



HAL
open science

A Time-Space Integro-Differential Economic Model of Epidemic Control

Carmen Camacho, Rodolphe Desbordes, Davide Torre

► **To cite this version:**

Carmen Camacho, Rodolphe Desbordes, Davide Torre. A Time-Space Integro-Differential Economic Model of Epidemic Control. 2022. hal-03693086

HAL Id: hal-03693086

<https://hal.science/hal-03693086>

Preprint submitted on 10 Jun 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Time-Space Integro-Differential Economic Model of Epidemic Control

Carmen Camacho · Rodolphe
Desbordes · Davide La Torre

Received: date / Accepted: date

Abstract In this paper we propose a time-space economic model to control the evolution and the spread of a disease. The underlying epidemiological model is formulated as a reaction-diffusion integro-differential partial differential equation. This specific model formulation, supported by empirical data, contains three different terms: a pure diffusion term, a linear growth term, and an integral term. These three terms capture different diffusion channels of a transmissible disease: a pure spatial diffusion effect, a local effect and a global effect. The decision maker aims at deciding the optimal effort to be implemented in order to control the number of infections and, at the same time, minimize the cost of treatment. We analyze both the finite and the infinite horizon cases, and provide the closed-form expression of the optimal policy to be implemented to control the epidemic while sustaining economic growth.

Keywords Epidemics · Macroeconomic Outcomes · Mitigation Policies

1 Introduction

The economic literature has demonstrated the crucial role played by communicable diseases in shaping economic development and how they have impacted on both industrialized and developing countries through a variety of different channels. The impact and severity of communicable diseases on the economy is amplified by the need to finance public health policies diverting

Carmen Camacho
CNRS at Paris School of Economics, France
E-mail: carmen.camacho@psemail.eu

Rodolphe Desbordes
SKEMA Business School, Paris, France. E-mail: rodolphe.desbordes@skema.edu

Davide La Torre
SKEMA Business School and Universite Cote d'Azur, Sophia Antipolis, France. E-mail: davide.latorre@skema.edu

resources from productive activities. Over the past years we have witnessed a growing economic epidemiology literature that tries to understand the mechanisms through which health policies may be used to contrast the spread of communicable diseases. COVID-19 is a recent example of highly contagious virus-induced communicable disease, transmitted via droplets and contaminated objects during close unprotected contact between a healthy and an infected person [25]. As infected people move away from the location in which they were contaminated, uncontaminated locations farther and farther away become centres of infection in their own right. Since the initial outbreak of the disease in China in late 2019, COVID-19 has generated more than 275 million cases and more than 5 million deaths at world level. A broad variety of policy measures have been implemented everywhere in the world in order to contain the spread of the disease, including lockdown, quarantines, social distancing, limitations on mobility ([7, 10]). COVID-19 has affected human activities, undermined the economy, turned the lives of billions of people around the globe upside down, and significantly affected the health, economic, environmental and social domains [22]. COVID-19 pandemic poses unprecedented challenges to the world health system, prompting academics and health professionals to develop appropriate solutions.

The present paper proposes a theoretical economic model to explain the evolution of a transmissible disease taking into account spatial diffusion, the local effect, and the global effect that the society may have on local contamination. Our proposal is not a merely theoretical abstract artifact. On the contrary, we initially empirically show with the help of a dynamic heterogeneous spatial econometric model with common factors¹ applied to data on the evolution of the COVID-19 pandemic in Europe, that all three aforementioned diffusion channels exist and represent distinct transmission mechanisms of the disease. Hence, although challenging, our approach to modelling COVID-19 proves to be comprehensive and complete. COVID-19 spread has been modelled by means of different classical compartmental epidemiological models such as SIS, SIR, SEIR, etc. and different optimal control models have been proposed to mitigate the spread of the virus while limiting the economic cost associated with isolation and treatment ([4, 14–16, 13, 17–21]). It has been observed by different contributions in the literature and by clinical evidence that recovered people becomes again susceptible after a relatively short period of time. For this reason, in our model formulation we do not stress on a specific compartmental model while, instead, focusing on the spread of infectives across space and time.

Our epidemiological model includes both diffusion and integral terms. The epidemic growth term is, instead, assumed to be linear with a time-dependent coefficient. This assumption encompasses two different phases of the epidemic dynamics: From one side, it is well known that the outbreak of a new epidemic does not generally translate in an immediate response from policymakers. Several contributions in the literature of mathematical epidemiology report that

¹ [11] describes this approach the "most advanced model currently available".

in the early phase of an epidemic the number of infectives tends to grow at a constant (or even decreasing) rate by showing an exponential (or even sub-exponential) growth dynamics [8]. Let us also notice that, from a mathematical point-of-view, the linearity assumption could also be intended to be the local first-order approximation of any Susceptible-Infected epidemiological model. The second justification of this assumption relies on the possibility of modeling infection waves. The COVID-19 pandemic has been so far a series of waves: surges in new cases followed by declines. Seasonal change in the incidence of infectious diseases is a common phenomenon and it is mainly due to significant seasonal variation in the transmission parameter [13]. Therefore, to capture this variability, we assume a time-dependent coefficient whose expression must be estimated from data.

We prove that there are important differences in the final state of the disease depending on whether the policy maker faces the problem as an everlasting or a temporary event. Indeed, if the policy maker believes that the disease will last forever, then it can be optimally eradicated if individuals are patient enough, or if the society effect dominates impatience. This result can have a great influence on the evolution of a pandemic, and it provides support for government intervention. Indeed, if a policy maker believes that a given epidemic has a chance to become endemic, then it can implement restrictive policies (increase the role of the global diffusion channel) to eradicate the disease. If individuals are either extremely impatient or the local and global effects are relative low, then the disease will affect an increasing share of individuals with time, and reach an endemic steady state. However, nothing ensures that this steady state is always stable. Indeed, if the long-term local effect is large enough, meaning here larger than population growth, then this steady state is unstable. We also consider the problem of a policy maker who considers the problem over a finite time horizon, which corresponds to the case in which a policy maker either believes that the disease is short-lived or he/she cannot implement long-term policies. We prove that in this case eradication of the disease is impossible. Besides, the disease could affect the entire population if the policy maker does not care enough about the final number of diseased or, if while considering a finite horizon problem, the horizon is far. Finally, let us underline that independently on whether the time horizon is finite or not, the amount devoted to treat the disease is the same at all locations and it can be adjusted to time changes in local diffusion. Hence locations facing a higher treatment cost will treat less diseased individuals.

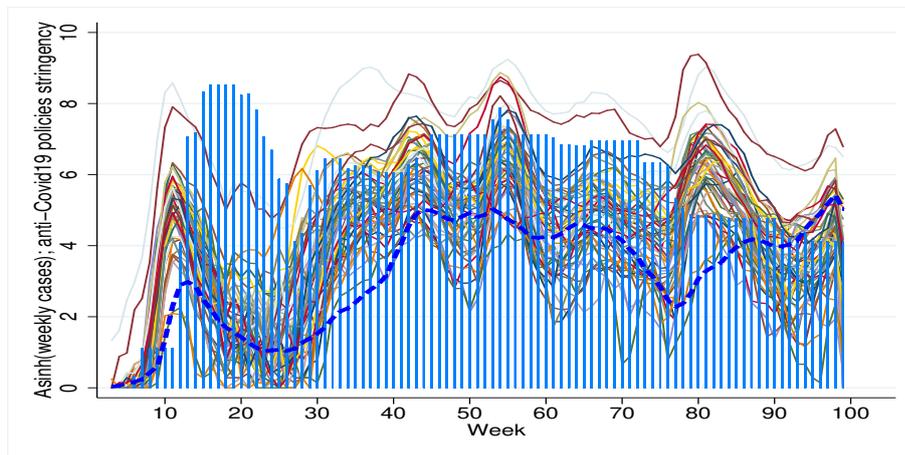
The paper is organized as follows. In Section 2 we conduct an empirical analysis which supports our model formulation. Section 3 is devoted to the presentation of the model formulation. In this section, both the underlying epidemiological model as well as the optimal decision making problem are discussed in details. Section 4 presents the infinite horizon case as well as the steady-state analysis while Section 5 analyzes the optimal control model in finite time. As usual, Section 6 concludes the paper by providing some policy recommendations.

2 Econometric model, data, and results

2.1 Econometric model

The illustrative case of the COVID-19 pandemic in Spain provides an initial motivation for the econometric model detailed below. As can be seen in Figure 1, the national pandemic path can be understood as the aggregation of the 51 regional pandemic paths.² Some regions have had on average, due their intrinsic characteristics (e.g. demographic and socio-economic factors), greater weekly cases than other regions. Each regional path appears to be relatively smooth suggesting substantial correlation over time between present and past infections. Weekly cases across regions also seems to fluctuate together over time, possibly implying substantial correlation across space. This spatial dependence may be the outcome of flows of infected people between regions but could also be driven by national factors, such as anti-COVID-19 government policies, or global factors at the European level, such as infections in other countries or the emergence of a new variant.

Fig. 1 Regional COVID-19 cases in Spain



Notes: Data transformed according to the inverse hyperbolic sine transformation (asinh). Coloured lines: 51 provinces; spikes: two-week COVID-19 Government Response Stringency Index (0-10); thick dashed line: average daily cases in all European regions.

