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Reconstruction of closely spaced small inhomogeneities via boundary measurements for the full time-dependent Maxwell's equations

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Abstract

We consider for the full time-dependent Maxwell's equations the inverse problem of identifying locations and certain properties of small electromagnetic inhomogeneities in a homogeneous background medium from dynamic boundary measurements on the boundary for a finite time interval.

Key words. Maxwell's equations, inhomogeneities, inverse problem, reconstruction, geometric control

2000 AMS subject classifications. 35R30, 35B40, 35B37, 78M35

1 Introduction

The ultimate objective of the work described in this paper is to determine locations and certain properties of the shapes of small electromagnetic inhomogeneities in a homogeneous background medium from dynamic boundary measurements on part of the boundary and for finite interval in time. Using as weights particular background solutions constructed by a geometrical control method we develop an asymptotic method based on appropriate averaging of the partial dynamic boundary measurements.

For stationary Maxwell's equations it has been known that the Dirichlet to Neumann map uniquely determines (smooth) isotropic electromagnetic parameters, see [16], [18], [20]. We will provide in this paper a rigorous derivation of the inverse Fourier transform of a linear combination of derivatives of point masses, located at the positions z_j of the inhomogeneities, as the leading order term of an appropriate averaging of (partial) dynamic boundary measurements of the tangential components of electric fields on part

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of the boundary. Our formulas may be used to determine properties (location, relative size) of the small inhomogeneities in case a single, or a few (tangential) boundary electric fields are known. Our approach differs from [1], [2], [3], [4], [22] and is expected to lead to very effective computational identification algorithms.

For discussions on closely related (stationary) identification problems we refer the reader to [19],[21], [6], and [10].

2 Problem formulation

Let Ω be a bounded C^2 -domain in \mathbb{R}^d , $d = 2, 3$. Assume that Ω contains a finite number of inhomogeneities, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbb{R}^d$ is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is $\mathcal{B}_\alpha = \cup_{j=1}^m (z_j + \alpha B_j)$. The points $z_j \in \Omega$, $j = 1, \dots, m$, which determine the location of the inhomogeneities, are assumed to satisfy the following inequalities:

$$|z_j - z_l| \geq c_0 > 0, \forall j \neq l \quad \text{and} \quad \text{dist}(z_j, \partial\Omega) \geq c_0 > 0, \forall j. \quad (1)$$

Assume that $\alpha > 0$, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to $\mathbb{R}^d \setminus \overline{\Omega}$ is larger than $c_0/2$. Let μ_0 and ε_0 denote the permeability and the permittivity of the background medium, and assume that $\mu_0 > 0$ and $\varepsilon_0 > 0$ are positive constants. Let $\mu_j > 0$ and $\varepsilon_j > 0$ denote the permeability and the permittivity of the j -th inhomogeneity, $z_j + \alpha B_j$, these are also assumed to be positive constants. Introduce the piecewise-constant magnetic permeability

$$\mu_\alpha(x) = \begin{cases} \mu_0, & x \in \Omega \setminus \mathcal{B}_\alpha, \\ \mu_j, & x \in z_j + \alpha B_j, \quad j = 1 \dots m. \end{cases} \quad (2)$$

If we allow the degenerate case $\alpha = 0$, then the function $\mu_0(x)$ equals the constant μ_0 . The electric permittivity is defined by $\varepsilon_\alpha(x) = \varepsilon_0$, for all $x \in \Omega$. Let $\mathbf{n} = \mathbf{n}(x)$ denote the outward unit normal vector to Ω at a point on $\partial\Omega$, $\partial_t u = \frac{\partial u}{\partial t}$ and Δ means the

Laplace operator defined by $\Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$.

In this paper, we will denote by bold letters the functional spaces for the vector fields. Thus $H^s(\Omega)$ denotes the usual Sobolev space on Ω and $\mathbf{H}^s(\Omega)$ denotes $(H^s(\Omega))^d$ and $\mathbf{L}^2(\Omega)$ denotes $(L^2(\Omega))^d$. As usual for Maxwell equations, we need spaces of fields with square integrable curls:

$$\mathbf{H}(\text{curl}; \Omega) = \{u \in \mathbf{L}^2(\Omega), \text{curl } u \in \mathbf{L}^2(\Omega)\},$$

and with square integrable divergences

$$\mathbf{H}(\text{div}; \Omega) = \{u \in \mathbf{L}^2(\Omega), \text{div } u \in L^2(\Omega)\}.$$

We will also need the following functional spaces:

$$Y(\Omega) = \{u \in \mathbf{L}^2(\Omega), \text{div } u = 0 \text{ in } \Omega\}, \quad X(\Omega) = \mathbf{H}^1(\Omega) \cap Y(\Omega),$$

and $TL^2(\partial\Omega)$ the space of vector fields on $\partial\Omega$ that lie in $\mathbf{L}^2(\partial\Omega)$. Finally, the "minimal" choice for the electric variational space would be

$$X_N(\Omega) = \{v \in \mathbf{H}(\text{curl}; \Omega) \cap \mathbf{H}(\text{div}; \Omega); \quad v \times \mathbf{n} = 0 \quad \text{on } \partial\Omega\}.$$

Now, we introduce the following time-dependent Maxwell equations (associated to the electric field)

$$\begin{cases} (\varepsilon_\alpha \partial_t^2 + \text{curl} \frac{1}{\mu_\alpha} \text{curl}) E_\alpha = 0 & \text{in } \Omega \times (0, T), \\ \text{div} (\varepsilon_\alpha E_\alpha) = 0 & \text{in } \Omega \times (0, T), \\ E_\alpha|_{t=0} = \varphi, \partial_t E_\alpha|_{t=0} = \psi & \text{in } \Omega, \\ E_\alpha \times \mathbf{n}|_{\partial\Omega \times (0, T)} = f, \end{cases} \quad (3)$$

where $E_\alpha \in \mathbb{R}^d$ is the electric field, f the boundary condition for $E_\alpha \times \mathbf{n}$, and φ and ψ the initial data.

