



**HAL**  
open science

## On neighborhoods of embedded complex tori

Xianghong Gong, Laurent Stolovitch

► **To cite this version:**

Xianghong Gong, Laurent Stolovitch. On neighborhoods of embedded complex tori. *Mathematische Annalen*, In press, <https://doi.org/10.1007/s00208-024-02975-w> . hal-03692172v2

**HAL Id: hal-03692172**

**<https://hal.science/hal-03692172v2>**

Submitted on 11 Aug 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON NEIGHBORHOODS OF EMBEDDED COMPLEX TORI

XIANGHONG GONG<sup>†</sup> AND LAURENT STOLOVITCH<sup>††</sup>

ABSTRACT. The goal of the article is to show that an  $n$ -dimensional complex torus embedded in a complex manifold of dimension  $n + d$ , with a split tangent bundle, has a neighborhood biholomorphic to a neighborhood of the zero section in its normal bundle, provided the latter has locally constant and diagonalizable transition functions and satisfies a *non-resonant Diophantine* condition.

*In memory of Jean-Pierre Demailly*

## 1. INTRODUCTION

To classify two neighborhoods  $U, U'$  of an embedded compact complex manifold  $C$  in complex manifolds  $M, M'$  up to biholomorphisms with a prescribed isomorphism of the normal bundles of  $C$  in  $M, M'$ , one may first try to compare them to a neighborhood of the zero section of  $N_C$ , the normal bundle of  $C$  in  $M$ , regarding  $N_C$  as a “first-order approximation” of the neighborhoods. When a neighborhood  $U$  is equivalent to a neighbourhood of the zero section of  $N_C$ ,  $U$  is said to be *linearizable*. However, not all neighborhoods are even formally linearizable. One way to measure the complexity of the embedding is to consider for each positive integer  $k$ , the  $k$ th-neighborhood of  $C$  in  $M$ ,  $\mathcal{O}_M/\mathcal{I}_C^{k+1}$ . Here,  $\mathcal{I}_C$  denotes the defining ideal of  $C$ . As already devised by Nirenberg-Spencer [NS60] and Morrow-Rossi [MR80, MR81] starting from the 60’s, for an embedding  $U'$  that is close to  $U$  up to order  $k$  the obstruction to  $U'$  being close to  $U$  up to order  $k + 1$  lies in  $H^1(C, (\Theta_M)|_C \otimes S^{k+1}(N_C^*))$ . Here,  $(\Theta_M)|_C$  denotes the sheaf of holomorphic tangent bundle of  $M$  restricted to  $C$  and  $S^{k+1}(N_C^*)$  is the  $(k + 1)$ -symmetric power of the dual bundle  $N_C^*$ . If the obstruction vanishes for each  $k$ , then there is a *formal equivalence* between the neighborhood  $U$  and  $U'$ . A natural question is the following: Does such a formal equivalence give rise to a genuine holomorphic equivalence between the (possibly smaller) neighborhoods? This is what Grauert called “Das Formale Prinzip” (e.g [Gra62, Kos88]).

Here the curvature in differential geometry enters into the play. Indeed, if the normal bundle  $N_C$  is *negative*, then Grauert [Gra62] and Hironaka-Rossi [HR64] proved a rigidity statement: formally equivalent neighborhoods of  $C$  in  $M$  and

---

2020 *Mathematics Subject Classification*. 32Q57, 32Q28, 32L10, 37F50.

*Key words and phrases*. Complex torus, neighborhood of a complex manifold, Stein manifold, Grauert’s Formal principle, normal bundle, small divisors, KAM scheme.

<sup>†</sup>Partially supported by a grant from the Simons Foundation (award number: 505027) and NSF grant DMS-2054989. <sup>††</sup>Research of L. Stolovitch has been supported by the French government through the UCAJEDI Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-15-IDEX-01.

$M'$  are actually biholomorphic. In the case the normal bundle is *2-positive*, Griffiths [Gri66] proved that there are only finitely many obstructions to being formally equivalent to  $U$ . He then proved, under some assumptions, that if the neighborhood  $U'$  is formally equivalent to  $U$ , then it is also biholomorphic to it (see [GS24] on finite dimensionality of the moduli space). When the normal bundle is *flat*, dynamical systems methods are more appropriate. Indeed, Arnol'd [Arn76] (see also [Arn88, Chap. 5, sect. 27]) studied the embedding of an elliptic curve into a complex surface when the normal bundle has zero self-intersection number (i.e. *flat*). He showed that under a *small divisors condition*, the neighborhood is biholomorphic to a (unspecific) neighborhood of the zero section of the normal bundle  $N_C$ . This was generalized by Ilyashenko and Pyartly [IP79] to direct product of 1 dimensional tori. In this paper we extend this last result to general complex tori :

**Theorem 1.1.** *Let  $C$  be an  $n$ -dimensional complex torus embedded in a complex manifold  $M$  of dimension  $n+d$ . Assume that  $T_C M$ , the restriction of  $TM$  on  $C$ , splits as  $TC \oplus N_C$ . Suppose that the normal bundle of  $C$  in  $M$  admits transition functions that are locally constant and diagonalizable matrices and satisfy a non-resonant Diophantine condition (see Definition 4.5). Then a neighborhood of  $C$  in  $M$  is biholomorphic to a neighborhood of the zero section in the normal bundle.*

Our *non-resonant Diophantine condition* on  $N_C$  is weaker than the one used by Il'yashenko-Pyartli in their restricted case. Of course, the study of neighborhood of embedded compact complex manifolds has a long history though mostly under curvature assumptions on the normal bundle. See some recent works in different context in [Hwa19, LTT19, Koi21, GS22].

We first describe the organization of the proof of our main theorem.

A complex torus  $C$  can be identified with the quotient of  $\mathbb{C}^n$  by a lattice  $\Lambda$  spanned by the standard unit vectors  $e_1, \dots, e_n$  in  $\mathbb{C}^n$  and  $n$  additional vectors  $e'_1, \dots, e'_n$  in  $\mathbb{C}^n$ , where  $\text{Im } e'_1, \dots, \text{Im } e'_n$  are linearly independent vectors in  $\mathbb{R}^n$ . Let  $\Lambda'$  be the lattice in the cylinder  $\tilde{C} := \mathbb{R}^n/\mathbb{Z}^n + i\mathbb{R}^n$  spanned by  $e'_1, \dots, e'_n \pmod{\mathbb{Z}^n}$ . There are two coverings for the torus  $C = \mathbb{C}^n/\Lambda = \tilde{C}/\Lambda'$ : the universal covering  $\pi: \mathbb{C}^n \rightarrow C$  and the covering by cylinder,  $\pi_{\tilde{C}}: \tilde{C} \rightarrow C$  that extends to a covering  $\mathcal{M}$  over  $M$ . In section two we recall some facts about factors of automorphy for vector bundles on  $C$  via the covering by  $\mathbb{C}^n$ . In section three, we study the flat vector bundles on  $C$ . The pull back of the flat vector bundle  $N_C$  to the cylinder  $\tilde{C}$  is the normal bundle  $N_{\tilde{C}}$  of  $\tilde{C}$  in  $\mathcal{M}$ . We show that  $N_{\tilde{C}}$  is always the holomorphically trivial vector bundle  $\tilde{C} \times \mathbb{C}^d$ . By “vertical coordinates”, we mean “coordinates on  $\mathbb{C}^d$ ”, the normal component of the normal bundle  $N_C$ , while “horizontal coordinates” mean the tangential components of  $N_C$ .

Since  $\tilde{C}$  is a Stein manifold, a theorem of Siu [Siu77] says that a neighborhood of  $\tilde{C}$  in  $\mathcal{M}$  is biholomorphic to a neighborhood of the zero section in its normal bundle, which is trivial as mentioned above. We show that the holomorphic classification of neighborhoods  $M$  of  $C$  with flat  $N_C$  is equivalent to the holomorphic classification of the family of the deck transformations of covering  $\mathcal{M}$  of  $M$  in a neighborhood of  $\tilde{C}$  (see Proposition 4.3 (ii)). These deck transformations are “higher-order” (in the vertical coordinates) perturbations  $\tau_1, \dots, \tau_n$  of  $\hat{\tau}_1, \dots, \hat{\tau}_n$ , where the latter are the deck transformations of the covering  $\tilde{N}_C$  over  $N_C$ . Therefore, classifying neighborhoods of a given complex torus  $C$  with flat  $N_C$  is equivalent to classifying the deck transformations  $\{\tau_1, \dots, \tau_n\}$  which are vertical higher-order perturbations

of  $\{\hat{\tau}_1, \dots, \hat{\tau}_n\}$ . As the latter can be interpreted as a dynamical system-like problem, this allows us to study the problem by similar methods as those used in normal form theory of dynamical systems as in [Gon12, Sto15, Yin15, GS19, Kri22, IY08].

There are two useful features. First, since the fundamental group of  $C$  is abelian, the deck transformations  $\tau_1, \dots, \tau_n$  commute pairwise. Second, we can also introduce suitable coordinates on  $\tilde{C}$  so that the "horizontal" components of deck transformations have diagonal linear parts. While the full theory for the above-mentioned deck transformation classification is out the scope of this paper, we study the case when  $N_C$  admits locally constant and diagonalizable transition functions. This is the case, in particular, when  $N_C$  admits Hermitian transition functions. Under these assumptions the deck transformations  $\hat{\tau}_j$  can be considered as diagonal linear maps. The convergence proof for Theorem 1.1 is given in section four and is similar, in essence, to the one used in KAM theory in [Sev03, EFK15, BKM21, SZ22]. It relies on a Newton rapid convergence scheme adapted to our situation based on an appropriate Diophantine condition among quantities defining the lattice and the normal bundle. At step  $k$  of the iteration scheme, let  $\delta_k$  be the error of the deck transformations  $\{\tau_1^{(k)}, \dots, \tau_n^{(k)}\}$  defined on domain  $D^{(k)}$  to  $\hat{\tau}_1, \dots, \hat{\tau}_n$  in suitable norms. By an appropriate transformation  $\Phi^{(k)}$ , we conjugate to a new set of deck transformations  $\{\tau_1^{(k+1)}, \dots, \tau_n^{(k+1)}\}$  of which the error to the linear ones is now  $\delta_{k+1}$  on a slightly smaller domain  $D^{(k+1)}$ . Using our Diophantine conditions, related to the lattice  $\Lambda$  and the normal bundle, we show that the sequence  $\Phi^{(k)} \circ \dots \circ \Phi^{(1)}$  converges to a holomorphic transformation  $\Phi$  on an open domain  $D^{(\infty)}$  where we linearize  $\{\tau_1, \dots, \tau_n\}$ .

**Acknowledgments.** This work benefited from helpful discussions with Jean-Pierre Demailly. Part of work was finished when X. G. was supported by CNRS and UCA for a visiting position at UCA. The authors thank the referee for helping improving their text.

## 2. VECTOR BUNDLES ON TORI AND FACTORS OF AUTOMORPHY

In this section, we identify vector bundles on a complex torus with factors of automorphy. The latter gives us a useful alternative definition of vector bundles on a higher dimensional complex torus  $C$  and the isomorphisms of two vector bundles. General references for line bundle on complex tori are [BL04, Deb05, Ien11] and [GH94, p. 307].

Let  $\Lambda$  be a  $2n$ -dimensional lattice in  $\mathbb{C}^n$ . We may assume that  $\Lambda$  is defined by  $2n$  vectors  $e_1, \dots, e_n, e'_1, \dots, e'_n$  of  $\mathbb{C}^n$ , where  $e_i = (0, \dots, 0, 1, 0, \dots, 0)$  with 1 being at the  $i$ -th place,  $e'_i = (e'_{i,1}, \dots, e'_{i,n})$  and the matrix  $(\text{Im } e'_{i,j})_{1 \leq i,j \leq n}$  is invertible [BL04, exerc. 2, p. 21]. The compact complex manifold  $C := \mathbb{C}^n / \Lambda$  is called an ( $n$ -dimensional) complex torus. Unless the lattice is equivalent to another one defined by a diagonal matrix  $e'$  whose rows are  $e'_1, \dots, e'_n$ ,  $C$  is not biholomorphic to a product of one-dimensional tori. Let  $\pi : \mathbb{C}^n \rightarrow C$  be the universal cover of  $C$ . Its group  $\Gamma$  of deck transformations consists of translations

$$T_\lambda : z \rightarrow z + \lambda, \quad \lambda \in \Lambda.$$

Note that  $\Gamma$  is abelian and is isomorphic to  $\mathbb{Z}^{2n}$ .  $\Gamma$  is also isomorphic to  $\pi_1(C, 0)$  since  $\mathbb{C}^n$  is a universal covering of  $C$ .

Next, we consider equivalence relations for holomorphic vectors bundles on  $C$  and  $\mathbb{C}^n$ , following the realization proof of Theorem 3.2 in [Ien11]. Let  $E$  be a

vector bundle of rank  $d$  over  $C$ . The pull-back bundle  $\pi^*E$  on  $\mathbb{C}^n$  is trivial and has global coordinates  $\hat{\xi}$ . Let  $\{U_j\}$  be an open covering of  $C$  so that coordinates  $\xi_j = (\xi_{j,1}, \dots, \xi_{j,d})^t$  of  $E$  are well-defined (injective) on  $U_j$ . Then we have

$$(2.1) \quad \hat{\xi} = h_j \xi_j(\pi),$$

where  $h_j$  is a non-singular holomorphic matrix on  $\pi^{-1}(U_j)$ . The transition functions  $g_{kj}$  satisfy

$$(2.2) \quad g_{kj}(\pi) = h_k^{-1} h_j, \quad \text{on } \pi^{-1}(U_k) \cap \pi^{-1}(U_j).$$

For any  $z, \lambda$ , we know that both  $\pi(z + \lambda)$  and  $\pi(z)$  are in the same  $U_j$  for some  $j$ . Then we have

$$\hat{\xi}(z + \lambda) = h_j(z + \lambda) \xi_j(\pi(z)) = h_j(z + \lambda) h_j(z)^{-1} \hat{\xi}(z).$$

We can define

$$(2.3) \quad \rho(\lambda, z) = h_j(z + \lambda) h_j(z)^{-1}, \quad z \in \pi^{-1}(U_j)$$

as the latter is independent of the choice of  $j$  by (2.2). Therefore,

$$(2.4) \quad \hat{\xi}(\lambda + z) = \rho(\lambda, z) \hat{\xi}(z), \quad \rho(\lambda, z) \in GL(d, \mathbb{C}),$$

$$(2.5) \quad \rho: \Lambda \times \mathbb{C}^n \rightarrow GL(d, \mathbb{C}).$$

Here  $\rho$  is called a *factor of automorphy*. We can verify that

$$(2.6) \quad \rho(\lambda + \mu, z) = \rho(\lambda, \mu + z) \rho(\mu, z).$$

In particular, if all  $\rho(\lambda, z) = \rho(\lambda)$  are independent of  $z$ , then  $\rho(\Lambda)$  is an abelian group.

The above construction from a vector bundle  $E$  on  $C$  to a factor of automorphy can be reversed. Namely, given (2.4)-(2.6), define the vector bundle  $E$  on  $C$  as the quotient vector space of  $\mathbb{C}^n \times \mathbb{C}^d$  via the equivalence relation

$$(2.7) \quad (z, \xi) \sim (z + \lambda, \rho(\lambda, z) \xi), \quad z \in \mathbb{C}^n, \xi \in \mathbb{C}^d, \lambda \in \Lambda.$$

We denote the projection from the cylinder  $\tilde{C} := \mathbb{R}^n / \mathbb{Z}^n + i\mathbb{R}^n = \mathbb{C}^n / \mathbb{Z}^n$  onto  $C$  by  $\pi_{\tilde{C}}$ . Therefore, we can define  $\pi_{\tilde{C}}^* E$  on the cylinder  $\tilde{C}$  by the equivalence relation

$$(2.8) \quad (z, \xi) \sim (z + \lambda, \rho(\lambda, z) \xi), \quad z \in \mathbb{C}^n, \xi \in \mathbb{C}^d, \lambda \in \mathbb{Z}^n.$$

Of course, global coordinates  $\hat{\xi}$ ,  $\rho$ , and  $g_{jk}$  are not uniquely determined by  $E$ . However, their equivalence classes are determined. Two vector bundles  $E, \tilde{E}$  are isomorphic if their corresponding transitions  $g_{jk}, \tilde{g}_{jk}$  satisfy  $\tilde{g}_{jk} = h_j^{-1} g_{jk} h_k$  where  $h_j$  are non-singular holomorphic matrices. Replacing global coordinates  $\hat{\xi}$  by  $\nu \hat{\xi}$  where  $\nu: \mathbb{C}^n \rightarrow GL(n, d)$  is holomorphic, we can verify that

$$(2.9) \quad \nu(\lambda + z) \rho(\lambda, z) \nu(z)^{-1} =: \tilde{\rho}(\lambda, z)$$

is also a factor of automorphy. Define two factors of automorphy  $\rho, \tilde{\rho}$  to be *equivalent* if (2.9) holds. Therefore, the classification of holomorphic vector bundles is identified with the classification of factors of automorphy.

