Principal spectrum point and application to nonlocal dispersal operators

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Abstract

We propose a general framework for a sufficient condition of the existence of the principal eigenpair to the eigenvalue problem $\mathcal{L}[u]+\mathcal{H}[u]=\lambda u,\ (\lambda,u)\in\mathbb{R}\times X$, where $\mathcal{L}:X\to X$ is a positive bounded linear operator and $\mathcal{H}:D(\mathcal{H})\subset X\to X$ a closed, possibly unbounded linear operator in the Banach space $(X,\|\cdot\|)$. Criteria for (i) the existence of a principal eigenpair, (ii) Asynchronous Exponential Growth (AEG) and, (iii) continuity result of the spectral bound are given, without necessarily specifying forms of operators for the above problem. The criteria are then applied to generalize some results in the existing literature in context of nonlocal dispersal operators. Our results are applied to a model of a chemostat and a model of a spatial evolution of a man-environment disease.

Keywords: Nonlocal diffusion operators; Principal eigenvalue, Resolvent positive operator; Asynchronous exponential growth.

1 Introduction

This work is devoted to a sufficient condition for the existence of the principal eigenpair of the following eigenvalue problem

$$\mathcal{L}[u]+\mathcal{H}[u]=\lambda u,\ (\lambda,u)\in\mathbb{R}\times X \tag{1}$$

where $\mathcal{L}:X\to X$ is a positive bounded linear operator and $\mathcal{H}:D(\mathcal{H})\subset X\to X$ a closed, possibly unbounded linear operator in the Banach space $(X,\|\cdot\|)$. Problem (1) was motivated by the emergence, during the last decade, of a trend in mathematical evolutionary epidemiology aiming to simultaneously model the epidemic and the evolutionary dynamics arising in population dynamics, essentially inspired by quantitative genetics and epidemiology, *eg.* [39, 34, 41, 3, 33, 23, 40, 42, 29, 11, 12]. Problem (1) is particularly relevant when dealing with the asymptotic behaviour of solutions for the model of interest involving nonlocal diffusion, where $\mathcal{L}$ usually represents a compact kernel operator and $\mathcal{H}$ a multiplication operator, *eg.* [18, 3, 23, 31, 43, 40, 10, 32, 2, 14, 9].

The existence and a variational characterisation of the principal eigenvalue of Problem (1) have been studied for particular cases of the form of operators $\mathcal{L}$ and $\mathcal{H}$. More precisely, in [18, 10, 2, 21, 9] authors proved, under certain conditions, the existence of a principal eigenvalue for the operator $\mathcal{L} + \mathcal{H}$, where $\mathcal{L}[u] = \int_\Omega k(x, y)u(y)dy$ and $\mathcal{H}[u] = b(x)u$, k being a probability distribution kernel, and b a continuous bounded function on $\Omega \subset \mathbb{R}^n$. While an assumption on the symmetry of the kernel k is required, for *eg.* by [21, 3], to characterise the principal eigenpair of Problem (1); such a symmetry property of the kernel is relaxed in [10]. In particular, in [10], the author established a rather general criterion for the existence of a principal eigenpair of Problem (1), and explore the relation between the sign of the largest element of the spectrum with a strong maximum property satisfied by the operator $\mathcal{L} + \mathcal{H}$.

In this work, we propose a more general framework for a sufficient condition of the existence of the principal eigenpair to the eigenvalue problem (1). Such an approach is not yet addressed in the above-mentioned works. Compared to the existing literature, which considers particular cases of operators for the eigenvalue problem (1), our results are more general without necessarily specifying the forms of operators for Problem (1). Importantly, such an approach can be applied to the evolutionary dynamics of various problems. More precisely, before stating our main results, we first introduce the below assumptions on operators $\mathcal{L}$ and $\mathcal{H}$ of Problem (1).

Assumption 1.1 *We assume that,*

1) $\mathcal{L} : X \to X$ *is a positive bounded linear operator that is not identically zero.*

2) The spectrum $\sigma(\mathcal{H})$ *of the operator* $\mathcal{H}$ *is such that* $\sigma(\mathcal{H}) \neq \emptyset$*, and* $\mathcal{H}$ *is resolvent positive i.e., there exists* $\lambda_0 \in \mathbb{R}$ *such that if* $\lambda \geq \lambda_0$ *then* $\lambda - \mathcal{H}$ *is invertible and* $(\lambda I_{\mathrm{d}} - \mathcal{H})^{-1} X_+ \subset X_+$*, where* X_+ *is the positive cone of the Banach space* $(X, \|\cdot\|)$*, assumed normal and reproducing.*

We recall that a positive cone X_+ is normal if there exists an equivalent norm $\|\cdot\|_1$ in X such that

$$x \leq y \Rightarrow \|x\|_1 \leq \|y\|_1;$$

while X_+ is reproducing if $X = X_+ - X_-$.

Denote by $s(\mathcal{H})$, the spectral bound of $\mathcal{H}$, defined by

$$s(\mathcal{H}) = \sup \{\Re(\lambda) : \lambda \in \sigma(\mathcal{H})\}.$$

Moreover, Assumption 1.1 mainly allows to state that if $\mathcal{H}$ is resolvent positive with $\sigma(\mathcal{H}) \neq \emptyset$ then, $(\lambda I_{\mathrm{d}} - \mathcal{H})^{-1}$ exists for all $\lambda > s(\mathcal{H})$ and $(\lambda I_{\mathrm{d}} - \mathcal{H})^{-1} X_+ \subset X_+$. The above statement is quite well known in the context of resolvent positive operators and will be more precise later (but, for *eg.*, see [36]).

Next, we also assume that

Assumption 1.2 X *is a Banach lattice, the operator* $\mathcal{L} : X \to X$ *is compact and,* *i)* *There exists* $\lambda > s(\mathcal{H})$ *such that, the spectral radius* $r\left(\mathcal{L}(\lambda I_{\mathrm{d}} - \mathcal{H})^{-1}\right)$, *of the operator* $\mathcal{L}(\lambda I_{\mathrm{d}} - \mathcal{H})^{-1}$, *satisfies* $r\left(\mathcal{L}(\lambda I_{\mathrm{d}} - \mathcal{H})^{-1}\right) > 1$.

ii) $\mathcal{H}$ *generates a strongly continuous semigroup of bounded linear operators.*

iii) *The Banach lattice* X *is either a) an abstract L space, b) an abstract M space, or c)* $X = L^p(\Omega, \mu)$ *with* $1 \le p < \infty$ *and some positive* σ*-finite measure* μ.

iv) *The strongly continuous semigroup* $\{T(t)\}_{t\ge 0}$ *of bounded linear operators generated by* $\mathcal{L} + \mathcal{H}$ *is irreducible.*

Let $\mathcal{B}(X)$ denotes the Banach space of bounded linear operators defined from X into X. We recall that a semigroup $\{T(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ is irreducible if for each $u \in X_+ \setminus \{0\}$ and $u^* \in X_+^* \setminus \{0\}$ (the dual of the positive cone X_+), there exists $t_0 \ge 0$ such that $(T(t_0)[u], u^*) > 0$.

We also recall the definition of the Asynchronous Exponential Growth (AEG) as follows (see also [38, 37])

Definition 1.3 *Let* $\{T(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ *be a strongly continuous semigroup.* $\{T(t)\}_{t\ge 0}$ *has asynchronous exponential growth with intrinsic growth constant* $\lambda_0 \in \mathbb{R}$ *if and only if there is a nonzero finite rank projection* $P_0 \in \mathcal{B}(X)$ *such that*

$$\lim_{t\to+\infty} e^{-\lambda_0 t} T(t) = P_0$$

where the above limit is in the operator norm topology.

We then have the following result

Theorem 1.4 *Let Assumptions 1.1 and 1.2 be satisfied. Then, the strongly continuous semigroup* $\{T(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ *generated by* $\mathcal{L}+\mathcal{H}$ *has an AEG with intrinsic growth constant* $\lambda_0 = s(\mathcal{L} + \mathcal{H})$. *Furthermore,* λ_0 *is the principal eigenvalue of* $\mathcal{L} + \mathcal{H}$.