Let E be the solution of the Maxwell's equations in the homogeneous domain:

$$\begin{cases} (\varepsilon_0 \partial_t^2 + \text{curl} \frac{1}{\mu_0} \text{curl}) E = 0 & \text{in } \Omega \times (0, T), \\ \text{div} (\varepsilon_0 E) = 0 & \text{in } \Omega \times (0, T), \\ E|_{t=0} = \varphi, \partial_t E|_{t=0} = \psi & \text{in } \Omega, \\ E \times \mathbf{n}|_{\partial\Omega \times (0, T)} = f. \end{cases} \quad (4)$$

Here $T > 0$ is a final observation time and $\varphi, \psi \in \mathcal{C}^\infty(\overline{\Omega})$ and $f \in \mathcal{C}^\infty(0, T; \mathcal{C}^\infty(\partial\Omega))$ are subject to the compatibility conditions

$$\partial_t^{2l} f|_{t=0} = (\Delta^l \varphi) \times \mathbf{n}|_{\partial\Omega} \text{ and } \partial_t^{2l+1} f|_{t=0} = (\Delta^l \psi) \times \mathbf{n}|_{\partial\Omega}, \quad l = 1, 2, \dots$$

it follows that (4) has a unique solution $E \in \mathcal{C}^\infty([0, T] \times \overline{\Omega})$. It is also known (see for example [17]) that since Ω is smooth (\mathcal{C}^2 -regularity would be sufficient) the non homogeneous Maxwell's equations (3) have a unique weak solution $E_\alpha \in \mathcal{C}^0(0, T; X(\Omega)) \cap \mathcal{C}^1(0, T; \mathbf{L}^2(\Omega))$. Indeed, $\text{curl } E_\alpha$ belongs to $\mathcal{C}^0(0, T; X(\Omega)) \cap \mathcal{C}^1(0, T; \mathbf{L}^2(\Omega))$.

3 Asymptotic formula

We start the derivation of the asymptotic formula for $\text{curl } E_\alpha \times \mathbf{n}$ with the following estimate.

Lemma 3.1 *The following estimate as $\alpha \rightarrow 0$ holds:*

$$\|\partial_t(E_\alpha - E)\|_{L^\infty(0, T; \mathbf{L}^2(\Omega))} + \|E_\alpha - E\|_{L^\infty(0, T; X_N(\Omega))} \leq C\alpha, \quad (5)$$

where the constant C is independent of α and the set of points $\{z_j\}_{j=1}^m$ provided that assumption (1) holds.

Proof. From (3)-(4), it is obvious that $E_\alpha - E \in X_N(\Omega)$, then due to the Green formula we have for any $\mathbf{v} \in X_N(\Omega)$:

$$\begin{aligned} \int_{\Omega} \varepsilon_0 \partial_t^2 (E_\alpha - E) \cdot \mathbf{v} \, dx + \int_{\Omega} \frac{1}{\mu_\alpha} \operatorname{curl} (E_\alpha - E) \cdot \operatorname{curl} \mathbf{v} \, dx = \\ \sum_{j=1}^m \left(\frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \int_{z_j + \alpha B_j} \operatorname{curl} E \cdot \operatorname{curl} \mathbf{v} \, dx. \end{aligned} \quad (6)$$

Let \mathbf{v}_α be defined by

$$\begin{cases} \mathbf{v}_\alpha \in X_N(\Omega), \\ \operatorname{curl} \frac{1}{\mu_\alpha} \operatorname{curl} \mathbf{v}_\alpha = \partial_t (E_\alpha - E) \quad \text{in } \Omega. \end{cases} \quad (7)$$

Then,

$$\begin{aligned} \int_{\Omega} \frac{1}{\mu_\alpha} \operatorname{curl} (E_\alpha - E) \cdot \operatorname{curl} \mathbf{v}_\alpha \, dx = - \int_{\Omega} \partial_t (E_\alpha - E) \cdot (E_\alpha - E) \, dx = \\ - \frac{1}{2} \partial_t \int_{\Omega} |E_\alpha - E|^2 \, dx \end{aligned}$$

and by Green formula, relation (7) gives:

$$\begin{aligned} \int_{\Omega} \partial_t^2 (E_\alpha - E) \cdot \mathbf{v}_\alpha \, dx &= \int_{\Omega} \operatorname{curl} \frac{1}{\mu_\alpha} \operatorname{curl} \partial_t \mathbf{v}_\alpha \cdot \mathbf{v}_\alpha \, dx \\ &= - \int_{\Omega} \frac{1}{\mu_\alpha} \operatorname{curl} \partial_t \mathbf{v}_\alpha \cdot \operatorname{curl} \mathbf{v}_\alpha \, dx \\ &= - \frac{1}{2} \partial_t \int_{\Omega} \frac{1}{\mu_\alpha} |\operatorname{curl} \mathbf{v}_\alpha|^2 \, dx. \end{aligned}$$

Thus, it follows from (6) that

$$\begin{aligned} \varepsilon_0 \partial_t \int_{\Omega} \frac{1}{\mu_\alpha} |\operatorname{curl} \mathbf{v}_\alpha|^2 \, dx + \partial_t \int_{\Omega} |E_\alpha - E|^2 \, dx = \\ - 2 \sum_{j=1}^m \left(\frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \int_{z_j + \alpha B_j} \operatorname{curl} E \cdot \operatorname{curl} \mathbf{v}_\alpha \, dx. \end{aligned}$$

Next,

$$\left| \sum_{j=1}^m \left(\frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \int_{z_j + \alpha B_j} \operatorname{curl} E \cdot \operatorname{curl} \mathbf{v}_\alpha \right| \leq C \|\operatorname{curl} E\|_{\mathbf{L}^2(\mathcal{B}_\alpha)} \|\operatorname{curl} \mathbf{v}_\alpha\|_{\mathbf{L}^2(\Omega)}.$$

Since $E \in \mathcal{C}^\infty([0, T] \times \overline{\Omega})$ we have

$$\|\operatorname{curl} E\|_{\mathbf{L}^2(\mathcal{B}_\alpha)} \leq \|\operatorname{curl} E\|_{L^\infty(\mathcal{B}_\alpha)} \alpha \left(\sum_{j=1}^m |B_j| \right)^{\frac{1}{2}} \leq C\alpha,$$

which gives

$$\left| \sum_{j=1}^m \left(\frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \int_{z_j + \alpha B_j} \operatorname{curl} E \cdot \operatorname{curl} \mathbf{v}_\alpha \, dx \right| \leq C\alpha \|\operatorname{curl} \mathbf{v}_\alpha\|_{\mathbf{L}^2(\Omega)}$$

and so,

$$\varepsilon_0 \partial_t \int_{\Omega} \frac{1}{\mu_{\alpha}} |\operatorname{curl} \mathbf{v}_{\alpha}|^2 dx + \partial_t \int_{\Omega} |E_{\alpha} - E|^2 dx \leq C\alpha \left(\int_{\Omega} \frac{1}{\mu_{\alpha}} |\operatorname{curl} \mathbf{v}_{\alpha}|^2 dx + \int_{\Omega} |E_{\alpha} - E|^2 dx \right)^{1/2}. \quad (8)$$

