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Corecursion up-to via Causal Transformations^{*}

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Abstract Up-to techniques are a widely used family of enhancements of corecursion and coinduction. The soundness of these techniques can be shown systematically through the use of distributive laws. In this paper we propose instead to present up-to techniques as causal transformations, which are a certain type of natural transformations over the final sequence of a functor. These generalise the approach to proving soundness via distributive laws, and inherit their good compositionality properties. We show how causal transformations give rise to valid up-to techniques both for corecursive definitions and coinductive proofs.

Keywords: Coalgebra · Corecursion · Coinduction · Final sequence

1 Introduction

We assume familiarity with the most basic concepts of universal coalgebra [21] in this introduction; we formally define them in Section 2.

Let us recall the corecursion up-to principle from [31,32], which encompasses (and is implicit in) various results from the literature [8,24,39,20,26,11].

Let B be a functor with a final coalgebra (Z, ζ) , and let F be a functor with an algebra $a: FZ \rightarrow Z$ on the final coalgebra. *Corecursion up to the algebra a is valid* if for every BF -coalgebra (X, f) , there exists a unique morphism $f^a: X \rightarrow Z$ making the following diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{f^a} & Z \\ f \downarrow & & \downarrow \zeta \\ BF X & \xrightarrow{BF f^a} BF Z \xrightarrow{Ba} & BZ \end{array} \quad (1)$$

When F is the identity functor (and a the identity morphism), this is just plain corecursion. Plain corecursion makes it possible, for instance, to define

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pointwise addition on streams. Indeed, streams (of real numbers, \mathbb{R}^ω) form the final coalgebra for the functor $BX = \mathbb{R} \times X$ on sets. Let us write x_0 for the first element of a stream x , and x' for its tail. If f is the following B -coalgebra structure on $(\mathbb{R}^\omega)^2$,

$$\begin{aligned} (\mathbb{R}^\omega)^2 &\rightarrow B((\mathbb{R}^\omega)^2) \\ (x, y) &\mapsto (x_0 + y_0, (x', y')) \end{aligned}$$

then $f^{\text{id}}: (\mathbb{R}^\omega)^2 \rightarrow \mathbb{R}^\omega$ is nothing but pointwise addition on streams: the only binary operation \oplus satisfying the following equations.

$$\begin{aligned} (x \oplus y)_0 &= x_0 + y_0 \\ (x \oplus y)' &= x' \oplus y' \end{aligned}$$

(Where x_0 and x' respectively denote the head and tail of a stream x .)

Corecursion up-to proves useful to define more complex operations like shuffle product (\otimes), satisfying the following equations:

$$\begin{aligned} (x \otimes y)_0 &= x_0 \times y_0 \\ (x \otimes y)' &= (x \otimes y') \oplus (x' \otimes y) \end{aligned}$$

Indeed, in such a situation we need to call a function (pointwise addition) on objects which are not fully defined yet (the two corecursive calls $x \otimes y'$ and $x' \otimes y$). Using the functor $FX = X^2$ and seeing \oplus as an F -algebra on \mathbb{R}^ω , we can define shuffle product using corecursion up-to (1) and the following BF -coalgebra:

$$\begin{aligned} (\mathbb{R}^\omega)^2 &\rightarrow BF((\mathbb{R}^\omega)^2) \\ (x, y) &\mapsto (x_0 \times y_0, ((x, y'), (x', y))) \end{aligned}$$

Here, the inner pairs $((x, y')$ and $(x', y))$ correspond to the corecursive calls to \otimes , while the intermediate pair $((x, y'), (x', y))$ corresponds to a call to the F -algebra, i.e., in this case, pointwise addition.

Of course, not every algebra on a final coalgebra yields a valid corecursion up-to principle. Here are two sufficient conditions:

1. [6,7] a is induced by a distributive law $\lambda: FB \Rightarrow BF$ and the base category has countable coproducts (or F is a monad and λ a distributive law of this monad over the functor B);
2. [31] B is a polynomial set functor, and a is a *causal* F -algebra.

While the first condition is nice and well-known, it requires the machinery of distributive laws, and it is not always easy to show that a given algebra arises from a distributive law. The second condition does not suffer from this: causality is rather simple to check in practice (for instance, an algebra on streams is causal if and only if the n -th element of its output only depends on the n first elements

of its inputs), but the condition that the starting functor is a polynomial set functor is often too strong (e.g., the finite powerset functor is not polynomial).

Shuffle product on streams is simple enough, so that the two approaches can both be used: the algebra we want to use, \oplus , arises from a simple distributive law, and it is obviously causal. In fact, the algebra induced by a distributive law on the final coalgebra of an ω -continuous³ set functor is always causal.

Causal transformations

Here we propose yet another condition based on the final sequence \overline{B} of the functor B . Assuming that B is an endofunctor on a complete category \mathcal{C} , recall that this final sequence is a sequence of objects indexed by ordinals, which, if it stabilises, yields a final coalgebra for B (cf. [5, Theorem 1.3], or earlier for the dual case of algebras [2]). Here we shall present this sequence as a functor $\overline{B}: \text{Ord}^{\text{op}} \rightarrow \mathcal{C}$ from the category of ordinals to the base category. We call a natural transformation of type $F\overline{B} \Rightarrow \overline{B}$ a *causal transformation for B* .

Assuming that the final sequence stabilises at ordinal κ , so that for all causal transformations α , α_κ is an algebra on the final coalgebra, our new condition is the following:

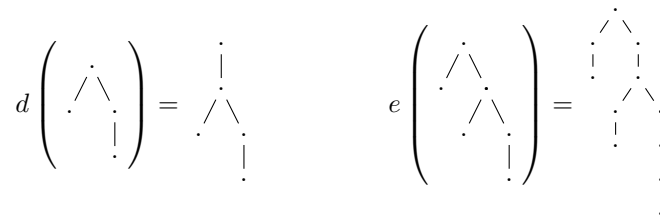
- 3. a is the κ -th element α_κ of a causal transformation α .

Intuitively, looking at the final sequence as a sequence of approximations of the final coalgebra, an algebra satisfying the above condition must be defined not only on the final coalgebra, but also on all its approximations.

Example 1.1. Let \mathcal{P}_f be the finite powerset functor, whose final coalgebra (\mathcal{T}_f, c) consists of all finitely branching trees quotiented by bisimilarity, and the function c mapping a tree to its finite set of children. Consider the “delay” function $d: \mathcal{T}_f \rightarrow \mathcal{T}_f$ that adds a unary node at the root of the given tree (formally, $d(t) = c^{-1}(\{t\})$), and suppose we want to define the function e that corecursively delays all inner nodes of a given tree. For all trees t , this function should satisfy

$$c(e(t)) = \mathcal{P}_f(d \circ e)(c(t)).$$

Graphically, on two examples, we have:



³ I.e., preserving limits of ω^{op} -chains.

We can obtain e by corecursion up-to, using the identity functor for F , d for the algebra a , and c as $\mathcal{P}_f\text{Id}$ -coalgebra: in this case, diagram (1) precisely yields the above equation for the solution $e = c^d$.