Remark 1.5 *Theorem 1.4 implies that* λ_0 *is the principal eigenvalue of* $\mathcal{L}+\mathcal{H}$ *with eigenfunction* $\varphi \in X_+$. *The projector* P_0 *associated to the AEG is of rank one with*

$$R(P_0) = N(\lambda_0 - \mathcal{L} - \mathcal{H}),$$

where $R(P_0)$ *is the range of* P_0 *and* $N(\lambda_0 - \mathcal{L} - \mathcal{H})$ *is the kernel of* $\lambda_0 - \mathcal{L} - \mathcal{H}$.

We also have $(P_0[u], u^*) > 0$ *for all* $u \in X_+$, $u^* \in X_+^*$ *both non-zero. Moreover we have the following properties [38]*

$$T(t)P_0 = e^{\lambda_0 t} P_0, \ \forall t \ge 0.$$

Compared to some works on eigenvalue problems which consider particular cases of operators (*eg.*, [18, 10, 21, 9]), Theorem 1.4 provides a quite general framework for the existence of a principal eigenpair for some class of operators as soon as Assumptions 1.1 and 1.2 are satisfied. More precisely, the results provided by Theorem 1.4 can be applied to some eigenvalue problem arising in evolutionary dynamics of some various phenomena. For instance, for all $p \in [1, \infty)$, let us introduce the following eigenvalue problem

$$\mathcal{M}_p[u] = \lambda u, \quad (\lambda, u) \in \mathbb{R} \times L^p(\Omega), \tag{2}$$

where $\mathcal{M}_p : L^p(\Omega)^m \to L^p(\Omega)^m$ (with $m \geq 1$) is a linear operator such that $\mathcal{M}_p = (\mathcal{M}_p^i)_{1 \leq i \leq m}$, and for $i \in \{1, \cdots, m\}$,

$$\mathcal{M}_p^i[u](x) = \int_\Omega J_i(x-y) u_i(y) \mathrm{d}y + \sum_{j=1}^m h_{ij}(x) u_j(x), \quad \text{a.e } x \in \Omega,$$

with $\Omega \subset \mathbb{R}^n$ an open, bounded, connected subset of $\mathbb{R}^n$; J_i,s and h_{ij},s are functions satisfying the following assumption

Assumption 1.6 *We assume that,*

(H1) *The functions J_i,s are Lipschitz continuous on $\overline{\Omega}$ for $i = 1, \ldots, m$ and $J_i(x) > 0$ for all $x \in \overline{\Omega}$.*

(H2) *The functions h_{ij},s are continuous on $\overline{\Omega}$ for $1 \leq i, j \leq m$,*

(H3) *The matrix $H(x) = \{h_{ij}(x)\}_{1 \leq i,j \leq m}$ satisfies*

$$h_{ij}(x) > 0, \ \forall x \in \overline{\Omega}, \ i \neq j.$$

Next, the operator $\mathcal{M} \equiv \mathcal{M}_p$ rewrites

$$\mathcal{M}_p = \mathcal{L}_p + \mathcal{H}_p, \tag{3}$$

with $\mathcal{L}_p[u](x) = \left(\int_\Omega J_i(x-y) u_i(y) \mathrm{d}y\right)_{1 \leq i \leq m}$, and $\mathcal{H}_p[u](x) = \left(\sum_{j=1}^m h_{ij}(x) u_j(x)\right)_{1 \leq i \leq m}$. Then, we have the following result

Theorem 1.7 *Let Assumption (1.6) be satisfied. Set*

$$\lambda_+(x) = \sup\left\{\Re(\lambda) : \lambda \in \sigma(H(x))\right\}, \ \forall x \in \overline{\Omega}, \quad \textit{and} \quad s_+ = \max_{x \in \overline{\Omega}} \lambda_+(x). \tag{4}$$

If $\frac{1}{s_+ - \lambda_+(x)} \notin L^1_{loc}(\overline{\Omega})$, then $\lambda_p = s(\mathcal{L}_p + \mathcal{H}_p)$ is the principal eigenvalue of $\mathcal{L}_p + \mathcal{H}_p$ with the associated eigenfunction $u_p = (u_p^i)_{1 \leq i \leq m} \in C_+(\Omega)^m$ such that, $u_p^i(x) > 0$ for all $x \in \Omega$. Moreover, the semigroup $\{T_p(t)\}_{t \geq 0}$ generated by $\mathcal{L}_p + \mathcal{H}_p$ on $L^p(\Omega)^m$ has AEG with the intrinsic growth constant λ_p.

The proof of the above Theorem 1.7 consists in showing that conditions of Theorem 1.4 hold true. Furthermore, Theorem 1.7 then allow to generalize the criterion for the existence of a principal eigenpair to Problem (2) in $L^p(\Omega)^m$, with $m \geq 1$, compared, for *eg.*, to [10] where the case $m = 1$ is addressed. A special case of the eigenvalue problem (2) has been considered [25] for $m = 2$ and $H(x)$ a symmetric matrix independent of x.

This article is organized as follows. Section 2 is devoted to some preliminary results including a sufficient condition on the existence of a positive eigenpair to Problem (1), and the convergence result of the spectral bound. In Section 3, we then state the result on a sufficient condition for the existence of a principal eigenpair for the eigenvalue problem (2). Finally, in Section 4, we discuss two case studies where our general results are applied. The first case concerned a model of a chemostat and the second a model of the spatial evolution of a man-environment disease.

2 Preliminaries

Let us first recall some definitions that will be used in the manuscript.

Definition 2.1 *A real number $\lambda \in \mathbb{R}$ is called principal spectrum point of $\mathcal{L} + \mathcal{H}$ if it is an eigenvalue of $\mathcal{L} + \mathcal{H}$ and*

$$\lambda = \sup \{\Re(\lambda) : \lambda \in \sigma(\mathcal{L} + \mathcal{H})\}$$

where $\sigma(\mathcal{L} + \mathcal{H})$ is the spectrum of $\mathcal{L} + \mathcal{H}$.

From the above definition, one can see that the existence of principal spectrum point is equivalent of showing that $s(\mathcal{L} + \mathcal{H})$ is an eigenvalue of $\mathcal{L} + \mathcal{H}$.

Definition 2.2 *A real number $\lambda \in \mathbb{R}$ is called principal eigenvalue of $\mathcal{L} + \mathcal{H}$ if it is an algebraically simple eigenvalue of $\mathcal{L} + \mathcal{H}$ with positive eigenfunction and for each $\mu \in \sigma(\mathcal{L} + \mathcal{H}) \setminus \{\lambda\}$ we have $\Re(\mu) < \lambda$.*

Definition 2.3 *Let $(X, \|\cdot\|)$ be a Banach space with the positive cone X_+.*

- *Let $A : D(A) \subset X \to X$ be a closed linear operator in the Banach space X. The resolvent $\rho(A)$, spectrum $\sigma(A)$, point spectrum $\sigma_p(A)$, spectral bound $s(A)$, and spectral radius $r(A)$ of A is respectively defined by*

$$\begin{aligned} \rho(A) &= \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is invertible}\}, \\ \sigma(A) &= \mathbb{C} \setminus \rho(A), \\ \sigma_p(A) &= \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is not injective}\}, \\ s(A) &= \sup \{\Re(\lambda) : \lambda \in \sigma(A)\}, \\ r(A) &= \sup \{|\lambda| : \lambda \in \sigma(A)\}. \end{aligned}$$

- *A closed linear operator $A : D(A) \subset X \to X$ is called resolvent positive if there exists $\lambda_0 \in \mathbb{R}$ such that if $\lambda \geq \lambda_0$ then $\lambda - A$ is invertible and $(\lambda - X)^{-1} X_+ \subset X_+$.*

Definition 2.4 *Let $A : D(A) \subset X \to X$ be a closed linear operator in the Banach space X. If $\lambda \in \sigma(A)$, then the generalized eigenspace of A with respect to λ, denoted by $N_\lambda(A)$, is the smallest closed subspace of X containing*

$$\bigcup_{k=1}^{\infty} N((\lambda - A)^k)$$

where $N((\lambda - A)^k)$ is the kernel of $(\lambda - A)^k$. The essential spectrum of A, denoted by $\sigma_{ess}(A)$ is the set of $\lambda \in \sigma(A)$ such that at least one of the following holds:

i) $R(\lambda - A)$ the range of $\lambda - A$ is not closed;

ii) λ is a limit point of $\sigma(A)$;

iii) $N_\lambda(A)$ is infinite dimensional.

