move on an extreme black vertex of the segment which has two white extremities. Playing so, Left takes also three vertices and this segment now has a black extremity. Once again, the invariant is still satisfied.

This strategy guarantees that the invariant is satisfied until all segments have size 3 or less. At this moment of the game, if it is Left's turn, by a greedy strategy, Left can ensure to take at least the same number of vertices than Right for the rest of the game. Thus, as she already has at least the same number of vertices that Right, Left is not losing.

If it is Right's turn, by the invariant property, Left has taken at least four vertices more than Right (the four vertices played at the beginning). Thus, regardless the move that Right does, Left will get more vertices than Right as any segment has size three or less. This strategy ensures $Ls(pS_n) \geq 0$.

By gluing the two above results, we get $0 \leq Ls(pS_n) \leq p + 8$, implying $0 \leq \frac{Ls(pS_n)}{p} \leq 1 + \frac{8}{p}$. By taking the limit, as we know from Theorem 14 that m_n exists, we have $0 \leq m_n \leq 1$. □

The above result about the mean is very useful to estimate the score of a sum of segments according to Corollary 23. Yet, it remains to have an upper bound on the temperature of a segment. The following theorem provides such a bound.

Theorem 46. *Let $n \in \mathbb{Z}$. We have $\sigma(S_n) \leq 4$.*

Proof. Up to switching Right and Left, we can assume without loss of generality that $n \geq 0$. We prove the result by induction on n . If $n \leq 4$, the result is straightforward as the temperature cannot be larger than the maximum score on the game. If $n = 5$, one can prove that $\sigma(S_n) = 4$ according to Figure 4.

Assume now that $n \geq 6$. By Definition 15 of the temperature, to prove that $\sigma(S_n) \leq 4$, it is enough to prove that for $t \geq 4$, $Ls((\widetilde{S}_n)_t) \leq Rs((\widetilde{S}_n)_t)$. Let $t \geq 4$. We recall that $(\widetilde{S}_n)_t = \langle (S_n^L)_t - t | (S_n^R)_t + t \rangle$. Consider a Left option g^L and a Right option g^R of S_n .

A move in S_n splits the segment into at most two parts, thus we can write $g^L = \alpha + S_{i_1^L} + S_{i_2^L}$, and $g^R = -\beta + S_{i_1^R} + S_{i_2^R}$ with $\alpha, \beta > 0$ the scores obtained by each move.

By induction, we have $\sigma(S_{i_1^L}), \sigma(S_{i_2^L}), \sigma(S_{i_1^R}), \sigma(S_{i_2^R}) \leq 4$. Thus, g_t^L and g_t^R are frozen for $t \geq 4$ and, by Proposition 21, equal to their means. In particular, since the mean is linear, we have $Rs(g_t^L) = \alpha + m(S_{i_1^L}) + m(S_{i_2^L})$ and $Ls(g_t^R) = -\beta + m(S_{i_1^R}) + m(S_{i_2^R})$.

Assume first that n is odd. Right can only play 3-moves from S_n , thus $\beta = 3$ and both $S_{i_1^R}$ and $S_{i_2^R}$ are of even length. Therefore, by Theorem 45, $m(S_{i_1^R}) = m(S_{i_2^R}) = 0$ and we have:

$$Ls(g_t^R) + t = -3 + t + m(S_{i_1^R}) + m(S_{i_2^R}) = -3 + t.$$

If Left made a 4-move, up to exchange $S_{i_1^L}$ and $S_{i_2^L}$, we can suppose $S_{i_2^L} = 0$. By Theorem 45, $m(S_{i_1^L}) \leq 1$ and thus:

$$Rs(g_t^L) - t = 4 - t + m(S_{i_1^L}) \leq 4 - t + 1 = 5 - t.$$

If Left made a 3-move, $m(S_{i_1^L}), m(S_{i_2^L}) \leq 1$ by Theorem 45, and we have:

$$Rs(g_t^L) - t = 3 - t + m(S_{i_1^L}) + m(S_{i_2^L}) \leq 3 - t + 1 + 1 = 5 - t.$$

Finally, we obtain, for any options g^L and g^R :

$$\begin{aligned} Rs(g_t^L) - Ls(g_t^R) &\leq 5 - t - (-3 + t) \\ &\leq 8 - 2t \\ &\leq 0 \end{aligned}$$

In particular, this is true if $g_t^L - t$ and $g_t^R + t$ are the best options of $(\widetilde{S}_n)_t$. Then we have:

$$Ls((\widetilde{S}_n)_t) - Rs((\widetilde{S}_n)_t) = Rs(g_t^L) - Ls(g_t^R) \leq 0.$$

