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MIXED DISCONTINUOUS GALERKIN METHOD FOR THE THREE-DIMENSIONAL ELECTROSTATIC PROBLEM

DAVEAU CHRISTIAN AND ABDELHAMID ZAGHDANI

Abstract. In this paper a new discontinuous Galerkin method for the three dimensional electrostatic problem is presented. The divergence contraint is taken into account by a regularized variational formulation and the tangential and normal jumps of the discrete solution at the element interface are penalized. Optimal error estimates in a discrete energy norm are proved. Some numerical experiments confirm the theoretical predictions.

1. Introduction

There exists several works about the discontinuous Galerkin method for the resolution of partial differential equations. For advection and diffusion problem, there are the works of Baumann (see [2]) and the paper with Oden, Babuška and Baumann (see [13]). Other versions of discontinuous Galerkin method are presented in the book edited by Cockburn, Karniadakis and Shu (see [4]). We also cite the work of S. Prudhomme, F.Pascal, J.T. Oden and A. Romkes (see [14]); they analyse different Galerkin discontinuous formulations for the Poisson problem. Besides, Ilaria Perugia and al. present different formulations for the Maxwell's equations, (see [15, 16]), and B. Rivière, and V. Girault and al. analyse and studied different formulations for the Navier Stokes problem, (see [10]).

Discontinuous Galerkin methods has several advantages over other types of finite element methods. For example, the trial and test spaces are very easy to construct; they can naturally handle inhomogeneous boundary conditions and curved conditions; and they allow the use of highly non uniform and unstructured meshes. In addition, the fact that the mass matrices are block diagonal is an attractive feature in the context of time-dependent problems, especially if explicit time discretizations are used.

In this paper, we present a new mixed discontinuous Galerkin method for the three dimensional time harmonic electrostatic problem:

$$\nabla \times (\nabla \times u) = J$$

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$$\nabla \cdot u = 0$$

with boundary conditions. Here u is related to electric field $\mathcal{E}$ by the relation $\mathcal{E}(x,t) = \mathcal{R}e(u(x)\exp(iwt))$, where $w \neq 0$ is a given frequency. This problem has also been studied, (see [16]), but we present a new discontinuous Galerkin method. There are substantial differences between the two approaches. The advantage of our method is that our primal formulation is consistent while in [16], the formulation is not consistent, due to the nature of the discrete lifting operator.

The outline of the paper is the following. Section 2, we introduce some notations and spaces. In section 3, we derive a discontinuous Galerkin formulation and prove the equivalence between the variational formulation and the original one. In section 4, we present the numerical method and prove the main properties of the discrete bilinear form. Section 5 is devoted to the convergence result derived from an *hp* analysis method and in section 6, we present numerical results.

2. Discontinuous Galerkin method

2.1. **Mathematical Model.** Let Ω be a bounded polyhedron domain included in $\mathbb{R}^3$. We also suppose that Ω and its boundary denoted by Γ is connected and simply connected. We deduce from Maxwell equations that the electric field u satisfies:

$$(1) \qquad \begin{cases} \nabla \times (\nabla \times u) &= -iwJ_s =: J & x \in \Omega \subset \mathbb{R}^3, \\ \nabla \cdot u &= 0 & x \in \Omega \\ n \times u &= 0 & x \in \Gamma. \end{cases}$$

where $\omega \geq 0$ is the frequency of the electromagnetic field and J_s is the impressed current density which we assume to be divergence free. We introduce a Lagrange multiplier p,the problem (1) is equivalent to the problem:

$$(2) \qquad \begin{cases} \nabla \times (\nabla \times u) - \nabla p = J & x \in \Omega \subset \mathbb{R}^3, \\ \nabla \cdot u = 0 & x \in \Omega \\ n \times u = 0 & x \in \Gamma. \end{cases}$$

Now, we introduce some notations. Let be Π_h a triangulation of Ω into tetrahedra such that:

Assumption (H)

(1) Two arbitrary tetrahedra K, $K' \in \Pi_h$ $(K \neq K')$ are either disjoint, or have a common vertex or a common edge or a common face. Further, we have

$$\overline{\Omega} = \cup_{K \in \Pi_h} K$$

(2) The triangulation is shape regular, i.e. if h_K denote the diameter of the element K and ρ_K the diameter of the largest sphere contained in

K, there exists a constant $\sigma > 0$, independent of K such that

$$\frac{h_K}{\rho_K} \leq \sigma.$$

(3) For all tetrahedron K of Π_h, K has not more than one face on Γ.

We finally denote by F_h, F_h^I and F_h^D the union of all faces of Π_h, the union of the internal faces and the union of the face supported by the boundary Γ respectively.

We also introduce some spaces. For a bounded domain $\mathcal{O} \in \mathbb{R}^3$, $C_0^\infty(\mathcal{O})$ is the set of $C^\infty(\mathcal{O})$ with support compact in $\mathcal{O}$, $L^2(\mathcal{O})$ is the square integrated function on $\mathcal{O}$, $H^s(\mathcal{O})$ is the usual Sobolev space for $s \in \mathbb{R}$ and $H_0^s(\mathcal{O})$ is the set of $H^s(\mathcal{O})$ whose trace is null on $\partial\mathcal{O}$ and we introduce

$$H(\nabla\cdot, \mathcal{O}) = \{u \in L^2(\mathcal{O})^3, \nabla \cdot u \in L^2(\mathcal{O})\},$$

$$H(\nabla \cdot 0, \mathcal{O}) = \{u \in H(\nabla\cdot, \mathcal{O}), \ \ \nabla \cdot u = 0 \text{ in } \mathcal{O}\},$$

$$H(\nabla\times, \mathcal{O}) = \{u \in L^2(\mathcal{O})^3, \nabla \times u \in L^2(\mathcal{O})^3\},$$

$$H_0(\nabla \times 0, \mathcal{O}) = \{u \in H(\nabla\times, \mathcal{O}), \ \ \nabla \times u = 0 \ \text{ in } \mathcal{O}, \ \mathrm{u} \times \mathrm{n} = 0 \quad \text{on } \partial\mathcal{O}\}.$$

The formulations involve the functional spaces

$$\mathcal{V}(h) := \{u \in L^2(\Omega)^3, \ \nabla \times u \in\in L^2(K)^3 \ \ \forall K \in \Pi_h\}$$

and

$$\mathcal{Q}(h) := \{u \in L^2(\Omega), \quad u_{|K} \in H^1(K) \ \ \forall K \in \Pi_h\}.$$

Let $K \in \Pi_h$ and n_K the unit exterior normal of ∂K. We multiply the first equation of the original problem (2) by a test function $v \in \mathcal{V}(h)$ and integrate on K and with the Stokes and Green Formulae we get:

$$\begin{aligned} \int_K (\nabla \times u) \cdot (\nabla \times v) \quad &+ \quad \int_K p\nabla \cdot v - \int_{\partial K} (v \cdot n_K)p \\ &- \quad \int_{\partial K} v \cdot ((\nabla \times u) \times n_K) = \int_K J \cdot v. \end{aligned} \tag{3}$$

We integrate on K the second equation of (2) and with Green formula, we obtain for $\psi \in \mathcal{Q}(h)$

$$-\int_K u \cdot \nabla\psi + \int_{\partial K} (u \cdot n_K)\psi = 0. \tag{4}$$

Then, we obtain the variational formulation: $\forall K \in \Pi_h$, find $(u,p) \in \mathcal{V}(h) \times \mathcal{Q}(h)$ satisfy

$$(5)\quad \begin{cases} \displaystyle\int_K (\nabla\times u)\cdot(\nabla\times v) & + & \displaystyle\int_K p\nabla\cdot v - \int_{\partial K}(v\cdot n_K)\widehat{p} \\ & - & \displaystyle\int_{\partial K} v\cdot((\widehat{\nabla\times u})\times n_K) \\ & = & \displaystyle\int_K J\cdot v \quad \forall v\in\mathcal{V}(h), \\ \displaystyle -\int_K u\cdot\nabla\psi + \int_{\partial K}(\widehat{u}\cdot n_K)\psi & = & 0 \quad \forall\psi\in\mathcal{Q}(h). \end{cases}$$

As the functions in the equations (3) and (4) are discontinuous at the element interfaces, we approximate the traces of the functions by numerical flux $\widehat{u}$, $\widehat{p}$ and $\widehat{\nabla\times u}$. In the next section, we give the definition for the different flux.