### 3. FLAT VECTOR BUNDLES

In this section, we will show that the pull-back of a flat vector bundle  $E$  on  $C$  to the cylinder  $\tilde{C} = \mathbb{C}^n/\mathbb{Z}^n$  is always trivial.

When  $E$  is flat, we can choose global coordinates as follows. We know that  $\pi^*E$  is also flat and we can choose its global flat basis, or global flat coordinates  $\hat{\xi}$  by using analytic continuation on  $\mathbb{C}^n$  and pulling back flat local coordinates of  $E$ . In other words, in (2.1)  $h_j$  are locally constants while  $\xi_j$  are locally flat coordinates. Then  $\rho(\lambda, z)$  depend only on  $\lambda$ , in which case we write  $\rho(\lambda)$  for  $\rho(\lambda, z)$ . As remarked above,  $\rho(\Lambda)$  is abelian. When  $E$  is unitary (flat), by the same reasoning  $\pi^*E$  is unitary and we can choose  $h_j$  and  $\rho(\lambda)$  to be unitary.

A  $d \times d$  Jordan block  $J_d(\lambda)$  is a matrix of the form  $\lambda I_d + N_d$ , where  $I_d$  is the  $d \times d$  identity matrix and  $N_d$  is the  $d \times d$  matrix with all entries being 0, except all the  $(i, i + 1)$ -th entries being 1. A matrix  $T$  commutes with  $J$  if and only if

$$T = T_d(a) := a_0 I_d + \sum_{i>0} a_i N_d^i.$$

Note that  $N_d^i = 0$  for  $i \geq d$ . Following [Gan98, p. 218], we call the above  $T$  as well as the following two types of matrices, *regular upper triangular* matrices w.r.t.  $J$ :

$$A = (0, T_d(a)), \quad \text{or} \quad B = \begin{pmatrix} T_{d'}(a') \\ 0 \end{pmatrix},$$

where 0 in  $A$  (resp.  $B$ ) denotes a 0 matrix of  $d$  rows (resp.  $d'$  columns). Given a Jordan matrix

$$\tilde{J} = \text{diag}(J_{d_1}(\lambda_1), \dots, J_{d_k}(\lambda_k)),$$

the matrices that commute with  $\tilde{J}$  are precisely the block matrices

$$X = (X_{\alpha\beta})_{m \times m}$$

where  $X_{\alpha\beta} = 0$  if  $\lambda_\alpha \neq \lambda_\beta$ , while  $X_{\alpha\beta}$  is a regular upper triangular ( $d_\alpha \times d_\beta$ ) matrix if  $\lambda_\alpha = \lambda_\beta$ . Such a matrix  $X$  is said to be a *regular upper triangular matrix* w.r.t.  $\tilde{J}$ .

From the structure of matrices commuting with a Jordan matrix, we can verify the following two results.

**Proposition 3.1.** *Let  $A_1, \dots, A_m$  be  $2 \times 2$  matrices commuting pairwise. Then there is a non-singular matrix  $S$  such that all  $S^{-1}A_jS$  are Jordan matrices.*

**Example 3.2.** *The  $3 \times 3$  matrices  $\lambda I_3 + N_3, \mu I_3 + N_3^2$  commute, but they cannot be transformed into the Jordan normal forms simultaneously.*

The following results on logarithms are likely classical. However, we cannot find a reference. Therefore, we give proofs emphasizing commutativity of logarithms of matrices.

We start with the following.

**Lemma 3.3.** *Let  $A_1, \dots, A_m$  be pairwise commuting matrices. Then there is a non-singular matrix  $S$  such that*

$$S^{-1}A_jS = (\hat{A}_{\alpha\beta})_{1 \leq \alpha, \beta \leq s} =: \hat{A}_j, \quad 1 \leq j \leq m$$

where  $\hat{A}_1$  is a Jordan matrix and all  $\hat{A}_j$  are upper triangular matrices. Assume further that each  $A_j$  is diagonalizable, then they are simultaneously diagonalizable.

*Proof.* We may assume that  $A_1$  is a Jordan matrix  $J = \text{diag}(J_{d_1}(\lambda_1), \dots, J_{d_s}(\lambda_s))$ . Then

$$A_j = (X_{\alpha\beta}^j)_{1 \leq \alpha, \beta \leq s}$$

are regular w.r.t  $J$ . Note that pairwise commuting non-singular matrices have non-trivial common eigenspaces. The eigenspace of  $J$  are spanned by  $e_{d'_1}, \dots, e_{d'_s}$  with  $d'_1 = 1$  and  $d'_j = d_1 + \dots + d_{j-1} + 1$ . To simplify the indices, we may assume that  $e_1$  is an eigenvector of all  $A_j$ . Then the first column of  $A_j$  is  $a_j e_1$  with  $a_j \neq 0$ . The new matrices  $\tilde{A}_j$ , obtained by removing all first rows and first columns, still commute pairwise. In particular all  $\tilde{A}_j$  are regular to the new Jordan matrix  $\tilde{J} = \tilde{A}_1$ . By induction on  $d$ , we can find a non-singular matrix  $\tilde{S}$  which is regular to  $\tilde{J}$  so that all  $\tilde{S}^{-1} \tilde{A}_j \tilde{S}$  are upper-triangular. Let  $S = \text{diag}(1, \tilde{S})$ . Now  $S^{-1} = \text{diag}(1, \tilde{S}^{-1})$ . We can check that all  $\hat{A}_j := S^{-1} A_j S$  are upper triangular. Then  $\hat{A}_1, J$  have the same entries, with only one possible exception

$$\hat{A}_{1;12} = s_{11} J_{12}.$$

If  $\hat{A}_{1;12} \neq 0$ , dilating the first coordinate can transform  $\hat{A}_1$  into the original  $J$ , while  $\hat{A}_j$  remain upper-triangular. If  $s_{12} J_{12} = 0$ , then  $J_{12}$  must be 0, i.e.  $\hat{A}_1 = J$ , because one cannot transform a Jordan matrix,  $A_1 = J$ , into a new Jordan matrix,  $\hat{A}_1$ , by changing an entry 1 to 0 and keeping other entries unchanged.

When each  $A_j$  is diagonalizable, an induction on  $m$  and the number of eigenspaces of  $A_1$  yields a proof.  $\square$

The above simultaneous normalization of upper-triangular matrices allows us to define the logarithms. The construction of logarithms of non-singular matrices can be found in [Gan98, p. 239]. Here we need to find a definition that is suitable to determine the commutativity of the logarithm of pairwise commuting non-singular matrices.

Recall that for a  $d \times d$  matrix  $A$ , the generalized eigenspace  $E_\lambda(A)$  with eigenvalue  $\lambda$  is the kernel of  $(A - \lambda I)^d$ , while  $\mathbb{C}^d$  is the direct sum of all  $E_\lambda(A)$ . A matrix  $B$  that commutes with  $A$  leaves each  $E_\lambda(A)$  invariant, i.e.  $B(E_\lambda(A)) \subset E_\lambda(A)$ . Thus if  $A_1, \dots, A_m$  commute pairwise, we can decompose  $\mathbb{C}^d$  as a direct sum of liner subspaces  $V_j$  such that each  $V_j$  is invariant by  $A_i$  and admits exactly one eigenvalue of  $A_i$ . Thus to define  $\ln A_j$ , we will assume that each  $A_j$  has a single eigenvalue on  $\mathbb{C}^d$  if we wish.

Given a non-singular matrix  $A$ , a logarithm of  $A$  is a matrix  $\ln A$  satisfying

$$e^{\ln A} = A$$

where the exponential matrix  $e^B = \sum \frac{B^n}{n!}$  is always well-defined. However,  $\ln A$  is not unique.

For a non-singular upper triangular matrix

$$(3.1) \quad A = \lambda I_d + a, \quad a := (a_{ij})_{1 \leq i, j \leq d}, \quad a_{ij} = 0 \quad \forall i \geq j,$$

we have  $a^k = 0$  for  $k \geq d$ . Using the identity  $e^{B+C} = e^B e^C$  for two commuting matrices  $B, C$ , we see that  $e^{\ln A} = A$  for

$$(3.2) \quad \ln A := (\ln \lambda) I_d - \sum_{k>0} \frac{(-\lambda^{-1} a)^k}{k}$$

with  $0 \leq \text{Im} \ln(\cdot) < 2\pi$ . For a non-singular Jordan matrix, we can define

$$\ln \text{diag}(J_{d_1}(\lambda_1), \dots, J_{d_m}(\lambda_m)) = \text{diag}(\ln J_{d_1}(\lambda_1), \dots, \ln J_{d_m}(\lambda_m)).$$

Note that  $\ln \lambda_\alpha = \ln \lambda_\beta$  if and only if  $\lambda_\alpha = \lambda_\beta$ . Since matrices that commute with a fixed matrix is closed under multiplication by a scalar, addition and multiplication, it is clear that if  $A$  is an upper triangular matrix that is regular to a non-singular Jordan matrix  $J = \text{diag}(J_{d_1}(\lambda_1), \dots, J_{d_s}(\lambda_s))$ , then  $\ln A$  remains regular to  $J$ . Equivalently and more importantly,  $\ln A$  is regular to the Jordan normal form of  $\ln J$ , which is

$$\text{diag}(J_{d_1}(\ln \lambda_1), \dots, J_{d_s}(\ln \lambda_s)).$$

**Proposition 3.4.** *Let  $A_1, \dots, A_m$  be pairwise commuting  $d \times d$  matrices. Then there is a non-singular matrix  $S$  such that  $\hat{A}_j := S^{-1}A_jS$  are block diagonal matrices of the form*

$$(3.3) \quad \hat{A}_j = \text{diag}(\hat{A}_{j,d_1}, \dots, \hat{A}_{j,d_k})$$

where all  $\hat{A}_{j,d_i}$  are upper triangular  $d_i \times d_i$  matrices, and each  $\hat{A}_{j,d_i}$  has only one eigenvalue  $\lambda_{j,i}$ . Assume further that all  $A_j$  are non-singular. Then

$$(3.4) \quad \ln A_j := S \text{diag}(\ln \hat{A}_{j,d_1}, \dots, \ln \hat{A}_{j,d_k})S^{-1}, \quad 1 \leq j \leq m$$

commute pairwise and  $e^{\ln A_j} = A_j$ , where  $\ln \hat{A}_{j,d_i}$  are defined by (3.1)-(3.2).

*Proof.* Note that for pairwise commuting matrices  $A_1, \dots, A_m$ , we have a decomposition  $\mathbb{C}^d = \bigoplus_{i=1}^t V_i$  where each  $A_j$  preserves  $V_i$  and has only one eigenvalue  $\lambda_{j,i}$ . Their restrictions of  $A_1, \dots, A_m$  on  $V_i$  remain commutative pairwise. Let  $\dim V_i = d_i, d'_0 = 0, d'_{i+1} - d'_i = d_i$ . By Lemma 3.3, we can find a basis

$$e_{d'_{i-1}+1}^*, \dots, e_{d'_{i-1}+d_i}^*$$

for  $V_i$  such that  $A_1|_{V_i}, \dots, A_m|_{V_i}$  are upper triangular matrices. Using the new basis  $e_1^*, \dots, e_d^*$  we can find the matrix  $S$  for the decomposition (3.3). Therefore,  $\hat{A}_{1,d_i}, \dots, \hat{A}_{m,d_i}$  commute pairwise.

Assume now that all  $\lambda_{j,i}$  are non-zero. It remains to show that  $\ln \hat{A}_{1,d_i}, \dots, \ln \hat{A}_{m,d_i}$  commute pairwise. Write

$$\hat{A}_{j,i} = \lambda_{j,i} I_{d_i} + W_{j,i}$$

where  $W_{j,i}$  are upper triangular matrices and  $W_{j,i}^{d_i} = 0$ . By a straightforward computation, we have

$$[W_{j,i}, W_{j',i}] = [\hat{A}_{j,i}, \hat{A}_{j',i}] = 0.$$

Therefore,  $W_{j,i}^k$  commutes with  $W_{j',i}^\ell$  for any  $k, \ell$ . Consequently, the finite sum

$$\ln \hat{A}_{j,i} = \ln \hat{\lambda}_{j,i} I_{d_i} - \sum_{k>0} \frac{(-\lambda_{j,i}^{-1} W_{j,i})^k}{k}$$

commutes with  $\ln \hat{A}_{j',i}$ . Therefore,  $\ln A_1, \dots, \ln A_m$  in (3.4) commute pairwise.  $\square$

**Lemma 3.5.** *Let  $A_1, \dots, A_n$  be non-singular upper triangular  $d \times d$  matrices. Suppose that  $A_1, \dots, A_n$  commute pairwise. There exists a linear mapping  $w \rightarrow \tilde{v}^z(w) := v(z)w$  in  $\mathbb{C}^d$ , that is linear in  $w \in \mathbb{C}^d$  and entire in  $z \in \mathbb{C}^n$  such that  $v(z) \in GL_d(\mathbb{C})$ ,  $v(0) = I_d$  and  $v(e_j) = A_j$  for  $j = 1, \dots, n$ . Furthermore,  $v(z+z') = v(z)v(z')$  for all  $z, z' \in \mathbb{C}^n$ .*



*Proof.* By Proposition 3.4, we define pairwise commuting matrices  $\ln A_1, \dots, \ln A_n$  such that  $e^{\ln A_j} = A_j$ . Then the Lie brackets of the vector fields for  $\dot{w} = \ln A_j w, j = 1, \dots, n$  vanish and their flows  $\varphi_j^t(w)$  commute pairwise. Note that

$$\varphi_j^0(w) = w, \quad \varphi_j^1(w) = e^{\ln A_j} w = A_j w.$$

Define

$$\tilde{v}^z(w) = \varphi_1^{z_1} \cdots \varphi_n^{z_n}(w).$$

We conclude  $\tilde{v}^z \tilde{v}^{z'}(w) = \tilde{v}^{z+z'}(w)$ , that is  $v(z+z') = v(z)v(z')$ .  $\square$

By Proposition 3.4, we define  $\ln \rho(e_1), \dots, \ln \rho(e_{2n})$  and they commute pairwise. We now define  $\ln \rho(\lambda)$  for all  $\lambda = \sum_{j=1}^{2n} m_j e_j \in \Lambda$  as follows

$$\ln \rho \left( \sum_{j=1}^{2n} m_j e_j \right) := \sum_{j=1}^{2n} m_j \ln \rho(e_j).$$

Thus the matrices  $\ln \rho(\lambda)$  for  $\lambda \in \Lambda$  commute pairwise.