In this example, condition 2/ is not satisfied: \mathcal{P}_f is not polynomial. Moreover, it is not obvious how to construct a distributive law yielding the algebra d , in order to fulfil condition 1/. In contrast, one can easily extend the algebra d into a causal transformation $\delta: \overline{\mathcal{P}_f} \Rightarrow \overline{\mathcal{P}_f}$. To this end, recall that the final sequence of \mathcal{P}_f stabilises at $\omega + \omega$ [41], and that it consists of all finite trees of depth at most n at all finite ordinals n , and of all *compactly branching* trees which are finitely branching up-to depth n at all ordinals $\omega + n$ for n finite. Intuitively, we can thus define a counterpart to the function d at all stages of this sequence: take a tree, add a unary node at the root, and, for the finite stages, truncate the resulting tree at the given depth. This is even easier formally: just set $\overline{\delta}_k(t) = \overline{\mathcal{P}_f}(k+1, k)(\{t\})$, where $\overline{\mathcal{P}_f}(k+1, k)$ is the morphism from $\overline{\mathcal{P}_f}(k+1)$ to $\overline{\mathcal{P}_f}(k)$ in the final sequence. \square

For polynomial set functors, the three conditions turn out to be equivalent (a consequence of [32, Theorem 8.6 and Corollary 9.6]) In the general case, condition 3/ is implied by condition 1/: every distributive law yields a causal transformation whose κ -th element is its induced algebra on the final coalgebra [32, Lemma 6.2]. The converse is not true, cf. end of Section 5, but it is if the functor B has a *companion* [32]. Condition 3/ also generalises condition 2/: on sets, when B is ω -continuous, there is a one-to-one correspondence between causal algebras and causal transformations [32, Theorem 8.6].

That correspondence is non-trivial to establish, like the fact that condition 2/ provides corecursion up-to (1). For the latter, the approach in [32] goes via the construction of a distributive law starting from a causal algebra.

We use a much simpler path here, and we prove directly that condition 3/ implies validity of corecursion up-to (1), without mentioning any distributive law. And we actually get more: we obtain a corecursion up-to principle even in those cases where the final sequence does not stabilise.

Recall the notion of *corecursive algebra* [12] (dual to recursive coalgebras [14]): a B -algebra (A, a) is said to be *corecursive* if for all B -coalgebras (X, f) , there is a unique morphism $f': X \rightarrow A$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f'} & A \\ f \downarrow & & \uparrow a \\ BX & \xrightarrow{Bf'} & BA \end{array}$$

Paul Levy observed that all elements of the final sequence (which are B -algebras by definition), are corecursive [25, p. 5, footnote 2].

When we have a causal transformation $\alpha: F\overline{B} \Rightarrow \overline{B}$, we prove here that for each stage k of the final sequence and for every BF -coalgebra (X, f) , there is a

unique morphism $f_k^\alpha: X \rightarrow \overline{B}(k)$ such that

$$\begin{array}{ccc} X & \xrightarrow{f_k^\alpha} & \overline{B}(k) \\ f \downarrow & & \uparrow \overline{B}(k+1,k) \\ BF X & \xrightarrow{BF f_k^\alpha} BF \overline{B}(k) \xrightarrow{B\alpha_k} & B\overline{B}(k) \end{array}$$

In other words, the BF -algebra $\overline{B}(k, k+1) \circ B\alpha_k$ is corecursive. This generalises Levy’s observation when F is the identity functor, and we recover that condition 3/ guarantees validity of corecursion up-to α_κ when the final sequence stabilises at κ (so that $\overline{B}(\kappa, \kappa+1)$ is an isomorphism, with inverse the final coalgebra).

This result shows that we can use up-to techniques to define operations on all approximations of the final coalgebra, even when this coalgebra does not exist!

Example 1.2. Consider the full powerset functor (\mathcal{P}), which does not admit a final coalgebra. At ordinal ω , its final sequence yields *compactly branching trees* $\mathcal{T}_\omega \triangleq \overline{\mathcal{P}}(\omega)$ [41]. Therefore, in order to define an operation on compactly branching trees, we can use $\mathcal{P}F$ -coalgebras for any causal transformation $F\overline{\mathcal{P}} \Rightarrow \overline{\mathcal{P}}$. For instance, we can define a similar operation as in Example 1.1. Call $c: \mathcal{T}_\omega \rightarrow \mathcal{P}\mathcal{T}_\omega$ the function mapping a (compactly branching) tree to its set of children, and define $\delta: \overline{\mathcal{P}} \Rightarrow \overline{\mathcal{P}}$ as before: $\delta_k(t) \triangleq \overline{\mathcal{P}}(k+1, k)(\{t\})$, so that $d \triangleq \delta_\omega: \mathcal{T}_\omega \rightarrow \mathcal{T}_\omega$ is a “delay” function on compactly branching trees. At ordinal ω , we get a unique function e such that

$$\begin{array}{ccc} \mathcal{T}_\omega & \xrightarrow{e} & \mathcal{T}_\omega \\ c \downarrow & & \uparrow \overline{\mathcal{P}}(\omega+1, \omega) \\ \mathcal{P}\mathcal{T}_\omega & \xrightarrow{\mathcal{P}e} \mathcal{P}\mathcal{T}_\omega \xrightarrow{\mathcal{P}d} & \mathcal{P}\mathcal{T}_\omega \end{array}$$

(Note however that here, unlike in Example 1.1, c is not an inverse of the morphism on the right: we only have $\overline{\mathcal{P}}(\omega+1, \omega) \circ c = \text{id}$. Therefore, this diagram is weaker than the one with a c going down on the right. \square)

We prove the aforementioned results in Section 3. Then we discuss compositionality of causal transformations (Section 4), and their lifting to coinductive predicates (Section 5).

Compositionality

A delicate point about up-to techniques for corecursion is compositionality. Indeed, given two algebras $a: FZ \rightarrow Z$ and $b: GZ \rightarrow Z$ on a final coalgebra, which are both valid for corecursion up-to (1), nothing guarantees that their composition ($a \circ Fb: FGZ \rightarrow Z$) or their coproduct ($[a; b]: (F+G)Z \rightarrow Z$) remains valid for corecursion up-to. For instance, knowing that for streams, both corecursion up-to \oplus and corecursion up-to \otimes are valid is not enough to deduce that corecursion up-to both \oplus and \otimes is valid too.

In the context of bisimilarity and coinductive predicates, such questions have been studied extensively by Davide Sangiorgi [36] early after the introduction of up-to techniques by Robin Milner [27]. This resulted in the concepts of *respectful functions* and then *compatible functions* [29], subclasses of valid up-to techniques enjoying nice compositionality properties.

In the context of categorical corecursion, distributive laws (condition 1/) and causal algebras (condition 2/) also enjoy such compositionality properties [11,32]. We show in Section 4 that the situation is similar with causal transformations (condition 3/): these can be organised as a category with arbitrary products, in which some generic and useful basic transformations always exist. As a consequence, simple causal transformations can be assembled into more complex ones, achieving the expected modularity for corecursion up-to.

Liftings and coinductive predicates

Corecursion (up-to) makes it possible to define operations on final coalgebras (e.g., \oplus and \otimes on streams). Once such operations have been defined, one often needs to reason about them, to establish some of their properties (e.g., both \oplus and \otimes are associative and commutative). This is why we also need to develop the theory of coinductive predicates, and to provide up-to proof techniques for those. Typically, reasoning about an operation defined by corecursion up-to requires related coinduction up-to techniques.

We exploit the fibrational approach to coinductive predicates [16,15,10] in Section 5, where we show how to get coinductive up-to techniques from causal transformations, provided that these causal transformation lift.

2 Preliminaries

We recall the basic categorical concepts we use in the paper. The reader familiar with universal coalgebra [21] may safely skip this section.

Coalgebras, algebras. For a category \mathcal{C} and a functor $F: \mathcal{C} \rightarrow \mathcal{C}$, an *F-coalgebra* is a pair (X, f) with X an object in \mathcal{C} and $f: X \rightarrow FX$. A *homomorphism* of *F-coalgebras* $h: (X, f) \rightarrow (Y, g)$ is a map $h: X \rightarrow Y$ such that $g \circ h = Fh \circ f$. Coalgebras for F form a category and an *F-coalgebra* is *final* if it is final in that category. An *F-algebra* is a pair (X, a) with X an object of \mathcal{C} and $a: FX \rightarrow X$.

Ordinals. We write ω for the first infinite ordinal. The category **Ord** of ordinals has as objects the ordinals themselves, and there is a unique arrow $j \rightarrow k$ iff $j \leq k$. This is similar to the usual view of a poset as a category, except that the ordinals do not form a set.