2.2. **Traces and numerical flux.** We introduce some notations for the traces of functions in $H^s(\Pi_h)^3 = (\Pi_{K\in\Pi_h} H^s(K))^3$ for $s > \frac{1}{2}$. To this end, let $e \in F_h^I$ be an interior face shared by the elements K_l and K_m. Let n_l (resp. n_m) be the outer unit normal vector on e with respect to K_l (resp. K_m). Let v be a vector belonging to $H^s(\Pi_h)^3$. We denote by v_l (resp. v_m) the restriction of v to K_l (resp. K_m). then, we define on e the average, the tangential and the normal jump of v by:

$$\{v\} = \frac{1}{2}(v_l|_e + v_m|_e),$$
$$[v]_T = v_l|_e \times n_l + v_m|_e \times n_m,$$
$$[v]_N = v_l|_e \cdot n_l + v_m|_e \cdot n_m.$$

Similary, we define the average and the normal jump for a scalar function $\varphi \in H^s(\Pi_h)$ by

$$\{\varphi\} = \frac{1}{2}(\varphi_l|_e + \varphi_m|_e),$$
$$[v]_N = \varphi_l|_e n_l + \varphi_m|_e n_m.$$

Finally for $e \in F_h^D$, we get:

$$\{v\} = v|_e,$$
$$[v]_T = v|_e \times n,$$
$$[v]_N = v|_e \cdot n.$$

For a vector $v \in H^s(\Pi_h)^3$ with $s > \frac{1}{2}$, numerical flux $\widehat{v}$ are functions of $L^2(F_h)^3$. It has an unique value on the element interface. It is the same for scalar function $\varrho \in H^s(\Pi_h)$ with $s > \frac{1}{2}$ numerical flux $\widehat{\varrho}$ are functions of $L^2(F_h)$. Following [1], we define the numerical flux such that:

• On the interior faces :

$$(6)\quad \begin{cases} \widehat{\nabla\times u} & = & \{\nabla\times u\} - \sigma_a[u]_T, \\ \widehat{u} & = & \{u\} - \sigma_c[p]_N, \\ \widehat{p} & = & \{p\} - \sigma_a[u]_N. \end{cases}$$

• On the faces supported by Γ :

$$\left\{\begin{array}{lcl} \widehat{\nabla\times u} & = & \{\nabla\times u\}-\sigma_a[u]_T, \\ \widehat{u} & = & u-\sigma_c p n, \\ \widehat{p} & = & 0 \end{array}\right. \tag{7}$$

with σ_a and σ_c which are stabilization parameters defined later.

2.3. **Discontinuous Galerkin formulation.** In the first time, we get the formula : $\forall v,t\in(\Pi_{K\in\pi_k}L^2(\partial K))^3$,$\forall\psi\in\Pi_{K\in\pi_k}L^2(\partial K)$ we have

$$\begin{array}{rl} \displaystyle\sum_{K\in\Pi_h}\int_{\partial K} v(t\times n_K)= & \displaystyle\int_{F_h}[v]_T\{t\}-\int_{F_h^I}[t]_T\{v\}; \\ \displaystyle\sum_{K\in\Pi_h}\int_{\partial K}\psi(v\cdot n_K) & \displaystyle=\int_{F_h^I}([v]_N\{\psi\}+[\psi]_N\{v\})+\int_{F_h^D}\psi(v\cdot n). \end{array} \tag{8}$$

Then we obtain with (8):

$$\begin{array}{rl} \displaystyle\sum_{K\in\Pi_h}\int_{\partial K} v\cdot((\widehat{\nabla\times u})\times n_K)= & \displaystyle\int_{F_h}[v]_T\{\widehat{\nabla\times u}\} \\ & \displaystyle-\int_{F_h^I}[\widehat{\nabla\times u}]_T\{v\}. \end{array} \tag{9}$$

We have with the definition of numerical flux:

$$\begin{array}{rcl} \displaystyle\sum_{K\in\Pi_h}\int_{\partial K} v\cdot((\widehat{\nabla\times u})\times n_K) & & \\ & = & \displaystyle\int_{F_h^I}[v]_T\{\nabla\times u\}-\int_{F_h^I}\sigma_a[u]_T[v]_T \\ & & \displaystyle+\int_{\Gamma}[v]_T\{\nabla\times u\}-\int_{\Gamma}\sigma_a[v]_T[u]_T \\ & = & \displaystyle\int_{F_h}[v]_T\{\nabla\times u\}-\int_{F_h}\sigma_a[v]_T[u]_T, \end{array} \tag{10}$$

$$\sum_{K\in\Pi_h}\int_{\partial K}(v\cdot n_K)\widehat{p}= -\int_{F_h^I}\sigma_a[u]_N[v]_N+\int_{F_h^I}\{p\}[v]_N. \tag{11}$$

and

$$\begin{array}{rl} \displaystyle\sum_{K\in\Pi_h}\int_{\partial K}((\widehat{u}-u)\cdot n_K)\psi= -\int_{F_h^I}\sigma_c[p]_N[\psi]_N & \displaystyle-\int_{F_h^I}[u]_N\{\psi\} \\ & \displaystyle-\int_{\Gamma}\sigma_c p\psi. \end{array} \tag{12}$$

With Green formula, we have

$$\int_K\psi\nabla\cdot u+\int_{\partial K}((\widehat{u}-u)\cdot n_K)\ \psi=0. \tag{13}$$

From (5), we add all elements of the triangulation $K \in \Pi_h$ and using $(8)-(12)$, we obtain the discontinuous formulation: find $(u,p) \in \mathcal{V}(h) \times \mathcal{Q}(h)$ satisfying:

$$\begin{aligned} &\int_\Omega (\nabla \times u)\cdot(\nabla \times v) + \int_\Omega p^h \nabla \cdot v - \int_{F_h} [v]_T\{\nabla \times u\} \\ &\quad + r\int_\Omega (\nabla \cdot u)(\nabla \cdot v) + \int_{F_h} \sigma_a [u]_T[v]_T \\ &\quad - \int_{F_h^I} [v]_N\{p\} + \int_{F_h^I} \sigma_a [u]_N[v]_N = \int_\Omega J \cdot v, \\ &\int_\Omega \psi \nabla \cdot u - \int_{F_h^I} \{\psi\}[u]_N - \int_{F_h} \sigma_c [p][\psi] = 0 \end{aligned} \tag{14}$$

for all test functions $(v,\psi) \in \mathcal{V}(h) \times \mathcal{Q}(h)$.

We add the penalty term to symmetrize the formulation

$$J(u,v) = \int_{F_h} [u]_T\{\nabla \times v\} \tag{15}$$

which is null for the exact solution of (2) and finally we have the following formulation:
Find $(u,p) \in \mathcal{V}(h) \times \mathcal{Q}(h)$ such that:

$$\begin{cases} A(u,v) \quad + \quad B(v,p) \quad = L(v) \ \ \forall v \in \mathcal{V}(h), \\ B(u,\psi) \quad - \quad C(p,\psi) \quad = 0 \ \ \forall \psi \in \mathcal{Q}(h) \end{cases} \tag{16}$$

where A, B and C are bilinear forms defined on $\mathcal{V}(h) \times \mathcal{V}(h)$, $\mathcal{V}(h) \times \mathcal{Q}(h)$ and $\mathcal{Q}(h) \times \mathcal{Q}(h)$ respectively by:

$$A(u,v) \quad := a(u,v) - J(v,u) - J(u,v). \tag{17}$$

$$\begin{aligned} a(u,v) \quad := &\int_\Omega (\nabla \times u)\cdot(\nabla \times v) + \int_{F_h} \sigma_a [u]_T[v]_T + \int_{F_h^I} \sigma_a [u]_N[v]_N \\ &+ r\int_\Omega (\nabla \cdot u)(\nabla \cdot v), \end{aligned} \tag{18}$$

$$B(v,p) := \int_\Omega p \nabla \cdot v - \int_{F_h^I} [v]_N\{p\} \tag{19}$$

and

$$C(p,\psi) \ := \int_{F_h} \sigma_c [p][\psi]. \tag{20}$$

Remark 2.1. We add the term

$$r\int_{\Omega}(\nabla\cdot u)(\nabla\cdot v),\ \text{with } r>0 \text{ and independent of } h \tag{21}$$

which is null for the exact solution to regularize the formulation and to penalize divergence free contraint.

We have the following result.

Theorem 1. *Let (u,p) be the exact solution of (2), then (u,p) is solution of (16). Conversely, if (u,p) is solution of (16), then (u,p) is solution of (2).*

Proof. If (u,p) is solution of (2) then (u,p) satisfy (16). Conversely, let (u,p) be a solution of (16). In the first time, we demonstrate that $\nabla\cdot u=0$ in Ω. Let $K\in\Pi_h$ and $\varphi\in H_0^1(K)$ extended by zero to Ω. We obtain with the second equality of (16)

$$\int_K \varphi\nabla\cdot u=0. \tag{22}$$

As $H_0^1(K)$ is dense in $L^2(K)$, we have

$$\nabla\cdot u=0 \ \text{ in } \ K. \tag{23}$$

Let f_{ij} be an element interface shared by the elements K_1 and K_2 of Π_h. We consider $\varphi\in H_0^1(K_1\cup K_2)$, which is extented to zero in Ω; then we have $\varphi_{/K_1}\in H^1(K_1)$ and $\varphi_{/K_2}\in H^1(K_2)$; we consider $(v,\psi)=(0,\varphi)$ in the second equality of (16)

$$\int_{K_1\cup K_2}\varphi\nabla\cdot u=0. \tag{24}$$

We multiply (23) by φ and integrate by parts on K:

$$\int_{K_i} u\cdot\nabla\varphi+\int_{f_{ij}}(u\cdot n)\varphi=0,\quad i\in\{1,2\}. \tag{25}$$