From the Gronwall Lemma it follows that

$$\left(\int_{\Omega} \frac{1}{\mu_{\alpha}} |\operatorname{curl} \mathbf{v}_{\alpha}|^2 dx \right)^{1/2} + \left(\int_{\Omega} |E_{\alpha} - E|^2 dx \right)^{1/2} \leq C\alpha. \quad (9)$$

Combining this last estimate (9) with the fact that

$$\|\partial_t(E_{\alpha} - E)\|_{L^{\infty}(0,T;H^{-1}(\Omega))} \leq C \|\operatorname{curl} \mathbf{v}_{\alpha}\|_{L^{\infty}(0,T;\mathbf{L}^2(\Omega))}$$

the following estimate holds

$$\|E_{\alpha} - E_0\|_{L^{\infty}(0,T;\mathbf{L}^2(\Omega))} + \|\partial_t(E_{\alpha} - E_0)\|_{L^{\infty}(0,T;\mathbf{L}^2(\Omega))} \leq C\alpha. \quad (10)$$

Now, taking (formally) $\mathbf{v} = \partial_t(E_{\alpha} - E)$ in (6) we arrive at

$$\begin{aligned} & \varepsilon_0 \partial_t \int_{\Omega} \left[|\partial_t(E_{\alpha} - E)|^2 + \frac{1}{\mu_{\alpha}} |\operatorname{curl}(E_{\alpha} - E)|^2 \right] dx = \\ & 2 \sum_{j=1}^m \left(\frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \int_{z_j + \alpha B_j} \operatorname{curl} E \cdot \operatorname{curl} \partial_t(E_{\alpha} - E) dx. \end{aligned}$$

By using the regularity of E in Ω and estimate (10) given above, we see that

$$\begin{aligned} \left| \sum_{j=1}^m \left(\frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \int_{z_j + \alpha B_j} \operatorname{curl} E \cdot \operatorname{curl} \partial_t(E_{\alpha} - E) dx \right| & \leq C \|\operatorname{curl} E\|_{\mathbf{H}^2(\mathcal{B}_{\alpha})} \|\partial_t(E_{\alpha} - E)\|_{\mathbf{H}^{-1}(\Omega)} \\ & \leq C\alpha^2, \end{aligned}$$

where C is independent of t and α , and so, we obtain

$$\partial_t \int_{\Omega} \left[|\partial_t(E_{\alpha} - E)|^2 + \frac{1}{\mu_{\alpha}} |\operatorname{curl}(E_{\alpha} - E)|^2 \right] dx \leq C\alpha^2$$

which yields the following estimate

$$\|\partial_t(E_{\alpha} - E)\|_{L^{\infty}(0,T;\mathbf{L}^2(\Omega))} + \|E_{\alpha} - E\|_{L^{\infty}(0,T;X_N(\Omega))} \leq C\alpha,$$

where C is independent of α and the points $\{z_j\}_{j=1}^m$.

□

Now, we can estimate $\operatorname{curl} E_{\alpha} - \operatorname{curl} E_0$ as follows.

Proposition 3.1 *Let E_{α} and E be solutions to the problems (3) and (4) respectively. There exist constants $0 < \alpha_0, C$ such that for $0 < \alpha < \alpha_0$ the following estimate holds:*

$$\|\operatorname{curl}(E_{\alpha} - E_0)\|_{L^{\infty}(0,T;\mathbf{L}^2(\Omega))} \leq C\alpha, \quad (11)$$

Proof. To prove estimate (11) it is useful to introduce the following function

$$\hat{v}(x) = \int_0^T v(x, t) z(t) dt \in L^2(\Omega), \quad (12)$$

where $v \in L^1(0, T; L^2(\Omega))$ and $z(t)$ is a given function in $\mathcal{C}_0^\infty([0, T])$. Then,

$$\hat{E}(x) = \int_0^T E(x, t) z(t) dt \text{ and } \hat{E}_\alpha(x) = \int_0^T E_\alpha(x, t) z(t) dt \in X(\Omega),$$

which by relation (5) give

$$\begin{cases} (\hat{E}_\alpha - \hat{E}) \in \mathbf{H}^1(\Omega), \\ \operatorname{curl} \operatorname{curl} (\hat{E}_\alpha - \hat{E}) = 0(\alpha) \quad \text{in } \Omega, \\ \operatorname{div} (\hat{E}_\alpha - \hat{E}) = 0 \quad \text{in } \Omega, \\ (\hat{E}_\alpha - \hat{E}) \times \mathbf{n}|_{\partial\Omega} = 0, \end{cases}$$

and so,

$$\|\operatorname{curl} (\hat{E}_\alpha - \hat{E})\|_{\mathbf{L}^2(\Omega)} = O(\alpha). \quad (13)$$

The fact that $\operatorname{curl} (E_\alpha - E)$ belongs to $L^\infty(0, T; \mathbf{L}^2(\Omega))$ and by using estimate (13) we deduce that

$$\int_\Omega |\operatorname{curl} E_\alpha(x, t) - \operatorname{curl} E(x, t)|^2 dx = O(\alpha^2) \quad \text{a.e. in } t \in (0, T),$$

which means that

$$\|\operatorname{curl} (E_\alpha - E)\|_{\mathbf{L}^2(\Omega)} = O(\alpha) \quad \text{a.e. in } t \in (0, T).$$

Thus, estimate (11) follows immediately if we take the sup on $t \in (0, T)$ in the last relation. \square

