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The logarithmic Bramson correction for Fisher-KPP equations on the lattice \mathbb{Z}

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Abstract

We establish in this paper the logarithmic Bramson correction for Fisher-KPP equations on the lattice \mathbb{Z} . The level sets of solutions with step-like initial conditions are located at position $c_*t - \frac{3}{2\lambda_*} \ln t + \mathcal{O}(1)$ as $t \rightarrow +\infty$ for some explicit positive constants c_* and λ_* . This extends a well-known result of Bramson in the continuous setting to the discrete case using only PDE arguments. A by-product of our analysis also gives that the solutions approach the family of logarithmically shifted traveling front solutions with minimal wave speed c_* uniformly on the positive integers, and that the solutions converge along their level sets to the minimal traveling front for large times.

Keywords: discrete Fisher-KPP equations; Bramson logarithmic corrections; Green's function.

1 Introduction

We consider the following Cauchy problem

$$\begin{cases} u'_j(t) = u_{j-1}(t) - 2u_j(t) + u_{j+1}(t) + f(u_j(t)), & t > 0, \quad j \in \mathbb{Z}, \\ u_j(0) = u_j^0, & j \in \mathbb{Z}, \end{cases} \quad (1.1)$$

for some nontrivial bounded initial sequence $(u_j^0)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. Throughout, we let $\ell^\infty(\mathbb{Z})$ denote the Banach space of bounded valued sequences indexed by \mathbb{Z} and equipped with the norm:

$$\|u\|_{\ell^\infty(\mathbb{Z})} := \sup_{j \in \mathbb{Z}} |u_j|.$$

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For future reference, we also let $\ell^q(\mathbb{Z})$ with $1 \leq q < +\infty$ denote the Banach space of sequences indexed by \mathbb{Z} such that the $\ell^q(\mathbb{Z})$ norm, defined for $u : \mathbb{Z} \rightarrow \mathbb{C}$ by

$$\|u\|_{\ell^q(\mathbb{Z})} := \left(\sum_{j \in \mathbb{Z}} |u_j|^q \right)^{1/q},$$

is finite. Here the reaction term $f \in \mathcal{C}^2([0, 1])$ is assumed to be of Fisher-KPP type, that is

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad \text{and } 0 < f(u) \leq f'(0)u \text{ for all } u \in (0, 1).$$

Without loss of generality, we extend f linearly on $(-\infty, 0) \cup (1, +\infty)$. Throughout, we will further assume that the initial sequence $(u_j^0)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ satisfies

$$0 \leq u_j^0 \leq 1, \quad \forall j \in \mathbb{Z}, \quad \text{and} \quad u_j^0 = 0, \quad j \geq J,$$

for some integer $J \in \mathbb{Z}$. The solution $u_j(t)$ to (1.1) is classical for $t > 0$ in the sense that $(u_j)_{j \in \mathbb{Z}} \in \mathcal{C}^1((0, \infty), \ell^\infty(\mathbb{Z})) \cap \mathcal{C}^0([0, \infty), \ell^\infty(\mathbb{Z}))$ and verifies $0 < u_j(t) < 1$ for each $t > 0$ and $j \in \mathbb{Z}$ thanks to the strong maximum principle which applies in the discrete setting.

The Cauchy problem (1.1) can be interpreted as the discrete version of the standard spatially-extended Fisher-KPP equation

$$\begin{cases} \partial_t u = \partial_x^2 u + f(u), & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u^0(x), & x \in \mathbb{R}, \end{cases} \quad (1.2)$$

for some step-like initial datum $0 \leq u^0 \leq 1$ with $u^0 \not\equiv 0$ and $u^0(x) = 0$ for $x \geq A$ with $A \in \mathbb{R}$. Such discrete and continuous equations arise in many mathematical models in biology, ecology, epidemiology or genetics, see e.g. [4, 18, 25, 32], and u typically stands for the density of a population. Much is known regarding the long time asymptotics of the solutions to (1.2) and a famous result due to Aronson-Weinberger [1] states that the solutions spread at speed $c_* = 2\sqrt{f'(0)}$. That is

$$\lim_{t \rightarrow +\infty} \min_{|x| \leq ct} u(t, x) = 1, \quad \text{for all } c \in (0, c_*),$$

and

$$\lim_{t \rightarrow +\infty} \max_{x \geq ct} u(t, x) = 0, \quad \text{for all } c > c_*.$$

In his celebrated papers [6, 7], Bramson obtained sharp asymptotics of the location of the level sets of $u(t, x)$ through probabilistic arguments using the relationship between the classical Fisher-KPP equation (1.2) and branching Brownian motion. More precisely, for any $m \in (0, 1)$, if $x_m(t) := \sup \{x \in \mathbb{R} \mid u(t, x) \geq m\}$ denotes the leading edge of the solution u at level m , then Bramson proved that

$$x_m(t) = c_* t - \frac{3}{2\lambda_*} \ln t + \sigma_m + \mathcal{O}(1), \quad \text{as } t \rightarrow +\infty,$$

for some shift σ_m depending on m and the initial datum with $\lambda_* := c_*/2$. There is actually a long history of works [25, 26, 31] regarding the large time behavior for the solutions of the Cauchy

problem (1.2) dating back to the pioneer work of Kolmogorov, Petrovskii and Piskunov which relate spreading properties of the solutions to the convergence towards a selected traveling front solution, often referred to as the critical pulled front. Indeed for (1.2), it is well-known [1, 25] that there exist traveling front solutions of the form $\varphi_c(x - ct)$ where $0 < \varphi_c < 1$ with limits $\varphi_c(-\infty) = 1$ and $\varphi_c(+\infty) = 0$ if and only if $c \geq c_* = 2\sqrt{f'(0)}$. Furthermore the profiles φ_c are decreasing, unique up to translations and verify

$$\varphi_c'' + c\varphi_c' + f(\varphi_c) = 0. \quad (1.3)$$

On the other hand, the logarithmic shift in KPP type equations has been much revisited in the recent years. Most notably, we mention the recent developments [22, 27, 28] which recover and extend Bramson's results using PDE arguments. As postulated in [15], the logarithmic shift has a universal character and has been retrieved in several other contexts: Fisher-KPP equations with nonlocal diffusion [20, 30] and nonlocal interactions [5], in a periodic medium [23] or in more general monostable reaction-diffusion equations [19]. It is also intimately linked to the selection of pulled fronts, and we refer to the recent work [2] and references therein. In this work, we show that the Bramson logarithmic correction also holds in the discrete setting of (1.1) rigorously justifying some formal asymptotic results from [16].

Main results. From [24, 32], we know that the solutions $u_j(t)$ to (1.1) satisfy the following spreading property:

$$\lim_{t \rightarrow +\infty} \min_{j \leq ct} u_j(t) = 1, \quad \text{for all } c \in (0, c_*), \quad (1.4)$$

and

$$\lim_{t \rightarrow +\infty} \max_{j \geq ct} u_j(t) = 0, \quad \text{for all } c > c_*. \quad (1.5)$$

Here, the spreading speed $c_* > 0$ is uniquely defined as

$$c_* := \min_{\lambda > 0} \frac{e^\lambda - 2 + e^{-\lambda} + f'(0)}{\lambda}. \quad (1.6)$$

Let us note that there is a unique $\lambda_* > 0$ where the above minimum is attained so that the couple (c_*, λ_*) is solution of the nonlinear problem

$$\begin{cases} c_* \lambda_* = e^{\lambda_*} - 2 + e^{-\lambda_*} + f'(0), \\ c_* = e^{\lambda_*} - e^{-\lambda_*}. \end{cases} \quad (1.7)$$

As previously explained, our aim is to provide sharp asymptotics of the level sets of $u_j(t)$. We define for any $m \in (0, 1)$ the quantity

$$j_m(t) := \sup \{j \in \mathbb{Z} \mid u_j(t) \geq m\}.$$

Our main result is the following.

