



On the multivariate Helly theorem*

B. B. Winter

► To cite this version:

| B. B. Winter. On the multivariate Helly theorem*. Annales de l'ISUP, 1984, XXIX (1), pp.93-99. <hal-03690286>

HAL Id: hal-03690286

<https://hal.science/hal-03690286v1>

Submitted on 8 Jun 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

ON THE MULTIVARIATE HELLY THEOREM*

B. B. WINTER

Department of Mathematics

University of Ottawa

Ottawa, Ont. Canada

*The preparation of this article was partly supported by the National Science and Engineering Research Council Canada.

The multivariate Helly selection theorem is an important tool in probability and statistics; for instance, it is used in proving the "continuity theorem" for multivariate characteristic functions. Proofs of the Helly theorem usually involve multivariate distribution functions (see, e.g., p. 235 in [2] or p. 227 in [1]). But some probabilists feel that distribution functions, though a very natural tool in the univariate case, are awkward and unattractive in the multivariate case. We therefore propose a simple proof in which multivariate distribution functions play no role. The proposed proof is accessible to any student who, as is usually the case, goes through a standard introduction to measure and integration prior to studying probability theory. The proof uses the Riesz representation theorem for Borel measures on \mathbb{R}^m (see p. 304 in [5]) and a simple separability lemma.

NOTATION. The set of functions $\mathbb{R}^m \rightarrow \mathbb{R}$ which are continuous and have compact support is denoted by $C_c(\mathbb{R}^m)$.

LEMMA. The set $C_c(\mathbb{R}^m)$ contains a countable subset which is sup-norm dense.

The proof of this lemma is left for the Appendix.

THEOREM (Helly). Suppose that $(\mu_n)_1^\infty$ is a sequence of measures on the Borel sets of \mathbb{R}^m , such that $\sup\{\mu_n(\mathbb{R}^m) : n \in \mathbb{N}^+\} < \infty$. Then $(\mu_n)_1^\infty$ has a vaguely convergent subsequence.

PROOF. The conclusion of the theorem means that one can find

$n_1 < n_2 < \dots$, and a measure μ , such that if $f \in C_c(\mathbb{R}^m)$ then

$$\int_{\mathbb{R}^m} f d\mu_{n_i} \rightarrow \int_{\mathbb{R}^m} f d\mu \quad \text{as } i \rightarrow \infty.$$

Consider $f \in C_c(\mathbb{R}^m)$ and $\varepsilon > 0$. Using the above separability lemma, it is easy to show that one can find $n_1 < n_2 < \dots$ such that

$$(*) \quad \text{if } f \in C_c(\mathbb{R}^m) \text{ then } \lim_{i \rightarrow \infty} \int f d\mu_{n_i} \text{ exists finitely ;}$$

see p. 235 in [2]. Put $\underline{T}f = \lim_{i \rightarrow \infty} \int f d\mu_{n_i}$. By (*), \underline{T} is real-valued; and it is easily seen that \underline{T} is a positive linear functional on $C_c(\mathbb{R}^m)$. By the Riesz representation theorem, there is a unique measure μ on the Borel sets of \mathbb{R}^m such that $\underline{T}f = \int f d\mu$ for all $f \in C_c$. Therefore $\int f d\mu_{n_i} \rightarrow \int f d\mu$ as $i \rightarrow \infty$, as alleged. \square

Appendix. If $a, b \in \mathbb{R}$ then $[a, b]$ will denote the "interval" $\{x \in \mathbb{R}^m : a \leq x_i \leq b \text{ for } i = 1, 2, \dots, m\}$. The unit cube $[0, 1]$ will be denoted by \underline{U} ; if $f: \mathbb{R}^m \rightarrow \mathbb{R}$ then $\|f\|_{\underline{U}}$ stands for $\sup_{p \in \underline{U}} |f(p)|$.

THEOREM (Weierstrass). If $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous on \underline{U} , and $\varepsilon > 0$, then there exists a polynomial g such that $\|f - g\|_{\underline{U}} < \varepsilon$.

This is but a particular case of the Stone-Weierstrass Theorem. It is worth noting that a direct and probabilistically flavored argument shows that g can always be taken to be a multivariate Bernstein polynomial, as follows.

Given $f: \mathbb{R}^m \rightarrow \mathbb{R}$, let $B_m(f)$ denote the set of all functions

$g: \mathbb{R}^m \rightarrow \mathbb{R}$ of the form

$$g(p) = \sum_{k_1=0}^n \dots \sum_{k_m=0}^n f\left(\frac{k_1}{n}, \dots, \frac{k_m}{n}\right) \prod_{i=1}^m \binom{n}{k_i} p_i^{k_i} (1-p_i)^{n-k_i},$$

where $p = (p_1, \dots, p_m) \in \mathbb{R}^m$, $n \in \mathbb{N}^+$, and $k_i \in \{0, 1, \dots, n\}$. In view of the terminology used, e.g., by Butzer [3], $B_m(f)$ is a proper subset of the set of m-dimensional Bernstein polynomials for f . Now, by an argument which is a natural extension of the argument for the univariate case (given, e.g., in [4]), one can establish the following.

THEOREM (Weirstrass-Bernstein). If $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous on \underline{U} , and $\varepsilon > 0$, then $\exists g \in B_m(f)$ such that $\|f - g\|_{\underline{U}} < \varepsilon$.