Before formulating our main result in this section, let us denote $\Phi_j, j = 1, \dots, m$ the unique vector-valued solution to

$$\begin{cases} \Delta \Phi_j = 0 \text{ in } B_j, \text{ and } \mathbb{R}^d \setminus \overline{B_j}, \\ \Phi_j \text{ is continuous across } \partial B_j, \\ \frac{\mu_j}{\mu_0} \frac{\partial \Phi_j}{\partial \nu_j} \Big|_+ - \frac{\partial \Phi_j}{\partial \nu_j} \Big|_- = -\nu_j, \\ \lim_{|y| \rightarrow +\infty} |\Phi_j(y)| = 0, \end{cases} \quad (14)$$

where ν_j denotes the outward unit normal to ∂B_j , and superscripts $-$ and $+$ indicate the limiting values as the point approaches ∂B_j from outside B_j , and from inside B_j , respectively. The existence and uniqueness of this Φ_j can be established using single layer potentials with suitably chosen densities, see [6] for the case of conductivity problem. For each inhomogeneity $z_j + \alpha B_j$ we introduce the polarizability tensor

M_j which is a $d \times d$, symmetric, positive definite matrix associated with the j -th inhomogeneity, given by

$$(M_j)_{k,l} = e_k \cdot \left(\int_{\partial B_j} (\nu_j + (\frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j}|_+(y)) y \cdot e_l \, d\sigma_j(y) \right). \quad (15)$$

Here (e_1, \dots, e_d) is an orthonormal basis of \mathbb{R}^d . In terms of this function we are able to prove the following result about the asymptotic behavior of $\text{curl } E_\alpha \cdot \nu_j|_{\partial(z_j + \alpha B_j)^+}$.

Theorem 3.1 *Suppose that (1) is satisfied and let $\Phi_j, j = 1, \dots, m$ be given as in (14). Then, for the solutions E_α, E of problems (3) and (4) respectively, and for $y \in \partial B_j$ we have*

$$\begin{aligned} (\text{curl } E_\alpha(z_j + \alpha y) \cdot \nu_j)|_{\partial(z_j + \alpha B_j)^+} &= \text{curl } E(z_j, t) \cdot \nu_j \\ &+ (1 - \frac{\mu_j}{\mu_0}) \frac{\partial \Phi_j}{\partial \nu_j}|_+(y) \cdot \text{curl } E(z_j, t) + o(1). \end{aligned} \quad (16)$$

The term $o(1)$ uniform in $y \in \partial B_j$ and $t \in (0, T)$ and depends on the shape of $\{B_j\}_{j=1}^m$ and Ω , the constants $c_0, T, \mu_0, \{\mu_j\}_{j=1}^m$, the data φ, ψ , and f , but is otherwise independent of the points $\{z_j\}_{j=1}^m$.

Proof.

Let $\mathcal{E}_\alpha = \text{curl } E_\alpha(x, t)$ and $\mathcal{E}_0 = \text{curl } E(x, t)$. Then, according to (3)-(4) we have

$$\varepsilon_0 \partial_t^2 E_\alpha - \text{curl } \frac{1}{\mu_\alpha} \mathcal{E}_\alpha = 0 \text{ and } \text{curl } \mathcal{E}_\alpha = 0, \text{ for } x \in \Omega. \quad (17)$$

We restrict, for simplicity, our attention to the case of a single inhomogeneity, i.e., the case $m = 1$. The proof for any fixed number m of well separated inhomogeneities follows by iteration of the argument that we will present for the case $m = 1$. In order to further simplify notation, we assume that the single inhomogeneity has the form αB , that is, we assume it is centered at the origin. We denote the electromagnetic permeability inside αB by μ_* and define Φ_* the same as Φ_j , defined in (14), but with B_j and μ_j replaced by B and μ_* , respectively. Define ν to be the outward unit normal to ∂B . Now, following a common practice in multiscale expansions we introduce the local variable $y = \frac{x}{\alpha}$, then the domain $\tilde{\Omega} = (\frac{\Omega}{\alpha})$ is well defined.

Next, let ϖ be given in $\mathcal{C}_0^\infty([0, T])$. For any function $v \in \mathbf{L}^1(0, T; \mathbf{L}^2(\Omega))$, we define

$$\hat{v}(x) = \int_0^T v(x, t) \varpi(t) \, dt \in \mathbf{L}^2(\Omega).$$

We remark that $\widehat{\partial_t v}(x) = - \int_0^T v(x, t) \varpi'(t) \, dt$. So that we deduce from (17) that $\hat{\mathcal{E}}_\alpha$ satisfies

$$\begin{cases} \text{curl } \frac{1}{\mu_\alpha} \hat{\mathcal{E}}_\alpha = \int_0^T E_\alpha \varpi''(t) \, dt & \text{in } \Omega, \\ \text{curl } \hat{\mathcal{E}}_\alpha = 0 & \text{in } \Omega. \end{cases}$$

Analogously, $\hat{\mathcal{E}}$ satisfies

$$\begin{cases} \frac{1}{\mu_0} \text{curl } \hat{\mathcal{E}} = \int_0^T E \varpi''(t) \, dt & \text{in } \Omega, \\ \text{curl } \hat{\mathcal{E}} = 0 & \text{in } \Omega. \end{cases}$$

Indeed, we have $\hat{\mathcal{E}}_\alpha \times \mathbf{n} = \hat{\mathcal{E}} \times \mathbf{n} = \text{curl}_{\partial\Omega} \hat{f} \times \mathbf{n}$ on the boundary $\partial\Omega$, where $\text{curl}_{\partial\Omega}$ is the tangential curl. Following [4] and [1], we introduce q_α^* as the unique solution to the following problem

$$\begin{cases} \Delta q_\alpha^* = 0 & \text{in } \tilde{\Omega} = (\frac{\Omega}{\alpha}) \setminus \overline{B} \text{ and in } B, \\ q_\alpha^* \text{ is continuous across } \partial B, \\ \mu_0 \frac{\partial q_\alpha^*}{\partial \nu}|_+ - \mu_* \frac{\partial q_\alpha^*}{\partial \nu}|_- = -(\mu_0 - \mu_*) \hat{\mathcal{E}}(\alpha y) \cdot \nu & \text{on } \partial B, \\ q_\alpha^* = 0 & \text{on } \partial\tilde{\Omega}. \end{cases}$$