Theorem 1. *Let $c_* > 0$ and $\lambda_* > 0$ be defined in (1.6) and (1.7). For each $m \in (0, 1)$ there exist $C > 1$ and $T > 1$, such that*

$$j_m(t) \in \mathbb{Z} \cap \left[c_* t - \frac{3}{2\lambda_*} \ln t - C, c_* t - \frac{3}{2\lambda_*} \ln t + C \right], \quad t \geq T.$$

Combining the above Theorem 1 with the spreading property (1.4), we get that

$$\liminf_{t \rightarrow +\infty} \left(\min_{0 \leq j \leq c_* t - \frac{3}{2\lambda_*} \ln t - C} u_j(t) \right) \rightarrow 1 \text{ as } C \rightarrow +\infty, \quad (1.8)$$

since $u_j(t) \rightarrow 1$ as $t \rightarrow +\infty$ locally uniformly in j . Furthermore, using that $\lim_{j \rightarrow +\infty} u_j(t) = 0$ for each $t \geq 0$, we also deduce that

$$\limsup_{t \rightarrow +\infty} \left(\max_{j \geq c_* t - \frac{3}{2\lambda_*} \ln t + C} u_j(t) \right) \rightarrow 0 \text{ as } C \rightarrow +\infty. \quad (1.9)$$

The limits (1.8) and (1.9) show that the region in the lattice where the solution $u_j(t)$ is bounded away from 0 and 1 is located around the position $c_* t - \frac{3}{2\lambda_*} \ln t$ and has a bounded width in the limit $t \rightarrow +\infty$.

Refining the arguments of Theorem 1, one can actually prove that the solution $u_j(t)$ approaches the family of shifted traveling fronts $U_{c_*} \left(j - c_* t + \frac{3}{2\lambda_*} \ln t + \zeta \right)$ uniformly in $j \geq 0$. Indeed, as in the continuous case, the spreading speed c_* is also the threshold to the existence of traveling front solutions to (1.1). More precisely, combining the results from [8–11, 33] for each $c \geq c_*$ there exists a unique (up to translation) monotone front solution $U_c \in \mathcal{C}^1(\mathbb{R})$ solution of

$$\begin{cases} 0 = cU'_c(x) + U_c(x+1) - 2U_c(x) + U_c(x-1) + f(U_c(x)), & x \in \mathbb{R}, \\ U_c(-\infty) = 1, \quad U_c(+\infty) = 0, \quad 0 < U_c < 1. \end{cases} \quad (1.10)$$

Our second main result reads as follows.

Theorem 2. *Let $c_* > 0$ and $\lambda_* > 0$ be defined in (1.6) and (1.7). There are some constant $C > 0$ and a function $\zeta : (0, +\infty) \rightarrow \mathbb{R}$ with $|\zeta(t)| \leq C$ such that*

$$\lim_{t \rightarrow +\infty} \sup_{j \geq 0} \left| u_j(t) - U_{c_*} \left(j - c_* t + \frac{3}{2\lambda_*} \ln t + \zeta(t) \right) \right| = 0. \quad (1.11)$$

Furthermore, for every $m \in (0, 1)$ and every sequence $(t_n, j_n)_{n \in \mathbb{N}}$ such that $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and $j_n \in j_m(t_n)$ for all $n \in \mathbb{N}$, there holds

$$u_{j+j_n}(t+t_n) \rightarrow U_{c_*}(j - c_* t + U_{c_*}^{-1}(m)) \quad \text{locally uniformly in } (t, j) \in \mathbb{R} \times \mathbb{Z},$$

where $U_{c_*}^{-1}$ denotes the inverse of the function U_{c_*} .

Strategy of proof. The strategy of our proof is inspired by the PDE arguments that were developed recently in the continuous setting [22]. The take home message from [22] is that the long time dynamics of (1.2) can be read out from the solutions of the linearized problem around the unstable state $u = 0$ with a Dirichlet boundary condition at $x = c_* t$. Indeed, these solutions can be compared to the critical front φ_{c_*} , appropriately shifted to the position $c_* t - \frac{3}{2\lambda_*} \ln t$, in the diffusive regime of the linearized equation that is for all $x \in [c_* t, c_* t + \vartheta \sqrt{t}]$ and for any $\vartheta > 0$. One of the main difficulty in our analysis comes from the discrete nature of our equation such that

the above argument has to be largely adapted. Our starting point is also the linearized equation around the unstable state 0. We first perform, in the linearized equation of $r_j(t)$, the change of variable $r_j(t) = e^{-\lambda_*(j-c_*t)}w_j(t)$ for some new sequence $(w_j(t))_{j \in \mathbb{Z}}$ which solves

$$w'_j(t) = e^{\lambda_*} (w_{j-1}(t) - 2w_j(t) + w_{j+1}(t)) - c_*(w_{j+1}(t) - w_j(t)), \quad t > 0, \quad j \in \mathbb{Z}.$$

Solutions of the above equation starting from some initial sequence $(w_j^0)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ are given through the representation formula

$$w_j(t) = \sum_{\ell \in \mathbb{Z}} \mathcal{G}_{j-\ell}(t) w_\ell^0, \quad t > 0, \quad j \in \mathbb{Z},$$

where $(\mathcal{G}_j(t))_{j \in \mathbb{Z}}$ stands for the temporal Green's function (see (3.3) below for a precise definition). The key step of the analysis is to obtain sharp pointwise estimates on the temporal Green's function. We establish that $\mathcal{G}_j(t)$ behaves like a Gaussian profile centered at $j = c_*t$ for all $|j - c_*t| \leq \vartheta t^\alpha$ for any $\vartheta > 0$, $\alpha \in (0, 1)$ and $t > 1$. Actually, we prove a sharper result, which is of independent interest, by showing that the temporal Green's function can be decomposed as a universal Gaussian profile plus some reminder term which can be bounded and which also satisfies a generalized Gaussian estimate. We refer to Proposition 4.1 for a precise statement. Let us note that similar generalized Gaussian bounds have been recently derived for discrete convolution powers [12–14] and are reminiscent of so-called local limit theorems in probability theory [29]. Owing to this sharp estimate on the temporal Green's function, we manage in a second step to construct appropriate sub and super solutions which allow us to precisely locate any level sets of the solutions to (1.1). For this part, we take benefit from the new results obtained by Roquejoffre [30] in the continuous setting with nonlocal diffusion. Let us finally emphasize that our analysis of the temporal Green's function does not rely on Fourier analysis but rather on a *spatial* point of view through Laplace inversion formula by defining each $\mathcal{G}_j(t)$ as

$$\forall t > 0, \quad \forall j \in \mathbb{Z}, \quad \mathcal{G}_j(t) = \frac{1}{2\pi \mathbf{i}} \int_{\Gamma} e^{\nu t} \mathbf{G}_j(\nu) d\nu,$$

where $\Gamma \subset \mathbb{C}$ is some well-chosen contour in the complex plane and $(\mathbf{G}_j(\nu))_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ is the associated spatial Green's function (see (3.5) below for a precise definition).

Compared to Bramson's result in the continuous setting, our main Theorem 1 only captures the logarithmic correction up to some $\mathcal{O}(1)$ terms as $t \rightarrow +\infty$. We expect that our logarithmic expansion could be refined along the lines of [27, 28] with convergence to a single traveling front solution. We leave it for a future work.

