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THE EXTREMAL POSITION OF A BRANCHING RANDOM WALK IN THE GENERAL LINEAR GROUP

ION GRAMA, SEBASTIAN MENTEMEIER, AND HUI XIAO

ABSTRACT. Consider a branching random walk $(G_u)_{u \in \mathbb{T}}$ on the general linear group $\mathrm{GL}(V)$ of a finite dimensional space V , where \mathbb{T} is the associated genealogical tree with nodes u . For any starting point $v \in V \setminus \{0\}$, let $M_n^v = \max_{|u|=n} \log \|G_u v\|$ denote the maximal position of the walk $\log \|G_u v\|$ in the generation n . We first show that under suitable conditions, $\lim_{n \rightarrow \infty} \frac{M_n^v}{n} = \gamma_+$ almost surely, where $\gamma_+ \in \mathbb{R}$ is a constant. Then, in the case when $\gamma_+ = 0$, under appropriate boundary conditions, we refine the last statement by determining the rate of convergence at which M_n^v converges to $-\infty$. We prove in particular that $\lim_{n \rightarrow \infty} \frac{M_n^v}{\log n} = -\frac{3}{2\alpha}$ in probability, where $\alpha > 0$ is a constant determined by the boundary conditions. Similar properties are established for the minimal position. As a consequence we derive the asymptotic speed of the maximal and minimal positions for the coefficients, the operator norm and the spectral radius of G_u .

1. INTRODUCTION

Let $V = \mathbb{R}^d$ be a d -dimensional Euclidean vector space equipped with the norm $\|\cdot\|$, where $d \geq 1$ is an integer. Denote by $\mathbb{G} = \mathrm{GL}(V)$ the general linear group of the vector space V . A branching random walk on the group \mathbb{G} is obtained by setting at time 0 its initial value to be the identity matrix, called also root particle. At time step 1, the root particle generates a random number of random elements of \mathbb{G} , called children. At subsequent time steps, this happens in an independent and identical manner for every obtained child. The above iterations generate a genealogical tree \mathbb{T} whose nodes are denoted by u . Starting with the root, by successive multiplication from the left of the random elements on the branch corresponding to a node $u \in \mathbb{T}$, we define a branching random walk in \mathbb{G} which we shall denote by $(G_u)_{u \in \mathbb{T}}$.

Let $v \in \mathbb{R}^d$ be a vector with $\|v\| = 1$ and let $x = \mathbb{R}v \in \mathbb{P}(V)$ be the direction of v . The first objective of the paper is to study the asymptotic behaviour of the maximal displacement $M_n^x = \max_{|u|=n} \log \|G_u v\|$ as $n \rightarrow \infty$, where we write $|u| = n$ when u is a node of generation $n \geq 1$. Moreover, we shall consider the following extension of this problem. Denote by $G_u \cdot x$ the Markov branching process associated to the projective action of the group \mathbb{G} on the projective space $\mathbb{P}(V)$, which describes the behaviour of the directions of the walk $(G_u v)_{u \in \mathbb{T}}$. Given a set of directions $A \subseteq \mathbb{P}(V)$, it is of interest to study the maximal position of $\log \|G_u v\|$ provided $G_u \cdot x \in A$, i.e. the directed maximal displacement $M_n^x(A) = \sup_{G_u \cdot x \in A, |u|=n} \log \|G_u v\|$, where, by convention $\sup \emptyset = -\infty$, to include the case when the condition $G_u \cdot x \in A$ is not satisfied. We shall as well study similar problems for the related characteristics of (G_u) such as the coefficients, the operator norm and the spectral radius.

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Let us describe briefly our main results in the particular case when $A = \mathbb{P}(V)$. Under suitable conditions, the following convergence holds (see Theorem 2.1): for any $x \in \mathbb{P}(V)$, conditionally on the system's survival,

$$\lim_{n \rightarrow \infty} \frac{M_n^x}{n} = \gamma_+ \quad \text{almost surely,} \quad (1.1)$$

where $\gamma_+ \in \mathbb{R}$ is a constant. When the branching product is in the *boundary case* (for a precise formulation see condition A4 formulated in the next section) we have that $\gamma_+ = 0$, so that $\lim_{n \rightarrow \infty} \frac{M_n^x}{n} = 0$ almost surely. It can be shown that in this case it still holds that $\lim_{n \rightarrow \infty} M_n^x = -\infty$. Our main interest is to refine the last statement by determining the rate of convergence at which M_n^x converges to $-\infty$. We prove in particular that, for any $x \in \mathbb{P}(V)$, conditionally on the system's survival,

$$\lim_{n \rightarrow \infty} \frac{M_n^x}{\log n} = -\frac{3}{2\alpha} \quad \text{in probability,}$$

where $\alpha > 0$ is a constant defined by the boundary condition A4. The full statement of our result for the maximal displacement is the content of Theorem 2.2 below.

The asymptotic behaviour of the maximal position of classical branching random walks in \mathbb{R}^1 has been investigated by many authors, see for example Kingman [24], Hammersley [19], Biggins [5]. Significant progress has been made by Hu and Shi [22], Addario-Berry and Reed [1], and Aïdékon [2], where the conditioned limit theorems for sums of independent and identically distributed (i.i.d.) random variables are used to establish a law of large numbers and limit theorems for its fluctuations.

The branching random walk on the general linear group \mathbb{G} studied here is a natural extension of the classical branching random walk in \mathbb{R}^1 and is of particular interest because the underlying group is non-commutative. The model was investigated in Buraczewski, Damek, Guivarc'h and Mentemeier [8], Mentemeier [25] and Bui, Grama and Liu [7]. In view of the one-dimensional results, it is natural to ask for the behaviour of M_n^x and, moreover, to study the maximal position of particles with a given direction $M_n^x(A)$. Both these questions have not been addressed in the works mentioned above and become a way more involved for branching random walks on groups. The difficulty in obtaining such type of the results is in a great part due to the lack of the corresponding conditioned local limit theory for products of random matrices, besides the inherently heavy argument in dealing with extremal position of the walk $(G_u v)_{u \in \mathbb{T}}$.

Recent progress for products of random matrices has been made in Grama, Le Page and Peigné [13] and Grama, Lauvergnat and Le Page [11], where some integral conditioned theorems have been established. Local limit theorems for finite Markov chains have been considered in Grama, Lauvergnat and Le Page [12], however, establishing convenient conditioned local limit theorems for products of random matrices is still an open problem. Following some recent developments in Grama, Quint and Xiao [15] and Grama and Xiao [16], in the present paper we shall establish some new conditioned limit theorems for products of random matrices under the assumption that the matrices have a density with respect to the Haar measure on \mathbb{G} . This conditioned local limit theory is the key point in studying the maximal displacement $M_n^x(A)$. Besides, it could be also useful for studying other problems like a version of the Seneta-Heyde theorem for branching random walks on groups. This question will be considered in a subsequent work.

2. THE SETUP AND MAIN RESULTS

2.1. Notations and Assumptions. Let $\mathbb{R}_+ = [0, \infty)$ and denote by \mathbb{N} the set of non-negative integers. For any integer $d \geq 1$, denote by $V = \mathbb{R}^d$ the d -dimensional Euclidean space. We fix a basis $(e_i)_{1 \leq i \leq d}$ of V and define the associated norm on V by $\|v\|^2 = \sum_{i=1}^d |v_i|^2$ for $v = \sum_{i=1}^d v_i e_i \in V$. Let $\mathbb{G} = GL(V)$ be the general linear group of V . The action of $g \in \mathbb{G}$ on a vector $v \in V$ is denoted by gv . For any $g \in \mathbb{G}$, let $\|g\| = \sup_{v \in V \setminus \{0\}} \frac{\|gv\|}{\|v\|}$.

Let $\mathbb{P}(V)$ be the projective space of V . The action of $g \in \mathbb{G}$ on $x = \mathbb{R}v \in \mathbb{P}(V)$ is defined by $g \cdot x = \mathbb{R}gv \in \mathbb{P}(V)$. For $g \in \mathbb{G}$ and $x = \mathbb{R}v \in \mathbb{P}(V)$ with $v \in V \setminus \{0\}$, introduce the norm cocycle

$$\sigma : \mathbb{G} \times \mathbb{P}(V) \rightarrow \mathbb{R}, \quad \sigma(g, x) = \log \frac{\|gv\|}{\|v\|}. \quad (2.1)$$

A branching random walk in the group \mathbb{G} is defined as follows. Assume that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a point process \mathcal{N} indexed with the Borel sets of \mathbb{G} . At the time 0, we pick up the identity matrix of the group \mathbb{G} . In the first generation, the children of the identity matrix, are formed by the collection of matrices in \mathbb{G} picked up according to the law of the point process \mathcal{N} . Each matrix in the first generation gives birth to new matrices according to the independent copies of the same point process \mathcal{N} , which form the second generation of matrices. The system goes on according to the same mechanism.

The genealogy of these matrices forms a Galton-Watson tree \mathbb{T} with the root \emptyset . We use Ulam-Harris notation, identifying \mathbb{T} as a random subset of $\bigcup_{n=0}^{\infty} \mathbb{N}^n$. Hence, each node $u \in \mathbb{T}$ of generation n will be identified with an n -tuple $u = (u_1, \dots, u_n)$, where $u_i \in \mathbb{N}$. For notational simplicity, we write $u = u_1 \dots u_n$ and further denote by $u|k := u_1 \dots u_k$ the restriction of u to its first k components. Given $u = u_1 \dots u_n$ and $v = v_1 \dots v_m$, we write $uv = u_1 \dots u_n v_1 \dots v_m$ for the concatenation. For a node u in \mathbb{T} by $|u|$ we denote its generation.

From the definition of the branching process above, to each node $u \in \mathbb{T}$ corresponds a random element $g_u \in \mathbb{G}$. Then, the branching random walk $(G_u)_{u \in \mathbb{T}}$ in the group \mathbb{G} is defined by taking the left product of the elements along the branch leading from \emptyset to $u \in \mathbb{T}$:

$$G_u := g_u g_{u|n-1} \dots g_{u|2} g_{u|1}, \quad u \in \mathbb{T}.$$

Note that these factors are independent, but not necessarily identically distributed. There is a natural filtration given by $\mathcal{F}_n := \sigma(\{g_u : |u| \leq n\})$. Obviously, the point process \mathcal{N} can be written as $\mathcal{N} = \sum_{|u|=1} \delta_{\{G_u\}}$. We denote by $N = \mathcal{N}(\mathbb{G})$ the number of particles in the first generation as well as by N_u the number of children of the particle at node $u \in \mathbb{T}$.

We define the following shift operator which will help us to use the branching property. For any function $F = F((g_u)_{u \in \mathbb{T}})$ of the branching process and any node $v \in \mathbb{T}$, we define

$$[F]_v = F((g_{vu})_{u \in \mathbb{T}_v}), \quad (2.2)$$

where \mathbb{T}_v denotes the random subtree rooted at v , its first generation being formed by the children of the particle at v . That is, $[F]_v$ is evaluated on the subtree started at v . For example, $[G_u]_v$ means the product $g_{vu} g_{vu|n-1} \dots g_{vu|2} g_{vu|1}$ for $u = u_1 u_2 \dots u_n \in \mathbb{T}_v$, i.e. the product of the elements of \mathbb{G} along the branch leading from v to vu .

Let \mathbb{E} be the expectation corresponding to the probability \mathbb{P} . We will need the following moment condition on N .

A1. *There exists a constant $\delta > 0$ such that $1 < \mathbb{E}N$ and $\mathbb{E}(N^{1+\delta}) < \infty$.*

This condition allows us to define a measure μ on \mathbb{G} by¹

$$\mu(B) := \frac{1}{\mathbb{E}N} \mathbb{E} \left[\sum_{|u|=1} \mathbb{1}_B(G_u) \right], \quad (2.3)$$

for any Borel measurable set $B \subseteq \mathbb{G}$. We will also make use of the following density and moment assumption.

A2. *The measure μ has a density with respect to the Haar measure on \mathbb{G} . Moreover, there exists a constant $c_0 > 0$ such that $\inf_{x \in \mathbb{P}(V)} \mu\{g \in \mathbb{G} : \sigma(g, x) > c_0\} > 0$ and there exists a constant $\eta_0 > 0$ such that*

$$\int_{\mathbb{G}} \max \{ \|g\|, \|g^{-1}\| \}^{\eta_0} \left(1 + \|g\|^d |\det(g^{-1})| \right) \mu(dg) < \infty. \quad (2.4)$$

The moment condition (2.4) is obviously satisfied when μ is compactly supported on \mathbb{G} . The condition $\inf_{x \in \mathbb{P}(V)} \mu\{g \in \mathbb{G} : \sigma(g, x) > c_0\} > 0$ is necessary to ensure that the harmonic function $V_s(x, y)$ (cf. Proposition 4.5) is strictly positive for any $x \in \mathbb{P}(V)$ and $y \geq 0$.

Condition **A2** plays an important role in the paper. It ensures that the spectral gap properties of the transfer operators P_s and P_s^* hold for $s > 0$ and $s < 0$ (see the precise definitions below). Besides, it is essential for the existence of the dual random walk and also opens the way to establish the conditioned local limit theory for products of random matrices. These facts allow us to study the asymptotics of both the maximal and the minimal positions.

Let $v \in \mathbb{R}^d$ be a non-zero vector. Our first goal is to study some extremal properties of the branching random walk $(G_u v)_{u \in \mathbb{T}}$ on \mathbb{R}^d defined by the action of G_u on v . We shall do it jointly with the walk $(G_u \cdot x)_{u \in \mathbb{T}}$ on $\mathbb{P}(V)$ defined by the action of G_u on the projective element $x = \mathbb{R}v \in \mathbb{P}(V)$. To be more precise, we shall study the couple

$$G_u \cdot x = \mathbb{R}G_u v \in \mathbb{P}(V), \quad \sigma(G_u, x) = \frac{\log \|G_u v\|}{\|v\|} \in \mathbb{R}, \quad u \in \mathbb{T}. \quad (2.5)$$

Given a Borel set $A \subseteq \mathbb{P}(V)$, define respectively the maximal and minimal positions of $\sigma(G_u, x) = \log \|G_u v\|$ provided that direction $G_u \cdot x$ belongs to the set A for nodes u in the n -th generation by setting

$$M_n^x(A) := \sup \{ \sigma(G_u, x) : G_u \cdot x \in A, |u| = n \}, \quad (2.6)$$

$$m_n^x(A) := \inf \{ \sigma(G_u, x) : G_u \cdot x \in A, |u| = n \}, \quad (2.7)$$

where $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$. From the results of [7, Theorem 2.1] it is easy to see that under conditions **A1** and **A2**, the sets in (2.6) and (2.7) become nonempty eventually for large n . The global maximal and minimal displacements in the n -th generation are defined by

$$M_n^x := M_n^x(\mathbb{P}(V)) = \sup \{ \sigma(G_u, x) : |u| = n \},$$

$$m_n^x := m_n^x(\mathbb{P}(V)) = \inf \{ \sigma(G_u, x) : |u| = n \},$$

with the same convention $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

¹Note that since $\|\cdot\|$ is submultiplicative, trying to avoid assumption **A1** by defining

$$\mu(B) = \frac{\mathbb{E} \left[\sum_{|u|=1} \|G_u\|^\theta \mathbb{1}_B(G_u) \right]}{\mathbb{E} \left[\sum_{|u|=1} \|G_u\|^\theta \right]}$$

for some $\theta > 0$ does not lead to a convolution-stable definition of a measure μ .

Set

$$I_\mu^+ = \left\{ s \geq 0 : \int_{\mathbb{G}} \|g\|^s \left(1 + \|g\|^d |\det(g^{-1})|\right) \mu(dg) < \infty \right\},$$

$$I_\mu^- = \left\{ s \leq 0 : \int_{\mathbb{G}} \|g^{-1}\|^{-s} \left(1 + \|g\|^d |\det(g^{-1})|\right) \mu(dg) < \infty \right\}.$$

By **A2** and Hölder's inequality, both I_μ^+ and I_μ^- are non-empty intervals of \mathbb{R} . Denote by $(I_\mu^+)^\circ$ the interior of I_μ^+ , and by $(I_\mu^-)^\circ$ the interior of I_μ^- . Let \mathcal{B} be the Banach space of real-valued continuous functions on $\mathbb{P}(V)$ endowed with the supremum norm $\|\cdot\|_{\mathcal{B}}$. For any $s \in I_\mu^+ \cup I_\mu^-$, define the transfer operator P_s as follows: for $\varphi \in \mathcal{B}$ and $x \in \mathbb{P}(V)$,

$$P_s \varphi(x) = \int_{\mathbb{G}} e^{s\sigma(g,x)} \varphi(g \cdot x) \mu(dg) = \frac{1}{\mathbb{E}(N)} \mathbb{E} \left(\sum_{|u|=1} e^{s\sigma(G_u, x)} \varphi(G_u \cdot x) \right), \quad (2.8)$$

where the second equality follows directly from the definition of μ . Using the quasi-compactness of the operator P_s , it will be shown in Lemma 5.1 that under condition **A2**, for any $s \in I_\mu^+ \cup I_\mu^-$,

$$P_s r_s = \kappa(s) r_s \quad \text{and} \quad \nu_s P_s = \kappa(s) \nu_s, \quad (2.9)$$

where κ is the unique dominant eigenvalue of P_s , r_s is the corresponding unique (up to a scaling constant) continuous eigenfunction on $\mathbb{P}(V)$, and ν_s is the unique probability eigenmeasure on $\mathbb{P}(V)$. When $s = 0$ we use the shortened notation $\nu := \nu_0$. We need the following condition.

A3. *The invariant measure ν has a strictly positive density with respect to the uniform probability measure on $\mathbb{P}(V)$.*

In particular, condition **A3** implies that ν_s also has a strictly positive density on $\mathbb{P}(V)$ (cf. [14]). Note that condition **A3** is satisfied for example when condition (id) from the paper [9, Section 2.3] holds, in which case it is also possible for μ to have a compact support in \mathbb{G} . Another example is when the density of μ from **A2** is strictly positive on the whole \mathbb{G} .

Denote

$$\mathfrak{m}(s) = \kappa(s) \mathbb{E}N, \quad s \in I_\mu^+ \cup I_\mu^-. \quad (2.10)$$

In order to formulate the asymptotic properties of the maximal positions $M_n^x(A)$ and M_n^x , we introduce the following boundary condition, where $\log_+ t = \max\{\log t, 0\}$ for $t > 0$.

A4. *There exists a constant $\alpha \in (I_\mu^+)^\circ$ with $\alpha > 0$ such that $\mathfrak{m}(\alpha) = 1$ and $\mathfrak{m}'(\alpha) = 0$. In addition,*

$$\mathbb{E} \left(\sum_{|u|=1} \|G_u\|^\alpha (1 + \log_+ \|G_u\| + \log_+ \|G_u^{-1}\|) \right)^2 < \infty.$$

The boundary condition **A4** guarantees that under some changed measure the behaviour of our branching random walk is similar to that of a left product of random matrices whose Lyapunov exponent is 0. The corresponding change of measure and a many-to-one formula will be introduced in Section 3. To study the asymptotic properties of the minimal positions $m_n^x(A)$ and m_n^x , we need a similar boundary condition.

A5. *There exists a constant $-\beta \in (I_\mu^-)^\circ$ with $\beta > 0$ such that $\mathfrak{m}(-\beta) = 1$ and $\mathfrak{m}'(-\beta) = 0$. In addition,*

$$\mathbb{E} \left(\sum_{|u|=1} \|G_u^{-1}\|^{-\beta} (1 + \log_+ \|G_u\| + \log_+ \|G_u^{-1}\|) \right)^2 < \infty.$$

Denote the event corresponding to the system's survival after n generations by $\mathcal{S}_n = \{\sum_{|u|=n} 1 > 0\}$. Then $\mathcal{S} = \cap_{n=1}^{\infty} \mathcal{S}_n$ is the event corresponding to the system's ultimate survival. Below, we say that a sequence of random variables $(\xi_n)_{n \geq 1}$ converges in probability to the random variable ξ under the system's survival if for any $\varepsilon > 0$ it holds

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\xi_n - \xi| \geq \varepsilon | \mathcal{S}) = 0,$$

where \mathcal{S} is the system's ultimate survival event. The convergence almost surely under the system's survival is defined in the same way:

$$\mathbb{P}(\lim_{n \rightarrow \infty} \xi_n = \xi | \mathcal{S}) = 1.$$

Note that by assumption **A1**, it holds that $\mathbb{P}(\mathcal{S}) > 0$.

2.2. Results for maximal and minimal positions. Now we state the main results of the paper. The next theorem gives the law of large numbers for maximal and minimal positions of our branching random walk. To state the corresponding results, we need the following additional conditions, where we denote $Y_s = \sum_{|u|=1} \|G_u\|^s$:

A6. For any $s \in I_\mu^+$, it holds $\mathbb{E}[Y_s \log_+ Y_s] < \infty$.

A7. For any $s \in I_\mu^-$, it holds $\mathbb{E}[Y_s \log_+ Y_s] < \infty$.

Note that if μ has a compact support on \mathbb{G} , then conditions **A6** and **A7** are satisfied due to condition **A1**. Denote

$$\gamma_+ = \inf_{s \in I_\mu^+} \frac{\log \mathbf{m}(s)}{s} \in \mathbb{R} \quad \text{and} \quad \gamma_- = \sup_{s \in I_\mu^-} \frac{\log \mathbf{m}(s)}{s} \in \mathbb{R}. \quad (2.11)$$

Theorem 2.1. Assume conditions **A1**, **A2** and **A6**. Let $x \in \mathbb{P}(V)$. Then, for any Borel set $A \subseteq \mathbb{P}(V)$ satisfying $\nu(A) > 0$ and $\nu(\partial A) = 0$, we have, conditionally on the system's survival,

$$\lim_{n \rightarrow \infty} \frac{M_n^x(A)}{n} = \gamma_+ \quad \text{almost surely.} \quad (2.12)$$

Similarly, under conditions **A1**, **A2** and **A7**, for any Borel set $A \subseteq \mathbb{P}(V)$ satisfying $\nu(A) > 0$ and $\nu(\partial A) = 0$, we have, conditionally on the system's survival,

$$\lim_{n \rightarrow \infty} \frac{m_n^x(A)}{n} = \gamma_- \quad \text{almost surely.} \quad (2.13)$$

For the proof of (2.12) and (2.13), we make use of the precise large deviation result for the counting measure which have been established recently in Bui, Grama and Liu [7]. We believe that if we do not account for the direction, with some extra effort conditions **A6** and **A7** can be removed, but we are refraining from doing so as our main goal is a second order asymptotic of the extreme position formulated below.

It is not difficult to verify that, under conditions of Theorem 2.1, the boundary assumption **A4** implies $\gamma_+ = 0$ and, similarly, the boundary assumption **A5** implies $\gamma_- = 0$. Of course the asymptotics (2.12) and (2.13) are not precise when $\gamma_+ = 0$ or $\gamma_- = 0$ respectively. This motivates our next theorem where we normalize $M_n^x(A)$ and $m_n^x(A)$ by $\log n$. On a rougher scale, it can be shown that under conditions **A1**, **A2**, **A4**, **A6**, for any $x \in \mathbb{P}(V)$ and any Borel set $A \subseteq \mathbb{P}(V)$ satisfying $\nu(A) > 0$,

$$\lim_{n \rightarrow \infty} M_n^x(A) = -\infty \quad \text{almost surely.} \quad (2.14)$$

Similarly, under conditions **A1**, **A2**, **A5**, **A7**, for any $x \in \mathbb{P}(V)$ and any Borel set $A \subseteq \mathbb{P}(V)$ satisfying $\nu(A) > 0$,

$$\lim_{n \rightarrow \infty} m_n^x(A) = \infty \quad \text{almost surely.} \quad (2.15)$$

This can be proved by adapting the method in Shi [26, Lemma 3.1]. We note that the divergences in probability in (2.14) and (2.15) are direct consequences of Theorems 2.2 and 2.3, respectively.

Our next result improves on (2.12) by showing that with the right normalization the maximal position M_n^x converges in probability to some constant, conditioned on the system's survival. In fact, we also show a stronger assertion for the joint behaviour of the maximal position and the direction $G_u \cdot x$.

Theorem 2.2. *Assume conditions **A1**, **A2**, **A3** and **A4**. Let $x \in \mathbb{P}(V)$. Then, for any Borel set $A \subseteq \mathbb{P}(V)$ satisfying $\nu(A) > 0$, we have, conditionally on the system's survival,*

$$\lim_{n \rightarrow \infty} \frac{M_n^x(A)}{\log n} = -\frac{3}{2\alpha} \quad \text{in probability.} \quad (2.16)$$

In particular, we have, conditionally on the system's survival,

$$\lim_{n \rightarrow \infty} \frac{M_n^x}{\log n} = -\frac{3}{2\alpha} \quad \text{in probability.} \quad (2.17)$$

Recall from the discussion above that the assumptions **A2** and **A3** are satisfied for example when μ has a compact support with a continuous density around the identity matrix.

Clearly the result (2.17) follows from (2.16) by taking $A = \mathbb{P}(V)$. It is worth mentioning that, even though the limits are the same, the first assertion (2.16) is much stronger than the second one, since it shows what is the limit behaviour of the maximal position $\sigma(G_u, x)$ over the subset of particles with the direction $G_u \cdot x \in A$. We also note that convergence in probability cannot be sharpened to an almost sure convergence. This will be considered in a forthcoming work.

Our second result proves similar properties for the minimal position m_n^x .

Theorem 2.3. *Assume conditions **A1**, **A2**, **A3** and **A5**. Let $x \in \mathbb{P}(V)$. Then, for any Borel set $A \subseteq \mathbb{P}(V)$ satisfying $\nu(A) > 0$, we have, conditionally on the system's survival,*

$$\lim_{n \rightarrow \infty} \frac{m_n^x(A)}{\log n} = \frac{3}{2\beta} \quad \text{in probability.} \quad (2.18)$$

In particular, we have, conditionally on the system's survival,

$$\lim_{n \rightarrow \infty} \frac{m_n^x}{\log n} = \frac{3}{2\beta} \quad \text{in probability.} \quad (2.19)$$

The appearance of α and β in the formulas (2.16), (2.17), (2.18) and (2.19) is due to the fact that a space transformation by an affine map of $\sigma(g, x)$ – as performed in the one-dimensional case to reduce to the case $\alpha = \beta = 1$ – is not possible, since it has no representation in terms of an affine transformation of the underlying product of random matrices.

Note that the limit in (2.18) and (2.19) is strictly positive, since $\beta > 0$ in **A5**, while in the case of the maximal position the limit in (2.16) and (2.17) is strictly negative, since $\alpha < 0$ in **A4**.

We complement the above results by proving that the asymptotic behaviour of maximal (minimal) value of the logarithm of a given coefficient, operator norm or spectral radius is similar to that of maximal (minimal) value of the logarithm of the vector norm. Let V^*

be the dual of V , i.e. the space of linear forms on V . For $v \in V$ and $f \in V^*$ denote by $\langle f, v \rangle = f(v)$ the duality bracket. Denote by $\|g\|$ and $\rho(g)$ respectively the operator norm and the spectral radius of the matrix $g \in \mathbb{G}$. Then we have the following results. Let $v \in V$, $f \in V^*$ and F_u be one of $|\langle f, G_u v \rangle|$, $\|G_u\|$ or $\rho(G_u)$. Theorems 2.1, 2.2 and 2.3 hold with M_n^x or m_n^x replaced by $\max_{|u|=n} \log F_u$ or $\min_{|u|=n} \log F_u$, respectively. We will only give precise formulations of the results corresponding to Theorems 2.2 and 2.3.

Theorem 2.4. *Assume conditions A1, A2, A3 and A4. Let $A \subseteq \mathbb{P}(V)$ be any Borel set satisfying $\nu(A) > 0$. Let $v \in V$, $f \in V^*$ and F_u be one of $|\langle f, G_u v \rangle|$, $\|G_u\|$ or $\rho(G_u)$. Then, we have, conditionally on the system's survival,*

$$\lim_{n \rightarrow \infty} \frac{\sup_{G_u \cdot x \in A, |u|=n} \log F_u}{\log n} = -\frac{3}{2\alpha} \quad \text{in probability.} \quad (2.20)$$

Similarly, under conditions A1, A2, A3 and A5, we have, conditionally on the system's survival,

$$\lim_{n \rightarrow \infty} \frac{\inf_{G_u \cdot x \in A, |u|=n} \log F_u}{\log n} = \frac{3}{2\beta} \quad \text{in probability.} \quad (2.21)$$

3. LAW OF LARGE NUMBERS FOR THE EXTREMAL POSITION

3.1. A change of measure formula. Let $(g_i)_{i \geq 1}$ be a sequence of i.i.d. random elements on \mathbb{G} with the law μ (defined by (2.3)) and denote by

$$G_n := g_n \dots g_1, \quad n \geq 1,$$

their left product. With any starting point $X_0 = x = \mathbb{R}v \in \mathbb{P}(V)$ and $S_0 = 0$, denote for $n \geq 1$,

$$X_n := g_n \cdot X_{n-1} = \mathbb{R}G_n v \quad \text{and} \quad S_n := \sum_{k=1}^n \sigma(g_k, X_{k-1}), \quad (3.1)$$

where the cocycle $\sigma(\cdot, \cdot)$ is defined in (2.1). Then the sequence $(X_n, S_n)_{n \geq 0}$ constitutes a Markov random walk on $\mathbb{P}(V) \times \mathbb{R}$. Denote by \mathbb{P}_x the probability measure on the canonical space $(\mathbb{P}(V))^{\mathbb{N}}$ induced by the Markov chain $(X_n)_{n \geq 0}$ with starting point $X_0 = x$. Let \mathbb{E}_x be the corresponding expectation. For any $s \in I_\mu^+ \cup I_\mu^-$, let $\kappa(s)$ and r_s be the eigenvalue and the eigenfunction given by (2.9). Since $\sigma(\cdot, \cdot)$ is a cocycle, one can check that the family of kernels

$$q_n^s(x, G_n) = \frac{1}{\kappa(s)^n} e^{s\sigma(G_n, x)} \frac{r_s(G_n \cdot x)}{r_s(x)}, \quad n \geq 1, \quad (3.2)$$

satisfies the property: for any $x \in \mathbb{P}(V)$ and $n, m \geq 1$,

$$q_n^s(x, G_n) q_m^s(G_n \cdot x, g_{n+m} \dots g_{n+1}) = q_{n+m}^s(x, G_{n+m}). \quad (3.3)$$

By (2.8) and (2.9), the sequence of the probability measures

$$\mathbb{Q}_{s,n}^x(dg_1, \dots, dg_n) := q_n^s(x, g_n \dots g_1) \mu(dg_1) \dots \mu(dg_n), \quad n \geq 1, \quad (3.4)$$

form a projective system on $\mathbb{G}^{\mathbb{N}}$, so that, by the Kolmogorov extension theorem, there is a unique probability measure \mathbb{Q}_s^x on $\mathbb{G}^{\mathbb{N}}$ with marginals $\mathbb{Q}_{s,n}^x$. Denote by $\mathbb{E}_{\mathbb{Q}_s^x}$ the corresponding expectation. All over the paper we use the convention that under the measure \mathbb{Q}_s^x , the Markov chain $(X_n)_{n \geq 0}$ defined by (3.1) starts with the point $x \in \mathbb{P}(V)$.