These observations lead us to the estimation of the following spatial dynamic (fixed effects) model with common factors for regions r located in coun-

² The COVID-19 European regional tracker provides data at the NUTS3 level for Spain, that we call regions to be coherent with the other countries and the database itself, although administratively speaking they are provinces.

try i at time t :

$$\begin{aligned} asinh(cases)_{rit} = & \lambda^i asinh(cases)_{rit-1} + \rho^i \sum_{s=1}^n w_{rs} asinh(cases)_{sit} \\ & \delta_r (\alpha_r * asinh(cases)_t) + \theta^i asinh(cum.cases)_{rit-1} \\ & + \beta^i P_{it-2} + \alpha_r + \epsilon_{rit} \end{aligned} \quad (1)$$

where $cases$ is the number of reported Covid-19 in region r of country i at time t , $\sum_{j=1}^n w_{rs} asinh(cases)_{sit}$ is a measure of ‘average’ outbreaks in other regions s , $(cum.cases)$ is a measure of cumulative cases, P_{it-2} is a lagged measure of the Covid-19 Government Response Stringency Index, α_r are time-invariant region fixed effects, $\alpha_r * asinh(cases)_t$ are the products of the interaction between region fixed effects and the average value of the incidence rate in all European countries, and ϵ_{rit} is the error term. $Asinh(\cdot)$ is the inverse hyperbolic sine transformation, which behaves similarly to a log transformation but allows retaining zero-valued observations [6, 23]. In line with our observations, equation 1 describes a rich model in which we allow for **local** regional effects through temporal dependence on lagged cases (λ) and lagged cumulative cases (θ), a **spatial** regional effect (ρ), a **country-level** policy effect which influences all regions (β_1), **global** time-varying shocks which have heterogeneous **local** impacts on each region (δ_r). More specifically, $\delta_r = \bar{\delta} + \xi_r$ can be understood as composed of a common time-effect $\bar{\delta}_i$ and a region-specific deviation ξ_{ir} . Equation 1 is estimated separately for each country i . Given that spatial dependence induces a simultaneity bias (cases in one region r depend on cases on other regions and vice-versa), estimation is achieved through a maximum likelihood (ML) estimator which accounts for the endogenous nature of the spatial lag term [1].³ Standard errors are clustered at the region level.

The pattern of *regional* spatial dependence is defined by the spatial weight matrix $\sum_{s=1}^n w_{rs}$. We follow [9] by considering that population movements between regions r and s (M_{ij}) can be modelled, and its volume approximated in the absence of comprehensive data, using a ‘gravity equation’: $M_{rs} = \frac{POP_r^{\theta_1} POP_s^{\theta_2}}{DIST^\alpha}$, where POP is population and $DIST$ is distance. The (row-normalised) spatial weights can then be $w_{rs} = \frac{M_{rs}}{\sum_{s=1}^n M_{rs}}$. As [9] in the context of international migration, we assume that $\theta_1 \approx \theta_2 \approx 0.70$ and $\alpha \approx -1.5$.⁴ Following [24] and [11], *global* cross-sectional dependence can be captured

³ The ML approach to estimate cross-sectional spatial models is easily extended to the estimation of fixed effects panel data models [12]. For example, prior to the estimation of a non-dynamic spatial fixed effects model, a within transformation (e.g. $y_{it}^* = y_{it} - \frac{1}{T} \sum_{t=1}^T y_{it}$) is applied to eliminate the fixed effects from the regression equation. The log-likelihood function is then: $\log L = -\frac{NT}{2} \log(2\pi\sigma^2) + T \log|I_n - \rho W| - \frac{1}{2\sigma^2} \sum_{i=1}^N \sum_{t=1}^T (y_{it} - \rho \sum_{j=1}^n w_{ij} y_{jt} - x_{it}\beta)^2$. Note the presence of the Jacobian term $\log|I_n - \rho W|$. It is the correction included, relative to a non-spatial model to deal with the endogeneity issue created by spatial simultaneity. We use the command for dynamic spatial fixed effects models -xsmle- coded by [2] in Stata.

⁴ Our results are robust to alternative values.

through the interaction of region fixed effects (α_r) with the cross-sectional average of the dependent variable ($\overline{asinh(cases)}_t$). In that way, global time-varying shocks (e.g. seasonality or emergence of a new COVID-19 variant) is allowed to influence all European regions simultaneously (a ‘common factor’) but heterogeneously, depending on the intrinsic characteristics of each region. The measure of the COVID-19 Government Response Stringency Index P is a time-varying *national* effect which is assumed to influence simultaneously and uniformly all regions in a given country. In many countries, the strength of this policy response has been determined by economy-wide disease conditions. Hence, we lag this index by two periods to reduce a simultaneity bias and to ensure by the same token that variations in this index determines, and is not caused, by variations in the contemporaneous number of cases. Note that the region fixed effects (α_r) control for all time-invariant regional factors (e.g. initial population size) which may determine the number of cases observed in a given region.

2.2 Data

Our COVID-19 data come from the COVID-19 European regional tracker which provides homogenised data on daily cases in 25 European countries at the lowest standardised administrative regional level available (NUTS3 or NUTS2) from January 2020 to November 2021.⁵ Values of the 2020 regional population are also provided. To smooth daily fluctuations, we take weekly averages. Data on the stringency index come from the Oxford COVID-19 Government Response Tracker.⁶ This index indicates the strictness of containment and closure policies (e.g. stay at home requirements). For the United Kingdom, regional index values have been used for Scotland and England.

Spatial ML estimators require the use of balanced panel data. We use the algorithm implemented by the Stata command `-xtbalance2-` to create a balanced subsample which maximises the number of observations across both time and units. Table 1 provides the list of countries in the sample, the number of regions involved for each country, and the periods covered.

2.3 Results

Our results are summarised, for ease of interpretation, in Figure 2. Across countries, we find strong evidence of local ($\hat{\gamma}$) and spatial ($\hat{\rho}$) regional effects. A high number of cases in the past tends to have a persistent positive effect over time while a high contemporaneous number of cases in ‘neighbouring’ regions tends to increase the number of cases in a given region. In most countries,

⁵ <https://github.com/asjadnaqvi/COVID19-European-Regional-Tracker>

⁶ <https://www.bsg.ox.ac.uk/research/research-projects/covid-19-government-response-tracker>

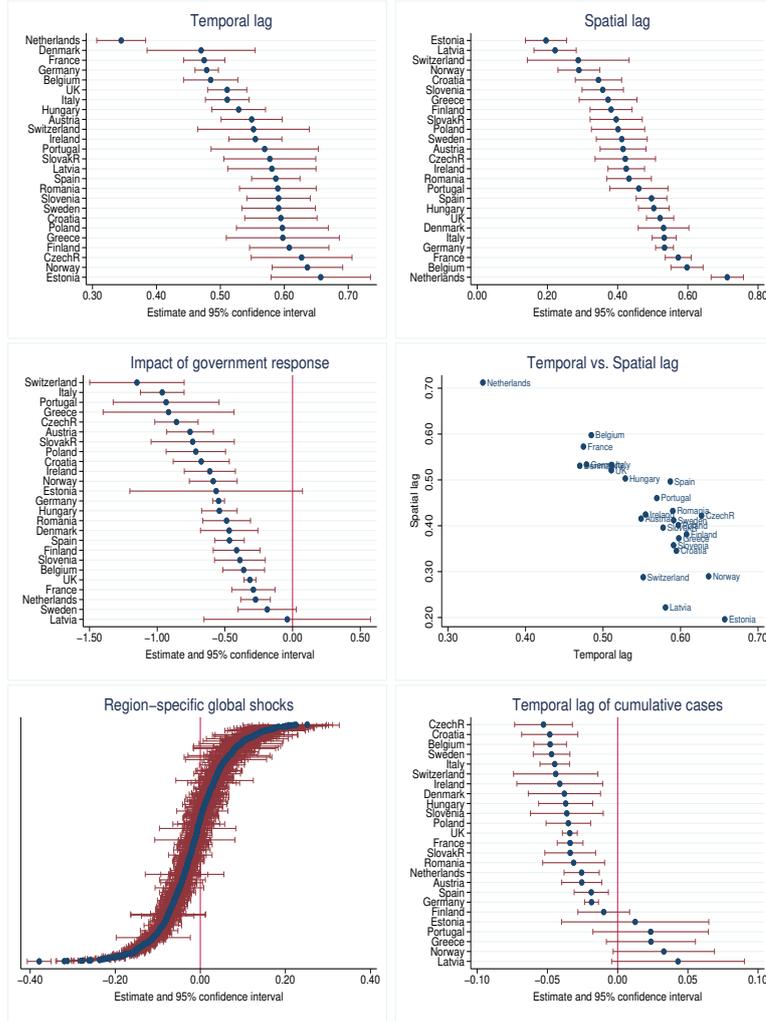
Table 1 Countries in the sample

Country	Number regions	First week	Last week
Austria	35	8	99
Belgium	35	10	91
Croatia	21	12	99
CzechR	14	9	99
Denmark	11	8	99
Estonia	5	10	99
Finland	19	4	99
France	96	10	99
Germany	401	9	99
Greece	13	8	99
Hungary	20	13	99
Ireland	8	9	99
Italy	106	8	99
Latvia	6	11	78
Netherlands	40	9	99
Norway	18	12	99
Poland	17	11	99
Portugal	14	12	98
Romania	18	15	99
SlovakR	6	10	90
Slovenia	12	9	99
Spain	51	3	99
Sweden	21	5	99
Switzerland	14	12	74
UK	156	10	92

Week 3 starts on 19/01/2020 and week 99 starts on 21/11/2021.

we also observe that strict anti-COVID-19 government policies are associated with a lower number of cases in all regions two periods later ($\hat{\beta}$). Figure 2 also reveals that the magnitude of these effects varies across countries. Interestingly, there is a negative relationship between the strength of the (autoregressive) local effect and that of the spatial effect. Values of $\hat{\delta}_r$ are jointly statistically significant and vary within (as well as across) countries, indicating that the interaction between region-specific characteristics and global shocks generate time-varying fluctuations in local infections. Lastly, in the majority of countries, but not all, we observe that the number of cases tend to decrease as the cumulative number of people infected rises ($\hat{\theta}$).

Having empirically shown that COVID-19 cases depend on regional, spatial, and global factors, we propose in the next Section a theoretical model which encompasses all these disease determinants.

Fig. 2 Spatio-temporal estimates including cumulative cases

Notes: Cluster-robust standard errors have been used. 95% confidence intervals.

3 Model set-up

3.1 The epidemiological model

We study the transmission of a disease over time and space. Time runs from 0 to $T \leq \infty$ and space is the unit circle, that we denote by \mathbb{T} . Population is uniformly distributed along the circle, and it grows with time at a constant rate $n \in \mathbb{R}^+$ at all locations. Then, denoting by $N(t, \theta)$ population at location θ at time t , we assume that population is homogeneously distributed in space, that

is, that $N(t, \theta) = N(t)$ for all $\theta \in \mathbb{T}$, and that $\dot{N}(t) = nN(t)$ with $n \in \mathbb{R}^+$. As a consequence, the population distribution function is $N(\theta, t) = N_0 e^{nt}$, where N_0 denotes the initial distribution of population. Without loss of generality, let us normalize N_0 to 1.

Note that aggregated population $\mathbf{N}(t)$ also grows at the constant rate n . Hence, if total initial population is $\mathbf{N}_0 = \int_{\mathbb{T}} N_0 d\theta = 2\pi$, then $\mathbf{N}(t) = 2\pi e^{nt}$.