Outline. The rest of the paper is mainly dedicated to the proof of Theorem 1. For expository reasons, instead of focusing directly on it, we first revisit in Section 2 the continuous case and provide an alternate proof of the Bramson's logarithmic correction up to some $\mathcal{O}(1)$ terms as $t \rightarrow +\infty$. Compared to [22], the novelty of this alternate proof, which takes its inspiration from the nonlocal continuous case [30], is to solely focus on the linearized problem around the unstable

state $u = 0$ with a Dirichlet boundary condition at $x = c_*t$ without relying on self-similar variables. Indeed, the use of self-similar variables for the discrete Fisher-KPP equation is prohibited. Then, in Section 3, we study the linearized problem for the discrete Fisher-KPP equation and, in a first step, we prove a generalized Gaussian bound for the associated temporal Green's function. In a second step, we provide in Section 4 a sharp asymptotic expansion for the temporal Green's function in the sub-linear regime which will be crucial to the proof of Theorem 1. Finally, in Section 5 we provide the lower and upper bounds in our main Theorem 1 following the strategy presented in the continuous case. In the last Section 6, we study the convergence to the logarithmically shifted minimal front and prove Theorem 2.

2 An alternative proof of the logarithmic Bramson correction in the continuous case

In this section, we revisit [22] and propose an alternative proof of the logarithmic Bramson correction in the continuous case. More precisely, the purpose of this section is to prove the following result.

Proposition 2.1. *Let u be the solution of the Cauchy problem (1.2) starting from some step-like initial datum $0 \leq u^0 \leq 1$ with $u^0 \not\equiv 0$ and $u^0(x) = 0$ for $x \geq A$ with $A \in \mathbb{R}$. Then, there exist $-\infty < b < a < +\infty$, $\eta \in (0, 1/2)$ and some large time $T_0 > 1$ such that*

$$\varphi_{c_*} \left(x - c_*t + \frac{3}{2\lambda_*} \ln t + a \right) \leq u(t, x) \leq \varphi_{c_*} \left(x - c_*t + \frac{3}{2\lambda_*} \ln t + b \right), \quad (2.1)$$

uniformly in $x \in \mathbb{R}$ with $|x - c_*t| \leq t^\eta$ for $t \geq T_0$, where $c_* = 2\sqrt{f'(0)}$, $\lambda_* = c_*/2$ and the minimal traveling front φ_{c_*} solution of (1.3) is normalized such that

$$\varphi_{c_*}(s) \underset{s \rightarrow +\infty}{\sim} se^{-\lambda_*s}.$$

A direct consequence of the above proposition is that for each $m \in (0, 1)$, there exists $C > 0$ such that

$$x_m(t) \in \left[c_*t - \frac{3}{2\lambda_*} \ln t - C, c_*t - \frac{3}{2\lambda_*} \ln t + C \right], \text{ as } t \rightarrow +\infty,$$

where $x_m(t)$ denotes the leading edge of the solution u at level m . Moreover, it is worth to note that, based on (2.1), the argument in [22] can be simplified by applying this time directly the Liouville type result Theorem 3.5 of [3] instead of using Lemma 4.1 in [22], to show the large time convergence of the solutions to the family of shifted traveling fronts as well as of the solutions along their level sets to the profile of the minimal traveling front, see Theorem 1.2 of [22].

2.1 Preliminaries

In a first step, we establish upper and lower barriers for a variant v (see (2.2) below) of the solution u , by using the solution for the linear equation (2.5) for t sufficiently large and $x \in \mathbb{R}$ ahead of the

position $x - c_*t \sim 0$ albeit with a small shift. These upper and lower bounds will play a crucial role in showing a refined estimate on the expansion of the level sets of u in the sequel.

Set

$$v(t, x) = e^{\lambda_*(x-c_*t)}u(t, x). \quad (2.2)$$

This leads to

$$\begin{cases} v_t - v_{xx} + c_*v_x + R(t, x; v) = 0, & t > 0, x \in \mathbb{R}, \\ v(0, x) = v^0(x) = e^{\lambda_*x}u^0, & x \in \mathbb{R}, \end{cases} \quad (2.3)$$

in which the nonlinear term $R(t, x; s)$ has the following precise form

$$R(t, x; s) := f'(0)s - e^{\lambda_*(x-c_*t)}f\left(e^{-\lambda_*(x-c_*t)}s\right) \geq 0, \quad (2.4)$$

due to the assumption on f that $0 < f(s) \leq f'(0)s$ for $s \in (0, 1)$ and the linear extension of f for $s \in (-\infty, 0) \cup (1, +\infty)$. One then has that $R(t, x; v) \geq 0$ for $(t, x) \in (0, +\infty) \times \mathbb{R}$.

Let us now consider the linear equation

$$(\partial_t - \mathcal{L})w := w_t - w_{xx} + c_*w_x = 0, \quad t > 0, x \in \mathbb{R}. \quad (2.5)$$

It is easy to see that the function $p(t, y) = w(t, y + c_*t)$ satisfies $p_t - p_{yy} = 0$ for $t > 0, y \in \mathbb{R}$. We impose an odd and compactly supported initial datum $p_0 \not\equiv 0$ in \mathbb{R} , then it follows that

$$w(t, y + c_*t) = p(t, y) = \frac{1}{\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{(y-z)^2}{4t}} - e^{-\frac{(y+z)^2}{4t}} \right) p_0(z) dz, \quad t > 0, y \in \mathbb{R}. \quad (2.6)$$

In particular, one has $p(t, y) > 0$ for $t > 0$ and $y > 0$, and the following asymptotic behavior holds true:

$$p(t, y) \sim Cy e^{-\frac{y^2}{4t}} t^{-\frac{3}{2}}, \quad \text{as } t \rightarrow +\infty,$$

for $-\sqrt{t} \leq y \leq \sqrt{t}$, where $C > 0$ depends only on p_0 . This then implies that $w(t, x)$ changes its sign exactly at $x - c_*t = 0$, i.e., $w(t, x) > 0$ for $t > 0$ and $x \in \mathbb{R}$ with $x - c_*t > 0$, whereas $w(t, x) < 0$ for $t > 0$ and $x \in \mathbb{R}$ with $x - c_*t < 0$. Moreover,

$$w(t, x) \sim C(x - c_*t)e^{-\frac{(x-c_*t)^2}{4t}} t^{-\frac{3}{2}}, \quad \text{as } t \rightarrow +\infty, \quad (2.7)$$

for $-\sqrt{t} \leq x - c_*t \leq \sqrt{t}$.

We are now in position to take advantage of $w(t, x)$ in the constructions of the upper and lower barriers for v .

2.2 Upper barrier for v

We start with the construction of a supersolution to (2.3) for all t large enough and $x \in \mathbb{R}$ ahead of $x - c_*t \sim 0$ by following the strategy recently proposed in [30] for the continuous setting with

nonlocal diffusion, which is itself reminiscent of the strategy used in [22]. Ideally, the solution $v(t, x)$ of (2.3) would be controlled from above by the function $w(t, x)$ solution of the linear equation (2.5) starting from an odd, nontrivial and compactly supported initial condition, since it is an actual supersolution by construction. However, as the initial condition is chosen to be odd, the function $w(t, x)$ is negative for $t > 0$ and $x - c_*t < 0$ which prevents us from readily comparing the two functions. The key observation from [30], very much in the spirit of Fife & McLeod [17], is that we can add a cosine perturbation to $w(t, x)$ which will eventually enable us to compare $v(t, x)$ to this new supersolution slightly to the left of $x - c_*t = 0$.