We now consider the separability lemma which was stated at the outset. For $C_c(\mathbb{R}^1)$, there is no problem. In this case it suffices to consider a class G defined as follows. A function $g: \mathbb{R} \rightarrow \mathbb{R}$ belongs to G iff there exist finitely many points $(x_1, y_1), \dots, (x_n, y_n)$ with rational coordinates, such that $x_1 < \dots < x_n$, $y_1 = 0 = y_n$, $g(x) = 0$ if $x < x_1$ or $x > x_n$, and the graph of $g|_{[x_1, x_n]}$ is a polygonal path joining the points $(x_1, y_1), \dots, (x_n, y_n)$. Clearly G is countable and it is easy to see that if $f \in C_c(\mathbb{R}^1)$ and $\varepsilon > 0$ then $(\exists g \in G) (\sup_{x \in \mathbb{R}} |f(x) - g(x)| < \varepsilon)$. If $m > 1$ then there is no obvious generalization of this simple construction; but now, as will be shown, one can obtain a dense set by suitable modification of those polynomials in m variables that have rational coefficients. (The basic idea of such an argument is mentioned, without any details, on p. 235 of [2].)

PROOF OF THE LEMMA. Let \tilde{G} be the set of all polynomials in m variables that have rational coefficients; clearly \tilde{G} is countable.

If $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous on \underline{U} then, by the Weierstrass theorem, there exists a polynomial g_0 such that $\|f - g_0\|_{\underline{U}} < \varepsilon/2$. For some $n \in \mathbb{N}^+$,

g can be written as $g_0(p) = \sum a_{i_1 \dots i_m} p_1^{i_1} \dots p_m^{i_m}$, $0 \leq i_j \leq n$.

Replace each $\tilde{a}_{i_1 \dots i_m}$ by a rational $\tilde{a}_{i_1 \dots i_m}$ which is within $\epsilon/\{2(n+1)^m\}$

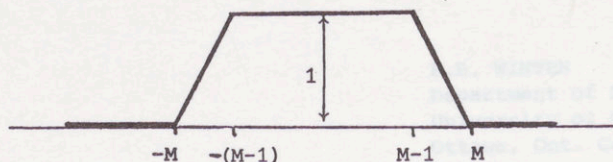
of the original coefficient; let g be a polynomial obtained from g_0 by

such a replacement of coefficients. If $p \in \underline{U}$ then, since g_0 has $(n+1)^m$ coefficients,

$$|g_0(p) - g(p)| \leq \sum |a_{i_1 \dots i_m} - \tilde{a}_{i_1 \dots i_m}| p_1^{i_1} \dots p_m^{i_m} \leq (n+1)^m \frac{\epsilon/2}{(n+1)^m} = \frac{\epsilon}{2};$$

it follows that $\|f - g\|_{\underline{U}} \leq \|f - g_0\|_{\underline{U}} + \|g_0 - g\|_{\underline{U}} < \epsilon$. Thus, if $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous on \underline{U} then f can be approximated on \underline{U} uniformly and arbitrarily closely, by an element of \tilde{G} .

Let (g_1, g_2, \dots) be an enumeration of \tilde{G} . For $M \in \mathbb{N}^+$, let ρ_M be the function with the following graph



and put

$$G_{nM}(x_1, \dots, x_m) = g_n\left(\frac{x_1 + M}{2M}, \dots, \frac{x_m + M}{2M}\right) \cdot \prod_{i=1}^m \rho_M(x_i).$$

Let $G = \{G_{nM} : (n, M) \in \mathbb{N}^+ \times \mathbb{N}^+\}$. Clearly G is countable and $G \subset C_c$. It

remains to show that G is sup-norm dense.



Given $f \in C_c$ and $\varepsilon > 0$, let $M \in \mathbb{N}^+$ be such that the support of f is contained in $[-(M-1), (M-1)] \subset [-M, M]$. For $p \in [0, 1]$ put

$$f^*(p_1, \dots, p_m) = f(M(2p_1 - 1), \dots, M(2p_m - 1))$$

and let g_n be a polynomial in \tilde{G} which approximates f^* to within ε on $\underline{U} = [0, 1]$. Given $x \in [-M, M]$, put $p_i = (x_i + M)/(2M)$; then $0 \leq p_i \leq 1$, $x_i = M(2p_i - 1)$, and $f(x) = f(x_1, \dots, x_m) = f^*(p_1, \dots, p_m) = f^*(p)$. If $x \in [-(M-1), (M-1)]$ then

$$|G_{nM}(x) - f(x)| = \left| g_n \left(\frac{x_1 + M}{2M}, \dots, \frac{x_m + M}{2M} \right) - f(x) \right| = |g_n(p) - f^*(p)| < \varepsilon.$$

If $x \in [-M, M] \setminus [-(M-1), (M-1)]$ then $f(x) = 0 = f^*(p)$ and

$$|G_{nM}(x) - f(x)| = |g_n(p_1, \dots, p_m)| \cdot \prod_{i=1}^m \rho_M(x_i) \leq |g_n(p) - f^*(p)| < \varepsilon.$$

Since $x \notin [-M, M] \implies G_{nM}(x) = 0 = f(x)$, it follows that $\|G_{nM} - f\| < \varepsilon$. \square

REFERENCES

1. P. Billingsley, Convergence of Probability Measures, Wiley, New York, 1968.
2. L. Breiman, Probability, Addison-Wesley, Reading (Mass.), 1968.
3. P.L. Butzer, On two-dimensional Bernstein polynomials, Can. J. of Math., 5 (1953) 107-113.
4. R.G. Kuller, Coin tossing, probability, and the Weierstrass approximation theorem, Math. Mag., 37 (1964) 262-265.
5. H.L. Royden, Real Analysis (2nd ed.), MacMillan, New York, 1968

Reçu en Mars 1982

B.B. WINTER
Department of Mathematics
University of Ottawa
Ottawa, Ont. Canada
K1N 9B4