Then, we have

$$\int_{K_1\cup K_2} u\cdot\nabla\varphi+\int_{f_{ij}}[u]_N\varphi=0. \tag{26}$$

We integrate by parts (24), we obtain

$$\int_{K_1\cup K_2} u\cdot\nabla\varphi=0. \tag{27}$$

Then, we have

$$\int_{f_{ij}}[u]_N\varphi=0 \tag{28}$$

which implies the continuity of the normal component of u. Then, we have

$$\nabla\cdot u=0 \quad \text{ in } \ \Omega. \tag{29}$$

Now, we show that the solution u of (16) satisfies

$$\nabla \times (\nabla \times u) - \nabla p = J \ \text{ in } \ \Omega. \tag{30}$$

Let $v \in C_0^\infty(K)^3$, then the first equality of (16) gives

$$\int_K (\nabla \times u) \cdot (\nabla \times v) + \int_K p \nabla \cdot v = \int_K J \cdot v. \tag{31}$$

Then, we integrate by parts and obtain

$$\nabla \times (\nabla \times u) - \nabla p = J \quad \text{in} \ \ K. \tag{32}$$

Since u satisfies (29) then $u \in H(\nabla \cdot, \Omega)$, we have $B(u,p) = 0$ and with $\psi = p$ we get

$$\int_{F_h} \sigma_c [p]^2 = 0 \tag{33}$$

then $p \in H^1(\Omega)$ and verifies $p = 0$ on Γ. Let f_{ij} be an element interface shared by the elements K_1, K_2 and $v \in H_0^2(K_1 \cup K_2)^3$, we obtain from the first equality of (16)

$$\int_{K_1 \cup K_2} (\nabla \times u) \cdot (\nabla \times v) + \int_{K_1 \cup K_2} p \nabla \cdot v = \int_{K_1 \cup K_2} J \cdot v. \tag{34}$$

We integrate by parts (32) on K_1 and K_2 and obtain

$$\int_{K_i} (\nabla \times u) \cdot (\nabla \times v) - \int_{K_i} v \cdot \nabla p + \int_{f_{ij}} (n \times \nabla \times u) \cdot v = \int_{K_i} J \cdot v, \quad i \in \{1,2\}.$$

Particularly, we have

$$\int_{K_1 \cup K_2} (\nabla \times u) \cdot (\nabla \times v) - \int_{K_1 \cup K_2} v \cdot \nabla p + \int_{f_{ij}} v \cdot [\nabla \times u]_T = \int_{K_1 \cup K_2} J \cdot v.$$

We have $[\nabla \times u]_T = 0$ on F_h^I and therefore (u,p) verifies (30).

Now we show that $u \times n = 0$ on Γ. Let $v \in H^2(\Omega)^3$ with $v \times n = 0$ on Γ. Then, we have

$$\int_\Omega (\nabla \times u) \cdot (\nabla \times v) + \int_\Omega p \nabla \cdot v + \int_\Gamma (n \times u) \cdot (\nabla \times v) = \int_\Omega J \cdot v.$$

We multiply (30) wit v and integrate by parts and using $p = 0$ on Γ we have:

$$\int_\Omega (\nabla \times u) \cdot (\nabla \times v) + \int_\Omega p \nabla \cdot v = \int_\Omega J \cdot v. \tag{35}$$

Then, we have:

$$\int_\Gamma (n \times u) \cdot (\nabla \times v) = 0 \quad \forall v \in H^2(\Pi_h)^3 \text{ with } \ v \times n = 0 \tag{36}$$

and we can conclude that u satisfies the boundary condition $u \times n = 0$ on Γ.

Since J is divergence free in Ω then (u,p) is solution of (16) and p is null in Ω and u is solution of the original problem (1). Indeed, we have that (u,p) solution of (16) satisfies $p \in H_0^1(\Omega)$ and

$$\int_\Omega (\nabla \times u)\cdot(\nabla \times v) + \int_\Omega p\nabla \cdot v = \int_\Omega J \cdot v \quad \forall v \in H_0(\nabla\times,\Omega) \cap H(\nabla\cdot,\Omega).$$

Let $v = \nabla\varphi$ with φ the solution of

$$\Delta\varphi = p \quad \text{and} \quad \varphi = 0 \quad \text{on } \Gamma, \tag{37}$$

We have $v \in H_0(\nabla \times 0, \Omega)$ which implies

$$\|p\|_{0,\Omega}^2 = \int_\Omega J\cdot\nabla\varphi = \int_\Omega \varphi\nabla\cdot J = 0.$$

□

3. Approximation of the problem

For $k \geq 1$, we denote by $P_k(K)$ the set of all polynomials of degree less than or equal to k. The approximation of the formulation (16) involves the discrete spaces:

$$\begin{aligned} \mathcal{V}_h &:= \quad \{u \in L^2(\Omega) \ : \ u_{|K} \in P_k(K)\}^3, \\ \mathcal{Q}_h &:= \quad \{q \in L^2(\Omega) \ : \ q_{|K} \in P_{k-1}(K)\}. \end{aligned}$$

The approximate formulation for the variational formulation (16) is
Fi nd $(u^h, p^h) \in \mathcal{V}_h \times \mathcal{Q}_h$ such that

$$\begin{cases} A(u^h,v) \quad + \quad B(v,p^h) \quad = L(v) \quad \forall v \in \mathcal{V}_h, \\ B(u^h,\psi) \quad - \quad C(p^h,\psi) \quad = 0 \quad \forall \psi \in \mathcal{Q}_h. \end{cases} \tag{38}$$

Remark 3.1. Another discontinuous Galerkin formulation is possible to resolve (2); we can propose a non symmetric formulation where the bilinear form A is defined by

$$A(u,v) := a(u,v) - J(v,u) + J(u,v). \tag{39}$$

The results obtained for the symmetric discontinuous formulation are also right for the non symmetric case.

We now precise the stabilization parameters in the next section.

3.1. Stabilization parameters.

The stabilization parameters depend on the mesh and we define them to have the stability of the method. In the mesh, the cell have different size. Let $h \in L^\infty(F_h)$ be a function such that

$$h = h(x) := \begin{cases} min(h_K, h_{K'}) & \text{si } x \in \partial K \cap \partial K' \ , \quad K, K' \in \Pi_h; \\ h_K & \text{si } x \in \partial K \cap \Gamma \ , \quad K \in \Pi_h. \end{cases}$$

Let $\kappa > 0$, we set

$$\sigma_a := \kappa h^{-1} \in L^\infty(F_h) \quad \text{and} \quad \sigma_c := \frac{1}{\sigma_a}. \tag{40}$$

4. Proprieties of bilinear forms A, B and C

In order to prove the continuity of the discrete forms A and B and the coercivity of A, we introduce the following discrete semi-norm on $\mathcal{V}(h)$, $\forall u \in \mathcal{V}(h)$

$$\begin{aligned} \|u\|^2_{\mathcal{V}(h)} := \quad & \|\nabla \times u\|^2_{0,\Omega} + \|\sqrt{\sigma_a}[u]_N\|^2_{0,F_h^I} + r\|\nabla \cdot u\|^2_{0,\Omega} \\ & + \|\sqrt{\sigma_a}[u]_T\|^2_{0,F_h} + \|\frac{1}{\sqrt{\sigma_a}}\{\nabla \times u\}\|^2_{0,F_h} \end{aligned} \tag{41}$$

and the following discrete semi-norm on $\mathcal{Q}(h)$, $\forall p \in \mathcal{Q}(h)$:

$$\|p\|^2_{\mathcal{Q}(h)} := \|p\|^2_{0,\Omega} + \|\sqrt{\sigma_c}[p]\|^2_{0,F_h}. \tag{42}$$

We have the following result.

Proposition 4.1. The semi-norms (42) and (41) are norms on $\mathcal{V}(h)$ and $\mathcal{Q}(h)$ respectively.

Proof. It is clear that (42) defined a norm on $\mathcal{Q}(h)$. Let $v \in \mathcal{V}(h)$ such that $\|v\|_{\mathcal{V}(h)} = 0$, then we have

$$\nabla \times v = 0 \quad \text{in} \quad K \quad \forall K \in \Pi_h$$

and

$$[v]_T = 0 \qquad \text{on} \quad F_h.$$

We deduce $v \in H_0(\nabla \times 0, \Omega)$ and we can write $v = \nabla\varphi$ with $\varphi \in H^1_0(\Omega)$ (see [5]). We also have

$$\nabla \cdot v = 0 \ \text{ in } \ K \ \ \forall K \in \Pi_h$$

and

$$[v]_N = 0 \quad \text{on} \quad F_h^I.$$

Therefore, we have $v \in H(\nabla \cdot 0, \Omega)$. φ is the solution of the following problem:

$$\begin{cases} \nabla \cdot (\nabla\varphi) &= 0 \quad \text{in} \ \ \Omega, \\ \varphi &= 0 \quad \text{on} \ \ \Gamma. \end{cases}$$

Then $\varphi = 0$ in Ω and therefore we have $v = 0$ in Ω. □

Remark 4.2. If we suppose that

$$r \geq 1 \quad \text{and} \quad \kappa \geq 1,$$

with the first inequality of [19], we have

$$\|u\|^2_{0,\Omega} \leq C\left(\|\nabla \times u\|^2_{0,\Omega} + r\|\nabla \cdot u\|^2_{0,\Omega} + \|\sqrt{\sigma_a}[u]_N\|^2_{0,F_h^I} + \|\sqrt{\sigma_a}[u]_T\|^2_{0,F_h}\right) \ \forall r \geq 1, \ \forall \kappa \geq 1.$$

We now prove that the discrete problem is well posed. Therefore, in a first time we prove that bilinear forms A, B and C are continuous. The coercivity of A on ker$B$ is not evident because of the term $\|\frac{1}{\sqrt{\sigma_a}}\{\nabla\times u\}\|_{0,F_h}$ in the norm of $\mathcal{V}(h)$. But, if we eliminate this term, we have difficulties to obtain the continuity of A. We meet the same problem with Poisson problem, [2, 13], [9] and [14].