Consider now $\delta \in (0, 1/3)$, that will be as small as needed. We now look for a barrier of $v(t, x)$ from above for t large enough and $x \in \mathbb{R}$ ahead of $x - c_*t = -t^\delta$. To do so, we construct a supersolution for (2.3) in the form

$$\bar{v}(t, x) = \bar{\xi}(t)w(t, x) + \frac{1}{(1+t)^{\frac{3}{2}-\beta}} \cos\left(\frac{x - c_*t}{(1+t)^\alpha}\right) \mathbb{1}_{\{x \in \mathbb{R} \mid -t^\delta \leq x - c_*t \leq \frac{3\pi}{2}(1+t)^\alpha\}}, \quad (2.8)$$

for t large enough and $x \in \mathbb{R}$ with $x - c_*t \geq -t^\delta$, where the unknown $\bar{\xi}(t) \in \mathcal{C}^1$ is assumed to be positive and bounded in $(0, +\infty)$, and $\bar{\xi}'(t) \geq 0$ in $(0, +\infty)$, $\alpha \in (1/3, 1/2)$ and $\beta > 0$, all of which will be chosen in the course of investigation. Let us now check that $\bar{v}(t, x)$ is a supersolution of (2.3) for t large enough and $x \in \mathbb{R}$ with $x - c_*t \geq -t^\delta$.

First of all, we look at the region $\{x \in \mathbb{R} \mid x - c_*t \geq \frac{3\pi}{2}(1+t)^\alpha\}$ for all t large enough, in which $\bar{v}(t, x) = \bar{\xi}(t)w(t, x)$. It is obvious to see that $(\partial_t - \mathcal{L})\bar{v}(t, x) = \bar{\xi}'(t)w(t, x) \geq 0$ since $w(t, x) > 0$ for all $t \geq 0$ in this area and since we assume that $\bar{\xi}'(t) \geq 0$ for $t > 0$. It remains to discuss the region $\{x \in \mathbb{R} \mid -t^\delta \leq x - c_*t \leq \frac{3\pi}{2}(1+t)^\alpha\}$ for all large times. We divide it into three zones:

$$R_1 := \left\{x \in \mathbb{R} \mid -t^\delta \leq x - c_*t \leq 1\right\}, \quad R_2 := \left\{x \in \mathbb{R} \mid 1 \leq x - c_*t \leq \frac{\pi}{4}(1+t)^\alpha\right\}, \\ R_3 := \left\{x \in \mathbb{R} \mid \frac{\pi}{4}(1+t)^\alpha \leq x - c_*t \leq \frac{3\pi}{2}(1+t)^\alpha\right\}.$$

In region R_1 . There holds

$$\frac{-t^\delta}{(1+t)^\alpha} \leq \frac{x - c_*t}{(1+t)^\alpha} \leq \frac{1}{(1+t)^\alpha}.$$

Thanks to (2.7), one gets

$$-\frac{1}{(1+t)^{\frac{3}{2}-\delta}} \lesssim w(t, x) \lesssim \frac{1}{(1+t)^{\frac{3}{2}}}.$$

Notice that $w(t, x)$ can be negative in this area, therefore the cosine perturbation needs to play a role here. We first require that $\beta > \delta > 0$ so that the cosine term will be the dominant term, that is,

$$\bar{v}(t, x) \sim \frac{1}{(1+t)^{\frac{3}{2}-\beta}} > 0 \quad (2.9)$$

for t large enough in this region. Moreover, a straightforward computation gives that

$$(\partial_t - \mathcal{L})(\bar{\xi}(t)w(t, x)) = \bar{\xi}'(t)w(t, x) \gtrsim -\frac{\bar{\xi}'(t)}{(1+t)^{\frac{3}{2}-\delta}}$$

and

$$\begin{aligned}
& (\partial_t - \mathcal{L}) \left(\frac{1}{(1+t)^{\frac{3}{2}-\beta}} \cos \left(\frac{x - c_* t}{(1+t)^\alpha} \right) \right) \\
&= \frac{\beta - \frac{3}{2}}{(1+t)^{\frac{5}{2}-\beta}} \cos \left(\frac{x - c_* t}{(1+t)^\alpha} \right) + \frac{1}{(1+t)^{\frac{3}{2}-\beta}} \left(\frac{c_*}{(1+t)^\alpha} + \frac{\alpha(x - c_* t)}{(1+t)^{\alpha+1}} \right) \sin \left(\frac{x - c_* t}{(1+t)^\alpha} \right) \\
&\quad + \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} \cos \left(\frac{x - c_* t}{(1+t)^\alpha} \right) - \frac{c_*}{(1+t)^{\frac{3}{2}-\beta+\alpha}} \sin \left(\frac{x - c_* t}{(1+t)^\alpha} \right) \\
&\sim \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}}.
\end{aligned}$$

Therefore, in order to ensure that $(\partial_t - \mathcal{L})\bar{v}(t, x) \geq 0$ for t large enough in this region, it suffices to impose the condition

$$\frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} \gg \frac{\bar{\xi}'(t)}{(1+t)^{\frac{3}{2}-\delta}} \quad \text{for all } t \text{ large enough,} \quad (2.10)$$

namely,

$$0 \leq \bar{\xi}'(t) \ll \frac{1}{(1+t)^{2\alpha+\delta-\beta}}. \quad (2.11)$$

In region R_2 . There holds

$$\frac{1}{(1+t)^\alpha} \leq \frac{x - c_* t}{(1+t)^\alpha} \leq \frac{\pi}{4}.$$

We notice that $w(t, x) > 0$ for all $t > 0$ in this region, so does the cosine perturbation. Since $\bar{\xi}(t)$ is assumed a priori to be a bounded positive function in $(0, +\infty)$, one has that $\bar{v}(t, x) > 0$ for all $t > 0$ in this region and

$$(\partial_t - \mathcal{L})(\bar{\xi}(t)w(t, x)) = \bar{\xi}'(t)w(t, x) \geq 0$$

for all $t > 0$ in this region, thanks to $\bar{\xi}'(t) \geq 0$ in $(0, +\infty)$. Moreover, by the same calculation as the previous region, one also obtains in R_2 that

$$(\partial_t - \mathcal{L}) \left(\frac{1}{(1+t)^{\frac{3}{2}-\beta}} \cos \left(\frac{x - c_* t}{(1+t)^\alpha} \right) \right) \sim \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} > 0 \quad \text{for } t \text{ large enough.}$$

Consequently, there obviously holds $(\partial_t - \mathcal{L})\bar{v}(t, x) \geq 0$ for t large enough in region R_2 .

In region R_3 . There holds

$$\frac{\pi}{4} \leq \frac{x - c_* t}{(1+t)^\alpha} \leq \frac{3\pi}{2}.$$

It is worth to note that the cosine perturbation may be negative in this area. Moreover, following an analogous procedure as in preceding cases, it follows that

$$(\partial_t - \mathcal{L})(\bar{\xi}(t)w(t, x)) = \bar{\xi}'(t)w(t, x) \sim \frac{\bar{\xi}'(t)}{(1+t)^{\frac{3}{2}-\alpha}} \geq 0 \quad \text{for } t \text{ large enough,}$$

and

$$(\partial_t - \mathcal{L}) \left(\frac{1}{(1+t)^{\frac{3}{2}-\beta}} \cos \left(\frac{x - c_* t}{(1+t)^\alpha} \right) \right) \geq \frac{-1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} \quad \text{for } t \text{ large enough.}$$

To ensure that $\bar{v}(t, x) > 0$ and $(\partial_t - \mathcal{L})\bar{v}(t, x) \geq 0$ for t large enough in this region, we require this time $\alpha > \beta$ and

$$\frac{\bar{\xi}'(t)}{(1+t)^{\frac{3}{2}-\alpha}} \gg \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} \quad \text{for } t \text{ large enough.} \quad (2.12)$$

Conclusion. Gathering (2.11) and (2.12), we should impose

$$\frac{1}{(1+t)^{3\alpha-\beta}} \ll \bar{\xi}'(t) \ll \frac{1}{(1+t)^{2\alpha+\delta-\beta}}.$$

This is possible so long as $\delta < \alpha$, which is exactly what we have assumed. Let us take

$$\bar{\xi}'(t) \sim \frac{1}{(1+t)^{3\alpha-2\beta}} \quad \text{and} \quad \bar{\xi}(t) = 1 - \frac{1}{(1+t)^{3\alpha-2\beta-1}} \quad \text{for } t \geq 0.$$