Final sequence. The *final sequence* of a functor $B: \mathcal{C} \rightarrow \mathcal{C}$ in a complete category \mathcal{C} is an ordinal indexed sequence of objects B_i with connecting morphisms $B_{j,i}: B_j \rightarrow B_i, i \leq j$ constructed in the following way. The first object is $B_0 \triangleq 1$,

the final object of \mathcal{C} and for a successor ordinal $j + 1$ we have $B_{j+1} \triangleq BB_j$. Further, $B_{i,i} \triangleq \text{id}$, $B_{i,0} \triangleq !_i: B_i \rightarrow 1$ and $B_{j+1,i+1} \triangleq BB_{j,i}$. For a limit ordinal λ we have $B_\lambda \triangleq \lim_{i < \lambda} B_i$ and $(B_{\lambda,i})_{i < \lambda}$ forms a limiting cone. We write \overline{B} for the final sequence of B , seen as a functor $\text{Ord}^{\text{op}} \rightarrow \mathcal{C}$. Accordingly, we write $\overline{B}(i)$ for B_i and $\overline{B}(j, i)$ for $B_{j,i}$.

We say that the final sequence \overline{B} stabilises at ordinal κ if $\overline{B}(\kappa + 1, \kappa)$ is an isomorphism. In this case, $\overline{B}(\kappa)$ is a final coalgebra [5, Theorem 1.3] (shown for the dual case of algebras in [2]). For ω -continuous functors (e.g., polynomial set functors), the final sequence stabilises at ω .

Fact 2.1. *Given a coalgebra $f: X \rightarrow BX$, we can construct a cone $f_i: X \rightarrow B_i$ over the final sequence, inductively. We start with the unique map $f_0 \triangleq !_X: X \rightarrow 1$ and for a successor we define $f_{i+1} = Bf_i \circ f$. For a limit ordinal k , the map $f_k: X \rightarrow B_k$ is the unique map obtained from the induction hypothesis and universality of B_k .*

Distributive laws. For functors $F, B: \mathcal{C} \rightarrow \mathcal{C}$, a distributive law of F over B is a natural transformation $\lambda: FB \Rightarrow BF$. When F comes with a monad structure (F, η, μ) , we call $\lambda: FB \Rightarrow BF$ a distributive law of a monad if it satisfies $B\eta = \lambda \circ \eta B$ and $\lambda \circ \mu B = B\mu \circ \lambda F \circ F\lambda$.

Every distributive law $\lambda: FB \Rightarrow BF$ over a functor B admitting a final coalgebra (Z, ζ) induces an algebra on Z by considering the coalgebra $\lambda_Z \circ F\zeta: FZ \rightarrow BFZ$ and using finality.

3 Corecursion up-to causal transformations

Our main result does not explicitly mention corecursion. Remember that we call a natural transformation of the form $F\overline{B} \Rightarrow \overline{B}$ a *causal transformation*. Given such a causal transformation, we inductively construct morphisms from any BF -coalgebra into each object of the final sequence \overline{B} . We get validity of corecursion up-to as a special case, when we have a final B -coalgebra at ordinal κ in the final sequence (Corollary 3.4 below).

Theorem 3.1. *Let $B, F: \mathcal{C} \rightarrow \mathcal{C}$ be endofunctors on a complete category \mathcal{C} and let $\alpha: F\overline{B} \Rightarrow \overline{B}$ be a causal transformation. For every BF -coalgebra $g: X \rightarrow BF X$ and every ordinal k , there is a unique map g_k^\dagger making the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{g_k^\dagger} & \overline{B}(k) \\ g \downarrow & & \uparrow \overline{B}(k+1, k) \\ BF X & \xrightarrow{BF g_k^\dagger} BF \overline{B}(k) \xrightarrow{B\alpha_k} & B\overline{B}(k) \end{array}$$

Proof. We proceed by transfinite induction on the ordinal k , additionally proving that for all ordinals $i < k$, we have

$$\overline{B}(k, i) \circ g_k^\dagger = g_i^\dagger \quad (2)$$

When $k = 0$, we have

$$\begin{array}{ccc} X & \xrightarrow{!_X} & 1 \\ g \downarrow & & \uparrow !_{B1} \\ BF X & \xrightarrow{BF!_X} BF 1 & \xrightarrow{B\alpha_0} B 1 \end{array}$$

Uniqueness and commutativity both follow from the uniqueness of the arrow $!_X$ from X into the final object. The property (2) holds trivially.

For the case $k = j + 1$ for some ordinal j , we assume the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{g_j^\dagger} & \overline{B}(j) \\ g \downarrow & & \uparrow \overline{B}(j+1,j) \\ BF X & \xrightarrow{BFg_j^\dagger} BF\overline{B}(j) & \xrightarrow{B\alpha_j} B\overline{B}(j) \end{array} \quad (3)$$

Now consider the following diagram, where g_{j+1}^\dagger is the map we wish to define:

$$\begin{array}{ccccc} X & \xrightarrow{g_{j+1}^\dagger} & & \xrightarrow{} & \overline{B}(j+1) \\ g \downarrow & & \nearrow BFg_j^\dagger & & \uparrow \overline{B}(j+2,j+1) \\ & & BF\overline{B}(j) & \xrightarrow{B\alpha_j} & \\ BF X & \xrightarrow{BFg_{j+1}^\dagger} & BF\overline{B}(j+1) & \xrightarrow{B\alpha_{j+1}} & B\overline{B}(j+1) \end{array} \quad (4)$$

The lower left triangle commutes as we have $\overline{B}(j+1,j) \circ g_{j+1}^\dagger = \overline{B}(j+1,j) \circ B\alpha_j \circ BFg_j^\dagger \circ g = g_j^\dagger$ by Eq. (3), meaning also $BF\overline{B}(j+1,j) \circ BFg_{j+1}^\dagger = BFg_j^\dagger$. The lower right trapezium commutes by naturality of α and functoriality of B :

$$\begin{array}{ccc} F\overline{B}(j) & \xrightarrow{\alpha_j} & \overline{B}(j) \\ \uparrow F\overline{B}(j+1,j) & & \uparrow \overline{B}(j+1,j) \\ F\overline{B}(j+1) & \xrightarrow{\alpha_{j+1}} & \overline{B}(j+1) \end{array} \quad \begin{array}{ccc} BF\overline{B}(j) & \xrightarrow{B\alpha_j} & B\overline{B}(j) \\ \uparrow BF\overline{B}(j+1,j) & & \uparrow B\overline{B}(j+1,j) \\ BF\overline{B}(j+1) & \xrightarrow{B\alpha_{j+1}} & B\overline{B}(j+1) \end{array}$$

Taking $g_{j+1}^\dagger \triangleq B\alpha_j \circ BFg_j^\dagger \circ g$, we see that this is the unique map we require, as it makes the diagram of Eq. (4) commute, and any such map making the successor case of (3) commute, must satisfy the above equation.

To show the property (2), let $i < k$ and consider the following diagram

$$\begin{array}{ccc}
 & & \overline{B}(i) \\
 & \nearrow^{g_i^\dagger} & \uparrow \overline{B}(j,i) \\
 X & \xrightarrow{g_j^\dagger} & \overline{B}(j) \\
 & \searrow_{g_k^\dagger} & \uparrow \overline{B}(k,j) \\
 & & \overline{B}(k = j + 1)
 \end{array}$$

Then (2) follows by the induction hypothesis and definition of g_k^\dagger .