By the definition of \mathbb{Q}_s^x , for any bounded measurable function $h : (\mathbb{P}(V) \times \mathbb{R})^n$, the following change of measure formula holds: under condition **A2**, for any $s \in I_\mu^+ \cup I_\mu^-$,

$$\frac{1}{\kappa(s)^n r_s(x)} \mathbb{E}_x \left[r_s(X_n) e^{sS_n} h(X_1, S_1, \dots, X_n, S_n) \right] = \mathbb{E}_{\mathbb{Q}_s^x} \left[h(X_1, S_1, \dots, X_n, S_n) \right]. \quad (3.5)$$

Set

$$\Lambda(s) = \log \kappa(s) \quad \text{and} \quad q = \Lambda'(s) = \frac{\kappa'(s)}{\kappa(s)}.$$

Following [17], one can verify that under condition **A2**, the strong law of large numbers holds: $\lim_{n \rightarrow \infty} \frac{S_n}{n} = q$, \mathbb{Q}_s^x -almost surely, for any $s \in I_\mu^+ \cup I_\mu^-$.

3.2. The many-to-one formula. In this section we recall the many-to-one formula which has been established in [25, Lemma 4.2] for the study of fixed points of multivariate smoothing transforms. In addition to (2.5), for $x \in \mathbb{P}(V)$ and a node $u \in \mathbb{T}$, we introduce the following functions of the branching process,

$$X_u^x = G_u \cdot x \quad \text{and} \quad S_u^x = \sigma(G_u, x). \quad (3.6)$$

Recall that the function \mathbf{m} is defined by (2.10) and that for a node $u \in \mathbb{T}$, $u|k$ is the restriction of u to its first k components, $1 \leq k \leq |u|$.

Lemma 3.1 (The many-to-one formula). *Assume condition **A2**. Then, for any $s \in I_\mu^+ \cup I_\mu^-$ and $x \in \mathbb{P}(V)$, $n \geq 1$ and any bounded measurable function $h : (\mathbb{P}(V) \times \mathbb{R})^n \rightarrow \mathbb{R}$,*

$$\begin{aligned} & \mathbb{E} \left[\sum_{|u|=n} h \left(X_{u|1}^x, S_{u|1}^x, \dots, X_u^x, S_u^x \right) \right] \\ &= r_s(x) \mathbf{m}(s)^n \mathbb{E}_{\mathbb{Q}_s^x} \left[\frac{1}{r_s(X_n)} e^{-sS_n} h(X_1, S_1, \dots, X_n, S_n) \right]. \end{aligned} \quad (3.7)$$

This formula allows us to reduce the study of the branching product of random matrices to that of the ordinary product of random matrices but under the changed probability measure \mathbb{Q}_s^x .

3.3. Proof of Theorem 2.1. We only show how to prove (2.12), the proof of (2.13) being similar.

We start by providing an equivalent definition of γ_+ . Let $\gamma_+ = \inf\{q : e^{-sq} \mathbf{m}(s) < 1\}$. Note that the function $s \in I_\mu^+ \mapsto \log \mathbf{m}(s) - sq$ is decreasing, as its derivative equals $-s\Lambda''(s)$ and $\log \mathbf{m}(s) = \Lambda(s) + \log \mathbb{E}N$ is convex. Since $q = q(s) = \Lambda'(s)$ is increasing with s , we conclude that $\gamma_+ = \sup\{q : e^{-sq} \mathbf{m}(s) > 1\}$. We show that γ_+ also satisfies (2.11). To see this, note that the derivative of $h(s) := s^{-1} \log \mathbf{m}(s)$ for $s > 0$ equals

$$h'(s) = \frac{1}{s} \left(q - \frac{\log \mathbf{m}(s)}{s} \right)$$

and we see that the function decreases as long as $e^{-sq} \mathbf{m}(s) > 1$ and increases if $e^{-sq} \mathbf{m}(s) < 1$. Hence h attains its minimum at s^* with the property that $q(s^*) = \Lambda'(s^*) = \gamma_+$. Since $h'(s^*) = 0$, we conclude that

$$\gamma_+ = q(s^*) = \frac{\log \mathbf{m}(s^*)}{s^*} = \inf_{s \in I_\mu^+} \frac{\log \mathbf{m}(s)}{s}.$$

In the sequel, we first prove that for any $\epsilon > 0$ and large n , no particle is above the threshold $n(\gamma_+ + \epsilon)$; then, by a similar argument, we show that there is a positive number of particles below the threshold $n(\gamma_+ - \epsilon)$.

Using the many-to-one formula (3.7), we get that for any $s \in (I_\mu^+)^{\circ}$, with $q = \Lambda'(s)$,

$$\begin{aligned} \mathbb{E}\left[\sum_{|u|=n} \mathbb{1}_{\{S_u^x > nq\}}\right] &\leq e^{-snq} \mathbb{E}\left[\sum_{|u|=n} e^{sS_u^x}\right] \\ &= e^{-snq} r_s(x) \mathbf{m}(s)^n \mathbb{E}_{\mathbb{Q}_s^x}\left[\frac{1}{r_s(X_n)}\right] \\ &\leq ce^{-snq} \mathbf{m}(s)^n, \end{aligned}$$

where in the last inequality we used the fact that the eigenfunction r_s is bounded from below and above by strictly positive constants. This implies that for any Borel set $A \subseteq \mathbb{P}(V)$,

$$\begin{aligned} \mathbb{P}\left(\sum_{|u|=n} \mathbb{1}_{\{X_u^x \in A\}} \mathbb{1}_{\{S_u^x > nq\}} > 0\right) &= \mathbb{P}\left(\sum_{|u|=n} \mathbb{1}_{\{X_u^x \in A\}} \mathbb{1}_{\{S_u^x > nq\}} \geq 1\right) \\ &\leq \mathbb{E}\left[\sum_{|u|=n} \mathbb{1}_{\{X_u^x \in A\}} \mathbb{1}_{\{S_u^x > nq\}}\right] \\ &\leq ce^{-snq} \mathbf{m}(s)^n. \end{aligned}$$

If $e^{-sq} \mathbf{m}(s) < 1$, then by Borel-Cantelli's lemma we get that for all but finitely many n ,

$$\sum_{|u|=n} \mathbb{1}_{\{X_u^x \in A\}} \mathbb{1}_{\{S_u^x > nq\}} = 0.$$

Hence, for any $\epsilon > 0$, it holds that for all but finitely many n ,

$$\sum_{|u|=n} \mathbb{1}_{\{X_u^x \in A\}} \mathbb{1}_{\{S_u^x > n(\gamma_+ + \epsilon)\}} = 0.$$

Since $\epsilon > 0$ can be arbitrary small, this implies that

$$\limsup_{n \rightarrow \infty} \frac{M_n^x(A)}{n} \leq \gamma_+. \quad (3.8)$$

On the other hand, under conditions **A1**, **A2** and **A6**, the following large deviation principle has been established in [7, Theorem 2.6]: for any $s \in (I_\mu^+)^{\circ}$ such that $e^{-sq} \mathbf{m}(s) > 1$, and any Borel set $A \subseteq \mathbb{P}(V)$ satisfying $\nu(A) > 0$ and $\nu(\partial A) = 0$ (noting that $\nu_s(A) > 0$ and $\nu_s(\partial A) = 0$ is equivalent to saying that $\nu(A) > 0$ and $\nu(\partial A) = 0$ under **A2**), we have \mathbb{P} -a.s.

$$\lim_{n \rightarrow \infty} \sqrt{2\pi n} \sigma_s e^{n\Lambda^*(q)} \frac{\sum_{|u|=n} \mathbb{1}_{\{X_u^x \in A\}} \mathbb{1}_{\{S_u^x \geq nq\}}}{(\mathbb{E}N)^n} = \frac{1}{s} W_s^x \frac{r_s(x)}{\nu_s(r_s)} \nu_s(A),$$

where $\Lambda^*(q) = sq - \Lambda(s)$, $q = \Lambda'(s)$ and W_s^x is a positive random variable on the survival event \mathcal{S} . This can be rewritten as: \mathbb{P} -a.s. on the survival event \mathcal{S}

$$\lim_{n \rightarrow \infty} \sqrt{2\pi n} \sigma_s e^{-n(\log(\mathbb{E}N) + \Lambda^*(q))} \sum_{|u|=n} \mathbb{1}_{\{X_u^x \in A\}} \mathbb{1}_{\{S_u^x \geq nq\}} = \frac{1}{s} W_s^x \frac{r_s(x)}{\nu_s(r_s)} \nu_s(A) > 0.$$

We choose $q > \gamma_+$ which is equivalent to $e^{-sq} \mathbf{m}(s) > 1$, which, in turn, is equivalent to

$$\log(\mathbb{E}N) - \Lambda^*(q) = \log(\mathbb{E}N) - sq + \Lambda(s) = \log \mathbf{m}(s) - sq > 0.$$

Then the number of nodes u at generation n such that $\sigma(G_u, x)$ is above the level nq and that $G_u \cdot x \in A$ explodes as $n \rightarrow \infty$ \mathbb{P} -a.s. on the survival event \mathcal{S} . It follows that

$$\liminf_{n \rightarrow \infty} \frac{M_n^x(A)}{n} \geq \gamma_+ \quad (3.9)$$

on the survival event \mathcal{S} .

Combining (3.8) and (3.9) concludes the proof of (2.12).

4. DUALITY AND CONDITIONED INTEGRAL LIMIT THEOREMS

In this section we make use of the density assumption **A2** to establish duality identities and state a series of conditioned limit theorems for products of random matrices under a change of measure.

4.1. Duality. Recall that \mathcal{B} is the Banach space of real-valued continuous functions on $\mathbb{P}(V)$ endowed with the supremum norm $\|\cdot\|_{\mathcal{B}}$. For any $s \in I_{\mu}^+ \cup I_{\mu}^-$ and $\varphi \in \mathcal{B}$, let

$$Q_s \varphi(x) = \frac{1}{\kappa(s)r_s(x)} P_s(\varphi r_s)(x), \quad x \in \mathbb{P}(V) \quad (4.1)$$

be the transfer operator of the Markov chain $(X_n)_{n \geq 0}$ under the changed measure \mathbb{Q}_s^x . It is well-known that under the density assumption **A2**, on the projective space $\mathbb{P}(V)$ there exists a unique invariant measure π_s of the Markov operator Q_s . Moreover, the measure π_s is absolutely continuous with respect to the uniform probability measure on $\mathbb{P}(V)$ (denoted by dx), i.e. $\pi_s(dx) = \dot{\pi}_s(x)dx$, where $\dot{\pi}_s$ is the corresponding density function of π_s . Recall that dx is invariant under the action of the orthogonal group $O(d)$. By condition **A3**, $\dot{\pi}_s$ is strictly positive on the projective space $\mathbb{P}(V)$. For any $s \in I_{\mu}^+ \cup I_{\mu}^-$ and $\varphi \in \mathcal{B}$, define the dual operator Q_s^* as follows:

$$Q_s^* \varphi(x) = \int_{\mathbb{G}} \varphi(g \cdot x) \frac{r_s(x) e^{-(s+d)\sigma(g,x)} \dot{\pi}_s(g \cdot x)}{\kappa(s)r_s(g \cdot x) \dot{\pi}_s(x)} |\det(g)| \check{\mu}(dg), \quad x \in \mathbb{P}(V), \quad (4.2)$$

where $\check{\mu}$ is the image of the measure μ by the map $g \mapsto g^{-1}$. It can be verified that, under **A2** and **A3**, the operator Q_s^* is well defined. The following result shows that Q_s^* is indeed the dual Markov operator of Q_s .

Lemma 4.1. *Assume conditions **A2** and **A3**. Then, for any $s \in I_{\mu}^+ \cup I_{\mu}^-$ and any $\varphi, \psi \in \mathcal{B}$, we have*

$$\int_{\mathbb{P}(V)} \varphi(x) Q_s \psi(x) \pi_s(dx) = \int_{\mathbb{P}(V)} \psi(x) Q_s^* \varphi(x) \pi_s(dx). \quad (4.3)$$

Proof. Since $\pi_s(dx) = \dot{\pi}_s(x)dx$, by the definition of \mathbb{Q}_s^x , we have that for any bounded measurable function $F : \mathbb{P}(V) \times \mathbb{G} \times \mathbb{P}(V) \mapsto \mathbb{R}$,

$$\begin{aligned} I &:= \int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^x} F(x, g_1, X_1) \pi_s(dx) \\ &= \int_{\mathbb{G}} \int_{\mathbb{P}(V)} \frac{r_s(g \cdot x) e^{s\sigma(g,x)}}{\kappa(s)r_s(x)} F(x, g, g \cdot x) \dot{\pi}_s(x) dx \mu(dg). \end{aligned} \quad (4.4)$$

Notice that for any bounded measurable function $\varphi : \mathbb{P}(V) \mapsto \mathbb{R}$ and any $g \in \mathbb{G}$, we have

$$\int_{\mathbb{P}(V)} \varphi(x) dx = \int_{\mathbb{P}(V)} \varphi(g \cdot x) |\det(g)| e^{-d\sigma(g,x)} dx. \quad (4.5)$$

Applying this formula to the integral in (4.4), we obtain that for any $g \in \mathbb{G}$,

$$\begin{aligned} & \int_{\mathbb{P}(V)} \frac{r_s(g \cdot x) e^{s\sigma(g,x)}}{r_s(x)} F(x, g, g \cdot x) \dot{\pi}_s(x) dx \\ &= \int_{\mathbb{P}(V)} \frac{r_s(x) e^{s\sigma(g, g^{-1} \cdot x)}}{r_s(g^{-1} \cdot x)} F(g^{-1} \cdot x, g, x) \dot{\pi}_s(g^{-1} \cdot x) |\det(g^{-1})| e^{-d\sigma(g^{-1}, x)} dx. \end{aligned}$$

Therefore,

$$I = \int_{\mathbb{G}} \int_{\mathbb{P}(V)} \frac{r_s(x) e^{s\sigma(g, g^{-1} \cdot x)}}{\kappa(s) r_s(g^{-1} \cdot x)} F(g^{-1} \cdot x, g, x) \frac{\dot{\pi}_s(g^{-1} \cdot x)}{\dot{\pi}_s(x)} |\det(g^{-1})| e^{-d\sigma(g^{-1}, x)} \pi_s(dx) \mu(dg).$$

Since $\sigma(g, g^{-1} \cdot x) = -\sigma(g^{-1}, x)$ and $\check{\mu}$ is the image of μ by $g \mapsto g^{-1}$, by a change of variable and Fubini's theorem, we obtain

$$I = \int_{\mathbb{P}(V)} \int_{\mathbb{G}} \frac{r_s(x) e^{-(s+d)\sigma(g,x)}}{\kappa(s) r_s(g \cdot x)} F(g \cdot x, g^{-1}, x) \frac{\dot{\pi}_s(g \cdot x)}{\dot{\pi}_s(x)} |\det(g)| \check{\mu}(dg) \pi_s(dx). \quad (4.6)$$

In particular, taking $F(x, g, x') = \varphi(x)\psi(x')$ for $x, x' \in \mathbb{P}(V)$, where φ and ψ are bounded measurable functions from $\mathbb{P}(V)$ to \mathbb{R} , using the definition of Q_s^* we get that

$$\begin{aligned} & \int_{\mathbb{P}(V)} \varphi(x) Q_s \psi(x) \pi_s(dx) = \int_{\mathbb{P}(V)} \varphi(x) \mathbb{E}_{\mathbb{Q}_s^x} \psi(X_1) \pi_s(dx) \\ &= \int_{\mathbb{P}(V)} \psi(x) \int_{\mathbb{G}} \frac{r_s(x) e^{-(s+d)\sigma(g,x)}}{\kappa(s) r_s(g \cdot x)} \varphi(g \cdot x) \frac{\dot{\pi}_s(g \cdot x)}{\dot{\pi}_s(x)} |\det(g)| \check{\mu}(dg) \pi_s(dx) \\ &= \int_{\mathbb{P}(V)} \psi(x) Q_s^* \varphi(x) \pi_s(dx), \end{aligned}$$

which ends the proof of the lemma. \square

Similarly to (3.2) and (3.3), using the fact that $\sigma(\cdot, \cdot)$ is a cocycle, one can verify that for any $s \in I_\mu^+ \cup I_\mu^-$, the family of kernels

$$q_n^{s,*}(x, G_n) = \frac{1}{\kappa(s)^n} e^{-(s+d)\sigma(G_n, x)} \frac{r_s(x)}{r_s(G_n \cdot x)} \frac{\dot{\pi}_s(G_n \cdot x)}{\dot{\pi}_s(x)} |\det(G_n)|, \quad n \geq 1, \quad (4.7)$$

satisfies the following property: for any $x \in \mathbb{P}(V)$,

$$q_n^{s,*}(x, G_n) q_m^{s,*}(G_n \cdot x, g_{n+m} \dots g_{n+1}) = q_{n+m}^{s,*}(x, G_{n+m}). \quad (4.8)$$

Recalling that $\check{\mu}$ is the image of μ by the map $g \mapsto g^{-1}$, using (2.8) and (2.9), one can verify that the sequence of probability measures

$$\mathbb{Q}_{s,n}^{x,*}(dg_1, \dots, dg_n) := q_n^{s,*}(x, G_n) \check{\mu}(dg_1) \dots \check{\mu}(dg_n), \quad n \geq 1, \quad (4.9)$$

form a projective system on $\Omega^* = \mathbb{G}^{\mathbb{N}}$. Hence, by the Kolmogorov extension theorem, there exists a unique probability measure $\mathbb{Q}_s^{x,*}$ on $\mathbb{G}^{\mathbb{N}}$ with marginals $\mathbb{Q}_{s,n}^{x,*}$. Denote by $\mathbb{E}_{\mathbb{Q}_s^{x,*}}$ the corresponding expectation.

For any $s \in I_\mu^+ \cup I_\mu^-$, consider the probability space $(\Omega^*, \mathcal{B}(\Omega^*), \mathbb{Q}_s^{x,*})$. Let g_1^*, g_2^*, \dots be coordinate maps of Ω^* , i.e. $g_k(\omega) = \omega_k$, where $\omega \in \Omega^*$ and ω_k is the k -th coordinate of ω . The dual Markov chain $(X_n^*)_{n \geq 0}$ with starting point $X_0 \in \mathbb{P}(V)$ is defined on the space $(\Omega^*, \mathbb{Q}_s^{x,*})$ by setting

$$X_n^* = (g_n^* \dots g_1^*) \cdot X_0, \quad n \geq 1. \quad (4.10)$$

For any starting point $X_0 = x \in \mathbb{P}(V)$ and any measurable set $A \subseteq \mathbb{P}(V)$, the transition probability of $(X_n^*)_{n \geq 0}$ is given by $\mathbb{Q}_s^*(x, A) = Q_s^* \mathbf{1}_A(x)$, where Q_s^* is defined by (4.2). It is easy to see that for any bounded measurable function φ on $\mathbb{P}(V)$,

$$\mathbb{E}_{\mathbb{Q}_s^{x,*}} \varphi(X_n^*) = (Q_s^*)^n \varphi(x), \quad x \in \mathbb{P}(V),$$

where

$$(Q_s^*)^n \varphi(x) = \int_{\mathbb{G}} \varphi(G_n \cdot x) \frac{r_s(x) e^{-(s+d)\sigma(G_n, x)}}{\kappa(s)^n r_s(G_n \cdot x)} \frac{\dot{\pi}_s(G_n \cdot x)}{\dot{\pi}_s(x)} |\det(G_n)| \check{\mu}(dg_1) \dots \check{\mu}(dg_n).$$

Lemma 4.2. *Assume conditions A2 and A3. Then, for any $s \in I_\mu^+ \cup I_\mu^-$, $n \geq 1$ and any measurable function $F : \mathbb{P}(V) \times (\mathbb{G} \times \mathbb{P}(V))^n \mapsto \mathbb{R}_+$, we have*

$$\begin{aligned} & \int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^x} F(x, g_1, X_1, \dots, g_n, X_n) \pi_s(dx) \\ &= \int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^{z,*}} F(X_n^*, (g_n^*)^{-1}, \dots, X_1^*, (g_1^*)^{-1}, z) \pi_s(dz). \end{aligned}$$

Proof. The assertion of the lemma for $n = 1$ follows from the identities (4.4) and (4.6), and the definition of the measure $\mathbb{Q}_s^{z,*}$. The case $n = 2$ is obtained from the case $n = 1$ by the following calculations:

$$\begin{aligned} J &:= \int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^x} F(x, g_1, X_1, g_2, X_2) \pi_s(dx) \\ &= \int_{\mathbb{G}} \int_{\mathbb{P}(V)} \int_{\mathbb{G}} F(x, g_1, X_1, g_2, g_2 \cdot X_1) \frac{r_s(g_2 \cdot X_1) e^{s\sigma(g_2, X_1)}}{\kappa(s) r_s(X_1)} \mathbb{Q}_{s,1}^x(dg_1) \pi_s(dx) \mu(dg_2), \end{aligned}$$

where $\mathbb{Q}_{s,1}^x$ is a probability measure defined by (3.4). From (4.4) and (4.6), and the definition of $\mathbb{Q}_s^{z,*}$, we see that for any measurable function $H : \mathbb{P}(V) \times \mathbb{G} \times \mathbb{P}(V) \mapsto \mathbb{R}_+$,

$$\int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^x} H(x, g_1, g_1 \cdot x) \pi_s(dx) = \int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^{z,*}} H(g_1^* \cdot z, (g_1^*)^{-1}, z) \pi_s(dz). \quad (4.11)$$

Using (4.11), we get that for any fixed $g_2 \in \mathbb{G}$,

$$\begin{aligned} & \int_{\mathbb{P}(V)} \int_{\mathbb{G}} F(x, g_1, X_1, g_2, g_2 \cdot X_1) \frac{r_s(g_2 \cdot X_1) e^{s\sigma(g_2, X_1)}}{\kappa(s) r_s(X_1)} \mathbb{Q}_{s,1}^x(dg_1) \pi_s(dx) \\ &= \int_{\mathbb{P}(V)} \int_{\mathbb{G}} F(g_1^* \cdot z, (g_1^*)^{-1}, z, g_2, g_2 \cdot z) \frac{r_s(g_2 \cdot z) e^{s\sigma(g_2, z)}}{\kappa(s) r_s(z)} \mathbb{Q}_{s,1}^{z,*}(dg_1) \pi_s(dz), \end{aligned}$$

where $\mathbb{Q}_{s,1}^{z,*}$ is a probability measure defined by (4.9). By a change of variable $z = g_2^{-1} \cdot z'$ and applying (4.5), we obtain

$$\begin{aligned} J &= \int_{\mathbb{G}} \int_{\mathbb{P}(V)} \int_{\mathbb{G}} F(g_1^* \cdot z, (g_1^*)^{-1}, z, g_2, g_2 \cdot z) \frac{r_s(g_2 \cdot z) e^{s\sigma(g_2, z)}}{\kappa(s) r_s(z)} \mathbb{Q}_{s,1}^{z,*}(dg_1) \dot{\pi}_s(z) dz \mu(dg_2) \\ &= \int_{\mathbb{G}} \int_{\mathbb{P}(V)} \int_{\mathbb{G}} F(g_1^* \cdot (g_2^{-1} \cdot z), (g_1^*)^{-1}, g_2^{-1} \cdot z, g_2, z) \frac{r_s(z) e^{s\sigma(g_2, g_2^{-1} \cdot z)}}{\kappa(s) r_s(g_2^{-1} \cdot z)} \\ &\quad \times |\det(g_2^{-1})| e^{-d\sigma(g_2^{-1}, z)} \mathbb{Q}_{s,1}^{g_2^{-1} \cdot z, *} (dg_1) \dot{\pi}_s(g_2^{-1} \cdot z) dz \mu(dg_2). \end{aligned}$$

Using the fact that $\sigma(g, g^{-1} \cdot z) = -\sigma(g^{-1}, z)$ and passing to the measure $\check{\mu}$, we get

$$\begin{aligned}
J &= \int_{\mathbb{G}} \int_{\mathbb{P}(V)} \int_{\mathbb{G}} F(g_1^* g_2 \cdot z, (g_1^*)^{-1}, g_2 \cdot z, g_2^{-1}, z) \\
&\quad \times \frac{r_s(z) e^{-(s+d)\sigma(g_2, z)}}{\kappa(s) r_s(g_2 \cdot z)} \mathbb{Q}_{s,1}^{g_2 \cdot z, *} (dg_1) \dot{\pi}_s(g_2 \cdot z) |\det(g_2)| dz \check{\mu}(dg_2) \\
&= \int_{\mathbb{G}} \int_{\mathbb{P}(V)} \int_{\mathbb{G}} F(g_1^* g_2 \cdot z, (g_1^*)^{-1}, g_2 \cdot z, g_2^{-1}, z) \\
&\quad \times \frac{r_s(z) e^{-(s+d)\sigma(g_2, z)}}{\kappa(s) r_s(g_2 \cdot z)} \mathbb{Q}_{s,1}^{g_2 \cdot z, *} (dg_1) \frac{\dot{\pi}_s(g_2 \cdot z)}{\dot{\pi}_s(z)} |\det(g_2)| \pi_s(dz) \check{\mu}(dg_2) \\
&= \int_{\mathbb{P}(V)} \int_{\mathbb{G}} \int_{\mathbb{G}} F(g_1^* g_2 \cdot z, (g_1^*)^{-1}, g_2 \cdot z, g_2^{-1}, z) \mathbb{Q}_{s,1}^{g_2 \cdot z, *} (dg_1) \mathbb{Q}_{s,1}^{z, *} (dg_2) \pi_s(dz),
\end{aligned}$$

where in the last equality we used (4.7) and (4.9). Using (4.8) and the fact that the law of (g_1^*, g_2^*) coincides with that of (g_2^*, g_1^*) under the measure $\mathbb{Q}_s^{z, *}$, we obtain

$$J = \int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^{z, *}} F(X_2^*, (g_2^*)^{-1}, X_1^*, (g_1^*)^{-1}, z) \pi_s(dz),$$

which concludes the proof of the case $n = 2$. The case of $n \geq 3$ is proved similarly. \square

From Lemma 4.2 we immediately get the following assertion:

Lemma 4.3. *Assume conditions A2 and A3. Then, for any $s \in I_\mu^+ \cup I_\mu^-$ and any measurable function $F : (\mathbb{P}(V))^{n+1} \mapsto \mathbb{R}_+$, we have*

$$\int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^x} F(x, X_1, \dots, X_n) \pi_s(dx) = \int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^{z, *}} F(X_n^*, \dots, X_1^*, z) \pi_s(dz).$$

Recall that the Markov random walk (X_n, S_n) is defined by (3.1). We consider the first time when the random walk $(y - S_n)_{n \geq 1}$ starting at the point $y \in \mathbb{R}_+$ becomes negative:

$$\tau_y = \inf\{k \geq 1 : y - S_k < 0\}. \quad (4.12)$$

Now we introduce the dual random walk $(S_n^*)_{n \geq 0}$ by setting $S_0^* = 0$ and for any $n \geq 1$,

$$S_n^* = \sum_{k=1}^n \sigma(g_k^*, X_{k-1}^*), \quad (4.13)$$

where $(X_n^*)_{n \geq 0}$ is defined by (4.10). The first time when the random walk $(y - S_n^*)_{n \geq 1}$ starting at the point $y \in \mathbb{R}_+$ becomes negative is defined by

$$\tau_y^* = \inf\{k \geq 1 : y - S_k^* < 0\}.$$

In its turn Lemma 4.3 implies a duality property of the Markov chains $(X_n, S_n)_{n \geq 0}$ and $(X_n^*, S_n^*)_{n \geq 0}$ conditioned to stay positive.