**Proposition 3.6.** *Let  $E$  be a flat vector bundle on  $C$ . Then  $\pi_C^* E$  admits a factor of automorphy  $\rho$  satisfying  $\rho(e_j) = Id$  for  $j = 1, \dots, n$ ; in particular,  $\pi_C^* E$  is holomorphically trivial.*

*Proof.* Let  $d$  be the rank of  $E$ . Let  $A_j = \rho(e_j)^{-1}$  for  $j = 1, \dots, n$ . With  $v(e_j) = A_j$  and  $v(0) = Id_d$ , we first see that

$$\tilde{\rho}(\lambda, z) := v(z + \lambda) \rho(\lambda) v(z)^{-1}$$

satisfies  $\tilde{\rho}(e_j, 0) = Id_d$ . We want to show that  $\tilde{\rho}(\lambda, z)$  depends only on  $\lambda$ . Fix  $\lambda = \sum_{j=1}^{2n} m_j e_j \in \Lambda$ . By definition, the matrix  $\ln \rho(\lambda)$  commutes with each  $\ln \rho(e_j)$ ,  $j = 1, \dots, 2n$ . Thus the flow  $\varphi_\lambda^t$  of  $\dot{w} = \ln \rho(\lambda) w$  commutes with the flows of  $\dot{w} = \ln \rho(e_j) w, j = 1, \dots, 2n$ . As in the proof of the previous lemma, we know that  $\varphi_\lambda^t(w)$  is linear in  $w$  and entire in  $t \in \mathbb{C}$ . For  $z \in \mathbb{C}^n$  and  $w \in \mathbb{C}^d$ , let  $\tilde{v}^z(w)$  be as defined in the previous lemma. Thus we have

$$\varphi_\lambda^t \tilde{v}^z(w) = \tilde{v}^z \varphi_\lambda^t(w).$$

Taking derivatives in  $w$  and plugging in  $t = 1$ , we get

$$\exp(\ln \rho(\lambda)) v(z) = v(z) \exp(\ln \rho(\lambda)).$$

Since  $\exp(\ln \rho(\lambda)) = \rho(\lambda)$ , we have  $\rho(\lambda) v(z) = v(z) \rho(\lambda)$  for all  $\lambda \in \Lambda, z \in \mathbb{C}^n$ . Considering  $z = z_1 e_1 + \dots + z_n e_n \in \Lambda$ , we have  $v(z + \lambda) = v(z) v(\lambda)$ . Hence,  $\tilde{\rho}(\lambda, z)$  is independent of  $z$ .

We have achieved  $\tilde{\rho}(\lambda) = v(z + \lambda) \rho(\lambda) v(z)^{-1}$  and  $\tilde{\rho}(e_j) = Id_d$  for  $j = 1, \dots, n$ . Therefore,  $\pi_C^* E$  is trivial, by the equivalence relation (2.8).  $\square$

It is known that there are Stein manifolds with non-trivial vector bundle [FR68]. Furthermore, we conclude the section by emphasizing that the triviality  $\pi_C^*|_{\tilde{C}} E$  relies on the extra assumption that it is a pull-back bundle. The following result is likely known, but we include a short proof for completeness.

**Proposition 3.7.** *The set of holomorphic equivalence classes of flat holomorphic line bundles on  $\tilde{C}$  can be identified with  $H^1(\tilde{C}, \mathbb{C}^*)$ . The latter is non-trivial.*

*Proof.* Each element  $\{c_{jk}\}$  in  $H^1(\tilde{C}, \mathbb{C}^*)$  is clearly an element in  $H^1(\tilde{C}, \mathcal{O}^*)$ . We want to show that if  $\{c_{jk}\}, \{\tilde{c}_{jk}\} \in H^1(\tilde{C}, \mathbb{C}^*)$  represent the same element in  $H^1(\tilde{C}, \mathcal{O}^*)$ , then they are also the same in  $H^1(\tilde{C}, \mathbb{C}^*)$ . Indeed, we can cover  $\tilde{C}$  by convex open sets  $U_1, U_2, U_3, U_4$  such that  $U_1 \cap U_2 \cap U_3 \cap U_4$  is non empty. Thus  $\{U_i\}$  is a Leray covering. If  $\tilde{c}_{jk} = h_j c_{jk} h_k^{-1}$ , where each  $h_j$  is a non-vanishing holomorphic function on  $U_j$ . Take  $p$  in all  $U_j$ . We get  $\tilde{c}_{jk} = c_j c_{jk} c_k^{-1}$  for  $c_j = h_j(p)$ .

Note that  $H^1(\tilde{C}, \mathbb{C}^*)$  is non-trivial. Otherwise, the exact sequences  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^* \rightarrow 0$  and  $0 \rightarrow H^0(\tilde{C}, \mathbb{Z}) \rightarrow H^0(\tilde{C}, \mathbb{C}) \rightarrow H^0(\tilde{C}, \mathbb{C}^*) \rightarrow 0$  would imply that  $H^1(\tilde{C}, \mathbb{Z}) \cong H^1(\tilde{C}, \mathbb{C})$ , a contradiction.  $\square$

#### 4. EQUIVALENCE OF NEIGHBORHOODS AND COMMUTING DECK TRANSFORMATIONS

In this section, we will discuss how the classification of neighborhoods  $U$  of a compact complex manifold  $C$  is related to the classification of deck transformations of a holomorphic covering  $\tilde{U} \rightarrow U$ , where  $U, \tilde{U}$  are chosen carefully and  $\tilde{U}$  contains  $C^*$  that covers  $C$ . When  $C^*$  is additionally Stein, we can choose  $\tilde{U}$  to be a neighborhood of  $C^*$  in its normal bundle  $N_{C^*}(\tilde{U})$  by applying a result of Siu. After preliminary results in Lemma 4.1 and Lemma 4.2, we will return to our previous study where  $C$  is a complex torus, its covering is the Stein manifold  $C^* = \tilde{C}$ , and  $N_C(M)$  is Hermitian flat. We then prove the main result of this paper by using a KAM rapid iteration scheme.

Let us start with  $\iota : C \hookrightarrow M$ , a holomorphic embedding of a compact complex manifold  $C$ . We shall still denote  $\iota(C)$  by  $C$ . Let  $U$  be a neighborhood of  $C$  in  $M$  such that  $U$  admits a smooth, possibly non-holomorphic, *deformation or strong retract*  $C$ ; namely there is a smooth mapping  $R : U \times [0, 1] \rightarrow U$  such that  $R(\cdot, 0) = Id$  on  $U$ ,  $R(\cdot, t) = Id$  on  $C$ , and  $R(\cdot, 1)(U) = C$ . Thus,  $\pi_1(U, x_0) = \pi_1(C, x_0)$  for  $x_0 \in C$  (see [Mun00, p. 361]). When  $M$  is  $N_C$ , we can find a *holomorphic deformation retraction* from a suitable neighborhood of its the zero section onto  $C$ , by using a Hermitian metric on  $N_C$ .

Let  $X$  be a complex manifold and  $\mathcal{X}$  be a universal covering of  $X$ . Then the group of deck transformations of the covering is identified with  $\pi_1(X, x_0)$ . The set of equivalence classes of coverings of  $X$  is identified with the conjugate classes of subgroups of  $\pi_1(X, x_0)$ ; see [Mun00, Thm. 79.4, p. 492]. Furthermore,  $\pi_1(X, x_0)$  acts transitively and freely on each fiber of the covering and  $X$  is the quotient of  $\mathcal{X}$  by  $\pi_1(X, x_0)$ .

**Lemma 4.1.** *Let  $C$  be a compact complex manifold. Let  $\pi : C^* \rightarrow C$  be a holomorphic covering and  $\pi(x_0^*) = x_0$ . Suppose that  $(M, C)$  (resp.  $(M', C)$ ) is a holomorphic neighborhood of  $C$ . There exist a neighborhood  $U$  in  $M$  (resp.  $U'$  in  $M'$ ) of  $C$  and a holomorphic neighborhood  $\tilde{U}$  (resp.  $\tilde{U}'$ ) of  $C^*$  such that  $p : \tilde{U} \rightarrow U$  (resp.  $p : \tilde{U}' \rightarrow U'$ ) is an extended covering of the covering  $\pi : C^* \rightarrow C$  and  $C$  (resp.  $C^*$ ) is a smooth strong retract of  $U, U'$  (resp.  $\tilde{U}, \tilde{U}'$ ). Consequently,*

$$\pi_1(\tilde{U}, x_0^*) = \pi_1(C^*, x_0^*), \quad \pi_1(U, x_0) = \pi_1(C, x_0).$$

*Suppose that  $(M, C)$  is biholomorphic to  $(M', C)$ . Then  $U, U', \tilde{U}, \tilde{U}'$  can be so chosen that there is a covering transformation sending  $\tilde{U}$  onto  $\tilde{U}'$  and fixing  $C^*$  pointwise. Conversely, if there is a covering transformation sending  $\tilde{U}$  onto  $\tilde{U}'$  fixing  $C^*$  pointwise, then  $(U, C), (U', C)$  are holomorphically equivalent.*

*Proof.* Since  $\pi: C^* \rightarrow C$  is a covering map, according to [Vic94, Thm. 4.9], it extends to a covering map  $p: \tilde{U} \rightarrow U$  such that  $\tilde{U}$  contains  $C^*$  and  $p|_{C^*} = \pi$ . Suppose that  $R$  is a strong retraction of  $U$  onto  $C$ . We can lift  $R(z, \cdot): [0, 1] \rightarrow U$  to a continuous mapping  $\tilde{R}(\tilde{z}, \cdot): [0, 1] \rightarrow \tilde{U}$  such that  $\tilde{R}(\tilde{z}, 0) = \tilde{z}$  and  $p\tilde{R}(\tilde{z}, \cdot) = R(p(\tilde{z}), \cdot)$  for all  $\tilde{z} \in \tilde{U}$ . One can verify that  $\tilde{R}$  is a strong retraction of  $\tilde{U}$  onto  $C^*$ .

Suppose that a biholomorphic map  $f$  sends  $(M, C)$  onto  $(M', C)$  fixing  $C$  pointwise. We may assume that  $f$  is a biholomorphic mapping from  $U$  onto  $U'$ . Then we can lift the mapping  $fp: \tilde{U} \rightarrow U'$  to obtain a desired covering biholomorphism  $F$ , since  $(fp)_*\pi_1(\tilde{U}, x_0^*) = \pi_*\pi_1(C^*, x_0^*)$ . Conversely, a covering biholomorphism from  $\tilde{U}$  onto  $\tilde{U}'$  fixing  $C^*$  pointwise clearly induces a biholomorphism from  $U$  onto  $U'$  fixing  $C$  pointwise.  $\square$

With the covering, we can identify  $(M, C)$  with  $(\tilde{M}, C^*)/\sim$  where  $\tilde{p} \sim p$  if and only if  $p, \tilde{p}$  are in the same stack of the covering  $\pi$ .

Applying the above to  $(N_C, C)$  and a covering  $\pi|_{C^*}: C^* \rightarrow C$ , we have a covering  $\hat{\pi}: \widetilde{N_C} \rightarrow N_C$  such that

$$C^* \subset \widetilde{N_C}, \quad \pi_1(\widetilde{N_C}, x_0^*) = \pi_1(C^*, x_0^*), \quad \pi_1(N_C, x_0) = \pi_1(C, x_0).$$

To simplify the notation, we denote  $U, \tilde{U}$  by  $M, \tilde{M}$  respectively. Thus we have commuting diagrams for the coverings:

$$\begin{array}{ccccc} \widetilde{N_C} & \leftarrow & C^* & \hookrightarrow & \tilde{M} \\ \hat{\pi} \downarrow & & \pi_{C^*} \downarrow & & p \downarrow \\ N_C & \leftarrow & C & \hookrightarrow & M \end{array}$$

The set of deck transformations of  $p$  (resp.  $\hat{\pi}$ ) will be denoted by  $\{\tau_1, \dots, \tau_n\}$  (resp.  $\{\hat{\tau}_1, \dots, \hat{\tau}_n\}$ ). If  $\pi: \tilde{U} \rightarrow U$  is a covering map,  $Deck(\tilde{U})$  denotes the set of deck transformations.

**Lemma 4.2.** *Let  $C, C^*, M$  be as in Lemma 4.1. Suppose that  $C^*$  is a Stein manifold. Let  $\omega_0^*$  be an open set of  $\widetilde{N_C}$  such that  $\hat{\pi}(\omega_0^*)$  contains  $C$ . Then there is an open subset  $\omega^*$  of  $\omega_0^*$  such that  $\hat{\pi}(\omega^*)$  contains  $C$  and  $(M, C)$  is holomorphically equivalent to the quotient space of  $\omega^*$  by  $Deck(\widetilde{N_C})$ .*

*Proof.* By a result of Siu [Siu77, Cor. 1], we find a biholomorphism  $L$  from a holomorphic strong retraction neighborhood of  $C^*$  in  $\tilde{M}$ , still denoted by  $\tilde{M}$  into  $N_{C^*}(\tilde{M})$  and a biholomorphism  $L'$  from a strong retraction neighborhood of  $C^*$  in  $\widetilde{N_C}$ , still denoted by  $\widetilde{N_C}$  into  $N_{C^*}(\widetilde{N_C})$ . Furthermore,  $L, L'$  fix  $C^*$  pointwise. We have

$$\begin{aligned} p_*L_*^{-1}\pi_1(N_{C^*}(\tilde{M}), x_0^*) &= p_*\pi_1(\tilde{M}, x_0^*) = \pi_1(C, x_0) = \hat{\pi}_*\pi_1(\widetilde{N_C}, x_0^*) \\ &= \hat{\pi}_*(L')_*^{-1}(\pi_1(N_{C^*}(\widetilde{N_C}), x_0^*)). \end{aligned}$$

Both  $\hat{\pi} \circ L'^{-1}: N_{C^*}(\widetilde{N_C}) \rightarrow N_C$  and  $p \circ L^{-1}: N_{C^*}(\tilde{M}) \rightarrow M$  are coverings and the above identifications show that the lifts of the two coverings yield a biholomorphism between neighborhoods of  $C^*$  in  $\tilde{M}$  and  $N_{C^*}(\widetilde{N_C})$  fixing  $C^*$  pointwise.  $\square$

Recall that  $C$  is the quotient by the lattice spanned by  $e_1, \dots, e_n, e'_1, \dots, e'_n$ . We define the fundamental domain  $\omega_0$  for  $C$ :

$$\omega_0 := \left\{ \sum_{j=1}^{2n} t_j e_j \in \mathbb{C}^n : t \in [0, 1)^{2n} \right\}, \quad e_{n+i} := e'_i, \quad i = 1, \dots, n.$$

We also define a fundamental domain  $\Omega_0$  for the covering  $\tilde{C}$  of  $C$  by

$$\Omega_0 := \{(e^{2\pi i \zeta_1}, \dots, e^{2\pi i \zeta_n}) : \zeta \in \omega_0\}, \quad \Omega_0^+ = \{(|z_1|, \dots, |z_n|) : z \in \Omega_0\}.$$

Thus  $\Omega_0$  is a Reinhardt domain, being  $\{(\nu_1 R_1, \dots, \nu_n R_n) : |\nu_j| = 1 : R \in \Omega_0^+\}$ . We have

$$\Omega_0^+ = \left\{ (e^{-2\pi R_1}, \dots, e^{-2\pi R_n}) : R = \sum_{i=1}^n t_i \operatorname{Im} e'_i, t \in [0, 1)^n \right\}.$$

For  $\epsilon > 0$ , define a (Reinhardt) neighborhood  $\Omega_\epsilon$  of  $\overline{\Omega_0}$  by

$$\omega_\epsilon := \left\{ \sum_{j=1}^{2n} t_j e_j : t \in [0, 1)^n \times (-\epsilon, 1 + \epsilon)^n \right\}, \quad \Omega_\epsilon := \{(e^{2\pi i \zeta_1}, \dots, e^{2\pi i \zeta_n}) : \zeta \in \omega_\epsilon\}.$$

With  $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ , we also define

$$(4.1) \quad \omega_{\epsilon, r} := \omega_\epsilon \times \Delta_r^d, \quad \Omega_{\epsilon, r} := \Omega_\epsilon \times \Delta_r^d.$$

Throughout the paper, a mapping  $(z', v') = \psi^0(z, v)$  from  $\omega_{\epsilon, r}$  into  $\mathbb{C}^{n+d}$  that commutes with  $z_j \rightarrow z_j + 1$  for  $j = 1, \dots, n$  will be identified with a well-defined mapping  $(h', v') = \psi(h, v)$  from  $\Omega_{\epsilon, r}$  into  $\mathbb{C}^{n+d}$ , where  $z, h$  and  $z', h'$  are related as in (4.6) below. A function on  $\omega_{\epsilon, r}$  that has period 1 in all  $z_j$  is identified with a function on  $\Omega_{\epsilon, r}$ . We shall use these identifications as we wish.