Let $I(t, \theta)$ denote the amount of infected individuals at time t and location θ , for $(t, \theta) \in [0, \infty) \times \mathbb{T}$. Then:

$$I(t, \theta) \leq N(t, \theta).$$

Three factors explain the evolution a disease in time and space. First, a local channel: the larger the number of infected individuals at a given location, the stronger is the disease transmission. Then, there is the infection local diffusion. Local diffusion captures the negative effect that encounters with infected individuals around a given location has on the total number of infected individuals at that specific location. And third, a global channel: individuals at each location can be influenced by the amount of infected population in the economy. This effect will be referred to as the global effect, and it is measured as a weighted integral on ill individuals. Namely, the society effect on each location θ is captured by the term $\int_{\mathbb{T}} \phi(\theta - \omega) I(\omega, t) d\omega$, where function ϕ is a real function of space such that

$$\int_{\mathbb{T}} \phi(\omega) d\omega = 1.$$

Function ϕ characterizes the type and extent of exchanges in the economy. If $\phi(\omega) = \frac{1}{2\pi}$ for all $\omega \in \mathbb{T}$, then all locations are connected with the same strength. If instead we had that $\phi(\omega) = \frac{1}{\xi}$ for $\omega < \xi$, and zero elsewhere, then each location in this economy interacts only with a small circle of locations.

The evolution of the number of infected individuals and its distribution in space and time is described by

$$\frac{\partial I(\theta, t)}{\partial t} = d_1 \frac{\partial^2 I(\theta, t)}{\partial \theta^2} + d_2(t) I(\theta, t) + d_3 \int_{\mathbb{T}} \phi(\theta - \omega) I(\omega, t) d\omega - M(\theta, t), \quad (2)$$

where $d_1 \in \mathbb{R}$ measures local diffusion, function $d_2(t)$ measures the local effect in time, and $d_3 \in \mathbb{R}$ measures the strength of society, respectively. $M(\theta, t)$ is the number of treated individuals at each location θ and time t . The initial distribution of I , $I_0 = I(0, \theta)$ is a known function, $I_0 : \mathbb{T} \rightarrow \mathbb{R}^+$.

3.2 The firm and the policy maker problem

There is a unique final good in the economy that can be used either for consumption or to treat infected individuals. Let us assume that only susceptible (healthy) individuals $N(\theta, t) - I(\theta, t)$ are productive, and that the production

function is linear in labor. Then, production Y of the final good at location $\theta \in \mathbb{T}$ and time $t \geq 0$ can be expressed as

$$Y(\theta, t) = A [N(\theta, t) - I(\theta, t)], \quad (3)$$

where $A \in \mathbb{R}^+$ is a scale parameter that describes the level of technology.

Let us write the number of treated individuals M as $M(\theta, t) = m(\theta, t)N(t)$ with $m(\theta, t)$ number of treated individuals per capita at location θ and time t .

We assume that the cost of curing M individuals is quadratic in m , that is $c(\theta)m(\theta, t)^2N(t)$, where $c(\cdot)$ is a positive, twice differentiable function in \mathbb{T} that models the treatment cost on the infected population resident at location θ . That is, we assume that the cost of treating one individual may vary across locations.

In this economy there exists a policy maker aiming at maximizing overall welfare. In turn, instantaneous utility depends uniquely on consumption, which equals production minus the cost of treatment: $A [N(\theta, t) - I(\theta, t)] - c(\theta)m^2(t, \theta)$.

We address the policy maker problem restricting our analysis to regular enough functions $i(\cdot, \cdot)$ and $m(\cdot, \cdot)$. Here regular enough means that for $t \geq 0$, the space functions $i(t, \cdot)$ and $m(t, \cdot)$ are elements of the Hilbert space $L^2(\mathbb{T})$. $L^2(\mathbb{T})$ is the set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ such that $\int_0^{2\pi} |f(\theta)|^2 d\theta < \infty$. Working in this space allows to use Dynamic Programming techniques and invoke classical results. Very briefly, we will define and solve the Hamilton-Jacobi-Bellman equation associated, to the policy maker problem, and use its solution to obtain the optimal control $m^*(t, \cdot)$.

4 The infinite horizon problem

We begin our analysis with the problem of a policy maker who believes that the epidemic will become endemic and last forever. Accordingly, we set $T = \infty$. The policy maker solves an infinite time horizon problem and her objective is to maximize aggregated discounted profits, that is, to solve

$$\max_m \int_0^\infty \int_{\mathbb{T}} (A [N(t) - I(\theta, t)] - c(\theta)m^2(t, \theta)N(t)) e^{-\rho t} d\theta dt, \quad (4)$$

subject to

$$\begin{cases} \frac{\partial I(\theta, t)}{\partial t} = d_1 \frac{\partial^2 I(\theta, t)}{\partial \theta^2} + d_2(t)I(\theta, t) + d_3 \int_{\mathbb{T}} \phi(\theta - \omega)I(\omega, t)d\omega - m(\theta, t)N(t), \\ I(\theta, t) \leq N(\theta, t), \forall \theta \in \mathbb{T}, \text{ and } t \geq 0. \end{cases},$$

where $e^{-\rho t}$ is the policy maker time discount function, with $\rho \geq 0$.

Since population grows at the same constant rate $n \in \mathbb{R}$ at all locations, we can detrend the number of infected, I , and write the problem in per capita terms. The detrended problem reads as

$$\max_m \int_0^\infty \int_{\mathbb{T}} (A[1 - i(\theta, t)] - c(\theta)m^2(t, \theta)) e^{-(\rho-n)t} d\theta dt, \quad (5)$$

subject to

$$\frac{\partial i(\theta, t)}{\partial t} = d_1 \frac{\partial^2 i(\theta, t)}{\partial \theta^2} + [d_2(t) - n] i(\theta, t) + d_3 \int_{\mathbb{T}} \phi(\theta - \omega) i(\omega, t) d\omega - m(\theta, t), \quad (6)$$

where $i(\theta, t) = \frac{I(\theta, t)}{N(t)}$, $i(\theta, t) \leq 1$ and the initial distribution of i , i_0 , is a known function on \mathbb{T} .

Let us make the following assumption on $d_2(t)$

Assumption 1. *Local transmission is measured by a real function $d_2(t) : [0, \infty) \rightarrow \mathbb{R}$, which reaches a constant value when time tends to infinite, $\lim_{t \rightarrow \infty} d_2(t) = \bar{d}_2$.*

The following assumption is required in order to ensure the convergence of the objective function:

Assumption 2. $n \leq \rho$.

Following [3] and [5], and using Dynamic Programming techniques in Hilbert spaces, we obtain the following result:

Theorem 1 *Under Assumptions 1 and 2, and if additionally $n < d_2(t) + d_3$ and $n + \rho < 2[d_2(t) + d_3]$ for all $t \geq 0$, then there exists an optimal solution m^* to problem (5)-(6)*

$$m^*(\theta, t) = \frac{A}{2c(\theta)} \frac{1}{d_2(t) + d_3 - n} + \frac{1}{c(\theta)} \frac{2[d_2(t) + d_3] - n - \rho}{\langle \frac{1}{c}, \mathbf{1} \rangle} \mathbf{i}(\mathbf{t}), \quad (7)$$

where $\mathbf{i}(\mathbf{t})$ denotes the spatial average of the share of infected individuals and $\langle \frac{1}{c}, \mathbf{1} \rangle = \int_{\mathbb{T}} \frac{1}{c(\theta)} d\theta$. Furthermore, the associated optimal trajectory for i is admissible.

Proof See Appendix A.1.

The expression for the optimal number of individuals to treat in (7) shows that the optimal trajectory for m is not homogeneous in space. The policy maker will take into account the location characteristics, namely the treatment cost. Indeed, we notice that the larger the cost at location θ , the less individuals will get treated. Worth mentioning, while m^* varies across space, total investment in treatment $m^*(\theta, t)c(\theta)$ is spatially homogeneous. All locations devote the same amount to treat the disease.

Very interestingly, (7) reveals that the local and the society effects are substitutes. Indeed, regarding diffusion, what determines the share of individuals to treat at every moment in time t is the term $d_2(t) + d_3$, which does not depend on the location itself. Furthermore, since $2[d_2(t) + d_3] - n - \rho > 0$, then the optimal amount of individuals to cure increases with total infected.

Substituting $m^*(\theta, t)$ in (6) and aggregating in space, the dynamics for the associated average share of infected obtain as⁷

$$\dot{\mathbf{i}}^*(t) = [\rho - d_2(t) - d_3] \mathbf{i}^*(t) - \frac{A}{2[d_2(t) + d_3 - n]} \frac{1}{\langle \frac{1}{c}, \mathbf{1} \rangle}. \quad (8)$$

Denoting by \mathbf{i}_0 the initial distribution of infected, we can solve (8) and obtain the optimal trajectory for \mathbf{i} :

$$\mathbf{i}^*(t) = \left[\mathbf{i}_0 - \frac{1}{c} \int_0^t \frac{A}{2(d_2(s) + d_3 - n)} e^{-\int_0^s [\rho - d_2(z) - d_3] dz} ds \right] e^{\int_0^t [\rho - d_2(s) - d_3] ds}, \quad (9)$$

where $\frac{1}{c} = \langle \frac{1}{c}, \mathbf{1} \rangle$. Note that if $\rho > d_2(t) + d_3$ for all t , then the growth rate of \mathbf{i} would always be positive and \mathbf{i} would increase until it reaches a steady state or until it reaches the value 1. If on the contrary, $\rho < d_2(t) + d_3$ for all t , then \mathbf{i} would continuously decrease, until it reaches a steady state or until the disease disappears.

Let us study next the steady state of the aggregate share of infected and under which conditions it exists:

Proposition 1 *Under the hypothesis of Theorem 1, and if additionally $A < \bar{d}_2 + d_3 - n$, $\bar{d}_2 + d_3 < \rho$ and $\frac{1}{c} < 2(\rho - \bar{d}_2 - d_3)$, then \mathbf{i}^* converges towards an interior steady state $\bar{\mathbf{i}}$ when time tends to infinite:*

$$\bar{\mathbf{i}} = \frac{A}{2(\bar{d}_2 + d_3 - n)} \frac{1}{\rho - \bar{d}_2 - d_3} \frac{1}{c} < \mathbf{1}. \quad (10)$$

Taking the limit when time tends to infinite in (7) and substituting $\bar{\mathbf{i}}$ using (10), the steady state of m^* obtains:

$$\bar{m}^*(\theta) = \frac{A}{2c(\theta)} \frac{1}{\rho - \bar{d}_2 - d_3} < 1.$$

Proof See Appendix A.2.