The essential spectral radius of A is defined as

$$r_{ess}(A) = \sup\{|\lambda| : \lambda \in \sigma_{ess}(A)\}.$$

Definition 2.5 *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators in the Banach space X with generator $A : D(A) \subset X \to X$. Then the growth bound of $\{T(t)\}_{t\geq 0}$ is*

$$\omega_0(A) := \lim_{t\to+\infty} \frac{\ln\left(\|T(t)\|_{\mathcal{B}(X)}\right)}{t}$$

and the essential growth bound of $\{T(t)\}_{t\geq 0}$ is

$$\omega_{0,ess}(A) := \lim_{t\to+\infty} \frac{\ln\left(\alpha[T(t)]\right)}{t}$$

where α is the Kuratowski measure of noncompactness (see [37]).

In the sequel, the positive cone X_+ of the Banach space $(X, \|\cdot\|)$ is always assumed normal and reproducing. The following theorem is well known in the context of resolvent positive operators (see *eg.* [36])

Theorem 2.6 *Let $A : D(A) \subset X \to X$ be a closed linear operator and resolvent positive in X. Assume that $\sigma(A)$ is non-empty. Then we have $s(A) > -\infty$ and $s(A) \in \sigma(A)$. Moreover $\lambda > s(A)$ if and only if $\lambda \in \rho(A)$ and $(\lambda - A)^{-1}X_+ \subset X_+$ i.e.,*

$$s(A) = \inf\left\{\lambda \in \mathbb{R} : \lambda \in \rho(A),\ (\lambda - A)^{-1}X_+ \subset X_+\right\}.$$

Next we derive the following condition that will be used in the proof of our main results.

Lemma 2.7 *Let Assumption 1.1 be satisfied. Then the following conditions are equivalent*

i) there exists $\lambda > s(\mathcal{H})$ such that $r\left(\mathcal{L}(\lambda - \mathcal{H})^{-1}\right) > 1$,

ii) $s(\mathcal{L} + \mathcal{H}) > s(\mathcal{H})$.

Proof. Let $\lambda > s(\mathcal{H})$ be given such that $r\left((\lambda - \mathcal{H})^{-1}\mathcal{L}\right) > 1$. Note that $\mathcal{H} - \lambda$ is resolvent positive with $s(\mathcal{H} - \lambda) = s(\mathcal{H}) - \lambda < 0$. Since $\mathcal{L}$ is a positive perturbation of $\mathcal{H} - \lambda$, it follows from [36, Theorem 3.5] that $r\left(\mathcal{L}(\lambda - \mathcal{H})^{-1}\right) - 1$ and $s(\mathcal{L} + \mathcal{H} - \lambda)$ have the same sign. Therefore

$$s(\mathcal{L} + \mathcal{H} - \lambda) > 0 \Rightarrow s(\mathcal{L} + \mathcal{H}) > \lambda > s(\mathcal{H}). \tag{5}$$

Assume that $s(\mathcal{L} + \mathcal{H}) > s(\mathcal{H})$ and let $\lambda_0 \in (s(\mathcal{H}), s(\mathcal{L} + \mathcal{H}))$ be fixed. Then we have $s(\mathcal{H} - \lambda_0) < 0$. Using again [36, Theorem 3.5] one knows that $r\left(\mathcal{L}(\lambda_0 - \mathcal{H})^{-1}\right) - 1$ and $s(\mathcal{L} + \mathcal{H} - \lambda_0)$ have the same sign. The result follows because $s(\mathcal{L} + \mathcal{H} - \lambda_0) > 0$. ■

Next, we give a simple criteria for $s(\mathcal{L} + \mathcal{H})$ to be an eigenvalue. Indeed, this is a consequence of [35] and Lemma 2.7.

Theorem 2.8 (Sufficient condition) *Let Assumption 1.1 be satisfied. Assume in addition that* $\mathcal{L}(\lambda-\mathcal{H})^{-1}$ *is compact for all* $\lambda > s(\mathcal{H})$ *and there exists* $\lambda > s(\mathcal{H})$ *such that* $r\left(\mathcal{L}(\lambda-\mathcal{H})^{-1}\right) \geq 1$. *Then* $\lambda_0 := s(\mathcal{L}+\mathcal{H})$ *is an eigenvalue of* $\mathcal{L}+\mathcal{H}$ *associated with* $u_0 \in X_+$ *and* $u_0 = (\lambda-\mathcal{H})^{-1}[\varphi]$ *for some* $\varphi \in X_+$ *satisfying*

$$\mathcal{L}(\lambda_0-\mathcal{H})^{-1}[\varphi] = \varphi.$$

Let us note that if there exists $\lambda > s(\mathcal{H})$ such that $r\left(\mathcal{L}(\lambda-\mathcal{H})^{-1}\right) > 1$ then by Lemma 2.7 we have $s(\mathcal{L}+\mathcal{H}) > s(\mathcal{H})$ and the result comes from [35, Theorems 4.7 and 4.9]. If $r\left(\mathcal{L}(\lambda-\mathcal{H})^{-1}\right) = 1$ for some $\lambda > s(\mathcal{H})$ then using similar arguments of the proof of Lemma 2.7 we get $s(\mathcal{L}+\mathcal{H}+\lambda) = 0$ *i.e.* $\lambda = s(\mathcal{L}+\mathcal{H})$ and the conclusion of Theorem 2.8 follows by applying Krein-Rutman's theorem to the compact positive operator $\mathcal{L}(\lambda-\mathcal{H})^{-1}$. It is worth mentioning that recently in [15], the authors discovered $s(\mathcal{L}+\mathcal{H}) > s(\mathcal{H})$ as a sufficient condition for the existence of a principal eigenpair. It is sometimes more simpler to estimate the spectral radius $r\left(\mathcal{L}(\lambda-\mathcal{H})^{-1}\right)$ rather that comparing the spectral bound of $s(\mathcal{L}+\mathcal{H})$ and $s(\mathcal{H})$. We can also note that in population dynamics models, such that $s(\mathcal{H}) < 0$ we can define the basic reproduction number by $\mathcal{R}_0 := r\left(\mathcal{L}(-\mathcal{H})^{-1}\right)$ according to [36]. Hence, we can automatically derive the existence of principal eigenpair when $\mathcal{R}_0 > 1$. Moreover, it is often important to have a convergence result on the spectral bound in order to perform asymptotic analysis such as global stability and uniform weak/strong persistence. This motivates the next investigations.

Theorem 2.9 *Let Assumption 1.1 be satisfied. Assume in addition that* $\mathcal{L} : X \to X$ *is compact and the following conditions are satisfied*

i) $\mathcal{H} \in \mathcal{B}(X)$ *and there exists* $\mu \geq 0$ *such that* $\mu+\mathcal{H}$ *is a positive operator.*

ii) There exists $\lambda > s(\mathcal{H})$ *such that* $r\left(\mathcal{L}(\lambda-\mathcal{H})^{-1}\right) > 1$ *or* $s(\mathcal{L}+\mathcal{H}) > s(\mathcal{H})$.

Then $\lambda_0 := s(\mathcal{L}+\mathcal{H})$ *is an eigenvalue of* $\mathcal{L}+\mathcal{H}$ *and a pole of the resolvent of* $\mathcal{L}+\mathcal{H}$ *with finite rank residuum.*

Proof. Since $\mu+\mathcal{H}$ is a bounded linear positive operator we have $r(\mu+\mathcal{H}) \in \sigma(\mu+\mathcal{H})$ so that

$$r(\mu+\mathcal{H}) = s(\mu+\mathcal{H}) = \mu + s(\mathcal{H}) \in \sigma(\mu+\mathcal{H}). \tag{6}$$

Moreover, because $\mu+\mathcal{L}+\mathcal{H}$ is a bounded linear positive operator, we also have

$$r(\mu+\mathcal{L}+\mathcal{H}) = s(\mu+\mathcal{L}+\mathcal{H}) = \mu + s(\mathcal{L}+\mathcal{H}) \in \sigma(\mu+\mathcal{L}+\mathcal{H}). \tag{7}$$

Next, using the fact that $\mathcal{L}$ is compact, we have

$$r_{ess}(\mu+\mathcal{H}) \geq r_{ess}(\mu+\mathcal{L}+\mathcal{H}). \tag{8}$$

Thus, it follows from item ii) and (6), (7) and (8) that

$$r(\mu+\mathcal{L}+\mathcal{H}) = \mu+s(\mathcal{L}+\mathcal{H}) > \mu+s(\mathcal{H}) = r(\mu+\mathcal{H}) \geq r_{ess}(\mu+\mathcal{H}) \geq r_{ess}(\mu+\mathcal{L}+\mathcal{H}). \tag{9}$$

Therefore, using (9), we infer from [37, Proposition 4.11] that $r(\mu + \mathcal{L} + \mathcal{H})$ is an eigenvalue of $\mu + \mathcal{L} + \mathcal{H}$ and a pole of the resolvent of $\mu + \mathcal{L} + \mathcal{H}$ with finite rank residuum. Recalling that $r(\mu + \mathcal{L} + \mathcal{H}) = \mu + s(\mathcal{L} + \mathcal{H})$, we conclude that $s(\mathcal{L} + \mathcal{H})$ is an eigenvalue of $\mathcal{L} + \mathcal{H}$ and a pole of the resolvent of $\mathcal{L} + \mathcal{H}$ with finite rank residuum. ∎

Using the foregoing Theorem 2.9, we can prove a convergence result of the spectral bounds. Such a convergence results will be very useful in analyzing the asymptotic behavior of certain epidemic models with nonlocal diffusion.