4.1. **Study of discrete problem.** We prove that the discrete forms are consistent with the partial differential operator involved in problem (16) and the existence and uniqueness of solution for the discrete problem (38).

In the sequel, the following inverse estimate will be useful (see [18] in two-dimensional and see [12] in three dimensional).

Lemma 4.3. For all $p\in P_k(K)$ we have

$$\|p\|^2_{0,\partial K}\leq C\frac{1}{h_K}\|p\|^2_{0,K}.$$

Theorem 2. *Let σ_a and σ_c are stabilization parameters defined by (40); Then there exists $\kappa_0>0$ such that $\forall\kappa>\kappa_0$, problem (38) is consistent and has a unique solution.*

Proof. . Theorem 1 implies the consistent. On other hand, the existence is equivalent to uniqueness. Set $j=0$ and let (u^h,p^h) is the solution of (38). We show that $(u^h,p^h)=(0,0)$.

Set $v=u^h$ and $\psi=p^h$, we obtain from (38)

$$A(u^h,u^h)+C(p^h,p^h)=0. \tag{43}$$

We have with the definitions of A and C,

$$\begin{aligned}\int_\Omega(\nabla\times u^h)^2 \quad &+\int_{F_h}\sigma_a[u^h]^2_T+r\int_\Omega(\nabla\cdot u^h)^2\\ &-2J(u^h,u^h)+\int_{F_h^I}\sigma_a[u^h]^2_N+\int_{F_h}\sigma_c[p^h]^2=0.\end{aligned} \tag{44}$$

We obtain with Cauchy-Schwarz inequality, $\forall\epsilon>0$,

$$2J(u^h,u^h)\leq 2\epsilon\int_{F_h}\sigma_a[u^h]^2_T+\frac{2}{\epsilon}\int_{F_h}\frac{1}{\sigma_a}|\{\nabla\times u^h\}|^2. \tag{45}$$

Using the definition of stabilisation parameter σ_a and after lemma 4.3, we have since, $\nabla\times\mathcal{V}_h\subset\mathcal{V}_h$

$$\int_{F_h}|\frac{1}{\sqrt{\sigma_a}}\{\nabla\times v\}|^2\leq\frac{C}{\kappa}\int_\Omega|\nabla\times v|^2\quad\forall v\in\mathcal{V}_h. \tag{46}$$

Then we have

$$A(u^h,u^h)+C(p^h,p^h) \geq (1-\tfrac{2C}{\epsilon\kappa})\int_\Omega (\nabla\times u^h)^2 + r\int_\Omega (\nabla\cdot u^h)^2 \\ +(1-2\epsilon)C\int_{F_h}\sigma_a[u^h]_T^2+\int_{F_h^I}\sigma_a[u^h]_N^2+\int_{F_h}\sigma_c[p^h]^2. \tag{47}$$

Now, we choose ϵ such that $2\frac{C}{\kappa}<\epsilon<\frac{1}{2}$ (such choice is possible if $\kappa>\kappa_0:=\frac{1}{2C}$) and if $A(u^h,u^h)+C(p^h,p^h)=0$, then all term in (47) are null:

$$\begin{aligned}&\nabla\times u^h=0 \text{ in } \Omega, \quad [u^h]_T=0 \text{ on } F_h,\\ &\nabla\cdot u^h=0 \ \text{ in } \Omega\ , \ [u^h]_N=0 \text{ on } F_h^I\end{aligned} \tag{48}$$

and

$$[p^h]=0 \ \text{ on } \ F_h. \tag{49}$$

We deduce from (48) and since $u^h\in\mathcal{V}_h$ satisfies

$$u^h\in H_0(\nabla\times 0,\Omega), \tag{50}$$

$$u^h\in H(\nabla\cdot 0,\Omega). \tag{51}$$

we have $\|u^h\|_{\mathcal{V}(h)}=0$. Therefore, u^h is null Ω. We deduce from (49) that $p^h\in H_0^1(\Omega)$. Then the jumps are null at element interfaces and with the second equality (38), after an integration by parts

$$-\int_\Omega v\cdot\nabla p^h=0 \quad \forall v\in\mathcal{V}_h. \tag{52}$$

Then, we have p^h is null in Ω. □

4.2. **Continuity of bilinear forms A, B and C.** We have the following result.

Proposition 4.4. The bilinear forms A, B and C are continuous on $\mathcal{V}(h)\times\mathcal{V}(h)$, $\mathcal{V}(h)\times\mathcal{Q}(h)$ and $\mathcal{Q}(h)\times\mathcal{Q}(h)$ respectively. There exists $C>0$ independent of h such that:

$$\begin{aligned}&|A(u,v)|\leq C\|u\|_{\mathcal{V}(h)}\|v\|_{\mathcal{V}(h)} \quad \forall u,v\in\mathcal{V}(h),\\ &|B(u,\psi)|\leq C\|u\|_{\mathcal{V}(h)}\|\psi\|_{\mathcal{Q}(h)} \quad \forall u\in\mathcal{V}(h),\ \ \forall\psi\in\mathcal{Q}(h),\\ &|C(p,q)|\leq C\|p\|_{\mathcal{Q}(h)}\|q\|_{\mathcal{Q}(h)} \quad \forall p,q\in\mathcal{Q}(h).\end{aligned} \tag{53}$$

Proof. . We only show the continuity of A, the method is the same to show the continuity of B and C.

Let $u, v \in \mathcal{V}(h)$, we have with Cauchy-Schwarz inequality

$$\begin{aligned}|A(u,v)| \leq \quad & C\ \{\|\nabla\times u\|_{0,\Omega}\|\nabla\times v\|_{0,\Omega} + r\|\nabla\cdot u\|_{0,\Omega}\|\nabla\cdot v\|_{0,\Omega} \\ & +\|\sqrt{\sigma_a}[u]_T\|_{0,F_h}\|\sqrt{\sigma_a}[v]_T\|_{0,F_h} \\ & +\|\sqrt{\sigma_a}[u]_N\|_{0,F_h^I}\|\sqrt{\sigma_a}[v]_N\|_{0,F_h^I} \\ & +\|\sqrt{\sigma_a}[v]_T\|_{0,F_h}\|\tfrac{1}{\sqrt{\sigma_a}}\{\nabla\times u\}\|_{0,F_h} \\ & +\|\sqrt{\sigma_a}[u]_T\|_{0,F_h}\|\tfrac{1}{\sqrt{\sigma_a}}\{\nabla\times v\}\|_{0,F_h}\Big\}.\end{aligned}$$

We use Cauchy-Schwartz discrete inequality

$$\begin{aligned}|A(u,v)| \leq \quad & C\Big\{\|\nabla\times u\|_{0,\Omega}^2 + \|\sqrt{\sigma_a}[u]_T\|_{0,F_h}^2 + \|\sqrt{\sigma_a}[u]_N\|_{0,F_h^I}^2 \\ & \quad + \sqrt{r}\|\nabla\cdot u\|_{0,\Omega}^2 + \|\tfrac{1}{\sqrt{\sigma_a}}\{\nabla\times u\}\|_{0,F_h}^2\Big\}^{\frac{1}{2}} \\ & \times\Big\{\|\nabla\times v\|_{0,\Omega}^2 + \|\sqrt{\sigma_a}[v]_T\|_{0,F_h}^2 + \|\sqrt{\sigma_a}[v]_N\|_{0,F_h^I}^2 \\ & \quad + \sqrt{r}\|\nabla\cdot v\|_{0,\Omega}^2 + \|\tfrac{1}{\sqrt{\sigma_a}}\{\nabla\times v\}\|_{0,F_h}^2\Big\}^{\frac{1}{2}} \\ \leq \quad & C\|u\|_{\mathcal{V}(h)}\|v\|_{\mathcal{V}(h)}.\end{aligned}$$

This implies the continuity of A on $\mathcal{V}(h)\times\mathcal{V}(h)$. □

The next result shows that the discrete bilinear form A is coercive on $\mathcal{V}_h\times\mathcal{V}_h$ with respect to the norm $\|.\|_{\mathcal{V}(h)}$

4.3. **Coercivity of A.** We know that if A is coercive on the kernel B then we can demonstrate a convergence result. Nevertheless, we show A is coercive on $\mathcal{V}_h\times\mathcal{V}_h$.

Proposition 4.5. Let σ_a the stabilisation parameter defined by (40). It exists $\kappa_0 > 0$ such that if $\kappa \geq \kappa_0$, we have

$$A(u,u) \geq \alpha_0\|u\|_{\mathcal{V}(h)}^2 \qquad \forall u \in \mathcal{V}_h \tag{54}$$

with $\alpha_0 > 0$ indépendent of h.