The jump condition

$$\mu_0 \frac{\partial q_\alpha^*}{\partial \nu}|_+ - \mu_* \frac{\partial q_\alpha^*}{\partial \nu}|_- = -(\mu_0 - \mu_*) \hat{\mathcal{E}}(\alpha y) \cdot \nu \quad \text{on } \partial B$$

guarantees that $\hat{\mathcal{E}}_\alpha(x) - \hat{\mathcal{E}}(x) - \text{grad}_y q_\alpha^*(\frac{x}{\alpha})$ belongs to the functional space $X_N(\Omega)$, where $\text{grad}_{\partial\Omega}$ is the tangential gradient. Since

$$\begin{cases} \text{curl} \frac{1}{\mu_\alpha} (\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{grad}_y q_\alpha^*(\frac{x}{\alpha})) = \int_0^T \left[E_\alpha - \chi(\Omega \setminus \overline{\alpha B}) E + \frac{\mu_*}{\mu_0} \chi(\alpha B) E \right] \varpi''(t) dt & \text{in } \Omega, \\ \text{curl} (\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{grad}_y q_\alpha^*(\frac{x}{\alpha})) = 0 & \text{in } \Omega, \\ (\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{grad}_y q_\alpha^*(\frac{x}{\alpha})) \times \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\chi(\omega)$ is the characteristic function of the domain ω , we arrive, as a consequence of the energy estimate given by Lemma 3.1, at the following

$$\begin{cases} (\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{grad}_y q_\alpha^*(\frac{x}{\alpha})) \in X_N(\Omega), \\ \text{curl} \frac{1}{\mu_\alpha} (\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{curl}_y q_\alpha^*(\frac{x}{\alpha})) = 0(\alpha) & \text{in } \Omega, \\ \text{curl} (\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{grad}_y q_\alpha^*(\frac{x}{\alpha})) = 0 & \text{in } \Omega, \\ (\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{grad}_y q_\alpha^*(\frac{x}{\alpha})) \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

From [4] we know that this yields the following estimate

$$\| \text{curl} \frac{1}{\mu_\alpha} (\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{grad}_y q_\alpha^*(\frac{x}{\alpha})) \|_{L^2(\Omega)} + \| \hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{grad}_y q_\alpha^*(\frac{x}{\alpha}) \|_{L^2(\Omega)} \leq C\alpha,$$

and so,

$$(\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \text{grad}_y q_\alpha^*(\frac{x}{\alpha})) \cdot \nu|_+ = 0(\alpha) \quad \text{on } \partial(\alpha B).$$

Now, we denote by q_* be the unique (scalar) solution to

$$\begin{cases} \Delta q_* = 0 & \text{in } \mathbb{R}^d \setminus \overline{B} \text{ and in } B, \\ q_* \text{ is continuous across } \partial B, \\ \mu_0 \frac{\partial q_*}{\partial \nu}|_+ - \mu_* \frac{\partial q_*}{\partial \nu}|_- = -(\mu_0 - \mu_*) \hat{\mathcal{E}}(0) \cdot \nu & \text{on } \partial B, \\ \lim_{|y| \rightarrow +\infty} q_* = 0. \end{cases}$$

In the spirit of Theorem 1 in [6] it follows that

$$\|(\operatorname{grad}_y q_* - \operatorname{grad}_y q_\alpha^*)(\frac{x}{\alpha})\|_{\mathbf{L}^2(\Omega)} \leq C\alpha^{1/2},$$

which yields

$$(\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}} - \operatorname{grad}_y q_*(\frac{x}{\alpha})) \cdot \nu = o(1) \quad \text{on } \partial(\alpha B).$$

Writing q_* in terms of Φ_* gives

$$\int_0^T \left[(\operatorname{curl} E_\alpha(\alpha y) \cdot \nu)|_{\partial(\alpha B)^+} - \nu \cdot \operatorname{curl} E(0, t) - (\frac{\mu_0}{\mu_*} - 1) \frac{\partial \Phi_*}{\partial \nu}|_+(y) \cdot \operatorname{curl} E(0, t) \right] \varpi(t) dt = o(1),$$

for any $\varpi \in \mathcal{C}_0^\infty([0, T])$, and so, by iterating the same argument for the case of m (well separated) inhomogeneities $z_j + \alpha B_j, j = 1, \dots, m$, we arrive at the promised asymptotic formula (16). \square

4 The identification procedure

Before describing our identification procedure, let us introduce the following cutoff function $\beta(x) \in \mathcal{C}_0^\infty(\Omega)$ such that $\beta \equiv 1$ in a subdomain Ω' of Ω that contains the inhomogeneities \mathcal{B}_α and let $\eta \in \mathbb{R}^d$. We will take in what follows $E(x, t) = \eta^\perp e^{i\eta \cdot x - i\sqrt{\mu_0}|\eta|t}$ where η^\perp is a unit vector that is orthogonal to η which corresponds to taking $\varphi(x) = \eta^\perp e^{i\eta \cdot x}$, $\psi(x) = -i\sqrt{\mu_0}|\eta|\eta^\perp e^{i\eta \cdot x}$, and $f(x, t) = \eta^\perp \times \mathbf{n} e^{i\eta \cdot x - i\sqrt{\mu_0}|\eta|t}$ and assume that we are in possession of the measurements of:

$$\operatorname{curl} E_\alpha \times \mathbf{n} \quad \text{on } \Gamma \times (0, T),$$

where Γ is an open part of $\partial\Omega$. Suppose now that T and the part Γ of the boundary $\partial\Omega$ are such that they geometrically control Ω which roughly means that every geometrical optic ray, starting at any point $x \in \Omega$ at time $t = 0$ hits Γ before time T at a non diffractive point, see [5]. It follows from [17] (see also [13], [11] and [12]) that there exists (a unique) $g_\eta \in H_0^1(0, T; TL^2(\Gamma))$ (constructed by the Hilbert Uniqueness Method) such that the unique weak solution w_η to

$$\begin{cases} (\partial_t^2 + \operatorname{curl} \operatorname{curl}) w_\eta = 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} w_\eta = 0 & \text{in } \Omega \times (0, T), \\ w_\eta|_{t=0} = \beta(x)\eta^\perp e^{i\eta \cdot x}, \partial_t w_\eta|_{t=0} = 0 & \text{in } \Omega, \\ w_\eta \times \mathbf{n}|_{\partial\Omega \setminus \bar{\Gamma} \times (0, T)} = 0, \\ w_\eta \times \mathbf{n}|_{\Gamma \times (0, T)} = g_\eta, \end{cases} \quad (18)$$

satisfies $w_\eta(T) = \partial_t w_\eta(T) = 0$ in Ω .