Theorem 2.10 (Convergence of the spectral bound) *Let Assumption 1.1 be satisfied. Assume in addition that X is a Banach lattice, $\mathcal{L} : X \to X$ is compact and the following conditions are satisfied*

i) $\mathcal{H} \in \mathcal{B}(X)$ and there exists $\mu \geq 0$ such that $\mu + \mathcal{H}$ is a positive operator.

ii) There exists $\lambda > s(\mathcal{H})$ such that $r\left(\mathcal{L}(\lambda - \mathcal{H})^{-1}\right) > 1$ or $s(\mathcal{L} + \mathcal{H}) > s(\mathcal{H})$.

Let $(\mathcal{L}_n) \subset \mathcal{B}(X)$ be a sequence of positive linear operators and $(\mathcal{H}_n) \subset \mathcal{B}(X)$ a sequence of linear operators with the following properties

iii) There exists $\mu \geq 0$ (independent of n) such that $(\mu + \mathcal{H}_n)$ is a sequence of positive linear operators

iv) $\mathcal{L}_n + \mathcal{H}_n \to \mathcal{L} + \mathcal{H}$, as $n \to +\infty$ in the operator norm topology.

Then we have

$$\lim_{n \to +\infty} s(\mathcal{L}_n + \mathcal{H}_n) = s(\mathcal{L} + \mathcal{H}).$$

Proof. Using similar arguments as for the proof of Theorem 2.9, one obtains that $r(\mu + \mathcal{L} + \mathcal{H})$ is a pole of the resolvent of $\mu + \mathcal{L} + \mathcal{H}$ with residuum of finite rank. Therefore using item iii) and iv), one can apply [1] to conclude that

$$\lim_{n \to +\infty} r(\mu + \mathcal{L}_n + \mathcal{H}_n) = r(\mu + \mathcal{L} + \mathcal{H}).$$

Since $\mu + \mathcal{L}_n + \mathcal{H}_n$ and $\mu + \mathcal{L} + \mathcal{H}$ are positive operators, we have $r(\mu + \mathcal{L}_n + \mathcal{H}_n) \in \sigma(\mu + \mathcal{L}_n + \mathcal{H}_n)$ so that

$$r(\mu + \mathcal{L}_n + \mathcal{H}_n) = s(\mu + \mathcal{L}_n + \mathcal{H}_n) = \mu + s(\mathcal{L}_n + \mathcal{H}_n)$$

and

$$r(\mu + \mathcal{L} + \mathcal{H}) = s(\mu + \mathcal{L} + \mathcal{H}) = \mu + s(\mathcal{L} + \mathcal{H}).$$

The result follows. ∎

The next result concerns AEG for the semigroup generated by $\mathcal{L} + \mathcal{H}$.

Theorem 2.11 (Asynchronous exponential growth) *Let Assumption 1.1 be satisfied. Assume in addition that X is a Banach lattice, $\mathcal{L} : X \to X$ is compact and the following conditions are satisfied*

i) There exists $\lambda > s(\mathcal{H})$ such that $r\left(\mathcal{L}(\lambda - \mathcal{H})^{-1}\right) > 1$ or $s(\mathcal{L} + \mathcal{H}) > s(\mathcal{H})$.

ii) $\mathcal{H}$ *generates a strongly continuous semigroup of bounded linear operators.*

iii) The Banach lattice X *is one of the following a) an abstract L space, b) an abstract M space, c)* $X = L^p(\Omega, \mu)$ *with* $1 \le p < \infty$ *and some positive* σ*-finite measure* μ*.*

iv) $\lambda_0 = s(\mathcal{L} + \mathcal{H})$ *is a simple pole of the resolvent of* $\mathcal{L} + \mathcal{H}$*.*

Then, the strongly continuous semigroup $\{T(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ *generated by* $\mathcal{L} + \mathcal{H}$ *has AEG with intrinsic growth constant* $\lambda_0 = s(\mathcal{L} + \mathcal{H})$*.*

Proof. We first note that condition iii) ensures (see [36, Theorem 3.14]) that

$$\omega_0(\mathcal{L} + \mathcal{H}) = s(\mathcal{L} + \mathcal{H}) \text{ and } s(\mathcal{H}) = \omega_0(\mathcal{H}). \tag{10}$$

Thus, it follows from condition ii) that

$$\omega_0(\mathcal{L} + \mathcal{H}) = s(\mathcal{L} + \mathcal{H}) > s(\mathcal{H}) = \omega_0(\mathcal{H}).$$

Thanks to ii), $\mathcal{H}$ generates a strongly continuous semigroup of bounded linear operators. By bounded linear perturbation theory, it follows that $\mathcal{L} + \mathcal{H}$ generates a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\ge 0} \subset \mathcal{B}(X)$. Since $\mathcal{L}$ is compact, one can use the result of [16][Theorem 1.2] to obtain

$$\omega_0(\mathcal{H}) \ge \omega_{0,ess}(\mathcal{L} + \mathcal{H}). \tag{11}$$

Thus, (10) and (11) imply that

$$\lambda_0 = s(\mathcal{L} + \mathcal{H}) > \omega_{0,ess}(\mathcal{L} + \mathcal{H}).$$

Since $\mathcal{L} + \mathcal{H}$ is a resolvent positive operator, it follows that the C_0-semigroup generated by $\mathcal{L} + \mathcal{H}$ is positive. Therefore, [38, Proposition 2.5] applies providing that $\{\lambda_0\} = \{\lambda \in \sigma(\mathcal{L} + \mathcal{H}) : \Re(\lambda) = \lambda_0\}$.

To conclude, we now claim that (see [38])

Claim 2.12 $\{T(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ *has AEG with intrinsic growth constant* $\lambda_0 \in \mathbb{R}$ *if and only if*

1) $\omega_{0,ess}(\mathcal{L} + \mathcal{H}) < \lambda_0$

2) $\{\lambda_0\} = \{\lambda \in \sigma(\mathcal{L} + \mathcal{H}) : \Re(\lambda) = \lambda_0\}$

3) λ_0 *is a simple pole of the resolvent of* $\mathcal{L} + \mathcal{H}$*.*

Consequently, by iv), we apply Claim 2.12 to obtain the result. ■

Note that Theorem 1.4 is a consequence of the above result. Furthermore, in condition iv) of the above Theorem 2.11, only the order of the pole is necessary. Indeed, the fact that λ_0 is a pole of the resolvent comes from the other hypothesis of the theorem. Therefore, condition iv) can be replaced by a irreducible property of the C_0-semigroup $\{T(t)\}_{t\ge 0} \subset \mathcal{B}(X)$.

3 Eigenvalue problem for nonlocal diffusion

In this section, we give a sufficient condition for the existence of principal eigenpair for the eigenvalue problem (2). The main result of such a problem is given by Theorem 1.7. The proof of Theorem 1.7 consists of showing that the conditions of Theorem 1.4 hold true. To this end, we divide the proof into subsections. In the first subsection, we give some properties concerning the multiplication operator $\mathcal{H}_p$. In the second subsection we show that there exists $\lambda > s(\mathcal{H})$ such that $r(\mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1}) > 1$. The third subsection is devoted to the proof of the irreducibility property of the positive semigroup generated by $\mathcal{L}_p + \mathcal{H}_p$.