Lemma 4.4. *Assume conditions A2 and A3. Then, for any $n \geq 1$, $s \in I_\mu^+ \cup I_\mu^-$ and any measurable functions $\varphi, \psi : \mathbb{P}(V) \times \mathbb{R} \rightarrow \mathbb{R}_+$, we have*

$$\begin{aligned}
&\int_{\mathbb{P}(V)} \int_{\mathbb{R}} \varphi(x, y) \mathbb{E}_{\mathbb{Q}_s^x} [\psi(X_n, y - S_n); \tau_y > n - 1] dy \pi_s(dx) \\
&= \int_{\mathbb{P}(V)} \int_{\mathbb{R}} \psi(z, t) \mathbb{E}_{\mathbb{Q}_s^{z, *}} [\varphi(X_n^*, t - S_n^*); \tau_t^* > n - 1] dt \pi_s(dz).
\end{aligned} \quad (4.14)$$

In particular, for any $n \geq 1$, $s \in I_\mu^+ \cup I_\mu^-$ and any measurable functions $\varphi, \psi : \mathbb{P}(V) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we have

$$\begin{aligned} & \int_{\mathbb{P}(V)} \int_{\mathbb{R}_+} \varphi(x, y) \mathbb{E}_{\mathbb{Q}_s^x} \left[\psi(X_n, y - S_n); \tau_y > n \right] dy \pi_s(dx) \\ &= \int_{\mathbb{P}(V)} \int_{\mathbb{R}_+} \psi(z, t) \mathbb{E}_{\mathbb{Q}_s^{z,*}} \left[\varphi(X_n^*, t - S_n^*); \tau_t^* > n \right] dt \pi_s(dz). \end{aligned} \quad (4.15)$$

Proof. Consider the function Ψ : for $x_0, x_n \in \mathbb{P}(V)$ and $y_0, y_1, \dots, y_n \in \mathbb{R}$,

$$\Psi(x_0, y_0, y_1, \dots, y_{n-1}, y_n, x_n) = \varphi(x_0, y_0) \psi(x_n, y_n) \mathbb{1}_{\{y_1 \geq 0, \dots, y_{n-1} \geq 0\}}.$$

By the definition of Ψ , It follows that

$$\begin{aligned} J &:= \int_{\mathbb{P}(V)} \int_{\mathbb{R}} \varphi(x, y) \mathbb{E}_{\mathbb{Q}_s^x} \left[\psi(X_n, y - S_n); \tau_y > n - 1 \right] dy \pi_s(dx) \\ &= \int_{\mathbb{P}(V)} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}_s^x} \Psi \left(x, y, y - \sigma(g_1, x), y - \sigma(g_1, x) - \sigma(g_2, X_1), \right. \\ &\quad \left. \dots, y - \sigma(g_1, x) - \dots - \sigma(g_n, X_{n-1}), X_n \right) dy \pi_s(dx). \end{aligned}$$

Applying Lemma 4.2, we obtain

$$\begin{aligned} J &= \int_{\mathbb{P}(V)} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}_s^{z,*}} \Psi \left(X_n^*, y, y - \sigma((g_n^*)^{-1}, X_n^*), y - \sum_{k=n}^{n-1} \sigma((g_k^*)^{-1}, X_k^*), \right. \\ &\quad \left. \dots, y - \sum_{k=n}^1 \sigma((g_k^*)^{-1}, X_k^*), z \right) dy \pi_s(dz). \end{aligned}$$

By a change of variable $y = t + \sum_{k=n}^1 \sigma((g_k^*)^{-1}, X_k^*)$, we see that

$$\begin{aligned} J &= \int_{\mathbb{P}(V)} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}_s^{z,*}} \Psi \left(X_n^*, t + \sum_{k=1}^n \sigma((g_k^*)^{-1}, X_k^*), t + \sum_{k=1}^{n-1} \sigma((g_k^*)^{-1}, X_k^*), \right. \\ &\quad \left. \dots, t + \sigma((g_1^*)^{-1}, X_1^*), t, z \right) dt \pi_s(dz). \end{aligned}$$

Since $\sigma((g_k^*)^{-1}, X_k^*) = -\sigma(g_k^*, X_{k-1}^*)$ for any $1 \leq k \leq n$ with $X_0^* = z$ under the measure $\mathbb{Q}_s^{z,*}$, we get

$$\begin{aligned} J &= \int_{\mathbb{P}(V)} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}_s^{z,*}} \Psi \left(X_n^*, t - \sum_{k=1}^n \sigma(g_k^*, X_{k-1}^*), t - \sum_{k=1}^{n-1} \sigma(g_k^*, X_{k-1}^*), \right. \\ &\quad \left. \dots, t - \sigma(g_1^*, z), t, z \right) dt \pi_s(dz) \\ &= \int_{\mathbb{P}(V)} \int_{\mathbb{R}} \psi(z, t) \mathbb{E}_{\mathbb{Q}_s^{z,*}} \left[\varphi(X_n^*, t - S_n^*); \tau_t^* > n - 1 \right] dt \pi_s(dz), \end{aligned}$$

which finishes the proof of (4.14). Identity (4.15) follows from (4.14) by taking $\varphi(x, y) = \varphi_1(x, y) \mathbb{1}_{\{y \geq 0\}}$ and $\psi(x, y) = \psi_1(x, y) \mathbb{1}_{\{y \geq 0\}}$. \square

4.2. Conditioned integral limit theorems. In this section we state several conditioned integral limit theorems, which will play important roles in Section 6 to establish the conditioned local limit theorems. In the following result we give the existence of the harmonic function V_s under the changed measure \mathbb{Q}_s^x , and states some of its properties.

Proposition 4.5. *Assume condition **A2** and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$.*

(1) *For any $x \in \mathbb{P}(V)$ and $y \geq 0$, the following limit exists:*

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^x}(y - S_n; \tau_y > n) =: V_s(x, y).$$

(2) *For any $x \in \mathbb{P}(V)$, the function $V_s(x, \cdot)$ is increasing on \mathbb{R}_+ and there exist constants $c_1, c_2 > 0$ such that for all $x \in \mathbb{P}(V)$ and $y \geq 0$,*

$$0 \vee (y - c_1) < V_s(x, y) \leq c_2(1 + y).$$

Moreover, $\inf_{x \in \mathbb{P}(V), y \geq 0} V_s(x, y) > 0$ and $\lim_{y \rightarrow \infty} \frac{V_s(x, y)}{y} = 1$.

(3) *The function V_s is harmonic, i.e., for any $x \in \mathbb{P}(V)$ and $y \geq 0$,*

$$\mathbb{E}_{\mathbb{Q}_s^x}(V_s(X_1, y - S_1); \tau_y > 1) = V_s(x, y).$$

The assertion $\inf_{x \in \mathbb{P}(V), y \geq 0} V_s(x, y) > 0$ is not stated in [13], but in fact its proof can be found in [13, Proposition 5.12].

The following result gives a uniform upper bound for $\mathbb{Q}_s^x(\tau_y > n)$.

Theorem 4.6. *Assume condition **A2** and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then*

$$\limsup_{n \rightarrow \infty} n^{1/2} \sup_{x \in \mathbb{P}(V)} \sup_{y \geq 0} \frac{1}{y + 1} \mathbb{Q}_s^x(\tau_y > n) < \infty.$$

In the following we formulate a conditional integral limit theorem for the random walk $(y - S_n)$ under the changed measure \mathbb{Q}_s^x . Denote $\sigma_s = \sqrt{\Lambda''(s)}$ and let $\Phi^+(t) = (1 - e^{-t^2/2})\mathbf{1}_{\{t \geq 0\}}$ be the Rayleigh distribution function on \mathbb{R} .

Theorem 4.7. *Assume condition **A2** and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Let $(\alpha_n)_{n \geq 1}$ be any sequence of positive numbers satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then, for any $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that for any $n \geq 1$, $x \in \mathbb{P}(V)$ and $y \in [0, \alpha_n \sqrt{n}]$,*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{Q}_s^x \left(\frac{y - S_n}{\sigma_s \sqrt{n}} \leq t, \tau_y > n \right) - \frac{2V_s(x, y)}{\sigma_s \sqrt{2\pi n}} \Phi^+(t) \right| \leq c_\varepsilon \frac{1 + y}{\sqrt{n}} (\alpha_n + n^{-\varepsilon}).$$

The proof of Proposition 4.5 and Theorems 4.6, 4.7 can be performed in the same way as the corresponding assertions in [11, Theorems 2.2, 2.4 and 2.5]. The key point is that the couple (X_n, S_n) is a Markov chain under the changed measure \mathbb{Q}_s^x and the spectral gap properties hold for the corresponding perturbed operator which allows us to obtain a martingale approximation for the Markov walk S_n . The rate of convergence in Theorem 4.7 can be obtained by using the techniques similar to that in [16].

Next we formulate similar results for the dual Markov random walk (X_n^*, S_n^*) . They are easy consequences of Theorems 4.6 and 4.7 due to the fact that the dual Markov random walk has spectral gap properties (see Section 5.1 below). For any $x \in \mathbb{P}(V)$ and $y \geq 0$, we define $V_s^*(x, y) := \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}_s^{x,*}}(y - S_n^*; \tau_y^* > n)$. Then V_s^* satisfies similar properties as V_s stated in Proposition 4.5.

Theorem 4.8. *Assume conditions **A2**, **A3**, and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then*

$$\limsup_{n \rightarrow \infty} n^{1/2} \sup_{x \in \mathbb{P}(V)} \sup_{y \geq 0} \frac{1}{y + 1} (\mathbb{Q}_s^{x,*})(\tau_y^* > n) < \infty.$$

Theorem 4.9. *Assume conditions [A2](#), [A3](#), and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Let $(\alpha_n)_{n \geq 1}$ be any sequence of positive numbers satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then, for any $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that for any $n \geq 1$, $x \in \mathbb{P}(V)$ and $y \in [0, \alpha_n \sqrt{n}]$,*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{Q}_s^{x,*} \left(\frac{y - S_n^*}{\sigma_s \sqrt{n}} \leq t, \tau_y^* > n \right) - \frac{2V_s^*(x, y)}{\sigma_s \sqrt{2\pi n}} \Phi^+(t) \right| \leq c_\varepsilon \frac{1+y}{\sqrt{n}} (\alpha_n + n^{-\varepsilon}).$$

5. SPECTRAL GAP, MARTINGALE APPROXIMATION AND LOCAL LIMIT THEOREMS

5.1. Spectral gap properties. The following result shows that the transfer operator P_s has spectral gap properties. Denote by $\mathcal{L}(\mathcal{B}, \mathcal{B})$ the set of all bounded linear operators from \mathcal{B} to \mathcal{B} equipped with the operator norm $\|\cdot\|_{\mathcal{B} \rightarrow \mathcal{B}}$.

Lemma 5.1. *Let $s \in I_\mu^+ \cup I_\mu^-$. Under condition [A2](#), there exists $\delta > 0$ such that for any $t \in (-\delta, \delta)$,*

$$P_s^n = \kappa(s)^n \Pi_s + N_s^n, \quad n \geq 1, \quad (5.1)$$

where the mappings $s \mapsto \Pi_s : (-\delta, \delta) \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{B})$ and $s \mapsto N_s : (-\delta, \delta) \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{B})$ are analytic in the strong operator sense, Π_s is a rank-one projection with $\Pi_s(\varphi)(x) = \pi_s(\varphi)$ for any $\varphi \in \mathcal{B}$ and $x \in \mathbb{P}(V)$, $\Pi_s N_s = N_s \Pi_s = 0$. Moreover, for any fixed integer $k \geq 0$, there exist constants $c > 0$ and $0 < a < 1$ such that

$$\sup_{|t| < \delta} \left\| \frac{d^k}{dz^k} N_s^n \right\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq ca^n, \quad n \geq 1. \quad (5.2)$$

Proof. We give the proof for negative s ($s \in I_\mu^-$), the proof for positive s ($s \in I_\mu^+$) being similar but easier.

We first prove that the operator P_s is quasi-compact on \mathcal{B} , i.e. the set $A := \{P_s \varphi : \|\varphi\|_{\mathcal{B}} \leq 1\}$ is a conditionally compact subset of \mathcal{B} . To show this, by the theorem of Arzela-Ascoli, it is enough to prove that the set A is uniformly bounded and uniformly equicontinuous. Since $\sigma(g, x) \geq \log \|g^{-1}\|^{-1}$, we have for any $s \in I_\mu^-$,

$$\|P_s \varphi\|_{\mathcal{B}} \leq \int_{\mathbb{G}} e^{s \log \|g^{-1}\|} \mu(dg),$$

uniformly in $\|\varphi\|_{\mathcal{B}} \leq 1$, so that the set A is uniformly bounded. Since μ has a density, we have $\mu(dg) = \dot{\mu}(g)dg$. For $x_1, x_2 \in \mathbb{P}(V)$, there exist $k_1, k_2 \in SO(d, \mathbb{R})$ such that $x_1 = \mathbb{R}k_1 e_1$ and $x_2 = \mathbb{R}k_2 e_1$. We equip $SO(d, \mathbb{R})$ with the metric $\mathbf{d}_{SO(d, \mathbb{R})}(k_1, k_2) = \mathbf{d}(x_1, x_2)$. It follows

that

$$\begin{aligned}
& |P_s\varphi(x_1) - P_s\varphi(x_2)| \\
&= \left| \int_{\mathbb{G}} e^{s\sigma(g,x_1)} \varphi(g \cdot x_1) \mu(dg) - \int_{\mathbb{G}} e^{s\sigma(g,x_2)} \varphi(g \cdot x_2) \mu(dg) \right| \\
&= \left| \int_{\mathbb{G}} e^{s \log \|gk_1 e_1\|} \varphi(\mathbb{R}gk_1 e_1) \mu(dg) - \int_{\mathbb{G}} e^{s \log \|gk_2 e_1\|} \varphi(\mathbb{R}gk_2 e_1) \mu(dg) \right| \\
&= \left| \int_{\mathbb{G}} e^{s \log \|gk_1 e_1\|} \varphi(\mathbb{R}gk_1 e_1) \dot{\mu}(g) dg - \int_{\mathbb{G}} e^{s \log \|gk_2 e_1\|} \varphi(\mathbb{R}gk_2 e_1) \dot{\mu}(g) dg \right| \\
&= \left| \int_{\mathbb{G}} e^{s \log \|ge_1\|} \varphi(\mathbb{R}ge_1) \dot{\mu}(gk_1^{-1}) dg - \int_{\mathbb{G}} e^{s \log \|ge_1\|} \varphi(\mathbb{R}ge_1) \dot{\mu}(gk_2^{-1}) dg \right| \\
&\leq \int_{\mathbb{G}} e^{s \log \|ge_1\|} |\varphi(\mathbb{R}ge_1)| \left| \dot{\mu}(gk_1^{-1}) - \dot{\mu}(gk_2^{-1}) \right| dg \\
&\leq \int_{\mathbb{G}} e^{-s \log \|g^{-1}\|} \left| \dot{\mu}(gk_1^{-1}) - \dot{\mu}(gk_2^{-1}) \right| dg. \tag{5.3}
\end{aligned}$$

This proves that the set of functions A is uniformly equicontinuous if $\dot{\mu}$ is bounded and continuous on \mathbb{G} and $\int_{\mathbb{G}} e^{-s \log \|g^{-1}\|} dg < \infty$.

Since $\dot{\mu} \in L^1$, there exists a sequence of bounded and continuous functions $\dot{\mu}_n$ on \mathbb{G} such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{G}} \left| \dot{\mu}_n(g) - \dot{\mu}(g) \right| dg = 0.$$

From (5.3), we get

$$\begin{aligned}
|P_s\varphi(x_1) - P_s\varphi(x_2)| &\leq \int_{\mathbb{G}} e^{-s \log \|g^{-1}\|} \left| \dot{\mu}(gk_1^{-1}) - \dot{\mu}(gk_2^{-1}) \right| dg \\
&\leq \int_{\mathbb{G}} e^{-s \log \|g^{-1}\|} \left| \dot{\mu}(gk_1^{-1}) - \dot{\mu}_n(gk_1^{-1}) \right| dg \\
&\quad + \int_{\mathbb{G}} e^{-s \log \|g^{-1}\|} \left| \dot{\mu}_n(gk_1^{-1}) - \dot{\mu}_n(gk_2^{-1}) \right| dg \\
&\quad + \int_{\mathbb{G}} e^{-s \log \|g^{-1}\|} \left| \dot{\mu}_n(gk_2^{-1}) - \dot{\mu}(gk_2^{-1}) \right| dg.
\end{aligned}$$

This proves the set of functions A is uniformly equicontinuous when $\dot{\mu} \in L^1$. By [6, Theorem III.4.3], this concludes the proof of Lemma 5.1. \square

For $s \in I_{\mu}^+ \cup I_{\mu}^-$ and $t \in \mathbb{R}$, define a family of perturbed operators $Q_{s,it}$ as follows: with $q = \Lambda'(s)$, for any $\varphi \in \mathcal{B}$,

$$Q_{s,it}\varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{it(S_1 - q)} \varphi(X_1) \right], \quad x \in \mathbb{P}(V). \tag{5.4}$$

It follows from the cocycle property (3.3) that

$$Q_{s,it}^n \varphi(x) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{it(S_n - nq)} \varphi(X_n) \right], \quad x \in \mathbb{P}(V).$$

The following result shows that the perturbed operator $Q_{s,it}$ has spectral gap properties.

Lemma 5.2. *Let $s \in I_{\mu}^+ \cup I_{\mu}^-$. Under condition A2, there exists $\delta > 0$ such that for any $t \in (-\delta, \delta)$,*

$$Q_{s,it}^n = \lambda_{s,it}^n \Pi_{s,it} + N_{s,it}^n, \quad n \geq 1, \tag{5.5}$$

where the mappings $t \mapsto \Pi_{s,it} : (-\delta, \delta) \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{B})$ and $z \mapsto N_{s,it} : (-\delta, \delta) \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{B})$ are analytic in the strong operator sense, $\Pi_{s,it}$ is a rank-one projection with $\Pi_{s,0}(\varphi)(x) = \pi_s(\varphi)$ for any $\varphi \in \mathcal{B}$ and $x \in \mathbb{P}(V)$, $\Pi_{s,it}N_{s,it} = N_{s,it}\Pi_{s,it} = 0$ and

$$\lambda_{s,it} = e^{-itq} \frac{\kappa(s+it)}{\kappa(s)}. \quad (5.6)$$

Moreover, for any fixed integer $k \geq 0$, there exist constants $c > 0$ and $0 < a < 1$ such that

$$\sup_{|t| < \delta} \left\| \frac{d^k}{dz^k} N_{s,it}^n \right\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq ca^n, \quad n \geq 1. \quad (5.7)$$

Using Lemma 5.1 and the perturbation theorem for linear operators [21, Theorem III.8], the proof of Lemma 5.2 can be performed in the same way as [9, Corollary 6.3] and [27, Proposition 3.3], and therefore we omit the details.

The eigenvalue $\lambda_{s,it}$ has the asymptotic expansion $\lambda_{s,it} = 1 - \frac{\sigma_s^2}{2}t^2 + o(|t|^2)$ as $t \rightarrow 0$, where $\sigma_s^2 = \Lambda''(s)$. Since condition A2 implies that the smallest semigroup spanned by the support of μ satisfies the strong irreducibility and proximality conditions in [17, 9, 27], the asymptotic variance σ_s^2 is strictly positive and moreover, the following assertion holds.

Lemma 5.3 ([17, 9, 27]). *Let $s \in I_\mu^+ \cup I_\mu^-$. Assume condition A2. Then for any compact set $K \subset \mathbb{R} \setminus \{0\}$, there exist constants $c, c_K > 0$ such that for any $n \geq 1$ and $\varphi \in \mathcal{B}$,*

$$\sup_{t \in K} \|Q_{s,it}^n \varphi\|_{\mathcal{B}} \leq ce^{-nc_K} \|\varphi\|_{\mathcal{B}}.$$

Similarly to (5.4), we define a family of dual perturbed operators $Q_{s,it}^*$ as follows: for $s \in I_\mu^+ \cup I_\mu^-$, $q = \Lambda'(s)$, $t \in \mathbb{R}$ and $\varphi \in \mathcal{B}$,

$$Q_{s,it}^* \varphi(x) = \mathbb{E}_{\mathbb{Q}_s^{x,*}} \left[e^{it(S_1 - q)} \varphi(X_1) \right], \quad x \in \mathbb{P}(V). \quad (5.8)$$

It follows from the cocycle property (3.3) that

$$(Q_{s,it}^*)^n \varphi(x) = \mathbb{E}_{\mathbb{Q}_s^{x,*}} \left[e^{it(S_n - nq)} \varphi(X_n) \right], \quad x \in \mathbb{P}(V).$$

The following results show that the dual perturbed operator $Q_{s,it}^*$ has spectral gap properties, which are similar to those for the operator $Q_{s,it}$, see Lemmas 5.2 and 5.3.

Lemma 5.4. *Let $s \in I_\mu^+ \cup I_\mu^-$. Under conditions A2 and A3, there exists $\delta > 0$ such that for any $t \in (-\delta, \delta)$,*

$$(Q_{s,it}^*)^n = \lambda_{s,it}^n \Pi_{s,it}^* + (N_{s,it}^*)^n, \quad n \geq 1, \quad (5.9)$$

where the mappings $t \mapsto \Pi_{s,it}^* : (-\delta, \delta) \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{B})$ and $z \mapsto N_{s,it}^* : (-\delta, \delta) \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{B})$ are analytic in the strong operator sense, $\Pi_{s,it}^*$ is a rank-one projection with $\Pi_{s,0}^*(\varphi)(x) = \pi_s(\varphi)$ for any $\varphi \in \mathcal{B}$ and $x \in \mathbb{P}(V)$, $\Pi_{s,it}^* N_{s,it}^* = N_{s,it}^* \Pi_{s,it}^* = 0$ and

$$\lambda_{s,it} = e^{-itq} \frac{\kappa(s+it)}{\kappa(s)}. \quad (5.10)$$

Moreover, for any fixed integer $k \geq 0$, there exist constants $c > 0$ and $0 < a < 1$ such that

$$\sup_{|t| < \delta} \left\| \frac{d^k}{dt^k} (N_{s,it}^*)^n \right\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq ca^n, \quad n \geq 1. \quad (5.11)$$

Lemma 5.5. *Let $s \in I_\mu^+ \cup I_\mu^-$. Assume conditions **A2** and **A3**. Then, for any compact set $K \subset \mathbb{R} \setminus \{0\}$, there exist constants $c, c_K > 0$ such that for any $n \geq 1$ and $\varphi \in \mathcal{B}$,*

$$\sup_{t \in K} \|(Q_{s,it}^*)^n \varphi\|_{\mathcal{B}} \leq c e^{-nc_K} \|\varphi\|_{\mathcal{B}}.$$

The proof of Lemmas 5.4 and 5.5 can be done using the techniques from [21, 9, 27] in the same way as Lemmas 5.2 and 5.3.

5.2. Martingale approximation. We shall use the strategy of Gordin [10] to construct a martingale approximation for the Markov walk S_n under the changed measure \mathbb{Q}_s^x . Assume that $q = \Lambda'(s) = 0$ (or equivalently $\kappa'(s) = 0$) for some $s \in I_\mu^+ \cup I_\mu^-$. Denote

$$\bar{\sigma}(x) := \mathbb{E}_{\mathbb{Q}_s^x} S_1 = \int_{\mathbb{G}} \sigma(g, x) \mathbb{Q}_{s,1}^x(dg), \quad x \in \mathbb{P}(V),$$

where $\mathbb{Q}_{s,1}^x$ is defined by (4.9). One can verify that $\bar{\sigma} \in \mathcal{B}$ and the cohomological equation

$$\bar{\sigma}(x) = \theta(x) - Q_s \theta(x), \quad x \in \mathbb{P}(V), \quad (5.12)$$

has a unique solution given by

$$\theta(x) = \bar{\sigma}(x) + \sum_{n=1}^{\infty} Q_s^n \bar{\sigma}(x), \quad x \in \mathbb{P}(V), \quad (5.13)$$

where Q_s is defined by (4.1). Indeed, using the spectral gap properties of Q_s (by Lemma 5.2 with $t = 0$),

$$Q_s^n \bar{\sigma}(x) = \pi_s(\bar{\sigma}) + N_{s,0}^n \bar{\sigma}(x). \quad (5.14)$$

In addition,

$$\Lambda'(s) = \int_{\mathbb{P}(V)} \int_{\mathbb{G}} \sigma(g, x) \mathbb{Q}_{s,1}^x(dg) \pi_s(dx). \quad (5.15)$$

Since $\pi_s(\bar{\sigma}) = 0$ (by (5.15) and the assumption $q = 0$) and $|N_{s,0}^n \bar{\sigma}(x)| \leq C e^{-cn}$, we get $|Q_s^n \bar{\sigma}(x)| \leq C e^{-cn}$. Therefore, the function θ in (5.13) is well defined and satisfies the cohomological equation (5.12).

Let \mathcal{F}_0 be the trivial σ -algebra and $\mathcal{F}_n = \sigma\{X_k : 1 \leq k \leq n\}$ for $n \geq 1$. For any $g \in \mathbb{G}$ and $x \in \mathbb{P}(V)$, let

$$\sigma_0(g, x) = \sigma(g, x) - \theta(x) + \theta(g \cdot x). \quad (5.16)$$

Then σ_0 satisfies the cocycle property $\sigma_0(g_2 g_1, x) = \sigma_0(g_2, g_1 \cdot x) + \sigma_0(g_1, x)$ for any $g_2, g_1 \in \mathbb{G}$ and $x \in \mathbb{P}(V)$. Define

$$M_0 = 0 \quad \text{and} \quad M_n = \sum_{k=1}^n \sigma_0(g_k, X_{k-1}), \quad n \geq 1. \quad (5.17)$$

By the Markov property, we have $\mathbb{E}_{\mathbb{Q}_s^x}(M_k | \mathcal{F}_{k-1}) = M_{k-1}$ and hence $(M_n, \mathcal{F}_n)_{n \geq 0}$ is a 0 mean \mathbb{Q}_s^x -martingale. The following lemma shows that the difference $S_n - M_n$ is bounded.

Lemma 5.6. *Assume conditions **A2**, **A3**, and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$,*

$$\sup_{n \geq 0} |S_n - M_n| \leq c, \quad \mathbb{Q}_s^x\text{-a.s.} \quad (5.18)$$

$$\sup_{n \geq 0} |S_n^* - M_n^*| \leq c, \quad \mathbb{Q}_s^{x,*}\text{-a.s.} \quad (5.19)$$

Proof. By (5.16), we have

$$\sum_{k=1}^n \sigma_0(g_k, X_{k-1}) = \sum_{k=1}^n \sigma(g_k, X_{k-1}) - \theta(X_0) + \theta(g_n \cdot X_{n-1}),$$

where $X_0 = x$. Taking into account (5.17), we get

$$M_n = S_n - \theta(X_0) + \theta(g_n \cdot X_{n-1}).$$

Since the function $\theta \in \mathcal{B}$ is bounded, we get (5.18). The proof of (5.19) is similar. \square

The following result is a consequence of Burkholder's inequality.

Lemma 5.7. *Assume conditions A2, A3, and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, for any $p > 2$, we have*

$$\begin{aligned} \sup_{n \geq 1} \frac{1}{n^{p/2}} \sup_{x \in \mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^x}(|M_n|^p) &< +\infty, \\ \sup_{n \geq 1} \frac{1}{n^{p/2}} \sup_{x \in \mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^{x,*}}(|M_n^*|^p) &< +\infty. \end{aligned}$$

Proof. Denote $\xi_k = \sigma_0(g_k, X_{k-1})$ for $1 \leq k \leq n$. By Burkholder's inequality and Hölder's inequality, we get

$$\mathbb{E}_{\mathbb{Q}_s^x}(|M_n|^p) \leq c_p \mathbb{E}_{\mathbb{Q}_s^x} \left| \sum_{k=1}^n \xi_k^2 \right|^{p/2} \leq c_p n^{p/2-1} \mathbb{E}_{\mathbb{Q}_s^x} \sum_{k=1}^n |\xi_k|^p \leq c_p n^{p/2} \sup_{1 \leq k \leq n} \mathbb{E}_{\mathbb{Q}_s^x}(|\xi_k|^p).$$

Since there exists a constant $c > 0$ such that for all $x \in \mathbb{P}(V)$,

$$\sup_{1 \leq k \leq n} \mathbb{E}_{\mathbb{Q}_s^x}(|\xi_k|^p) \leq \mathbb{E}_{\mathbb{Q}_s^x}(\log^p N(g_1)) \leq c \mathbb{E}[\|g_1\|^s \log^p N(g_1)],$$

the first inequality follows. The second inequality can be proved in the same way. \square

5.3. Non-asymptotic local limit theorems. In the following we establish effective local limit theorems for products of random matrices under a change of measure \mathbb{Q}_s^x . Our results are non-asymptotic, i.e. they are written in the form of precise upper and lower bounds which hold for any fixed $n \geq 1$. Besides, we consider a general target function h on the couple (X_n, S_n) and this plays a crucial role for establishing conditioned local limit theorems in Section 6.2. The main difficulty is to give the explicit dependence of the remainder terms on the target function h . The following lemma is taken from [15].

Lemma 5.8 ([15]). *Let h be a real-valued function on $\mathbb{P}(V) \times \mathbb{R}$ such that*

- (1) *For any $t \in \mathbb{R}$, the function $x \mapsto h(x, t)$ is continuous on $\mathbb{P}(V)$.*
- (2) *For any $x \in \mathbb{P}(V)$, the function $t \mapsto h(x, t)$ is measurable on \mathbb{R} .*

Then, the function $(x, t) \mapsto h(x, t)$ is measurable on $\mathbb{P}(V) \times \mathbb{R}$ and the function $t \mapsto \|h(\cdot, t)\|_{\mathcal{B}}$ is measurable on \mathbb{R} . Moreover, if the integral $\int_{\mathbb{R}} \|h(\cdot, t)\|_{\mathcal{B}} dt$ is finite, we define the partial Fourier transform \widehat{h} of h by setting for any $x \in \mathbb{P}(V)$ and $u \in \mathbb{R}$,

$$\widehat{h}(x, u) = \int_{\mathbb{R}} e^{-itu} h(x, t) dt. \quad (5.20)$$

This is a continuous function on $\mathbb{P}(V) \times \mathbb{R}$. In addition, for every $u \in \mathbb{R}$, the function $x \mapsto \widehat{h}(x, u)$ is continuous and $\|\widehat{h}(\cdot, u)\|_{\mathcal{B}} \leq \int_{\mathbb{R}} \|h(\cdot, t)\|_{\mathcal{B}} dt$.