Assume now that n is even. If Left made a 3-move, one of the remaining path is even and the other is odd. Without loss of generality, we can assume that $S_{i_2^L}$ has even length. By Theorem 45, $m(S_{i_2^L}) = 0$ and $|m(S_{i_1^L})| \leq 1$. This implies that

$$Rs(g_t^L) - t = 3 + m(S_{i_1^L}) + m(S_{i_2^L}) - t \leq 4 - t.$$

Otherwise, Left made a 4-move and only one path is remaining, and has even length. Thus we also have:

$$Rs(g_t^L) - t = 4 + m(S_{n-4}) - t \leq 4 - t.$$

Symmetrically, $Ls(G_t^R) + t \leq -4 + t$. Therefore, if $t \geq 4$, we have for any options g^L and g^R , $Rs(g_t^L) - Ls(g_t^R) \leq 8 - 2t \leq 0$ and so, as before, $Ls((\widetilde{S}_n)_t) - Rs((\widetilde{S}_n)_t) \leq 0$.

Finally, regardless if n is odd or even, we have $Ls((\widetilde{S}_n)_t) - Rs((\widetilde{S}_n)_t) \leq 0$ for any $t \geq 4$. In particular, this means that $\sigma(S_n) \leq 4$. \square

We now have all the material to give an estimate of the score on a sum of segments.

Corollary 47. *Let $n_1, \dots, n_p \in \mathbb{Z}$, with exactly k of the n_i of odd value. We have*

$$-k - 4 \leq Rs(S_{n_1} + \dots + S_{n_p}) \leq Ls(S_{n_1} + \dots + S_{n_p}) \leq k + 4$$

Proof. This result is directly deduced from Theorem 46, Theorem 45 and Corollary 23. \square

Note that the bounds of Corollary 47 are tight. Consider for example a sum of $4n + 1$ segments S_5 . Since $4S_5 = 4$ according to Example 12, we have $Ls((4n + 1)S_5) = 4n + Ls(S_5) = 4n + 5$.

6.3 Additional results about the periodicity conjecture of segments

In order to explore the validity of Conjecture 43, more values are required. Recall that in [DGP⁺21], only the first 80 values of the score has been computed. The computation of the scores of segments can be boosted by using equivalences between sums. Recall that in a sum of games, any component can be replaced by another component equivalent to it.

The following result shows that two consecutive segments of even size (starting by the one that is a multiple of 4) can be replaced by the segment S_2 in a sum.

Theorem 48. *Let $k \in \mathbb{N}$, we have $S_{4k} + S_{4k+2} = S_2$.*

Proof. According to Lemma 8, it suffices to show that $Ls(S_{4k} + S_{4k+2} + S_2) = 0$.

We will prove by induction on k that for any $k \in \mathbb{N}$, we have :

$$\begin{aligned} Ls(S_{-(4k+5)} + S_{4k+3} + S_2) &= 0 \\ Ls(S_{-(4k+3)} + S_{4k+1} + S_2) &= 0 \\ Ls(S_{4k} + S_{4k+2} + S_2) &= 0 \end{aligned}$$

First, we give strategies for Left to ensure the following inequalities:

$$\begin{aligned} Ls(S_{-(4k+5)} + S_{4k+3} + S_2) &\geq 0 \\ Ls(S_{4k+3} + S_{-(4k+1)} + S_2) &\geq 0 \\ Ls(S_{4k} + S_{4k+2} + S_2) &\geq 0 \end{aligned}$$

In $S_{4k} + S_{4k+2} + S_2$, Left can play a move on a extremal vertex in S_{4k+2} that takes exactly two vertices. After that, it is Right's turn and the game is $S_{4k} + S_{4k} + S_2$ that is equivalent to S_2 . Therefore, Right also scores two points and we have $Ls(S_{4k+2} + S_{4k} + S_2) \geq 0$.

In $S_{-(4k+5)} + S_{4k+3} + S_2$, Left can first play S_2 and scores two points. Then, by Theorem 25 (third inequality), imagine that Left gives the second vertex of $S_{-(4k+5)}$ to Right. After this move, the game is equivalent to $S_{-(4k+3)} + S_{4k+3} + S_{-1} = -1$. Once again, we have $Ls(S_{-(4k+5)} + S_{4k+3} + S_2) \geq 0$.

In $S_{4k+3} + S_{-(4k+1)} + S_2$, Left can first play an extremal move on S_{4k+3} that takes exactly two vertices. After that, it is Right's turn and the game is $S_{4k+1} + S_{-(4k+1)} + S_2 = S_2$. Therefore, Right scores two points on it and we have $Ls(S_{4k+3} + S_{-(4k+1)} + S_2) \geq 0$.