**Proposition 4.3.** *Let  $C$  be the complex torus and  $\pi_{\tilde{C}} : \tilde{C} = \mathbb{C}^n / \mathbb{Z}^n \rightarrow C$  be the covering. Let  $(M, C)$  be a neighborhood of  $C$ . Assume that  $N_C$  is flat.*

- (i) *Then one can take  $\omega_{\epsilon_0, r_0} = \omega_{\epsilon_0} \times \Delta_{r_0}^d$  such that  $(M, C)$  is biholomorphic to the quotient of  $\omega_{\epsilon_0, r_0}$  by  $\tau_1^0, \dots, \tau_n^0$ . Let  $\tau_j$  be the mapping defined on  $\Omega_{\epsilon_0, r_0}$  corresponding to  $\tau_j^0$ . Then  $\tau_1, \dots, \tau_n$  commute pairwise wherever they are defined, i.e.*

$$\tau_i \tau_j(h, v) = \tau_j \tau_i(h, v) \quad \forall i \neq j$$

for  $(h, v) \in \Omega_{\epsilon_0, r_0} \cap \tau_i^{-1} \Omega_{\epsilon_0, r_0} \cap \tau_j^{-1} \Omega_{\epsilon_0, r_0}$ .

- (ii) *Let  $(\tilde{M}, C)$  be another such neighborhood having the corresponding generators  $\tilde{\tau}_1, \dots, \tilde{\tau}_n$  of deck transformations defined on  $\Omega_{\tilde{\epsilon}_0, \tilde{r}_0}$ . Then  $(M, C)$  and  $(\tilde{M}, C)$  are holomorphically equivalent if and only if there is a biholomorphic mapping  $F$  from  $\Omega_{\epsilon, r}$  into  $\Omega_{\tilde{\epsilon}, \tilde{r}}$  for some positive  $\epsilon, r, \tilde{\epsilon}, \tilde{r}$  such that*

$$F \tau_j(h, v) = \tilde{\tau}_j F(h, v), \quad j = 1, \dots, n,$$

wherever both sides are defined, i.e.  $(h, v) \in \tau_j^{-1}(\Omega_{\epsilon, r}) \cap \Omega_{\epsilon, r} \cap F^{-1}(\Omega_{\tilde{\epsilon}, \tilde{r}})$ .

*Proof.* We now apply Lemma 4.2, in which  $C^*$  is replaced by  $\tilde{C} = \mathbb{R}^n / \mathbb{Z}^n + i\mathbb{R}^n$  is a Stein manifold. Assume that  $N_C$  is flat. Then according to Proposition 3.6,  $N_{\tilde{C}}(\tilde{N}_C) = N_{\tilde{C}}(\tilde{M}) = \pi_{\tilde{C}}^*(N_C)$  is the trivial vector bundle  $\tilde{C} \times \mathbb{C}^d$  with coordinates  $(h, v)$ , while  $\tilde{C} \times \{0\}$  is defined by  $v = 0$ . The proposition then follows from Lemma 4.1.  $\square$

Set

$$(4.2) \quad \mathcal{P}_\epsilon^+ = \left\{ \sum_{i=1}^n t_i \operatorname{Im} e'_i \in \mathbb{R}^n : t \in (-\epsilon, 1 + \epsilon)^n \right\},$$

$$(4.3) \quad \Omega_\epsilon^+ = \{(e^{-2\pi R_1}, \dots, e^{-2\pi R_n}) : R \in \mathcal{P}_\epsilon^+\}.$$

Note that  $\mathcal{P}_\epsilon^+, \mathcal{P}_0^+$  are  $n$ -dimensional parallelotopes, and  $\Omega_\epsilon^+$  contains  $(1, \dots, 1)$ , the image of  $0 \in \mathcal{P}_\epsilon^+$ , corresponding to the real torus  $(S^1)^n$ .

Since  $\Omega_\epsilon$  is Reinhardt, we have

$$(4.4) \quad \Omega_\epsilon \supset \Omega_\epsilon^+, \quad (\partial\Omega_\epsilon)^+ = \partial(\Omega_\epsilon^+), \quad (\partial\Omega_\epsilon)^+ := \{(|h_1|, \dots, |h_n|) : h \in \partial\Omega_\epsilon\}.$$

We now apply the above general results to the case where  $C$  is a complex torus,  $C^* = \tilde{C}$  and  $N_C(M)$  is Hermitian flat.

As in [IP79], the deck transformations of  $(\tilde{N}_C, \tilde{C})$  are generated by  $n$  biholomorphisms  $\hat{\tau}_1, \dots, \hat{\tau}_n$  that preserve  $\tilde{C}$ . Write

$$(4.5) \quad \hat{\tau}_j(h, v) = (T_j h, M_j v), \quad M_j := \operatorname{diag}(\mu_{j,1}, \dots, \mu_{j,d})$$

with  $h, T_j$  being defined by :

$$(4.6) \quad h = (e^{2\pi i z_1}, \dots, e^{2\pi i z_n}), \quad T_j := \operatorname{diag}(\lambda_{j,1}, \dots, \lambda_{j,n}), \quad \lambda_{j,k} := e^{2\pi i e'_{jk}}.$$

Here the invertible  $(d \times d)$ -matrix  $M_j$  is the factor of automorphy  $\rho(e_{n+j})$  of  $\pi_{\tilde{C}}^* N_C$ .

Recall that each deck transformation  $\tau_j^0(z, v)$ ,  $j = 1, \dots, n$ , is a holomorphic map defined on  $\overline{\omega}_{\epsilon,r}$ . In coordinates  $(h, v)$ ,  $\tau_j^0$  becomes  $\tau_j$  defined on  $\overline{\Omega}_{\epsilon,r}$ . Since  $T_C M$  splits, it is a higher-order perturbation of  $\hat{\tau}_j$  with

$$\hat{\tau}_j(h, 0) = (T_j h, 0).$$

The above computation is based on the assumption that  $N_C$  is flat. We now assume that  $N_C$  admits locally constant and diagonalizable transition matrices as it is the case whenever  $N_C$  is *Hermitian and flat*. Then all  $\rho(e_{n+j})$ ,  $1 \leq j \leq n$  are actually simultaneously diagonalizable by Lemma 3.3 since they pairwise commute. Hence, we can assume that  $M_j = \operatorname{diag}(\mu_{j,1}, \dots, \mu_{j,d})$ . We recall from (4.6) that  $T_j = \operatorname{diag}(\lambda_{j,1}, \dots, \lambda_{j,n})$  are already diagonal.

**Definition 4.4.** *The normal bundle  $N_C$  is said to be non-resonant if, for each  $(Q, P) \in \mathbb{N}^d \times \mathbb{Z}^n$  with  $|Q| > 1$ , each  $i = 1, \dots, n$ , and each  $j = 1, \dots, d$ , there exist  $i_h := i_h(Q, P, i)$  and  $i_v := i_v(Q, P, j)$  that are in  $\{1, \dots, n\}$  such that*

$$\lambda_{i_h}^P \mu_{i_h}^Q - \lambda_{i_h, i} \neq 0 \quad \text{and} \quad \lambda_{i_v}^P \mu_{i_v}^Q - \mu_{i_v, j} \neq 0.$$

**Definition 4.5.** *The normal bundle  $N_C$  is said to be Diophantine if there exist positive constants  $D, \tau$  such that for all  $(Q, P) \in \mathbb{N}^d \times \mathbb{Z}^n$ ,  $|Q| > 1$  and all  $i = 1, \dots, n$ , and  $j = 1, \dots, d$ , we have*

$$(4.7) \quad \max_{\ell \in \{1, \dots, n\}} \left| \lambda_\ell^P \mu_\ell^Q - \lambda_{\ell, i} \right| > \frac{D}{(|P| + |Q|)^\tau},$$

$$(4.8) \quad \max_{\ell \in \{1, \dots, n\}} \left| \lambda_\ell^P \mu_\ell^Q - \mu_{\ell, j} \right| > \frac{D}{(|P| + |Q|)^\tau}.$$

We shall choose  $i_h$  (resp.  $i_v$ ) to be the index that realizes the maximum of (4.7) (resp. (4.8)).

**Remark 4.6.** *If the right-hand sides of (4.7)-(4.8) are replaced by 0, then it means that  $N_C$  is non-resonant in the sense of Definition 4.4. The small divisor condition in Arnold [Arn76] corresponds to  $n = 1 = d$ . The condition used in [IP79] is : for all  $1 \leq \ell \leq n$ ,*

$$\min \left\{ |\lambda_\ell^P \mu_\ell^Q - \lambda_{\ell,i}|, |\lambda_\ell^P \mu_\ell^Q - \mu_{\ell,j}| \right\} > \frac{D}{(|P| + |Q|)^\tau}.$$

*It is stronger than our condition. Indeed, the lower bound might hold only for an index  $\ell$  that depends on  $P$  and  $Q$  and that index might change when  $P$  and  $Q$  change. In a similar but different context of germs of vector fields at a fixed point (e.g. [Sto00]), there are examples such that for each  $\ell$ , the inequality for all  $Q$  ( $P = 0$  in this situation) is not satisfied while it is satisfied with the max over the family.*

**Proposition 4.7.** *The properties of being non-resonant and Diophantine is a property of the (abelian) group generated by  $\{\hat{\tau}_1, \dots, \hat{\tau}_n\}$  and not the choice of the generators.*

*Proof.* Recall that

$$\lambda_\ell = (\lambda_{\ell,1}, \dots, \lambda_{\ell,n}) = (e^{2\pi i e'_{\ell,1}}, \dots, e^{2\pi i e'_{\ell,n}}).$$

Let  $G$  be the group generated by the  $\hat{\tau}_\ell$ 's. Then,  $\{\tilde{\tau}_\ell\}_\ell$  defines another set of generators of  $G$  if  $\tilde{\tau}_\ell = \hat{\tau}_1^{a_{\ell,1}} \cdots \hat{\tau}_n^{a_{\ell,n}}$ ,  $\ell = 1, \dots, n$  where  $A = (a_{i,j})_{1 \leq i,j \leq n} \in GL_n(\mathbb{Z})$  with  $\det A = \pm 1$ . Then, the eigenvalues of  $\tilde{\tau}_\ell$  are

$$\tilde{\lambda}_{\ell,i} = \prod_{k=1}^n \lambda_{k,i}^{a_{\ell,k}}, \quad \tilde{\mu}_{\ell,j} = \prod_{k=1}^d \mu_{k,j}^{a_{\ell,k}}.$$

Hence, we have

$$\tilde{\lambda}_\ell^P \tilde{\mu}_\ell^Q \tilde{\lambda}_{\ell,i}^{-1} = (\lambda_1^P \mu_1^Q \lambda_{1,i}^{-1})^{a_{\ell,1}} \cdots (\lambda_n^P \mu_n^Q \lambda_{n,i}^{-1})^{a_{\ell,n}}.$$

Fix  $P, Q$  and  $i$ . Taking the logarithm, we have as  $n$ -vectors

$$(4.9) \quad \left( \ln \tilde{\lambda}_\ell^P \tilde{\mu}_\ell^Q \tilde{\lambda}_{\ell,i}^{-1} \right)_{\ell=1, \dots, n} = A \left( \ln \lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1} \right)_{\ell=1, \dots, n}, \quad \text{mod } 2\pi i.$$

Since  $A, A^{-1} \in GL_n(\mathbb{Z})$ , then given  $P, Q$  and  $i$ ,

$$\left( \ln \tilde{\lambda}_\ell^P \tilde{\mu}_\ell^Q \tilde{\lambda}_{\ell,i}^{-1} \right)_{\ell=1, \dots, n} = 0 \quad \text{mod } 2\pi i$$

holds if and only if  $\left( \ln \lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1} \right)_{\ell=1, \dots, n} = 0 \quad \text{mod } 2\pi i$ . Similarly, by considering  $\ln \tilde{\lambda}_\ell^P \tilde{\mu}_\ell^Q \tilde{\mu}_{\ell,i}^{-1}$ , we obtain that the non-resonant condition does not depend on the choice of generators. Given  $P, Q, i$ , if one of the  $\lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1}$ 's is not close to 1, then  $\left\| \left( \ln \tilde{\lambda}_\ell^P \tilde{\mu}_\ell^Q \tilde{\lambda}_{\ell,i}^{-1} \right)_\ell \right\|$  is bounded way from 0. On the other hand, if all  $\lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1}$ 's are close to 1, then  $|\ln \lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1}|$  (with  $\text{Im } \ln$  in  $(-\pi, \pi]$ ) is comparable to  $|\lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1} - 1|$ . Furthermore, taking the module of (4.9), we obtain

$$\|A^{-1}\|^{-1} \left\| \left( \ln \lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1} \right)_\ell \right\| \leq \left\| \left( \ln \tilde{\lambda}_\ell^P \tilde{\mu}_\ell^Q \tilde{\lambda}_{\ell,i}^{-1} \right)_\ell \right\| \leq \|A\| \left\| \left( \ln \lambda_\ell^P \mu_\ell^Q \lambda_{\ell,i}^{-1} \right)_\ell \right\|,$$

where  $\|(a_\ell)_\ell\| = \max_\ell |a_\ell|$ . If the latter is bounded below by  $\frac{C}{(|P|+|Q|)^\tau}$ , so is

$$\left\| \left( \ln \tilde{\lambda}_\ell^P \tilde{\mu}_\ell^Q \tilde{\lambda}_{\ell,i}^{-1} \right)_\ell \right\|. \quad \square$$

**Theorem 4.8.** *Let  $C$  be an  $n$ -dimensional complex torus, holomorphically embedded into a complex manifold  $M_{n+d}$ . Assume that  $T_C M$  splits. Assume the normal bundle  $N_C$  has (locally constant) Hermitian transition functions. Assume that  $N_C$  is Diophantine. Then some neighborhood of  $C$  is biholomorphic to a neighborhood of the zero section in the normal bundle.*

**Remark 4.9.** *When  $C$  is a product of 1-dimensional tori with normal bundle which is a direct sum of line bundles, the above result is due to Il'yashenko-Pyatli [IP79] (under a stronger small-division condition when  $n > 1$ ).*

We have  $\tau_j = \hat{\tau}_j + \tau_j^\bullet = \hat{\tau}_j + (\tau_j^h, \tau_j^v)$ . Here, functions

$$\tau_j^\bullet(h, v) = \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} \tau_{j,Q}^\bullet(h) v^Q$$

are holomorphic in  $(h, v)$  in a neighborhood of  $\Omega_{\epsilon,r}$  with values in  $\mathbb{C}^{n+d}$ .

To work with a function  $f$  that involving expressions  $f \circ \hat{\tau}_i$  for  $i = 1, \dots, n$  simultaneously, we need to define a norm of  $f, f \circ \hat{\tau}_i$  on the same domain  $\Omega_\epsilon \times \Delta_r^d$ , i.e. the norm  $\|f\|_{\epsilon,r}$  of the  $f$  on the union of  $\Omega_\epsilon \times \Delta_r^d$  with all its images under  $\hat{\tau}_i$ . We further remark that the definition of  $f$  that appears as coordinate functions on the union is more subtle. This will be treated with care at the end of the proof of Proposition 4.17. Thus, we will use the following notations.