3.1 Some properties of the multiplication of operator $\mathcal{H}_p$

It is well known that, under the condition **(H2)**, the multiplication operator $\mathcal{H}_p$ is a bounded linear operator on $X = L^p(\Omega)^m$. More precisely we have

$$\|\mathcal{H}_p\|_{\mathcal{B}(X)} = \sup_{x\in\Omega} \|H(x)\| \tag{12}$$

where $\|H(x)\|$ is the usual matrix norm. Moreover, using [20, Proposition 1] and [20, Corollary 2], one knows that

$$\sigma(\mathcal{H}_p) = \overline{\bigcup_{x\in\Omega} \sigma(H(x))} \tag{13}$$

and

$$s(\mathcal{H}_p) = \sup_{x\in\Omega} s(H(x)). \tag{14}$$

The below proposition is due to [19] and summarize some spectral properties of $\mathcal{H}_p$.

Proposition 3.1 *The following statements are equivalent*

i) $\lambda \in \rho(\mathcal{H}_p)$,

ii) $\lambda - H(x)$ *is invertible for all* $x \in \Omega$ *and*

$$\sup\left\{\|\lambda - H(x)\|^{-1} : x \in \Omega\right\} < +\infty$$

iii) $\inf\{|\det(\lambda - H(x))| : x \in \Omega\} > 0$.

The next lemma shows that $\mathcal{H}_p$ is resolvent positive.

Lemma 3.2 *Let Assumptions* **(H2)-(H3)** *be satisfied. Then* $\mathcal{H}_p$ *is resolvent positive. Moreover for all* $\lambda > s(\mathcal{H}_p)$, $(\lambda - H(x))^{-1}$ *exists for all* $x \in \Omega$, *is a matrix with strictly positive entries, and*

$$(\lambda - \mathcal{H}_p)^{-1}[u](x) = (\lambda - H(x))^{-1}u(x), \quad a.e.\ x \in \Omega,\ u \in L^p(\Omega)^m.$$

Proof. The first part of the lemma follows from Theorem 2.6 by combing the equality $s(\mathcal{H}_p) = \sup_{x\in\overline{\Omega}} s(H(x))$ together with the fact that $H(x)$ is quasipositive and irreducible for each $x \in \overline{\Omega}$. ■

3.2 Existence of the principal spectrum point

In this section we proceed to the proof of Theorem 1.7. The proof is given throughout several steps.

Step 0:

Lemma 3.3 *Let Assumption* **(H1)-(H3)** *be satisfied. Then the following properties hold true:*

i) $s(\mathcal{H}_p) = s_+ = \max_{x\in\overline{\Omega}} \lambda_+(x)$*;*

ii) $\mathcal{L}_p$ *is a compact positive operator;*

iii) For all $\lambda > s_+$, $\mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1}$ *is nonsupporting in particular*

$$u \in L^p_+(\Omega)^m \setminus \{0_{L^p}\} \implies \mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1}[u](x) \in \text{int}(\mathbb{R}^m_+), \ \ a.e\ x \in \Omega;$$

iv) If $\lambda_p = s(\mathcal{L}_p + \mathcal{H}_p)$ *is the solution of* $r(\mathcal{L}_p(\lambda_p - \mathcal{H}_p)^{-1}) = 1$ *then it is an eigenvalue of* $\mathcal{L}_p + \mathcal{H}_p$ *associated with eigenfunction* $u_p = (u_p^i)_{1\le i\le m} \in C_+(\Omega)^m$, $u_p^i(x) > 0$ *for all* $x \in \Omega$*.*

Proof.

i). Thanks to **(H3)**, $H(x)$ is resolvent positive so that $\lambda_+(x) = s(H(x))$ for all $x \in \overline{\Omega}$. Thus, we infer from (14) that $s_+ = s(\mathcal{H}_p)$.

ii). This follows from the fact that $\mathcal{L}_p$ is a diagonal matrix of operators where each components is compact as an operator form $L^p(\Omega)$ into $L^p(\Omega)$ [13, 16].

iii). This comes from Lemma 3.3 and **(H1)**.

iv). Assume that $\lambda_p = s(\mathcal{L}_p + \mathcal{H}_p)$ satisfies $r(\mathcal{L}_p(\lambda_p - \mathcal{H}_p)^{-1}) = 1$. Since $\mathcal{L}_p$ is compact, it follows that $\mathcal{L}_p(\lambda_p - \mathcal{H}_p)^{-1}$ is also compact. Therefore, we infer from the Krein-Rutman theorem that 1 is an eigenvalue of $\mathcal{L}_p(\lambda_p - \mathcal{H}_p)^{-1}$ with eigenfunction $\varphi_p \in L^p_+(\Omega)^m$, that is

$$\varphi_p(x) = \mathcal{L}_p(\lambda_p - \mathcal{H}_p)^{-1}[\varphi_p](x), \ \ \text{a.e } x \in \Omega \Leftrightarrow \lambda_p u_p(x) = (\mathcal{L}_p + \mathcal{H}_p)[u_p](x), \ \ \text{a.e } x \in \Omega$$

with

$$u_p(x) = (\lambda_p - H(x))^{-1}\varphi_p(x), \ \ \text{a.e } x \in \Omega.$$

The property iii) implies that $u_p(x) \in \text{int}(\mathbb{R}^m_+)$ for a.e $x \in \Omega$. Moreover, the Lipschitz continuity of the kernels J_k together with the uniform boundedness of $(\lambda - H(y))^{-1}$ (see Proposition 3.1) easily imply that u_p is continuous in Ω. ■

Lemma 3.4 *Let Assumption* **(H2)-(H3)** *be satisfied. Then, there exists* $\alpha_0 > 0$ *and a* $m \times m$ *matrix* C *with strictly positive entries such that*

$$(\lambda - H(x))^{-1} \ge \frac{\alpha_0}{\lambda - \lambda_+(x)} C, \ \forall x \in \overline{\Omega}, \ \forall \lambda \in (s_+, s_+ + 1]. \tag{15}$$

Proof. Let us first note that

$$(\lambda - H(x))^{-1} = \frac{1}{\det(\lambda - H(x))}\mathrm{adj}(\lambda - H(x)),\ \forall x \in \overline{\Omega} \tag{16}$$

with $\mathrm{adj}(\lambda - H(x))$ the adjugate of $\lambda - H(x)$ i.e. the transpose of its cofactor matrix. Recalling that $H(x)$, $x \in \overline{\Omega}$ has positive off-diagonal entries for all $x \in \overline{\Omega}$ we infer from [28, Theorem A.2] that

1. For each $\lambda > s_+ \geq \lambda_+(x) = s(H(x))$, $(\lambda - H(x))^{-1}$ has strictly positive entries for all $x \in \overline{\Omega}$.

2. For each $\lambda > s_+ \geq \lambda_+(x) = s(H(x))$, $\det(\lambda - H(x)) > 0$ for all $x \in \overline{\Omega}$.

3. $\mathrm{adj}(\lambda - H(x))$ has strictly positive entries for all $x \in \overline{\Omega}$ and $\lambda \geq \lambda_+(x)$.

Step 1: Our goal is to prove that $\frac{\lambda-\lambda_+(x)}{\det(\lambda-H(x))}$ is uniformly bounded from below with respect to $\lambda \in (s_+, s_+ + 1]$ and $x \in \overline{\Omega}$ by a positive constant. Since $H(x)$ has strictly positive off diagonal entries for $x \in \overline{\Omega}$, it follows that $\lambda_+(x) = s(H(x))$ is algebraically simple so that

$$\det(\lambda - H(x)) = (\lambda - \lambda_+(x)) \prod_{k=1}^{m-1} (\lambda - \lambda_k(x))^{n_k(x)},\ \forall x \in \overline{\Omega} \tag{17}$$

for all $\lambda > s_+ \geq \lambda_+(x)$. Hence, using the fact that $\frac{\det(\lambda-H(x))}{\lambda-\lambda_+(x)}$ is real positive it follows from (17) that the map

$$\lambda \in (s_+, +\infty) \mapsto \frac{\det(\lambda - H(x))}{\lambda - \lambda_+(x)}$$

is increasing for each fixed $x \in \overline{\Omega}$. Therefore, thanks to item 2. we obtain

$$\frac{\lambda - \lambda_+(x)}{\det(\lambda - H(x))} \geq \alpha_0 = \inf_{x\in\overline{\Omega}} \frac{s_+ + 1 - \lambda_+(x)}{\det(s_+ + 1 - H(x))} > 0,\ \forall \lambda \in (s_+, s_+ + 1],\ \forall x \in \overline{\Omega}. \tag{18}$$