Now, we give strategies for Right that ensure the other inequalities (i.e. Right is the second player and is not losing).

If $k = 0$ or $k = 1$, the induction hypothesis is true by checking by hand all the possible options. Now assume $k \geq 2$ and that the induction hypothesis is true for all $0 \leq k' < k$.

First, remark that no 5-move is available in any of these games for Left as there is no S_5 component.

Secondly, if Left plays a 4-move in any of these games, Right can answer by a 4-move in the other component that is not S_2 and then obtains an instance that is strictly smaller than the initial one and where he will not be losing by induction.

Thirdly, since 2-moves are dominated, one can suppose, without loss of generality, that Left never plays one of them.

It thus remains to consider which strategy can be used by Right when Left starts with a 3-move.

If the game is $S_{4k} + S_{4k+2} + S_2$:

- If Left plays a 3-move in S_{4k} :

- if this move creates two instances $S_{4m+1} + S_{4n}$ (with $0 < n, m < k$), Right can answer by playing a 3-move in S_{4k+2} that creates $S_{-(4m+1)} + S_{4n+2}$. The resulting game is $S_{4m+1} + S_{4n} + S_{-(4m+1)} + S_{4n+2} + S_2 = S_{4n} + S_{4n+2} + S_2$ that is not losing for Right by induction hypothesis.
- if this move creates two instances $S_{4m+3} + S_{4n+2}$, Right can answer by playing a 3-move in S_{4k+2} that creates $S_{-(4m+5)} + S_{4n+2}$. The resulting game becomes $S_{-(4m+5)} + S_{4n+2} + S_{4m+3} + S_{4n+2} + S_2 = S_{4m+3} + S_{-(4m+5)} + S_2$ that is also not losing for Right by induction hypothesis.
- If Left plays a 3-move in S_{4k+2} :
 - if she creates two instances $S_{4m+2} + S_{4n+1}$, Right can answer by playing a 3-move in S_{4k} that creates $S_{4m} + S_{-(4n+1)}$. The game becomes $S_{4m+2} + S_{4n+1} + S_{4m} + S_{-(4n+1)} + S_2 = S_{4m} + S_{4m+2} + S_2$ and we conclude as previously by induction.
 - if she creates two instances $S_{4m} + S_{4n+3}$, Right can answer by playing a 3-move in S_{4k} that creates $S_{4m} + S_{-(4n+1)}$. The game becomes $S_{4m} + S_{4n+3} + S_{4m} + S_{-(4n+1)} + S_2 = S_{4n+3} + S_{-(4n+1)} + S_2$ and we can conclude by induction.

If the game is $S_{4k+3} + S_{-(4k+5)} + S_2$:

- if Left makes a 3-move in S_{4k+3} , it necessarily creates two instances S_{4m+1} and S_{4n+3} . Right can answer by playing a 3-move in $S_{-(4k+5)}$ that creates $S_{-(4m+1)}$ and $S_{-(4n+5)}$. The resulting game is $S_{4m+1} + S_{4n+3} + S_{-(4m+1)} + S_{-(4n+5)} + S_2 = S_{4n+3} + S_{-(4n+5)} + S_2$, and thus not losing for Right by induction hypothesis.
- if Left makes a 3-move in $S_{-(4k+5)}$, it creates two instances S_{4m} and S_{4n+2} . Right can answer by playing a 3-move in $S_{-(4k+3)}$ that creates S_{4m} and S_{4n} . The game becomes $S_{4m} + S_{4n+2} + S_{4m} + S_{4n} + S_2 = S_{4n} + S_{4n+2} + S_2$, and we can conclude by induction.

If the game is $S_{4k+3} + S_{-(4k+1)}$:

- if Left makes a 3-move in S_{4k+3} , she creates two instances S_{4m+1} and S_{4n+3} . Right can answer by playing a 3-move in $S_{-(4k+1)}$ that creates $S_{-(4m+1)}$ and $S_{-(4n+1)}$. The resulting game is $S_{4m+1} + S_{4n+3} + S_{-(4m+1)} + S_{-(4n+1)} + S_2 = S_{4n+3} + S_{-(4n+1)} + S_2$, and hence not losing for Right by induction.
- if Left makes a 3-move in $S_{-(4k+1)}$, she creates two instances S_{4m+2} and S_{4n} . Right can answer by playing a 3-move in S_{4k+3} that creates S_{4m+2} and S_{4n+2} . The resulting game becomes $S_{4m+2} + S_{4n} + S_{4m+2} + S_{4n+2} + S_2 = S_{4n} + S_{4n+2} + S_2$ and we can conclude by induction.