We denote by \mathcal{H} the set of real-valued functions on $\mathbb{P}(V) \times \mathbb{R}$ such that conditions (1) and (2) of Lemma 5.8 hold and the integral $\int_{\mathbb{R}} \|h(\cdot, t)\|_{\mathcal{B}} dt$ is finite. For any compact set $K \subset \mathbb{R}$, denote by \mathcal{H}_K the set of functions $h \in \mathcal{H}$ such that the Fourier transform $\widehat{h}(x, \cdot)$ has a support contained in K for any $x \in \mathbb{P}(V)$. Below, for any function $h \in \mathcal{H}$, we use the notation

$$\|h\|_{\mathcal{H}} = \int_{\mathbb{R}} \|h(\cdot, u)\|_{\mathcal{B}} du, \quad \|h\|_{\pi_s \otimes \text{Leb}} = \int_{\mathbb{P}(V) \times \mathbb{R}} |h(x, u)| \pi_s(dx) du.$$

Let $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$, $y \in \mathbb{R}$ be the standard normal density function.

Theorem 5.9. *Assume condition A2 and $\kappa'(s) = 0$ for some $s \in I_{\mu}^+ \cup I_{\mu}^-$. Let $K \subset \mathbb{R}$ be a compact set of \mathbb{R} . Then there exists a constant $c_K > 0$ such that for any $h \in \mathcal{H}_K$, $n \geq 1$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}$,*

$$\left| \sigma_s \sqrt{n} \mathbb{E}_{\mathbb{Q}_s^x} h(X_n, y - S_n) - \int_{\mathbb{P}(V) \times \mathbb{R}} h(x', y') \phi\left(\frac{y - y'}{\sigma_s \sqrt{n}}\right) \pi_s(dx') dy' \right| \leq \frac{c_K}{\sqrt{n}} \|h\|_{\mathcal{H}}. \quad (5.21)$$

Proof. For the sake of ease in exposition, we assume that $\sigma_s = 1$. By the Fourier inversion formula, with the notation (5.20) it holds that

$$h(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ity} \widehat{h}(x, t) dt, \quad x \in \mathbb{P}(V), y \in \mathbb{R}.$$

By Fubini's theorem, this implies that for any $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}$,

$$\mathbb{E}_{\mathbb{Q}_s^x} h(X_n, y - S_n) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ity} \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{-itS_n} \widehat{h}(X_n, t) \right] dt.$$

By a change of variable $t = \frac{u}{\sqrt{n}}$, we obtain

$$\begin{aligned} \sqrt{n} \mathbb{E}_{\mathbb{Q}_s^x} h(X_n, y - S_n) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu \frac{y}{\sqrt{n}}} \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{-iu \frac{S_n}{\sqrt{n}}} \widehat{h}\left(X_n, \frac{u}{\sqrt{n}}\right) \right] du \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu \frac{y}{\sqrt{n}}} \widehat{\phi}(u) \left[\int_{\mathbb{P}(V)} \widehat{h}\left(x', \frac{u}{\sqrt{n}}\right) \pi_s(dx') \right] du + I(x, y), \end{aligned} \quad (5.22)$$

where $\widehat{\phi}(u) = e^{-u^2/2}$ and

$$I(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu \frac{y}{\sqrt{n}}} J(x, u) du, \quad x \in \mathbb{P}(V), y \in \mathbb{R},$$

with

$$J(x, u) = \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{-iu \frac{S_n}{\sqrt{n}}} \widehat{h}\left(X_n, \frac{u}{\sqrt{n}}\right) \right] - \widehat{\phi}(u) \int_{\mathbb{P}(V)} \widehat{h}\left(x', \frac{u}{\sqrt{n}}\right) \pi_s(dx').$$

For the first term in (5.22), since $\sqrt{n} \widehat{\phi}(\sqrt{n} \cdot)$ is the Fourier transform of $\phi(\frac{\cdot}{\sqrt{n}})$. using the change of variable $u' = \frac{u}{\sqrt{n}}$ and the Fourier inversion formula, we get

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu \frac{y}{\sqrt{n}}} \widehat{h}\left(x', \frac{u}{\sqrt{n}}\right) \widehat{\phi}(u) du &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{iu' y} \widehat{h}(x', u') \sqrt{n} \widehat{\phi}(\sqrt{n} u') du' \\ &= \int_{\mathbb{R}} h(x', y') \phi\left(\frac{y - y'}{\sqrt{n}}\right) dy'. \end{aligned} \quad (5.23)$$

Therefore, to establish (5.21), it remains to prove that there exists a constant $c_K > 0$ such that for any $h \in \mathcal{H}_K$, $n \geq 1$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}$,

$$|I(x, y)| \leq \frac{c_K}{\sqrt{n}} \|h\|_{\mathcal{H}}. \quad (5.24)$$

Now we are going to show (5.24). Let $\delta > 0$ be a sufficiently small constant. We decompose the integral $I(x, y)$ into three parts: $I(x, y) = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{|u| \leq \delta\sqrt{n}} e^{iu\frac{y}{\sqrt{n}}} J(x, u) du, \\ I_2 &= -\frac{1}{2\pi} \int_{|u| > \delta\sqrt{n}} e^{iu\frac{y}{\sqrt{n}}} \widehat{\phi}(u) \left[\int_{\mathbb{P}(V)} \widehat{h}\left(x', \frac{u}{\sqrt{n}}\right) \pi_s(dx') \right] du, \\ I_3 &= \frac{1}{2\pi} \int_{|u| > \delta\sqrt{n}} e^{iu\frac{y}{\sqrt{n}}} \mathbb{E}_{\mathbb{Q}_s^x} \left[e^{-iu\frac{S_n}{\sqrt{n}}} \widehat{h}\left(X_n, \frac{u}{\sqrt{n}}\right) \right] du. \end{aligned}$$

Bound of I_1 . By Lemma 5.2, we have that for any $|u| \leq \delta\sqrt{n}$,

$$\begin{aligned} J(x, u) &= Q_{s, \frac{-iu}{\sqrt{n}}}^n \widehat{h}\left(\cdot, \frac{u}{\sqrt{n}}\right)(x) - \widehat{\phi}(u) \int_{\mathbb{P}(V)} \widehat{h}\left(x', \frac{u}{\sqrt{n}}\right) \pi_s(dx') \\ &= J_1(x, u) + J_2(x, u), \end{aligned}$$

where

$$\begin{aligned} J_1(x, u) &= \lambda_{s, \frac{-iu}{\sqrt{n}}}^n \Pi_{s, \frac{-iu}{\sqrt{n}}} \widehat{h}\left(\cdot, \frac{u}{\sqrt{n}}\right)(x) - \widehat{\phi}(u) \int_{\mathbb{P}(V)} \widehat{h}\left(x', \frac{u}{\sqrt{n}}\right) \pi_s(dx'), \\ J_2(x, u) &= N_{s, \frac{-iu}{\sqrt{n}}}^n \widehat{h}\left(\cdot, \frac{u}{\sqrt{n}}\right)(x). \end{aligned}$$

For the first term $J_1(x, u)$, since $\Pi_{s,0}(\varphi) = \pi_s(\varphi)$, we have

$$\begin{aligned} J_1(x, u) &= \left(\lambda_{s, \frac{-iu}{\sqrt{n}}}^n - \widehat{\phi}(u) \right) \Pi_{s, \frac{-iu}{\sqrt{n}}} \widehat{h}\left(\cdot, \frac{u}{\sqrt{n}}\right)(x) \\ &\quad + \widehat{\phi}(u) \left(\Pi_{s, \frac{-iu}{\sqrt{n}}} - \Pi_{s,0} \right) \widehat{h}\left(\cdot, \frac{u}{\sqrt{n}}\right)(x) \\ &=: K_1 + K_2. \end{aligned}$$

For K_1 , since $\widehat{\phi}(u) = e^{-u^2/2}$, using (5.6), one can verify that there exists a constant $c > 0$ such that for all $|u| \leq \delta\sqrt{n}$ and $n \geq 1$,

$$\left| \lambda_{s, \frac{-iu}{\sqrt{n}}}^n - \widehat{\phi}(u) \right| \leq \frac{c}{\sqrt{n}} e^{-\frac{u^2}{4}},$$

see [12]. By Lemma 5.2, the mapping $t \mapsto \Pi_{s,it} : (-\delta, \delta) \rightarrow \mathcal{L}(\mathcal{B}, \mathcal{B})$ is analytic, hence there exists a constant $c > 0$ such that

$$\sup_{x \in \mathbb{P}(V)} \sup_{|u| \leq \delta\sqrt{n}} \left| \Pi_{s, \frac{-iu}{\sqrt{n}}} \widehat{h}\left(\cdot, \frac{u}{\sqrt{n}}\right)(x) \right| \leq \sup_{|y'| \leq \delta} \|\Pi_{s, iy'}\|_{\mathcal{B} \rightarrow \mathcal{B}} \left\| \widehat{h}\left(\cdot, y'\right) \right\|_{\mathcal{B}} \leq c \|h\|_{\mathcal{H}}.$$

It follows that $K_1 \leq \frac{c}{\sqrt{n}} e^{-\frac{u^2}{4}} \|h\|_{\mathcal{H}}$. For K_2 , by Lemma 5.2, we get $K_2 \leq c \frac{|u|}{\sqrt{n}} e^{-\frac{u^2}{2}} \|h\|_{\mathcal{H}} \leq \frac{c}{\sqrt{n}} e^{-\frac{u^2}{4}} \|h\|_{\mathcal{H}}$ and hence there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$ and

$$|u| \leq \delta\sqrt{n},$$

$$J_1(x, u) \leq \frac{c}{\sqrt{n}} e^{-\frac{u^2}{4}} \|h\|_{\mathcal{H}}. \quad (5.25)$$

For the second term $J_2(x, u)$, using (5.7) we get that there exist constants $c, c' > 0$ such that for any $x \in \mathbb{P}(V)$ and $|u| \leq \delta\sqrt{n}$,

$$J_2(x, u) \leq ce^{-c'n} \sup_{|y'| \leq \delta} \left\| \widehat{h}(\cdot, y') \right\|_{\mathcal{B}} \leq ce^{-c'n} \|h\|_{\mathcal{H}}. \quad (5.26)$$

Therefore, combining (5.25) and (5.26), we obtain the upper bound for I_1 :

$$|I_1| \leq \frac{1}{2\pi} \int_{|u| \leq \delta\sqrt{n}} |J(x, u)| du \leq c \left(\frac{1}{\sqrt{n}} + e^{-c'n} \right) \|h\|_{\mathcal{H}} \leq \frac{c}{\sqrt{n}} \|h\|_{\mathcal{H}}. \quad (5.27)$$

Bound of I_2 . Since $h \in \mathcal{H}_K$, we have

$$|I_2| \leq \frac{1}{2\pi} \int_{|u| > \delta\sqrt{n}} \widehat{\phi}(u) \int_{\mathbb{P}(V)} \left| \widehat{h} \left(x', \frac{u}{\sqrt{n}} \right) \right| \pi_s(dx') du \leq ce^{-\delta^2 n/4} \|h\|_{\mathcal{H}}. \quad (5.28)$$

Bound of I_3 . Since $h \in \mathcal{H}_K$, the Fourier transform $\widehat{h}(x, \cdot)$ has a support contained in K for any $x \in \mathbb{P}(V)$. Applying Lemma 5.3, we get that there exist constants $c_K, c'_K > 0$ such that

$$|I_3| \leq c_K e^{-nc'_K} \|h\|_{\mathcal{H}}. \quad (5.29)$$

Collecting the bounds (5.27), (5.28) and (5.29), we obtain (5.24) and thus conclude the proof of (5.21). \square

Now we give an extension of Theorem 5.9 for functions h with non-integrable Fourier transforms. For any $\varepsilon > 0$ and any non-negative measurable function $h \in \mathcal{H}$, we denote by h_ε a measurable function such that for any $x \in \mathbb{P}(V)$ and $t \in \mathbb{R}$, it holds that $h(x, t) \leq h_\varepsilon(x, t+v)$ for all $|v| \leq \varepsilon$. In this case we simply write $h \leq_\varepsilon h_\varepsilon$ or $h_\varepsilon \geq_\varepsilon h$. Similarly, we denote by $h_{-\varepsilon}$ a measurable function such that $h(x, t) \geq h_{-\varepsilon}(x, t+v)$ for any $x \in \mathbb{P}(V)$, $t \in \mathbb{R}$ and $|v| \leq \varepsilon$, and we write $h_{-\varepsilon} \leq_\varepsilon h$ or $h \geq_\varepsilon h_{-\varepsilon}$.

In the proofs we make use of the following smoothing inequality (cf. [12, 16]). Denote by ρ the non-negative density function on \mathbb{R} , which is the Fourier transform of the function $(1 - |t|)\mathbb{1}_{\{|t| \leq 1\}}$ for $t \in \mathbb{R}$. Set $\rho_\varepsilon(u) = \frac{1}{\varepsilon} \rho(\frac{u}{\varepsilon})$ for $u \in \mathbb{R}$ and $\varepsilon > 0$.

Lemma 5.10. *Let $\varepsilon \in (0, \frac{1}{8})$. Let $h : \mathbb{R} \rightarrow \mathbb{R}_+$ be an integrable function and let $h_{-\varepsilon}$ and h_ε be any measurable functions such that $h_{-\varepsilon} \leq_\varepsilon h \leq_\varepsilon h_\varepsilon$. Then for any $u \in \mathbb{R}$,*

$$h(u) \leq (1 + 4\varepsilon)h_\varepsilon * \rho_{\varepsilon^2}(u), \quad h(u) \geq h_{-\varepsilon} * \rho_{\varepsilon^2}(u) - \int_{|v| > \varepsilon} h_{-\varepsilon}(u - v) \rho_{\varepsilon^2}(v) dv.$$

Below, for any $h \in \mathcal{H}$, we use the notation

$$h * \rho_{\varepsilon^2}(x, t) = \int_{\mathbb{R}} h(x, t - v) \rho_{\varepsilon^2}(v) dv, \quad x \in \mathbb{P}(V), \quad t \in \mathbb{R}.$$

Theorem 5.11. *Assume condition A2 and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, for any $\varepsilon \in (0, \frac{1}{8})$, there exist constants $c, c_\varepsilon > 0$ such that for any non-negative function h and any function $h_\varepsilon \in \mathcal{H}$, $n \geq 1$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}$,*

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_x} h(X_n, y - S_n) &\leq \frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x', y') \phi\left(\frac{y - y'}{\sigma_s \sqrt{n}}\right) \pi_s(dx') dy' \\ &\quad + \frac{c\varepsilon}{\sqrt{n}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon}{n} \|h_\varepsilon\|_{\mathcal{H}} \end{aligned} \quad (5.30)$$

and

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^x} h(X_n, y - S_n) &\geq \frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_{-\varepsilon}(x', y') \phi\left(\frac{y - y'}{\sigma_s \sqrt{n}}\right) \pi_s(dx') dy' \\ &\quad - \frac{c\varepsilon}{\sqrt{n}} \|h\|_{\pi_s \otimes \text{Leb}} - \frac{c\varepsilon}{n} \|h\|_{\mathcal{H}}. \end{aligned} \quad (5.31)$$

Proof. We first prove the upper bound (5.30). By Lemma 5.10, we have $h \leq (1 + 4\varepsilon)h_\varepsilon * \rho_{\varepsilon^2}$ and hence

$$\mathbb{E}_{\mathbb{Q}_s^x} h(X_n, y - S_n) \leq (1 + 4\varepsilon) \mathbb{E}_{\mathbb{Q}_s^x} h_\varepsilon * \rho_{\varepsilon^2}(X_n, y - S_n). \quad (5.32)$$

Since the support of the function $\widehat{h_\varepsilon * \rho_{\varepsilon^2}}(x, \cdot) = \widehat{h_\varepsilon}(x, \cdot) \widehat{\rho_{\varepsilon^2}}(\cdot)$ is included in $[-\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon^2}]$ for any $x \in \mathbb{P}(V)$, by Theorem 5.9, there exists $c_\varepsilon > 0$ such that for all $n \geq 1$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_s^x} h_\varepsilon * \rho_{\varepsilon^2}(X_n, y - S_n) \\ &\leq \frac{1 + 4\varepsilon}{\sigma_s \sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon * \rho_{\varepsilon^2}(x', y') \phi\left(\frac{y' - y}{\sigma_s \sqrt{n}}\right) \pi_s(dx') dy' + \frac{c_\varepsilon}{n} \|h_\varepsilon\|_{\mathcal{H}} \\ &= \frac{1 + 4\varepsilon}{\sigma_s \sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon * \rho_{\varepsilon^2}(x, u + y) \phi\left(\frac{u}{\sigma_s \sqrt{n}}\right) \pi_s(dx) du + \frac{c_\varepsilon}{n} \|h_\varepsilon\|_{\mathcal{H}}. \end{aligned} \quad (5.33)$$

By a change of variable and Fubini's theorem, we have for any $x \in \mathbb{P}(V)$,

$$\frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{R}} h_\varepsilon * \rho_{\varepsilon^2}(x, u + y) \phi\left(\frac{u}{\sigma_s \sqrt{n}}\right) du = \int_{\mathbb{R}} h_\varepsilon(x, t + y) \phi_{\sigma_s \sqrt{n}} * \rho_{\varepsilon^2}(t) dt, \quad (5.34)$$

where $\phi_{\sigma_s \sqrt{n}}(t) = \frac{1}{\sigma_s \sqrt{2\pi n}} e^{-\frac{t^2}{2\sigma_s^2 n}}$, $t \in \mathbb{R}$. For brevity, denote $\psi(t) = \sup_{|v| \leq \varepsilon} \phi_{\sigma_s \sqrt{n}}(t + v)$, $t \in \mathbb{R}$. Using the second inequality in Lemma 5.10, we have

$$\begin{aligned} &\int_{\mathbb{R}} h_\varepsilon(x, t + y) \phi_{\sigma_s \sqrt{n}} * \rho_{\varepsilon^2}(t) dt \\ &\leq \int_{\mathbb{R}} h_\varepsilon(x, t + y) \psi(t) dt + \int_{\mathbb{R}} h_\varepsilon(x, t + y) \left[\int_{|v| \geq \varepsilon} \phi_{\sigma_s \sqrt{n}}(t - v) \rho_{\varepsilon^2}(v) dv \right] dt =: J_1 + J_2. \end{aligned}$$

For J_1 , by Taylor's expansion and the fact that the function ϕ' is bounded on \mathbb{R} , we derive that

$$J_1 \leq \frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{R}} h_\varepsilon(x, t + y) \phi\left(\frac{t}{\sigma_s \sqrt{n}}\right) dt + \frac{c\varepsilon}{\sqrt{n}} \int_{\mathbb{R}} h_\varepsilon(x, t) dt. \quad (5.35)$$

For J_2 , since $\phi_{\sigma_s \sqrt{n}} \leq \frac{c}{\sqrt{n}}$ and $\int_{|v| \geq \varepsilon} \rho_{\varepsilon^2}(v) dv \leq c\varepsilon$, we get

$$J_2 \leq \frac{c}{\sqrt{n}} \int_{\mathbb{R}} h_\varepsilon(x, t + y) \left[\int_{|v| \geq \varepsilon} \rho_{\varepsilon^2}(v) dv \right] dt \leq \frac{c\varepsilon}{\sqrt{n}} \int_{\mathbb{R}} h_\varepsilon(x, t) dt. \quad (5.36)$$

Putting together (5.32), (5.33), (5.34), (5.35) and (5.36), we get (5.30).

We next prove the lower bound (5.31). Since $h \geq_\varepsilon h_{-\varepsilon}$, using the second inequality in Lemma 5.10, we get

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^x} h(X_n, y - S_n) &\geq \mathbb{E}_{\mathbb{Q}_s^x} h_{-\varepsilon} * \rho_{\varepsilon^2}(X_n, y - S_n) \\ &\quad - \int_{|v| \geq \varepsilon} \mathbb{E}_{\mathbb{Q}_s^x} h_{-\varepsilon}(X_n, y - S_n - v) \rho_{\varepsilon^2}(v) dv. \end{aligned} \quad (5.37)$$

For the first term, by Theorem 5.9, there exists $c_\varepsilon > 0$ such that for all $n \geq 1$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^x} h_{-\varepsilon} * \rho_{\varepsilon^2}(X_n, y - S_n) \\ & \geq \frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_{-\varepsilon} * \rho_{\varepsilon^2}(x', y') \phi\left(\frac{y' - y}{\sigma_s \sqrt{n}}\right) \pi_s(dx') dy' - \frac{c_\varepsilon}{n} \|h_{-\varepsilon}\|_{\mathcal{H}} \\ & = \frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_{-\varepsilon} * \rho_{\varepsilon^2}(x, u + y) \phi\left(\frac{u}{\sigma_s \sqrt{n}}\right) \pi_s(dx) du - \frac{c_\varepsilon}{n} \|h_{-\varepsilon}\|_{\mathcal{H}}. \end{aligned} \quad (5.38)$$

In the same way as in (5.34), we have

$$\frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{R}} h_{-\varepsilon} * \rho_{\varepsilon^2}(x, u + y) \phi\left(\frac{u}{\sigma_s \sqrt{n}}\right) du = \int_{\mathbb{R}} h_{-\varepsilon}(x, t + y) \phi_{\sigma_s \sqrt{n}} * \rho_{\varepsilon^2}(t) dt. \quad (5.39)$$

Using the first inequality in Lemma 5.10, we have $\phi_{\sigma_s \sqrt{n}} * \rho_{\varepsilon^2}(t) \geq (1 - c\varepsilon)\psi(t)$, for $t \in \mathbb{R}$, where $\psi(t) = \inf_{|v| \leq \varepsilon} \phi_{\sigma_s \sqrt{n}}(t + v)$. Proceeding in the same way as in (5.35) and (5.36), we obtain that

$$\begin{aligned} & \int_{\mathbb{R}} h_{-\varepsilon}(x, t + y) \phi_{\sigma_s \sqrt{n}} * \rho_{\varepsilon^2}(t) dt \\ & \geq \frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{R}} h_{-\varepsilon}(x, t + y) \phi\left(\frac{t}{\sigma_s \sqrt{n}}\right) dt - \frac{c\varepsilon}{\sqrt{n}} \int_{\mathbb{R}} h_{-\varepsilon}(x, t) dt. \end{aligned} \quad (5.40)$$

Therefore, combining (5.38), (5.39) and (5.40), we get

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_s^x} h_{-\varepsilon} * \rho_{\varepsilon^2}(X_n, y - S_n) & \geq \frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_{-\varepsilon}(x, t + y) \phi\left(\frac{t}{\sigma_s \sqrt{n}}\right) \pi_s(dx) dt \\ & \quad - \frac{c\varepsilon}{\sqrt{n}} \|h_{-\varepsilon}\|_{\pi_s \otimes \text{Leb}} - \frac{c_\varepsilon}{n} \|h_{-\varepsilon}\|_{\mathcal{H}}. \end{aligned} \quad (5.41)$$

For the second term on the right hand side of (5.37), using the upper bound (5.30) and the fact that $h_{-\varepsilon} \leq_\varepsilon h$ and $\phi \leq 1$, we get that there exist constants $c, c_\varepsilon > 0$ such that for any $v \in \mathbb{R}$ and $n \geq 1$,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^x} h_{-\varepsilon}(X_n, y - S_n - v) \\ & \leq \frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}} h(x, t) \phi\left(\frac{y - v - t}{\sigma_s \sqrt{n}}\right) \pi_s(dx) dt + \frac{c\varepsilon}{\sqrt{n}} \|h\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon}{n} \|h\|_{\mathcal{H}} \\ & \leq \frac{c}{\sqrt{n}} \|h\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon}{n} \|h\|_{\mathcal{H}}. \end{aligned}$$

This, together with the fact that $\int_{|v| \geq \varepsilon} \rho_{\varepsilon^2}(v) dv \leq c\varepsilon$, implies

$$\int_{|v| \geq \varepsilon} \mathbb{E}_{\mathbb{Q}_s^x} h_{-\varepsilon}(X_n, y - S_n - v) \rho_{\varepsilon^2}(v) dv \leq \frac{c\varepsilon}{\sqrt{n}} \|h\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon}{n} \|h\|_{\mathcal{H}}. \quad (5.42)$$

Substituting (5.41) and (5.42) into (5.37) and using the fact that $\|h_{-\varepsilon}\|_{\pi_s \otimes \text{Leb}} \leq \|h\|_{\pi_s \otimes \text{Leb}}$ and $\|h_{-\varepsilon}\|_{\mathcal{H}} \leq \|h\|_{\mathcal{H}}$, we obtain the lower bound (5.31). \square

6. CONDITIONED LOCAL LIMIT THEOREMS

6.1. Bounds in the conditioned local limit theorems. The next lemma shows that the $\|\cdot\|_{\pi_s \otimes \text{Leb}}$ norm of the probability $\mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n)$ is of order $n^{-1/2}$. This turns out to be one of the key points in the sequel. The proof is based upon the duality lemma (Lemma 4.4) and the bound for the exit time τ_y^* for the dual random walk S_n^* (Theorem 4.8).

Lemma 6.1. *Assume conditions [A2](#), [A3](#), and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, there exists a constant $c > 0$ such that for any $n \geq 1$ and $0 \leq a < b < \infty$,*

$$\int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n) \pi_s(dx) dy \leq \frac{c}{\sqrt{n}}(b - a)(b + a + 1).$$

Proof. Using the duality lemma (Lemma [4.4](#)) and Fubini's theorem, we get that for $n \geq 1$,

$$\begin{aligned} J &:= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n) \pi_s(dx) dy \\ &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbf{1}_{[a, b]}(z) \mathbb{Q}_s^{x, *}(\tau_z^* > n) \pi_s(dx) dz. \end{aligned}$$

By Theorem [4.8](#), there exists a constant c such that for any $x \in \mathbb{P}(V)$, $z \geq 0$ and $n \geq 1$,

$$\mathbb{Q}_s^{x, *}(\tau_z^* > n) \leq c \frac{1 + z}{\sqrt{n}}.$$

Therefore,

$$J \leq \frac{c}{\sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbf{1}_{[a, b]}(z)(1 + z) \pi_s(dx) dz = \frac{c}{\sqrt{n}}(b - a)(b + a + 1),$$

which ends the proof of the lemma. \square

Lemma 6.2. *Assume condition [A2](#) and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, for any $\varepsilon \in [0, \frac{1}{2})$, there exists a constant $c_\varepsilon > 0$ such that for any $n \geq 2$ and $-\sqrt{n} \log^{1-\varepsilon} n \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,*

$$\int_{\mathbb{R}} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy \leq c_\varepsilon(b - a + 1) \log^{1-\varepsilon} n.$$

Proof. We first decompose the integral into three parts:

$$\int_{\mathbb{R}} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy = J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= \int_{|y| \leq 2\sqrt{n} \log^{1-\varepsilon} n} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy, \\ J_2 &= \int_{2\sqrt{n} \log^{1-\varepsilon} n < |y| \leq n^2} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy, \\ J_3 &= \int_{|y| > n^2} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy. \end{aligned}$$

Bound of J_1 . By the local limit theorem ([5.30](#)), there exists a constant $c > 0$ such that for any $-\infty < a < b < \infty$ and $n \geq 1$,

$$\begin{aligned} J_1 &\leq 4 \log^{1-\varepsilon} n \sup_{x \in \mathbb{P}(V)} \sup_{y \in \mathbb{R}} \sqrt{n} \mathbb{Q}_s^x(y - S_n \in [a, b]) \\ &\leq 4 \log^{1-\varepsilon} n \left[(b - a + c\varepsilon) + \frac{c_\varepsilon}{\sqrt{n}}(b - a + c\varepsilon) \right] \\ &\leq c(b - a + 1) \log^{1-\varepsilon} n. \end{aligned} \tag{6.1}$$

Bound of J_2 . When $y \in [2\sqrt{n} \log^{1-\varepsilon} n, n^2]$ and $b \leq \sqrt{n} \log^{1-\varepsilon} n$, there exist constant $c, c' > 0$ such that for any $x \in \mathbb{P}(V)$ and $y \in [2\sqrt{n} \log^{1-\varepsilon} n, n^2]$,

$$\mathbb{Q}_s^x(y - S_n \in [a, b]) \leq \mathbb{Q}_s^x(S_n \geq \sqrt{n} \log^{1-\varepsilon} n) \leq ce^{-c' \log^{2-2\varepsilon} n}, \quad (6.2)$$

where in the last inequality we used the upper tail moderate deviation asymptotic for S_n under the changed measure \mathbb{Q}_s^x (cf. [28]). Since $\varepsilon \in [0, \frac{1}{2})$, it follows that

$$\int_{2\sqrt{n} \log^{1-\varepsilon} n}^{n^2} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy \leq ce^{-c' \log^{2-2\varepsilon} n}. \quad (6.3)$$

When $y \in [-n^2, -2\sqrt{n} \log^{1-\varepsilon} n]$ and $a \geq -\sqrt{n} \log^{1-\varepsilon} n$, there exist constant $c, c' > 0$ such that for any $x \in \mathbb{P}(V)$ and $y \in [-n^2, -2\sqrt{n} \log^{1-\varepsilon} n]$,

$$\mathbb{Q}_s^x(y - S_n \in [a, b]) \leq \mathbb{Q}_s^x(S_n \leq -\sqrt{n} \log^{1-\varepsilon} n) \leq ce^{-c' \log^{2-2\varepsilon} n},$$

where in the last inequality we used the lower tail moderate deviation asymptotic for S_n under the changed measure \mathbb{Q}_s^x (cf. [28]). It follows that

$$\int_{-n^2}^{-2\sqrt{n} \log^{1-\varepsilon} n} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy \leq ce^{-c' \log^{2-2\varepsilon} n}. \quad (6.4)$$

Combining (6.3) and (6.4), we get that there exist constants $c, c' > 0$ such that for any $-\sqrt{n} \log^{1-\varepsilon} n \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,

$$J_2 \leq ce^{-c' \log^{2-2\varepsilon} n}. \quad (6.5)$$

Bound of J_3 . Since $y > n^2$ and $b \leq \sqrt{n} \log^{1-\varepsilon} n$, by the Markov inequality, we have for sufficiently small $\delta > 0$,

$$\mathbb{Q}_s^x(y - S_n \in [a, b]) \leq \mathbb{Q}_s^x\left(S_n \geq \frac{y}{2}\right) \leq e^{-\frac{\delta}{2}y} \mathbb{E}_{\mathbb{Q}_s^x} e^{\delta S_n}.$$

Using (3.5), we get that for $s \in I_\mu^+$,

$$\begin{aligned} \sup_{x \in \mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^x} e^{\delta S_n} &= \sup_{x \in \mathbb{P}(V)} \frac{1}{\kappa(s)^n r_s(x)} \mathbb{E}_x[r_s(X_n) e^{(s+\delta)S_n}] \\ &\leq ce^{n\Lambda(s)} \sup_{x \in \mathbb{P}(V)} \mathbb{E}_x(e^{(s+\delta)S_n}) \leq ce^{n\Lambda(s)} \left[\mathbb{E}(e^{(s+\delta) \log \|g_1\|})\right]^n \leq ce^{c'n}. \end{aligned}$$

Similarly, for $s \in I_\mu^-$, it also holds that $\sup_{x \in \mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^x} e^{\delta S_n} \leq ce^{c'n}$ by using the fact that $\sup_{x \in \mathbb{P}(V)} \mathbb{E}_x(e^{(s+\delta)S_n}) \leq [\mathbb{E}(e^{-(s+\delta) \log \|g_1^{-1}\|})]^n$. Hence there exists a constant $\delta > 0$ such that for any $s \in I_\mu^+ \cup I_\mu^-$,

$$\int_{n^2}^{\infty} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy \leq e^{-\frac{\delta}{2}n^2} \sup_{x \in \mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^x} e^{\delta S_n} \leq e^{-\frac{\delta}{4}n^2}.$$

In the same way, we can show that

$$\int_{-\infty}^{-n^2} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy \leq e^{-\frac{\delta}{4}n^2}.$$

Therefore, there exists a constant $\delta > 0$ such that for any $-\sqrt{n} \log^{1-\varepsilon} n \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,

$$J_3 \leq 2e^{-\frac{\delta}{4}n^2}. \quad (6.6)$$

Putting together (6.1), (6.5) and (6.6) concludes the proof of the lemma. \square

Below we state two lemmata based upon each other and providing successively improved bounds of the integral

$$I_n := \int_{\mathbb{R}} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x (y - S_n \in [a, b], \tau_y > n) dy. \quad (6.7)$$

The estimate of I_n is one of the difficult points of the paper.