Step 2: Noting that for all $\lambda > s_+$ we have

$$(\lambda - H(x))^{-1} = \frac{1}{\lambda - \lambda_+(x)} \frac{\lambda - \lambda_+(x)}{\det(\lambda - H(x))}\mathrm{adj}(\lambda - H(x)),\ \forall x \in \overline{\Omega}$$

it follows from Step 1 that

$$(\lambda - H(x))^{-1} \geq \frac{\alpha_0}{\lambda - \lambda_+(x)}\mathrm{adj}(\lambda - H(x)),\ \forall x \in \overline{\Omega},\ \forall \lambda \in (s_+, s_+ + 1].$$

Moreover, thanks to [24, Theorem 2] and [4, Corollary 2.13 and Theorem 3.1]], for each fixed $x \in \overline{\Omega}$, the map $\lambda \mapsto \mathrm{adj}(\lambda - H(x))$ is increasing in $[s_+, s_+ + 1]$ so that

$$\mathrm{adj}(\lambda - H(x)) \geq \mathrm{adj}(s_+ - H(x)),\ \forall x \in \overline{\Omega},\ \forall \lambda \in [s_+, s_+ + 1].$$

The proof is completed because $\mathrm{adj}(s_+ - H(x))$ has strictly positive entries that are continuous in $\overline{\Omega}$. ∎

Lemma 3.5 *Let Assumption* **(H2)-(H3)** *be satisfied. Then, there exists* $\lambda > s(\mathcal{H}_p)$ *such that*

$$r(\mathcal{L}(\lambda - \mathcal{H}_p)^{-1}) > 1.$$

Proof. The idea of the proof is a somehow generalization of the idea in [10]. Thanks to Lemma 3.3 and Theorem 2.8 we only need to prove that there exists $\lambda > s_+$ such that $r(\mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1}) > 1$. We start with some observations. Let $\lambda > s_+$ be given and fixed. Then we have

$$(\lambda - \mathcal{H}_p)^{-1}[u](x) = (\lambda - H(x))^{-1}u(x), \ \forall x \in \Omega, \ \forall u \in L^p(\Omega)^m. \tag{19}$$

Thanks to Lemma 3.4, there exists a matrix C with strictly positive entries and $\alpha_0 > 0$ such that

$$(\lambda - H(x))^{-1} \geq \frac{\alpha_0}{\lambda - \lambda_+(x)} C, \ \forall x \in \overline{\Omega}, \ \forall \lambda \in (s_+, s_+ + 1].$$

Since C is a primitive matrix, it admits a positive eigenvalue $\mu_0 > 0$ associated with eigenfunction $\mathbf{e} \in \text{int}(\mathbb{R}^n_+)$. By identifying $\mathbf{e}$ with the constant function in $\overline{\Omega}$ we obtain

$$\int_\Omega J(x-y)(\lambda - H(y))^{-1}\mathbf{e}\ \mathrm{d}y \geq \mu_0\alpha_0 \int_\Omega \frac{1}{\lambda - \lambda_+(y)} J(x-y)\mathrm{d}y\ \mathbf{e}, \ \forall x \in \overline{\Omega}\ \forall \lambda \in (s_+, s_+ + 1]. \tag{20}$$

Recalling that $J_k(0) > 0$ and $x \mapsto J_k(x)$ is continuous in $\overline{\Omega}$ for $k = 1, \ldots, m$ we deduce that there exists a constant $c_0 > 0$ and $r > 0$ such that

$$|z| < r, \ z \in \overline{\Omega} \Rightarrow J_k(z) > c_0. \tag{21}$$

Next, using the fact that $\dfrac{1}{s_+ - \lambda_+(y)} \notin L^1_{loc}(\overline{\Omega})$, we deduce that there exists $x_0 \in \overline{\Omega}$ and $\epsilon \in (0, 1)$, small enough, such that

$$\mu_0 c_0 \alpha_0 \int_{B(x_0,r)\cap\overline{\Omega}} \frac{1}{\epsilon + s_+ - \lambda_+(y)} \mathrm{d}y > 1. \tag{22}$$

Combining (21) and (22) it comes

$$\int_{B(x_0,r)\cap\overline{\Omega}} \frac{\mu_0\alpha_0 c_0}{\epsilon + s_+ - \lambda_+(y)} J_k(x-y)\mathrm{d}y > 1, \ \forall x \in \overline{\Omega}, \ k = 1, 2 \tag{23}$$

and therefore, setting $\lambda = s_+ + \epsilon$ we deduce from (20) and (23) that

$$\int_\Omega J(x-y)(\lambda - H(y))^{-1}\mathbf{e}\ \mathrm{d}y > \mathbf{e}, \ \forall x \in \overline{\Omega}. \tag{24}$$

Since $u \mapsto \mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1}[u]$ is non supporting, one can use the results in [22] to deduce from (24) that $r(\mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1}) > 1$. The proof is completed. ■

3.3 The irreducibility of the semigroup generated by $\mathcal{L}_p + \mathcal{H}_p$

In this section, we prove that the semigroup generated by $\mathcal{L}_p + \mathcal{H}_p$ is irreducible. This will be performed by using the well known properties of positive semigroup in Banach lattice. Let us recall J is a closed ideal of $L^p(\Omega)$ if and only if (see [30]) there exists a subset $\Omega_0 \subset \Omega$ such that $J = L^p(\Omega_0)$ with

$$L^p(\Omega_0) = \{u \in L^p(\Omega) : u \text{ vanishes on } \Omega \setminus \Omega_0\}.$$

A bounded linear operator $\mathcal{K}$ on $L^p(\Omega)$ is irreducible if for any closed ideal J of $L^p(\Omega)$ we have

$$\mathcal{K}(J) \subset J \Leftrightarrow J = \{0_{L^p}\} \text{ or } \ J = L^p(\Omega).$$

Since $L^p(\Omega)$ is a Banach lattice and the semigroup generated by $\mathcal{L}_p + \mathcal{H}_p$ is positive, it suffices to prove that (see [8, Proposition 7.6]) the resolvent of $\mathcal{L}_p + \mathcal{H}_p$ is irreducible for some $\lambda > s(\mathcal{L}_p + \mathcal{H}_p)$.

Lemma 3.6 *Let Assumption* **(H2)-(H3)** *be satisfied. Then the semigroup generated by $\mathcal{L}_p + \mathcal{H}_p$ is irreducible.*

Proof. Using Lemma 2.7 and 3.5 one knows that $s(\mathcal{L}_p + \mathcal{H}_p) > s(\mathcal{H}_p)$. Then we have

$$\lambda - \mathcal{L}_p - \mathcal{H}_p = (id - \mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1})(\lambda - \mathcal{H}_p), \ \forall \lambda > s(\mathcal{L}_p + \mathcal{H}_p).$$

Since $\mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1}$ tends to 0 when $\lambda \to +\infty$ in the operator norm topology, we deduce that for $\lambda > s(\mathcal{L}_p + \mathcal{H}_p)$ large enough

$$(id - \mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1})^{-1} = \sum_{k=0}^{\infty} (\mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1})^k$$

so that

$$(\lambda - \mathcal{L}_p - \mathcal{H}_p)^{-1} = (\lambda - \mathcal{H}_p)^{-1} \sum_{k=0}^{\infty} (\mathcal{L}_p(\lambda - \mathcal{H}_p)^{-1})^k.$$

Hence, for $\lambda > s(\mathcal{L}_p + \mathcal{H}_p)$ large enough we have

$$(\lambda - \mathcal{L}_p - \mathcal{H}_p)^{-1} \geq (\lambda - \mathcal{H}_p)^{-1} \mathcal{L}_p (\lambda - \mathcal{H}_p)^{-1}$$

and the result follows from Lemma 3.2 and 3.3. ■

4 Case study

In this section, we provided two cases study where the results presented here can be applied to easily study some properties of evolution problems. Indeed, Theorem 1.7 can be used to study the stability of the zero solution and existence of positive solutions of the following evolution problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathcal{M}[u(t)], \ t > 0 \\ u(0) = \bar{u} \in L^p_+(\Omega)^m. \end{cases} \tag{25}$$

where $\mathcal{M}$ is the operator defined by (3) and $m \geq 1$. One consequence of Theorem 1.7 on Problem (25) reads