Note that the share of treated individuals at the steady state also depends on the cost of treatment. However, total investment in treatment at the steady state $c(\theta)\bar{m}^*(\theta)$ is still homogeneous in space.

Proposition 1 builds on the hypothesis that individuals are relatively impatient, namely that $\bar{d}_2 + d_3 < \rho$. As already commented, this hypothesis implies that \mathbf{i}^* is increasing in time at least from a moment in time onwards, when d_2 gets close enough to \bar{d}_2 . Hence, Proposition 1 shows is that if individuals are impatient enough and treatment costs relatively high on average, then the average share of infected increases until it reaches a steady state. Maybe even more important, the proposition shows that if $\bar{d}_2 + d_3 > \rho$, not only \mathbf{i}^* is decreasing in time from one point in time onwards, but there is no steady

⁷ All computational details can be found in Appendix A.1, at the end of Step 1.

state. Hence, if $\bar{d}_2 + d_3 > \rho$, then \mathbf{i}^* will decrease until it reaches 0. Therefore, total eradication of the disease is possible when individuals are patient enough or when the mixed effect of both local and global diffusion are high. Let us summarize this result in the following corollary:

Corollary 1 *If $\rho < \bar{d}_2 + d_3$, then \mathbf{i}^* decreases with time from one moment onwards until eventually reaching zero.*

Corollary 1 shows that when individuals are patient enough, then the disease is optimally eradicated with time.

Note that diffusion, captured by d_1 does not play any role neither in the dynamics of \mathbf{i}^* nor in its steady state solution. Next section shows that diffusion does play a role in the stability of the steady state of i .

4.1 Steady state stability analysis

Proposition 1 has shown that under certain assumptions there exists a steady state for the average number of infected individuals, $\bar{\mathbf{i}}$. Under the same set of assumptions, let us prove next that there exists a spatially homogeneous steady state for the distribution of the share of infectives on space, i , that we will denote by \bar{i} . Further, we shall prove that the homogeneous steady state of the local number of infected individuals may be unstable for a large family of reasonable spatial aggregators ϕ .

If a steady state \bar{i} exists, then by definition $\lim_{t \rightarrow \infty} i(\theta, t) = \bar{i}(\theta)$ for every $\theta \in \mathbb{T}$ and $\bar{i}_t(\theta, t) = 0$ for all t when t tends to infinity. \bar{i} verifies then

$$0 = d_1 \frac{\partial^2 \bar{i}(\theta)}{\partial \theta^2} + (\bar{d}_2 - n)\bar{i}(\theta) + d_3 \int_{\mathbb{T}} \phi(\theta - \omega) \bar{i}^*(\omega) d\omega - \bar{m}^*(\theta). \quad (11)$$

Under Proposition 1's assumptions, there exists a steady state, \bar{i} given by

$$\bar{i}(\theta) = \frac{A}{2c(\theta)} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3}.$$

Note that integrating \bar{i} over space yields $\bar{\mathbf{i}}$.

Recall that spatial diffusion of infection is captured by the second spatial derivative, and its strength is captured by parameter d_1 . If this homogeneous steady state was stable for (6), then spatial diffusion would not add anything to our understanding of disease transmission in the long-run since d_1 does not affect \bar{i} . Nevertheless, the stability of \bar{i} can crucially depend on the diffusion coefficient d_1 as the following proposition shows

Proposition 2 *Let us consider the special case in which $\bar{d}_2 + d_3 = 1 + n$, and let us focus on the family of solutions i such that $\nabla i|_0 = \nabla i|_{2\pi}$ and $i(2\pi) = i(0) = \bar{i}$. Let $\hat{\phi}$ denote the Fourier transform of ϕ . The spatially*

homogeneous steady state \bar{i} is unstable if there exists a single value of $\xi \in \mathbb{T}$ such that

$$\tilde{\phi}(\xi) > \frac{d_1 \xi^2 + n - \bar{d}_2}{d_3}.$$

In particular, \bar{i} is unstable if $n < \bar{d}_2$ since we obtain for $\xi = 0$ that $\tilde{\phi}(0) > 0 > \frac{n - \bar{d}_2}{d_3}$.

Proof See Appendix A.3.

Up to this last result, our analysis has depended on the relative size of diffusion and patience. Proposition 2 reveals the stabilizer role of d_1 , showing that the stronger the spatial diffusion of the disease, d_1 , the more likely it is that the homogeneous steady state is stable. Proposition 2 also proves that if \bar{d}_2 , the steady state of local disease persistence is high enough, larger than n , then the steady state is unstable independently of diffusion, of patience, and most importantly, of the spatial aggregator ϕ . In this case, we expect the dynamics of the disease to be driven by local characteristics.

Additionally, Proposition 2 allows us to unveil other extremely interesting behaviours. In the case where the steady state is unstable, that is where $\bar{d}_2 + d_3 = 1 + n$ and $n < \bar{d}_2$, we have $d_3 = 1 + n - \bar{d}_2 < 1$. Hence, unstable long-term steady states correspond to economies where the global effect is not powerful enough.

5 The finite horizon problem

If the policy maker believes that the epidemic is temporary, then instead of considering an infinite time horizon as in Section 4 she will consider a finite horizon. Consequently, she will solve a finite horizon problem from 0, the moment the epidemic is taken into consideration until T . The policy maker solves the following problem

$$\max_m \int_0^T \int_{\mathbb{T}} (A[1 - i(\theta, t)] - c(\theta)m^2(t, \theta)) e^{-(\rho-n)t} d\theta dt - \Phi e^{-(\rho-n)T} \int_{\mathbb{T}} i(\theta, T) d\theta, \quad (12)$$

subject to

$$\frac{\partial i(\theta, t)}{\partial t} = d_1 \frac{\partial^2 i(\theta, t)}{\partial \theta^2} + d_2 i(\theta, t) + d_3 \int_{\mathbb{T}} \phi(\theta - \omega) i(\omega, t) d\omega - m(\theta, t), \quad (13)$$

where $\rho > 0$ and where the initial distribution of i , i_0 is a known function on $[0, 1]$, $i_0 : \mathbb{T} \rightarrow [0, 1]$. Worth to note, we assume in this finite time horizon problem that d_2 , which measures the strength of local transmission is constant in time, which is not shocking if T is short enough.

Here the policy maker suffers from the final state of the epidemics, which is measured using the scrap function $\Phi e^{-(\rho-n)T} \int_{\mathbb{T}} i(\theta, T) d\theta$ in (12). The scrap

function conveys the idea that the more infected in the economy at time T , the lower welfare. This disutility increases with parameter $\Phi \geq 0$ and it decreases with the time horizon. That is, the larger the time horizon, T , the less the policy maker cares about the overall final number of diseased.

Under Theorem 1's hypothesis, the optimal share of treated individuals is as in the previous section given by

$$m^*(\theta, t) = \frac{A}{2c(\theta)} \frac{1}{d_2 + d_3 - n} + \frac{1}{c(\theta)} \frac{2[d_2 + d_3] - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(t). \quad (14)$$

Here again $\mathbf{i}(t) = \langle i(t), \mathbb{1} \rangle$ is the spatial aggregate of $i(\theta, t)$ at time t .

Although m^* is the same function of \mathbf{i} as in Section 4, it does not mean that they both take the same values at any point $(\theta, t) \in \mathbb{T} \times [0, T]$. The difference between this and the optimal share in the infinite horizon case stems from the terminal condition. Indeed, using the scrap function, we obtain that the share of individuals to treat at the terminal time T is⁸

$$m^*(\theta, T) = \frac{1}{2c(\theta)} \frac{A}{d_2 + d_3 - n} + \frac{1}{2c(\theta)} \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(T), \quad (15)$$

and $\mathbf{i}(T)$ itself obtains using the terminal condition and the explicit solution for the value function of the HJB equation of this problem.

The following proposition describes the optimal average share of infected at time T depending on diffusion, impatience and the disutility from the ill as measured by Φ .

Proposition 3 *Suppose that diffusion is large, namely that $n < d_2 + d_3$, $n + \rho < 2(d_2 + d_3)$ and that Assumption 2 holds. Then $\mathbf{i}(T)$ will never be zero. Furthermore, a non zero solution exists for $\mathbf{i}(T)$ if the following condition holds*

$$\Phi e^{-(\rho-n)T} \geq \frac{2\pi A}{\rho - n} + \frac{1}{4(\rho - n)} \frac{A^2}{(d_2 + d_3 - n)^2} \langle \frac{1}{c}, \mathbb{1} \rangle - \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} - \frac{A}{d_2 + d_3 - n} \quad (16)$$

More specifically,

- i) $0 < \mathbf{i}(T) < 1$ if condition (16) holds with strict inequality. Besides, $i(T)$ decreases with T ;
- ii) $\mathbf{i}(T) = 1$ if and only if condition (16) holds with equality.

Proof See Appendix A.5.

Proposition 3 contains important results. First, when the time horizon is finite, then the share of infected is never zero, the disease is never optimally eradicated. Second, if households care enough about the final share of diseased and choose a high Φ , then their final share will be below one. If not, then all population will get infected by T . Also note that the larger T , the smaller the left hand side, and the more likely is full infection.

⁸ See Appendix A.4 for a proof.