Let $\theta_\eta \in H^1(0, T; TL^2(\Gamma))$ denote the unique solution of the Volterra equation of second kind

$$\begin{cases} \partial_t \theta_\eta(x, t) + \int_t^T e^{-i|\eta|(s-t)} (\theta_\eta(x, s) - i|\eta| \partial_t \theta_\eta(x, s)) ds = g_\eta(x, t) & \text{for } x \in \Gamma, t \in (0, T), \\ \theta_\eta(x, 0) = 0 & \text{for } x \in \Gamma. \end{cases} \quad (19)$$

The existence and uniqueness of this θ_η in $H^1(0, T; TL^2(\Gamma))$ for any $\eta \in \mathbb{R}^d$ can be established using the resolvent kernel. However, observing from differentiation of (19) with respect to t that θ_η is the unique solution of the ODE:

$$\begin{cases} \partial_t^2 \theta_\eta - \theta_\eta = e^{i|\eta|t} \partial_t (e^{-i|\eta|t} g_\eta) & \text{for } x \in \Gamma, t \in (0, T), \\ \theta_\eta(x, 0) = 0, \partial_t \theta_\eta(x, T) = 0 & \text{for } x \in \Gamma, \end{cases} \quad (20)$$

the function θ_η may be find (in practice) explicitly with variation of parameters and it also immediately follows from this observation that θ_η belongs to $H^2(0, T; TL^2(\Gamma))$. We introduce v_η as the unique weak solution (obtained by transposition as done in [15] and in [14] [Theorem 4.2, page 46] for the scalar function) in $\mathcal{C}^0(0, T; X(\Omega)) \cap \mathcal{C}^1(0, T; L^2(\Omega))$ to the following problem

$$\begin{cases} (\partial_t^2 + \operatorname{curl} \operatorname{curl}) v_\eta = 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} v_\eta = 0 & \text{in } \Omega \times (0, T), \\ v_\eta|_{t=0} = 0 & \text{in } \Omega, \\ \partial_t v_\eta|_{t=0} = \sum_{j=1}^m i(1 - \frac{\mu_0}{\mu_j}) \eta \times (\nu_j + (\frac{\mu_0}{\mu_j} - 1) \frac{\partial \Phi_j}{\partial \nu_j}|_+) e^{i\eta \cdot z_j} \delta_{\partial(z_j + \alpha B_j)} \in Y(\Omega) & \text{in } \Omega, \\ v_\eta \times \mathbf{n}|_{\partial\Omega \times (0, T)} = 0. \end{cases}$$

Then, the following holds.

Proposition 4.1 *Suppose that Γ and T geometrically control Ω . For any $\eta \in \mathbb{R}^d$ we have*

$$\begin{aligned} \int_0^T \int_\Gamma g_\eta \cdot (\operatorname{curl} v_\eta \times \mathbf{n}) \, d\sigma(x) dt &= \alpha^2 \sum_{j=1}^m \mu_0 (1 - \frac{\mu_j}{\mu_0}) e^{2i\eta \cdot z_j} \eta \cdot \int_{\partial B_j} (\nu_j \\ &+ (\frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j}|_+(y)) \eta \cdot y \, ds_j(y) + o(\alpha^2). \end{aligned} \quad (21)$$

Proof. Multiply the equation $(\partial_t^2 + \operatorname{curl} \operatorname{curl}) v_\eta = 0$ by w_η and integrating by parts over $(0, T) \times \Omega$, for any $\eta \in \mathbb{R}^d$ we have

$$\begin{aligned} \alpha \sum_{j=1}^m i(1 - \frac{\mu_j}{\mu_0}) e^{2i\eta \cdot z_j} \eta \cdot \int_{\partial B_j} (\nu_j + (\frac{\mu_j}{\mu_0} - 1) \frac{\partial \Phi_j}{\partial \nu_j}|_+(y)) e^{i\alpha \eta \cdot y} \, ds(y) = \\ -\mu_0^{-1} \int_0^T \int_\Gamma g_\eta \cdot (\operatorname{curl} v_\eta \times \mathbf{n}) \, d\sigma(x) dt. \end{aligned}$$

Now, we take the Taylor expansion of $\alpha e^{i\alpha \eta \cdot y}$ in the left side of the last equation, we obtain the convenient asymptotic formula (21). \square

To identify the locations and certain properties of the small inhomogeneities \mathcal{B}_α let us view the averaging of the boundary measurements

$$\operatorname{curl} E_\alpha \times \mathbf{n}|_{\Gamma \times (0, T)},$$

using the solution θ_η to the Volterra equation (19) or equivalently the ODE (20), as a function of η . The following holds.

Theorem 4.1 *Let $\eta \in \mathbb{R}^d$. Let E_α be the unique solution in $\mathcal{C}^0(0, T; X(\Omega)) \cap \mathcal{C}^1(0, T; L^2(\Omega))$ to the Maxwell's equations (3) with $\varphi(x) = \eta^\perp e^{i\eta \cdot x}$, $\psi(x) = -i\sqrt{\mu_0}|\eta|\eta^\perp e^{i\eta \cdot x}$, and $f(x, t) = \eta^\perp e^{i\eta \cdot x - i\sqrt{\mu_0}|\eta|t}$. Suppose that Γ and T geometrically control Ω , then we have*

$$\begin{aligned} & \int_0^T \int_\Gamma \left[\theta_\eta \cdot (\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n}) + \partial_t \theta_\eta \cdot \partial_t (\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n}) \right] d\sigma(x) dt = \\ & \alpha^2 \sum_{j=1}^m (\mu_0 - \mu_j) e^{2i\eta \cdot z_j} M_j(\eta) \cdot \eta + O(\alpha^2), \end{aligned} \quad (22)$$

where θ_η is the unique solution to the Volterra equation (20) with g_η defined as the boundary control in (18) and M_j is the polarization tensor of B_j , defined by

$$(M_j)_{k,l} = e_k \cdot \left(\int_{\partial B_j} \left(\nu_j + \left(\frac{\mu_j}{\mu_0} - 1 \right) \frac{\partial \Phi_j}{\partial \nu_j} \Big|_{+(y)} \right) y \cdot e_l ds_j(y) \right). \quad (23)$$

Here (e_1, e_2) is an orthonormal basis of \mathbb{R}^d . The term $O(\alpha^2)$ is independent of the points $\{z_j, j = 1, \dots, m\}$.