**Definition 4.10.** *Set  $\Omega_{\epsilon,r} := \Omega_\epsilon \times \Delta_r^d$ ,  $\tilde{\Omega}_{\epsilon,r} := \overline{\Omega}_{\epsilon,r} \cup \bigcup_{i=1}^n \hat{\tau}_i(\overline{\Omega}_{\epsilon,r})$ . Denote by  $\mathcal{A}_{\epsilon,r}$  (resp.  $\tilde{\mathcal{A}}_{\epsilon,r}$ ) the set of holomorphic functions on  $\overline{\Omega}_{\epsilon,r}$  (resp.  $\overline{\tilde{\Omega}}_{\epsilon,r}$ ). If  $f \in \mathcal{A}_{\epsilon,r}$  (resp.  $\tilde{f} \in \tilde{\mathcal{A}}_{\epsilon,r}$ ), we set*

$$(4.10) \quad \|f\|_{\epsilon,r} := \sup_{(h,v) \in \Omega_{\epsilon,r}} |f(h,v)|, \quad \|\tilde{f}\|_{\epsilon,r} := \sup_{(h,v) \in \tilde{\Omega}_{\epsilon,r}} |\tilde{f}(h,v)|.$$

Each  $f \in \mathcal{A}_{\epsilon,r}$  can be expressed as a convergent Taylor-Laurent series

$$f(h, v) = \sum_{P \in \mathbb{Z}^n, Q \in \mathbb{N}^d} f_{Q,P} h^P v^Q$$

for  $(h, v) \in \Omega_{\epsilon,r} = \Omega_\epsilon \times \Delta_r^d$ .

We need to introduce more subdomains of  $\Omega_\epsilon$ . For an  $\ell$ -tuple  $J = (j_1, \dots, j_\ell)$  of indices in  $\{1, \dots, n\}$  and  $t = (t', t'')$  with  $t', t'' \in \mathbb{R}^n$  we define

$$(4.11) \quad \omega_\epsilon^J := \left\{ \sum_{j=1}^{2n} t_j e_j : t' \in [0, 1)^n, t''_{j_i} - 1 \in (-\epsilon, \epsilon), t''_k \in (-\epsilon, 1 + \epsilon), k \neq j_i, \forall i \right\},$$

$$(4.12) \quad \tilde{\omega}_\epsilon^J := \left\{ \sum_{j=1}^{2n} t_j e_j : t' \in [0, 1)^n, t''_{j_i} \in (-\epsilon, \epsilon), t''_k \in (-\epsilon, 1 + \epsilon), k \neq j_i, \forall i \right\}.$$

Note that  $\tilde{\omega}_\epsilon^{j_1 \dots j_\ell}$  and  $\omega_\epsilon^{j_1 \dots j_\ell}$  are subsets of  $\omega_\epsilon$ , and  $\omega_\epsilon^{1 \dots n} = \{\sum_{j=1}^{2n} t_j e_j \in \omega_\epsilon : t \in [0, 1)^n \times (-\epsilon, \epsilon)^n\}$ . Then

$$(4.13) \quad \Omega_\epsilon^{j_1 \dots j_\ell} := \Omega_\epsilon \cap \bigcap_{k=1}^{\ell} T_{j_k}^{-1} \Omega_\epsilon = \{(e^{2\pi i \zeta_1}, \dots, e^{2\pi i \zeta_n}) : \zeta \in \omega_\epsilon^{j_1 \dots j_\ell}\},$$

$$(4.14) \quad \tilde{\Omega}_\epsilon^{j_1 \dots j_\ell} := \Omega_\epsilon \cap \bigcap_{k=1}^{\ell} T_{j_k} \Omega_\epsilon = \{(e^{2\pi i \zeta_1}, \dots, e^{2\pi i \zeta_n}) : \zeta \in \tilde{\omega}_\epsilon^{j_1 \dots j_\ell}\}$$

are connected non-empty Reinhardt domains. Moreover,  $\Omega_0^{1 \cdots n} := \cap_{\epsilon > 0} \Omega_\epsilon^{1 \cdots n}$  and  $\tilde{\Omega}_0^{1 \cdots n} = \cap_{\epsilon > 0} \tilde{\Omega}_\epsilon^{1 \cdots n}$  are diffeomorphic to the real torus  $(S^1)^n$ . We remark that  $T_i \circ T_j$  maps  $\Omega_\epsilon^{ij}$  into  $\Omega_\epsilon$  for  $i \neq j$ , while  $T_i \circ T_i$  does not map  $\Omega_\epsilon^{ii}$  into  $\Omega_\epsilon$ . Since  $\hat{\tau}_1, \dots, \hat{\tau}_n$  are diagonal, we obtain analogously

$$\begin{aligned} \Omega_{\epsilon,r}^{j_1 \cdots j_\ell} &:= \Omega_{\epsilon,r} \cap_{k=1}^\ell \hat{\tau}_{j_k}^{-1} \Omega_{\epsilon,r}^{j_1 \cdots j_\ell} = \Omega_\epsilon^{j_1 \cdots j_\ell} \times \Delta_{r'_1} \times \cdots \times \Delta_{r'_n}, \\ \tilde{\Omega}_{\epsilon,r}^{j_1 \cdots j_\ell} &:= \Omega_{\epsilon,r} \cap_{k=1}^\ell \hat{\tau}_{j_k} \Omega_{\epsilon,r}^{j_1 \cdots j_\ell} = \tilde{\Omega}_\epsilon^{j_1 \cdots j_\ell} \times \Delta_{r''_1} \times \cdots \times \Delta_{r''_n} \end{aligned}$$

where  $r', r''$  depend on  $j_1, \dots, j_\ell$  and  $r$ . We can record the following, for later use :

**Lemma 4.11.** *For  $i \neq j$ , the set  $\hat{\tau}_j \bar{\Omega}_{\epsilon,r} \cap \hat{\tau}_i(\bar{\Omega}_{\epsilon,r})$  is a connected Reinhardt domain containing  $\tilde{\Omega}_\epsilon^{1 \cdots n} \times \Delta_{r'}^d$  in  $\mathbb{C}^{n+d}$  when  $r' > 0$  is sufficiently small.*

**4.1. Holomorphic functions on  $\Omega_{\epsilon,r}$ .** In this section, we study elementary properties and estimate holomorphic functions  $f$  on  $\bar{\Omega}_{\epsilon,r}$ .

**Lemma 4.12.** *An element  $f(h)$  of  $\mathcal{A}_\epsilon$ , that is a holomorphic function in a neighborhood of  $\bar{\Omega}_\epsilon$ , admits a Laurent series expansion in  $h$*

$$(4.15) \quad f(h) = \sum_{P \in \mathbb{Z}^n} c_P h^P.$$

The series converges normally on  $\Omega_\epsilon$ . Moreover, the Laurent coefficients

$$(4.16) \quad c_P = \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=s_1, \dots, |\zeta_n|=s_n} f(\zeta) \zeta^{-P-(1, \dots, 1)} d\zeta_1 \wedge \cdots \wedge d\zeta_n$$

are independent of  $s \in \Omega_\epsilon^+$  and

$$(4.17) \quad |c_P| \leq \sup_{\Omega_\epsilon} |f| \inf_{s \in \Omega_\epsilon^+} s^{-P}.$$

*Proof.* Obviously, estimate (4.17) follows from (4.16). Define  $A(a, b) = \{w \in \mathbb{C} : a < |w| < b\}$ . Fix  $h \in \Omega_\epsilon$ . Then  $|h| := (|h_1|, \dots, |h_n|) \in \Omega_\epsilon^+$ . The latter is an open set and we have for a small positive number  $\epsilon = \epsilon(h)$ ,

$$f(h) = \frac{1}{(2\pi i)^n} \int_{\partial A(|h_1|-\epsilon, |h_1|+\epsilon)} \cdots \int_{\partial A(|h_n|-\epsilon, |h_n|+\epsilon)} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - h_1) \cdots (\zeta_n - h_n)}.$$

By Laurent expansion in one-variable, we get the expansion (4.15) in which  $c_P$  are given by (4.16) if we take  $s_j = |h_j| \pm \epsilon$  according to the sign of  $p_j \in \mathbb{Z}^\mp$ .

We want to show that  $c_P$  is independent of  $s \in \Omega_\epsilon^+$ . Note that  $\Omega_\epsilon^+$  is a connected open set. For any two points  $s, \tilde{s}$  can be connected by a union of line segments in  $\Omega_\epsilon^+$  which are parallel to coordinate axes in  $\mathbb{R}^n$ . Using such line segments, say a line segment  $[a, b] \times (s_2, \dots, s_n)$  in  $\Omega_\epsilon^+$ , we know that  $f(\zeta) \zeta^{-P-(1, \dots, 1)}$  is holomorphic in  $\zeta_1$  in the closure of  $A(a, b)$  when  $|\zeta_2| = s_2, \dots, |\zeta_n| = s_n$ . By Cauchy theorem, the integrals are independent of  $s_1 \in [a, b]$  for these  $s_2, \dots, s_n$ . This shows that (4.16) is independent of  $s$ .

Finally the series converges uniformly on each compact subset of  $\Omega_\epsilon$ . Indeed, for a small perturbation of  $h$ , we can choose  $\epsilon(h)$  to be independent of  $h$ . Then we can see easily that the series converges locally uniformly in  $h$ .  $\square$

Recall that

$$\begin{aligned} \mathcal{P}_\epsilon^+ &:= \left\{ \sum_{i=1}^n t_i \operatorname{Im} e'_i : t \in (-\epsilon, 1 + \epsilon)^n \right\}, \\ \Omega_\epsilon^+ &:= \{ (e^{-2\pi R_1}, \dots, e^{-2\pi R_n}) : R \in \mathcal{P}_\epsilon^+ \}. \end{aligned}$$



**Lemma 4.13.** *There is a constant  $\kappa_0 > 0$  that depends only on  $\text{Im } e'_1, \dots, \text{Im } e'_n$  such that if  $P \in \mathbb{R}^n$  and  $\epsilon > \epsilon'$ , there exists  $R \in \overline{\mathcal{P}_\epsilon^+}$  such that for all  $R' \in \mathcal{P}_{\epsilon'}^+$  we have*

$$(4.18) \quad (R' - R) \cdot P := \sum_{j=1}^n (R'_j - R_j) P_j \leq -\kappa_0(\epsilon - \epsilon')|P|.$$

*Proof.* Without loss of generality, we may assume that  $P$  is a unit vector. Let  $\pi(x)P$  with  $\pi(x) \in \mathbb{R}$  be the orthogonal projection from  $x \in \mathbb{R}^n$  onto the line spanned by  $P$ . Choose  $R \in \overline{\mathcal{P}_\epsilon^+}$  so that  $\pi(R)$  has the largest value for  $R \in \overline{\mathcal{P}_\epsilon^+}$ . Note that  $R$  must be on the boundary of  $\mathcal{P}_\epsilon^+$  and latter is contained in the half-space  $H$  defined by  $\pi(y) \leq \pi(R)$  for  $y \in \mathbb{R}^n$ . Hence,  $\partial H$  is orthogonal to  $P$ . Then for any  $R' \in \mathcal{P}_{\epsilon'}^+$ ,

$$\pi(R) - \pi(R') = \text{dist}(R', \partial H) \geq \text{dist}(\partial \mathcal{P}_\epsilon^+, \mathcal{P}_{\epsilon'}^+) \geq (\epsilon - \epsilon')/C.$$

Therefore, we obtain  $(R' - R) \cdot P = -(\pi(R) - \pi(R')) \leq -(\epsilon - \epsilon')/C$ .  $\square$

**Remark 4.14.** *Since  $\mathcal{P}_\epsilon^+$  is a parallelotope, we can choose  $R$  to be a vertex of  $\mathcal{P}_\epsilon^+$ .*

In what follows, we denote the fixed constant :

$$(4.19) \quad \kappa := 2\pi\kappa_0.$$

**Lemma 4.15** (Cauchy estimates). *If  $f = \sum_{Q \in \mathbf{N}^d, P \in \mathbb{Z}^n} f_{QP} h^P v^Q \in \mathcal{A}_{\epsilon, r}$ , then for all  $(P, Q) \in \mathbb{Z}^n \times \mathbb{N}^d$ ,*

$$(4.20) \quad |f_{Q, P}| \leq \frac{\|f\|_{\epsilon, r}}{r^{|Q|} \max_{s \in \Omega_\epsilon^+} s^P},$$

where  $\Omega_\epsilon^+ = \{(e^{-2\pi R_1}, \dots, e^{-2\pi R_n}) : R \in \mathcal{P}_\epsilon^+\}$  (see (4.3)). If  $0 < \delta < \kappa\epsilon$  additionally, then

$$\|f\|_{\epsilon - \delta/\kappa, r e^{-\delta}} \leq \frac{C\|f\|_{\epsilon, r}}{\delta^\nu},$$

where  $C$  and  $\nu$  depend only on  $n$  and  $d$ .

*Proof.* According to Lemma 4.12 and Cauchy estimates on polydiscs, we have for any  $s \in \Omega_\epsilon^+$ ,

$$|f_{Q, P}| \leq \frac{\sup_{\Omega_\epsilon} |f_Q(h)|}{s^P} \leq \frac{\sup_{\Omega_{\epsilon, r}} |f|}{s^P r^{|Q|}}.$$

According to Lemma 4.12 and Cauchy estimates for polydiscs, we have if  $(h, v) \in \Omega_{\epsilon - \delta/\kappa, r e^{-\delta}}$ , then for all  $s \in \Omega_\epsilon^+$ ,

$$|f_Q(h)| \leq \frac{\sup_{v \in \Delta_r^d} |f(h, v)|}{r^{|Q|}},$$

$$|f_{Q, P} h^P| \leq \left| \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=s_1, \dots, |\zeta_n|=s_n} f_Q(\zeta) \frac{h^P}{\zeta^P} \frac{d\zeta_1 \wedge \dots \wedge d\zeta_n}{\zeta_1 \dots \zeta_n} \right|.$$

Set  $s_j = e^{-2\pi R_j}$ ,  $|h_j| = e^{-2\pi R'_j}$ ,  $R = (R_1, \dots, R_n)$  and  $R' = (R'_1, \dots, R'_n)$ . By Lemma 4.13 in which  $\epsilon', P$  are  $\epsilon - \delta, -P$  respectively, we obtain

$$(4.21) \quad \inf_{(|\zeta_1|, \dots, |\zeta_n|) = s \in \Omega_\epsilon^+} \sup_{h \in \Omega_{\epsilon - \delta/\kappa, r e^{-\delta}}} \left| \frac{h^P}{\zeta^P} \right| = \inf_{R \in \mathcal{P}_\epsilon^+} \sup_{R' \in \mathcal{P}_{\epsilon - \delta/\kappa}^+} e^{2\pi(R' - R) \cdot (-P)} \leq e^{-\kappa\delta'|P|},$$

where the positive constant  $\kappa$ , defined by Lemma 4.13 and (4.19), is independent of  $P, \zeta, h$ . Thus

$$(4.22) \quad |f_{Q,P} h^P v^Q| \leq \frac{\sup_{\Omega_{\epsilon,r}} |f| e^{-\kappa \delta' |P| r^{|Q|}} e^{-\delta |Q|}}{r^{|Q|}} \leq \sup_{\Omega_{\epsilon,r}} |f| e^{-\delta' \kappa |P|} e^{-\delta |Q|}.$$

Hence, setting  $\delta' := \delta/\kappa$ , we have

$$\|f\|_{\epsilon-\delta/\kappa, r e^{-\delta}} \leq \sum_{Q \in \mathbf{N}^d, P \in \mathbf{Z}^n} |f_{Q,P} h^P v^Q| \leq \frac{C \sup_{\Omega_{\epsilon,r}} |f|}{\delta^\nu},$$

where  $C$  and  $\nu$  depend only on  $n$  and  $d$ .  $\square$

**4.2. Conjugacy of the deck transformations.** Let us show that there is a bi-holomorphism  $\Phi = (h, v) + \phi(h, v)$  of some neighborhood  $\Omega_{\tilde{\epsilon}, \tilde{r}}$ , fixing  $\tilde{C}$  pointwise (i.e.  $\Phi(h, 0) = (h, 0)$ ) conjugating the deck transformations  $\{\tau_i\}_{i=1, \dots, n}$  of  $\tilde{M}$  to the deck transformations  $\{\hat{\tau}_i\}_{i=1, \dots, n}$  of  $\tilde{N}_C$ , that is :

$$\Phi \circ \hat{\tau}_i = \tau_i \circ \Phi, \quad i = 1, \dots, n.$$

This reads  $\hat{\tau}_i + \phi(\hat{\tau}_i) = \hat{\tau}_i (Id + \phi) + \tau_i^\bullet (Id + \phi)$ , that is, for all  $i = 1, \dots, n$ ,

$$(4.23) \quad \mathcal{L}_i(\phi) = \tau_i^\bullet (Id + \phi) + (\hat{\tau}_i (Id + \phi) - \hat{\tau}_i - D\hat{\tau}_i \cdot \phi) = \tau_i^\bullet (Id + \phi).$$