Finally, we have the case of a limit ordinal k . In this case, we assume we have the diagram as in (3) commuting for all $j < k$ and note that, by definition of the final sequence, the maps $\overline{B}(k, j)$ for $j < k$ form a limiting cone. We would like to use the universal property of such a limiting cone to construct a map into its apex, the object $\overline{B}(k)$. To do this, we require that the maps $g_j^\dagger: X \rightarrow \overline{B}(j)$ form a cone over the final sequence. This holds by the induction hypothesis, specifically the property of (2) for all $j < k$, and so by the universal property, we have a map $g_k^\dagger: X \rightarrow \overline{B}(k)$. This also immediately establishes the limit case of property (2).

Now consider the following diagram:

$$\begin{array}{ccccc}
 & & \overline{B}(j) & & \\
 & \nearrow^{g_j^\dagger} & \uparrow & \longleftarrow \overline{B}(k,j) & \\
 X & \xrightarrow{g_k^\dagger} & \overline{B}(k) & & \\
 \downarrow g & \nearrow^{BFg_j^\dagger} & \uparrow \overline{B}(j+1,j) & \longleftarrow \overline{B}(k,j+1) & \\
 BFX & \xrightarrow{BFg_j^\dagger} & BF\overline{B}(j) & \xrightarrow{B\alpha_j} & \overline{B}(j+1) \\
 \parallel & \nearrow^{BF\overline{B}(k,j)} & \uparrow & \longleftarrow \overline{B}(k,j) & \\
 BFX & \xrightarrow{BFg_k^\dagger} & BF\overline{B}(k) & \xrightarrow{B\alpha_k} & \overline{B}(k+1)
 \end{array} \tag{5}$$

We would like, for all $j < k$, the outer route from X to $\overline{B}(j)$ to be equal to g_j^\dagger (in an equation: $g_j^\dagger = \overline{B}(k, j) \circ \overline{B}(k+1, k) \circ B\alpha_k \circ BFg_k^\dagger \circ g$). Then, by definition of g_k^\dagger as the unique map such that $\overline{B}(k, j) \circ g_k^\dagger = g_j^\dagger$ we will have commutativity of diagram (3) for the ordinal k , i.e., $g_k^\dagger = \overline{B}(k+1, k) \circ B\alpha_k \circ BFg_k^\dagger \circ g$.

Equationally, the proof goes as follows:

$$\overline{B}(k, j) \circ \overline{B}(k+1, k) \circ B\alpha_k \circ BFg_k^\dagger \circ g \quad (6)$$

$$= \overline{B}(j+1, j) \circ \overline{B}(k, j+1) \circ \overline{B}(k+1, k) \circ B\alpha_k \circ BFg_k^\dagger \circ g \quad (7)$$

$$= \overline{B}(j+1, j) \circ B\overline{B}(k, j) \circ B\alpha_k \circ BFg_k^\dagger \circ g \quad (8)$$

$$= \overline{B}(j+1, j) \circ B\alpha_j \circ BF\overline{B}(k, j) \circ BFg_k^\dagger \circ g \quad (9)$$

$$= \overline{B}(j+1, j) \circ B\alpha_j \circ BFg_j^\dagger \circ g \quad (10)$$

$$\stackrel{I.H.}{=} g_j^\dagger \quad (11)$$

To show this diagrammatically, we have included the inner part in Eq. (5). Then the upper right and right-hand triangles commute by definition of the final sequence (7), (8). The lower right trapezium commutes by naturality of α and functoriality of B (9). Commutativity of the lower left square follows by property (2) as well as functoriality of BF (10). The final equality (11) holds by the induction hypothesis. Together, this gives the required commutativity, and uniqueness of g_k^\dagger . \square

As announced in the introduction, the above theorem actually generalises some well-known facts:

Corollary 3.2 ([25, p. 5, footnote 2]). *Every element $\overline{B}(k)$ of the final sequence \overline{B} , seen as a B -algebra with structure map $\overline{B}(k+1, k)$, is corecursive.*

Proof. This is the special case of Theorem 3.1 where we take the identity functor for F , and the identity causal transformation for α . \square

Corollary 3.3. [2,5] *If the final sequence \overline{B} stabilises at ordinal κ , then $\overline{B}(\kappa)$ is a final coalgebra (with structure map $\overline{B}(\kappa+1, \kappa)^{-1}$).*

Proof. This is the special case of the previous corollary where we select the κ -th element of the final sequence. \square

More importantly for the present paper, Theorem 3.1 justifies condition 3/ from the introduction, for corecursion up to an algebra.

Corollary 3.4 (Corecursion up-to from causal transformations). *If the final sequence \overline{B} stabilises at ordinal κ , and if α is a causal transformation for B , then corecursion up-to the algebra α_κ is valid for B .*

The above corollary does not require any distributive law to start with, only a causal transformation. This is the point we want to emphasise in the present work, and we shall study causal transformations in the following sections. Nevertheless, this result also gives a new way to get validity of corecursion up-to from a distributive law, which we discuss in the remainder of this section:

Corollary 3.5 (Corecursion up-to from distributive laws). *Let $\lambda: FB \Rightarrow BF$ be a distributive law. If the final sequence \overline{B} yields a final coalgebra at ordinal κ , then corecursion up-to the algebra induced by λ on this final coalgebra is valid.*

Proof. It is shown in [32, Lemma 6.2] that such a distributive law induces a unique $\alpha: F\overline{B} \Rightarrow \overline{B}$ with the property that $\alpha_\kappa: F\overline{B}(\kappa) \rightarrow \overline{B}(\kappa)$ is the algebra induced by the distributive law on $\overline{B}(\kappa)$. These properties allow us to apply the above corollary, giving the required result. \square

The statement of this last corollary is very close to the work of Falk Bartels on generalised coinduction [6,7]: in our terminology, Theorems 3.8 and 3.9 of [6] can be summarised as follows:

Theorem 3.6. *Let $\lambda: FB \Rightarrow BF$ be a distributive law. If B has a final coalgebra and either of the following two conditions holds:*

- *the category \mathcal{C} has countable coproducts, or*
- *F is a monad and λ is a distributive law of a monad,*

then corecursion up-to the algebra induced by λ on the final coalgebra is valid.

In [32, Theorem 9.2] we showed a result analogous to Corollary 3.4 for the case of polynomial functors on \mathbf{Set} (with $\kappa = \omega$). The proof uses the *companion* of B (the final distributive law in a suitable sense) and uses Theorem 3.6 to conclude. The above direct proof from causal transformations is much simpler.

Let us highlight the differences between Theorem 3.6 and Corollary 3.5, first concerning the statements: 1/ we require a complete category where Bartels only needs countable coproducts (if any); and 2/ we require that the final sequence stabilises where he only needs the existence of a final coalgebra. Those differences disappear in the category of sets, which is complete and where the mere existence of a final coalgebra ensures that the final sequence stabilises [3].

Now let us compare the two proofs.

Under the first assumption of Theorem 3.6, Bartels uses the given distributive law to construct a B -coalgebra with the countable coproduct $\sum_{i=0}^{\infty} F^i X$ as carrier. There is a unique map from that coalgebra into the final B -coalgebra, which is used to obtain the unique solution for the BF -coalgebra structure. Under the second assumption, the monad structure on F can be used to construct more directly a B -coalgebra with carrier FX . This idea also underlies the generalised powerset construction developed in [37]: one determinises the given BF -coalgebra into a B -coalgebra with a larger carrier.

Interestingly, we never construct such a B -coalgebra in our proof, instead giving a direct construction of the required map into the final coalgebra, by transfinite induction. The downside is that we need ordinals and transfinite induction, where Falk Bartels does not. Therefore, in a sense his argument is more constructive (e.g., it can be formalised in type theory).

4 Compositionality

In this section, we show that the natural transformations of the form $F\overline{A} \Rightarrow \overline{B}$, which we will call *causal transformations from A to B* , enjoy good compositionality properties. This generalises the transformations $F\overline{B} \Rightarrow \overline{B}$ of earlier sections and in

fact, we define a category in which such causal transformations are the morphisms. This category has all products, and we obtain compositionality properties which are similar to those established in [11, Proposition 3.3] for compatible functors.