Lemma 4.1 *Let Assumptions* **(H1)-(H3)** *be satisfied. Then the zero solution to* (25) *is globally asymptotically stable if* $s(\mathcal{L}_p + \mathcal{H}_p) < 0$ *and unstable if* $s(\mathcal{L}_p + \mathcal{H}_p) > 0$*. Moreover, if* $s(\mathcal{L}_p + \mathcal{H}_p) > 0$ *then for each* $\bar{u} \in L^p_+(\Omega)^m \setminus \{0_{L^p}\}$ *we have*

$$\lim_{t \to +\infty} \int_\Omega \|u(t,x)\|^p_{\mathbb{R}^m} \mathrm{d}x = +\infty.$$

Proof. Let $\bar{u} \in L^p_+(\Omega)^m$ be given. Since the semigroup $\{T_p(t)\}_{t \geq 0}$ generated by $\mathcal{L}_p + \mathcal{H}_p$ on $L^p(\Omega)^m$ has AEG with intrinsic growth constant $\lambda_p = s(\mathcal{L}_p + \mathcal{H}_p)$ associated with projector P, by Remark 1.5, it follows that

$$u(t) = T_p(t)[\bar{u}] = T_p(t)[P\bar{u}] + T_p(t)[(id - P)\bar{u}] = e^{\lambda_p t} P[\bar{u}] + T_p(t)[(id - P)\bar{u}], \ \forall t \geq 0.$$

The statement of the theorem follows since $T_p(t) \circ (id - P)$ goes to 0 when $t \to +\infty$ in the operator norm topology and $P[\bar{u}] \in L^p_+(\Omega)^m \setminus \{0_{L^p}\}$ for all $\bar{u} \in L^p_+(\Omega)^m \setminus \{0_{L^p}\}$. ■

We now introduce two cases to further discuss our results.

4.1 Model of a chemostat

In the first case study, we consider a model of a chemostat where the phenomenon of attachment and detachment to the wall (or other marked surface) is allowed. More precisely, it is assumed that the flow does not prevent the growth of microorganisms on the wall. The chemostat is subject to a constant dilution rate $d > 0$ with an input concentration of substrate $s_{in} > 0$. To this possible phenomenon of the chemostat we add the possibility of having a biomass composed of a continuum phenotypic traits represented by a variable $x \in \Omega$, an open connected set of $\mathbb{R}$. Let $s(t)$ denote the quantity of substrate at each instant t, $v_1(t,x)$ and $v_2(t,x)$ the concentration of microorganisms at time t with phenotypic trait x in the flow media and on the wall, respectively. The mathematical model of interest is then given by

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} = d(s_{in} - s(t)) - \mu(s(t)) \displaystyle\int_\Omega \frac{\beta_1(y)}{\gamma_1} v_1(t,y)\mathrm{d}y - \mu(s(t)) \int_\Omega \frac{\beta_2(y)}{\gamma_2} v_2(t,y)\mathrm{d}y \\ \dfrac{\partial v_1(t,x)}{\partial t} = \mu(s(t)) \displaystyle\int_\Omega J(x-y)\beta_1(y)v_1(t,y)\mathrm{d}y - D_1 v_1(t,x) - \alpha_1(x)v_1(t,x) + \alpha_2(x)v_2(t,x) \\ \dfrac{\partial v_2(t,x)}{\partial t} = \mu(s(t)) \displaystyle\int_\Omega J(x-y)\beta_2(y)v_2(t,y)\mathrm{d}y - D_2 v_2(t,x) + \alpha_1(x)v_1(t,x) - \alpha_2(x)v_2(t,x) \end{cases} \tag{26}$$

In the foregoing model, the term $-\alpha_2(x)v_2(t,x)$ account for the shearing of the microorganism from the wall while $+\alpha_1(x)v_1(t,x)$ is for the adhesion to the wall. The kernel function $J(x-y)$ represents the probability of mutation from phenotype y to phenotype x, $\beta_i(x)$,s are the ability of a microorganism of trait x to consume the substrate s, γ_i,s are yield constants, and D_i,s the removal rates of the microorganisms. Note that the model (26) without the phenotypic traits has been considered in [17, 27] where a system of ordinary differential equations has been used. We assume that μ takes the form $\mu(s) = \dfrac{\mu_0 s}{\kappa + s}$, with μ_0, and κ positive constants. We also assume that the kernel function $J(x-y)$ satisfies conditions (H1) and (H2) of Assumption 1.6.

Setting $v = (v_1, v_2)$, $\gamma = \operatorname{diag}(\gamma_1, \gamma_2)$, $\beta = \operatorname{diag}(\beta_1, \beta_2)$, and $H = \begin{pmatrix} -D_1 - \alpha_1 & \alpha_2 \\ \alpha_1 & -D_2 - \alpha_2 \end{pmatrix}$, the system (26) rewrites

$$\begin{cases} \dfrac{\mathrm{d}s(t)}{\mathrm{d}t} = d(s_{in} - s(t)) - \mu(s(t)) \displaystyle\int_\Omega \left(\gamma^{-1}\beta(y), v(t,y)\right) \mathrm{d}y, \\ \dfrac{\partial v(t,x)}{\partial t} = \mu(s(t))\mathcal{L}[v(t,\cdot)](x) + \mathcal{H}[v(t,\cdot)](x), \end{cases} \tag{27}$$

with $\mathcal{L}[v](x) = \int_{\mathbb{R}} J(x-y)\beta(y)\bar{v}(y)\mathrm{d}y$ and $\mathcal{H}[v](x) = H(x)v(x)$. We have the following result

Theorem 4.2 *Let $\mathcal{R}_0 = r\left(\mu(s_{in})\mathcal{L}(-\mathcal{H})^{-1}\right)$.*

1. *If $\mathcal{R}_0 < 1$ then, $v(t,\cdot) \to 0$ when $t \to +\infty$ in $L^1(\Omega)$. That is, the washout equilibrium of System (26) is globally asymptotically stable.*

2. *If $\mathcal{R}_0 > 1$ then, there exists a unique positive equilibrium $(\bar{s}, \bar{v}(\cdot))$ of System (26). Furthermore, we can find $\eta > 0$, such that for any nonnegative initial condition, with $\int_\Omega (v_1(0,x) + v_2(0,x))\mathrm{d}x > 0$ we have*

$$\limsup_{t+\infty} \int_\Omega (v_1(t,x) + v_2(t,x))\mathrm{d}x > \eta. \tag{28}$$

Proof.

Global stability of the washout equilibrium. Since

$$s(\mathcal{H}) = \sup_{x\in\overline{\Omega}} \lambda_+(x) < 0,$$

with

$$\lambda_+(x) := \frac{1}{2}\left[\sqrt{(D_1 + \alpha_1(x) - D_2 - \alpha_2(x))^2 + 4\alpha_1(x)\alpha_2(x)} - (D_1 + D_2 + \alpha_1(x) + \alpha_2(x))\right] < 0$$

it comes,

$$\operatorname{sign}\left(s(\mu(s_{in})\mathcal{L} + \mathcal{H})\right) = \operatorname{sign}\left(r\left(\mu(s_{in})\mathcal{L}(-\mathcal{H})^{-1}\right) - 1\right). \tag{29}$$

Assuming that $\mathcal{R}_0 = r\left(\mu(s_{in})\mathcal{L}(-\mathcal{H})^{-1}\right) < 1$ then leads to $s(\mu(s_{in})\mathcal{L} + \mathcal{H}) < 0$. Let (s_0, v_0) be a nonnegative initial condition of System (26). Since $s'(t) \leq d(s_{in} - s(t))$, it follows that for each $\epsilon > 0$ there exists $t_0 := t_0(\epsilon, s_0) > 0$ such that $s(t) \leq s_{in} + \epsilon$ for all $t \geq t_0$. Therefore, we have

$$\frac{\partial v(t,x)}{\partial t} \leq \mu(s_{in} + \epsilon) \int_\Omega J(x-y)\beta(y)v(t,y)\mathrm{d}y + H(x)v(t,x), \ \forall t \geq t_0. \tag{30}$$

Using the upper semicontinuity of the spectral radius together with the equality (29), it is straightforward that $s(\mu(s_{in} + \epsilon)\mathcal{L} + \mathcal{H}) < 0$, for $\epsilon > 0$ small enough. Using comparison principle, we infer from Lemma 4.1 that $v(t,\cdot) \to 0$ when $t \to +\infty$ in $L^1(\Omega)$.