□

The above result yields a very significant way to make the computation of the values faster. In fact, as our program is based on a dynamic programming approach, all sums of segments need to be stored. Thanks to this result, all sums containing a S_{4k+2} can be simplified by replacing it with a $S_{4k} + S_2$.

Table 11 expands the result presented in [DGP⁺21] by giving the first 120 values of the score (instead of 80). A careful analysis of this table suggests that Conjecture 43 holds, with a period of 40 and a preperiod of 30. In addition, one can remark that after the preperiod, the structure of the values is very regular (almost of period 8), except for $n = 37, 77, 117$ that correspond to the rare values satisfying $R_s(S_n) = -5$.

This new table is however not sufficient to prove the ultimate periodicity of the structure, as the conjectured period of 40 does not appear frequently enough to be used in a general proof. In addition, the segments S_n and S_{n+40} are not equivalent for almost all values of (up to $n = 40$), which is another pitfall to prove the conjecture. A first step towards it would be to prove that the values of $L_s(S_n)$ are equal to 2 for all even $n \geq 32$.

n	001	002	003	004	005	006	007	008	009	010	011	012	013	014	015	016	017	018	019	020
Ls(S_n)	1	2	3	4	5	2	1	2	3	2	1	2	3	4	3	2	3	2	3	2
Rs(S_n)	1	-2	-3	-4	-1	-2	-3	-2	-1	-2	-3	-2	-1	-4	-3	-2	-1	-2	-3	-2
n	021	022	023	024	025	026	027	028	029	030	031	032	033	034	035	036	037	038	039	040
Ls(S_n)	5	4	3	2	3	2	3	2	5	4	3	2	3	2	3	2	5	2	3	2
Rs(S_n)	-1	-4	-3	-2	-1	-2	-3	-2	-1	-4	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2
n	041	042	043	044	045	046	047	048	049	050	051	052	053	054	055	056	057	058	059	060
Ls(S_n)	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
Rs(S_n)	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2
n	061	062	063	064	065	066	067	068	069	070	071	072	073	074	075	076	077	078	079	080
Ls(S_n)	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	5	2	3	2
Rs(S_n)	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2
n	081	082	083	084	085	086	087	088	089	090	091	092	093	094	095	096	097	098	099	100
Ls(S_n)	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
Rs(S_n)	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2
n	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
Ls(S_n)	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	5	2	3	2
Rs(S_n)	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2	-1	-2	-3	-2

Figure 11: New values on Segments

7 Grids

As a natural extension to paths, we consider in this section rectangular grids. We denote by $G_{n,m}$ the grid with n rows and m columns, with alternated black and white vertices and a black vertex in the top left corner. See Figure 12 for an illustration of $G_{2,7}$.

Despite their symmetry, there is no BW-automorphism in rectangular grids. Indeed, the vertices close to the center of the grid are necessarily sent to each other by any automorphism, and thus could not be sufficiently distant. The results of Section 5 cannot therefore be applied. Moreover, it seems that there are neither draws nor games favourable to the second player in rectangular grids. Actually, all the computations made so far lead to a victory of the first player. This induces the following conjecture.

Conjecture 49. *Let $n, m \geq 2$. We have $Ls(G_{n,m}) > 0 > Rs(G_{n,m})$. In particular, the first player wins and there is no draw.*

Note that if n or m is even, the grid is symmetric by exchanging the roles of Left and Right. Since the game is nonzugzwang, $Ls(G_{n,m}) = -Rs(G_{n,m})$. Moreover, as $Ls(G_{n,m}) \geq Rs(G_{n,m})$, it implies directly that $Ls(G_{n,m}) \geq 0 \geq Rs(G_{n,m})$. However, if n and m are odd, this argument cannot be used.

In the rest of the section, we give some partial results supporting the conjecture for rectangular grids with two and three rows.

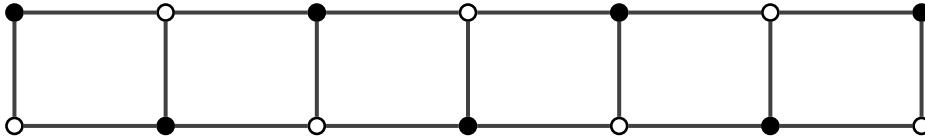


Figure 12: Grid $G_{2,7}$

7.1 Grids with two rows

In this subsection, we prove Conjecture 49 on grids with only two rows. As said before, since the number of rows is even, we just need to consider the Left score. We actually prove a stronger result when the number of columns is odd, by giving the exact value of $Ls(G_{2,m})$.