Lemma 6.3. *Assume conditions A2, A3 and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, for any $\varepsilon \in [0, \frac{1}{2})$, there exists a constant $c_\varepsilon > 0$ such that for any $n \geq 2$ and $-\sqrt{n} \log^{1-\varepsilon} n \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,*

$$I_n \leq c_\varepsilon \frac{\log^{2-2\varepsilon} n}{\sqrt{n}} (b - a + 1)(b + a + 1).$$

Proof. In view of (6.7), we split the integral I_n into two parts:

$$\begin{aligned} I_n &= \int_{|y| \leq 2\sqrt{n} \log^{1-\varepsilon} n} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x (y - S_n \in [a, b], \tau_y > n) dy \\ &\quad + \int_{|y| > 2\sqrt{n} \log^{1-\varepsilon} n} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x (y - S_n \in [a, b], \tau_y > n) dy \\ &=: I_{n,1} + I_{n,2}. \end{aligned} \quad (6.8)$$

For the first term $I_{n,1}$, we use the Markov property to get that for any $m = \lfloor \frac{n}{2} \rfloor$ and $k = n - m$,

$$\begin{aligned} &\mathbb{Q}_s^x (y - S_n \in [a, b], \tau_y > n) \\ &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{Q}_s^{x'} (y' - S_m \in [a, b], \tau_{y'} > m) \mathbb{Q}_s^x (X_k \in dx', y - S_k \in dy', \tau_y > k) \\ &\leq \int_{\mathbb{P}(V) \times \mathbb{R}_+} h(x', y') \mathbb{Q}_s^x (X_k \in dx', y - S_k \in dy') =: J_n(x, y), \end{aligned} \quad (6.9)$$

where we denoted

$$h(x', y') = \begin{cases} \mathbb{Q}_s^{x'} (y' - S_m \in [a, b], \tau_{y'} > m), & x' \in \mathbb{P}(V), y' > 0 \\ 0, & x' \in \mathbb{P}(V), y' \leq 0. \end{cases} \quad (6.10)$$

By the local limit theorem ((5.30) of Theorem 5.11), we get

$$\begin{aligned} \sup_{x \in \mathbb{P}(V)} J_n(x, y) &\leq \frac{1}{\sigma_s \sqrt{k}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x', y') \phi\left(\frac{y - y'}{\sigma_s \sqrt{k}}\right) \pi_s(dx') dy' \\ &\quad + \frac{c_\varepsilon}{\sqrt{k}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon}{k} \|h_\varepsilon\|_{\mathcal{H}}, \end{aligned} \quad (6.11)$$

where we choose

$$h_\varepsilon(x', y') = \begin{cases} \mathbb{Q}_s^{x'} (y' - S_m \in [a - \varepsilon, b + \varepsilon], \tau_{y'+\varepsilon} > m), & x' \in \mathbb{P}(V), y' > -\varepsilon \\ 0, & x' \in \mathbb{P}(V), y' \leq -\varepsilon. \end{cases} \quad (6.12)$$

Using the fact $\phi \leq 1$ and $h_\varepsilon(x', y') = 0$ when $y' \leq -\varepsilon$, we get an upper bound for the first term in the right hand side of (6.11):

$$\begin{aligned}
& \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x', y') \phi\left(\frac{y - y'}{\sigma_s \sqrt{n}}\right) \pi_s(dx') dy' \\
& \leq \int_{\mathbb{P}(V)} \int_{-\varepsilon}^{\infty} h_\varepsilon(x', y') \pi_s(dx') dy' = \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_\varepsilon(x', t - \varepsilon) \pi_s(dx') dt \\
& = \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{Q}_s^{x'}(t - S_m \in [a, b + 2\varepsilon], \tau_t > m) \pi_s(dx') dt \\
& \leq \frac{c}{\sqrt{m}}(b - a + 1)(b + a + 1), \tag{6.13}
\end{aligned}$$

where in the last inequality we used Lemma 6.1.

For the second term in the right hand side of (6.11), we proceed in the same way as for the first one to get that

$$\frac{c\varepsilon}{\sqrt{k}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} \leq \frac{c\varepsilon}{\sqrt{km}}(b - a + \varepsilon)(b + a + 1). \tag{6.14}$$

For the third term in the right hand side of (6.11), by (6.12) and Lemma 6.2, there exists a constant $c > 0$ such that for all $n \geq 2$ and $-\sqrt{n} \log^{1-\varepsilon} n \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,

$$\begin{aligned}
\|h_\varepsilon\|_{\mathcal{H}} &= \int_{-\varepsilon}^{\infty} \sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'}(y' - S_m \in [a - \varepsilon, b + \varepsilon], \tau_{y'+\varepsilon} > m) dy' \\
&= \int_{\mathbb{R}_+} \sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'}(t - S_m \in [a, b + 2\varepsilon], \tau_t > m) dt \\
&\leq c_\varepsilon(b - a + 1) \log^{1-\varepsilon} n, \tag{6.15}
\end{aligned}$$

where we used the fact that $m = \lfloor \frac{n}{2} \rfloor$ and the inequality in Lemma 6.2 still holds when a and b are replaced by their constant multiples. Substituting (6.13), (6.14) and (6.15) into (6.11), and taking into account that $m = \lfloor \frac{n}{2} \rfloor$ and $k = n - m$, one has

$$\begin{aligned}
\sup_{x \in \mathbb{P}(V)} J_n(x, y) &\leq \frac{c}{n}(b - a + 1)(b + a + 1) + \frac{c\varepsilon}{n}(b - a + 1)(b + a + 1) + c_\varepsilon \frac{\log^{1-\varepsilon} n}{n}(b - a + 1) \\
&\leq c_\varepsilon \frac{\log^{1-\varepsilon} n}{n}(b - a + 1)(b + a + 1),
\end{aligned}$$

from which we get

$$I_{n,1} \leq \int_{|y| \leq 2\sqrt{n} \log^{1-\varepsilon} n} \sup_{x \in \mathbb{P}(V)} J_n(x, y) dy \leq c_\varepsilon \frac{\log^{2-2\varepsilon} n}{\sqrt{n}}(b - a + 1)(b + a + 1). \tag{6.16}$$

It was shown in the proof of Lemma 6.2 (cf. (6.5) and (6.6)) that there exist constants $c, c' > 0$ such that for any $-\sqrt{n} \log^{1-\varepsilon} n \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,

$$I_{n,2} \leq \int_{|y| > 2\sqrt{n} \log^{1-\varepsilon} n} \sup_{x \in \mathbb{P}(V)} \mathbb{Q}_s^x(y - S_n \in [a, b]) dy \leq ce^{-c' \log^{2-2\varepsilon} n}. \tag{6.17}$$

Putting together (6.16) and (6.17), we conclude the proof of the lemma. \square

The convergence rate in Lemma 6.3 can be improved by repeating the same proof. Recall that I_n is given by (6.7).

Lemma 6.4. *Assume conditions [A2](#), [A3](#) and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, for any $\varepsilon \in [0, \frac{1}{2})$, there exists a constant $c_\varepsilon > 0$ such that for any $n \geq 2$ and $-\sqrt{n} \log^{1-\varepsilon} n \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,*

$$I_n \leq c_\varepsilon \frac{\log^{1-\varepsilon} n}{\sqrt{n}} (b - a + 1)(b + a + 1).$$

Proof. We repeat the same proof as in [Lemma 6.3](#). The only difference is that in [\(6.15\)](#), we apply [Lemma 6.3](#) instead of [Lemma 6.2](#) to get that there exists a constant $c > 0$ such that for all $m \geq 2$ and $-\sqrt{m} \log^{1-\varepsilon'} m \leq a < b \leq \sqrt{m} \log^{1-\varepsilon'} m$ with $\varepsilon' \in (0, \varepsilon)$,

$$\begin{aligned} \|h_\varepsilon\|_{\mathcal{H}} &= \int_{\mathbb{R}} \sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'}(y' - S_m \in [a - \varepsilon, b + \varepsilon], \tau_{y'} > m) dy' \\ &\leq c_\varepsilon \frac{\log^{2-2\varepsilon'} m}{\sqrt{m}} (b - a + 1)(b + a + 1). \end{aligned} \quad (6.18)$$

Substituting [\(6.13\)](#), [\(6.14\)](#) and [\(6.18\)](#) into [\(6.11\)](#), and taking into account that $\varepsilon' \in (0, \varepsilon)$, $m = \lfloor \frac{n}{2} \rfloor$ and $k = n - m$, one has for any $n \geq 2$ and $-\sqrt{n} \log^{1-\varepsilon} n \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,

$$\begin{aligned} \sup_{x \in \mathbb{P}(V)} J_n(x) &\leq \frac{c}{n} (b - a + 1)(b + a + 1) + \frac{c\varepsilon}{n} (b - a + 1)(b + a + 1) + c_\varepsilon \frac{\log^{2-2\varepsilon'} n}{n^{3/2}} (b - a + 1) \\ &\leq \frac{c}{n} (b - a + 1)(b + a + 1), \end{aligned}$$

from which we get that for any $n \geq 2$ and $-\sqrt{n} \log^{1-\varepsilon} n \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,

$$I_{n,1} \leq \int_{|y| \leq 2\sqrt{n} \log^{1-\varepsilon} n} \sup_{x \in \mathbb{P}(V)} J_n(x) dy \leq c \frac{\log^{1-\varepsilon} n}{\sqrt{n}} (b - a + 1)(b + a + 1).$$

Combining this with [\(6.8\)](#) and [\(6.17\)](#) ends the proof of the lemma. \square

Using [Lemma 6.4](#), we can now establish an upper bound for the conditioned local probability $\mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n)$. Note that this bound does not depend on the starting point $y \geq 0$ and the convergence rate is of order $O(\frac{1}{n})$.

Lemma 6.5. *Assume conditions [A2](#), [A3](#) and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$, $y \geq 0$, $n \geq 1$ and $0 \leq a < b \leq \sqrt{n} \log n$,*

$$\mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n) \leq \frac{c}{n} (b - a + 1)(b + a + 1).$$

Proof. Let $k = \lfloor n/2 \rfloor$ and $m = n - k$. It is shown in [\(6.9\)](#) that

$$\begin{aligned} &\mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n) \\ &\leq \int_{\mathbb{P}(V) \times \mathbb{R}_+} h(x', y') \mathbb{Q}_s^x(X_k \in dx', y - S_k \in dy') =: J_n(x, y), \end{aligned}$$

where h is defined by [\(6.10\)](#). It is shown in [\(6.11\)](#) and [\(6.12\)](#) that

$$\begin{aligned} \sup_{x \in \mathbb{P}(V)} J_n(x, y) &\leq \frac{1}{\sigma_s \sqrt{k}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x', y') \phi\left(\frac{y - y'}{\sigma_s \sqrt{k}}\right) dy' \pi_s(dx') \\ &\quad + \frac{c\varepsilon}{\sqrt{k}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon}{k} \|h_\varepsilon\|_{\mathcal{H}} \\ &=: J_{n,1}(y) + J_{n,2} + J_{n,3}, \end{aligned}$$

where

$$h_\varepsilon(x', y') = \mathbb{Q}_s^{x'}(y' - S_m \in [a - \varepsilon, b + \varepsilon], \tau_{y'+\varepsilon} > m). \quad (6.19)$$

By (6.13) and (6.14), we have

$$J_{n,1}(y) \leq \frac{c}{n}(b - a + 1)(b + a + 1), \quad J_{n,2} \leq \frac{c\varepsilon}{n}(b - a + \varepsilon)(b + a + 1). \quad (6.20)$$

For $J_{n,3}$, using (6.19) and Lemma 6.4, we get that for any $\varepsilon \in [0, \frac{1}{2})$, there exists a constant $c_\varepsilon > 0$ such that for all $m \geq 1$ and $0 \leq a < b \leq \sqrt{n} \log^{1-\varepsilon} n$,

$$\begin{aligned} J_{n,3} &= \frac{c_\varepsilon}{k} \int_{\mathbb{R}} \sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'}(y' - S_m \in [a - \varepsilon, b + \varepsilon], \tau_{y'} > m) dy' \\ &\leq c_\varepsilon \frac{\log^{1-\varepsilon} n}{n^{3/2}} (b - a + \varepsilon)(b + a + 1). \end{aligned} \quad (6.21)$$

Putting together (6.20) and (6.21) concludes the proof of the lemma. \square

From Lemma 6.5, we further prove that the convergence rate for the conditioned local probability $\mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n)$ can be improved to $O(\frac{1}{n^{3/2}})$, whereas the upper bound depends on the starting point y .

Lemma 6.6. *Assume conditions A2, A3 and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$, $y \geq 0$, $n \geq 1$ and $0 \leq a < b \leq \sqrt{n} \log n$,*

$$\mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n) \leq \frac{c}{n^{3/2}}(1 + y)(b - a + 1)(b + a + 1).$$

Proof. As in (6.9), we use the Markov property to get that for any $m = \lfloor \frac{n}{2} \rfloor$ and $k = n - m$,

$$\begin{aligned} \mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n) &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{Q}_s^{x'}(y' - S_m \in [a, b], \tau_{y'} > m) \\ &\quad \times \mathbb{Q}_s^x(X_k \in dx', y - S_k \in dy', \tau_y > k). \end{aligned} \quad (6.22)$$

By Lemma 6.5, there exists a constant $c > 0$ such that for any $x' \in \mathbb{P}(V)$, $y' \geq 0$, $m \geq 1$ and $0 \leq a < b \leq \sqrt{n} \log n$,

$$\mathbb{Q}_s^{x'}(y' - S_m \in [a, b], \tau_{y'} > m) \leq \frac{c}{n}(b - a + 1)(b + a + 1), \quad (6.23)$$

where we used the fact that $m = \lfloor \frac{n}{2} \rfloor$ and the inequality in Lemma 6.5 still holds when b is replaced by its constant multiple. By Theorem 4.6, there exist constants $c, c' > 0$ such that for any $x \in \mathbb{P}(V)$ and $y \geq 0$,

$$\mathbb{Q}_s^x(\tau_y > k) \leq c \frac{1 + y}{\sqrt{k}} = c' \frac{1 + y}{\sqrt{n}}. \quad (6.24)$$

Combining (6.22) and (6.23) and (6.24) concludes the proof of the lemma. \square

The following assertion is a combination of Lemmas 6.5 and 6.6.

Lemma 6.7. *Assume conditions A2, A3 and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$, $y \geq 0$, $n \geq 1$ and $0 \leq a < b \leq \sqrt{n} \log n$,*

$$\mathbb{Q}_s^x(y - S_n \in [a, b], \tau_y > n) \leq c \frac{(1 + y) \wedge n^{1/2}}{n^{3/2}} (b - a + 1)(b + a + 1).$$

6.2. Effective conditioned local limit theorems. A central point to establish Theorems 2.2 and 2.3 is a conditioned local limit theorem for random walks on the general linear group \mathbb{G} . For sums of i.i.d. real-valued random variables conditioned local limit theorems has been well-known in the literature based on the Wiener-Hopf factorization and the duality argument. For Markov chains the obtention of such exact asymptotics turns out to be much more complicated and has been recently done in [12] in the particular case when the Markov chain has a finite state space. For random walks on groups the problem is still open. In this section we shall establish such kind of results under the additional assumption that the matrix law μ admits a density with respect to the Haar measure on \mathbb{G} .

To state the effective conditioned local limit theorem, we need to introduce the Rayleigh density function $\phi^+(t) = te^{-t^2/2}$, $t \in \mathbb{R}_+$. Below we assume that the function h is supported on $\mathbb{P}(V) \times [0, \infty)$, h_ε on $\mathbb{P}(V) \times [-\varepsilon, \infty)$ and $h_{-\varepsilon}$ on $\mathbb{P}(V) \times [\varepsilon, \infty)$.

Theorem 6.8. *Assume conditions A2, A3 and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Let $(\alpha_n)_{n \geq 1}$ be any sequence of positive numbers satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then, there exist constants $c, c_\varepsilon > 0$ such that for any $\varepsilon \in (0, \frac{1}{8})$, $x \in \mathbb{P}(V)$, $y \in [0, \alpha_n \sqrt{n}]$, $n \geq 1$, $h \in \mathcal{H}$ and $h_\varepsilon \in \mathcal{H}$ satisfying $h \leq_\varepsilon h_\varepsilon$,*

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^x} [h(X_n, y - S_n); \tau_y > n] \\ & \leq \frac{2V_s(x, y)}{n\sigma_s^2\sqrt{2\pi}} \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_\varepsilon(x, t) \phi^+\left(\frac{t}{\sigma_s\sqrt{n}}\right) \pi_s(dx) dt \\ & \quad + \left(c\varepsilon^{1/4} + c_\varepsilon\alpha_n + c_\varepsilon n^{-\varepsilon}\right) \frac{1+y}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon(1+y)}{n^{3/2}} \|h_\varepsilon\|_{\mathcal{H}} \end{aligned} \quad (6.25)$$

and any $h, h_{-\varepsilon}, h_\varepsilon \in \mathcal{H}$ satisfying $h_{-\varepsilon} \leq_\varepsilon h \leq_\varepsilon h_\varepsilon$,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}_s^x} [h(X_n, y - S_n); \tau_y > n] \\ & \geq \frac{2V_s(x, y)}{n\sigma_s^2\sqrt{2\pi}} \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_{-\varepsilon}(x, t) \phi^+\left(\frac{t}{\sigma_s\sqrt{n}}\right) \pi_s(dx) dt \\ & \quad - \left(c\varepsilon^{1/12} + c_\varepsilon\alpha_n + c_\varepsilon n^{-\varepsilon}\right) \frac{1+y}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} - \frac{c_\varepsilon(1+y)}{n^{3/2}} \|h_\varepsilon\|_{\mathcal{H}} \\ & \quad - \frac{c_\varepsilon(1+y)}{n} \sup_{x \in \mathbb{P}(V)} \sup_{y \in \mathbb{R}} h_\varepsilon(x, y). \end{aligned} \quad (6.26)$$

Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$ be the standard normal density function. Let

$$\phi_v(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{x^2}{2v}}, \quad \phi_v^+(x) = \frac{x}{v} e^{-\frac{x^2}{2v}} \mathbf{1}_{\mathbb{R}_+}(x), \quad x \in \mathbb{R},$$

be the normal density of variance $v > 0$ and the Rayleigh density with scale parameter \sqrt{v} , respectively. Clearly we have $\phi = \phi_1$ and $\phi^+ = \phi_1^+$. The following lemma, which will be used in the proof of Theorem 6.8, shows that when v is small the convolution $\phi_v * \phi_{1-v}^+$ behaves like the Rayleigh density.

Lemma 6.9 ([16]). *For any $v \in (0, 1/2]$ and $t \in \mathbb{R}$, it holds that*

$$\sqrt{1-v}\phi^+(t) \leq \phi_v * \phi_{1-v}^+(t) \leq \sqrt{1-v}\phi^+(t) + \sqrt{v}e^{-\frac{t^2}{2v}} + |t|e^{-\frac{t^2}{2}} \mathbf{1}_{\{t < 0\}}.$$

To establish Theorem 6.8, we also need the following inequality of Haeusler [18, Lemma 1], which is a generalisation of Fuk's inequality for martingales.

Lemma 6.10 ([18]). *Let ξ_1, \dots, ξ_n be a martingale difference sequence with respect to the non-decreasing σ -fields $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n$. Then, for any $u, v, w > 0$ and $n \geq 1$,*

$$\mathbb{P} \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k \xi_i \right| \geq u \right) \leq \sum_{i=1}^n \mathbb{P} (|\xi_i| > v) + 2\mathbb{P} \left(\sum_{i=1}^n \mathbb{E} \left(\xi_i^2 \middle| \mathcal{F}_{i-1} \right) > w \right) + 2e^{\frac{u}{v}(1 - \log \frac{uv}{w})}.$$

We first give a proof of the upper bound (6.25) of Theorem 6.8.

Proof of (6.25). It is sufficient to prove (6.25) for large enough $n > n_0(\varepsilon)$, where $n_0(\varepsilon)$ depends on ε , since otherwise the bound becomes trivial.

For any $\varepsilon \in (0, \frac{1}{8})$, let $\delta = \sqrt{\varepsilon}$, $m = \lceil \delta n \rceil$ and $k = n - m$. Then, for n sufficiently large, we have $\frac{1}{2}\delta \leq \frac{\lceil \delta n \rceil}{n} \leq \frac{m}{k} \leq \frac{\delta}{1-\delta}$. Using the Markov property of the couple (X_n, S_n) , we get that for any $n \geq 1$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}_+$,

$$\begin{aligned} I_n(x, y) &:= \mathbb{E}_{\mathbb{Q}_s^x} [h(X_n, y - S_n); \tau_y > n] \\ &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m); \tau_{y'} > n] \mathbb{Q}_s^x (X_n \in dx', y - S_k \in dy', \tau_y > k) \\ &\leq \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m)] \mathbb{Q}_s^x (X_k \in dx', y - S_k \in dy', \tau_y > k). \end{aligned} \quad (6.27)$$

By the local limit theorem ((5.30) of Theorem 5.11), for any $\varepsilon \in (0, \frac{1}{8})$, there exist constants $c, c_\varepsilon > 0$ such that for any $m \geq 1$, $x' \in \mathbb{P}(V)$ and $y' \in \mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m)] \leq H_m(y') + \frac{c\varepsilon}{\sqrt{m}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon}{m} \|h_\varepsilon\|_{\mathcal{H}}, \quad (6.28)$$

where $h \leq_\varepsilon h_\varepsilon$ and

$$H_m(y') = \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, y) \frac{1}{\sigma_s \sqrt{m}} \phi \left(\frac{y' - y}{\sigma_s \sqrt{m}} \right) \pi_s(dx) dy. \quad (6.29)$$

From (6.27) and (6.28), using Theorem 4.6 we get that for any $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}_+$,

$$\begin{aligned} I_n(x, y) &\leq J_n(x, y) + \left(\frac{c\varepsilon}{\sqrt{m}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon}{m} \|h_\varepsilon\|_{\mathcal{H}} \right) \mathbb{Q}_s^x (\tau_y > k) \\ &\leq J_n(x, y) + \left(\frac{c\varepsilon}{\sqrt{mk}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon}{m\sqrt{k}} \|h_\varepsilon\|_{\mathcal{H}} \right) (1 + y), \end{aligned} \quad (6.30)$$

where

$$J_n(x, y) = \int_{\mathbb{R}_+} H_m(y') \mathbb{Q}_s^x (y - S_k \in dy', \tau_y > k). \quad (6.31)$$

Now we deal with the first term $J_n(x, y)$. By a change of variable, we have for any $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}_+$,

$$J_n(x, y) = \int_{\mathbb{R}_+} F_m(t) \mathbb{Q}_s^x \left(\frac{y - S_k}{\sigma_s \sqrt{k}} \in dt, \tau_y > k \right),$$

where $F_m(t) = H_m(\sigma_s \sqrt{kt})$, $t \in \mathbb{R}$. Since the function $t \mapsto F_m(t)$ is differentiable on \mathbb{R} and vanishes as $t \rightarrow \pm\infty$, using integration by parts, it follows that for any $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}_+$,

$$J_n(x, y) = \int_{\mathbb{R}_+} F'_m(t) \mathbb{Q}_s^x \left(\frac{y - S_k}{\sigma_s \sqrt{k}} \geq t, \tau_y > k \right) dt.$$

Applying the conditioned integral limit theorem (Theorem 4.7), we get that there exists a constant $c_\varepsilon > 0$ such that for any $x \in \mathbb{P}(V)$ and $y \in [0, \alpha_n \sqrt{n}]$,

$$J_n(x, y) \leq \frac{2V_s(x, y)}{\sigma_s \sqrt{2\pi k}} \int_{\mathbb{R}_+} F'_m(t) (1 - \Phi^+(t)) dt + c_\varepsilon \frac{1+y}{\sqrt{n}} (\alpha_n + n^{-\varepsilon}) \int_{\mathbb{R}_+} |F'_m(t)| dt. \quad (6.32)$$

Since $F_m(t) = H_m(\sigma_s \sqrt{kt})$, by (6.29) and a change of variable, it holds that

$$F_m(t) = \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, \sigma_s \sqrt{ky}) \phi\left(\frac{t-y}{\sqrt{m/k}}\right) \pi_s(dx) \frac{dy}{\sqrt{m/k}}. \quad (6.33)$$

Therefore, by a change of variable and Fubini's theorem, we get

$$\begin{aligned} \int_{\mathbb{R}_+} |F'_m(t)| dt &\leq \int_{\mathbb{R}_+} \left[\int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, \sigma_s \sqrt{ky}) \left| \phi'\left(\frac{t-y}{\sqrt{m/k}}\right) \right| \pi_s(dx) \frac{dy}{\sqrt{m/k}} \right] \frac{dt}{\sqrt{m/k}} \\ &= \int_{\mathbb{R}_+} \left[\int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, \sigma_s \sqrt{my'}) |\phi'(t'-y')| \pi_s(dx) dy' \right] dt' \\ &\leq c \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, \sigma_s \sqrt{my'}) \pi_s(dx) dy' \\ &= \frac{c}{\sqrt{m}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, y) \pi_s(dx) dy = \frac{c}{\sqrt{m}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}. \end{aligned} \quad (6.34)$$

For the first term in the right hand side of (6.32), by the definition of F_m (cf. (6.33)), using integration by parts, a change of variable and Fubini's theorem, we deduce that, with $\delta_n = \frac{m}{n}$,

$$\begin{aligned} \int_{\mathbb{R}_+} F'_m(t) (1 - \Phi^+(t)) dt &= \int_{\mathbb{R}_+} F_m(t) \phi^+(t) dt \\ &= \int_{\mathbb{R}_+} \left[\int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, \sigma_s \sqrt{ky}) \phi\left(\frac{t-y}{\sqrt{m/k}}\right) \pi_s(dx) \frac{dy}{\sqrt{m/k}} \right] \phi^+(t) dt \\ &= \int_{\mathbb{R}_+} \left[\int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, \sigma_s \sqrt{ny'}) \frac{1}{\sqrt{m/k}} \phi\left(\frac{t'-y'}{\sqrt{m/n}}\right) \pi_s(dx) \frac{dy'}{\sqrt{k/n}} \right] \phi^+\left(\frac{t'}{\sqrt{k/n}}\right) \frac{dt'}{\sqrt{k/n}} \\ &= \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, \sigma_s \sqrt{ny'}) \phi_{\delta_n} * \phi_{1-\delta_n}^+(y') \pi_s(dx) dy' \\ &= \frac{1}{\sigma_s \sqrt{n}} \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, t) \phi_{\delta_n} * \phi_{1-\delta_n}^+\left(\frac{t}{\sigma_s \sqrt{n}}\right) \pi_s(dx) dt. \end{aligned} \quad (6.35)$$

Using Lemma 6.9 with $v = \delta_n$ and recalling that $\delta_n = \frac{m}{n}$ and $1 - \delta_n = \frac{k}{n}$, we have, for any $t \in \mathbb{R}$,

$$\phi_{\delta_n} * \phi_{1-\delta_n}^+\left(\frac{t}{\sigma_s \sqrt{n}}\right) \leq \sqrt{\frac{k}{n}} \phi^+\left(\frac{t}{\sigma_s \sqrt{n}}\right) + \sqrt{\frac{m}{n}} + \frac{|t|}{\sigma_s \sqrt{n}} \mathbb{1}_{\{t < 0\}}.$$

Implementing this bound into (6.35) and using the fact that $\phi^+(u) = 0$ for $u \leq 0$, we get

$$\begin{aligned} \int_{\mathbb{R}_+} F'_m(t)(1 - \Phi^+(t))dt &\leq \frac{\sqrt{k}}{\sigma_s n} \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_\varepsilon(x, t) \phi^+ \left(\frac{t}{\sigma_s \sqrt{n}} \right) \pi_s(dx) dt \\ &\quad + \frac{\sqrt{m}}{\sigma_s n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c}{n} \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x, t) |t| \mathbf{1}_{\{t < 0\}} \pi_s(dx) dt \\ &\leq \frac{\sqrt{k}}{\sigma_s n} \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_\varepsilon(x, t) \phi^+ \left(\frac{t}{\sigma_s \sqrt{n}} \right) \pi_s(dx) dt + \frac{c\varepsilon^{1/4}}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}, \end{aligned} \quad (6.36)$$

where in the last inequality we used the fact that $m = \lceil \varepsilon^{1/2} n \rceil$ and the function h_ε is compactly supported on $\mathbb{P}(V) \times [-\varepsilon, \infty)$. Combining (6.32), (6.34) and (6.36), and using the fact that $V_s(x, y) \leq c(1 + y)$, we derive that

$$\begin{aligned} J_n(x, y) &\leq \frac{2V_s(x, y)}{n\sigma_s^2\sqrt{2\pi}} \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_\varepsilon(x, t) \phi^+ \left(\frac{t}{\sigma_s \sqrt{n}} \right) \pi_s(dx) dt \\ &\quad + \frac{c\varepsilon^{1/4}(1 + y)}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon(1 + y)}{n} (\alpha_n + n^{-\varepsilon}) \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}. \end{aligned} \quad (6.37)$$

Substituting this into (6.30) ends the proof of the upper bound (6.25). \square

We next establish the lower bound (6.26) of Theorem 6.8.