6 Policy recommendations and conclusions

This paper has proposed a time-space economic model to control the evolution and the spread of a disease. The underlying epidemiological model is formulated as a reaction-diffusion integro-differential partial differential equation. Ours is a novel formulation which captures three different diffusion channels of a transmissible disease: a pure spatial diffusion effect, a local effect, and a global diffusion effect. As mentioned, this model is not a mere mathematical artifact but it is supported by recent evidence on COVID-19. Using data on the evolution of the COVID-19 pandemic in Europe, we show that all three diffusion channels are present and are significant to explain the evolution of COVID-19. That is, the local effect, the diffusion effect and the global effect are significant and represent different transmission mechanisms of the disease. Hence, we are confident that although challenging, our modelling of COVID-19 is throughout. Within this time-space framework, two versions of the policy maker problem have been presented. The first one, the infinite horizon case, can be understood as the problem of a policy maker who understands that the epidemic can last forever and become endemic. The second one, instead, considers that the time horizon is finite and we can consider this as the problem of a policy maker who faces the epidemic as a finite-lived phenomenon. Thanks to the linear quadratic expression of the overall cost function, we have been able to provide the closed-form expression of the optimal policy and the economy's steady state. After solving and discussing both models, some conclusions and policy recommendations can be drawn. In the finite horizon scenario, the disease will never be fully eradicated. Moreover, the less the policy maker cares about the final state of the disease or the longer the decision horizon, the more likely it is that the entire population will be infected. In the infinite horizon case, the disease can be eradicated with time either if individuals are relatively patient or if the joint effect of the local and global diffusion effects are sufficiently strong. Otherwise, the disease will evolve and reach an endemic steady state. In this case, if the local persistence is too strong, $\bar{d}_2 > n$, then the steady state is not stable. Throughout the paper, we have delved into epidemics investigating the roles of the aforementioned three diffusion channels. Obviously, our approach encompasses situations in which one of the channels is not relevant to a given epidemic or to a specific economy. Finally, let us discuss further applications and avenues of our model. Indeed, although we have motivated the construction of our model using the current COVID-19 pandemic, our model can be used to describe the evolution of other epidemics or transmissible diseases, caused by external agents like viruses or other novel pathogens. The model could also be enriched by adding nonlinear terms. This would lead to more complicated models that could not be solved in closed form and for which only numerical solutions could be provided.

References

1. Anselin, L. (1988) *Spatial econometrics: methods and model*, Springer.

2. Belotti, F., Hughes, G., Mortari, A. P. (2017) Spatial panel-data models using Stata, *The Stata Journal*, 17, 139–180.
3. Bensoussan, A., Da Prato, G., Delfour, M.C., Mitter, S.K. (2007) *Representation and control of infinite dimensional system*, second edition, Birkhäuser, Boston.
4. Bosi, S., Camacho, C., Desmarchelier, D. (2021) Optimal lockdown in altruistic economies, *Journal of Mathematical Economics*, 93, 102488.
5. Boucekkine, R., Camacho, C., Fabbri, G. (2013) Spatial dynamics and convergence: The spatial AK model, *Journal of Economic Theory*, 148(6), 2719–2736.
6. Burbidge, J. B., Magee, L., Robb, A. L. (1988) Alternative transformations to handle extreme values of the dependent variable, *Journal of the American Statistical Association*, 83, 123–127.
7. Cheng, C., Barceló, J., Hartnett, A.S., Kubinec, R., Messerschmidt, L. (2020) COVID-19 government response event dataset (CoronaNet v.1.0), *Nature Human Behavior*, 4, 756–768
8. Chowell, G., Sattenspiel, L., Bansal, S., Viboud, C. (2016) Mathematical models to characterize early epidemic growth: A review, *Physics of Life Reviews*, 18, 66–97.
9. Desbordes, R. (2021) Spatial dynamics of major infectious diseases outbreaks: A global empirical assessment, *Journal of Mathematical Economics*, 93, 102493.
10. Dong, E., Du, H., Gardner, L. (2020) An interactive web-based dashboard to track COVID-19 in real time, *Lancet Infectious Diseases*, 20, 533–534.
11. Elhorst, J. P. (2021) The dynamic general nesting spatial econometric model for spatial panels with common factors: further raising the bar, *Review of Regional Research*, 1–19, <https://doi.org/10.1007/s10037-021-00163-w>.
12. Elhorst, P. J. (2014) *Spatial econometrics. From cross-sectional data to spatial panels*, Springer, London.
13. Grassly, N.C., Fraser, C. (2006) Seasonal infectious disease epidemiology, *Proceedings. Biological sciences*, 273, 1600, 2541–50.
14. Goenka, A., Liu, L. (2012) Infectious diseases and endogenous fluctuations, *Economic Theory*, 50, 125–149.
15. Goenka, A., Liu, L. (2020) Infectious diseases, human capital and economic growth, *Economic Theory*, 70, 1–47.
16. Goenka, A., Liu, L., Nguyen, M.H. (2014) Infectious diseases and economic growth, *Journal of Mathematical Economics*, 50, 34–53.
17. Kermack, W.O., McKendrick, A.G. (1927) A contribution to the mathematical theory of epidemics, *Proceedings of the Royal Society A*, 115, 700–721.
18. La Torre, D., Malik, T., Marsiglio, S. (2020) Optimal control of prevention and treatment in a basic macroeconomic-epidemiological model, *Mathematical Social Sciences*, 100, 100–108.
19. La Torre, D., Liuzzi, D., Marsiglio, S. (2021) Epidemics and macroeconomic outcomes: social distancing intensity and duration, *Journal of Mathematical Economics*, 93, 102473.
20. La Torre, D., Liuzzi, D., Muggisto, R., Marsiglio, S. (2021) Mobility Choices and Strategic Interactions in a Two-Group Macroeconomic–Epidemiological Model, *Dynamic Games and Applications*, <https://doi.org/10.1007/s13235-021-00413-z>
21. La Torre, D., Liuzzi, D., Marsiglio, S. (2022) Geographical heterogeneities and externalities in an epidemiological-macroeconomic framework, *Journal of Public Economic Theory*, 1–28, <https://doi.org/10.1111/jpet.12576>
22. Mofijur, M. et al. (2021) Impact of COVID-19 on the social, economic, environmental and energy domains: Lessons learnt from a global pandemic, *Sustainable Production and Consumption*, 26, 343–359.
23. MacKinnon, J. G., Magee, L. (1990) Transforming the dependent variable in regression models, *International Economic Review*, 31, 315–339.
24. Pesaran, M. H. (2006) Estimation and inference in large heterogeneous panels with a multifactor error structure, *Econometrica*, 74, 967–1012.
25. World Health Organization (2020) “Immunity passports” in the context of COVID-19 - *Scientific brief*, available at: <https://www.who.int/news-room/commentaries/detail/immunity-passports-in-the-context-of-covid-19><https://www.who.int/news-room/commentaries/detail/immunity-passports-in-the-context-of-covid-19>

A Mathematical Results

A.1 The Hamilton-Jacobi-Bellman equation and the optimal solution

Before defining the Hamilton Jacobian Bellman equation and applying dynamic programming techniques, we need to write the policy maker problem differently. Note that

$$\max_m \int_0^\infty \int_{\mathbb{T}} \{A[1 - i(\theta, t)] - c(\theta)m^2(t, \theta)\} e^{-(\rho-n)t} d\theta dt, \quad (17)$$

is equivalent to

$$- \min_m \int_0^\infty \int_{\mathbb{T}} \{A[1 - i(\theta, t)] - c(\theta)m^2(t, \theta)\} e^{-(\rho-n)t} d\theta dt. \quad (18)$$

That is,

$$\min_m \int_0^\infty \int_{\mathbb{T}} \{c(\theta)m^2(t, \theta) - A[1 - i(\theta, t)]\} e^{-(\rho-n)t} d\theta dt. \quad (19)$$

Let us define the objective functional associated to I_0 as a function of M as

$$J(I_0; M) := \int_0^\infty e^{-\rho t} U(I, M) dt,$$

where

$$U(I, M) = \min_M \int_0^\infty \int_{\mathbb{T}} \{A[N(t) - I(\theta, t)] - c(\theta)m^2(t, \theta)N(t)\} e^{-\rho t} d\theta dt.$$

The value function associated to our problem is then

$$V(I_0) := \inf_M J(I_0; M).$$

The objective functional and the value function associated to the detrended problem are

$$j(i_0; m) := \int_0^\infty u(i, m) e^{-(\rho-n)t} dt,$$

where

$$\min_m \int_0^\infty \int_{\mathbb{T}} \{A[1 - i(\theta, t)] - c(\theta)m^2(t, \theta)\} e^{-(\rho-n)t} d\theta,$$

and

$$v(i_0) := \inf_m j(i_0; m).$$

For simplicity of exposition, let us define the second order linear operator \mathcal{L} as the second spatial derivative, that is $\mathcal{L} : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ and $\mathcal{L}v(\theta, t) = v_{\theta\theta}(\theta, t)$, for any function v twice differentiable in θ .

The HJB equation associated to the detrended problem is

$$\begin{aligned} \rho v(i) = & \langle d_1 \mathcal{L}i, \nabla v(i) \rangle + \langle (d_2 - n)i, \nabla v(i) \rangle \\ & + \langle d_3 \int_{\mathbb{T}} \phi(\theta - \omega) i(\omega) d\omega, \nabla v(i) \rangle \\ & + \inf_m \{ \langle cm^2 - A[1 - i], \mathbb{1} \rangle - \langle m, \nabla v(i) \rangle \}, \end{aligned} \quad (20)$$

where we have omitted the time index of the diffusion coefficient function d_2 for simplicity reasons.

Let us first define some notions and important spaces. As already mentioned, we will search for solutions to our problem in the Hilbert space $\mathbb{L}^2(\mathbb{T})$. Also recall that our operator

\mathcal{L} was defined as $\mathcal{L}i = \frac{\partial^2 i}{\partial \theta^2}$, and let us denote its domain by $D(\mathcal{L})$. We will require that $D(\mathcal{L})$ is a subspace of $L^2(\mathbb{T})$, satisfying that for any function $i \in D(\mathcal{L})$, its first and second derivative remain in $L^2(\mathbb{T})$.

The HJB equation in (21) is infinite dimensional, which means that for every function i , $i \in L^2(\mathbb{T})$, $v(i)$ is a real number, $v : L^2(\mathbb{T}) \rightarrow \mathbb{R}$. Let Ω be the open set $\Omega = \{i \in L^2(\mathbb{T}) : \langle i, \mathbb{1} \rangle > 0\}$. Function v needs to be sufficiently regular. Indeed, v needs to be continuous with a continuous differential, i.e. $v \in C^1(\Omega)$. That is, both v and ∇v move continuously with the distance in $L^2(\mathbb{T})$. Note that this assumption involves a second derivative, \mathcal{L} . We need then that $\nabla v(i) \in D(\mathcal{L})$, and $\mathcal{L}\nabla v(i)$ to be continuous for all $i \in \Omega$.

The rest of the proof is structured in three steps. We will first provide with the explicit solution to the HJB equation. Then, we will prove the feasibility of the induced trajectory for m . Finally, we will prove the optimality of the induced trajectory for i .