Due to our assumption that the function $\bar{\xi}(t)$ is positive and bounded in $(0, +\infty)$, it suffices to require $3\alpha - 2\beta - 1 > 0$. Hence, we can fix $\delta \in (0, 1/4)$ very small, then there exist $\alpha \in (1/3, 1/2)$ and $\beta > 0$ such that $3\alpha - 2\beta > 2\alpha + \delta - \beta$. To be more precise, the parameters δ, β, α is chosen such that

$$0 < \delta < \beta < \min \left(\alpha - \delta, \frac{3\alpha - 1}{2} \right) < \alpha < \frac{1}{2}. \quad (2.13)$$

As a consequence, there is $T_0 > 0$ large enough such that $(\partial_t - \mathcal{L})\bar{v}(t, x) \geq 0$ for $t \geq T_0$ and $x \in \mathbb{R}$ with $x - c_* t \geq -t^\delta$, whence $(\partial_t - \mathcal{L})\bar{v}(t, x) + R(t, x; \bar{v}(t, x)) \geq 0$ for $t \geq T_0$ and $x \in \mathbb{R}$ with $x - c_* t \geq -t^\delta$, since $R(t, x; \bar{v}(t, x)) \geq 0$ for $t \geq T_0$ and $x \in \mathbb{R}$ with $x - c_* t \geq -t^\delta$ by applying (2.4) and $\bar{v}(t, x) > 0$ for $t \geq T_0$ and $x \in \mathbb{R}$ with $x \geq c_* t - t^\delta$, and such that $c_* T_0 - T_0^\delta > A$, where we recall that $A \in \mathbb{R}$ is beyond the support of u^0 . Hence, there holds $\bar{v}(T_0, x) > 0 = e^{\lambda_* x} u^0(x) = v(0, x)$ for all $x - c_* T_0 \geq -T_0^\delta$. For $x - c_* t = -t^\delta$, we observe that $v(t - T_0, x) = e^{\lambda_*(x - c_*(t - T_0))} u(t - T_0, x) \leq e^{\lambda_*(c_* T_0 - t^\delta)}$ since $0 \leq u(t, x) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{R}$, while $\bar{v}(t, x) \sim (1+t)^{-3/2+\beta}$ for $t \geq T_0$ by (2.9), up to increasing T_0 . Then, there is $T_1 > T_0$ sufficiently large such that $e^{\lambda_*(c_* T_0 - t^\delta)} < (1+t)^{-3/2+\beta}$ for all $t \geq T_1$, which will yield that $\bar{v}(t, x) > v(t - T_0, x)$ at $x - c_* t = t^\delta$ for all $t \geq T_1$. On the other hand, one can choose $K > 0$ large enough such that $K(1+t)^{-3/2+\beta} > e^{\lambda_*(c_* T_0 - t^\delta)}$, that is, $K\bar{v}(t, x) > v(t - T_0, x)$ at $x - c_* t = t^\delta$ for $t \in [T_0, T_1]$. Therefore, there holds $K\bar{v}(t, x) > v(t - T_0, x)$ for all $t \geq T_0$ and $x \in \mathbb{R}$ with $x - c_* t = -t^\delta$. We then conclude that $K\bar{v}(t, x)$ is a supersolution of (2.3) for all $t \geq T_0$ and $x \in \mathbb{R}$ with $x - c_* t \geq -t^\delta$. It follows from the strong maximum principle that $K\bar{v}(t, x) > v(t - T_0, x)$ for all $t > T_0$ and $x \in \mathbb{R}$ with $x - c_* t > -t^\delta$.

2.3 Lower barrier for $v(t, x)$

Let δ , β and α be fixed as in (2.13). The idea is to estimate $v(t, x)$ ahead of $x - c_*t = t^\delta$. To do so, let construct a lower barrier as follows:

$$\underline{v}(t, x) = \begin{cases} 0, & x - c_*t \leq 0, \\ \underline{\xi}(t)w(t, x), & x - c_*t > 0, \end{cases} \quad (2.14)$$

for $t > 0$ and $x \in \mathbb{R}$, where we assume that the unknown $\underline{\xi}(t) \in \mathcal{C}^1$ is positive and bounded away from 0 in $(0, +\infty)$ and satisfies $\underline{\xi}'(t) \leq 0$ in $(0, +\infty)$, which will be made clear in the sequel. We notice that $\underline{v}(t, x)$ is nonnegative and continuous for $t > 0$ and $x \in \mathbb{R}$, thanks to (2.6). Let now identify that $\underline{v}(t, x)$ is a generalized subsolution of (2.3) for $t > 0$ and $x \in \mathbb{R}$.

It is sufficient to look at the region $\{(t, x) \in (0, +\infty) \times \mathbb{R} : x > c_*t\}$. We first claim that, there is $C > 0$ sufficiently large such that

$$e^{-\lambda_*(x-c_*t)}w(t, x) \leq C(1+t)^{-3/2}, \quad (2.15)$$

for $t > 0$ and $x - c_*t > 0$. Indeed, it follows from (2.7) that there exist t_0 large enough and $C > 0$ large enough such that (2.15) is true for $t \geq t_0$ and $0 < x - c_*t \leq \sqrt{t}$. Since $w(t, x)$ is a bounded function for $t > 0$ and $x \in \mathbb{R}$, we derive that, up to increasing C , (2.15) holds true for $t \in [0, t_0]$ and $0 < x - c_*t \leq \sqrt{t}$. In a similar way, we see that, by increasing C if needed, there holds $e^{-\lambda_*(x-c_*t)}w(t, x) < e^{-\lambda_*\sqrt{t}}w(t, x) \leq C(1+t)^{-3/2}$ for $t \geq 0$ and $x - c_*t > \sqrt{t}$. Therefore, our claim (2.15) is proved for $t > 0$ and $x - c_*t > 0$. On the other hand, since $f \in \mathcal{C}^2([0, 1])$, there exist $M > 0$ and $s_0 \in (0, 1)$ such that $f(s) - f'(0)s \geq -Ms^2$ for $s \in [0, s_0]$. Eventually, let us require $\underline{\xi}(t)$ to solve

$$\underline{\xi}'(t) = -CM\underline{\xi}^2(t)(1+t)^{-3/2}, \quad \text{for } t > 0, \quad (2.16)$$

starting from $\underline{\xi}(0) = \underline{\xi}_0 > 0$ which is set very small such that $\underline{\xi}_0 w(t, x) < s_0$ for $t > 0$ and $x - c_*t > 0$. We then note that $\underline{\xi}(t)$ is positive and uniformly bounded from above and below such that

$$0 < \frac{\underline{\xi}_0}{1 + 2\underline{\xi}_0 CM} \leq \underline{\xi}(t) \leq \underline{\xi}_0 < +\infty \quad \text{for } t \geq 0.$$

Moreover, for $t > 0$ and $x - c_*t > 0$, the function $\underline{v}(t, x) = \underline{\xi}(t)w(t, x)$ satisfies

$$\begin{aligned} (\partial_t - \mathcal{L})\underline{v}(t, x) + R(t, x; \underline{v}) &= \underline{\xi}'(t)w(t, x) + f'(0)\underline{v}(t, x) - e^{\lambda_*(x-c_*t)}f(e^{-\lambda_*(x-c_*t)}\underline{v}(t, x)) \\ &\leq \underline{\xi}'(t)w(t, x) + Me^{-\lambda_*(x-c_*t)}\underline{\xi}^2(t)(w(t, x))^2 \\ &= \left(-CM\underline{\xi}^2(t)(1+t)^{-3/2} + M\underline{\xi}^2(t)e^{-\lambda_*(x-c_*t)}w(t, x)\right)w(t, x) \leq 0, \end{aligned}$$

thanks to (2.15) and (2.16).