Proof of (6.26). Let us keep the notation used in the proof of the upper bound (6.25). By the Markov property of the couple (X_n, S_n) , we get

$$I_n(x, y) := \mathbb{E}_{\mathbb{Q}_s^x} [h(X_n, y - S_n); \tau_y > n] = I_{n,1}(x, y) - I_{n,2}(x, y), \quad (6.38)$$

where

$$\begin{aligned} I_{n,1}(x, y) &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m)] \\ &\quad \times \mathbb{Q}_s^x (X_k \in dx', y - S_k \in dy', \tau_y > k), \end{aligned} \quad (6.39)$$

$$\begin{aligned} I_{n,2}(x, y) &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m); \tau_{y'} \leq m] \\ &\quad \times \mathbb{Q}_s^x (X_k \in dx', y - S_k \in dy', \tau_y > k). \end{aligned} \quad (6.40)$$

Lower bound of $I_{n,1}(x, y)$. By the local limit theorem ((5.31) of Theorem 5.11), there exist constants $c, c_\varepsilon > 0$ such that for any $m \geq 1$, $x' \in \mathbb{P}(V)$ and $y' \in \mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m)] \geq H_m^-(y') - \frac{c\varepsilon}{\sqrt{m}} \|h\|_{\pi_s \otimes \text{Leb}} - \frac{c_\varepsilon}{m} \|h\|_{\mathcal{H}}, \quad (6.41)$$

where

$$H_m^-(y') = \int_{\mathbb{P}(V) \times \mathbb{R}} h_{-\varepsilon}(x, y) \frac{1}{\sigma_s \sqrt{m}} \phi \left(\frac{y' - y}{\sigma_s \sqrt{m}} \right) \pi_s(dx) dy.$$

Following the proof of the upper bound of $J_n(x, y)$ (cf. (6.31) and (6.37)) and using the lower bound in Lemma 6.9 instead of the upper one, one has, uniformly in $x \in \mathbb{P}(V)$ and

$y \in [0, \alpha_n \sqrt{n}]$,

$$\begin{aligned} J_n^-(x, y) &:= \int_{\mathbb{R}_+} H_m^-(y') \mathbb{Q}_s^x (y - S_k \in dy', \tau_y > k) \\ &\geq \frac{2V_s(x, y)}{n\sigma_s^2 \sqrt{2\pi}} \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_{-\varepsilon}(x, t) \phi^+ \left(\frac{t}{\sigma_s \sqrt{n}} \right) \pi_s(dx) dt \\ &\quad - \frac{c\varepsilon^{1/4}(1+y)}{n} \|h_{-\varepsilon}\|_{\pi_s \otimes \text{Leb}} - \frac{c_\varepsilon(1+y)}{n} (\alpha_n + n^{-\varepsilon}) \|h_{-\varepsilon}\|_{\pi_s \otimes \text{Leb}}. \end{aligned} \quad (6.42)$$

Substituting (6.41) into (6.39), using (6.42), the bound $\mathbb{Q}_s^x(\tau_y > k) \leq \frac{c(1+y)}{\sqrt{n}}$ and the fact that $\|h_{-\varepsilon}\|_{\pi_s \otimes \text{Leb}} \leq \|h\|_{\pi_s \otimes \text{Leb}}$ and $\|h_{-\varepsilon}\|_{\mathcal{H}} \leq \|h\|_{\mathcal{H}}$, we get

$$\begin{aligned} I_{n,1}(x, y) &\geq \frac{2V_s(x, y)}{n\sigma_s^2 \sqrt{2\pi}} \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_{-\varepsilon}(x, t) \phi^+ \left(\frac{t}{\sigma_s \sqrt{n}} \right) \pi_s(dx) dt \\ &\quad - \left(c\varepsilon^{1/4} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{1+y}{n} \|h\|_{\pi_s \otimes \text{Leb}} - \frac{c_\varepsilon(1+y)}{n^{3/2}} \|h\|_{\mathcal{H}}. \end{aligned} \quad (6.43)$$

Upper bound of $I_{n,2}(x, y)$. According to whether the value of y' in the integral is less or greater than $\varepsilon^{1/6} \sqrt{n}$, we decompose $I_{n,2}(x, y)$ into two terms:

$$I_{n,2}(x, y) = J_{n,1}(x, y) + J_{n,2}(x, y), \quad (6.44)$$

where

$$\begin{aligned} J_{n,1}(x, y) &= \int_0^{\varepsilon^{1/6} \sqrt{n}} \int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m); \tau_{y'} \leq m] \\ &\quad \times \mathbb{Q}_s^x (X_k \in dx', y - S_k \in dy', \tau_y > k), \\ J_{n,2}(x, y) &= \int_{\varepsilon^{1/6} \sqrt{n}}^\infty \int_{\mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m); \tau_{y'} \leq m] \\ &\quad \times \mathbb{Q}_s^x (X_k \in dx', y - S_k \in dy', \tau_y > k). \end{aligned}$$

For $J_{n,1}(x, y)$, since $\mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m); \tau_{y'} \leq m] \leq \mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m)]$, using Theorem 5.11 and proceeding in the same way as in (6.28) and (6.30), we get

$$J_{n,1}(x, y) \leq K_n(x, y) + \frac{c\varepsilon^{3/4}(1+y)}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon(1+y)}{n^{3/2}} \|h_\varepsilon\|_{\mathcal{H}}, \quad (6.45)$$

where

$$K_n(x, y) = \int_0^{\varepsilon^{1/6} \sqrt{n}} H_m(y') \mathbb{Q}_s^x (y - S_k \in dy', \tau_y > k)$$

and H_m is defined by (6.29). By integration by parts, it follows that

$$\begin{aligned} K_n(x, y) &= \int_0^\infty H_m(y') \mathbb{Q}_s^x (y - S_k \in dy', y - S_k \leq \varepsilon^{1/6} \sqrt{n}, \tau_y > k) \\ &= \int_0^\infty H'_m(y') \mathbb{Q}_s^x (y - S_k \in [0, \varepsilon^{1/6} \sqrt{n}], y - S_k > y', \tau_y > k) dy' \\ &= \int_0^{\varepsilon^{1/6} \sqrt{n}} H'_m(t) \mathbb{Q}_s^x \left(\frac{y - S_k}{\sigma \sqrt{k}} \in \left[\frac{t}{\sigma_s \sqrt{k}}, \frac{\varepsilon^{1/6} \sqrt{n}}{\sigma_s \sqrt{k}} \right], \tau_y > k \right) dt. \end{aligned} \quad (6.46)$$

Applying the conditioned integral limit theorem (Theorem 4.7), we derive that uniformly in $t \in \mathbb{R}_+$, $x \in \mathbb{P}(V)$ and $y \in [0, \alpha_n \sqrt{n}]$,

$$\left| \mathbb{Q}_s^x \left(\frac{y - S_k}{\sigma_s \sqrt{k}} \in \left[\frac{t}{\sigma_s \sqrt{k}}, \frac{\varepsilon^{1/6} \sqrt{n}}{\sigma_s \sqrt{k}} \right], \tau_y > k \right) - \frac{2V_s(x, y)}{\sigma_s \sqrt{2\pi k}} \left[\Phi^+ \left(\frac{\varepsilon^{1/6} \sqrt{n}}{\sigma_s \sqrt{k}} \right) - \Phi^+ \left(\frac{t}{\sigma_s \sqrt{k}} \right) \right] \right| \leq c_\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{1+y}{\sqrt{n}}.$$

By the definition of H_m (cf. (6.29)) and the fact that ϕ' is bounded, it holds that

$$\int_0^{\varepsilon^{1/6} \sqrt{n}} |H'_m(t)| dt \leq \varepsilon^{1/6} \sqrt{n} \sup_{t \in \mathbb{R}} |H'_m(t)| \leq \frac{c\varepsilon^{1/6} \sqrt{n}}{m} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} \leq \frac{c_\varepsilon}{\sqrt{n}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}.$$

Therefore, we obtain

$$\begin{aligned} K_n(x, y) &\leq \frac{2V_s(x, y)}{\sigma_s \sqrt{2\pi k}} \int_0^{\varepsilon^{1/6} \sqrt{n}} H'_m(t) \left[\Phi^+ \left(\frac{\varepsilon^{1/6} \sqrt{n}}{\sigma_s \sqrt{k}} \right) - \Phi^+ \left(\frac{t}{\sigma_s \sqrt{k}} \right) \right] dt \\ &\quad + c_\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{1+y}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}. \end{aligned} \quad (6.47)$$

Using integration by parts and the fact that $H_m(0) \geq 0$ and $\sup_{t \in \mathbb{R}} H_m(t) \leq \frac{1}{\sigma_s \sqrt{m}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}$ (cf. (6.29)), we get

$$\begin{aligned} &\int_0^{\varepsilon^{1/6} \sqrt{n}} H'_m(t) \left[\Phi^+ \left(\frac{\varepsilon^{1/6} \sqrt{n}}{\sigma_s \sqrt{k}} \right) - \Phi^+ \left(\frac{t}{\sigma_s \sqrt{k}} \right) \right] dt \\ &= -H_m(0) \Phi^+ \left(\frac{\varepsilon^{1/6} \sqrt{n}}{\sigma_s \sqrt{k}} \right) + \frac{1}{\sigma_s \sqrt{k}} \int_0^{\varepsilon^{1/6} \sqrt{n}} H_m(t) \phi^+ \left(\frac{t}{\sigma_s \sqrt{k}} \right) dt \\ &\leq \frac{1}{\sigma_s^2 \sqrt{mk}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} \int_0^{\varepsilon^{1/6} \sqrt{n}} \phi^+ \left(\frac{t}{\sigma_s \sqrt{k}} \right) dt =: A_n. \end{aligned} \quad (6.48)$$

By a change of variable and the fact that $\phi^+(u) \leq u$ for $u \geq 0$, it follows that

$$A_n = \frac{1}{\sigma_s \sqrt{m}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} \int_0^{\frac{\varepsilon^{1/6}}{\sigma_s} \sqrt{n/k}} \phi^+(u) du \leq \frac{c\varepsilon^{1/3}}{\sqrt{m}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} \leq \frac{c\varepsilon^{1/12}}{\sqrt{n}} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}.$$

This, together with (6.47) and (6.48), implies that

$$\begin{aligned} K_n(x, y) &\leq c\varepsilon^{1/12} \frac{V_s(x, y)}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + c_\varepsilon (\alpha_n + n^{-\varepsilon}) \frac{1+y}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} \\ &\leq \left(c\varepsilon^{1/12} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{1+y}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}. \end{aligned}$$

Combining this with (6.45), we obtain

$$J_{n,1}(x, y) \leq \left(c\varepsilon^{1/12} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{1+y}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon (1+y)}{n^{3/2}} \|h_\varepsilon\|_{\mathcal{H}}. \quad (6.49)$$

Now we proceed to give an upper bound for $J_{n,2}(x, y)$, which can be rewritten as

$$J_{n,2}(x, y) = \int_{\mathbb{P}(V) \times \mathbb{R}} L(x', y') \mathbb{Q}_s^x (X_k \in dx', y - S_k \in dy', \tau_y > k), \quad (6.50)$$

where, for $x' \in \mathbb{P}(V)$ and $y' \in \mathbb{R}$,

$$L(x', y') := \mathbf{1}_{\{y' > \varepsilon^{1/6} \sqrt{n}\}} \mathbb{E}_{\mathbb{Q}_s^{x'}} [h(X_m, y' - S_m); \tau_{y'} \leq m]. \quad (6.51)$$

Since $h(x', \cdot)$ is integrable on \mathbb{R} , by Fubini's theorem and a change of variable, it is easy to check that the function $y' \mapsto L(x', y')$ is integrable on \mathbb{R} , for any $x' \in \mathbb{P}(V)$. Denote for any $x' \in \mathbb{P}(V)$ and $y' \in \mathbb{R}$,

$$L_\varepsilon(x', y') := \mathbf{1}_{\{y'+\varepsilon > \varepsilon^{1/6} \sqrt{n}\}} \mathbb{E}_{\mathbb{Q}_s^{x'}} [h_\varepsilon(X_m, y' - S_m); \tau_{y'-\varepsilon} \leq m]. \quad (6.52)$$

Then we have $L \leq_\varepsilon L_\varepsilon$. Using the upper bound (6.25) of Theorem 6.8 and the fact that $\|L_\varepsilon\|_{\pi_s \otimes \text{Leb}} \leq \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}$, we obtain that uniformly in $x \in \mathbb{P}(V)$ and $y \in [0, \alpha_n \sqrt{n}]$,

$$\begin{aligned} J_{n,2}(x, y) &\leq \frac{2V_s(x, y)}{k\sigma_s^2 \sqrt{2\pi}} \int_{\mathbb{P}(V) \times \mathbb{R}_+} L_\varepsilon(x', y') \phi^+ \left(\frac{y'}{\sigma_s \sqrt{k}} \right) \pi_s(dx') dy' \\ &\quad + \left(c\varepsilon^{1/4} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{(1+y)}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon(1+y)}{n^{3/2}} \|L_\varepsilon\|_{\mathcal{H}}. \end{aligned} \quad (6.53)$$

For the first term, in view of (6.52), we use the duality formula (Lemma 4.4) to derive that

$$\begin{aligned} &\int_{\mathbb{P}(V) \times \mathbb{R}_+} L_\varepsilon(x, y) \phi^+ \left(\frac{y}{\sigma_s \sqrt{k}} \right) \pi_s(dx) dy \\ &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{E}_{\mathbb{Q}_s^x} [h_\varepsilon(X_m, y - S_m); \tau_{y-\varepsilon} \leq m] \mathbf{1}_{\{y+\varepsilon > \varepsilon^{1/6} \sqrt{n}\}} \phi^+ \left(\frac{y}{\sigma_s \sqrt{k}} \right) \pi_s(dx) dy \\ &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{E}_{\mathbb{Q}_s^x} [h_\varepsilon(X_m, y + \varepsilon - S_m); \tau_y \leq m] \mathbf{1}_{\{y+2\varepsilon > \varepsilon^{1/6} \sqrt{n}\}} \phi^+ \left(\frac{y+\varepsilon}{\sigma_s \sqrt{k}} \right) \pi_s(dx) dy \\ &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{E}_{\mathbb{Q}_s^{x,*}} \left[\phi^+ \left(\frac{z - S_m^* + \varepsilon}{\sigma_s \sqrt{k}} \right) \mathbf{1}_{\{z - S_m^* + 2\varepsilon > \varepsilon^{1/6} \sqrt{n}\}}; \tau_z^* \leq m \right] h_\varepsilon(x, z) \pi_s(dx) dz, \end{aligned} \quad (6.54)$$

where (S_m^*) is the dual random walk defined by (4.13). Since ϕ^+ is bounded by 1, by the martingale approximation (Lemma 5.6) and the fact that n is large enough so that $\varepsilon^{1/6} \sqrt{n} \geq c$ for some constant $c > 0$, we get

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_s^{x,*}} \left[\phi^+ \left(\frac{z - S_m^* + \varepsilon}{\sigma_s \sqrt{k}} \right) \mathbf{1}_{\{z - S_m^* + 2\varepsilon > \varepsilon^{1/6} \sqrt{n}\}}; \tau_z^* \leq m \right] \\ &\leq \mathbb{Q}_s^{x,*} \left(z - S_m^* + 2\varepsilon > \varepsilon^{1/6} \sqrt{n}, \min_{1 \leq j \leq m} (z - S_j^*) < 0 \right) \\ &\leq \mathbb{Q}_s^{x,*} \left(z - S_m^* - \min_{1 \leq j \leq m} (z - S_j^*) > \varepsilon^{1/6} \sqrt{n} - 2\varepsilon \right) \\ &\leq \mathbb{Q}_s^{x,*} \left(\max_{1 \leq j \leq m} (S_j^* - S_m^*) > \varepsilon^{1/6} \sqrt{n} - 2\varepsilon \right) \\ &\leq \mathbb{Q}_s^{x,*} \left(\max_{1 \leq j \leq m} |M_j^*| > \frac{1}{2} \varepsilon^{1/6} \sqrt{n} \right). \end{aligned} \quad (6.55)$$

Using Lemma 6.10 with n replaced by $m = \lceil \varepsilon^{1/2} n \rceil$, $u = v = \frac{1}{2} \varepsilon^{1/6} \sqrt{n}$ and $w = \varepsilon^{5/12} n$, Markov's inequality and the fact that $\sup_{1 \leq i \leq m} \mathbb{E}_{\mathbb{Q}_s^{x,*}} (\xi_i^2) \leq c$ for some constant $c > 0$, we

obtain

$$\begin{aligned}
& \mathbb{Q}_s^{x,*} \left(\max_{1 \leq j \leq m} |M_j^*| > \frac{1}{2} \varepsilon^{1/6} \sqrt{n} \right) \\
& \leq \sum_{i=1}^m \mathbb{Q}_s^{x,*} \left(|\xi_i^*| > \frac{1}{2} \varepsilon^{1/6} \sqrt{n} \right) + 2 \mathbb{Q}_s^{x,*} \left(\sum_{i=1}^m \mathbb{E}_{\mathbb{Q}_s^{x,*}} \left(\xi_i^2 \middle| \mathcal{F}_{i-1} \right) > \varepsilon^{5/12} n \right) + c\varepsilon^{1/12} \\
& \leq 4 \frac{m}{\varepsilon^{1/3} n} \sup_{1 \leq i \leq m} \mathbb{E}_{\mathbb{Q}_s^{x,*}} (\xi_i^2) + 2 \frac{m}{\varepsilon^{5/12} n} \sup_{1 \leq i \leq m} \mathbb{E}_{\mathbb{Q}_s^{x,*}} (\xi_i^2) + c\varepsilon^{1/12} \leq c\varepsilon^{1/12}.
\end{aligned} \tag{6.56}$$

Combining (6.54), (6.55) and (6.56), we get

$$\int_{\mathbb{P}(V) \times \mathbb{R}_+} L_\varepsilon(x, y) \phi^+ \left(\frac{y}{\sigma_s \sqrt{k}} \right) \pi_s(dx) dy \leq c\varepsilon^{1/12} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}. \tag{6.57}$$

For the last term in (6.53), by the definition of L_ε (cf. (6.52)) and τ_y (cf. (4.12)), we have

$$\begin{aligned}
\|L_\varepsilon\|_{\mathcal{H}} &= \int_{\mathbb{R}} \mathbf{1}_{\{y'+\varepsilon > \varepsilon^{1/6} \sqrt{n}\}} \sup_{x' \in \mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^{x'}} [h_\varepsilon(X_m, y' - S_m); \tau_{y'-\varepsilon} \leq m] dy' \\
&\leq \sup_{x \in \mathbb{P}(V)} \sup_{y \in \mathbb{R}} h_\varepsilon(x, y) \int_{\frac{1}{2} \varepsilon^{1/6} \sqrt{n}}^\infty \sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'} (\tau_{y'-\varepsilon} \leq m) dy' \\
&\leq 2 \sup_{x \in \mathbb{P}(V)} \sup_{y \in \mathbb{R}} h_\varepsilon(x, y) \int_{\frac{1}{4} \varepsilon^{1/6} \sqrt{n}}^\infty \sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'} \left(\max_{1 \leq j \leq m} |S_j| \geq t \right) dt.
\end{aligned}$$

By the martingale approximation (Lemma 5.6), Doob's martingale maximal inequality and Lemma 5.7, we get that for $t \geq \frac{1}{4} \varepsilon^{1/6} \sqrt{n}$ and $\delta > 0$,

$$\begin{aligned}
\sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'} \left(\max_{1 \leq j \leq m} |S_j| \geq t \right) &\leq \sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'} \left(\max_{1 \leq j \leq m} |M_j| \geq \frac{t}{2} \right) \\
&\leq \frac{c}{t^{2+\delta}} \sup_{x' \in \mathbb{P}(V)} \mathbb{E}_{\mathbb{Q}_s^{x'}} (M_m^{2+\delta}) \leq \frac{c' m^{(2+\delta)/2}}{t^{2+\delta}} \leq \frac{c' (\varepsilon^{1/2} n)^{1+\frac{\delta}{2}}}{t^{2+\delta}},
\end{aligned}$$

so that

$$\|L_\varepsilon\|_{\mathcal{H}} \leq c' (\varepsilon^{1/2} n)^{1+\frac{\delta}{2}} \sup_{x \in \mathbb{P}(V)} \sup_{y \in \mathbb{R}} h_\varepsilon(x, y) \int_{\frac{1}{4} \varepsilon^{1/6} \sqrt{n}}^\infty \frac{1}{t^{2+\delta}} dt \leq c\varepsilon^{1/3} \sqrt{n} \sup_{x \in \mathbb{P}(V)} \sup_{y \in \mathbb{R}} h_\varepsilon(x, y). \tag{6.58}$$

Substituting (6.57) and (6.58) into (6.53) gives

$$J_{n,2}(x, y) \leq \left(c\varepsilon^{1/12} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{1+y}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon(1+y)}{n} \sup_{x \in \mathbb{P}(V)} \sup_{y \in \mathbb{R}} h_\varepsilon(x, y). \tag{6.59}$$

From (6.44), (6.49) and (6.59), we get the upper bound for $I_{n,2}(x, y)$:

$$\begin{aligned}
I_{n,2}(x, y) &\leq \left(c\varepsilon^{1/12} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{1+y}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon(1+y)}{n^{3/2}} \|h_\varepsilon\|_{\mathcal{H}} \\
&\quad + \frac{c_\varepsilon(1+y)}{n} \sup_{x \in \mathbb{P}(V)} \sup_{y \in \mathbb{R}} h_\varepsilon(x, y).
\end{aligned}$$

Combining this with (6.38) and (6.43), and using the fact that $\|h\|_{\pi_s \otimes \text{Leb}} \leq \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}$ and $\|h\|_{\mathcal{H}} \leq \|h_\varepsilon\|_{\mathcal{H}}$, we conclude the proof of the lower bound (6.26). \square

From Theorem 6.8, we shall establish the following result, which plays a crucial role for proving Theorem 2.2.

Proposition 6.11. *Assume conditions [A2](#), [A3](#) and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Let $(\alpha_n)_{n \geq 1}$ be any sequence of positive numbers satisfying $\lim_{n \rightarrow \infty} \alpha_n = 0$. Let $(a_n)_{n \geq 1}$ be any sequence of nonnegative numbers satisfying $\limsup_{n \rightarrow \infty} \frac{a_n}{\sqrt{n}} < \infty$. Then, for any $\Delta_0 > 0$ and $b \in \mathbb{R}_+$, there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$, $y \in [0, \alpha_n \sqrt{n}]$, $\Delta_n \in [\Delta_0, b\sqrt{n}]$ and any measurable set $A \subseteq \mathbb{P}(V)$ satisfying $\pi_s(\partial A) = 0$,*

$$\begin{aligned} J(x, y) &:= \mathbb{Q}_s^x \left(X_{2n} \in A, \min_{n < j \leq 2n} (y - S_j) \geq a_n, y - S_{2n} \in [a_n, a_n + \Delta_n], \tau_y > n \right) \\ &\geq \frac{c(1+y)}{n^{3/2}} \Delta_n^2 \pi_s(A). \end{aligned}$$

Proof. By the Markov property of the couple (X_n, S_n) , we have for any $n \geq 1$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{R}$,

$$\begin{aligned} J(x, y) &= \int_{\mathbb{P}(V) \times \mathbb{R}_+} h(x', y') \mathbb{Q}_s^x (X_n \in dx', y - S_n \in dy', \tau_y > n) \\ &= \mathbb{E}_{\mathbb{Q}_s^x} [h(X_n, y - S_n); \tau_y > n], \end{aligned}$$

where for brevity we denote for any $x' \in \mathbb{P}(V)$ and $y' \geq 0$,

$$h(x', y') = \mathbb{Q}_s^{x'} \left(X_n \in A, \min_{1 \leq j \leq n} (y' - S_j) \geq a_n, y' - S_n \in [a_n, a_n + \Delta_n] \right). \quad (6.60)$$

Applying the conditioned local limit theorem (cf. [\(6.26\)](#) of [Theorem 6.8](#)), we get

$$J(x, y) \geq J_1(x, y) - J_2(y) - J_3(y) - J_4(y), \quad (6.61)$$

where, with $h_{-\varepsilon} \leq_\varepsilon h \leq_\varepsilon h_\varepsilon$,

$$\begin{aligned} J_1(x, y) &= \frac{2V_s(x, y)}{n\sigma_s^2\sqrt{2\pi}} \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_{-\varepsilon}(x', y') \phi^+ \left(\frac{y'}{\sigma_s \sqrt{n}} \right) \pi_s(dx') dy', \\ J_2(y) &= \left(c\varepsilon^{1/12} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{1+y}{n} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}}, \\ J_3(y) &= \frac{c_\varepsilon(1+y)}{n^{3/2}} \|h_\varepsilon\|_{\mathcal{H}}, \\ J_4(y) &= \frac{c_\varepsilon(1+y)}{n} \sup_{x \in \mathbb{P}(V)} \sup_{y \in \mathbb{R}} h_\varepsilon(x, y). \end{aligned}$$

Lower bound of $J_1(x, y)$. By the definition of h (cf. [\(6.60\)](#)), we have

$$\begin{aligned} h_{-\varepsilon}(x', y') &\geq \mathbb{Q}_s^{x'} \left(X_n \in A, y' - S_n \in [a_n + \varepsilon, a_n + \Delta_n - \varepsilon], \min_{1 \leq j \leq n} (y' - S_j) \geq a_n + \varepsilon \right) \\ &= \mathbb{Q}_s^{x'} (X_n \in A, y' - a_n - \varepsilon - S_n \in [0, \Delta_n - 2\varepsilon], \tau_{y' - a_n - \varepsilon} > n). \end{aligned}$$

Hence, by a change of variable $y' - a_n - \varepsilon = t$ and the duality lemma (cf. Lemma 4.4), we get

$$\begin{aligned}
& \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_{-\varepsilon}(x', y') \phi^+ \left(\frac{y'}{\sigma_s \sqrt{n}} \right) \pi_s(dx') dy' \\
& \geq \int_{\mathbb{P}(V)} \int_{-a_n - \varepsilon}^{\infty} \mathbb{Q}_s^{x'} (X_n \in A, t - S_n \in [0, \Delta_n - 2\varepsilon], \tau_t > n) \phi^+ \left(\frac{t + a_n + \varepsilon}{\sigma_s \sqrt{n}} \right) dt \pi_s(dx') \\
& \geq \int_{\mathbb{P}(V) \times \mathbb{R}_+} \mathbb{Q}_s^{x'} (X_n \in A, t - S_n \in [0, \Delta_n - 2\varepsilon], \tau_t > n) \phi^+ \left(\frac{t + a_n + \varepsilon}{\sigma_s \sqrt{n}} \right) dt \pi_s(dx') \\
& = \int_A \int_0^{\Delta_n - 2\varepsilon} \mathbb{E}_{\mathbb{Q}_s^{x,*}} \left[\phi^+ \left(\frac{z - S_n^* + a_n + \varepsilon}{\sigma_s \sqrt{n}} \right); \tau_z^* > n \right] dz \pi_s(dx). \tag{6.62}
\end{aligned}$$