Theorem 50. *Let $m \geq 3$ be an odd integer. We have $Ls(G_{2,m}) = -Rs(G_{2,m}) = 4$ if $m \geq 5$ and $Ls(G_{2,3}) = -Rs(G_{2,3}) = 6$.*

For even m , we just manage to prove a lower bound on the score.

Lemma 51. *Let $m \geq 4$ be an even integer. We have $Ls(G_{2,m}) \geq 2$.*

Before proving Theorem 50 and Lemma 51, we introduce some notations that will be used in both proofs. As each column contains at most one white and one black vertex, we will say playing i for playing the vertex of the current player's color in column i . Consider a connected component G at some point of a game played in a grid with two rows. Note that G can only have columns with one vertex at its extremities. Thus the full columns are consecutive. We encode G as a triplet (x, n, y) with $x, y \in \{0, -1, 1\}$ and n an integer. The integer n corresponds to the number of consecutive full columns, x represents the first column of G and y the last one with the following convention: $x = 0$ ($y = 0$ resp.) if the column is full, $x = 1$ if there is a black vertex and $x = -1$ if there is a white vertex. In Figure 14, the two grids have respectively forms $(-1, 2, -1)$ and $(1, 4, 1)$. Note that the form of a partial grid completely characterizes it and that (x, n, y) and (y, n, x) correspond to the same graph.

Proof of Theorem 50. For $m = 3$, the first player can take the whole grid in one move, which implies that $Ls(G_{2,3}) = -Rs(G_{2,3}) = 6$.

Let $m \geq 5$ be an odd integer. We first prove that $Ls(G) \geq 4$. Indeed, Left can play $\frac{m+1}{2}$, i.e in the middle column. This way, she takes four vertices and cuts the game in two parts of form $(0, \frac{n-3}{2}, 1)$ (see Figure 13). Then, each time Right decides to play on column i , Left plays symmetrically on column $m - i + 1$. If Right plays $\frac{m+1}{2} - 2$ (respectively $\frac{m+1}{2} + 2$) and removes the single black vertex of column $\frac{m+1}{2} - 1$ (resp. $\frac{m+1}{2} + 1$), the symmetric move of Left will remove the symmetric vertices of Right's move plus the other single black vertex. In all the other cases, Left's move removes exactly the symmetric vertices (considering a central symmetry) that Right has taken. This way, Left can finish the game without losing any vertex and $Ls(G_n) \geq 4$.

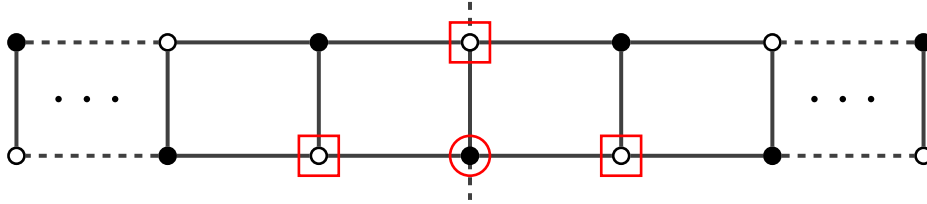


Figure 13: An optimal strategy for Left when the number of columns is odd, is to take the central black vertex (marked with a red circle) and then playing the symmetric column. This strategy gives a final score of 4.

We now prove that $Ls(G_{2,m}) \leq 4$ by induction. First, we prove that if G has the form $G_1 + G_2$ with $G_1 = (x, n, y)$ and $G_2 = (-x, n + 2, -y)$, then $Ls(G) \leq 4$ (see Figure 14).

We prove this result by induction on n the number of full columns of G_1 (G_2 has $n+2$ full columns). Note that G_2 has four more vertices than G_1 . Consider the first move of Left. Assume first that Left did not take a full component and that Left plays in G_1 . By symmetry, we can assume Left has play in some column i with $i \leq \frac{n+1}{2}$ (starting to count from the first full column). The whole graph after this move can be written as $R + (1, n - i - 1, y) + G_2$ with R that might be empty (and necessarily, $i + 1 \leq n$). Then Right can answer by playing in column i of G_2 . He takes the same number of vertices than Left. The remaining graph after these two moves has the form $R + (1, n - i - 1, y) + R' + (-1, n - i + 1, -y)$ where R and R' are opposite (and might be empty). Hence, R and R' cancelled each other. By induction, Left can score at most 4 in $(1, n - i - 1, y) + (-1, n - i + 1, -y)$, and thus can score at most 4 in total.