Since $v(1, x) = e^{\lambda_*(x-c_*)}u(1, x) > 0$ in \mathbb{R} and since $\underline{\xi}_0 w(0, x) = \underline{\xi}_0 p_0$ is bounded and compactly supported in $(0, +\infty)$, there exists $\kappa > 0$ very small such that $\kappa \underline{v}(0, x) < v(1, x)$ in \mathbb{R} . Therefore, $\kappa \underline{v}(t, x)$ is a subsolution of (2.3) for all $t \geq 0$ and $x \in \mathbb{R}$. By the strong maximum principle, we then conclude that $\kappa \underline{v}(t, x) < v(t+1, x)$ for $t > 0$ and $x \in \mathbb{R}$.

2.4 Proof of Proposition 2.1

We have shown in the previous sections that the function v , solution of (2.3), has an upper barrier and a lower barrier given respectively by (2.8) and (2.14). We are now in position to prove Proposition 2.1. The proof is based on the comparison between $v(t, x)$ and a variant of the shifted critical KPP traveling front with logarithmic correction in a well-chosen moving zone $|x - c_*t| \leq t^\eta$ with certain small η for all large times. Let δ, β, α be chosen as in (2.13). We notice that $\beta < 1/4$. Fix now

$$\eta = \beta + \varepsilon < \alpha \quad (2.17)$$

for $\varepsilon > 0$ small enough.

Step 1: Upper bound. Let $K\bar{v}(t, x)$, where $\bar{v}(t, x)$ is given in (2.8), be a supersolution of (2.3) for t large enough and $x \in \mathbb{R}$ with $x - c_*t \geq -t^\delta$. Define a function $\psi(t, x)$ by

$$\psi(t, x) = e^{\lambda_*(x-c_*t)} \varphi_{c_*} \left(x - c_*t + \frac{3}{2\lambda_*} \ln t + b \right) \quad \text{for } t \text{ large enough, } |x - c_*t| \leq t^\eta,$$

where $b \in \mathbb{R}$ is fixed such that $\psi(t, x) = K\bar{v}(t+1, x)$ for t large enough and $x \in \mathbb{R}$ with $x - c_*t = t^\eta$. Substituting $\psi(t, x)$ into the equation of $v(t, x)$ yields

$$\psi_t - \psi_{xx} + c_*\psi_x + R(t, x; \psi) = \frac{3}{2\lambda_*t} e^{\lambda_*(x-c_*t)} \varphi'_{c_*} \left(x - c_*t + \frac{3}{2\lambda_*} \ln t + b \right) \sim \frac{1}{t^{\frac{5}{2}-\eta}} < O\left(t^{-(1-\eta)}\right)$$

for t large enough and $x \in \mathbb{R}$ with $|x - c_*t| \leq t^\eta$. Now, set $s(t, x) := (v(t, x) - \psi(t, x))^+$, being the positive part of $v(t, x) - \psi(t, x)$. We are then led to the following problem

$$\begin{cases} s_t - s_{xx} + c_*s_x + Q(t, x; s) < O\left(t^{-(1-\eta)}\right), & |x - c_*t| \leq t^\eta, \\ s(t, x) = O\left(e^{-\lambda_*t^\eta}\right), & x - c_*t = -t^\eta, \\ s(t, x) = 0, & x - c_*t = t^\eta, \end{cases} \quad (2.18)$$

for t large enough. Here, $Q(t, x; s) = 0$ if $s(t, x) = 0$; otherwise,

$$\begin{aligned} Q(t, x; s) &= R(t, x; v) - R(t, x; \psi) \\ &= f'(0)s - e^{\lambda_*(x-c_*t)} \left(f(e^{-\lambda_*(x-c_*t)}v) - f(e^{-\lambda_*(x-c_*t)}\psi) \right) \\ &= f'(0)s - d(t, x)s \geq 0, \end{aligned}$$

in which $d(t, x)$ is continuous and bounded in L^∞ norm by $f'(0)$ since $0 < f(s) \leq f'(0)s$ for $s \in (0, 1)$. We claim that, there holds

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}, |x - c_*t| \leq t^\eta} s(t, x) = 0. \quad (2.19)$$

We use a comparison argument to verify this. Define

$$\bar{s}(t, x) = \frac{1}{t^\lambda} \cos\left(\frac{x - c_*t}{t^\gamma}\right) \quad \text{for } t \text{ large enough and } |x - c_*t| \leq t^\eta,$$

where $\eta < \gamma < 1$ such that $2\gamma + \eta < 1$, which is possible by choosing, for instance, $\varepsilon > 0$ is sufficiently small in (2.17) such that $\eta \sim \beta < \frac{1}{4}$ due to the observation from (2.13) and choosing $\gamma > \frac{1}{3}$ close to $\frac{1}{3}$, and the parameter $\lambda > 0$ is to be chosen later. One observes that $\bar{s}(t, x) \sim t^{-\lambda}$ for t large enough and $x \in \mathbb{R}$ with $|x - c_*t| \leq t^\eta$. It follows from a direct computation that

$$\bar{s}_t - \bar{s}_{xx} + c_*\bar{s}_x \sim \frac{1}{t^{2\gamma+\lambda}}$$

for t large enough and $|x - c_*t| \leq t^\eta$. Since $Q(t, x; \bar{s}) \geq 0$, one concludes that $\bar{s}(t, x)$ is a supersolution of (2.18) for t large enough and $x \in \mathbb{R}$ with $|x - c_*t| \leq t^\eta$ (up to multiplying a large constant) so long as

$$\frac{1}{t^{2\gamma+\lambda}} \geq \frac{1}{t^{1-\eta}},$$

that is, $2\gamma + \lambda < 1 - \eta$, which implies $0 < \lambda < 1 - 2\gamma - \eta$. Our claim (2.19) is then reached by noticing that

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}, |x - c_*t| \leq t^\eta} \bar{s}(t, x) = 0.$$

Then, it follows from the comparison principle that $v(t, x) \leq \psi(t, x)$ uniformly in $x \in \mathbb{R}$ with $|x - c_*t| \leq t^\eta$ for t large enough, that is,

$$u(t, x) \leq \varphi_{c_*} \left(x - c_*t + \frac{3}{2\lambda_*} \ln t + b \right), \quad (2.20)$$

uniformly in $x \in \mathbb{R}$ with $|x - c_*t| \leq t^\eta$ for t large enough, which gives the upper bound in Proposition 2.1.

Step 2: Lower bound. The proof of this part is similar to Step 1. We sketch it for the sake of completeness. Let $\kappa \underline{v}(t, x)$, where $\underline{v}(t, x)$ is given in (2.14), be a subsolution of (2.3) for $t > 0$ and $x \in \mathbb{R}$. Define a function $\gamma(t, x)$ by

$$\gamma(t, x) = e^{\lambda_*(x - c_*t)} \varphi_{c_*} \left(x - c_*t + \frac{3}{2\lambda_*} \ln t + a \right) \quad \text{for } t \text{ large enough, } |x - c_*t| \leq t^\eta.$$

Here, we fix $a \in \mathbb{R}$ such that $\gamma(t, x) = \kappa \underline{v}(t - T_0, x)$ for t large enough and $x \in \mathbb{R}$ with $x - c_*t = t^\eta$. It is also noticed that $a > b$ since the relation

$$\gamma(t, x) = \kappa \underline{v}(t - T_0, x) < v(t - T_0 + 1, x) < K \bar{v}(t + 1, x) = \psi(t, x)$$

for t large enough and $x \in \mathbb{R}$ with $x - c_*t = t^\eta$ implies that $\varphi_{c_*} \left(x - c_*t + \frac{3}{2\lambda_*} \ln t + a \right) < \varphi_{c_*} \left(x - c_*t + \frac{3}{2\lambda_*} \ln t + b \right)$ for t large enough and $x \in \mathbb{R}$ with $x - c_*t = t^\eta$.