For brevity we denote for $z \geq 0$,

$$\psi(z) = \phi^+ \left(\frac{z + a_n + \varepsilon}{\sigma_s \sqrt{n}} \right).$$

Applying again (6.26) of Theorem 6.8 (for the dual random walk S_n^*), we obtain that for any $z \geq 0$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_s^{x,*}} \left[\phi^+ \left(\frac{z - S_n^* + a_n + \varepsilon}{\sigma_s \sqrt{n}} \right); \tau_z^* > n \right] &= \mathbb{E}_{\mathbb{Q}_s^{x,*}} [\psi(z - S_n^*); \tau_z^* > n] \\
&\geq K_1(x, z) - K_2(z), \tag{6.63}
\end{aligned}$$

where

$$\begin{aligned}
K_1(x, z) &= \frac{2V_s^*(x, z)}{n\sigma_s^2\sqrt{2\pi}} \int_{\mathbb{R}_+} \psi_{-\varepsilon}(t) \phi^+ \left(\frac{t}{\sigma_s \sqrt{n}} \right) dt \\
K_2(z) &= \left(c\varepsilon^{1/4} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{1+z}{n} \|\psi_\varepsilon\|_{\pi_s \otimes \text{Leb}} + \frac{c_\varepsilon(1+z)}{n^{3/2}} \|\psi_\varepsilon\|_{\mathcal{H}} + \frac{c_\varepsilon(1+z)}{n} \sup_{y \in \mathbb{R}} \psi_\varepsilon(y).
\end{aligned}$$

For the first term $K_1(x, z)$, since

$$\begin{aligned}
\int_{\mathbb{R}_+} \psi_{-\varepsilon}(t) \phi^+ \left(\frac{t}{\sigma_s \sqrt{n}} \right) dt &= \int_{\mathbb{R}_+} \phi_{-\varepsilon}^+ \left(\frac{t + a_n + \varepsilon}{\sigma_s \sqrt{n}} \right) \phi^+ \left(\frac{t}{\sigma_s \sqrt{n}} \right) dt \\
&= \sigma_s \sqrt{n} \int_{\mathbb{R}_+} \phi_{-\varepsilon}^+ \left(y + \frac{a_n + \varepsilon}{\sqrt{n}} \right) \phi^+(y) dy \\
&\geq c\sqrt{n},
\end{aligned}$$

we get that there exists a constant $c' > 0$ such that for any $x \in \mathbb{P}(V)$ and $z \in \mathbb{R}_+$,

$$K_1(x, z) \geq \frac{c'}{\sqrt{n}} V_s^*(x, z). \tag{6.64}$$

For the second term $K_2(z)$, since

$$\|\psi_\varepsilon\|_{\pi_s \otimes \text{Leb}} = \|\psi_\varepsilon\|_{\mathcal{H}} = \int_{\mathbb{R}} \phi_\varepsilon^+ \left(\frac{z + a_n + \varepsilon}{\sigma_s \sqrt{n}} \right) dz \leq c\sqrt{n}$$

and $\sup_{y \in \mathbb{R}} \psi_\varepsilon(y) \leq 1$, we have

$$K_2(z) \leq \left(c\varepsilon^{1/4} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{1+z}{\sqrt{n}}. \tag{6.65}$$

Substituting (6.63), (6.64) and (6.65) into (6.62), we get

$$\begin{aligned} \int_{\mathbb{P}(V) \times \mathbb{R}_+} h_{-\varepsilon}(x', y') \phi^+ \left(\frac{y'}{\sigma_s \sqrt{n}} \right) \pi_s(dx') dy' &\geq \frac{c}{\sqrt{n}} \int_A \int_0^{\Delta_n - 2\varepsilon} V_s^*(x, z) dz \pi_s(dx) \\ &\geq \frac{c'}{\sqrt{n}} \int_A \int_0^{\Delta_n/2} z dz \pi_s(dx) = \frac{c''}{\sqrt{n}} \Delta_n^2 \pi_s(A), \end{aligned}$$

where in the second inequality we used the fact that $\Delta_n > 4\varepsilon$ by taking $\varepsilon > 0$ sufficiently small and there exists a constant $c_1 > 0$ such that $\inf_{x \in \mathbb{P}(V)} V^*(x, z) > c_1 z$ for any $z \geq 0$. Thus, we get

$$J_1(x, y) \geq c \frac{1+y}{n^{3/2}} \Delta_n^2 \pi_s(A). \quad (6.66)$$

Upper bound of $J_2(y)$. By the definition of h (cf. (6.60)), we have

$$h_\varepsilon(x', y') \leq \mathbb{Q}_s^{x'}(X_n \in A, y' - a_n + \varepsilon - S_n \in [0, \Delta_n + 2\varepsilon], \tau_{y' - a_n + \varepsilon} > n). \quad (6.67)$$

By a change of variable $y' - a_n + \varepsilon = t$ and the duality lemma (cf. Lemma 4.4), we get

$$\begin{aligned} \|h_\varepsilon\|_{\pi_s \otimes \text{Leb}} &= \int_{\mathbb{P}(V) \times \mathbb{R}} h_\varepsilon(x', y') \pi_s(dx') dy' \\ &\leq \int_{\mathbb{P}(V)} \int_{-a_n}^{\infty} \mathbb{Q}_s^{x'}(X_n \in A, t - S_n \in [0, \Delta_n + 2\varepsilon], \tau_t > n) \pi_s(dx') dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{P}(V)} \mathbb{1}_{\{t \in [-a_n, \infty)\}} \mathbb{Q}_s^{x'}(X_n \in A, t - S_n \in [0, \Delta_n + 2\varepsilon], \tau_t > n - 1) \pi_s(dx') dt \\ &= \int_A \int_0^{\Delta_n + 2\varepsilon} \mathbb{Q}_s^{x,*}(z - S_n^* \in [-a_n, \infty), \tau_z^* > n - 1) dz \pi_s(dx) \\ &\leq \int_A \int_0^{\Delta_n + 2\varepsilon} \mathbb{Q}_s^{x,*}(\tau_z^* > n - 1) dz \pi_s(dx) \\ &\leq \frac{c}{\sqrt{n}} \Delta_n^2 \pi_s(A), \end{aligned}$$

where in the last inequality we used Theorem 4.8 and the fact that $\Delta_n \geq \Delta_0 > 0$ and $\varepsilon > 0$ is sufficiently small. Thus

$$J_2(y) \leq \left(c\varepsilon^{1/12} + c_\varepsilon \alpha_n + c_\varepsilon n^{-\varepsilon} \right) \frac{1+y}{n^{3/2}} \Delta_n^2 \pi_s(A). \quad (6.68)$$

Upper bound of $J_3(y)$. Using (6.67), a change of variable $y' - a_n + \varepsilon = t$ and Lemma 6.4, it follows that

$$\begin{aligned} \|h_\varepsilon\|_{\mathcal{H}} &= \int_{\mathbb{R}} \sup_{x' \in \mathbb{P}(V)} h_\varepsilon(x', y') dy' \\ &\leq \int_{\mathbb{R}} \sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'}(y' - a_n + \varepsilon - S_n \in [0, \Delta_n + 2\varepsilon], \tau_{y' - a_n + \varepsilon} > n) dy' \\ &= \int_{\mathbb{R}} \sup_{x' \in \mathbb{P}(V)} \mathbb{Q}_s^{x'}(t - S_n \in [0, \Delta_n + 2\varepsilon], \tau_t > n) dt \\ &\leq c_\varepsilon \frac{\Delta_n^2 \log^{1-\varepsilon} n}{\sqrt{n}}, \end{aligned}$$

where $\varepsilon \in [0, \frac{1}{2})$. Therefore,

$$J_3(y) = \frac{c_\varepsilon(1+y)}{n^{3/2}} \|h_\varepsilon\|_{\mathcal{H}} \leq c_\varepsilon(1+y) \frac{\Delta_n^2 \log^{1-\varepsilon} n}{n^{5/2}}. \quad (6.69)$$

Upper bound of $J_4(y)$. To give an upper bound for $h_\varepsilon(x', y')$, we consider two cases: when $y' \in [-\varepsilon, \eta\sqrt{n \log n}]$ and $y' \in (\eta\sqrt{n \log n}, \infty)$ with $\eta > 0$ whose value will be chosen to be sufficiently large. Using (6.67) and Lemma 6.7, we get that there exists a constant $c > 0$ such that for any $x' \in \mathbb{P}(V)$ and $y' \in [-\varepsilon, \eta\sqrt{n \log n}]$,

$$\begin{aligned} h_\varepsilon(x', y') &\leq \mathbb{Q}_s^{x'}(y' - a_n + \varepsilon - S_n \in [0, \Delta_n + 2\varepsilon], \tau_{y' - a_n + \varepsilon} > n) \\ &\leq c_{\varepsilon, \eta} \frac{\sqrt{n \log n}}{n^{3/2}} \Delta_n^2 = c_{\varepsilon, \eta} \frac{\sqrt{\log n}}{n} \Delta_n^2. \end{aligned} \quad (6.70)$$

In a similar way as in (6.2), by taking $\eta > 0$ to be sufficiently large and using the lower tail moderate deviation asymptotic for $-S_n$ under the changed measure $\mathbb{Q}_s^{x'}$, we derive that there exists a constants $c_\eta > 0$ such that for any $x' \in \mathbb{P}(V)$ and $y' \in (\eta\sqrt{n \log n}, \infty)$

$$h_\varepsilon(x', y') \leq \mathbb{Q}_s^{x'}\left(-S_n \leq -\frac{1}{2}\eta\sqrt{n \log n}\right) \leq \frac{c_\eta}{n}. \quad (6.71)$$

Combining (6.70) and (6.71) gives

$$J_4(y) \leq c_\varepsilon(1+y) \frac{\sqrt{\log n}}{n^2} \Delta_n^2. \quad (6.72)$$

Putting together (6.61), (6.66), (6.68), (6.69) and (6.72) concludes the proof of Proposition 6.11. \square

7. PROOF OF THEOREM 2.2

The proof of Theorem 2.2 is splitted into two parts: the upper bound (see Lemma 7.1 below) and the lower bound (see Lemma 7.4 below).

7.1. Proof of the upper bound. In this section we shall obtain the upper bound in Theorem 2.2. We only give a proof of (2.16) since the proof of (2.17) follows from that of (2.16) by taking $A = \mathbb{P}(V)$.

Lemma 7.1. *Assume conditions A1, A2, A3 and A4. Let $x \in \mathbb{P}(V)$. Then, there exists a constant $c > 0$ such that for any $\varepsilon \in (0, \frac{3}{2\alpha})$ and any Borel set $A \subseteq \mathbb{P}(V)$,*

$$I := \mathbb{P}\left(\frac{M_n^x(A)}{\log n} \geq -\frac{3}{2\alpha} + \varepsilon \mid \mathcal{S}\right) \leq c \frac{\log^3 n}{n^{\varepsilon\alpha}}. \quad (7.1)$$

Proof. Let $K > 1$. We write $I = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \mathbb{P}\left(\frac{M_n^x(A)}{\log n} \geq -\frac{3}{2\alpha} + \varepsilon, \max_{1 \leq i \leq n} M_i^x(A) \leq K \mid \mathcal{S}\right), \\ I_2 &= \mathbb{P}\left(\frac{M_n^x(A)}{\log n} \geq -\frac{3}{2\alpha} + \varepsilon, \max_{1 \leq i \leq n} M_i^x(A) > K \mid \mathcal{S}\right), \end{aligned}$$

with the notation $M_i^x(A) = \sup\{S_{u|i}^x : |u| = i, X_u^x \in A\}$ for $x \in \mathbb{P}(V)$ and $1 \leq i \leq n$. For the first term I_1 , we have $I_1 \leq \mathbb{E}(Z_n^x(A) \mid \mathcal{S})$, where, for $x \in \mathbb{P}(V)$,

$$Z_n^x(A) = \sum_{|u|=n} \mathbf{1}\left\{X_u^x \in A, \frac{S_u^x}{\log n} \geq -\frac{3}{2\alpha} + \varepsilon, S_{u|i}^x \leq K, \forall 1 \leq i \leq n\right\}.$$

Using the many-to-one formula (3.7), the fact that $\mathbf{m}(\alpha) = 1$ (cf. condition A4) and r_α is bounded and strictly positive on $\mathbb{P}(V)$, we get that there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$,

$$\begin{aligned} \mathbb{E}(Z_n^x(A)) &= r_\alpha(x) \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[\frac{1}{r_\alpha(X_n)} e^{-\alpha S_n} \mathbf{1}_{\{X_n \in A, S_n \geq (-\frac{3}{2\alpha} + \varepsilon) \log n, S_i \leq K, \forall 1 \leq i \leq n\}} \right] \\ &\leq cn^{-(\frac{3}{2\alpha} + \varepsilon)\alpha} \mathbb{Q}_\alpha^x \left(S_n \geq \left(-\frac{3}{2\alpha} + \varepsilon \right) \log n, S_i \leq K, \forall 1 \leq i \leq n \right) \\ &= cn^{\frac{3}{2} - \varepsilon\alpha} \mathbb{Q}_\alpha^x \left(K - S_n \in \left[0, K + \left(\frac{3}{2\alpha} - \varepsilon \right) \log n \right], \tau_K > n \right). \end{aligned}$$

Applying Lemma 6.7 and taking $K = a \log n$ with $a > 0$ (whose value will be chosen to be sufficiently large), we obtain

$$\mathbb{E}(Z_n^x(A)) \leq cn^{\frac{3}{2} - \varepsilon\alpha} (1 + K) \frac{[K + (\frac{3}{2\alpha} - \varepsilon) \log n]^2}{n^{3/2}} \leq c \frac{a^3 \log^3 n}{n^{\varepsilon\alpha}}.$$

Since $\mathbb{P}(\mathcal{S}) > 0$, it follows that

$$I_1 \leq \mathbb{E}(Z_n^x(A) \mid \mathcal{S}) \leq c \frac{a^3 \log^3 n}{n^{\varepsilon\alpha}}. \quad (7.2)$$

For the second term I_2 , by Markov's inequality, the many-to-one formula (3.7) and the fact that $\mathbf{m}(\alpha) = 1$, we have

$$\begin{aligned} I_2 &\leq c \mathbb{P} \left(\max_{1 \leq i \leq n} M_i^x(A) > K \right) \leq c \mathbb{P} \left(\max_{1 \leq i \leq n} \max_{|u|=i} S_{u|i}^x > K \right) \\ &\leq ce^{-\alpha K} \mathbb{E} \left(e^{\alpha \max_{1 \leq i \leq n} \max_{|u|=i} S_{u|i}^x} \right) \leq ce^{-\alpha K} \mathbb{E} \left(\max_{1 \leq i \leq n} \sum_{|u|=i} e^{\alpha S_{u|i}^x} \right) \\ &\leq ce^{-\alpha K} \sum_{i=1}^n \mathbb{E} \left(\sum_{|u|=i} e^{\alpha S_{u|i}^x} \right) = ce^{-\alpha K} r_\alpha(x) \sum_{i=1}^n \mathbb{E}_{\mathbb{Q}_\alpha^x} \left(\frac{1}{r_\alpha(X_i)} \right) \leq \frac{c}{n^{a\alpha-1}} \leq \frac{c}{n^{a\alpha/2}}, \quad (7.3) \end{aligned}$$

by taking $K = a \log n$ and $a > 0$ sufficiently large. From (7.2) and (7.3), we get (7.1). \square

7.2. Proof of the lower bound. In the proof of the lower bound we will use the following auxiliary assertion for the minimal position $m_n^x(A)$ defined in (2.7).

Lemma 7.2. *Assume conditions A1, A2 and A4. Let $x \in \mathbb{P}(V)$. Then, for any Borel set $A \subseteq \mathbb{P}(V)$ satisfying $\nu(A) > 0$, there exists a constant $c_0 > 0$ such that*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} m_n^x(A) \geq -c_0 \quad a.s. \quad (7.4)$$

Proof. Consider the random variable

$$Y_n = \sum_{|u|=n} \mathbf{1}_{\{X_u^x \in A, S_u^x \leq -c_0 n\}},$$

where $c_0 > 0$ will be chosen sufficiently large. Clearly, $Y_n \leq e^{-c_0 \delta n} \sum_{|u|=n} e^{-\delta S_u^x}$ for some constant $\delta > 0$. By Markov's inequality, the many-to-one formula (3.7) and the fact that

$\mathfrak{m}(\alpha) = 1$ (cf. condition **A4**), we get

$$\begin{aligned} \mathbb{P}(Y_n > 0) &= \mathbb{P}(Y_n \geq 1) \leq \mathbb{E}(Y_n) \leq e^{-c_0 \delta n} \mathbb{E} \left(\sum_{|u|=n} e^{-\delta S_u^x} \right) \\ &= e^{-c_0 \delta n} r_\alpha(x) \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[\frac{1}{r_\alpha(X_n)} e^{-(\alpha+\delta)S_n} \right] =: J. \end{aligned}$$

Using (3.5) and taking $\delta > 0$ sufficiently small and then $c_0 > 0$ sufficiently large, we obtain that there exists a constant $c > 0$ such that

$$J = e^{-c_0 \delta n} \frac{1}{\kappa(\alpha)^n} \mathbb{E}_x \left(e^{-\delta S_n} \right) \leq e^{-c_0 \delta n} \frac{1}{\kappa(\alpha)^n} \left(\mathbb{E} e^{\delta \log \|g^{-1}\|} \right)^n \leq e^{-cn}.$$

By the Borel-Cantelli lemma, it follows that $\lim_{n \rightarrow \infty} Y_n = 0$, a.s., and hence (7.4) holds. \square

We next proceed to give a lower bound for the lower tail of $M_n^x(A)$.

Lemma 7.3. *Assume conditions **A1**, **A2**, **A3** and **A4**. Then, for any $\Delta > 0$, there exists a constant $c > 0$ such that for any Borel set $A \subseteq \mathbb{P}(V)$ and $n \geq 2$,*

$$\mathbb{P} \left(M_n^x(A) \geq -\frac{3}{2\alpha} \log n - \Delta \right) \geq \frac{1}{c \log^3 n}. \quad (7.5)$$

Proof. Let $\Delta > 0$ be a fixed constant and set

$$\begin{aligned} U_n^x(A) &:= \sum_{|u|=2n} \mathbb{1}_{\{X_u^x \in A\}} \mathbb{1}_{\{S_u^x \in [-\frac{3}{2\alpha} \log n - \Delta, -\frac{3}{2\alpha} \log n]\}} \\ &\quad \times \mathbb{1}_{\left\{ \max_{1 \leq i \leq n} S_{u_i}^x \leq 0, \max_{n < j \leq 2n} S_{u_j}^x \leq -\frac{3}{2\alpha} \log n \right\}}. \end{aligned}$$

By the definition of $U_n^x(A)$ and $M_{2n}^x(A)$, it holds that

$$\mathbb{P} \left(M_{2n}^x(A) \geq -\frac{3}{2\alpha} \log n - \Delta \right) \geq \mathbb{P}(U_n^x(A) > 0). \quad (7.6)$$

For simplicity, denote for any integer $n \geq 1$,

$$I_k = I_k(n) = \begin{cases} (-\infty, 0] & 1 \leq k \leq n \\ (-\infty, -\frac{3}{2\alpha} \log n] & n < k < 2n \\ [-\frac{3}{2\alpha} \log n - \Delta, -\frac{3}{2\alpha} \log n] & k = 2n. \end{cases} \quad (7.7)$$

Then $U_n^x(A)$ can be rewritten as

$$U_n^x(A) = \sum_{|u|=2n} \mathbb{1}_{\{X_u^x \in A\}} \mathbb{1}_{\{S_{u|k}^x \in I_k, \forall 1 \leq k \leq 2n\}}.$$

Since $\mathbb{E}(U_n^x(A)) \geq 0$, by the Cauchy-Schwarz inequality, it holds that

$$\mathbb{P}(U_n^x(A) > 0) \geq \frac{(\mathbb{E}U_n^x(A))^2}{\mathbb{E}[(U_n^x(A))^2]}.$$

Hence, in order to give a lower bound for $\mathbb{P}(U_n^x(A) > 0)$, it suffices to provide a lower bound for $\mathbb{E}(U_n^x(A))$ and an upper bound for $\mathbb{E}[(U_n^x(A))^2]$. To deal with $\mathbb{E}(U_n^x(A))$, by the many-to-one formula (3.7) and the fact that $\mathfrak{m}(\alpha) = 1$ (cf. condition **A4**), we get

$$\mathbb{E}(U_n^x(A)) = r_\alpha(x) \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[\frac{1}{r_\alpha(X_{2n})} e^{-\alpha S_{2n}} \mathbb{1}_{\{X_{2n} \in A\}} \mathbb{1}_{\{S_k \in I_k, \forall 1 \leq k \leq 2n\}} \right].$$

Since r_α is strictly positive and bounded on $\mathbb{P}(V)$ and $S_{2n} \leq -\frac{3}{2\alpha} \log n$, there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$,

$$\begin{aligned} \mathbb{E}(U_n^x(A)) &\geq c \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[e^{-\alpha S_{2n}} \mathbb{1}_{\{X_{2n} \in A\}} \mathbb{1}_{\{S_k \in I_k, \forall 1 \leq k \leq 2n\}} \right] \\ &\geq cn^{3/2} \mathbb{Q}_\alpha^x(X_{2n} \in A, S_k \in I_k, \forall 1 \leq k \leq 2n) \\ &\geq c\Delta^2 \pi_s(A), \end{aligned} \tag{7.8}$$

where in the last inequality we used Proposition 6.11.

Next we are going to give an upper bound for $\mathbb{E}[(U_n^x(A))^2]$. By the definition of $U_n^x(A)$, we have

$$(U_n^x(A))^2 = \sum_{|u|=2n} \sum_{|w|=2n} \mathbb{1}_{\left\{X_u^x \in A, X_w^x \in A, S_{u|k}^x \in I_k, S_{w|k}^x \in I_k, \forall 1 \leq k \leq 2n\right\}}.$$

We can rearrange this sum by summing over the generation j of the last common ancestor z (with $|z| = j$) of u and w , to obtain

$$\mathbb{E}[(U_n^x(A))^2] = \mathbb{E}(U_n^x(A)) + Y_n(A).$$

The first summand corresponds to the case $u = w$, while the second summand is given by

$$\begin{aligned} Y_n(A) &= 2\mathbb{E} \left[\sum_{j=1}^{2n-1} \sum_{|z|=j} \mathbb{1}_{\left\{S_{z|i}^x \in I_i, \forall 1 \leq i \leq j\right\}} \right. \\ &\quad \left. \times \sum_{1 \leq l < m \leq N_z} \sum_{|u|=2n-j-1} \sum_{|w|=2n-j-1} \mathbb{1}_{\left\{X_{zlu}^x \in A, X_{zmw}^x \in A, S_{zlu|k}^x \in I_k, S_{zmw|k}^x \in I_k, \forall j < k \leq 2n\right\}} \right]. \end{aligned}$$

Here N_z denotes the number of children of the particle z , l and m denote different children of the last common ancestor z , the sums over u and w correspond to the paths leading from z to their children in generation $2n$. Note that N_z is independent of \mathcal{F}_j . Taking conditional expectation with respect to \mathcal{F}_{j+1} , we get

$$\begin{aligned} Y_n(A) &= 2\mathbb{E} \left[\sum_{j=1}^{2n-1} \sum_{|z|=j} \mathbb{1}_{\left\{S_{z|i}^x \in I_i, \forall 1 \leq i \leq j\right\}} \sum_{1 \leq l < m \leq N_z} \mathbb{1}_{\left\{S_{zl}^x \in I_{j+1}, S_{zm}^x \in I_{j+1}\right\}} \right. \\ &\quad \left. \times K_{j,n}(X_{zl}^x, S_{zl}^x) K_{j,n}(X_{zm}^x, S_{zm}^x) \right], \end{aligned} \tag{7.9}$$

where for $x' \in \mathbb{P}(V)$ and $y \in I_{j+1}$,

$$K_{j,n}(x', y) = \mathbb{E} \left[\sum_{|u|=2n-j-1} \mathbb{1}_{\left\{X_u^{x'} \in A, y + S_{u|l}^x \in I_{l+j+1}, \forall 1 \leq l \leq 2n-j-1\right\}} \right].$$

By the many-to-one formula (3.7) and the fact that $\mathfrak{m}(\alpha) = 1$ (cf. condition A4), we get

$$K_{j,n}(x', y) = r_\alpha(x') \mathbb{E}_{\mathbb{Q}_\alpha^{x'}} \left[(r_\alpha^{-1} \mathbb{1}_A)(X_{2n-j-1}) e^{-\alpha S_{2n-j-1}} \mathbb{1}_{\{y + S_l \in I_{l+j+1}, \forall 1 \leq l \leq 2n-j-1\}} \right].$$

Since $y + S_{2n-j-1} \in I_{2n}$, in view of (7.7), we have $S_{2n-j-1} \geq -\frac{3}{2\alpha} \log n - \Delta - y$. Since the eigenfunction r_α is strictly positive and bounded on $\mathbb{P}(V)$, there exists a constant $c > 0$ such that

$$\begin{aligned} K_{j,n}(x', y) &\leq cn^{3/2} e^{\alpha(\Delta+y)} \mathbb{Q}_\alpha^{x'}(y + S_l \in I_{l+j+1}, \forall 1 \leq l \leq 2n-j-1) \\ &\leq cn^{3/2} e^{\alpha y} L_{j,n}(x', y), \end{aligned}$$

where for $x' \in \mathbb{P}(V)$ and $y \in I_{j+1}$,

$$L_{j,n}(x', y) = \mathbb{Q}_\alpha^{x'} \left(-y - S_{2n-j-1} \in \left[\frac{3}{2\alpha} \log n, \frac{3}{2\alpha} \log n + \Delta \right], \tau_{-y} > 2n - j - 1 \right).$$

When $1 \leq j \leq n$, applying Lemma 6.6 with $n' = 2n - j - 1$, $a = \frac{3}{2\alpha} \log n$ and $b = \frac{3}{2\alpha} \log n + \Delta$, we obtain that there exists a constant $c > 0$ such that for any $x' \in \mathbb{P}(V)$ and $y \in I_{j+1}$,

$$L_{j,n}(x', y) \leq \frac{c}{(n')^{3/2}} (1-y)(b-a+1)(b+a+1) \leq \frac{c \log n}{n^{3/2}} (1-y).$$

When $n < j \leq 2n - 1$, we use the local limit theorem (5.30) to get that there exists a constant $c > 0$ such that for any $x' \in \mathbb{P}(V)$ and $y \in I_{j+1}$,

$$L_{j,n}(x', y) \leq \mathbb{Q}_\alpha^{x'} \left(-y - S_{2n-j-1} \in \left[\frac{3}{2\alpha} \log n, \frac{3}{2\alpha} \log n + \Delta \right] \right) \leq \frac{c}{(2n-j)^{1/2}}.$$

Therefore, we obtain that for $S_{zl}^x \in I_{j+1}$,

$$K_{j,n}(X_{zl}^x, S_{zl}^x) \leq \begin{cases} c \log n e^{\alpha S_{zl}^x} |1 - S_{zl}^x|, & 1 \leq j \leq n \\ c \frac{n^{3/2}}{(2n-j)^{1/2}} e^{\alpha S_{zl}^x}, & n < j \leq 2n - 1. \end{cases} \quad (7.10)$$

Similarly, for $K_{j,n}(X_{zm}^x, S_{zm}^x)$ we also have

$$K_{j,n}(X_{zm}^x, S_{zm}^x) \leq \begin{cases} c \log n e^{\alpha S_{zm}^x} |1 - S_{zm}^x|, & 1 \leq j \leq n \\ c \frac{n^{3/2}}{(2n-j)^{1/2}} e^{\alpha S_{zm}^x}, & n < j \leq 2n - 1. \end{cases} \quad (7.11)$$

Therefore, substituting (7.10) and (7.11) into (7.9), we obtain

$$\begin{aligned} Y_n(A) &\leq c \log^2 n \, 2\mathbb{E} \left\{ \sum_{j=1}^n \sum_{|z|=j} \mathbb{1}_{\{S_{zi}^x \in I_i, \forall 1 \leq i \leq j\}} \sum_{1 \leq l < m \leq N_z} \mathbb{1}_{\{S_{zl}^x \in I_{j+1}, S_{zm}^x \in I_{j+1}\}} \right. \\ &\quad \left. \times e^{\alpha S_{zl}^x + \alpha S_{zm}^x} |1 - S_{zl}^x| |1 - S_{zm}^x| \right\} \\ &\quad + 2cn^3 \mathbb{E} \left\{ \sum_{j=n+1}^{2n-1} \frac{1}{2n-j} \sum_{|z|=j} \mathbb{1}_{\{S_{zi}^x \in I_i, \forall 1 \leq i \leq j\}} \sum_{1 \leq l < m \leq N_z} e^{\alpha S_{zl}^x + \alpha S_{zm}^x} \right\}. \end{aligned}$$