Proof. . We have

$$\|u\|_{\mathcal{V}(h)}^2 = a(u,u) + \|\frac{1}{\sqrt{\sigma_a}}\{\nabla\times u\}\|_{0,F_h}^2. \tag{55}$$

Then, we obtain

$$\begin{aligned}A(u,u) - \alpha\|u\|_{\mathcal{V}(h)}^2 = \quad & (1-\alpha)a(u,u) - 2J(u,u) \\ & -\alpha\|\tfrac{1}{\sqrt{\sigma_a}}\{\nabla\times u\}\|_{0,F_h}^2.\end{aligned} \tag{56}$$

By lemma 4.3, we have

$$\|\frac{1}{\sqrt{\sigma_a}}\{\nabla\times u\}\|^2_{0,F_h}\leq\frac{C}{\kappa}\|\nabla\times u\|^2_{0,\Omega}\quad\forall u\in\mathcal{V}_h. \tag{57}$$

This implies that

$$A(u,u)-\alpha\|u\|^2_{\mathcal{V}(h)}\geq(1-\alpha)\,a(u,u)-2J(u,u)-\alpha\frac{C}{\kappa}\|\nabla\times u\|^2_{0,\Omega}$$

$$=(1-\alpha-\alpha C)\left[\|\nabla\times u\|^2_{0,\Omega}+\|\sqrt{\sigma_a}[u]_T\|^2_{0,F_h}+\|\sqrt{\sigma_a}[u]_N\|^2_{0,F_h^I}+r\|\nabla\cdot u\|^2_{0,\Omega}\right]$$

$$-2J(u,u)-\alpha\frac{C}{\kappa}\|\nabla\times u\|^2_{0,\Omega}$$

With the inequalities (45) and (57) and if α is such that

$$1-\alpha>0,$$

we have

$$\begin{aligned}A(u,u)-\alpha\|u\|^2_{\mathcal{V}(h)}\quad&\geq(1-\alpha-\tfrac{2C}{\epsilon\kappa})\|\nabla\times u\|^2_{0,\Omega}\\&\quad+(1-\alpha-\alpha C-2\epsilon)C\|\sqrt{\sigma_a}[u]_T\|^2_{0,F_h}.\end{aligned} \tag{58}$$

It suffies to find $\alpha>0$ such that

$$1-\alpha>0\ ,\ 1-\alpha-\frac{2C}{\epsilon\kappa}>0\ \text{ and }\ 1-\alpha-\alpha C-2\epsilon>0. \tag{59}$$

The second inequality implies the first one, and the third inequality is satisfied if

$$0<\alpha\leq\frac{1-2\epsilon}{1+C}, \tag{60}$$

which implies

$$\epsilon<\frac{1}{2}. \tag{61}$$

Besides, the second inequality is satisfied if

$$0<\alpha\leq1-2C/\epsilon\kappa\quad\leq\quad1-C/\kappa. \tag{62}$$

Then, if we have $1-\frac{C}{\kappa}>0$ or $\kappa>\kappa_0$ with $\kappa_0>C$, there exists $\alpha_0>0$ such that

$$A(u,u)-\alpha_0\|u\|^2_{\mathcal{V}(h)}\geq0 \tag{63}$$

Therefore A is coercive on $\mathcal{V}(h)$. □

4.4. **Condition inf-sup.** In this section, we show that B satisfies an inf-sup condition. In the first time, by lemma 4.3 we have

$$\|q\|_{0,\Omega}^2 \geq \frac{1}{2}\|q\|_{0,\Omega}^2+\frac{1}{2}\|q\|_{0,\Omega}^2 \geq \frac{1}{2}\|q\|_{0,\Omega}^2+C\|\frac{1}{\sqrt{\sigma_a}}[q]\|_{0,F_h}^2 \geq C\|q\|_{\mathcal{Q}(h)}^2 \quad \forall q \in \mathcal{Q}_h.$$

We have the following result.

Proposition 4.6. If $k \in \{1,2\}$, we define

$$\widetilde{\mathcal{V}_h} := \{v_h \in \mathcal{V}_h \ : \forall f \subset F_h, \ \int_f q_h \cdot [v_h] = 0 \ \ \forall q_h \in P_{k-1}(f)^3 \ \},$$

There exists an interpolation operator $\mathbf{R_h} : H^1(\Omega)^3 \longrightarrow \widetilde{\mathcal{V}_h}$ which is continuous such that

$$\forall v \in H_0^1(\Omega)^3, \ \ \forall q_h \in P_{k-1}(K), \ \ \int_K q_h \nabla \cdot (\mathbf{R_h}(v) - v) = 0,$$
$$\forall v \in H_0^1(\Omega)^3, \ \ \forall e \subset F_h, \ \ \forall q_h \in P_{k-1}(e)^3, \ \ \int_e q_h \cdot [\mathbf{R_h}(v)] = 0,$$
$$\forall v \in W^{s,t}(\Omega)^3, \ \ \forall t \geq 0, \ \ \forall s \in [1, k+1], \ \ \forall m \in \{0,1\},$$
$$|v - \mathbf{R_h}(v)|_{W^{m,t}(K)} \leq Ch^{s-m}|v|_{W^{s,t}(\Delta_K)}$$

with Δ_K are macro-elements which contain K, (see [11]).

Proof. For the two-dimensional and the three-dimensional case see [7] and [8] respectively. □

We have the result, (see [11]).

Proposition 4.7. If $k \geq 3$, there exists an interpolation operator $\widetilde{\mathbf{R_h}} : H_0^1(\Omega)^3 \longrightarrow \mathcal{V}_h \cap H_0^1(\Omega)^3$ continuous which satisfies

$$\forall \ \ v \in H_0^1(\Omega)^3, \ \ \forall q_h \in \mathcal{Q}_h \cap L_0^2(\Omega), \ \ \int_\Omega q_h \nabla \cdot (\widetilde{\mathbf{R_h}}(v) - v) = 0,$$
$$\forall \ \ v \in W^{s,p}(\Omega)^3, \ \ \forall T \in \Pi_h,$$
$$|\widetilde{\mathbf{R_h}}(v) - v|_{W^{m,q}(T)} \leq Ch^{s-m+3(\frac{1}{q}-\frac{1}{p})}|v|_{W^{s,p}(\Delta_T)}$$
$$\forall \ \ s \in [1, k+1], \ \ \forall \ \ 1 \leq p, q \leq \infty, \ \forall m \in \{0,1\} \ \ \text{verifying} \ \ W^{s,p}(\Omega) \subset W^{m,q}(\Omega).$$

We give this last result useful for this section, (see [6]).

Proposition 4.8. We suppose that the **assumption (H)** holds. $\forall f \subset \Gamma$, there exists a function ρ_h such that the support $\rho_{h|\Gamma}$ is in f and

$$\vec{\rho_h} := \rho_h \vec{n_f} \in Y_h := \{q_h \in \mathcal{C}^0(\overline{\Omega}) \ \ s.t. \ \ q_{h|K} \in P_1(K) \ \ \forall K \in \Pi_h\}^3 \cap H_0(\nabla\times, \Omega) \cap H(\nabla\cdot, \Omega).$$

and verifies

$$\int_f \rho_h = \int_f \vec{\rho_h} \cdot \vec{n_f} = 1 \ ; \ \ |\rho_h|_{1,\Omega} \leq K_2$$

with $K_2 > 0$ indépendent of h and $\vec{n_f}$ is the restriction of n :, the unit normal on f, a face supported by Γ, *i.e.* $\vec{n_f} = n_{|f}$.

Proof. . Let $\Pi_{\frac{h}{2}}$ be the mesh obtained by cutting every $K \in \Pi_h$ into eight equal tetrahedra whose vertices are the middles of edges of K. Then, in [6], they show that such a function ρ_h exists and $\rho_h \in Y_{\frac{h}{2}}$. We obtain the result with scaling argument. □

Now, we can demonstrate the inf-sup condition.

Proposition 4.9. Bilinear form B verifies the condition

$$\inf_{q \in \mathcal{Q}_h \setminus \{0\}} \sup_{v \in \mathcal{V}_h \setminus \{0\}} \frac{B(v,q)}{\|q\|_{\mathcal{Q}(h)} \|v\|_{\mathcal{V}(h)}} \geq \beta > 0 \tag{64}$$

with $\beta > 0$ is independent of h.