Assume now that Left plays in G_2 but did not take the whole graph. Again, by symmetry, we can assume Left has play in some column i with $i \leq \frac{n+3}{2}$ and that it remains $G_1 + R' + (1, n + 1 - i, -y)$, where R' might be empty but where $(1, n + 1 - i, -y)$ contains at least two vertices. If Right can take fully G_1 , he takes it and at the end, the

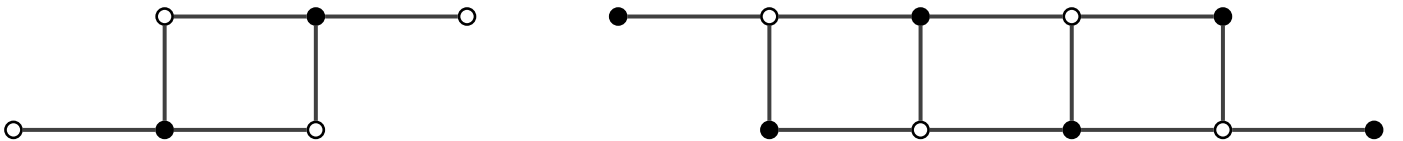


Figure 14: In $(-1, 2, -1) + (1, 4, 1)$, Left can score at most 4.

score will be at most 4. Otherwise, we must have $n \geq 2$ and then we have $i \leq n$. Right plays in column i of G_1 and take as many vertices as Left. Then it is remaining after the two moves $R + (1, n - i - 1, y) + R' + (-1, n - i + 1, -y)$ and we can conclude as before.

Finally, assume that Left takes a full component. If Right can take the other component, the game ends with a score of at least 4. Otherwise, it means that Left played in G_1 and that $n \leq 3$. Let $i \leq 2$ be the column where Left played. If $n = 1$, the only case where Right cannot take the whole G_2 is when G_2 has form $(1, 3, 1)$ or $(0, 3, 1)$. In the first case, Right can take four vertices and then both players take two vertices, ending in a draw. In the second case, Right takes five vertices, Left can take the two remaining vertices, and the games end in a draw. If $n = 2$, Left took at most six vertices. If x or y is equal to -1 , then Right can take four vertices in G_2 and leaves a graph with two components of size at least 2 by playing in column 2 or 3. Thus at the end, Right will win at least six vertices. If $x = y = 1$, Right can directly take six vertices in G_2 by playing in column 2. In both cases, Left takes at most ten vertices, ending with a score of at most 4. If $x = 0$ or $y = 0$, Right can take 5 vertices and there are at most 14 vertices in total, thus the score is also at most 4. Finally, if $n = 3$, Left took at most eight vertices. As before, Right can take at least $|G_1|$ vertices by taking four vertices in his first move and splitting G_2 in two components of size at most 3 (if $|G_1| = 7$) or 4 (if $|G_1| = 8$).

Turn back to our original graph $G_{2,m}$. Right can follow the following strategy :

- While Left does not play in the five columns in the middle, play the symmetric. The most left and most right graphs are opposite and thus simplify each other. The remaining graph has form $(1, m', -1)$ with m' odd. In particular, there is still a symmetry center.
- If Left plays column $(m + 1)/2 - 2$ (or the symmetric), play the symmetric. The components on the left and the right simplify. Only four vertices are left in the middle, so Left win at most four vertices. We have $Ls(G_{2,m}) \leq 4$.
- If Left plays at distance 1 to the central column, i.e. she plays $(m + 1)/2 - 1$, then play the symmetric. The two remaining graphs are opposite. So $Ls(G_{2,m}) \leq 0$.
- If Left plays the central column $(m + 1)/2$:
 - If the vertices in column $(m + 1)/2 - 3$ have been played, the graph was (before the move of Left) of the form $(1, 3, -1)$. Thus Left took six vertices and let a S_2 . Right can take it. The final score is $6 - 2 = 4$.
 - If the vertices in column $(m + 1)/2 - 3$ have not been played, Right plays in column $(m + 1)/2 - 2$ and takes four vertices (like Left did). Then the remaining graph has form $(x, n, 1) + (-1, n + 2, -x)$. Using the result proved above (recall that $(-1, n + 2, -x) = (-x, n + 2, -1)$, the score is at most 4 at the end.

Finally, $Ls(G_{2,m}) \leq 4$, and the theorem is proven. □

Proof of Lemma 51. Let $m \geq 4$ be an even integer. We prove that $Ls(G_{2,m}) \geq 2$ by giving a strategy for Left to score 2. Left can start by playing in column $m/2$ (it corresponds to the vertex marked by a circle on Figure 15). Denote by G' the remaining graph after this first move. Left can then decide to "give" the two vertices marked by blue triangles in the graph of Figure 15. More precisely, we apply Theorem 25 with $B_0 = \{x_1, x_2\}$ where x_1 and x_2 are the black vertices of columns $m/2 + 1$ and $m/2 + 2$. We obtain $R_s(G') \geq R_s(G' \setminus B_0) - |B_0| \geq -2$, since $G' \setminus B_0$ is the sum of a game and its opposite, and thus a score 0. Finally, $Ls(G_{2,m}) \geq 4 - R_s(G') \geq 2$.