Definition 4.1. *For functors $F: \mathcal{C} \rightarrow \mathcal{D}$, $A: \mathcal{C} \rightarrow \mathcal{C}$ and $B: \mathcal{D} \rightarrow \mathcal{D}$, a causal transformation from A to B is a natural transformation $\alpha: F\overline{A} \Rightarrow \overline{B}$.*

Theorem 4.2. *We have a category \mathcal{K} with the following data:*

- objects are pairs (A, \mathcal{C}) with $A: \mathcal{C} \rightarrow \mathcal{C}$ a functor on a complete category \mathcal{C} .
- morphisms from (A, \mathcal{C}) to (B, \mathcal{D}) are pairs (F, α) with $\alpha: F\overline{A} \Rightarrow \overline{B}$.

Proof. The identity on an object (A, \mathcal{C}) is the pair (Id, id) . The composition of two morphisms $(F, \alpha): (X, \mathcal{C}) \rightarrow (Y, \mathcal{D})$ and $(G, \beta): (Y, \mathcal{D}) \rightarrow (Z, \mathcal{E})$ is given by

$$(G \circ F)\overline{X} \xrightarrow{G\alpha} G\overline{Y} \xrightarrow{\beta} \overline{Z} \quad \square$$

When the categories \mathcal{C} and \mathcal{D} are clear from the context, we write $A \dot{\rightarrow} B$ for the homset from (A, \mathcal{C}) to (B, \mathcal{D}) in \mathcal{K} .

Theorem 4.3. *The category \mathcal{K} has all products.*

Proof. We only deal with binary products to ease notation, leaving the general case to the reader.

For objects $(A_1, \mathcal{C}_1), (A_2, \mathcal{C}_2)$ in \mathcal{K} we construct the pair $(A_1 \times A_2, \mathcal{C}_1 \times \mathcal{C}_2)$, taking products in the category **Cat**. The projections must be causal transformations $p_i: F_i\overline{A_1 \times A_2} \Rightarrow \overline{A_i}$. To obtain these we take F_i to be exactly the projection π_i in **Cat**, together with families of maps $p_i(k): \pi_i \circ \overline{A_1 \times A_2}(k) \rightarrow \overline{A_i}(k)$ which necessarily consist of identity maps, as we show now.

First, we claim that $\overline{A_1 \times A_2} = \langle \overline{A_1}, \overline{A_2} \rangle$. To prove this we use transfinite induction. For the successor case, assume that $\overline{A_1 \times A_2}(i) = \langle \overline{A_1}, \overline{A_2} \rangle(i) = (\overline{A_1}(i), \overline{A_2}(i))$ for some i . Then, we have

$$\overline{A_1 \times A_2}(i+1) = (A_1 \times A_2)(\overline{A_1 \times A_2}(i)) \quad (12)$$

$$= (A_1 \times A_2)(\overline{A_1}(i), \overline{A_2}(i)) \quad (13)$$

$$= (A_1(\overline{A_1}(i)), A_2(\overline{A_2}(i))) \quad (14)$$

$$= (\overline{A_1}(i+1), \overline{A_2}(i+1)) \quad (15)$$

$$= \langle \overline{A_1}, \overline{A_2} \rangle(i+1) \quad (16)$$

where Eq. (13) follows from the induction hypothesis.

For the limit case, let k be some limit ordinal and assume that $\overline{A_1 \times A_2}(l) = \langle \overline{A_1}, \overline{A_2} \rangle(l)$ holds for all $l < k$. Now, we have

$$\overline{A_1 \times A_2}(k) = \lim_{l < k} \overline{A_1 \times A_2}(l) \quad (17)$$

$$= \lim_{l < k} \langle \overline{A_1}, \overline{A_2} \rangle(l) \quad (18)$$

$$= \langle \overline{A_1}, \overline{A_2} \rangle(k) \quad (19)$$

where Eq. (18) follows from the induction hypothesis. The last step, Eq. (19), follows from how limits are computed in a product category; the important point being that cones over a pairing of functors are the same as pairs of cones over the component functors.

From the established equality $\overline{A_1 \times A_2} = \langle \overline{A_1}, \overline{A_2} \rangle$, we conclude that $\pi_i \circ \overline{A_1 \times A_2}(k) = \overline{A_i}(k)$ so that we can take each component $p_i(k)$ of our projections to be the identity. Naturality then holds trivially. It remains to show that the above construction is universal.

Suppose we have an object $(Q: \mathcal{D} \rightarrow \mathcal{D}, \mathcal{D}) \in \mathcal{K}$ with morphisms $(F_i, \alpha_i): Q \rightarrow (A_i, \mathcal{C}_i)$ for $i = 1, 2$. Then we can construct the pair $(\langle F_1, F_2 \rangle, (\alpha_1, \alpha_2))$ of the functors and causal transformations so that $(\alpha_1, \alpha_2): \langle F_1, F_2 \rangle \overline{Q} \rightarrow \overline{A_1 \times A_2}$ is a map in $\mathcal{C}_1 \times \mathcal{C}_2$. As this is a product in Cat , we have the required property that for all $j = 1, 2$

$$(\pi_j, \text{id}) \circ (\langle F_1, F_2 \rangle, (\alpha_1, \alpha_2)) = (F_j, \alpha_j)$$

More concretely, the components of the causal transformations are maps of type

$$\langle F_1, F_2 \rangle \overline{Q}(l) \rightarrow \langle \overline{A_1}(l), \overline{A_2}(l) \rangle = (F_1 \overline{Q}(l), F_2 \overline{Q}(l)) \rightarrow (\overline{A_1}(l), \overline{A_2}(l))$$

which, by definition of the product category, consist of maps in \mathcal{C}_1 and \mathcal{C}_2 which we consider in parallel. Further, uniqueness follows from the definition of the pairing $\langle F_1, F_2 \rangle$ and of maps in the product category. Finally, naturality follows by assumption on the α_i , thus, we indeed have a categorical product. \square

Finally, we have the following basic morphisms in \mathcal{K} , giving access to up-to constant techniques and to coproduct of up-to techniques.

Proposition 4.4. *For every endofunctor $B: \mathcal{C} \rightarrow \mathcal{C}$, we have the following morphisms in \mathcal{K} :*

1. $(\Delta_X, \delta_f): B \dot{\rightarrow} B$ where X is the carrier of a coalgebra $f: X \rightarrow BX$ and Δ_X is the constant functor associated to X .
2. $(\coprod: \mathcal{C}^I \rightarrow \mathcal{C}, \gamma): B^I \dot{\rightarrow} B$, assuming that coproducts in \mathcal{C} exist.

Proof. For Item 1, we define δ_f as the cone given by Fact 2.1. Item 2 follows from the universal property of coproduct. \square

Together with Theorem 4.2 and Theorem 4.3, the above proposition makes it possible to define complex causal transformations out of basic ones, thus enabling compositional proofs of validity for complex corecursion up-to schemes.

Example 4.5. Suppose we work with streams, and we want to define a unary operation f satisfying the following equations

$$\begin{aligned} f(x)_0 &= x_0 \\ f(x)' &= (x \oplus f(x')) \oplus f(x'') \end{aligned}$$

In order to use corecursion up-to, we need an algebra combining the ability to call \oplus twice in a row, with arguments which are either an existing stream (x) , or corecursive calls to f on some existing streams $(f(x')$ and $f(x''))$.

We can use for that the functor $F(F(\mathbb{R}^\omega + \text{Id}) + \text{Id})$ where $FX = X^2$, and the associated algebra $\oplus \circ [\oplus \circ [\text{id}; \text{id}]; \text{id}]$ on \mathbb{R}^ω .