Step 1: Under Theorem 1's assumptions, all terms in (21) can be computed. Let us start by solving the inf problem in (21) for any value function v . We obtain that the optimal choice for m is

$$m^*(\theta, t) = \frac{1}{2c(\theta)} \nabla v(i). \quad (21)$$

We propose the following function as the value function of our problem:

$$v(i) = a_0 + a_1 \langle i, \mathbb{1} \rangle + a_2 \langle i, \mathbb{1} \rangle^2, \quad (22)$$

where $a_0, a_1, a_2 \in \mathbb{R}$, and they are to be determined. If v is defined as in (22), we have that

$$\nabla v(i) = a_1 + 2a_2 \langle i, \mathbb{1} \rangle. \quad (23)$$

We obtain next the values of a_0, a_1, a_2 , equating the coefficients that are constant, those that multiply $\langle i, \mathbb{1} \rangle$ and those that multiply $\langle i, \mathbb{1} \rangle^2$ in the HJB equation. First, note that $\langle \mathcal{L}i, \nabla v(i) \rangle = 0$. Second, let us compute the value of the inf in (21) using the optimal m . Let us define

$$f(m) := \langle cm^2 - A[1 - i], \mathbb{1} \rangle - \langle m, \nabla v(i) \rangle \quad (24)$$

then we can easily compute:

$$\begin{aligned} \inf_m f(m) &= \langle c \frac{1}{4c^2} \nabla^2 v(i) - A[1 - i], \mathbb{1} \rangle - \langle \frac{1}{2c} \nabla v(i), \nabla v(i) \rangle \\ &= -\frac{1}{4} \langle \frac{1}{c}, \nabla^2 v(i) \rangle - \langle A, \mathbb{1} \rangle + A \langle i, \mathbb{1} \rangle \\ &= -\frac{1}{4} \langle \frac{1}{c}, a_1^2 + 4a_2^2 \langle i, \mathbb{1} \rangle^2 + 4a_1 a_2 \langle i, \mathbb{1} \rangle \rangle - 2\pi A + A \langle i, \mathbb{1} \rangle \quad (25) \\ &= -a_1^2 \frac{1}{4} \langle \frac{1}{c}, \mathbb{1} \rangle - 4a_2^2 \langle i, \mathbb{1} \rangle^2 \langle \frac{1}{c}, \mathbb{1} \rangle - a_1 a_2 \langle i, \mathbb{1} \rangle \langle \frac{1}{c}, \mathbb{1} \rangle \\ &\quad - 2\pi A + A \langle i, \mathbb{1} \rangle \\ &= -a_1^2 \frac{1}{4} \langle \frac{1}{c}, \mathbb{1} \rangle - 2\pi A - a_1 a_2 \langle i, \mathbb{1} \rangle \langle \frac{1}{c}, \mathbb{1} \rangle + A \langle i, \mathbb{1} \rangle \\ &\quad - 4a_2^2 \langle i, \mathbb{1} \rangle^2 \langle \frac{1}{c}, \mathbb{1} \rangle. \end{aligned}$$

Then we compute all remaining terms in the HJB equation in (21):

$$\begin{aligned} \langle (d_2 - n)i, \nabla v(i) \rangle &= \langle (d_2 - n)i, a_1 + 2a_2 \langle i, \mathbb{1} \rangle \rangle = \\ &= a_1(d_2 - n) \langle i, \mathbb{1} \rangle + 2a_2(d_2 - n) \langle i, \mathbb{1} \rangle^2. \end{aligned}$$

In order to solve for the the last term, recall that $\int_{\mathbb{T}} \phi(\omega) d\omega = 1$. Then, since

$$\langle \int_{\mathbb{T}} \phi(\theta - \omega) i(\omega) d\omega, \mathbb{1} \rangle = \int_{\mathbb{T}} \int_{\mathbb{T}} \phi(\theta - \omega) i(\omega) d\omega d\theta = \int_{\mathbb{T}} i(\omega) d\omega = \langle i, \mathbb{1} \rangle.$$

Using this result, we obtain that

$$\langle d_3 \int_{\mathbb{T}} \phi(\theta - \omega) i(\omega) d\omega, \nabla v(i) \rangle = a_1 d_3 \langle i, \mathbb{1} \rangle + 2a_2 d_3 \langle i, \mathbb{1} \rangle^2.$$

Then,

$$(\rho - n)a_0 = -\frac{a_1^2}{4} \langle \frac{1}{c}, \mathbb{1} \rangle - 2\pi A;$$

$$(\rho - n)a_1 = a_1(d_2 - n) + a_1 d_3 - a_1 a_2 \langle \frac{1}{c}, \mathbb{1} \rangle + A,$$

$$(\rho - n)a_2 = 2a_2(d_2 - n) + 2a_2 d_3 - \langle \frac{1}{c}, \mathbb{1} \rangle a_2^2.$$

Solving this system, and assuming that $a_2 \neq 0$:

$$a_0 = -\frac{2\pi A}{\rho - n} - \frac{1}{4(\rho - n)} \langle \frac{1}{c}, \mathbb{1} \rangle \frac{A^2}{(d_2 + d_3 - n)^2},$$

$$a_1 = \frac{A}{d_2 + d_3 - n},$$

$$a_2 = \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle}.$$

As a result, the value function v is:

$$v(i) = -\frac{2\pi A}{\rho - n} - \frac{1}{4(\rho - n)} \frac{A^2}{(d_2 + d_3 - n)^2} \langle \frac{1}{c}, \mathbb{1} \rangle + \quad (26)$$

$$\frac{A}{d_2 + d_3 - n} \langle i, \mathbb{1} \rangle + \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \langle i, \mathbb{1} \rangle^2,$$

and

$$\nabla v(i) = \frac{A}{d_2 + d_3 - n} + 2 \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \langle i, \mathbb{1} \rangle. \quad (27)$$

We can write it in a simplified manner denoting using that $\mathbf{i}(t) = \langle i, \mathbb{1} \rangle$.

$$v(i) = -\frac{2\pi A}{\rho - n} - \frac{1}{4(\rho - n)} \frac{A^2}{(d_2 + d_3 - n)^2} \langle \frac{1}{c}, \mathbb{1} \rangle + \frac{A}{d_2 + d_3 - n} \mathbf{i}(t) + \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(t)^2,$$

$$\nabla v(i) = \frac{A}{d_2 + d_3 - n} + 2 \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(t).$$

Then, using (21), the optimal amount of treated individuals per capita in location θ at time t , m^* , obtains as a function of the total number of infected:

$$m^*(\theta, t) = \frac{1}{2c(\theta)} \frac{A}{d_2(t) + d_3 - n} + \frac{1}{c(\theta)} \frac{2(d_2(t) + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(t). \quad (28)$$

Since c is a positive function, then $m^*(\theta, t)$ is positive if $d_2(t) + d_3 - n > 0$ and $2[d_2(t) + d_3] - n - \rho > 0$.

We still need to check that $m^*(\theta, t)$ is smaller than 1:

$$\frac{1}{2c(\theta)} \frac{A}{d_2(t) + d_3 - n} + \frac{1}{c(\theta)} \frac{2(d_2(t) + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(t) < 1,$$

Note that since $\mathbf{i}(t) < 1$,

$$\begin{aligned} & \frac{1}{2c(\theta)} \frac{A}{d_2(t) + d_3 - n} + \frac{1}{c(\theta)} \frac{2(d_2(t) + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(t) \\ & < \frac{1}{2c(\theta)} \frac{A}{d_2(t) + d_3 - n} + \frac{1}{c(\theta)} \frac{2(d_2(t) + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle}, \end{aligned}$$

and since by assumption $A < d_2(t) + d_3 - n$, then

$$\frac{1}{2c(\theta)} \frac{A}{d_2(t) + d_3 - n} + \frac{1}{c(\theta)} \frac{2(d_2(t) + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} < \frac{1}{2} + \frac{1}{c(\theta)} \frac{2(d_2(t) + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle}.$$

If

$$\frac{1}{2} + \frac{1}{c(\theta)} \frac{2(d_2(t) + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} < 1,$$

then $m^*(\theta, t) < 1$. The condition above is equivalent to

$$2(d_2(t) + d_3) - n - \rho < \frac{1}{2} \langle \frac{1}{c}, \mathbb{1} \rangle.$$

Before passing to Step 2 and prove that v is the value function of our problem, we need to obtain the analytical expression for \mathbf{i} .

One can substitute the optimal solution m^* into (6):

$$\begin{aligned} \frac{\partial \mathbf{i}(\theta, t)}{\partial t} &= d_1 \frac{\partial^2 \mathbf{i}}{\partial \theta^2}(\theta, t) + (d_2 - n)\mathbf{i}(\theta, t) + d_3 \int_{\mathbb{T}} \phi(\theta - \omega) \mathbf{i}(\omega, t) d\omega \\ &- \frac{A}{2c(\theta)} \frac{1}{d_2 + d_3 - n} - \frac{1}{c(\theta)} \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(t). \end{aligned} \quad (29)$$

From (29), we can integrate with respect to space and the dynamics of $\mathbf{i}(t) = \langle \mathbf{i}, \mathbb{1} \rangle$ obtains:

$$\begin{aligned} < \frac{\partial \mathbf{i}}{\partial t}, \mathbb{1} \rangle = < d_1 \frac{\partial^2 \mathbf{i}}{\partial \theta^2}, \mathbb{1} \rangle + < (d_2 - n)\mathbf{i}, \mathbb{1} \rangle + < d_3 \int_{\mathbb{T}} \phi(\theta - \omega) \mathbf{i}(\omega, t) d\omega, \mathbb{1} \rangle \\ - < \frac{A}{2c(\theta)} \frac{1}{d_2 + d_3 - n}, \mathbb{1} \rangle - < \frac{1}{c(\theta)} \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}, \mathbb{1} \rangle. \end{aligned} \quad (30)$$

We can rewrite (30) as:

$$\frac{\partial \mathbf{i}}{\partial t} = (d_2 - n)\mathbf{i} + d_3 \mathbf{i} - \frac{A}{2(d_2 + d_3 - n)} \langle \frac{1}{c}, \mathbb{1} \rangle - [2(d_2 + d_3) - n - \rho] \mathbf{i}. \quad (31)$$

or

$$\mathbf{i}'(t) = [\rho - (d_2(t) + d_3)] \mathbf{i}(t) - \frac{A}{2(d_2(t) + d_3 - n)} \langle \frac{1}{c}, \mathbb{1} \rangle. \quad (32)$$

Denoting by \mathbf{i}_0 the initial average of ill individuals in \mathbb{T} :

$$\mathbf{i}^*(t) = \left[\mathbf{i}_0 - \frac{1}{c} \int_0^t \frac{A}{2(d_2(s) + d_3 - n)} e^{\int_0^s [\rho - d_2(z) - d_3] dz} ds \right] e^{-\int_0^t [\rho - d_2(s) - d_3] ds}. \quad (33)$$

Step 2: We need to prove that v is indeed the value function of the Hamilton-Bellman-Jacobi equation associated to the detrended problem, that is, the function v which solves (21).