Proof. . Following [10], let $q_h \in \mathcal{Q}_h \setminus \{0\}$, we look for a function $v_h \in \mathcal{V}_h \setminus \{0\}$ and a positive constant C independent of h such that

$$B(v_h, q_h) \geq C \|q_h\|_{\mathcal{Q}(h)} \|v_h\|_{\mathcal{V}(h)}. \tag{65}$$

We consider two cases.
• First case $k \geq 3$.
We can write

$$q_h = \tilde{q}_h + \overline{q}_h \ \text{ with } \tilde{q}_h = q_h - \frac{1}{mes(\Omega)} \int_\Omega q_h. \tag{66}$$

As $\tilde{q}_h \in L^2_0(\Omega)$, [5], there exists $v_h \in H^1_0(\Omega)^3$ such as

$$\nabla \cdot v_h = \tilde{q}_h \quad \text{and} \quad \|v_h\|_{1,\Omega} \leq C \|\tilde{q}_h\|_{0,\Omega}. \tag{67}$$

Let $\tilde{v}_h = \widetilde{\mathbf{R}_\mathbf{h}}(v_h)$ with $\widetilde{\mathbf{R}_\mathbf{h}}$ the interpolation operator from proposition 4.7 We can write

$$v_h = \alpha \tilde{v}_h + \overline{v}_h \ \text{ and } \ \overline{v}_h = \overline{q}_h \vec{\rho_h} \tag{68}$$

with $\vec{\rho_h}$ is given by the proposition 4.7 and $\alpha > 0$ to choose. We have

$$\begin{array}{rcl} B(v_h, q_h) = & B(\tilde{v}_h + \overline{v}_h, \tilde{q}_h + \overline{q}_h) \\ = & B(\tilde{v}_h, \tilde{q}_h) + B(\tilde{v}_h, \overline{q}_h) + B(\overline{v}_h, \tilde{q}_h) + B(\overline{v}_h, \overline{q}_h). \end{array} \tag{69}$$

Since $\tilde{v}_h \in H^1_0(\Omega)^3$ and $\overline{q}_h \in \mathbb{R}$, we have

$$B(\tilde{v}_h, \overline{q}_h) = 0. \tag{70}$$

On other hand, we have by proposition 4.8

$$B(\overline{v}_h, \overline{q}_h) = \overline{q}_h^2 \int_f \rho_h \vec{n_f} \cdot \vec{n_f} = \overline{q}_h^2. \tag{71}$$

We obtain with the definition of $\overline{v}_h$

$$B(\overline{v}_h, \tilde{q}_h) = \int_\Omega \tilde{q}_h \overline{q}_h \nabla \cdot \vec{\rho_h} \leq K_2 \|\tilde{q}_h\|_{0,\Omega} \|\overline{q}_h\|_{0,\Omega}. \tag{72}$$

Using (67) and the proposition 4.7, we have

$$B(\tilde{v}_h, \tilde{q}_h) = \|\tilde{q}_h\|_{0,\Omega}^2.$$

We deduce

$$\begin{aligned} B(v_h, q_h) \geq & \quad \alpha \|\tilde{q}_h\|_{0,\Omega}^2 + \frac{1}{mes(\Omega)} \|\overline{q}_h\|_{0,\Omega}^2 - \frac{K_2}{mes(\Omega)^{\frac{1}{2}}} \|\overline{q}_h\|_{0,\Omega} \|\tilde{q}_h\|_{0,\Omega} \\ \geq & \quad \alpha \|\tilde{q}_h\|_{0,\Omega}^2 + \frac{1}{mes(\Omega)} \|\overline{q}_h\|_{0,\Omega}^2 - \frac{1}{2\epsilon} \frac{K_2^2}{mes(\Omega)} \|\overline{q}_h\|_{0,\Omega}^2 \\ & -\frac{\epsilon}{2} \|\tilde{q}_h\|_{0,\Omega}^2, \quad \forall \epsilon > 0. \end{aligned} \tag{73}$$

If we choose $\epsilon = \alpha = K_2^2$, we can write

$$\begin{aligned} B(v_h, q_h) \geq & \quad \alpha \|\tilde{q}\|_{0,\Omega}^2 + \frac{1}{mes(\Omega)} \|\overline{q}_h\|_{0,\Omega}^2 - \frac{1}{2mes(\Omega)} \|\overline{q}_h\|_{0,\Omega}^2 \\ & -\frac{\alpha}{2} \|\tilde{q}_h\|_{0,\Omega}^2 \\ \geq & \quad \frac{K_2^2}{2} \|\tilde{q}_h\|_{0,\Omega}^2 + \frac{1}{2mes(\Omega)} \|\overline{q}_h\|_{0,\Omega}^2 \\ \geq & \quad C \|q_h\|_{0,\Omega}^2 \\ \geq & \quad C \|q_h\|_{\mathcal{Q}(h)}^2. \end{aligned} \tag{74}$$

Then we obtain the inf-sup condition:

$$\begin{aligned} \|v_h\|_{\mathcal{V}(h)} \leq & \quad C \|v_h\|_{1,\Omega} \\ \leq & \quad C \left(\|\tilde{v}_h\|_{1,\Omega} + \|\overline{v}_h\|_{1,\Omega} \right) \\ \leq & \quad C \left(\|\tilde{q}_h\|_{0,\Omega} + \|\overline{q}_h\|_{0,\Omega} \|\rho_h\|_{1,\Omega} \right) \\ \leq & \quad C \left(\|\tilde{q}_h\|_{0,\Omega} + \|\overline{q}_h\|_{0,\Omega} \right) \\ \leq & \quad C \left(\|\tilde{q}_h\|_{0,\Omega} + \|\overline{q}_h\|_{0,\Omega} \right) \\ \leq & \quad C \|q_h\|_{0,\Omega} \\ \leq & \quad C \|q_h\|_{\mathcal{Q}(h)}. \end{aligned}$$

• Second case $k \in \{1, 2\}$, let $q_h \in \mathcal{Q}_h$ then there exists $\tilde{v}_h \in H^1(\Omega)^3$ such as

$$\nabla \cdot \tilde{v}_h = q_h \quad \text{and} \quad \|\tilde{v}_h\|_{1,\Omega} \leq C \|q_h\|_{0,\Omega}. \tag{75}$$

We set $v_h = \mathbf{R_h}(\tilde{v}_h)$ with $\mathbf{R_h}$ the interpolation operator given by the proposition 4.6 We deduce

$$B(v_h, q_h) = \sum_{K\in\Pi_h} \int_K q_h \nabla\cdot \mathbf{R_h}(\tilde{v}_h) = \sum_{K\in\Pi_h} q_h \nabla\cdot v_h = \|q_h\|^2_{0,\Omega} \geq C\|q_h\|^2_{\mathcal{Q}(h)}.$$

As $\mathbf{R_h}$ is continuous, we have

$$\begin{aligned} \|v_h\|_{\mathcal{V}(h)} &= \|\mathbf{R_h}(\tilde{v}_h)\|_{\mathcal{V}(h)} \\ &\leq C\|\tilde{v}_h\|_{1,\Omega} \\ &\leq C\|q_h\|_{0,\Omega} \\ &\leq C\|q_h\|_{\mathcal{Q}(h)}. \end{aligned}$$

□

5. Interpolation error estimate

We denote by (u, p) the exacte solution of (2) and (u^h, p^h) the solution of the discrete problem (38). Let z_u be the interpolation operator associated to the discretization of u and z_p associated to discretization of p. We denote by

$$e := u - u^h, \quad e' := p - p^h$$

and we write e, e' such that:

$$\begin{aligned} e &= \eta - \xi \quad \text{with} \quad \xi := u^h - z_u \quad \text{and} \quad \eta := u - z_u \\ e' &= \eta' - \xi' \quad \text{with} \quad \xi' := p^h - z_p \quad \text{and} \quad \eta' := p - z_p. \end{aligned} \tag{76}$$

We have with triangular inequality

$$\|e\|_{\mathcal{V}(h)} + \|e'\|_{\mathcal{Q}(h)} \leq \|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)} + \|\xi\|_{\mathcal{V}(h)} + \|\xi'\|_{\mathcal{Q}(h)}. \tag{77}$$

Next, we show that

$$\|\xi\|_{\mathcal{V}(h)} + \|\xi'\|_{\mathcal{Q}(h)} \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]. \tag{78}$$

Remark 5.1. If the bilinear form C is positive, we can write

$$C(p, q) \leq C(p, p)^{\frac{1}{2}} C(q, q)^{\frac{1}{2}}. \tag{79}$$

Indeed, the polynomial function $P(t) := C(p+tq, p+tq)$ verifies $P(t) \geq 0 \quad \forall t \in \mathbb{R}$.
We deduce there exists a constant $M > O$ independent of h such as

$$C(p, q) \leq MC(p, p)^{\frac{1}{2}} \|q\|_{\mathcal{Q}(h)} \quad \forall p, q \in \mathcal{Q}(h). \tag{80}$$

As the formulation is consistent, the solution (u^h, p^h) of (38), the errors e and e' verify

$$\begin{cases} A(e, v) + B(v, e') = 0 \quad \forall v \in \mathcal{V}_h, \\ B(e, \psi) - C(e', q) = 0 \quad \forall \psi \in \mathcal{Q}_h. \end{cases} \tag{81}$$

We deduce ξ and ξ' verify

$$\left\{\begin{array}{lcll} A(\xi,v) & + & B(v,\xi') & = L(v) \ \ \forall v \in \mathcal{V}_h, \\ B(\xi,\psi) & - & C(\xi',q) & = g(\psi) \ \ \forall \psi \in \mathcal{Q}_h \end{array}\right. \tag{82}$$

with L and g are linear applications defined on $\mathcal{V}_h$ and $\mathcal{Q}_h$ respectively by

$$L(v) = A(\eta,v) + B(v,\eta') \quad \text{and} \quad g(\psi) = B(\eta,\psi) - C(\eta',\psi). \tag{83}$$

By the proposition 4.9, we have

$$\begin{array}{rcl} \beta\|\xi'\|_{\mathcal{Q}(h)} & \leq & \displaystyle\sup_{v\in\mathcal{V}_h\setminus\{0\}} \frac{B(v,\xi')}{\|v\|_{\mathcal{V}(h)}} \\ & \leq & \displaystyle\sup_{v\in\mathcal{V}_h\setminus\{0\}} \frac{A(\xi,v)-L(v)}{\|v\|_{\mathcal{V}(h)}}. \end{array} \tag{84}$$