Note that

$$\begin{aligned} &2 \sum_{1 \leq l < m \leq N_z} \mathbb{1}_{\{S_{zl}^x \in I_{j+1}, S_{zm}^x \in I_{j+1}\}} e^{\alpha S_{zl}^x + \alpha S_{zm}^x} |1 - S_{zl}^x| |1 - S_{zm}^x| \\ &\leq \left(\sum_{1 \leq l \leq N_z} \mathbb{1}_{\{S_{zl}^x \in I_{j+1}\}} e^{\alpha S_{zl}^x} |1 - S_{zl}^x| \right)^2 \end{aligned}$$

and

$$2 \sum_{1 \leq l < m \leq N_z} \mathbb{1}_{\{S_{zl}^x \in I_{j+1}, S_{zm}^x \in I_{j+1}\}} e^{\alpha S_{zl}^x + \alpha S_{zm}^x} \leq \left(\sum_{1 \leq l \leq N_z} \mathbb{1}_{\{S_{zl}^x \in I_{j+1}\}} e^{\alpha S_{zl}^x} \right)^2.$$

Taking conditional expectation, we get for any $1 \leq j \leq n$, $|z| = j$,

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{1 \leq l \leq N_z} \mathbf{1}_{\{S_{zl}^x \in I_{j+1}\}} e^{\alpha S_{zl}^x} (1 + |S_{zl}^x|) \right)^2 \middle| \mathcal{F}_j \right] \\ & \leq e^{2\alpha S_z^x} \mathbb{E} \left[\left(\sum_{1 \leq l \leq N_z} e^{\alpha(S_{zl}^x - S_z^x)} (1 + |S_z^x| + |S_{zl}^x - S_z^x|) \right)^2 \middle| \mathcal{F}_j \right] \\ & = e^{2\alpha S_z^x} M(S_z^x, X_z^x), \end{aligned}$$

where, for $s \in \mathbb{R}$ and $y \in \mathbb{P}(V)$,

$$\begin{aligned} M(s, y) &= \mathbb{E} \left(\sum_{|u|=1} e^{\alpha S_u^y} (1 + |s| + |S_u^y|) \right)^2 \\ &\leq 2(1 + |s|)^2 \mathbb{E} \left(\sum_{|u|=1} e^{\alpha S_u^y} \right)^2 + 2 \mathbb{E} \left(\sum_{|u|=1} e^{\alpha S_u^y} |S_u^y| \right)^2. \end{aligned}$$

By condition [A4](#), we have $M(s, y) \leq c(1 + |s|)^2$ and consequently

$$\mathbb{E} \left[\left(\sum_{1 \leq l \leq N_z} \mathbf{1}_{\{S_{zl}^x \in I_{j+1}\}} e^{\alpha S_{zl}^x} (1 + |S_{zl}^x|) \right)^2 \middle| \mathcal{F}_j \right] \leq c e^{2\alpha S_z^x} (1 + |S_z^x|)^2.$$

Similarly, it holds

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{1 \leq l \leq N_z} \mathbf{1}_{\{S_{zl}^x \in I_{j+1}\}} e^{\alpha S_{zl}^x} \right)^2 \middle| \mathcal{F}_j \right] &= e^{2\alpha S_z^x} \mathbb{E} \left[\left(\sum_{1 \leq l \leq N_z} \mathbf{1}_{\{S_{zl}^x \in I_{j+1}\}} e^{\alpha(S_{zl}^x - S_z^x)} \right)^2 \middle| \mathcal{F}_j \right] \\ &\leq c e^{2\alpha S_z^x}. \end{aligned}$$

It follows that

$$\begin{aligned} Y_n(A) &\leq c \log^2 n \sum_{j=1}^n \mathbb{E} \left\{ \sum_{|z|=j} \mathbf{1}_{\{S_{zi}^x \in I_i, \forall 1 \leq i \leq j\}} e^{2\alpha S_z^x} (1 + |S_z^x|)^2 \right\} \\ &\quad + c n^3 \sum_{j=n+1}^{2n-1} \frac{1}{2n-j} \mathbb{E} \left\{ \sum_{|z|=j} \mathbf{1}_{\{S_{zi}^x \in I_i, \forall 1 \leq i \leq j\}} e^{2\alpha S_z^x} \right\}. \end{aligned}$$

Using the many-to-one formula [\(3.7\)](#), we get

$$Y_n(A) \leq Y_{n,1}(A) + Y_{n,2}(A),$$

where

$$\begin{aligned} Y_{n,1}(A) &= c \log^2 n \sum_{j=1}^n r_\alpha(x) \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[\frac{1}{r_\alpha(X_j)} e^{\alpha S_j} (1 - S_j)^2 \mathbf{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} \right], \\ Y_{n,2}(A) &= c n^3 \sum_{j=n+1}^{2n-1} \frac{1}{2n-j} r_\alpha(x) \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[\frac{1}{r_\alpha(X_j)} e^{\alpha S_j} \mathbf{1}_{\{S_i \in I_i, \forall 1 \leq i \leq j\}} \right]. \end{aligned}$$

Since the eigenfunction r_α is strictly positive and bounded, we get

$$\begin{aligned} Y_{n,1}(A) &\leq c \log^2 n \sum_{j=1}^n \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[e^{\alpha S_j} (1 - S_j)^2 \mathbb{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} \right], \\ Y_{n,2}(A) &\leq cn^3 \sum_{j=n+1}^{2n-1} \frac{1}{2n-j} \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[e^{\alpha S_j} \mathbb{1}_{\{S_i \in I_i, \forall 1 \leq i \leq j\}} \right]. \end{aligned}$$

For $Y_{n,1}(A)$, since

$$\begin{aligned} e^{\alpha S_j} (1 - S_j)^2 \mathbb{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} &= e^{\alpha S_j} (1 - S_j)^2 \mathbb{1}_{\{S_j \geq -\frac{4}{\alpha} \log j\}} \mathbb{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} \\ &\quad + e^{\alpha S_j} (1 - S_j)^2 \mathbb{1}_{\{S_j < -\frac{4}{\alpha} \log j\}} \mathbb{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} \\ &\leq c \mathbb{1}_{\{S_j \in [-\frac{4}{\alpha} \log j, 0]\}} \mathbb{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} + \frac{c}{j^2}, \end{aligned}$$

applying Lemma 6.7 we obtain

$$\begin{aligned} Y_{n,1}(A) &\leq c \log^2 n \sum_{j=1}^n \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[\mathbb{1}_{\{S_j \in [-\frac{4}{\alpha} \log j, 0]\}} \mathbb{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} \right] + c \log^2 n \\ &\leq c \log^2 n \sum_{j=1}^n \frac{\log^2 j}{(j+1)^{3/2}} + c \log^2 n \\ &\leq c \log^2 n. \end{aligned}$$

For $Y_{n,2}(A)$, since $S_j \leq -\frac{3}{2\alpha} \log n$ for any $j \in [n+1, 2n-1]$, using Lemma 6.7 we get that for any $j \in [n+1, 2n-1]$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[e^{\alpha S_j} \mathbb{1}_{\{S_i \in I_i, \forall 1 \leq i \leq j\}} \right] &\leq \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[e^{\alpha S_j} \mathbb{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} \right] \\ &= \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[e^{\alpha S_j} \mathbb{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} \mathbb{1}_{\{S_j \in [-\frac{3}{\alpha} \log n, -\frac{3}{2\alpha} \log n]\}} \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}_\alpha^x} \left[e^{\alpha S_j} \mathbb{1}_{\{S_i \leq 0, \forall 1 \leq i \leq j\}} \mathbb{1}_{\{S_j < -\frac{3}{\alpha} \log n\}} \right] \\ &\leq \frac{1}{n^{3/2}} \mathbb{Q}_\alpha^x \left(\max_{1 \leq i \leq j} S_i \leq 0, S_j \in \left[-\frac{3}{\alpha} \log n, -\frac{3}{2\alpha} \log n \right] \right) + \frac{1}{n^3} \\ &\leq c \frac{1}{n^{3/2}} \frac{\log^2 n}{j^{3/2}} + \frac{1}{n^3}, \end{aligned}$$

so that

$$\begin{aligned} Y_{n,2}(A) &\leq cn^3 \sum_{j=n+1}^{2n-1} \frac{1}{2n-j} \left(\frac{1}{n^{3/2}} \frac{\log^2 n}{j^{3/2}} + \frac{1}{n^3} \right) \\ &= cn^{3/2} \log^2 n \sum_{j=n+1}^{2n-1} \frac{1}{2n-j} \frac{1}{j^{3/2}} + c \sum_{j=1}^{n-1} \frac{1}{j} \leq c \log^3 n. \end{aligned}$$

Hence

$$Y_n(A) \leq Y_{n,1}(A) + Y_{n,2}(A) \leq c \log^3 n. \quad (7.12)$$

Since $\mathbb{E}((U_n^x(A))^2) = \mathbb{E}(U_n^x(A)) + Y_n(A)$ and $\mathbb{E}(U_n^x(A)) \geq c\pi_s(A) > 0$ (cf. (7.8)), using (7.12) we get

$$\mathbb{E}[(U_n^x(A))^2] \leq \mathbb{E}(U_n^x(A)) + c \log^3 n \leq c \log^3 n (\mathbb{E}U_n^x(A))^2.$$

Hence

$$\mathbb{P}(U_n^x(A) > 0) \geq \frac{(\mathbb{E}U_n^x(A))^2}{\mathbb{E}[(U_n^x(A))^2]} \geq \frac{1}{c \log^3 n},$$

which, by the definition of $U_n^x(A)$, implies that

$$\mathbb{P}\left(M_{2n}^x(A) \geq -\frac{3}{2\alpha} \log n - \Delta\right) \geq \mathbb{P}(U_n^x(A) > 0) \geq \frac{1}{c \log^3 n}. \quad (7.13)$$

In the same way, one can also prove such a lower bound for the random variable $M_{2n-1}^x(A)$. The assertion of the lemma follows. \square

Using Lemma 7.3, we now give a proof of the following upper bound for $M_n^x(A)$.

Lemma 7.4. *Assume conditions A1, A2, A3 and A4. Let $x \in \mathbb{P}(V)$. Then, for any Borel set $A \subseteq \mathbb{P}(V)$ and any $b < -\frac{3}{2\alpha}$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{M_n^x(A)}{\log n} \leq b \mid \mathcal{S}\right) = 0. \quad (7.14)$$

Proof. Let $\varepsilon > 0$ and let η_n be the first time when the number of particles exceeds n^ε , i.e.

$$\eta_n = \inf \left\{ k \geq 1 : \left(\sum_{|u|=k} 1 \right) \geq [n^\varepsilon] \right\}.$$

Denote $m = \mathbb{E}N$. On the system's ultimate survival, by the Kesten-Stigum theorem, we have that $\frac{1}{m^n} \sum_{|u|=n} 1$ converges almost surely as $n \rightarrow \infty$ to a strictly positive random variable, say $W > 0$. Hence $\limsup_{n \rightarrow \infty} \frac{\eta_n}{\log n} = \frac{\varepsilon}{\log m}$ almost surely. For brevity, denote for any constant $\Delta > 0$,

$$A_n = \left\{ M_n^x(A) \geq -c\eta_n - \frac{3}{2\alpha} \log n - \Delta \right\}.$$

We will prove that

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} A_n\right) = 1, \quad (7.15)$$

which in turn will imply (7.14). Indeed, if (7.15) holds, then, almost surely, on the system's survival,

$$M_n^x(A) \geq -c\eta_n - \frac{3}{2\alpha} \log n - \Delta,$$

which implies that, on the system's survival,

$$\liminf_{n \rightarrow \infty} \frac{M_n^x(A)}{\log n} \geq -c \limsup_{n \rightarrow \infty} \frac{\eta_n}{\log n} - \frac{3}{2\alpha} = -c \frac{\varepsilon}{\log m} - \frac{3}{2\alpha}.$$

Since $\varepsilon > 0$ can be arbitrary small, we get (7.14).

In order to prove (7.15), we introduce sets C_n with the property that if $\mathbb{P}(\liminf_{n \rightarrow \infty} C_n^c) = 1$ then $\mathbb{P}(\liminf_{n \rightarrow \infty} A_n) = 1$, and use the Borel-Cantelli lemma to show that $\mathbb{P}(\limsup_{n \rightarrow \infty} C_n) = 0$.

The details are as follows. Since η_n grows like $\log n$, it holds that $\mathbb{P}(\limsup_{n \rightarrow \infty} \{\eta_n > n/2\}) = 0$, so that in order to show (7.15), it is enough to prove that

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} (A_n \cup \{\eta_n > n/2\})\right) = 1. \quad (7.16)$$

Denote, additionally, for any constant $\Delta > 0$,

$$B_n = \left\{ M_n^x(A) < m_{\eta_n}^x - \frac{3}{2\alpha} \log n - \Delta, \eta_n \leq n/2 \right\},$$

$$C_n = \left\{ \min_{k \in [\frac{n}{2}, n]} M_{k+\eta_n}^x(A) < m_{\eta_n}^x - \frac{3}{2\alpha} \log n - \Delta, \eta_n \leq n/2 \right\},$$

where $m_{\eta_n}^x$ is defined by (2.7) with $A = \mathbb{P}(V)$. Since on the set $\{\eta_n \leq n/2\}$, it holds that $\min_{k \in [\frac{n}{2}, n]} M_{k+\eta_n}^x(A) \leq M_n^x(A)$ and therefore $B_n \subset C_n$. Note that

$$\liminf_{n \rightarrow \infty} (A_n \cup \{\eta_n > n/2\}) \supset \liminf_{n \rightarrow \infty} B_n^c \cap \liminf_{n \rightarrow \infty} \{m_{\eta_n}^x \geq -c\eta_n\}.$$

By Lemma 7.2, it holds that $\mathbb{P}(\liminf_{n \rightarrow \infty} \{m_{\eta_n}^x \geq -c\eta_n\}) = 1$. Since $B_n^c \supset C_n^c$, to prove (7.16) it suffices to show that

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} C_n^c\right) = 1. \quad (7.17)$$

As said before, (7.17) will be proved by using the Borel-Cantelli lemma. For this we give an upper bound for $\mathbb{P}(C_n)$. Clearly $\mathbb{P}(C_n) \leq \sum_{k \in [\frac{n}{2}, n]} R_k$, where

$$R_k = \mathbb{P}\left\{ M_{k+\eta_n}^x(A) < m_{\eta_n}^x - \frac{3}{2\alpha} \log n - \Delta, \eta_n \leq \frac{n}{2} \right\}.$$

By the definition of $M_{k+\eta_n}^x(A)$ and $m_{\eta_n}^x$, we have

$$R_k = \mathbb{P}\left(\max_{|u|=k+\eta_n, X_u^x \in A} S_u^x < \min_{|v|=\eta_n} S_v^x - \frac{3}{2\alpha} \log n - \Delta, \eta_n \leq \frac{n}{2}\right)$$

$$= \sum_{j=1}^{[n/2]} \mathbb{P}\left(\max_{|u|=j+k, X_u^x \in A} S_u^x < \min_{|v|=j} S_v^x - \frac{3}{2\alpha} \log n - \Delta, \eta_n = j\right).$$

Recall the notion of the shift operator $[\cdot]_v$ from (2.2). By the cocycle property, for any $v \in \mathbb{T}$ and $u \in \mathbb{T}_v$,

$$S_{vu}^x = S_v^x + [S_u^{X_v^x}]_v. \quad (7.18)$$

Since for any v with $|v| = j$, $S_v^x \geq \min_{|v'|=j} S_{v'}^x$, from (7.18) we have for any $u \in \mathbb{T}_v$ with $|u| = k$,

$$S_{vu}^x \geq \min_{|v'|=j} S_{v'}^x + [S_u^{X_v^x}]_v.$$

Hence

$$\max_{|v|=j, |u|=k, X_{vu}^x \in A} S_{vu}^x \geq \min_{|v'|=j} S_{v'}^x + \max_{|v|=j, |u|=k, X_{vu}^x \in A} [S_u^{X_v^x}]_v. \quad (7.19)$$

It follows that

$$R_k \leq \sum_{j=1}^{[n/2]} \mathbb{P}\left(\max_{|v|=j, |u|=k, X_{vu}^x \in A} [S_u^{X_v^x}]_v < -\frac{3}{2\alpha} \log n - \Delta, \eta_n = j\right). \quad (7.20)$$

Since

$$\max_{|v|=j, |u|=k, X_{vu}^x \in A} [S_u^{X_v^x}]_v = \max_{|v|=j} [M_k^{X_v^x}(A)]_v$$

it holds that

$$\begin{aligned} & \mathbb{P} \left(\max_{|v|=j, |u|=k, X_{vu}^x \in A} [S_u^{X_v^x}]_v < -\frac{3}{2\alpha} \log n - \Delta, \eta_n = j \right) \\ &= \mathbb{P} \left([M_k^{X_v^x}(A)]_v < -\frac{3}{2\alpha} \log n - \Delta, \forall |v| = j, \eta_n = j \right). \end{aligned}$$

Using the independence of the branching property of each node, from (7.20) we get, upon conditioning on \mathcal{F}_{η_n} in the last step,

$$\begin{aligned} R_k &\leq \sum_{j=1}^{\lfloor n/2 \rfloor} \mathbb{P} \left([M_k^{X_v^x}(A)]_v < -\frac{3}{2\alpha} \log n - \Delta, \forall |v| = j, \eta_n = j \right) \\ &= \mathbb{P} \left([M_k^{X_v^x}(A)]_v < -\frac{3}{2\alpha} \log n - \Delta, \forall |v| = \eta_n, \eta_n \leq \lfloor n/2 \rfloor \right) \\ &\leq \mathbb{P} \left([M_k^{X_v^x}(A)]_v < -\frac{3}{2\alpha} \log n - \Delta, \forall |v| = \eta_n \right) \\ &= \mathbb{E} \left[\prod_{|v|=\eta_n} \mathbf{1}_{\{ [M_k^{X_v^x}(A)]_v < -\frac{3}{2\alpha} \log n - \Delta \}} \right] \\ &= \mathbb{E} \left[\prod_{|v|=\eta_n} F(X_v^x) \right]. \end{aligned}$$

Here $F(x) = \mathbb{P}(M_k^x < -\frac{3}{2\alpha} \log n - \Delta)$, for which we have found a uniform bound in Lemma 7.3. Using further the definition of η_n and the inequality $\log(1-t) \leq -t$ for $t \in (0, 1)$, we get

$$R_k \leq \mathbb{E} \left[\prod_{|u|=\eta_n} \left(1 - \frac{1}{c \log^3 n} \right) \right] \leq \left(1 - \frac{1}{c \log^3 n} \right)^{[n^\varepsilon]} \leq e^{-[n^\varepsilon] \frac{1}{\log^3 n}}.$$

This implies

$$\mathbb{P}(C_n) \leq \sum_{k \in [\frac{n}{2}, n]} R_k \leq \sum_{k \in [\frac{n}{2}, n]} e^{-[n^\varepsilon] \frac{1}{\log^3 n}} \leq n e^{-[n^\varepsilon/2]}.$$

Since $\sum_{n=1}^{\infty} n e^{-[n^\varepsilon]/2} < \infty$, we get that $\sum_{n=1}^{\infty} \mathbb{P}(C_n) < \infty$. By the Borel-Cantelli lemma, we have $\mathbb{P}(\limsup_{n \rightarrow \infty} C_n) = 0$ and hence (7.15) holds. This completes the proof of the lemma. \square

8. PROOF OF THEOREMS 2.3 AND 2.4

8.1. Proof of Theorem 2.3. Consider the first time when the random walk $(y + S_n)_{n \geq 1}$ starting at the point $y \in \mathbb{R}_+$ becomes negative:

$$\tilde{\tau}_y = \inf\{k \geq 1 : y + S_k < 0\}.$$

The following result is an analog of Lemma 6.7.

Lemma 8.1. *Assume condition A2, A3 and $\kappa'(s) = 0$ for some $s \in I_\mu^+ \cup I_\mu^-$. Then, there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$, $y \geq 0$, $n \geq 1$ and $0 \leq a < b \leq \sqrt{n} \log n$,*

$$\mathbb{Q}_s^x(y + S_n \in [a, b], \tilde{\tau}_y > n) \leq c \frac{(1+y) \wedge n^{1/2}}{n^{3/2}} (b-a+1)(b+a+1).$$

Since the proof of Lemma 8.1 can be carried out in the same way as that of Lemma 6.7, we omit the details.

The following result is similar to Lemma 7.1.

Lemma 8.2. *Assume conditions A1, A2, A3 and A5. Let $x \in \mathbb{P}(V)$. Then, there exists a constant $c > 0$ such that for any $\varepsilon \in (0, \frac{3}{2\beta})$ and any Borel set $A \subseteq \mathbb{P}(V)$,*

$$I := \mathbb{P} \left(\frac{m_n^x(A)}{\log n} \leq \frac{3}{2\beta} - \varepsilon \mid \mathcal{S} \right) \leq c \frac{\log^3 n}{n^{\varepsilon\beta}}.$$

Proof. Let $K > 1$. We write $I = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \mathbb{P} \left(\frac{m_n^x(A)}{\log n} \leq \frac{3}{2\beta} - \varepsilon, \min_{1 \leq i \leq n} m_i^x(A) \geq -K \mid \mathcal{S} \right), \\ I_2 &= \mathbb{P} \left(\frac{m_n^x(A)}{\log n} \leq \frac{3}{2\beta} - \varepsilon, \min_{1 \leq i \leq n} m_i^x(A) < -K \mid \mathcal{S} \right), \end{aligned}$$

with the notation $m_i^x(A) = \min \{S_u^x : |u| = i, X_u^x \in A\}$ for $1 \leq i \leq n$. For the first term I_1 , we have $I_1 \leq \mathbb{E}(Z_n^x(A) \mid \mathcal{S})$, where, for $x \in \mathbb{P}(V)$,

$$Z_n^x(A) = \sum_{|u|=n} \mathbf{1}_{\left\{ X_u^x \in A, \frac{S_u^x}{\log n} \leq \frac{3}{2\beta} - \varepsilon, S_{u|i}^x \geq -K, \forall 1 \leq i \leq n \right\}}.$$

Using the many-to-one formula (3.7), the fact that $\mathfrak{m}(-\beta) = 1$ (cf. condition A5) and $r_{-\beta}$ is bounded and strictly positive on $\mathbb{P}(V)$, we get that there exists a constant $c > 0$ such that for any $x \in \mathbb{P}(V)$,

$$\begin{aligned} \mathbb{E}(Z_n^x(A)) &= r_{-\beta}(x) \mathbb{E}_{\mathbb{Q}_{-\beta}^x} \left[\frac{1}{r_{-\beta}(X_n)} e^{\beta S_n} \mathbf{1}_{\{X_n \in A, S_n \leq (\frac{3}{2\beta} - \varepsilon) \log n, S_i \geq -K, \forall 1 \leq i \leq n\}} \right] \\ &\leq cn^{\frac{3}{2} - \varepsilon\beta} \mathbb{Q}_{-\beta}^x \left\{ S_n \leq \left(\frac{3}{2\beta} - \varepsilon \right) \log n, S_i \geq -K, \forall 1 \leq i \leq n \right\} \\ &= cn^{\frac{3}{2} - \varepsilon\beta} \mathbb{Q}_{-\beta}^x \left\{ K + S_n \in \left[0, K + \left(\frac{3}{2\beta} - \varepsilon \right) \log n \right], \tilde{\tau}_K > n \right\}. \end{aligned}$$

Applying Lemma 8.1 and taking $K = a \log n$ with $a > 0$ (whose value will be chosen to be sufficiently large), we obtain

$$\mathbb{E}(Z_n^x(A)) \leq cn^{\frac{3}{2} - \varepsilon\beta} (1 + K) \frac{[K + (\frac{3}{2\beta} - \varepsilon) \log n]^2}{n^{3/2}} \leq c \frac{a^3 \log^3 n}{n^{\varepsilon\beta}}.$$

It follows that

$$I_1 \leq \mathbb{E}(Z_n^x(A) \mid \mathcal{S}) \leq c \frac{a^3 \log^3 n}{n^{\varepsilon\beta}}. \quad (8.1)$$

For the second term I_2 , by Markov's inequality, the many-to-one formula (3.7) and the fact that $\mathfrak{m}(-\beta) = 1$ (cf. condition A5), we have

$$\begin{aligned} I_2 &\leq c\mathbb{P}\left(\min_{1 \leq i \leq n} m_i^x(A) < -K\right) \leq c\mathbb{P}\left(\min_{1 \leq i \leq n} \min_{|u|=i} S_u^x < -K\right) \\ &\leq ce^{-\beta K} \mathbb{E}\left(e^{-\beta \min_{1 \leq i \leq n} \min_{|u|=i} S_u^x}\right) \leq ce^{-\beta K} \mathbb{E}\left(\max_{1 \leq i \leq n} \sum_{|u|=i} e^{-\beta S_u^x}\right) \\ &\leq ce^{-\beta K} \sum_{i=1}^n \mathbb{E}\left(\sum_{|u|=i} e^{-\beta S_u^x}\right) = ce^{-\beta K} r_{-\beta}(x) \sum_{i=1}^n \mathbb{E}_{\mathbb{Q}_{-\beta}^x}\left(\frac{1}{r_{-\beta}(X_i)}\right) \leq \frac{c}{n^{a\beta-1}} \leq \frac{c}{n^{a\beta/2}}, \end{aligned} \tag{8.2}$$

by taking $K = a \log n$ and $a > 0$ sufficiently large. Combining (8.1) and (8.2) ends the proof of Lemma 8.2. \square

The following result is an analog of Lemma 7.4 and its proof can be carried out in the same way.

Lemma 8.3. *Assume conditions A1, A2, A3 and A5. Let $x \in \mathbb{P}(V)$. Then, for any Borel set $A \subseteq \mathbb{P}(V)$ and any $b > \frac{3}{2\beta}$, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{m_n^x(A)}{\log n} > b \mid \mathcal{S}\right) = 0.$$

Theorem 2.3 follows from Lemmas 8.2 and 8.3.

8.2. Proof of Theorem 2.4. We first prove (2.20) for the coefficients $\langle f, G_u v \rangle$. Since $|\langle f, G_u v \rangle| \leq \|G_u v\|$, using Theorem 2.2, we easily get that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{\sup_{G_u \cdot x \in A, |u|=n} \log |\langle f, G_u v \rangle|}{\log n} + \frac{3}{2\alpha} > \varepsilon \mid \mathcal{S}\right) = 0.$$

On the other hand, we denote $A_n = \{\log |\langle f, G_u v \rangle| - \log \|G_u v\| \leq -\frac{\varepsilon}{2} \log n\}$ and by A_n^c its complement. By [4, Lemma 14.11], we have $\mathbb{P}(A_n) \leq n^{-c}$ for some constant $c > 0$. This, together with the fact that $\mathbb{P}(\mathcal{S}) > 0$, implies that

$$\begin{aligned} &\mathbb{P}\left(\frac{\sup_{G_u \cdot x \in A, |u|=n} \log |\langle f, G_u v \rangle|}{\log n} + \frac{3}{2\alpha} < -\varepsilon \mid \mathcal{S}\right) \\ &\leq \frac{1}{n^c} + \mathbb{P}\left(\frac{\sup_{G_u \cdot x \in A, |u|=n} \log \|G_u v\|}{\log n} + \frac{3}{2\alpha} < -\frac{\varepsilon}{2} \mid \mathcal{S}\right), \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$, using Theorem 2.2. The proof of (2.20) for the coefficients $\langle f, G_u v \rangle$ is complete.

We next prove (2.20) for the operator norm $\|G_u\|$. Since $\|G_u\| \geq \|G_u v\|$, using Theorem 2.2, we easily get that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{\sup_{G_u \cdot x \in A, |u|=n} \log \|G_u\|}{\log n} + \frac{3}{2\alpha} < -\varepsilon \mid \mathcal{S}\right) = 0.$$

Denote $A_n = \{\log \|G_u v\| - \log \|G_u\| \leq -\frac{\varepsilon}{2} \log n\}$ and by A_n^c its complement. By [20, Proposition 3.11], we see that $\mathbb{P}(A_n) \leq n^{-c}$ for some constant $c > 0$. Then, we obtain

$$\begin{aligned} & \mathbb{P} \left(\frac{\sup_{G_u \cdot x \in A, |u|=n} \log \|G_u\|}{\log n} + \frac{3}{2\alpha} > \varepsilon \middle| \mathcal{S} \right) \\ & \leq \frac{1}{n^c} + \mathbb{P} \left(\frac{\sup_{G_u \cdot x \in A, |u|=n} \log \|G_u v\|}{\log n} + \frac{3}{2\alpha} > \frac{\varepsilon}{2} \middle| \mathcal{S} \right), \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$, using Theorem 2.2. The proof of (2.20) for the operator norm $\|G_u\|$ is complete.

We finally prove (2.20) for the spectral radius $\rho(G_u)$. Since $\rho(G_u) \leq \|G_u\|$, using the law of large numbers (2.20) for the operator norm $\|G_u\|$, we easily get that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{\sup_{G_u \cdot x \in A, |u|=n} \log \rho(G_n)}{\log n} + \frac{3}{2\alpha} > \varepsilon \middle| \mathcal{S} \right) = 0.$$

Denote $A_n = \{\log \rho(G_n) - \log \|G_u\| \leq -\frac{\varepsilon}{2} \log n\}$ and by A_n^c its complement. By [4, Lemma 14.13], we see that $\mathbb{P}(A_n) \leq n^{-c}$ for some constant $c > 0$. Then, we obtain

$$\begin{aligned} & \mathbb{P} \left(\frac{\sup_{G_u \cdot x \in A, |u|=n} \log \rho(G_n)}{\log n} + \frac{3}{2\alpha} < -\varepsilon \middle| \mathcal{S} \right) \\ & \leq \frac{1}{n^c} + \mathbb{P} \left(\frac{\sup_{G_u \cdot x \in A, |u|=n} \log \|G_u\|}{\log n} + \frac{3}{2\alpha} < -\frac{\varepsilon}{2} \middle| \mathcal{S} \right), \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$, using the law of large numbers (2.20) for the operator norm $\|G_u\|$. The proof of (2.20) for the spectral radius $\rho(G_u)$ is complete.

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