Here we define  $\mathcal{L}_i(\phi) := \phi(\hat{\tau}_i) - D\hat{\tau}_i \cdot \phi$  and  $(\mathcal{L}_i^h(\phi^h), \mathcal{L}_i^v(\phi^v)) := \mathcal{L}_i(\phi)$ . Since  $\hat{\tau}_j$  are linear, then  $D\hat{\tau}_j = \hat{\tau}_j$ . We have

$$(\mathcal{L}_i^h(\phi^h), \mathcal{L}_i^v(\phi^v)) = (\phi^h(T_i h, M_i v) - T_i \phi^h(h, v), \phi^v(T_i h, M_i v) - M_i \phi^v(h, v)).$$

Expand the latter in Taylor-Laurent expansions as

$$\begin{aligned} \mathcal{L}_i^h(\phi^h) &= \sum_{Q \in \mathbf{N}^d, |Q| > 1} \left( \sum_{P \in \mathbf{Z}^n} (\lambda_i^P \mu_i^Q \times Id_n - T_i) \phi_{Q,P}^h h^P \right) v^Q, \quad \phi_{Q,P}^h \in \mathbb{C}^n, \\ \mathcal{L}_i^v(\phi^v) &= \sum_{Q \in \mathbf{N}^d, |Q| > 1} \left( \sum_{P \in \mathbf{Z}^n} (\lambda_i^P \mu_i^Q \times Id_d - M_i) \phi_{Q,P}^v h^P \right) v^Q, \quad \phi_{Q,P}^v \in \mathbb{C}^d. \end{aligned}$$

Recall the notations  $\lambda_\ell = (\lambda_{\ell,1}, \dots, \lambda_{\ell,n})$  and  $\mu_\ell = (\mu_{\ell,1}, \dots, \mu_{\ell,d})$ . With  $P = (p_1, \dots, p_n) \in \mathbf{Z}^n$  and  $Q = (q_1, \dots, q_d) \in \mathbf{N}^d$ , we have

$$\lambda_\ell^P \mu_\ell^Q := \prod_{i=1}^n \lambda_{\ell,i}^{p_i} \prod_{j=1}^d \mu_{\ell,j}^{q_j}.$$

**Lemma 4.16.** *Let  $\tau_j \in \mathcal{A}_{\epsilon,r}^{n+d}$  with  $\tau_j = \hat{\tau}_j - \tilde{F}_j$  and  $\tilde{F}_j(h, v) = O(|v|^{q+1})$  with  $q \geq 1$ . Suppose that  $\tau_i \tau_j = \tau_j \tau_i$  in a neighborhood of  $\Omega_0^{1 \cdots n} \times \{0\}$  in  $\mathbb{C}^{n+d}$  and  $i \neq j$ . Then  $\mathcal{L}_i(\tilde{F}_j) - \mathcal{L}_j(\tilde{F}_i) = O(|v|^{2q+1})$ .*

*Proof.* Recall that  $\hat{\tau}_i(h, v)$  are linear maps in  $h, v$ . Also,  $\hat{\tau}_i \hat{\tau}_j$  sends  $\Omega_\epsilon^j \times \Delta_\epsilon^d$  into  $\mathbb{C}^{n+d}$ . Since  $\tau_j \in \mathcal{A}_{\epsilon,r}^{n+d}$  and  $\tau_j = \hat{\tau}_j + O(|v|^2)$ , the continuity implies that  $\tau_i \tau_j$  is well-defined on the product domain  $\Omega_{\epsilon'}^j \times \Delta_{r'}^d$ , when  $\epsilon', r'$  are sufficiently small. Fix  $h \in \Omega_{\epsilon'}^j$ . By Taylor expansions in  $v$ , we obtain

$$\tau_i \tau_j(h, v) = \hat{\tau}_i \hat{\tau}_j(h, v) + \hat{\tau}_i \circ \tilde{F}_j(h, v) + \tilde{F}_i \circ \hat{\tau}_j(h, v) + O(|v|^{2q+1}).$$

Since  $\hat{\tau}_i \hat{\tau}_j = \hat{\tau}_j \hat{\tau}_i$  and  $\tau_i \tau_j = \tau_j \tau_i$  in a neighborhood of  $\Omega_0^{1 \cdots n} \times \{0\}$ , we get  $\mathcal{L}_i(\tilde{F}_j) = \mathcal{L}_j(\tilde{F}_i) + O(|v|^{2q+1})$  in a possibly smaller neighborhood of  $\Omega_0^{1 \cdots n} \times \{0\}$ .  $\square$

We will apply the following result to  $F_j = \hat{\tau}_j - J^{2q}(\tau_j)$ , where  $J^{2q}(\tau_j)$  denotes the  $2q$ -jet at 0 in the variable  $v$ . They are holomorphic on  $\Omega_{\epsilon, r}$ . Recall that

$$\Omega_{\epsilon', r'}^{ij} := \Omega_{\epsilon', r'} \cap \hat{\tau}_i^{-1}(\Omega_{\epsilon', r'}) \cap \hat{\tau}_j^{-1}(\Omega_{\epsilon', r'}).$$

Recall that  $\|f\|_{\epsilon, r}$  is defined in (4.10) for a holomorphic function  $f \in \mathcal{A}_{\epsilon, r}$  (see Definition 4.10). For a holomorphic mapping  $F \in \mathcal{A}_{\epsilon, r}^\ell$ , define  $\|F\|_{\epsilon, r} = \max\{\|F_1\|_{\epsilon, r}, \dots, \|F_\ell\|_{\epsilon, r}\}$ .

**Proposition 4.17.** *Assume  $N_C$  is Diophantine. Fix  $\epsilon_0, r_0, \delta_0$  in  $(0, 1)$ . Let  $0 < \epsilon' < \epsilon < \epsilon_0$ ,  $0 < r' < r < r_0$ ,  $0 < \delta < \delta_0$ , and  $\frac{\delta}{\kappa} < \epsilon$ . Suppose that  $F_i = O(|v|^2) \in \mathcal{A}_{\epsilon, r}^{n+d}$ ,  $i = 1, \dots, n$ , satisfy*

$$(4.24) \quad \mathcal{L}_i(F_j) - \mathcal{L}_j(F_i) = 0 \quad \text{on } \Omega_{\epsilon', r'}^{ij}.$$

There exists a mapping  $G \in \tilde{\mathcal{A}}_{\epsilon-\delta/\kappa, r\epsilon^{-\delta}}^{n+d}$  (see Definition 4.10) such that

$$(4.25) \quad \mathcal{L}_i(G) = F_i \quad \text{on } \Omega_{\epsilon-\delta/\kappa, r\epsilon^{-\delta}}.$$

Furthermore, the  $G$  satisfies

$$(4.26) \quad \|G\|_{\epsilon-\delta/\kappa, r\epsilon^{-\delta}} \leq \max_i \|F_i\|_{\epsilon, r} \frac{C'}{\delta\tau+\nu},$$

$$(4.27) \quad \|G \circ \hat{\tau}_i\|_{\epsilon-\delta/\kappa, r\epsilon^{-\delta}} \leq \max_i \|F_i\|_{\epsilon, r} \frac{C'}{\delta\tau+\nu}$$

for some constant  $C'$  that is independent of  $F, q, \delta, r, \epsilon$  and  $\nu$  that depends only on  $n$  and  $d$ . Furthermore, if  $F_j(h, v) = J^{2q}F_j(h, v) = O(|v|^{q+1})$  for all  $j$ , then  $G$  can be chosen so that

$$(4.28) \quad G(h, v) = O(|v|^{q+1}), \quad G(h, v) = J^{2q}G(h, v).$$

*Proof.* Since  $F_i \in \mathcal{A}_{\epsilon, r}^{n+d}$ , we can write

$$F_i(h, v) = \sum_{Q \in \mathbb{N}^d, |Q| \geq 2} \sum_{P \in \mathbb{Z}^n} F_{i, Q, P} h^P v^Q,$$

which converges normally for  $(h, v) \in \Omega_{\epsilon, r}$ . We emphasize that  $F_{i, Q, P}$  are vectors, and its  $k$ th component is denoted by  $F_{i, k, Q, P}$ . For each  $(Q, P) \in \mathbb{N}^d \times \mathbb{Z}^n$ , each  $i = 1, \dots, n$ , and each  $j = 1, \dots, d$ , let  $i_h := i_h(Q, P, i)$ ,  $i_v := i_v(Q, P, j)$  be in  $\{1, \dots, n\}$  as in Definition 4.4. Let us set

$$(4.29) \quad G_i^h := \sum_{Q \in \mathbb{N}^d, 2 \leq |Q| \leq 2q} \sum_{P \in \mathbb{Z}^n} \frac{F_{i_h, i, Q, P}^h}{\lambda_{i_h}^P \mu_{i_h}^Q - \lambda_{i_h, i}} h^P v^Q, \quad i = 1, \dots, n$$

$$(4.30) \quad G_j^v := \sum_{Q \in \mathbb{N}^d, 2 \leq |Q| \leq 2q} \sum_{P \in \mathbb{Z}^n} \frac{F_{i_v, j, Q, P}^v}{\lambda_{i_v}^P \lambda_{i_v}^Q - \mu_{i_v, j}} h^P v^Q, \quad j = 1, \dots, d.$$

According to (4.24), we have

$$(4.31) \quad (\lambda_{i_h}^P \mu_{i_h}^Q - \lambda_{i_h, i}) F_{m, i, Q, P}^h = (\lambda_m^P \mu_m^Q - \lambda_{m, i}) F_{i_h, i, Q, P}^h, \quad 2 \leq |Q| \leq 2q.$$

Therefore, using (4.31), the  $i$ th-component of  $\mathcal{L}_m^h(G^h)$  reads

$$\begin{aligned} (\mathcal{L}_m^h(G^h))_i &= \sum_{Q \in \mathbb{N}^d, 2 \leq |Q| \leq 2q} \sum_{P \in \mathbb{Z}^n} (\lambda_m^P \mu_m^Q - \lambda_{m, i}) \frac{F_{i_h, i, Q, P}^h}{(\lambda_{i_h}^P \mu_{i_h}^Q - \lambda_{i_h, i})} h^P v^Q \\ &= \sum_{Q \in \mathbb{N}^d, 2 \leq |Q| \leq 2q} \sum_{P \in \mathbb{Z}^n} F_{m, i, Q, P}^h h^P v^Q. \end{aligned}$$

Proceeding similarly for the vertical components, we have obtained the formal solution  $G$  to the following equations :

$$(4.32) \quad \mathcal{L}_m(G) = F_m, \quad m = 1, \dots, n.$$

Let us estimate these solutions. According to Definition 4.5 and formulas (4.29)-(4.30), we have

$$(4.33) \quad \max_{i,j} (|G_{i,Q,P}^h|, |G_{j,Q,P}^v|) \leq \max_i |F_{i,Q,P}| \frac{(|P| + |Q|)^\tau}{D}.$$

Let  $(h, v) \in \Omega_{\epsilon-\delta/\kappa, re^{-\delta}}$ . According to (4.22), we have

$$\begin{aligned} \|G_{Q,P}^h h^P v^Q\| &\leq \max_i \|F_i\|_{\epsilon,r} e^{-\delta(|P|+|Q|)} \frac{(|P| + |Q|)^\tau}{D} \\ &\leq \max_i \|F_i\|_{\epsilon,r} e^{-\delta/2(|P|+|Q|)} \frac{(2\tau)^\tau}{D(e\delta)^\tau}, \end{aligned}$$

where the last inequality is obtained by using the maximum value of  $y^\tau e^{-\delta y}$  for  $y > 0$  at the critical point  $\tau/\delta$ . Summing over  $P$  and  $Q$ , we obtain

$$\|G\|_{\epsilon-\delta/\kappa, re^{-\delta}} \leq \max_i \|F_i\|_{\epsilon,r} \frac{C'}{\delta^{\tau+\nu}},$$

for some constants  $C', \nu$  that are independent of  $F, \epsilon, \delta$ . Hence,  $G \in \mathcal{A}_{\epsilon-\delta/\kappa, re^{-\delta}}^{n+d}$ .

Let us prove (4.27). Let  $B := 2 \max_{k,i,j} (|\lambda_{k,i}|, |\mu_{k,j}|)$ . Then, there is a constant  $D'$  such that

$$(4.34) \quad \max_{\ell \in \{1, \dots, n\}} \left| \lambda_\ell^P \mu_\ell^Q - \lambda_{\ell,i} \right| \geq \frac{D' \max_k |\lambda_k^P \mu_k^Q|}{(|P| + |Q|)^\tau},$$

$$(4.35) \quad \max_{\ell \in \{1, \dots, n\}} \left| \lambda_\ell^P \mu_\ell^Q - \mu_{\ell,j} \right| \geq \frac{D' \max_k |\lambda_k^P \mu_k^Q|}{(|P| + |Q|)^\tau}.$$

Indeed, if  $\max_k |\lambda_k^P \mu_k^Q| < B$ , then Definition 4.5 gives (4.34) with  $D' := \frac{D}{B}$ . Otherwise, if

$$|\lambda_{k_0}^P \mu_{k_0}^Q| := \max_k |\lambda_k^P \mu_k^Q| \geq B,$$

then  $|\lambda_{k_0,i}| \leq \frac{B}{2} \leq \frac{|\lambda_{k_0}^P \mu_{k_0}^Q|}{2}$ . Hence, we have

$$\left| \lambda_{k_0}^P \mu_{k_0}^Q - \lambda_{k_0,i} \right| \geq \left| |\lambda_{k_0}^P \mu_{k_0}^Q| - |\lambda_{k_0,i}| \right| \geq \frac{|\lambda_{k_0}^P \mu_{k_0}^Q|}{2}.$$

Finally, we define  $D' = \min\{\frac{D}{B}, \frac{1}{2}\}$ . We have verified (4.34). Similarly, we can verified (4.35). Finally, combining all cases gives us, for  $m = 1, \dots, n$ ,

$$\begin{aligned} |[G \circ \hat{\tau}_m]_{QP}| &= |G_{Q,P} \lambda_m^P \mu_m^Q| \leq \max_\ell |F_{\ell,Q,P}| \frac{|\lambda_m^P \mu_m^Q|}{|\lambda_{i_h}^P \mu_{i_h}^Q - \lambda_{i_h,i}|} \\ &\leq \max_\ell |F_{\ell,Q,P}| \frac{|\lambda_m^P \mu_m^Q| (|P| + |Q|)^\tau}{D' \max_k |\lambda_k^P \mu_k^Q|} \\ &\leq \max_\ell |F_{\ell,Q,P}| \frac{(|P| + |Q|)^\tau}{D'}. \end{aligned}$$

Hence,  $\tilde{G}_m := G \circ \hat{\tau}_m \in \mathcal{A}_{\epsilon-\delta/\kappa, re^{-\delta}}^{n+d}$ . We can define  $\tilde{G} \in \tilde{\mathcal{A}}_{\epsilon-\delta/\kappa, re^{-\delta}}^{n+d}$  such that  $\tilde{G} = \tilde{G}_m \hat{\tau}_m^{-1}$  on  $\hat{\tau}_m \Omega_{\epsilon,r}$ . We verify that  $\tilde{G}$  extends to a single-valued holomorphic function of class  $\tilde{\mathcal{A}}_{\epsilon,r}$ . Indeed,  $\tilde{G}_i \hat{\tau}_i^{-1} = \tilde{G}_j \hat{\tau}_j^{-1}$  on  $\hat{\tau}_i \Omega_{\epsilon,r} \cap \hat{\tau}_j \Omega_{\epsilon,r}$ , since the latter is

connected by Lemma 4.11 and the two functions agree with  $G$  on  $\hat{\tau}_i \Omega_{\epsilon, r} \cap \hat{\tau}_j \Omega_{\epsilon, r} \cap \Omega_{\epsilon, r}$  that contains a neighborhood of  $\tilde{\Omega}_\epsilon \times \{0\}$  in  $\mathbb{C}^{n+d}$ .