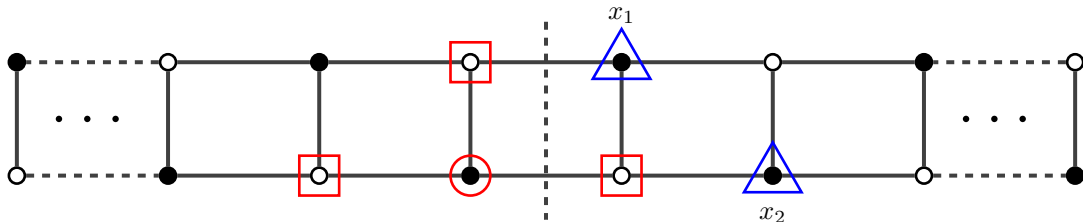


Figure 15: Strategy if the number of columns is even

□

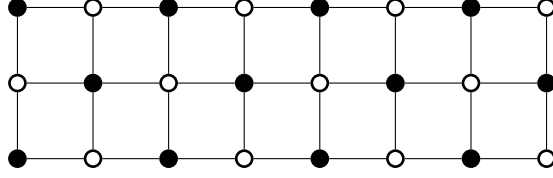


Figure 16: Example of grid of three rows and eight columns.

Theorem 50 and Lemma 51 actually suggest the following conjecture that gives the complete values for grids of two rows. It only remains to prove that $Ls(G_{2,m}) \leq 2$ for even m . We have checked this conjecture up to $m = 30$.

Conjecture 52. *Let $m \geq 2$. We have*

$$Ls(G_{2,m}) = -Rs(G_{2,m}) = \begin{cases} 6 & \text{if } m = 3; \\ 4 & \text{if } m \equiv 1 \pmod{2} \text{ and } m \geq 5 \text{ or } m = 2; \\ 2 & \text{if } m \equiv 0 \pmod{2} \text{ and } m \geq 4. \end{cases}$$

7.2 Grids of three rows

As detailed in the previous sections, in the cases of segments and grids of two rows, a common technique to prove that the first player has a winning strategy consists in cutting the graph into two "similar" parts and then guarantee that he will not lose many points on the rest of the game. Unfortunately, this technique is no more available for grids having at least four rows, as single moves generally keep the graph connected. Yet, in the case of grids of three rows, some moves split the grid into two parts, but other moves keep it connected. This is sufficient to yield strategies for the first player in almost all configurations.

First note that if m is odd, as the grid is not symmetric, having $Ls(G_{3,m}) \geq 0$ or $Rs(G_{3,m}) \leq 0$ is not straightforward at first sight. However, we will prove Conjecture 49 for grids of three rows in almost all cases.

Theorem 53. *Let $G_{3,m}$ be a grid of three rows and m columns, with a black vertex at the top left. We have $Ls(G_{3,m}) > 0$, and if $m \not\equiv 3 \pmod{4}$, $Rs(G_{3,m}) < 0$.*

Proof. To make the proof more understandable, in the following figures, we will mark by a red circle the vertex played by the first player, by red squares the vertices taken by this move, and by blue triangles the vertices freely given to the opponent using Theorem 25.

Let $G = G_{3,m}$ be a grid of three rows.

- If $m \equiv 0 \pmod{2}$. We already know by symmetry that $Ls(G) = -Rs(G)$. So, without loss of generality, it is sufficient to prove that $Ls(G) > 0$.

Left can adopt the following strategy: play the black vertex on column $\frac{m}{2}$ marked by a red circle (see Figure 17). Left takes all vertices marked by red squares and scores 5 points. Then, she can give the two black vertices x_1 and x_2 of column $\frac{m}{2} + 1$ to Right, and apply Theorem 25 with $B_0 = \{x_1, x_2\}$.

The remaining graph is composed of two grids $G_{3, \frac{m}{2} - 1}$. But the left part has one missing white vertex marked by a red square. Add it to the graph by giving one more point to Right once again according to Theorem 25 with $W_0 = \{x_3\}$. Note H_{sym} the symmetric graph obtained. The final score thus satisfies $Ls(G) \geq 5 + Rs(H_{sym}) - |W_0| - |B_0| \geq 5 - 1 - 2 = 2$.