Thanks to the above results, showing that such an algebra arises from a causal transformation amounts to showing that \oplus arises from a causal transformation, which is straightforward. This is sufficient because we know that we have identity morphisms in \mathcal{K} and we can cope with the constant \mathbb{R}^ω functor via Item 1 of Proposition 4.4 since \mathbb{R}^ω is a coalgebra (the final one). Since we can take coproducts in Set , we can finally apply Item 2 of Proposition 4.4 and compose these constructions in \mathcal{K} to obtain the required causal transformation. \square

As is the case for distributive laws [34,40,24,22,32], we can also define maps between causal transformations.

Definition 4.6. *Given two causal transformations $(F, \alpha), (G, \beta): A \dot{\rightarrow} B$, an arrow from (F, α) to (G, β) is a natural transformation $\kappa: F \Rightarrow G$ such that $\beta \circ \kappa \bar{A} = \alpha$. These arrows turn \mathcal{K} into a 2-category.*

We claim that the arrows of the above definition turn \mathcal{K} into a 2-category, however we have not checked the details and the consequences of this are currently unclear. Of most interest is a possible correspondence with an analogous category DL with endofunctors as objects and distributive laws as maps [32, Definition 6.1]. The definition of maps between distributive laws again yields a 2-category, from which there may be a 2-functor whose image on distributive laws are exactly the causal transformations obtained via the construction in [32, Lemma 6.2]. There, it is already shown that the construction extends to a functor from distributive laws of A over B to causal transformations from A to B , so giving a 2-functor should generalise this result to the setting where A and B are not fixed. It is further known that, under certain conditions (e.g. the existence of the companion), we can also go back from causal transformations to distributive laws. We would like to further investigate this correspondence in the 2-categorical context and its relation to up-to techniques.

5 Up-to techniques for coinductive proofs

As explained in the previous sections, a causal transformation $\alpha: F\bar{B} \Rightarrow \bar{B}$ gives rise to a valid corecursion-up-to principle. In this section we show how it also induces up-to techniques for *coinductive proofs*. We take a fibrational view on coinductive predicates [16,15], where coalgebras in the base category are viewed as state-based systems, and coinductive proofs arise as coalgebras in the *fibre* above the state space. These fibres are often assumed to be a complete lattices in this setting. The key technical result in this section is that any causal transformation in the base category gives rise to a causal transformation in the fibre above the final coalgebra (assumed to be a complete lattice), which enables its use as an up-to technique for coinductive proofs.

5.1 Coinduction up-to in a lattice

Let us first briefly recall the basic notions of coinduction and up-to techniques in complete lattices [33]. This can be viewed as a special case of the theory of coalgebras and corecursion (up-to), by instantiating the base category with a complete lattice viewed as a posetal category.

Let $b: L \rightarrow L$ be a monotone map on a complete lattice L . By the Knaster-Tarski theorem, b has a greatest fixed point νb , which is also the greatest post-fixed point. This gives a coinduction principle: if $x \leq b(x)$, then $x \leq \nu b$.

The definition of corecursion up-to given in the Introduction (1) instantiates to the following coinduction up-to principle. Given a function $f: L \rightarrow L$ such that $f(\nu b) \leq \nu b$ (this is the algebra structure), coinduction up to f is valid if $x \leq bf(x)$ implies $x \leq \nu b$. This amounts to the established notion of soundness in this setting, with the additional requirement that f preserves νb .

Conditions similar to 1/ from the Introduction have been developed independently in this setting [36,29]: a function f is called *b-compatible* if $fb \leq bf$, that is, there is a distributive law of f over b . Compatibility implies validity in the above sense, and enjoys good compositionality properties (which the class of valid or sound functions does not).

Condition 2/ has no clear counterpart in this setting. In contrast, we considered condition 3/ in previous work [31,32]. In this case, a causal transformation $f\bar{b} \Rightarrow \bar{b}$ is just a function which preserves the final sequence at any point, i.e., $fb^i(\top) \leq b^i(\top)$ for every ordinal i . In *op. cit.* we show that f satisfies this property if and only if $f \leq t$, where t is the *companion* of b , that is, the greatest compatible function [30]. Such a property is very close to Parrow and Weber’s characterisation of the greatest respectful function [28] (which happens to coincide with the greatest compatible function—the companion). A constructive version was also used later in the context of Agda [13].

In the lattice-theoretic setting, these causal transformations form a class of valid enhancements which can be more convenient to work with than the stricter requirement of being compatible. In the remainder of this section, we show how to move from “categorical” causal transformations, in a base category where coalgebras are state-based systems, to these lattice-theoretic causal transformations. This is enabled by the use of fibrations, which provide us precisely with the infrastructure to move from coalgebras (as state-based systems) to coinductive proofs thereon.

5.2 Background on coinduction in a fibrational setting

We recall the basics of coinductive predicates in a fibration, but only briefly; see, for instance, [15] for a detailed introduction. First, let $B: \mathbf{Set} \rightarrow \mathbf{Set}$ and recall that *relation lifting* assigns to every relation $R \subseteq X \times X$ a relation $\text{Rel}(B)(R)$ on BX , defined by

$$\text{Rel}(B)(R) = \{(B\pi_1(t), B\pi_2(t)) \mid t \in BR\}.$$

A *bisimulation* on a B -coalgebra (X, f) is then a relation $R \subseteq X \times X$ such that $R \subseteq (f \times f)^{-1}(\text{Rel}(B)(R))$, and bisimilarity is the greatest fixed point of the monotone map $(f \times f)^{-1}(\text{Rel}(B)(-)) : \text{Rel}_X \rightarrow \text{Rel}_X$, where Rel_X is the lattice of relations on X . It arises as the limit of the final sequence of $(f \times f)^{-1}(\text{Rel}(B)(-))$.

The situation can be massively generalised by moving from relations on sets to a *fibration* $p: \mathcal{E} \rightarrow \mathcal{C}$, and from the relation lifting to arbitrary liftings of endofunctors on \mathcal{C} . We omit the definition of fibration here (see [19]); we however recall the key notions associated to them.

For such a fibration p , we say an object R in \mathcal{E} is *above* an object X in \mathcal{C} if $p(R) = X$, and similarly for morphisms. Further, the *fibre* \mathcal{E}_X above an object X in \mathcal{C} is the subcategory of \mathcal{E} consisting of all objects above X , and all morphisms above id_X . For every arrow $f: X \rightarrow Y$ there is a *reindexing functor* $f^*: \mathcal{E}_Y \rightarrow \mathcal{E}_X$.

Throughout this section, we assume that $p: \mathcal{E} \rightarrow \mathcal{C}$ is a CLat_\wedge -*fibration* (e.g., [38]), which means that each fibre \mathcal{E}_X is a complete lattice, and reindexing preserves arbitrary meets. Below, we shall often refer explicitly to this poset structure, by writing $R \leq S$ if there exists an arrow from R to S in \mathcal{E}_X . These are instances of topological functors [17]. Every CLat_\wedge -fibration is a bifibration, which means every reindexing functor f^* has a left adjoint \coprod_f .

A *lifting* of a functor $B: \mathcal{C} \rightarrow \mathcal{C}$ is a functor $\mathbb{B}: \mathcal{E} \rightarrow \mathcal{E}$ such that $p \circ \mathbb{B} = B \circ p$. For such a lifting and an object X in \mathcal{C} , the functor \mathbb{B} restricts to a functor between fibres $\mathbb{B}_X: \mathcal{E}_X \rightarrow \mathcal{E}_{BX}$. A lifting (B, \mathbb{B}) is a *fibration map* if

$$(Bf)^* \circ \mathbb{B}_Y = \mathbb{B}_X \circ f^*$$

for any arrow $f: X \rightarrow Y$ in \mathcal{C} (the inequality from right to left holds for any lifting).