Finally, we remark that our result is mainly on the existence of  $G$  and its estimates. The  $G = O(|v_j|^2)$  is unique under the non-resonance condition. In fact, the  $G = O(2)$  is given by (4.29)-(4.30). The uniqueness also implies that  $G(h, v) = J^{2q}G(h, v) = O(|v|^{q+1})$  if  $F_j(h, v) = J^{2q}F_j(h, v) = O(|v|^{q+1})$  for all  $j$ .  $\square$

In what follows, we shall set  $\hat{\tau}_0 = Id$  and for any  $f \in (\tilde{A}_{\epsilon, r})^{n+d}$  and  $F \in (\tilde{A}_{\epsilon, r})^{n+d}$ , we modify the norm  $||| \cdot |||_{\epsilon, r}$  in (4.10) for a function to define a norm for a mapping  $f$  by the following

$$|||f|||_{\epsilon, r} := \|f\|_{\epsilon, r} + \sum_{i=1}^n \|\hat{\tau}_i^{-1} \circ f \circ \hat{\tau}_i\|_{\epsilon, r} = \sum_{i=0}^n \|\hat{\tau}_i^{-1} \circ f \circ \hat{\tau}_i\|_{\epsilon, r},$$

$$F_{(i)} := \hat{\tau}_i^{-1} \circ F \circ \hat{\tau}_i \in \mathcal{A}_{\epsilon, r}, \quad F_{(0)} := F, \quad i = 1, \dots, n.$$

**4.3. Iteration scheme.** We shall prove the main result through a Newton scheme. For  $\delta_0, r_0, \epsilon_0$  to be determined in  $(0, 1)$ , let us define sequences of positive real numbers

$$\delta_k := \frac{\delta_0}{(k+1)^2}, \quad r_{k+1} := r_k e^{-5\delta_k}, \quad \epsilon_{k+1} := \epsilon_k - \frac{5\delta_k}{\kappa}$$

such that

$$(4.36) \quad \sigma := \sum_{k \geq 0} \delta_k < 2\delta_0.$$

We assume the following conditions hold :

$$(4.37) \quad \delta_0 < \frac{\kappa}{20} \epsilon_0,$$

$$(4.38) \quad \delta_0 < \frac{\ln 2}{10}.$$

Condition (4.37) ensures that

$$\frac{5}{\kappa} \sigma < \frac{10}{\kappa} \delta_0 < \frac{\epsilon_0}{2}, \quad \text{so that} \quad \epsilon_k > \frac{\epsilon_0}{2}, \quad k \geq 0.$$

Condition (4.38) ensures that

$$e^{-5\sigma} > e^{-10\delta_0} > \frac{1}{2}, \quad \text{so that} \quad r_k > \frac{r_0}{2}, \quad k \geq 0.$$

Let  $m = 5$  be fixed. We define  $\epsilon_{k+1} \leq \epsilon_k^{(\ell)} < \epsilon_k$  and  $r_{k+1} \leq r_k^{(\ell)} < r_k$ ,  $\ell = 1, \dots, m$  as follows :

$$\begin{aligned} \epsilon_k^{(\ell)} &= \epsilon_k - \frac{\ell \delta_k}{\kappa}, & \epsilon_{k+1} &= \epsilon_k^{(m)}, \\ r_k^{(\ell)} &= r_k e^{-\ell \delta_k}, & r_{k+1} &= r_k^{(m)}. \end{aligned}$$

We emphasize that condition (4.37) ensures  $\epsilon_k^{(\ell)} > 0$ . Let us assume that for each  $i = 1, \dots, n$ ,  $\tau_i^{(k)} = \hat{\tau}_i + \tau_i^{\bullet(k)}$ , is holomorphic and for  $\mu > 0$  to be chosen at the end of the proof of Proposition 4.19 on  $\Omega_{\epsilon_k, r_k}$  we have

$$(4.39) \quad |||\tau^{\bullet(k)}|||_{\epsilon_k, r_k} < \delta_k^\mu, \quad \tau^{\bullet(k)}(h, v) = O(|v|^{q_k+1}).$$

We further assume that  $\tau_i^{(k)}, \tau_j^{(k)}$  commute on  $\Omega_{\epsilon', r'}^{ij} \subset \Omega_{\epsilon_k, r_k}$  for some positive  $\epsilon' < \epsilon_k, r' < r_k$ . We take

$$q_{k+1} = 2q_k + 1 \geq q_0 2^k$$

for  $q_0 \geq 1$  to be determined.

We will define a sequence  $\Phi^{(k)}$  with  $\Phi^{(k)}(h, 0) = (h, 0)$ . Let us write on appropriate domains

$$\begin{aligned} \Phi^{(k)} &= Id + \phi^{(k)}, \quad (\Phi^{(k)})^{-1} := Id - \psi^{(k)} =: \Psi^{(k)}, \\ \tau_i^{(k+1)} &= \Phi^{(k)} \circ \tau_i^{(k)} \circ (\Phi^{(k)})^{-1}, \quad i = 1, \dots, n. \end{aligned}$$

Note that there is a constant  $C > 0$  (depending only on the  $\mu_{i,j}$ 's) such that if  $\epsilon'' < \epsilon', Cr'' < r'$  then

$$\Omega_{\epsilon'', r''}^{ij} \subset \Omega_{\epsilon'} \times \Delta_{r'}^d, \quad \Omega_{\epsilon''} \times \Delta_{r''}^d \subset \Omega_{\epsilon', r'}^{ij}.$$

Since the  $\tau_i^{(k)}, \tau_j^{(k)}$  commute on  $\Omega_{\epsilon'} \times \Delta_{\epsilon'}^d$  for some  $\epsilon' > 0$  and  $\Phi^{(k)}(h, v) = (h, v) + O(|v|^2)$ , then  $\tau_i^{(k+1)}, \tau_j^{(k+1)}$  still commute on the same kind of domains for a possibly smaller  $\epsilon'$ . By Lemma 4.16, we obtain :

$$(4.40) \quad \mathcal{L}_j(\tau_i^{\bullet(k)}) - \mathcal{L}_i(\tau_j^{\bullet(k)}) = O(|v|^{2q_k+1}).$$

We want to find  $\phi^{(k)}$  so that

$$(4.41) \quad \begin{aligned} \tau_i^{(k+1)} &:= \hat{\tau}_i + \tau_i^{\bullet(k+1)} = \Phi^{(k)} \circ \tau_i^{(k)} \circ (\Phi^{(k)})^{-1} \\ &= \hat{\tau}_i (Id - \psi^{(k)}) + \tau_i^{\bullet(k)} (Id - \psi^{(k)}) \\ &\quad + \phi^{(k)} \left( \hat{\tau}_i (Id - \psi^{(k)}) + \tau_i^{\bullet(k)} (Id - \psi^{(k)}) \right) \end{aligned}$$

is defined and bounded on  $\Omega_{\epsilon_{k+1}, r_{k+1}}$ .

We now define  $\Phi^{(k)}$  by applying Proposition 4.17 with these  $F_i := -J^{2q_k}(\tau_i^{\bullet(k)})$ . Let  $\phi^{(k)}$  stand for  $G$ . Therefore, given  $0 < \delta < \kappa \epsilon_k^{(1)}$ ,  $\phi^{(k)}$  is holomorphic and bounded on  $\tilde{\Omega}_{\epsilon_k - \frac{\delta}{\kappa}, r_k e^{-\delta}}$  and it satisfies on  $\Omega_{\epsilon_k^{(1)} - \frac{\delta}{\kappa}, r_k^{(1)} e^{-\delta}}$ ,

$$(4.42) \quad \mathcal{L}_i(\phi^{(k)}) := \phi^{(k)}(\hat{\tau}_i) - D\hat{\tau}_i \cdot \phi^{(k)} = -J^{2q_k}(\tau_i^{\bullet(k)}), \quad i = 1, \dots, n.$$

Writing formally  $\Phi^{(k)} \circ \Psi^{(k)} = Id$  and using linearity of  $\hat{\tau}_i$ , we obtain

$$(4.43) \quad \psi_{(i)}^{(k)} = \phi_{(i)}^{(k)} (Id - \psi_{(i)}^{(k)}), \quad i = 1, \dots, n,$$

defining the notation  $\phi_{(i)}^{(k)} = \hat{\tau}_i^{-1} \circ \phi^{(k)} \circ \hat{\tau}_i$ . According to Proposition 4.17, we have

$$(4.44) \quad |||\phi^{(k)}|||_{\epsilon_k - \frac{\delta}{\kappa}, r_k e^{-\delta}} \leq ||\tau^{\bullet(k)}||_{\epsilon_k, r_k} \frac{C'}{\delta^{\tau+\nu}} \leq C' \delta_k^\mu \delta^{-\tau-\nu}.$$

We recall that the constant  $C'$  does not depend on  $k$ , nor on  $\tau^{\bullet(k)}$ .

**Lemma 4.18.** *There is constant  $\tilde{D} > 0$  (independent of  $k$ ) such that if positive numbers  $\mu$  and  $\delta < \min\{\tilde{D}, \ln 2, \frac{\tau_0}{2}\}$  satisfy*

$$(4.45) \quad 2^{m+2} C' \delta_k^\mu \delta^{-\tau-\nu-3} < 1,$$

where  $m = 5$ , then for all  $0 \leq \ell \leq m$  and  $i = 0, \dots, n$ , the maps  $\Phi_{(i)}^{(k)} := Id + \phi_{(i)}^{(k)}$  are biholomorphisms satisfying

$$\Phi_{(i)}^{(k)} : \Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}, r_k e^{-(\ell+1)\delta}} \rightarrow \Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}, r_k e^{-\ell\delta}}$$

with inverse  $\Psi_{(i)}^{(k)} := Id - \psi_{(i)}^{(k)} : \Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}} \rightarrow \Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}, r_k e^{-(\ell+1)\delta}}$  satisfying

$$\Phi_{(i)}^{(k)} \circ \Psi_{(i)}^{(k)} = \hat{\tau}_i^{-1} \circ (\Phi^{(k)} \circ \Psi^{(k)}) \circ \hat{\tau}_i = Id$$

on  $\Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}$ .

*Proof.* Before we compute, we mention that the lemma will be applied to  $\delta = \delta_k$  where the  $\delta_k$  is the decreasing sequence in (4.36) with  $\delta_0 < 1$  to be further determined. Thus  $\mu$  will be a fixed number bigger than  $\tau + \nu + 3$  so that  $\delta_k^{\mu - \tau - \nu - 3} < \frac{1}{2^{\tau} C'}$ .

We have  $1 - e^{-\delta} > \delta/2$  and  $\frac{1}{2} < e^{-\delta}$  for  $0 < \delta < \ln 2$ . Assuming  $\delta < \frac{r_0}{2}$ , we have  $\delta < r_k$  so that

$$\text{dist}(\Delta_{r_k e^{-(\ell+1)\delta}}^d, \partial \Delta_{r_k e^{-\ell\delta}}^d) = r_k e^{-\ell\delta} (1 - e^{-\delta}) > \frac{\delta^2}{2^{\ell+1}}.$$

On the other hand, by (4.4), we have

$$\text{dist}(\Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}}^+, \partial \Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}}^+) = \text{dist}(\Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}}^+, \partial \Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}}^+).$$

Let  $(e^{-2\pi R}, e^{-2\pi R'}) \in \Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}}^+ \times \partial \Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}}^+$ . By (4.2)-(4.3),  $(R, R') \in \mathcal{P}_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}}^+ \times \partial \mathcal{P}_{\epsilon_k - \frac{\ell\delta}{\kappa}}^+$ . Since the matrix  $(\text{Im } e'_{ij})_{1 \leq i, j \leq n}$  is invertible, there is a constant  $\tilde{C}_0$  (independent of  $k$ ) such that

$$\text{dist}(\mathcal{P}_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}}^+, \partial \mathcal{P}_{\epsilon_k - \frac{\ell\delta}{\kappa}}^+) > \tilde{C}_0^{-1} \frac{\delta}{\kappa}.$$

Since  $R \rightarrow (e^{-2\pi R_1}, \dots, e^{-2\pi R_d})$  is a diffeomorphism between bounded sets, then there exists  $\tilde{C}$  (independent of  $k$ ) such that

$$|e^{-2\pi R} - e^{-2\pi R'}| \geq \tilde{C}^{-1} |R' - R| > \tilde{C}_0^{-1} \tilde{C}^{-1} \frac{\delta}{\kappa}.$$

Let us set  $\tilde{D} := \frac{1}{C_0 \tilde{C} \kappa}$ . Assuming  $\delta < \tilde{D}$ , we have  $\delta < \tilde{D} < 2^{\ell+2} \tilde{D} \leq 2^{m+2} \tilde{D}$ . Assume  $2^{m+2} C' \delta_k^{\mu} \delta^{-\tau-\nu-3} < 1$ . Then according to (4.44), we have, for  $(h, v) \in \Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}, r_k e^{-\ell\delta}}$ ,

$$(4.46) \quad |\phi_{(i)}^{(k)}(h, v)| \leq C' \delta_k^{\mu} \delta^{-\tau-\nu} < \frac{\delta^3}{2^{m+2}} < \frac{\delta}{2} \frac{\delta^2}{2^{\ell+1}} < \frac{\delta}{2} \text{dist}(\Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}, r_k e^{-(\ell+1)\delta}}, \partial \Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}, r_k e^{-\ell\delta}}).$$

By the Cauchy inequality, we have, for  $(h, v) \in \Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}$

$$(4.47) \quad |D\phi_{(i)}^{(k)}(h, v)| \leq \frac{C' \delta_k^{\mu} \delta^{-\tau-\nu}}{\text{dist}(\Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}, \partial \Omega_{\epsilon_k - \frac{(\ell+1)\delta}{\kappa}, r_k e^{-(\ell+1)\delta}})} \leq \frac{\delta}{2} < \frac{1}{2}.$$

We can apply the contraction mapping theorem to (4.43) together with the last inequality of (4.46). We find a holomorphic solution  $\psi_{(i)}^{(k)}$  such that for  $(h, v) \in \Omega_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}}$

$$(4.48) \quad |\psi_{(i)}^{(k)}(h, v)| \leq \|\phi_{(i)}^{(k)}\|_{\epsilon_k - \frac{\delta(\ell+2)}{\kappa}, r_k e^{-(\ell+2)\delta}} \leq C' \delta_k^{\mu} \delta^{-\tau-\nu} \leq \frac{\delta^3}{2^{m+2}} < \frac{\delta}{2^{m-\ell-1}} \text{dist}(\Omega_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}}, \partial \Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}).$$

Hence, we have found a mapping  $\Psi_{(i)}^{(k)} := Id - \psi_{(i)}^{(k)}$  such that

$$\hat{\Omega}_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}} := \Psi_{(i)}^{(k)}(\Omega_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}}) \subset \Omega_{\epsilon_k - \frac{(\ell+2)\delta}{\kappa}, r_k e^{-(\ell+2)\delta}}.$$

Also,  $\Phi_{(i)}^{(k)} \circ \Psi_{(i)}^{(k)} = Id$  on  $\Omega_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}}$ . Therefore

$$\Phi_{(i)}^{(k)} : \hat{\Omega}_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}} \rightarrow \Omega_{\epsilon_k - \frac{(\ell+3)\delta}{\kappa}, r_k e^{-(\ell+3)\delta}}$$

is an (onto) biholomorphism such that  $(\Phi_{(i)}^{(k)})^{-1} = \Psi_{(i)}^{(k)}$ .  $\square$

**Proposition 4.19.** *Keep conditions on  $\delta, \mu$  in Lemma 4.18 as well as conditions (4.37), (4.38). If  $\delta_0$  is small enough there is possibly larger  $\mu > 0$  such that if for all  $i = 1, \dots, n$ ,  $\tau_i^{(0)} \in (\mathcal{A}_{\epsilon_0, r_0})^{n+d}$  satisfy  $\|\tau_i^{(0)}\|_{\epsilon_0, r_0} \leq \delta_0^\mu$ , then for all  $k \geq 0$  we have the following :*

$$(4.49) \quad \begin{aligned} \tau_i^{\bullet(k+1)} &\in (\mathcal{A}_{\epsilon_{k+1}, r_{k+1}})^{n+d}, \quad \tau_i^{\bullet(k+1)} = O(|v|^{2q_k+1}), \\ \|\tau_i^{\bullet(k+1)}\|_{\epsilon_{k+1}, r_{k+1}} &\leq \delta_{k+1}^\mu, \end{aligned}$$

provided  $q_0 > C(\delta_0, \mu)$ .