We deduce from the propositions 4.4

$$\beta\|\xi'\|^2_{\mathcal{Q}(h)} \leq C\left[\|\xi\|_{\mathcal{V}(h)} + \|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]. \tag{85}$$

We now estimate $\|\xi\|_{\mathcal{V}(h)}$. We have the decomposition

$$\mathcal{V}_h = KerB \oplus (KerB)^{\perp}, \tag{86}$$

and we can write

$$\xi = \xi^c + \xi^{c\perp} \quad \text{with} \quad \xi^c \in KerB \ , \xi^{c\perp} \in (KerB)^{\perp}. \tag{87}$$

By the proposition 4.9, we obtain

$$\begin{array}{rcl} \beta\|\xi^{c\perp}\|_{\mathcal{V}(h)} & \leq & \displaystyle\sup_{q\in\mathcal{Q}_h\setminus\{0\}} \frac{B(\xi^{c\perp},q)}{\|q\|_{\mathcal{Q}(h)}} \\ & \leq & \displaystyle\sup_{q\in\mathcal{Q}_h\setminus\{0\}} \frac{C(\xi',q)-g(q)}{\|q\|_{\mathcal{Q}(h)}}. \end{array} \tag{88}$$

By the definition of g, we have

$$\beta\|\xi^{c\perp}\|_{\mathcal{V}(h)} \leq C\left[C(\xi',\xi')^{\frac{1}{2}} + \|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]. \tag{89}$$

On the other hand, using proposition 4.5, we obtain

$$\begin{array}{rcl} \alpha_0\|\xi^c\|_{\mathcal{V}(h)} \leq & \displaystyle\sup_{v_0\in KerB} \frac{A(\xi,v_0)}{\|v_0\|_{\mathcal{V}(h)}} \\ = & \displaystyle\sup_{v_0\in KerB} \frac{(\xi,v_0)-A(\xi^{c\perp},v_0)}{\|v_0\|_{\mathcal{V}(h)}} \\ \leq & C\left[\|\xi^{c\perp}\|_{\mathcal{V}(h)} + \|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right] \\ \leq & C\left[C(\xi',\xi')^{\frac{1}{2}} + \|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]. \end{array} \tag{90}$$

Then, we have

$$\|\xi^c\|_{\mathcal{V}(h)} \leq \ C\left[C(\xi',\xi')^{\frac{1}{2}} + \|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]. \tag{91}$$

Now, let us consider $v = \xi$ and $\psi = \xi'$, we have from (82)

$$A(\xi,\xi) + C(\xi',\xi') = L(\xi) - g(\xi'). \tag{92}$$

On the other hand, we have

$$A(\xi,\xi) + C(\xi',\xi') \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]\left[\|\xi\|_{\mathcal{V}(h)} + \|\xi'\|_{\mathcal{Q}(h)}\right]. \tag{93}$$

We obtain with (89)

$$\begin{aligned} & A(\xi,\xi) + C(\xi',\xi') \\ & \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]\left[\|\xi\|_{\mathcal{V}(h)} + \|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)} + \|\xi\|_{\mathcal{V}(h)}\right] \\ & \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)} + \|\xi\|_{\mathcal{V}(h)}\right] \\ & \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right] C(\xi',\xi')^{\frac{1}{2}} + C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]^2. \end{aligned} \tag{94}$$

Therefore, there exists

$$C(\xi',\xi') \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right] C(\xi',\xi')^{\frac{1}{2}} + C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]^2.$$

In particular, we have

$$C(\xi',\xi')^{\frac{1}{2}} \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]. \tag{95}$$

We then deduce

$$\|\xi\|_{\mathcal{V}(h)} \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right] \tag{96}$$

and therefore with (89), we have

$$\|\xi'\|_{\mathcal{Q}(h)} \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]. \tag{97}$$

From (96) and (97), we have

$$\begin{aligned} \|\xi\|_{\mathcal{V}(h)} + \|\xi'\|_{\mathcal{Q}(h)} & \leq \|\xi\|_{\mathcal{V}(h)} + \|\xi'\|_{\mathcal{Q}(h)} \\ & \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right]. \end{aligned} \tag{98}$$

We conclude

$$\|e\|_{\mathcal{V}(h)} + \|e'\|_{\mathcal{Q}(h)} \leq C\left[\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}\right] \tag{99}$$

and we must estimate $\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}$. We then give the following interpolation result.

Theorem 3. *Let $K \in \Pi_h$ and we suppose that $u \in H^{t_K}(K)$ and $t_K \geq 0$, then there exists a sequence of polynomial functions $\pi^{h_K} \in P_k(K)$ such that*

$$\|u - \pi^{h_K}(u)\|_{q,K} \leq C h_K^{min(k+1,t_K)-q} \|u\|_{t_K,K} \quad \forall\ 0 \leq q \leq t_K. \tag{100}$$

If $t_K \geq 1$, then we have

$$\|u - \pi^{h_K}(u)\|_{0,\partial K} \leq C h_K^{min(k+1,t_K)-\frac{1}{2}} \|u\|_{t_K,K}. \tag{101}$$

The constant C is independent of u, and h_K but depends on k, the mesh regularity and $t = \max\limits_{K\in\Pi_h} t_K$.

Proof. [16]. □

We interpolate vector functions. We denote par π^h the interpolation operator defined by $\pi^h(u)|_K = \pi^{h_K}(u|_K)$, then we have for vector function, $\mathbf{\Pi}^h$ the interpolation operator defined by $\mathbf{\Pi}^h(u) := (\pi^h(u_1), \pi^h(u_2), \pi^h(u_3))$ if $u = (u_1, u_2, u_3)$.

We give a interpolation result.

Theorem 4. *Let (u^h, p^h) the solution of (38) and (u, p) the solution of (2). We suppose that $u \in H^{t+1}(\Pi_h)^3$, $p \in H^{s-1}(\Pi_h)$, $t \geq 1$, $s \geq 2$; then, we have*

$$\|e\|^2_{\mathcal{V}(h)} + \|e'\|^2_{\mathcal{Q}(h)} \leq \quad C\left[h^{2min(k,t)}\ \|u\|^2_{t+1,\Pi_h} + h^{2min(k,s)-2}\|p\|^2_{s,\Pi_h}\right] \tag{102}$$

with C is a positive constant independent of h.

Proof. . We have $\|e\|_{\mathcal{V}(h)} + \|e'\|_{\mathcal{Q}(h)} \leq C(\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)})$. We can estimate $\|\eta\|_{\mathcal{V}(h)} + \|\eta'\|_{\mathcal{Q}(h)}$.

We first consider $\|\eta\|_{\mathcal{V}(h)}$. We have

$$\|\eta\|^2_{0,\Omega} = \sum_{K\in\Pi_h} \|\eta\|^2_{0,K}. \tag{103}$$

We deduce from theorem 3

$$\begin{aligned} \|\eta\|^2_{0,\Omega} &\leq C \sum_{K\in\Pi_h} h^{2min(k,t)+2}\|u\|^2_{t+1,K} \\ &\leq C\left(h^{2min(k,t)+2}\|u\|^2_{t+1,\Pi_h}\right). \end{aligned} \tag{104}$$

We also have

$$\begin{aligned} \|\nabla\times\eta\|^2_{0,\Omega} &\leq C \sum_{K\in\Pi_h} h^{2min(k,t)}\|u\|^2_{t+1,K} \\ &\leq C\left(h^{2min(k,t)}\|u\|^2_{t+1,\Pi_h}\right). \end{aligned} \tag{105}$$

We obtain with the definition of stabilization parameter σ_a and with an inequality trace

$$\begin{aligned} \|\frac{1}{\sqrt{\sigma_a}}\{\nabla\times\eta\}\|_{0,F_h} &\leq C \sum_{K\in\Pi_h} \|\nabla\times\eta\|^2_{0,K} \\ &\leq C \sum_{K\in\Pi_h} h^{2min(k,t)}\|u\|^2_{t+1,K} \\ &\leq Ch^{2min(k,t)}\|u\|^2_{t+1,\Pi_h}. \end{aligned} \tag{106}$$

With theorem 3, we also have

$$\begin{aligned} \|\sqrt{\sigma_a}[\eta]_N\|^2_{0,F_h^I} &\leq C \sum_{K\in\Pi_h} h^{2min(k,t)}\|u\|^2_{t+1,K} \\ &\leq Ch^{2min(k,t)+2}\|u\|^2_{t+1,\Pi_h}. \end{aligned} \tag{107}$$

We have

$$\begin{aligned}\|\frac{1}{\sqrt{\sigma_a}}\{\eta\}\|^2_{0,F_h} &\leq C\sum_{K\in\Pi_h}\|\eta\|^2_{1,K}\\ &\leq C\sum_{K\in\Pi_h}h^{2min(k,t)}\|u\|^2_{t+1,K}\\ &\leq Ch^{2min(k,t)}\|u\|^2_{t+1,\Pi_h}.\end{aligned}$$

We deduce

$$\|\eta\|^2_{\mathcal{V}(h)}\leq C\left(h^{2min(k,t)}\|u\|^2_{t+1,\Pi_h}\right).$$

Similary, we obtain

$$\|\eta'\|^2_{\mathcal{Q}(h)}\leq C\left(h^{2min(k-1,s-1)}\|p\|^2_{s,\Pi_h}\right)$$

Hence we have the result.