Analogously to the previous step, substituting $\gamma(t, x)$ into the equation of v yields

$$\gamma_t - \gamma_{xx} + c_*\gamma_x + R(t, x; \gamma) < O \left(t^{-(1-\eta)} \right)$$

for t large enough and $x \in \mathbb{R}$ with $|x - c_*t| \leq t^\eta$.

Set $z(t, x) := (v(t, x) - \gamma(t, x))^-$, being the negative part of $v(t, x) - \gamma(t, x)$. Then, the function z satisfies

$$\begin{cases} z_t - z_{xx} + c_* z_x + H(t, x; z) < \mathcal{O}\left(t^{-(1-\eta)}\right), & |x - c_* t| \leq t^\eta, \\ z(t, x) = \mathcal{O}\left(e^{-\lambda_* t^\eta}\right), & x - c_* t = -t^\eta, \\ z(t, x) = 0, & x - c_* t = t^\eta, \end{cases} \quad (2.21)$$

for t large enough. Here, $H(t, x; z) = 0$ when $z(t, x) = 0$; otherwise,

$$\begin{aligned} H(t, x; z) &= R(t, x; v) - R(t, x; \varphi) \\ &= f'(0)z - e^{\lambda_*(x-c_*t)} \left(f(e^{-\lambda_*(x-c_*t)} v) - f(e^{-\lambda_*(x-c_*t)} \varphi) \right) \\ &= f'(0)z - h(t, x)z \geq 0, \end{aligned}$$

in which $h(t, x)$ is continuous and bounded in L^∞ norm by $f'(0)$ since $0 < f(s) \leq f'(0)s$ for $s \in (0, 1)$. Following the proof of (2.19) in Step 1, one can show that, there holds

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}, |x - c_* t| \leq t^\eta} z(t, x) = 0. \quad (2.22)$$

Then the comparison principle implies that $v(t, x) \geq \gamma(t, x)$ uniformly in $x \in \mathbb{R}$ with $|x - c_* t| \leq t^\eta$ for t large enough, namely,

$$u(t, x) \geq \varphi_{c_*} \left(x - c_* t + \frac{3}{2\lambda_*} \ln t + a \right), \quad (2.23)$$

uniformly in $x \in \mathbb{R}$ with $|x - c_* t| \leq t^\eta$ for t large enough, which gives the lower bound and concludes the proof of Proposition 2.1.

3 The linearized problem on \mathbb{Z}

One of the key message from Section 2 in the continuous case is that solutions $w(t, x)$ of the linear equation (2.5) starting from odd and compactly supported initial conditions play a crucial role in designing accurate upper and lower barriers for the full nonlinear problem. In fact, the asymptotic expansion (2.7) is the corner stone of the proof. We will dedicate all our efforts in proving an equivalent expansion in the discrete case. More precisely, we consider the linearized problem

$$\begin{cases} r'_j(t) = r_{j-1}(t) - 2r_j(t) + r_{j+1}(t) + f'(0)r_j(t), & t > 0, \quad j \in \mathbb{Z}, \\ r_j(0) = r_j^0, & j \in \mathbb{Z}, \end{cases} \quad (3.1)$$

for some nontrivial bounded initial sequence $(r_j^0)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. First, we perform the change of variable $r_j(t) = e^{-\lambda_*(j-c_*t)} w_j(t)$ for some new sequence $(w_j(t))_{j \in \mathbb{Z}}$, which now satisfies

$$w'_j(t) = e^{\lambda_*} (w_{j-1}(t) - 2w_j(t) + w_{j+1}(t)) - c_*(w_{j+1}(t) - w_j(t)), \quad t > 0, \quad j \in \mathbb{Z}. \quad (3.2)$$

And we recall that (c_*, λ_*) are defined through (1.7).

Our aim in this section is to study the temporal Green's function $(\mathcal{G}_j(t))_{j \in \mathbb{Z}}$ which is solution of (3.2) starting from the initial sequence $\boldsymbol{\delta}$ which is defined as $\delta_j = 1$ if $j = 0$ and $\delta_j = 0$ otherwise, that is

$$\begin{cases} \mathcal{G}'_j(t) = e^{\lambda_*} (\mathcal{G}_{j-1}(t) - 2\mathcal{G}_j(t) + \mathcal{G}_{j+1}(t)) - c_*(\mathcal{G}_{j+1}(t) - \mathcal{G}_j(t)), & t > 0, \quad j \in \mathbb{Z}. \\ \mathcal{G}_j(0) = \delta_j, & j \in \mathbb{Z}. \end{cases} \quad (3.3)$$

Unlike the continuous case, there does not exist explicit representation formula for the temporal Green's function $(\mathcal{G}_j(t))_{j \in \mathbb{Z}}$. Nevertheless, we will obtain pointwise estimates for each $t > 1$ and $j \in \mathbb{Z}$, see Propositions 3.9 and 4.1. Roughly speaking, in a first step, we will prove in Proposition 3.9 that for $|j - c_*t| > \theta_*t$ the temporal Green's function is both exponentially localized in space and time, whereas for $|j - c_*t| \leq \theta_*t$ it behaves as a Gaussian profile centered at $j = c_*t$, for t large enough and some universal constant $\theta_* > 0$. However, such a generalized Gaussian estimate will not be enough to obtain an equivalent asymptotic expansion of (2.7) in our discrete setting. We thus need to refine our asymptotics and this is the key result of Proposition 4.1 where we provide a full asymptotic expansion of the temporal Green's function up to Gaussian corrections of order $t^{-3/2}$. We really enforce that this result is of independent interest and should be compared to existing results in the fully discrete case for discrete convolution powers [12–14] and to local limit theorems in probability theory [29]. Eventually, we will show in Section 4.2 how sharp asymptotics of $w_j(t)$ emanating from odd and finitely supported initial data can be derived from this improved result.

3.1 The spatial Green's function

The starting point of our approach is the study of the so-called spatial Green's function which we now introduce. Let \mathcal{L} denote the operator acting on a sequence $\mathbf{r} = (r_j)_{j \in \mathbb{Z}}$ as

$$\forall j \in \mathbb{Z}, \quad (\mathcal{L}\mathbf{r})_j := e^{\lambda_*} (r_{j-1} - 2r_j + r_{j+1}) - c_*(r_{j+1} - r_j), \quad (3.4)$$

then its spectrum on $\ell^2(\mathbb{Z})$ is given by the parametrized closed curve

$$\sigma(\mathcal{L}) = \left\{ e^{\lambda_*} \left(e^{-i\xi} - 2 + e^{i\xi} \right) - c_*(e^{-i\xi} - 1) \mid \xi \in [-\pi, \pi] \right\}.$$