First, as an intermediate step, we prove here that $|e^{-(\rho-n)t} v[\mathbf{i}(t)]| \rightarrow_{t \rightarrow \infty} 0$:

$$\begin{aligned} |e^{-(\rho-n)t} v[\mathbf{i}(t)]| &= |e^{-(\rho-n)t} (a_0 + a_1 \langle \mathbf{i}, \mathbb{1} \rangle + a_2 \langle \mathbf{i}, \mathbb{1} \rangle^2)| \\ &\leq |a_0| e^{-(\rho-n)t} + |a_1| e^{-(\rho-n)t} \mathbf{i} + |a_2| e^{-(\rho-n)t} \mathbf{i}^2. \end{aligned}$$

If each of the three terms at the right of the expression above tend to zero, so will $|e^{-(\rho-n)t} v[\mathbf{i}(t)]|$. First, note that under Assumption 1, $\lim_{t \rightarrow \infty} |a_0| e^{-(\rho-n)t} = 0$.

Next, we need to study $\lim_{t \rightarrow \infty} e^{-(\rho-n)t} \mathbf{i}(t)$ and $\lim_{t \rightarrow \infty} e^{-(\rho-n)t} \mathbf{i}^2(t)$.

First, let us solve (8) from time τ to t , where τ is large enough so that we can assume that $d_2(t)$ is close enough to its steady state. Then

$$\begin{aligned} i(t) &= i(\tau) e^{(\rho - \bar{d}_2 - d_3)(t - \tau)} + \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \left(1 - e^{(\rho - \bar{d}_2 - d_3)(t - \tau)} \right) \\ &= \left(i(\tau) - \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right) e^{(\rho - \bar{d}_2 - d_3)(t - \tau)} + \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3}. \end{aligned}$$

Let us compute the first limit we need to compute:

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-(\rho-n)t} \mathbf{i}(t) &= \\ \lim_{t \rightarrow \infty} \left(i(\tau) - \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right) e^{(n - \bar{d}_2 - d_3)(t - \tau)} &+ \\ \lim_{t \rightarrow \infty} \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} e^{-(\rho-n)t} &= 0, \end{aligned}$$

if $n < \rho$ and $n < \bar{d}_2 + d_3$.

Then, let us focus on the second limit. Let us first compute $\mathbf{i}^2(t)$:

$$\begin{aligned} \mathbf{i}^2(t) &= \left(i(\tau) - \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right)^2 e^{2(\rho - \bar{d}_2 - d_3)(t - \tau)} + \left(\frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right)^2 \\ &+ 2 \left(i(\tau) - \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right) e^{(\rho - \bar{d}_2 - d_3)(t - \tau)} \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \end{aligned}$$

Accordingly,

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-(\rho-n)t} \mathbf{i}^2(t) &= \\ \lim_{t \rightarrow \infty} \left(i(\tau) - \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right)^2 e^{2(\rho - \bar{d}_2 - d_3)(t - \tau)} e^{-(\rho-n)t} &+ \\ \lim_{t \rightarrow \infty} e^{-(\rho-n)t} \left(\frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right)^2 &+ \\ + \lim_{t \rightarrow \infty} e^{-(\rho-n)t} 2 \left(i(\tau) - \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right) e^{(\rho - \bar{d}_2 - d_3)(t - \tau)} \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} &= \\ = \lim_{t \rightarrow \infty} \left(i(\tau) - \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right)^2 e^{-2(\rho - \bar{d}_2 - d_3)\tau} e^{(\rho+n-2(\bar{d}_2 - d_3))t} &+ \\ + \lim_{t \rightarrow \infty} e^{-(\rho-n)t} \left(\frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right)^2 &+ \\ \lim_{t \rightarrow \infty} 2 \left(i(\tau) - \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} \right) \frac{1}{c} \frac{1}{\bar{d}_2 + d_3 - n} \frac{1}{\rho - \bar{d}_2 - d_3} e^{-(\rho - \bar{d}_2 - d_3)\tau} e^{(n - \bar{d}_2 - d_3)t} &= 0, \end{aligned}$$

if $n < \rho$, $n < \bar{d}_2 + d_3$ and $\rho + n < 2(\bar{d}_2 + d_3)$, which is the case under the theorem's assumptions.

Step 3: Finally, let us finally prove that v is the value function so that m is optimal. This will also prove that the solution v to the HJB equation is the value function. By definition, m^* is an optimal control if for any other admissible control \tilde{m} we have that

$$j(i_0, m^*) \leq j(i_0, \tilde{m}).$$

Hence, it is our objective to prove that the above inequality holds for any admissible control \tilde{m} .

To start, let us denote \tilde{i} the trajectory associated to \tilde{m} , and let us denote by $w(t, i) : \mathbb{R} \times L^2(\mathbb{T}) \rightarrow \mathbb{R}$:

$$w(t, i) = e^{-(\rho-n)t} v(i).$$

Then

$$\begin{aligned}
v(i_0) - w(T, \tilde{i}(T)) &= w(0, \tilde{i}_0) - w(T, \tilde{i}(T)) = - \int_0^T \frac{d}{dt} w(t, \tilde{i}(t)) dt = \\
&= \int_0^T \left[(\rho - n) e^{-(\rho-n)t} v[\tilde{i}(t)] - e^{-(\rho-n)t} \frac{d}{dt} v[\tilde{i}(t)] \right] dt \\
&= \int_0^T \left[(\rho - n) e^{-(\rho-n)t} v[\tilde{i}(t)] - e^{-(\rho-n)t} \langle i_t, \nabla v[\tilde{i}(t)] \rangle \right] dt \\
&= \int_0^T (\rho - n) e^{-(\rho-n)t} v[\tilde{i}(t)] dt \\
&\quad - \int_0^T e^{-(\rho-n)t} \left\{ \langle d_1 \mathcal{L} \tilde{i} + (d_2(t) - n) \tilde{i} + d_3 \int_{\mathbb{T}} \phi(\theta - \omega) \tilde{i}(\omega) d\omega - \tilde{m} N_0, \nabla v[\tilde{i}(t)] \rangle \right\} dt.
\end{aligned}$$

We pass to the limit when $T \rightarrow \infty$, knowing that $\langle d_1 \mathcal{L} \tilde{i}, \nabla v(\tilde{i}(t)) \rangle = 0$ and that $\lim_{T \rightarrow \infty} w(T, \tilde{i}(T)) = 0$:

$$\begin{aligned}
v(i_0) &= \int_0^\infty e^{-(\rho-n)t} \left\{ (\rho - n) v[\tilde{i}(t)] - \langle (d_2(t) - n) \tilde{i} + d_3 \int_{\mathbb{T}} \phi(\theta - \omega) \tilde{i}(\omega) d\omega - \tilde{m}, \nabla v[\tilde{i}(t)] \rangle \right\} dt \\
v(i_0) - j(i_0, \tilde{m}) &= \\
\int_0^\infty e^{-(\rho-n)t} \left\{ (\rho - n) v[\tilde{i}(t)] - \langle (d_2(t) - n) \tilde{i} + d_3 \int_{\mathbb{T}} \phi(\theta - \omega) \tilde{i}(\omega) d\omega - \tilde{m}, \nabla v[\tilde{i}(t)] \rangle \right\} dt &- \langle u(\tilde{i}, \tilde{m}), \mathbb{1} \rangle.
\end{aligned}$$

We know that $\tilde{m}(\cdot)$ is an admissible control, so that v satisfies (21) by construction. Hence

$$v(i_0) - j(i_0, \tilde{m}) = \int_0^\infty e^{-\rho t} \left(\inf_m u(\tilde{i}, m) - \langle \tilde{m}, \nabla v[\tilde{i}(t)] \rangle \right) dt - \langle u(\tilde{i}, \tilde{m}), \mathbb{1} \rangle.$$

Let us prove that $v(i_0) - j(i_0, \tilde{m}) \leq 0$. First and since we know that by definition $\inf_m f(m) \leq f(\tilde{m})$ for any function f and any \tilde{m} . Then

$$\int_0^\infty e^{-\rho t} \inf_m u(\tilde{i}, m) dt - \langle u(\tilde{i}, \tilde{m}), \mathbb{1} \rangle \leq 0.$$

Hence

$$\begin{aligned}
\int_0^\infty e^{-\rho t} \left(\inf_m u(\tilde{i}, m) - \langle \tilde{m}, \nabla v[\tilde{i}(t)] \rangle \right) dt &- \langle u(\tilde{i}, \tilde{m}), \mathbb{1} \rangle \leq - \int_0^\infty e^{-\rho t} \\
&\langle \tilde{m}, \nabla v[\tilde{i}(t)] \rangle dt \leq 0.
\end{aligned}$$

Consequently $v(i_0) - j(i_0, \tilde{m}) \leq 0$. Note that $v(i_0) - j(i_0, m^*) = 0$, so that

$$\Rightarrow v(i_0) - j(i_0, \tilde{m}) \leq v(i_0) - j(i_0, m^*) \Rightarrow j(i_0, m^*) \leq j(i_0, \tilde{m}).$$

This implies the optimality of m^* . Furthermore, since $v(i_0) = j(i_0, m^*) \Rightarrow v(i_0)$ is the value function for i_0 . Uniqueness follows from standard convexity considerations.