□

6. Numerical results

In this section, we present numerical results obtained for the three-dimensional problem (1) with the density current:

$$J(x,y,z):=\begin{pmatrix}J_1(x,y,z)\\J_2(x,y,z)\\J_3(x,y,z)\end{pmatrix}$$

with

$$\begin{aligned}J_1(x,y,z)=\ &-\exp(yz)\left[(z^2-z)\left(2+2z(2y-1)+z^2(y^2-y)\right)\right]\\ &-\exp(yz)\left[(y^2-y)\left(2+2y(2z-1)+y^2(z^2-z)\right)\right]\\ &-\exp(xyz)\left[(2x-1)(y^2-y)(z^2-z)+yz(x^2-x)(y^2-y)(z^2-z)\right],\end{aligned}$$

$$\begin{aligned}J_2(x,y,z)=\ &-\exp(xz)\left[(x^2-x)\left(2+2x(2z-1)+x^2(z^2-z)\right)\right]\\ &-\exp(yz)\left[(z^2-z)\left(2+2z(2x-1)+z^2(x^2-x)\right)\right]\\ &-\exp(xyz)\left[(2y-1)(x^2-x)(z^2-z)+xz(x^2-x)(y^2-y)(z^2-z)\right]\end{aligned}$$

and

$$\begin{aligned}J_3(x,y,z)=\ &-\exp(xy)\left[(y^2-y)\left(2+2y(2x-1)+y^2(x^2-x)\right)\right]\\ &-\exp(yx)\left[(x^2-x)\left(2+2x(2y-1)+x^2(y^2-y)\right)\right]\\ &-\exp(xyz)\left[(2z-1)(x^2-x)(y^2-y)+xy(x^2-x)(y^2-y)(z^2-z)\right].\end{aligned}$$

Then, the exact solution (u,p) is :

$$u(x,y,z)=\begin{pmatrix}(y^2-y)(z^2-z)\exp(yz)\\(z^2-z)(x^2-x)\exp(xz)\\(y^2-y)(x^2-x)\exp(yx)\end{pmatrix},\tag{108}$$

$$p(x,y,z)=(y^2-y)(z^2-z)(x^2-x)\exp(xyz).$$

In the experiments, Ω is the cube $[0,1] \times [0,1] \times [0,1]$. We choose $\kappa = 100$, it can't be chosen too big otherwise the matrix associated to discrete bilinear form A is ill conditionned. We set $r = 1$ and we choose $k = 2$.

We use Uzawa Algorithme, [3], we eliminate u

$$u = A^{-1}(f - Bp)$$

and we solve linear system with conjugated gradient

$$(B^t A^{-1} B + C)p = B^t A^{-1} f. \tag{109}$$

since matrix $B^t A^{-1} B + C$ is symmetric and positive definite.

We denote by **Nbte** the number of tetraedron of the mesh of Ω and by **Nbtr** the number of triangles supported by Γ.

h	**Nbte**	**Nbtr**	$\|u - u^h\|_{\mathcal{V}(h)}$	$\|p - p^h\|_{\mathcal{Q}(h)}$	$\|\nabla \cdot u^h\|_{0,\Omega}$
0.4367	12	30	0.2380E+00	0.8891E+00	0.1603E-01
0.2184	96	216	0.8899E-01	0.2229E+00	0.1641E-01
0.1733	192	432	0.7307E-01	0.1700E+00	0.2176E-01
0.1694	371	826	0.4589E-01	0.7804E-01	0.1027E-01
0.1379	660	1416	0.2952E-01	0.6283E-01	0.6412E-02
9.26E-02	2631	5502	0.1381E-01	0.4660E-01	0.3417E-02

TABLE 1. Errors table

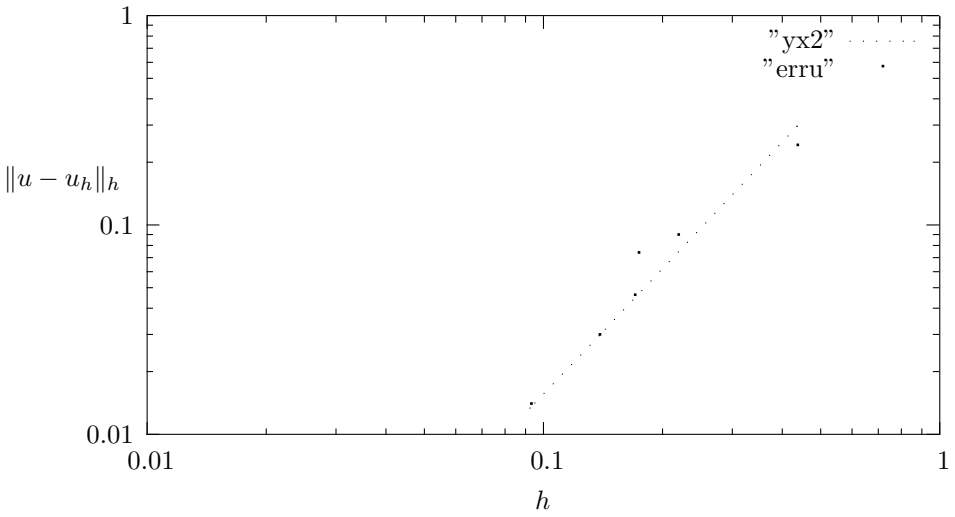
h	**Nbte**	**Nbtr**	$\|u - u^h\|_{L^1(\Omega)^3}$	$\|p - p^h\|_{L^1(\Omega)}$	$\|u - u^h\|_{L^2(\Omega)^3}$	$\|p - p^h\|_{L^2(\Omega)}$
0.4367	12	30	0.3471E-01	0.7633E-01	0.4517E-01	0.1045E+00
0.2184	96	216	0.6913E-02	0.1325E-01	0.8761E-02	0.1700E-01
0.1733	192	432	0.3846E-02	0.8522E-02	0.5215E-02	0.1134E-01
0.1694	371	826	0.2350E-02	0.5901E-02	0.3046E-02	0.7447E-02
0.1379	660	1416	0.1434E-02	0.5440E-02	0.1857E-02	0.7059E-02
9.26E-02	2631	5502	0.6172E-03	0.5277E-02	0.7924E-03	0.6941E-02

TABLE 2. Errors table

We remark that errors decrease when the mean mesh decreases and the quantity $\|\nabla \cdot u^h\|_{0,\Omega}$ is small even with coarse mesh. We obtain the exact solution u to 10^{-3} and p to 10^{-2} with $L^1(\Omega)$ norm and $L^2(\Omega)$ norm respectively after tables 4.1 and 4.2.

Errors $\|u - u^h\|_{\mathcal{V}(h)}$ $\|p - p^h\|_{\mathcal{Q}(h)}$ are plotted in Figures 4.1 and 4.2. These plots highlight the convergence of the numerical solution towards exact solution according to the rate $O(h^2)$ and $O(h)$ for $\|u - u^h\|_{\mathcal{V}(h)}$ and $\|p - p^h\|_{\mathcal{Q}(h)}$

Figure 4.1



respectively.

Remark 6.1. We have tested our method with $k = 1$ and the $P_1 - P_0$ elements is also convergent.

7. Conclusion

We presented and analysed a new discontinuous Galerkin method to resolve the three-dimensional electrostatic problem. An error a priori estimate is derived and we present numerical results to validate the convergence result. In the future, we study this problem in the case where the exact solution is singular using edge element of the first kind in the discontinuous Galerking method as in [17].

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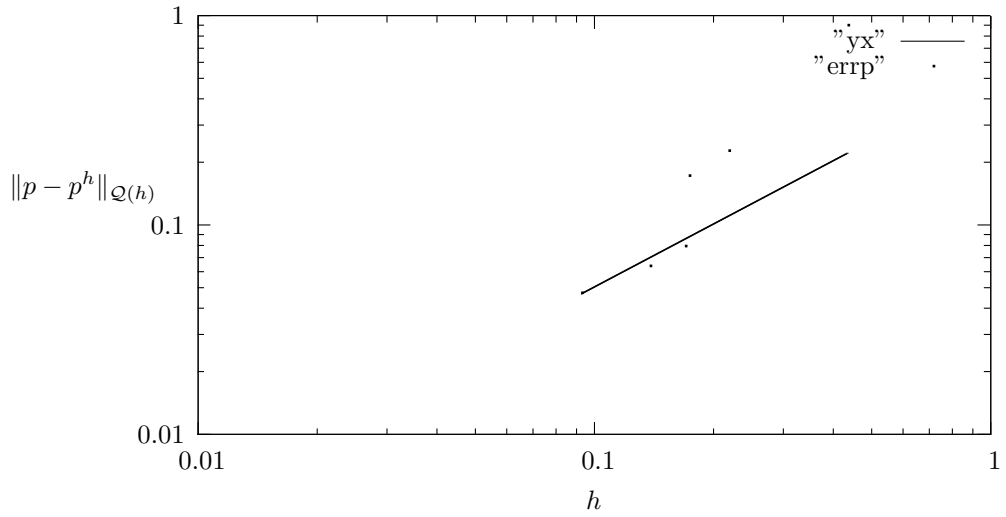
Figure 4.2



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