Positive equilibrium of System (26). Let $(\bar{s}, \bar{v}(\cdot))$ be a positive equilibrium of System (26). Then, for all $x \in \mathbb{R}$, $\mu(\bar{s})\mathcal{L}[\bar{v}](x) + H(x)\bar{v}(x) = 0$, *i.e.*,

$$\mu(\bar{s})\mathcal{L}(-\mathcal{H})^{-1}[\bar{v}](x) = \bar{v}(x). \tag{31}$$

By (31) and noting that $\mu(\bar{s})\mathcal{L}(-\mathcal{H})^{-1}$ is a positive compact operator, we infer from the Krein-Rutman's theorem that $\bar{v} = c_0\bar{u}$, where c_0 is a positive constant, and $\bar{u} \in C_+(\Omega)^2$ is the normalized positive eigenfunction of $\mu(\bar{s})\mathcal{L}(-\mathcal{H})^{-1}$ associated to eigenvalue $\mu(\bar{s})r\left(\mathcal{L}(-\mathcal{H})^{-1}\right) = 1$. Let $s_0 > 0$ be given and fixed. By the compacity of the operator $\mu(s_0)\mathcal{L}(-\mathcal{H})^{-1}$, the map $g : s_0 \mapsto r\left(\mu(s_0)\mathcal{L}(-\mathcal{H})^{-1}\right)$ is continuous. Note that $g(0) = 0$, $g(s_{in}) = r\left(\mu(s_{in})\mathcal{L}(-\mathcal{H})^{-1}\right)$, and $\mu(\cdot)$ is an increasing function. Therefore, $g(s_{in}) > 1$ gives $\bar{s} \in (0, s_{in})$. To conclude, it remain to determine the constant c_0. By the s-equation of (27), we have

$$c_0 = \frac{D(s_{in} - \bar{s})}{\mu(\bar{s})\int_\Omega (\gamma^{-1}\beta(y), \bar{u}(y))\,\mathrm{d}y} > 0.$$

Weak uniform persistence. We argue by contradiction for the proof of estimate (28). We assume that

$$\limsup_{t+\infty} \int_\Omega (v_1(t,x) + v_2(t,x))\mathrm{d}x \le \eta \tag{32}$$

with $\eta > 0$ a positive constant to be fixed later on. Assume that $\mathcal{R}_0 > 1$ *i.e.*, $s(\mu(s_{in})\mathcal{L} + \mathcal{H}) > 0$. Since we are in the situation of Theorem 2.10, there exists $\epsilon > 0$ small enough such that $\lambda_\epsilon := s(\mu(s_{in} - \epsilon)\mathcal{L} + \mathcal{H}) > 0$. Noting that $\mu(s) = \frac{\mu_0 s}{\kappa + s}$, we can fix $\eta > 0$ small enough such that $\frac{\eta}{1+\eta} \in (0, \frac{\epsilon}{s_{in}})$ and if (32) is satisfied with initial condition (s_0, v_0) then

$$s'(t) \ge d(s_{in} - s) - d\eta s.$$

Hence, there exists $t_0 := t_0(\epsilon, s_0)$ such that $s(t) \ge s_{in} - \epsilon$ for all $t \ge t_0$. As a consequence, we get

$$\frac{\partial v(t,x)}{\partial t} \ge \mu(s_{in} - \epsilon)\int_\Omega J(x-y)\beta(y)v(t,y)\mathrm{d}y + H(x)v(t,x),\ \forall t \ge t_0. \tag{33}$$

Using comparison principles, we infer from Lemma 4.1 that

$$\lim_{t\to+\infty} \int_\Omega (v_1(t,x) + v_2(t,x))\mathrm{d}x = +\infty \text{ if } \int_\Omega (v_1(0,x) + v_2(0,x))\mathrm{d}x > 0 \tag{34}$$

which is a contradiction to (32). ■

4.2 Model of the spatial evolution of a man-environment disease

Here, we introduce a model for the spatial evolution of the man-environment-man epidemic, *i.e.* infected human acts as a multiplier of the infectious agent which is returned to the environment, and the infectious agent is transmitted to the human via the infected environment (*eg.* see [6, 5, 7]). The model reads,

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \displaystyle\int_\Omega J_1(x-y)u(t,y) - u(t,x))\mathrm{d}y - a_1(x)u + b_1(x)v \\ \dfrac{\partial v(t,x)}{\partial t} = \displaystyle\int_\Omega J_2(x-y)(v(t,y) - v(t,x))\mathrm{d}y - a_2(x)v + b_2(x)u \\ u(0,\cdot) = u_0 \in L^2_+(\Omega),\ v(0,\cdot) = v_0 \in L^2_+(\Omega), \end{cases} \tag{35}$$

where $u(t,x)$ denotes the spatial density the infectious agent in the environment, and $v(t,x)$ denotes the spatial density of infectious human at time t and at location $x \in \Omega = (0,l)$, $l > 0$. The diffusion process within the human population and infectious agents in the environment is captured by probability density functions J_2 and J_1 respectively. Human and infectious agents death rate at the location x are $a_1(x)$ and $a_2(x)$. The term $b_1(x)v$ accounts for the contribution of infectious humans to the density of infectious agents in the environment at the location x. Finally, the term $b_2(x)u$ gives the 'force of infection' on the human population due to the concentration of the infectious agent in the environment. Note that System (35) has been considered in [26] for a non fixed domain but with coefficient a_k,s and b_k,s independent of the location x.

Setting, $\mu_k(x) = \int_\Omega J_k(x-y)\mathrm{d}y - a_k(x)$, $w = (u,v)$, $J = \mathrm{diag}(J_1, J_2)$, and $H(x) = \begin{pmatrix} -\mu_1(x) & b_1(x) \\ b_2(x) & -\mu_2(x) \end{pmatrix}$, the system (35) rewrites

$$\frac{\partial w(t,x)}{\partial t} = \mathcal{L}[w(t,\cdot)](x) + \mathcal{H}[w(t,\cdot)](x), \tag{36}$$

with $\mathcal{L}[w](x) = \int_\Omega J(x-y)w(y)\mathrm{d}y$, and $\mathcal{H}[w](x) = H[x]w(x)$. Let us note that, in (36), the trace of the matrix $H(x)$ is negative and its determinant is $\det(H(x)) = \mu_1(x)\mu_2(x) - b_1(x)b_2(x)$. Therefore, the dominant eigenvalue is positive if $\det(H(x)) < 0$, for some $x \in \overline{\Omega} = (0,l)$. Consequently, in such a case, and compared to the case study in Section 4.1, it becomes impossible to inverse the operator $(-\mathcal{H})$. Thus, it is more difficult in having the equivalence between threshold conditions, $s(\mathcal{L}+\mathcal{H}) < 0$ and $r(\mathcal{L}(-\mathcal{H})^{-1}) < 1$, by applying classical results based Krein-Rutman's type theorem. In order to apply our result, we assume that a_k, b_k, $k = 1,2$ are Lipschitz continuous in $\overline{\Omega} = [0,l]$ and the principal eigenvalue of $H(x)$ reach its maximum in $\Omega = (0,l)$. This implies that the conditions of Theorem 1.7 is satisfied (see [10, Theorem 1.2]).

We then have the following result.

Theorem 4.3 *The zero solution to* (35) *is globally asymptotically stable if* $s(\mathcal{L}+\mathcal{H}) < 0$ *and unstable if* $s(\mathcal{L}+\mathcal{H}) > 0$. *Furthermore, when* $s(\mathcal{L}+\mathcal{H}) > 0$, *for any nonnegative initial condition with* $\int_\Omega w(0,x)\mathrm{d}x > 0$, *it comes*

$$\lim_{t\to+\infty} \int_\Omega \|w(t,x)\|\mathrm{d}x = +\infty.$$

The proof of the above theorem is a direct consequence of Lemma 4.1.

Note that the study of the eigenvalue problem of (35) is of great importance as can be seen in [26]. More precisely, the study of the principal eigenvalue problem of (35) combined with maximum principles will allows to construct subsolution and supersolution in order to prove the existence and nonexistence of endemic equilibrium to (35). However, these facts are out of the scope of the paper.

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