A.2 Proof of Proposition 1

If we let time tend to ∞ in equation (33), and denoting by $\bar{\mathbf{i}}$ the steady state value of $\mathbf{i}(t)$, we obtain the result in Proposition 1:

$$\bar{\mathbf{i}} = \frac{A}{2(\bar{d}_2 + d_3 - n)} \frac{1}{\rho - (\bar{d}_2 + d_3)} \langle \frac{1}{c}, \mathbb{1} \rangle,$$

and $\bar{m} = \frac{A}{\bar{d}_2 + d_3 - n}$. $\bar{m} < 1$ if $A < \bar{d}_2 + d_3 - n$, so that $\bar{\mathbf{i}} < 1$ if

$$\frac{1}{2} \frac{1}{\rho - (\bar{d}_2 + d_3)} \langle \frac{1}{c}, \mathbb{1} \rangle < 1,$$

or

$$\langle \frac{1}{c}, \mathbb{1} \rangle < 2(\rho - (\bar{d}_2 + d_3)),$$

A.3 Proof of Proposition 2

Let us define $z = i - \bar{i}$, then under the proposition assumptions z verifies that

$$\frac{\partial z(\theta, t)}{\partial t} = d_1 \frac{\partial^2 z}{\partial \theta^2}(\theta, t) + (d_2(t) - n)z(\theta, t) + d_3 \int_{\mathbb{T}} \phi(\theta - \omega)z(\omega, t)d\omega. \quad (34)$$

Let us then define the stationary problem associated to (34):

$$0 = \frac{\partial \bar{z}(\theta, t)}{\partial t} = d_1 \frac{\partial^2 \bar{z}}{\partial \theta^2}(\theta, t) + (\bar{d}_2 - n)\bar{z}(\theta, t) + d_3 \int_{\mathbb{T}} \phi(\theta - \omega)\bar{z}(\omega, t)d\omega. \quad (35)$$

The eigenvalue problem associated to the stationary problem (35) is to find the set of values λ such that

$$\lambda \bar{z} = d_1 \mathcal{L} \bar{z} + (\bar{d}_2 - n)\bar{z} + d_3 \int_{\mathbb{T}} \phi(\theta - \omega)\bar{z}(\omega)d\omega.$$

We apply the Fourier transform on both sides

$$\begin{aligned} \int_{\mathbb{T}} \lambda \bar{z}(\theta)e^{-i\xi\theta}d\theta &= d_1 \int_{\mathbb{T}} \mathcal{L} \bar{z}(\theta)e^{-i\xi\theta}d\theta + (\bar{d}_2 - n) \int_{\mathbb{T}} \bar{z}(\theta)e^{-i\xi\theta}d\theta \\ &+ d_3 \int_{\mathbb{T}} \int_{\mathbb{T}} \phi(\theta - \omega)\bar{z}(\omega)e^{-i\xi\theta}d\omega d\theta. \end{aligned} \quad (36)$$

Note that

$$\begin{aligned} \int_{\mathbb{T}} \mathcal{L} \bar{z}(\theta)e^{-i\xi\theta}d\theta &= e^{-i\xi\theta} \nabla \bar{z} |_{\mathbb{T}} + \int_{\mathbb{T}} i\xi \nabla \bar{z}(\theta)e^{-i\xi\theta}d\theta \\ &= e^{-i\xi\theta} \nabla \bar{z} |_{\mathbb{T}} + i\xi \bar{z} e^{-i\xi\theta} |_{\mathbb{T}} + (i\xi)^2 \int_{\mathbb{T}} \bar{z}(\theta)e^{-i\xi\theta}d\theta. \end{aligned}$$

If $\nabla \bar{z} |_{0} = \nabla \bar{z} |_{2\pi}$ and $\bar{z}(2\pi) = \bar{z}(0) = 0$, then

$$\int_{\mathbb{T}} \mathcal{L} \bar{z}(\theta)e^{-i\xi\theta}d\theta = -\xi^2 \int_{\mathbb{T}} \bar{z}(\theta)e^{-i\xi\theta}d\theta.$$

Defining χ as the Fourier transform of \bar{z} , that is $\chi(\xi) = \int_{\mathbb{T}} \bar{z}(\theta)e^{-i\xi\theta}d\theta$, we obtain from (36) that

$$(\lambda + n - \bar{d}_2)\chi(\xi) = d_3 \tilde{\phi}(\xi)\chi(\xi) - d_1 \xi^2 \chi(\xi),$$

that is

$$\left(\lambda + n + d_1 \xi^2 - \bar{d}_2 - d_3 \tilde{\phi}(\xi) \right) \chi(\xi) = 0.$$

Since we are looking for non trivial steady state solutions, we assume that $\chi(\xi) \neq 0$. Hence, defining λ as a function of ξ :

$$\lambda(\xi) = \bar{d}_2 + d_3 \tilde{\phi}(\xi) - n - d_1 \xi^2.$$

If $\tilde{\phi}(\cdot)$ is such that for one single value of $\xi \in [0, 2\pi]$, $\lambda(\xi) > 0$, then the steady state is unstable. Note that if $\bar{d}_2 > n$, then the steady state is always unstable because for $\xi = 0$, we would have that $\lambda(0) = \bar{d}_2 + d_3 \tilde{\phi}(0) - n > 0$.

If on the contrary $\bar{d}_2 < n$, \bar{z} is unstable if there exists one value of $\xi \in \mathbb{T}$ such that

$$\tilde{\phi}(\xi) > \frac{d_1 \xi^2 + n - \bar{d}_2}{d_3}.$$

This condition obviously depends on $\tilde{\phi}$ and ϕ . Note also that d_1 plays a major role since it tends to stabilize the steady state. Indeed, the larger d_1 , the higher the value of $\tilde{\phi}$ required for instability.

A.4 The terminal condition in the finite horizon problem

The value function of the HJB equation associated to the finite time horizon problem coincides with the value function of the infinite horizon problem. As proven in [3], the only difference is the change of the transversality condition by the following terminal condition. When time is finite the value function satisfies that

$$v(i(T)) = -\Phi e^{-(\rho-n)T} \int_{\mathbb{T}} i(\theta, T) d\theta = -\Phi e^{-(\rho-n)T} \mathbf{i}(T). \quad (37)$$

Since $m_F^*(\theta, t) = \frac{1}{2c(\theta)} \nabla v(i)$, for every t , in particular,

$$m_F^*(\theta, T) = \frac{1}{2c(\theta)} \nabla v(\mathbf{i}(T)) \quad (38)$$

Using (27), we have that

$$m_F^*(\theta, T) = \frac{1}{2c(\theta)} \left(\frac{A}{d_2 + d_3 - n} + 2 \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(T) \right), \quad (39)$$

where $\mathbf{i}(T)$ is still unknown.

A.5 Proof of Proposition 3

To obtain $\mathbf{i}(T)$, one can equate the value for $m_F^*(\theta, T)$ obtained in (39) and that provided by the (38), where we use the terminal condition (37) to replace $v(i(T))$:

$$\begin{aligned} -\frac{2\pi A}{\rho - n} - \frac{1}{4(\rho - n)} \frac{A^2}{(d_2 + d_3 - n)^2} \langle \frac{1}{c}, \mathbb{1} \rangle + \frac{A}{d_2 + d_3 - n} \mathbf{i}(T) + \\ \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(T)^2 = -\Phi e^{-(\rho-n)T} \mathbf{i}(T), \end{aligned}$$

which can be rewritten as a second order polynomial in $\mathbf{i}(T)$:

$$\begin{aligned} \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(T)^2 + \mathbf{i}(T) \left(\frac{A}{d_2 + d_3 - n} + \Phi e^{-(\rho-n)T} \right) - \\ \frac{2\pi A}{\rho - n} - \frac{1}{4(\rho - n)} \frac{A^2}{(d_2 + d_3 - n)^2} \langle \frac{1}{c}, \mathbb{1} \rangle = 0. \end{aligned}$$

The polynomial above always has two roots. Note that $\mathbf{i}(T)$ can never be 0. If it was, then $\frac{2\pi A}{\rho - n} + \frac{1}{4(\rho - n)} \frac{A^2}{(d_2 + d_3 - n)^2} \langle \frac{1}{c}, \mathbb{1} \rangle$ should be zero, which is impossible under the model assumptions.

Denoting by $a = \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle}$ the second order coefficient, by $b = \frac{A}{d_2 + d_3 - n} + \Phi e^{-(\rho-n)T}$ and $c = -\frac{2\pi A}{\rho - n} - \frac{1}{4(\rho - n)} \frac{A^2}{(d_2 + d_3 - n)^2} \langle \frac{1}{c}, \mathbb{1} \rangle$, the polynomial roots are

$$\mathbf{i}(T) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence, depending on the relative importance of diffusion, of impatience and population growth, there will be a corner solution for $\mathbf{i}(T)$ or an interior solution, or no solution at all. Note first that in this case $b > 0$, so that there is only one positive root if and only if $ac < 0$, which is true under the proposition's assumptions. Then

- i) $\mathbf{i}(T) = 1$ if and only if $a + b + c = 0$, which is what the third point in Proposition 3.
- ii) $0 < \mathbf{i}(T) < 1$ if $a + b + c > 0$.

To prove that $\mathbf{i}(T)$ decreases with T , let us define the following function F as

$$F(i_T, T) = \frac{2(d_2 + d_3) - n - \rho}{\langle \frac{1}{c}, \mathbb{1} \rangle} \mathbf{i}(T)^2 + \mathbf{i}(T) \left(\frac{A}{d_2 + d_3 - n} + \Phi e^{-(\rho-n)T} \right) - \frac{2\pi A}{\rho - n} - \frac{1}{4(\rho - n)} \frac{A^2}{(d_2 + d_3 - n)^2} < \frac{1}{c}, \mathbb{1} \rangle.$$

We know that $F(i_T, T) = 0$. Using the Implicit Function Theorem:

$$\frac{\partial F(i_T, T)}{\partial i_T} \frac{\partial i_T}{\partial T} + \frac{\partial F(i_T, T)}{\partial T} = 0,$$

so that $\frac{\partial i_T}{\partial T} = -\frac{\frac{\partial F(i_T, T)}{\partial T}}{\frac{\partial F(i_T, T)}{\partial i_T}}$, if $\frac{\partial F(i_T, T)}{\partial i_T} \neq 0$. We can compute the partial derivatives of F :

$$\frac{\partial F(i_T, T)}{\partial T} = \phi(\rho - n)i_T e^{-(\rho-n)T} > 0,$$

and

$$\frac{\partial F(i_T, T)}{\partial i_T} = 2ai_T + b,$$

where a and b were previously defined and they are both positive. Hence, we conclude that $\frac{\partial i_T}{\partial T} < 0$.