Proof. From $\partial_t \theta_\eta(T) = 0$ and $(\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n})|_{t=0} = 0$ the term $\int_0^T \int_\Gamma \partial_t \theta_\eta \cdot \partial_t (\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n}) d\sigma(x) dt$ has to be interpreted as follows

$$\int_0^T \int_\Gamma \partial_t \theta_\eta \cdot \partial_t (\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n}) = - \int_0^T \int_\Gamma \partial_t^2 \theta_\eta \cdot (\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n}). \quad (24)$$

Next, introduce

$$\tilde{E}_{\alpha, \eta}(x, t) = E(x, t) + \int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_\eta(x, t-s) ds, \quad x \in \Omega, t \in (0, T). \quad (25)$$

We have

$$\begin{aligned} & \int_0^T \int_\Gamma \left[\theta_\eta \cdot (\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n}) + \partial_t \theta_\eta \cdot \partial_t (\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n}) \right] = \\ & \int_0^T \int_\Gamma \left[\theta_\eta \cdot (\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} \tilde{E}_{\alpha, \eta} \times \mathbf{n}) + \partial_t \theta_\eta \cdot \partial_t (\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} \tilde{E}_{\alpha, \eta} \times \mathbf{n}) \right] \\ & + \int_0^T \int_\Gamma \left[\theta_\eta \cdot \int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_\eta(x, t-s) \times \mathbf{n} ds + \partial_t \theta_\eta \cdot \partial_t \int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_\eta(x, t-s) \times \mathbf{n} ds \right]. \end{aligned}$$

Since θ_η satisfies the Volterra equation (20) and

$$\begin{aligned} & \partial_t \left(\int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_\eta(x, t-s) \times \mathbf{n} ds \right) = \partial_t \left(-e^{-i\sqrt{\mu_0}|\eta|t} \int_0^t e^{i\sqrt{\mu_0}|\eta|s} v_\eta(x, s) \times \mathbf{n} ds \right) \\ & = i\sqrt{\mu_0}|\eta| e^{-i\sqrt{\mu_0}|\eta|t} \int_0^t e^{i\sqrt{\mu_0}|\eta|s} v_\eta(x, s) \times \mathbf{n} ds + v_\eta(x, t) \times \mathbf{n}, \end{aligned}$$

we obtain by integrating by parts over $(0, T)$ that

$$\begin{aligned}
& \int_0^T \int_{\Gamma} \left[\theta_{\eta} \cdot \int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_{\eta}(x, t-s) \times \mathbf{n} \, ds + \partial_t \theta_{\eta} \cdot \partial_t \int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_{\eta}(x, t-s) \times \mathbf{n} \, ds \right] \\
&= \int_0^T \int_{\Gamma} (v_{\eta}(x, t) \times \mathbf{n}) \cdot (\partial_t \theta_{\eta} + \int_t^T \theta_{\eta}(s) e^{i\sqrt{\mu_0}|\eta|(t-s)} \, ds) \\
&\quad - i\sqrt{\mu_0}|\eta| (e^{-i\sqrt{\mu_0}|\eta|t} \partial_t \theta_{\eta}(t)) \cdot \int_0^t e^{i\sqrt{\mu_0}|\eta|s} v_{\eta}(x, s) \times \mathbf{n} \, ds \, dt \\
&= \int_0^T \int_{\Gamma} v_{\eta}(x, t) \times \mathbf{n} \cdot (\partial_t \theta_{\eta} + \int_t^T (\theta_{\eta}(s) - i\sqrt{\mu_0}|\eta| \partial_t \theta_{\eta}(s)) e^{i\sqrt{\mu_0}|\eta|(t-s)} \, ds) \, dt \\
&= \int_0^T \int_{\Gamma} g_{\eta}(x, t) \cdot (\operatorname{curl} v_{\eta}(x, t) \times \mathbf{n}) \, dt
\end{aligned}$$

and so, from Proposition 4.1 we obtain

$$\begin{aligned}
& \int_0^T \int_{\Gamma} \left[\theta_{\eta} \cdot (\operatorname{curl} E_{\alpha} \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n}) + \partial_t \theta_{\eta} \cdot \partial_t (\operatorname{curl} E_{\alpha} \times \mathbf{n} - \operatorname{curl} E \times \mathbf{n}) \right] = \\
& \alpha^2 \sum_{j=1}^m \left(1 - \frac{\mu_j}{\mu_0} \right) e^{2i\eta \cdot z_j} \eta \cdot \int_{\partial B_j} \left(\nu_j + \left(\frac{\mu_j}{\mu_0} - 1 \right) \frac{\partial \Phi_j}{\partial \nu_j} \Big|_+(y) \right) \eta \cdot y \, ds_j(y) \\
& + \int_0^T \int_{\Gamma} \left[\theta_{\eta} \cdot (\operatorname{curl} E_{\alpha} \times \mathbf{n} - \operatorname{curl} \tilde{E}_{\alpha, \eta} \times \mathbf{n}) + \partial_t \theta_{\eta} \cdot \partial_t (\operatorname{curl} E_{\alpha} \times \mathbf{n} \right. \\
& \quad \left. - \operatorname{curl} \tilde{E}_{\alpha, \eta} \times \mathbf{n}) \right] + o(\alpha^2).
\end{aligned}$$

In order to prove Theorem 4.1 it suffices then to show that

$$\int_0^T \int_{\Gamma} \left[\theta_{\eta} \cdot (\operatorname{curl} E_{\alpha} \times \mathbf{n} - \operatorname{curl} \tilde{E}_{\alpha, \eta} \times \mathbf{n}) + \partial_t \theta_{\eta} \cdot \partial_t (\operatorname{curl} E_{\alpha} \times \mathbf{n} - \operatorname{curl} \tilde{E}_{\alpha, \eta} \times \mathbf{n}) \right] = o(\alpha^2). \quad (26)$$