Given a B -coalgebra (X, g) and a lifting (B, \mathbb{B}) , we define the functor (that is, monotone map)

$$g^* \circ \mathbb{B}_X: \mathcal{E}_X \rightarrow \mathcal{E}_X.$$

Its final coalgebra (greatest fixed point) $\nu(g^* \circ \mathbb{B}_X)$ exists by the assumption that each fibre is a complete lattice, and is referred to as the *coinductive predicate* defined by \mathbb{B}_X . It is the greatest post-fixed point (coalgebra) of $g^* \circ \mathbb{B}_X$; such post-fixed points are called *invariants* in [15]. This gives rise to the lattice-theoretic coinductive proof technique: to prove that an object R in \mathcal{E} is below the greatest fixed point, it suffices to show it is a post-fixed point of $g^* \circ \mathbb{B}_X$.

Example 5.1. Consider the category Rel where an object is a pair (R, X) of sets with $R \subseteq X \times X$, and an arrow from (R, X) to (S, Y) is a map $f: X \rightarrow Y$ such that $f(R) \subseteq S$. Reindexing is given by inverse image. The forgetful functor $p: \text{Rel} \rightarrow \text{Set}$ mapping (R, X) to X is a CLat_\wedge -fibration. The relation lifting $\text{Rel}(B)$ is a lifting of B , often referred to as the *canonical* lifting in this general setting. For a coalgebra (X, g) , $g^* \circ \text{Rel}(B)_X$ is precisely the monotone map described at the beginning of this subsection, whose greatest fixed point is bisimilarity on (X, g) . \square

Other liftings of B give rise to other coinductive predicates. For instance, for the powerset functor $\mathcal{P}: \mathbf{Set} \rightarrow \mathbf{Set}$, consider the lifting

$$\mathbf{Rel}_{\leq}(\mathcal{P})(R) = \{(S, U) \mid \forall x \in S. \exists y \in U. (x, y) \in R\}.$$

Coalgebras for \mathcal{P} are transition systems, post-fixed points of $g^* \circ \mathbf{Rel}_{\leq}(\mathcal{P})$ are simulations, and its greatest fixed point is similarity. This is an instance of a much more general fibrational characterisation of similarity [18]. Other examples of coinductive predicates that have been explored in a fibrational setting are behavioural distances [4,9,38] and various unary predicates and invariants [15,10] in the fibration of predicates over sets.

In the abstract setting of coinductive predicates via liftings, we can consider *up-to techniques* in the fibre as well, basically by instantiating the setting in Section 3. A systematic construction of such up-to techniques in a fibration is in [11]. Of particular interest is the *contextual closure*: given a lifting \mathbb{F} of F and an algebra $a: FX \rightarrow X$, it is defined as the map $\coprod_a \circ \mathbb{F}_X: \mathcal{E}_X \rightarrow \mathcal{E}_X$.

Example 5.2. On streams, the algebra $\oplus: F\mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ for the squaring functor $FX = X^2$, together with the canonical lifting of F , gives rise to the following monotone function on relations on streams:

$$\begin{aligned} [\oplus]: \mathcal{P}(\mathbb{R}^\omega \times \mathbb{R}^\omega) &\rightarrow \mathcal{P}(\mathbb{R}^\omega \times \mathbb{R}^\omega) \\ R &\mapsto \{(x \oplus y, z \oplus t) \mid x R y \text{ and } z R t\} \end{aligned}$$

Such a function often proves useful as an up-to technique in bisimulation proofs on streams: it makes it possible to use the coinductive hypothesis under calls to pointwise addition, and to get rid of common sub-expressions. This is typically convenient to reason about operations defined by corecursion up to \oplus , like shuffle product (see, e.g., the example in [30, Section 5]). \square

In [10, Theorem 6.7], it is shown that $\coprod_a \circ \mathbb{F}_X$ is valid (even compatible) if there is a distributive law $\lambda: FB \Rightarrow BF$ such that (X, a, g) is a λ -bialgebra, and λ lifts to a distributive law $\mathbb{F}\mathbb{B} \Rightarrow \mathbb{B}\mathbb{F}$.

5.3 Causal transformations in the fibre

We now show how to move from a causal transformation in the base category of a fibration to one in the fibre above the state space of the final coalgebra.

Assumption 5.3. *Throughout this subsection, we assume:*

- a \mathbf{CLat}_\wedge -fibration $p: \mathcal{E} \rightarrow \mathcal{C}$ into a complete category \mathcal{C} ;
- endofunctors $B, F: \mathcal{C} \rightarrow \mathcal{C}$ such that the final sequence of B stabilises at some ordinal κ ; thus the final coalgebra is given by $(\overline{B}(\kappa), \zeta)$;
- liftings \mathbb{B}, \mathbb{F} of B and F respectively;
- a causal transformation $\alpha: F\overline{B} \Rightarrow \overline{B}$.

Now, consider the final B -coalgebra $(\overline{B}(\kappa), \zeta)$. The key idea is to extract from the above data a causal transformation

$$\coprod_{\alpha_\kappa} \circ \mathbb{F}_{\overline{B}(\kappa)} \circ \overline{(\zeta^* \circ \mathbb{B}_{\overline{B}(\kappa)})} \leq \overline{(\zeta^* \circ \mathbb{B}_{\overline{B}(\kappa)})}.$$

Here, $f \triangleq \coprod_{\alpha_\kappa} \circ \mathbb{F}_{\overline{B}(\kappa)} : \mathcal{E}_{\overline{B}(\kappa)} \rightarrow \mathcal{E}_{\overline{B}(\kappa)}$ is the up-to technique induced by the lifting \mathbb{F} and the algebra α_κ , whereas $b \triangleq (\zeta^* \circ \mathbb{B}_{\overline{B}(\kappa)}) : \mathcal{E}_{\overline{B}(\kappa)} \rightarrow \mathcal{E}_{\overline{B}(\kappa)}$ is the monotone map in the fibre above $\overline{B}(\kappa)$, whose greatest fixed point (final coalgebra) is the coinductive predicate defined by \mathbb{B} . Having such a causal transformation means that the up-to technique f is a valid enhancement for b . This is the contents of Theorem 5.5.

In the proof of Theorem 5.5, we use the following lemma, which relates the final sequence of $\zeta^* \circ \mathbb{B}_{\overline{B}(\kappa)}$ in the fibre to the final sequence of the lifting \mathbb{B} . Recall from Fact 2.1 that every coalgebra (X, g) induces a cone $g_i : X \rightarrow \overline{B}(i)$ over the final sequence.

Lemma 5.4. *Suppose (B, \mathbb{B}) is a fibration map, and let $g : X \rightarrow BX$ be a coalgebra. For any ordinal i , we have $(g^* \circ \mathbb{B}_X)(i) = g_i^* \circ \mathbb{B}(i)$.*

Proof. By transfinite induction on i . The base and successor case are shown in [23, Lemma 5.4]. For k a limit ordinal, we compute:

$$\begin{aligned} g_k^*(\lim_{i < k} \overline{\mathbb{B}}(i)) &= g_k^*(\bigwedge_{i < k} B_{k,i}^*(\overline{\mathbb{B}}(i))) \\ &= \bigwedge_{i < k} g_k^* \circ \overline{B}(k, i)^*(\overline{\mathbb{B}}(i)) \\ &= \bigwedge_{i < k} (\overline{B}(k, i) \circ g_k)^*(\overline{\mathbb{B}}(i)) \\ &= \bigwedge_{i < k} g_i^*(\overline{\mathbb{B}}(i)) \\ &= \bigwedge_{i < k} \overline{(g^* \circ \mathbb{B}_X)}(i). \end{aligned}$$

The first step follows from the computation of limits in \mathbf{CLat}_\wedge -fibrations, see, e.g., [38]; this is a consequence of [19, Prop. 9.2.1]. The second step follows from the definition of \mathbf{CLat}_\wedge -fibrations. The third since \mathbf{CLat}_\wedge -fibrations are split. The fourth since the g_i 's form a cone over the final sequence. And the last by the induction hypothesis. \square

This brings us to the main result of this section:

Theorem 5.5. *Suppose (B, \mathbb{B}) is a fibration map, and suppose that for every ordinal i , we have $\coprod_{\alpha_i} (\mathbb{F}\overline{\mathbb{B}}(i)) \leq \overline{\mathbb{B}}(i)$. Then for every ordinal i :*

$$\coprod_{\alpha_\kappa} \circ \mathbb{F}_{\overline{B}(\kappa)} \circ \overline{(\zeta^* \circ \mathbb{B}_{\overline{B}(\kappa)})}(i) \leq \overline{(\zeta^* \circ \mathbb{B}_{\overline{B}(\kappa)})}(i).$$

Proof. For any i , we have

$$\begin{aligned}
 & \prod_{\alpha_\kappa} \circ \mathbb{F}_{\overline{B}(\kappa)} \circ \overline{(\zeta^* \circ \mathbb{B}_{\overline{B}(\kappa)})}(i) \\
 &= \prod_{\alpha_\kappa} \circ \mathbb{F}_{\overline{B}(\kappa)} \circ \zeta_i^*(\overline{\mathbb{B}}(i)) && \text{(Lemma 5.4)} \\
 &\leq \prod_{\alpha_\kappa} \circ (F\zeta_i)^* \circ \mathbb{F}_{\overline{B}(i)}(\overline{\mathbb{B}}(i)) && \text{(basic property liftings)} \\
 &\leq \zeta_i^* \circ \prod_{\alpha_i} \circ \mathbb{F}_{\overline{B}(i)}(\overline{\mathbb{B}}(i)) && \text{(follows from naturality } \alpha) \\
 &\leq \zeta_i^*(\overline{\mathbb{B}}(i)) && \text{(by assumption)}
 \end{aligned}$$

For the one-but-last step, note that $\zeta_i = \overline{B}(\kappa, i)$, and thus naturality of α implies $\alpha_i \circ F\zeta_i = \zeta_i \circ \alpha_\kappa$, and hence $(F\zeta_i)^* \circ \alpha_i^* = \alpha_\kappa^* \circ \zeta_i^*$. The desired inequality is obtained as the mate:

$$\prod_{\alpha_\kappa} \circ (F\zeta_i)^* \leq \prod_{\alpha_\kappa} \circ (F\zeta_i)^* \circ \alpha_i^* \circ \prod_{\alpha_i} = \prod_{\alpha_\kappa} \circ \alpha_\kappa^* \circ \zeta_i^* \circ \prod_{\alpha_i} \leq \zeta_i^* \circ \prod_{\alpha_i},$$

using the unit of the adjunction $\prod_{\alpha_i} \dashv \alpha_i^*$ in the first step, and the counit of $\prod_{\alpha_\kappa} \dashv \alpha_\kappa^*$ in the last step. \square

The condition that $\prod_{\alpha_i}(\mathbb{F}\overline{\mathbb{B}}(i)) \leq \overline{\mathbb{B}}(i)$ holds for all i is equivalent to the requirement that α lifts to a natural transformation $\mathbb{F}\overline{\mathbb{B}} \Rightarrow \overline{\mathbb{B}}$. In our setting of \mathbf{CLat}_\wedge -fibrations, it basically says that \mathbb{F} and α need to preserve all approximations of the coinductive predicate of interest. For instance, for the lifting for similarity of transition systems defined below Example 5.1, the i -th component of the final sequence of $\mathbf{Rel}_{\leq}(\mathcal{P})$ consists of the i -steps similarity relation \leq_i ; and the condition that α lifts means that the direct image of $\mathbb{F}(\leq_i)$ under $\alpha \times \alpha$ is contained in \leq_i . This is the case, for instance, if \mathbb{F} takes the transitive closure (and F is the identity functor, α the identity map); in that case, the requirement simply amounts to the fact that each element in the final sequence of $\mathbf{Rel}_{\leq}(\mathcal{P})$ is transitive. Indeed, up-to transitive closure is valid for similarity.

If $\mathbb{F} = \mathbf{Rel}(F)$ and $\mathbb{B} = \mathbf{Rel}(B)$ in the relation fibration $\mathbf{Rel} \rightarrow \mathbf{Set}$, the condition that α lifts vacuously holds. This follows, for instance, by the following lemma, which is closely related to the coinduction principle in [16]. It makes use of the equality functor $\mathbf{Eq}: \mathbf{Set} \rightarrow \mathbf{Rel}$, which maps a set X to the diagonal $\{(x, x) \mid x \in X\}$.

Lemma 5.6. *For any functor $B: \mathbf{Set} \rightarrow \mathbf{Set}$, we have $\overline{\mathbf{Rel}(B)} = \mathbf{Eq} \circ \overline{B}$.*

Proof. By transfinite induction. For a limit ordinal k , use that \mathbf{Eq} has a left adjoint (quotients, see [16]) so that it preserves limits. Then

$$\overline{\mathbf{Rel}(B)}(k) = \lim_{i < k} \overline{\mathbf{Rel}(B)}(i) = \lim_{i < k} \mathbf{Eq}(\overline{B}(i)) = \mathbf{Eq}(\lim_{i < k} \overline{B}(i)) = \mathbf{Eq} \circ \overline{B}(k).$$

For a successor ordinal, we use that relation lifting preserves equality [21], that is, $\text{Rel}(B) \circ \text{Eq} = \text{Eq} \circ B$:

$$\begin{aligned} \overline{\text{Rel}(B)}(i+1) &= \text{Rel}(B)(\overline{\text{Rel}(B)}(i)) \\ &= \text{Rel}(B)(\text{Eq}(\overline{B}(i))) \\ &= \text{Eq}(B\overline{B}(i)) \\ &= \text{Eq}(\overline{B}(i+1)). \quad \square \end{aligned}$$

Using the above lemma we get $\text{Rel}(F) \circ \overline{\text{Rel}(B)} = \text{Rel}(F) \circ \text{Eq} \circ \overline{B} = \text{Eq} \circ F \circ \overline{B}$, again using that relation lifting preserves equality, and thus we can define the lifting of α simply as $\text{Eq}(\alpha)$. Alternatively, we expect the fact that any causal transformation lifts in this way also follows similarly to the fact that any distributive law lifts in this case [10], using that Rel is a 2-functor [21].

Curiously, in Theorem 5.5 we assumed that (B, \mathbb{B}) is a fibration map. For the canonical relation lifting $\text{Rel}(B)$, this means that B needs to preserve weak pullbacks for the above theorem to apply. This is in contrast to the result in [10], which does not make this requirement.

However, Theorem 5.5 cannot be easily generalised to functors that do not preserve weak pullbacks. Consider, for instance, the case where $F = \text{Id}$, $\alpha = \text{id}$, and $\mathbb{F}(R)$ is the least equivalence relation containing R . The requirement that α lifts then says that each element of the final sequence of \mathbb{B} is an equivalence relation.

A classical simple example of a functor $B: \text{Set} \rightarrow \text{Set}$ which does not preserve pullbacks is the one defined on objects as $B(X) = \{(x, y, z) \mid |\{x, y, z\}| \leq 2\}$, see [1]. The final sequence of B stabilises immediately. The final sequence of its canonical lifting $\text{Rel}(B)$ consists simply of the equality relation on a singleton (cf. Lemma 5.6). This is clearly an equivalence relation., and therefore the condition that α lifts holds in this case (thus taking \mathbb{B} to be $\text{Rel}(B)$). But up-to-equivalence is not sound for this functor (a counterexample for up-to-bisimilarity is given in [35], this can be adapted).

Indeed, the condition from [10] that the distributive law between F and B lifts, is much stronger: it says that $\text{Rel}(B)$ should commute with the equivalence closure functor \mathbb{F} . This is not the case in general, if B does not preserve weak pullbacks. In fact, this example shows that not all causal transformations are definable by a distributive law.

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