- If $m \equiv 1 \pmod{4}$ and Left starts, she can play the black vertex at the bottom of column $\frac{m+1}{2}$ marked by a red circle (see Figure 18). Then, by giving to Right the two black vertices in top corners and the black vertex in the middle of the first row, she creates two identical symmetric graphs. Note this set of three vertices $B_0 = \{x_1, x_2, x_3\}$ and the resulting graph H_{sym} . Therefore, the score satisfies $Ls(G) \geq 4 + Rs(H_{sym}) - |B_0| = 4 + 0 - 3 = 1$.

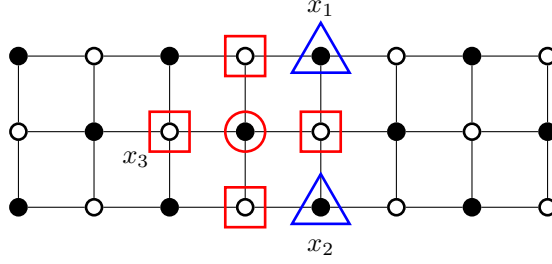


Figure 17: Strategy if $m \equiv 0 \pmod 2$

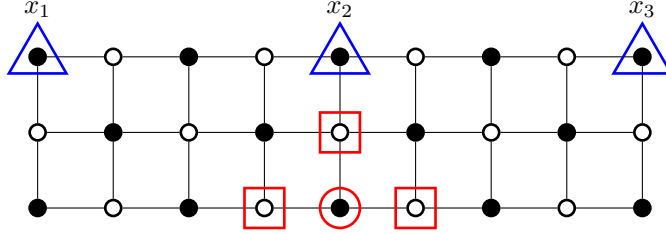


Figure 18: Strategy for Left if $m \equiv 1 \pmod 4$.

- If $m \equiv 1 \pmod 4$ and Right starts, he can play the vertex in the middle of the graph (see Figure 19), and then give the two black vertices x_1 and x_2 of the columns $\frac{m-1}{2}$ and $\frac{m+3}{2}$ to Left. Note them $B_0 = \{x_1, x_2\}$. The remaining graph is composed of two instances of $G_{3, \frac{m-1}{2}}$, and is symmetric as $m-1 \equiv 0 \pmod 4$. Note it $H_{sym} = 2 * G_{3, \frac{m-1}{2}}$. Thus, by Theorem 25, the score verifies $Rs(G) \leq -5 + Ls(H_{sym}) + |B_0| = -5 + 0 + 2 = -3$

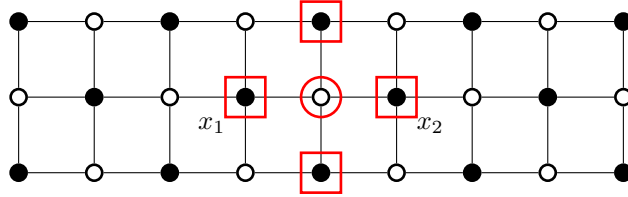


Figure 19: Strategy for Right if $m \equiv 1 \pmod 4$.

- If $m \equiv 3 \pmod 4$ and Left starts, she can play the middle vertex of the graph and give the four black vertices at its diagonals to Right (see Figure 20). Note these four vertices $B_0 = \{x_1, x_2, x_3, x_4\}$. The remaining graph H_{sym} is composed of two instances of $G_{3, \frac{m-3}{2}}$, so it is equivalent to 0 as $m-3 \equiv 0 \pmod 4$. Finally, the score satisfies $Ls(G) \geq 5 + Rs(H_{sym}) - |B_0| = 5 + 0 - 4 = 1$.

□

The case of grids of three rows is almost solved from the above result. As an open problem, it remains to show that if $m \equiv 3 \pmod 4$ and Right starts, he has a winning strategy.

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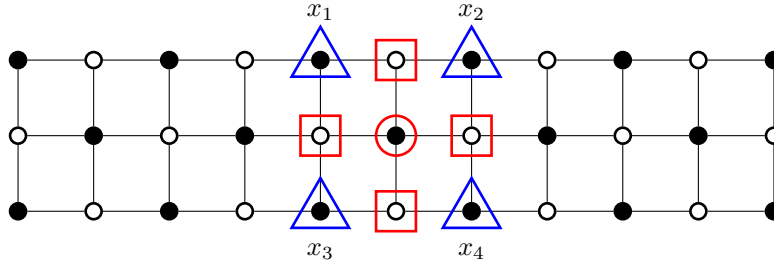


Figure 20: Strategy for Left if $m \equiv 3 \pmod{4}$.

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