Since

$$\left\{ \begin{aligned} & (\partial_t^2 - \operatorname{curl} \frac{1}{\mu_0} \operatorname{curl}) \left(\int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_{\eta}(x, t-s) \, ds \right) \\ &= \sum_{j=1}^m i \left(1 - \frac{\mu_j}{\mu_0} \right) \eta \times \left(\nu_j + \left(\frac{\mu_j}{\mu_0} - 1 \right) \frac{\partial \Phi_j}{\partial \nu_j} \Big|_+(y) \right) e^{i\eta \cdot z_j} \delta_{\partial(z_j + \alpha B_j)} e^{-i\sqrt{\mu_0}|\eta|t} \quad \text{in } \Omega \times (0, T), \\ & \left(\int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_{\eta}(x, t-s) \, ds \right) \Big|_{t=0} = 0, \partial_t \left(\int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_{\eta}(x, t-s) \, ds \right) \Big|_{t=0} = 0 \quad \text{in } \Omega, \\ & \left(\int_0^t e^{-i\sqrt{\mu_0}|\eta|s} v_{\eta}(x, t-s) \, ds \right) \times \mathbf{n} \Big|_{\partial \Omega \times (0, T)} = 0, \end{aligned} \right.$$

it follows from Theorem 3.1 that

$$\left\{ \begin{aligned} & (\partial_t^2 - \operatorname{curl} \frac{1}{\mu_0} \operatorname{curl}) (E_{\alpha} - \tilde{E}_{\alpha, \eta}) = o(\alpha^2) \quad \text{in } \Omega \times (0, T), \\ & (E_{\alpha} - \tilde{E}_{\alpha, \eta}) \Big|_{t=0} = 0, \partial_t (E_{\alpha} - \tilde{E}_{\alpha, \eta}) \Big|_{t=0} = 0 \quad \text{in } \Omega, \\ & (E_{\alpha} - \tilde{E}_{\alpha, \eta}) \times \mathbf{n} \Big|_{\partial \Omega \times (0, T)} = 0. \end{aligned} \right.$$

Following the proof of Proposition 3.1, we immediately obtain

$$\|E_\alpha - \tilde{E}_{\alpha,\eta}\|_{L^2(\Omega)} = o(\alpha^2), t \in (0, T), x \in \Omega,$$

where $o(\alpha^2)$ is independent of the points $\{z_j\}_{j=1}^m$. To prove (26) it suffices then from (24) to show that the following estimate holds

$$\|\operatorname{curl} E_\alpha \times \mathbf{n} - \operatorname{curl} \tilde{E}_{\alpha,\eta} \times \mathbf{n}\|_{L^2(0,T;TL^2(\Gamma))} = o(\alpha^2).$$

Let θ be given in $\mathcal{C}_0^\infty(]0, T[)$ and define

$$\hat{\tilde{E}}_{\alpha,\eta}(x) = \int_0^T \tilde{E}_{\alpha,\eta}(x, t) \theta(t) dt$$

and

$$\hat{E}_\alpha(x) = \int_0^T E_\alpha(x, t) \theta(t) dt.$$

From definition (25) we can write

$$\left\{ \begin{array}{l} (\hat{E}_\alpha - \hat{\tilde{E}}_\alpha) \in \mathbf{H}^1(\Omega), \\ \operatorname{curl} \operatorname{curl} (\hat{E}_\alpha - \hat{\tilde{E}}_\alpha) = 0(\alpha) \in Y(\Omega) \quad \text{in } \Omega, \\ \operatorname{div} (\hat{E}_\alpha - \hat{\tilde{E}}_\alpha) = 0 \quad \text{in } \Omega, \\ (\hat{E}_\alpha - \hat{\tilde{E}}_\alpha) \times \mathbf{n}|_{\partial\Omega} = 0. \end{array} \right. \quad (27)$$

In the spirit of the standard elliptic regularity [9] we deduce for the boundary value problem (27) that

$$\|\operatorname{curl} (\hat{E}_\alpha - \hat{\tilde{E}}_\alpha) \times \mathbf{n}\|_{\mathbf{L}^2(\Gamma)} = O(\alpha^2),$$

for all $\theta \in \mathcal{C}_0^\infty(]0, T[)$; whence

$$\|\operatorname{curl} (E_\alpha - \hat{E}_\alpha) \times \mathbf{n}\|_{\mathbf{L}^2(\Gamma)} = o(\alpha^2) \text{ a. e. in } t \in (0, T),$$

and so, the desired estimate (22) holds. The proof of Theorem 4.1 is then over. \square

Our identification procedure is deeply based on Theorem 4.1. Let us neglect the asymptotically small remainder in the asymptotic formula (22), and define $\aleph_\alpha(\eta)$ by

$$\aleph_\alpha(\eta) = \int_0^T \int_\Gamma \left[\theta_\eta \cdot (\operatorname{curl} (E_\alpha - E) \times \mathbf{n}) + \partial_t \theta_\eta \cdot \partial_t (\operatorname{curl} (E_\alpha - E) \times \mathbf{n}) \right].$$

Recall that the function $e^{2i\eta \cdot z_j}$ is exactly the Fourier Transform (up to a multiplicative constant) of the Dirac function δ_{-2z_j} (a point mass located at $-2z_j$). From Theorem 4.1 it follows that the function $e^{2i\eta \cdot z_j}$ is (approximately) the Fourier Transform of a linear combination of derivatives of point masses, or

$$\aleph_\alpha(\eta) \approx \alpha^2 \sum_{j=1}^m L_j \delta_{-2z_j},$$

where L_j is a second order constant coefficient, differential operator whose coefficients depend on the polarization tensor M_j defined by (23) (see [6] for its properties) and

$\check{\aleph}_\alpha(\eta)$ represents the inverse Fourier Transform of $\aleph_\alpha(\eta)$. The reader is referred to [6] for properties of the tensor polarization M_j .

The method of reconstruction consists in sampling values of $\check{\aleph}_\alpha(\eta)$ at some discrete set of points and then calculating the corresponding discrete inverse Fourier Transform. After a rescaling the support of this discrete inverse Fourier Transform yields the location of the small inhomogeneities \mathcal{B}_α . Once the locations are known we may calculate the polarization tensors $(M_j)_{j=1}^m$ by solving an appropriate linear system arising from (22). This procedure generalizes the approach developed in [3] for the two-dimensional (time-independent) inverse conductivity problem and generalize the results in [1] to the full time-dependent Maxwell's equations.

5 Conclusion

In this paper, we are convinced that the use of approximate formulae such as (22) represents a very promising approach to the dynamical identification of small inhomogeneities that are embedded in a homogeneous medium. We also believe that our method yields a good approximation to small amplitude perturbations in the electromagnetic parameters (for the example of electric permittivity $\varepsilon_\alpha(x) = \varepsilon_0 + \alpha\varepsilon(x)$) from the measurements:

$$\operatorname{curl} H_\alpha \times \mathbf{n} \quad \text{on } \Gamma \times (0, T).$$

Our method may yield the Fourier transform of the amplitude perturbation $\varepsilon(x)$. This issue will be considered in a forthcoming work [7].

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