*Proof.* Let us first show that  $\tau_i^{(k+1)} := \Phi_{(i)}^{(k)} \circ \tau_i^{(k)} \circ (\Phi_{(i)}^{(k)})^{-1}$  is well defined on  $\Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}, r_k e^{-\ell\delta}}$  for  $m \geq \ell \geq 0$  and all  $i = 1, \dots, n$ . Here  $m$  is fixed from Lemma 4.18. Indeed, we have

$$\tau_i^{(k+1)} = \hat{\tau}_i(I + \phi_{(i)}^{(k)}) \circ (I + \hat{\tau}_i^{-1} \tau_i^{\bullet(k)}) \circ (\Phi_{(i)}^{(k)})^{-1}.$$

Since  $\tau_i^{\bullet(k)}$  is of order  $\geq 2q_{k-1} + 1$ , the Schwarz inequality gives

$$\|\hat{\tau}_i^{-1} \tau_i^{\bullet(k)}\|_{\epsilon_k - \frac{(\ell-1)\delta}{\kappa}, r_k e^{-(\ell-1)\delta}} \leq \max_i \|\hat{\tau}_i^{-1}\|_{\epsilon_0, r_0} C e^{-(2q_{k-1}+1)\delta} \|\tau_i^{\bullet(k)}\|_{\epsilon_k - \frac{(\ell-2)\delta}{\kappa}, r_k e^{-(\ell-2)\delta}}.$$

We recall that  $\delta_0$  satisfies (4.37) and (4.38). Setting  $\delta := \delta_k$  and if  $q_0$  large enough, we have

$$\max_i \|\hat{\tau}_i^{-1}\|_{\epsilon_0, r_0} C e^{-(2q_{k-1}+1)\delta} \leq \max_i \|\hat{\tau}_i^{-1}\|_{\epsilon_0, r_0} C e^{-\frac{2k-1}{(k+1)^2} q_0 \delta_0} < 1.$$

According to Lemma 4.18, (4.45) and the distance estimate in (4.46), we have

$$(I + \hat{\tau}_i^{-1} \tau_i^{\bullet(k)}) \circ (\Phi_{(i)}^{(k)})^{-1}(\Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}, r_k e^{-\ell\delta}}) \subset \Omega_{\epsilon_k - \frac{(\ell-3)\delta}{\kappa}, r_k e^{-(\ell-3)\delta}}.$$

Thus,  $\tau_i^{(k+1)}$  is defined on  $\Omega_{\epsilon_k - \frac{\ell\delta}{\kappa}, r_k e^{-\ell\delta}}$  for  $\ell = 4 \leq m$  since  $\phi_{(i)}^{(k)}$  is defined on  $\Omega_{\epsilon_k - \delta/\kappa, r_k e^{-\delta/\kappa}}$ . For the rest of the proof, we fix  $\ell = 4$ . From the argument above, we have

$$\tau_i^{(k+1)}(\Omega_{\epsilon_k - 4\delta/\kappa, r_k e^{-4\delta/\kappa}}) \subset \hat{\tau}_i(\Phi_{(i)}^k(\Omega_{\epsilon_k - \delta/\kappa, r_k e^{-\delta/\kappa}})) \subset \hat{\tau}_i(\Omega_{\epsilon_k, r_k}) \subset \hat{\tau}_i(\Omega_{\epsilon_0, r_0}).$$

Hence,  $\tau_i^{\bullet(k+1)}$  is uniformly bounded on  $\Omega_{\epsilon_k - 4\delta/\kappa, r_k e^{-4\delta/\kappa}}$  w.r.t  $k$  :

$$\|\tau_i^{\bullet(k+1)}\|_{\epsilon_k - 4\delta/\kappa, r_k e^{-4\delta/\kappa}} \leq C.$$

On the other hand, on  $\Omega_{\epsilon_k - 4\delta/\kappa, r_k e^{-4\delta/\kappa}}$ , we have

$$(4.50) \quad \begin{aligned} \tau_i^{\bullet(k+1)} &= \hat{\tau}_i(\phi^{(k)} - \psi^{(k)}) + \left( \tau_i^{\bullet(k)}(Id - \psi^{(k)}) - \tau_i^{\bullet(k)} \right) \\ &\quad + \left( \phi^{(k)} \left( \hat{\tau}_i - \hat{\tau}_i \psi^{(k)} + \tau_i^{\bullet(k)}(Id - \psi^{(k)}) \right) - \phi^{(k)}(\hat{\tau}_i) \right) \\ &\quad + \left( \phi^{(k)}(\hat{\tau}_i) - \hat{\tau}_i \phi^{(k)} + \tau_i^{\bullet(k)} \right). \end{aligned}$$



The last term is equal to  $\tau_i^{\bullet(k)} - J^{2q_k}(\tau_i^{\bullet(k)}) = O(|v|^{2q_k+1})$ . We also have  $\phi^{(k)} - \psi^{(k)} = O(|v|^{2q_k+1})$ ,  $\phi^{(k)} = O(|v|^{q_k+1})$ . Thus

$$(4.51) \quad \tau_i^{\bullet(k+1)} = O(|v|^{2q_k+1}).$$

Improving the estimate by the Schwarz inequality, we obtain

$$\|\tau_i^{\bullet(k+1)}\|_{\epsilon_k - \frac{5\delta}{\kappa}, r_k e^{-5\delta}} \leq C e^{-q_{k+1}\delta}.$$

We have  $e^{-x} < 1/x$  for  $x > 0$ . For  $\delta := \delta_k < \min\{\tilde{D}, \ln 2, \frac{r_0}{2}\}$ , let  $\mu$  satisfy  $2^{m+2} C' \delta_k^{\mu-\tau-\nu-3} < 1$ , in which case assumptions of Lemma 4.18 are satisfied. We then obtain

$$C \delta_{k+1}^{-\mu} e^{-q_{k+1}\delta} \leq C \left( \frac{(k+2)^2}{\delta_0} \right)^{\mu+1} \frac{1}{q_0 2^k} < 1$$

provided  $q_0 > C(\delta_0, \mu) := \sup_k C \left( \frac{(k+2)^2}{\delta_0} \right)^{\mu+1} \frac{1}{2^k}$ . The last two displayed inequalities for the chosen  $\delta = \delta_k$  yield (4.49).  $\square$

Finally, quite classically, since (4.47) and (4.36), the sequence of diffeomorphisms  $\{\Phi_k \circ \Phi_{k-1} \circ \dots \circ \Phi_1\}_k$  converges uniformly on the open set  $\Omega_{\frac{\epsilon_0}{2}, r_0 e^{-\sigma}}$  to a diffeomorphism  $\Phi$  which satisfies

$$\Phi \circ \hat{\tau}_i = \tau_i \circ \Phi, \quad i = 1, \dots, n$$

provided  $q_0 \geq C(\delta_0, \mu)$  and  $\|\tau_i^{\bullet(0)}\|_{\epsilon_0, r_0} \leq \delta_0^\mu$  for some  $\delta_0, \mu$  are fixed. The condition  $q_0 > C(\delta_0, \mu, \tau, \nu)$  can be achieved by using finitely many  $\Phi_0, \dots, \Phi_m$ . Then initial condition  $\|\tau_i^{\bullet(0)}\|_{\epsilon_0, r_0} \leq \delta_0^\mu$  can be achieved easily by a dilation in the  $v$  variable. Indeed, we apply the dilation in  $v$  to the original  $\tau_1, \dots, \tau_n$  with  $q_0 = 1$ . This allows us to construct  $\Phi_0$  in Proposition 4.19 with  $k = 0$  and define  $\Phi_0 \tau_j \Phi_0^{-1}$  to achieve  $q_1 \geq 2$ . Then  $\Phi_0 \tau_j \Phi_0^{-1}$ ,  $j = 1, \dots, n$  still commute pairwise on  $\Omega_{\epsilon'} \times \Delta_{\epsilon'}^d$  for some  $\epsilon' > 0$ . Applying the procedure again, this allows us to find  $\Phi_1, \dots, \Phi_{k-1}$  to achieve  $q_k \geq 2^k > C(\delta_0, \mu)$ . Finally using dilation we can apply the full version of Proposition 4.19 for all  $k$  to construction a new sequence of desired mapping  $\Phi_0, \Phi_1, \dots$ . Hence, the torus  $C$  has a neighborhood in  $M$  biholomorphic to a neighborhood of the zero section in its normal bundle  $N_C$  since there is a biholomorphism fixing  $\tilde{C}$  that conjugates the deck transformations of the covering of the latter to those of the former.

**Remark 4.20.** *The assumption “ $T_C M$  splits” can be replaced by the condition that there exist positive constant  $D, \tau$  such that for all  $P \in \mathbb{Z}^n$ , all  $Q \in \mathbb{N}^d$  with  $|Q| = 1$ , and all  $i = 1, \dots, n$  we have*

$$\max_{\ell \in \{1, \dots, n\}} \left| \lambda_\ell^P \mu_\ell^Q - \lambda_{\ell, i} \right| > \frac{D}{(|P| + 1)^\tau}.$$

**Data Availability:** Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

**Conflict of interest:** On behalf of all authors, the corresponding author states that there is no conflict of interest.

## REFERENCES

- [Arn76] V. I. Arnol'd. Bifurcations of invariant manifolds of differential equations, and normal forms of neighborhoods of elliptic curves. *Funkcional. Anal. i Priložen.*, 10(4):1–12, 1976.
- [Arn88] V. I. Arnold. *Geometrical methods in the theory of ordinary differential equations*, volume 250 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1988. Translated from the Russian by Joseph Szücs [József M. Szücs].
- [BKM21] Massimiliano Berti, Thomas Kappeler, and Riccardo Montalto. Large KAM tori for quasi-linear perturbations of KdV. *Arch. Ration. Mech. Anal.*, 239(3):1395–1500, 2021.
- [BL04] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.
- [Deb05] Olivier Debarre. *Complex tori and abelian varieties*, volume 11 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2005. Translated from the 1999 French edition by Philippe Mazaud.
- [EFK15] L. H. Eliasson, B. Fayad, and R. Krikorian. Around the stability of KAM tori. *Duke Math. J.*, 164(9):1733–1775, 2015.
- [FR68] O. Forster and K. J. Ramsrott. Über die Anzahl der Erzeugenden von projektiven Steinschen Moduln. *Arch. Math. (Basel)*, 19:417–422, 1968.
- [Gan98] F. R. Gantmacher. *The theory of matrices. Vol. 1*. AMS Chelsea Publishing, Providence, RI, 1998. Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation.
- [GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.
- [Gon12] Xianghong Gong. Existence of divergent Birkhoff normal forms of Hamiltonian functions. *Illinois J. Math.*, 56(1):85–94 (2013), 2012.
- [Gra62] Hans Grauert. über Modifikationen und exzeptionelle analytische Mengen. *Math. Ann.*, 146:331–368, 1962.
- [Gri66] Phillip A. Griffiths. The extension problem in complex analysis. II. Embeddings with positive normal bundle. *Amer. J. Math.*, 88:366–446, 1966.
- [GS19] X. Gong and L. Stolovitch. Real submanifolds of maximum complex tangent space at a CR singular point II. *J. Differential Geometry*, 112:121–198, 2019.
- [GS22] Xianghong Gong and Laurent Stolovitch. Equivalence of Neighborhoods of Embedded Compact Complex Manifolds and Higher Codimension Foliations. *Arnold Math. J.*, 8(1):61–145, 2022.
- [GS24] X. Gong and L. Stolovitch. A structure theorem for neighborhoods of compact complex manifolds. *J. Geom. Anal.*, 34(133):p.1–30, 2024.
- [HR64] H. Hironaka and H. Rossi. On the equivalence of imbeddings of exceptional complex spaces. *Math. Ann.*, 156:313–333, 1964.
- [Hwa19] Jun-Muk Hwang. An application of Cartan's equivalence method to Hirschowitz's conjecture on the formal principle. *Ann. of Math. (2)*, 189(3):979–1000, 2019.
- [Ien11] Oleksandr Iena. Vector bundles on elliptic curves and factors of automorphy. *Rend. Istit. Mat. Univ. Trieste*, 43:61–94, 2011.
- [IP79] Ju. S. Il'jashenko and A. S. Pjartli. Neighborhoods of zero type imbeddings of complex tori. *Trudy Sem. Petrovsk.*, (5):85–95, 1979.
- [IY08] Yulij Ilyashenko and Sergei Yakovenko. *Lectures on analytic differential equations*, volume 86 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
- [Koi21] Takayuki Koike. Linearization of transition functions of a semi-positive line bundle along a certain submanifold. *Ann. Inst. Fourier (Grenoble)*, 71(5):2237–2271, 2021.
- [Kos88] Siegmund Kosarew. Ein allgemeines Kriterium für das formale Prinzip. *J. Reine Angew. Math.*, 388:18–39, 1988.
- [Kri22] Raphaël Krikorian. On the divergence of Birkhoff normal forms. *Publ. Math. Inst. Hautes Études Sci.*, 135:1–181, 2022.
- [LTT19] Frank Loray, Olivier Thom, and Frédéric Touzet. Two-dimensional neighborhoods of elliptic curves: formal classification and foliations. *Mosc. Math. J.*, 19(2):357–392, 2019.

- [MR80] J. Morrow and H. Rossi. Canonical embeddings. *Trans. Amer. Math. Soc.*, 261(2):547–565, 1980.
- [MR81] J. Morrow and H. Rossi. Pluricanonical embeddings. *Compositio Math.*, 44(1-3):219–239, 1981.
- [Mun00] James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [ MR0464128].
- [NS60] L. Nirenberg and D. C. Spencer. On rigidity of holomorphic imbeddings. In *Contributions to function theory (Internat. Colloq. Function Theory, Bombay, 1960)*, pages 133–137. Tata Institute of Fundamental Research, Bombay, 1960.
- [Sev03] Mikhail B. Sevryuk. The classical KAM theory at the dawn of the twenty-first century. volume 3, pages 1113–1144, 1201–1202. 2003. {Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday}.
- [Siu77] Yum Tong Siu. Every Stein subvariety admits a Stein neighborhood. *Invent. Math.*, 38(1):89–100, 1976/77.
- [Sto00] L. Stolovitch. Singular complete integrability. *Publ. Math. I.H.E.S.*, 91:133–210, 2000.
- [Sto15] L. Stolovitch. Family of intersecting totally real manifolds of  $(\mathbb{C}^n, 0)$  and germs of holomorphic diffeomorphisms. *Bull. Soc. math. France*, 143(1):247–263, 2015.
- [SZ22] L. Stolovitch and Z. Zhao. Geometry of hyperbolic Cauchy-Riemann singularities and KAM-like theory for holomorphic involutions. *Math. Ann.*, pages 1–86, 2022.
- [Vic94] James W. Vick. *Homology theory*, volume 145 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1994. An introduction to algebraic topology.
- [Yin15] Wanke Yin. Divergent Birkhoff normal forms of real analytic area preserving maps. *Math. Z.*, 280(3-4):1005–1014, 2015.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, MADISON, WI 53706, U.S.A.

*Email address:* `gong@math.wisc.edu`

CNRS AND LABORATOIRE J.-A. DIEUDONNÉ U.M.R. 7351, UNIVERSITÉ CÔTE D’AZUR, PARC VALROSE 06108 NICE CEDEX 02, FRANCE

*Email address:* `Laurent.stolovitch@univ-cotedazur.fr`