For each $\mu \in \sigma(\mathcal{L})$, we have $\operatorname{Re}(\mu) = (e^{\lambda_*} + e^{-\lambda_*}) (\cos(\xi) - 1) \leq 0$ such that the curve $\sigma(\mathcal{L})$ touches the imaginary axis only at the origin and is located in the left half plane otherwise. The resolvent set of \mathcal{L} is given by the union of two sets defined as the interior and the exterior of $\sigma(\mathcal{L})$. We denote by \mathcal{U} the exterior which at least contains the set $\{\nu \in \mathbb{C} \mid \operatorname{Re}(\nu) \geq 0\} \setminus \{0\}$. For each $\nu \in \mathcal{U}$, we can define the spatial Green's function as the sequence $\mathbf{G}(\nu) \in \ell^2(\mathbb{Z})$ solution of

$$(\nu \operatorname{Id} - \mathcal{L}) \mathbf{G}(\nu) = \boldsymbol{\delta}, \quad (3.5)$$

where we recall that the sequence $\boldsymbol{\delta}$ is defined as $\delta_j = 1$ if $j = 0$ and $\delta_j = 0$ otherwise, and that we have denoted Id the identity operator acting on $\ell^2(\mathbb{Z})$. Next, we introduce the vector $\mathbf{W}_j(\nu) = (\mathbf{G}_{j-1}(\nu), \mathbf{G}_j(\nu))^{\mathbf{T}}$ for $j \in \mathbb{Z}$ such that the above relation (3.5) rewrites

$$\mathbf{W}_{j+1}(\nu) = \mathbb{A}(\nu) \mathbf{W}_j(\nu) - e^{\lambda_*} \delta_j \mathbf{e}, \quad j \in \mathbb{Z}, \quad (3.6)$$

where $\mathbf{e} := (0, 1)^{\mathbf{T}}$ and the matrix $\mathbb{A}(\nu)$ is given by

$$\mathbb{A}(\nu) := \begin{pmatrix} 0 & 1 \\ -e^{2\lambda_*} & \frac{\nu + e^{\lambda_*} + e^{-\lambda_*}}{e^{-\lambda_*}} \end{pmatrix}.$$

The eigenvalues of the above matrix are given by

$$\rho_{\pm}(\nu) = \frac{\nu + e^{\lambda_*} + e^{-\lambda_*}}{2e^{-\lambda_*}} \pm \frac{\sqrt{(\nu + e^{\lambda_*} + e^{-\lambda_*})^2 - 4}}{2e^{-\lambda_*}}. \quad (3.7)$$

For each $\nu \in \mathcal{U}$, we readily remark that we have the spectral splitting $|\rho_+(\nu)| > 1$ and $|\rho_-(\nu)| < 1$. With these notations in hands, we start to estimate the spatial Green's function away from the tangency point at the origin.

Lemma 3.1 (Local bounds). *Let $\underline{\nu} \in \mathcal{U}$ and $\delta > 0$ small enough such that $B_{\delta}(\underline{\nu}) \subset \mathcal{U}$. There exist two constants $C > 0$ and $\beta > 0$ such that*

$$\forall \nu \in B_{\delta}(\underline{\nu}), \quad \forall j \in \mathbb{Z}, \quad |\mathbf{G}_j(\nu)| \leq C e^{-\beta|j|}.$$

Proof. We let $\underline{\nu} \in \mathcal{U}$ be fixed. We know that the matrix $\mathbb{A}(\nu)$ above is well defined and holomorphic in a sufficiently small neighborhood of $\underline{\nu}$. Since we have an explicit expression of the two eigenvalues of the matrix $\mathbb{A}(\nu)$, we trivially have a consistent splitting for all ν close to $\underline{\nu}$. We define $\mathbb{E}^{s,u}(\nu)$ the stable/unstable subspaces of $\mathbb{A}(\nu)$ for all ν close to $\underline{\nu}$. As a consequence, for $\delta > 0$ small enough, we have the decomposition

$$\mathbb{C}^2 = \mathbb{E}^s(\nu) \oplus \mathbb{E}^u(\nu), \quad \forall \nu \in B_{\delta}(\underline{\nu}).$$

And we denote $\pi^{s,u}(\nu)$ the associated projectors, which are given by contour integrals. For instance, we have

$$\pi^s(\nu) = \frac{1}{2\pi\mathbf{i}} \int_{\gamma} (zI_2 - \mathbb{A}(\nu))^{-1} dz,$$

where γ is a contour that encloses the stable eigenvalue $\rho_-(\nu)$ and $I_2 \in \mathcal{M}_2(\mathbb{R})$ is the 2×2 identity matrix. A similar formula holds for $\pi^u(\nu)$, with this time a contour that encloses the unstable eigenvalue $\rho_+(\nu)$. This shows that $\pi^{s,u}(\nu)$ depends holomorphically on ν in $B_{\delta}(\underline{\nu})$ and, consequently, the stable and unstable subspaces $\mathbb{E}^{s,u}(\nu)$ also depend holomorphically on ν in $B_{\delta}(\underline{\nu})$.

Up to taking δ even smaller, we can ensure that any $\nu \in B_{\delta}(\underline{\nu})$ lies in the exterior \mathcal{U} of the resolvent set of \mathcal{L} , hence there exists a unique sequence $(\mathbf{W}_j(\nu))_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ solution to (3.6). We can easily compute the stable and unstable components of each $\mathbf{W}_j(\nu)$ for $j \in \mathbb{Z}$ as

$$\pi^s(\nu) \mathbf{W}_j(\nu) = -e^{\lambda_*} \sum_{\ell=-\infty}^{j-1} \delta_{\ell} \rho_-(\nu)^{j-1-\ell} \pi^s(\nu) \mathbf{e}, \quad j \in \mathbb{Z},$$

for the stable component, and

$$\pi^u(\nu) \mathbf{W}_j(\nu) = e^{\lambda_*} \sum_{\ell=0}^{\infty} \delta_{j+\ell} \rho_+(\nu)^{-1-\ell} \pi^u(\nu) \mathbf{e}, \quad j \in \mathbb{Z},$$

for the unstable one. It is important to note that the sequence δ has only one nonzero coefficient at $j = 0$. Hence, we deduce that

$$\pi^s(\nu)\mathbf{W}_j(\nu) = \begin{cases} 0 & \text{for } j \leq 0, \\ -e^{\lambda_*}\rho_-(\nu)^{j-1}\pi^s(\nu)\mathbf{e} & \text{for } j \geq 1, \end{cases}$$

while

$$\pi^u(\nu)\mathbf{W}_j(\nu) = \begin{cases} 0 & \text{for } j \geq 1, \\ e^{\lambda_*}\rho_+(\nu)^{-1+j}\pi^u(\nu)\mathbf{e} & \text{for } j \leq 0. \end{cases}$$

Next, we remark that there exists $\beta > 0$ such that

$$\forall \nu \in B_\delta(\underline{\nu}), \quad |\rho_-(\nu)| \leq e^{-\beta}, \quad \text{and} \quad |\rho_+(\nu)| \geq e^\beta.$$

As a consequence, we have

$$\forall \nu \in B_\delta(\underline{\nu}) \quad |\pi^s(\nu)\mathbf{W}_j(\nu)| \leq \begin{cases} 0 & \text{for } j \leq 0, \\ Ce^{-\beta(j-1)} & \text{for } j \geq 1, \end{cases} \quad (3.8)$$

while

$$\forall \nu \in B_\delta(\underline{\nu}) \quad |\pi^u(\nu)\mathbf{W}_j(\nu)| \leq \begin{cases} 0 & \text{for } j \geq 1, \\ Ce^{\beta(j+1)} & \text{for } j \leq 0, \end{cases} \quad (3.9)$$

where $C > 0$ is a uniform bound on $B_\delta(\underline{\nu})$ of $|e^{\lambda_*}\pi^{s,u}(\nu)\mathbf{e}|$. Finally, adding (3.8) and (3.9) concludes the proof of the lemma, since the spatial Green's function $\mathbf{G}_j(\nu)$ is the second coordinate of the vector $\mathbf{W}_j(\nu) \in \mathbb{C}^2$. \square

Lemma 3.2 (Bounds at infinity). *There exist a radius $R > 1$ and two constants $C > 0$ and $\kappa > 0$ such that $\{\nu \in \mathbb{C} \mid |\nu| \geq R\} \subset \mathcal{U}$ and*

$$\forall |\nu| \geq R, \quad \forall j \in \mathbb{Z}, \quad |\mathbf{G}_j(\nu)| \leq Ce^{-\kappa|j|}.$$

Proof. The proof is similar to Lemma 3.1 and simply relies on the fact that from their definition (3.7) we have

$$\lim_{|\nu| \rightarrow +\infty} |\nu| |\rho_-(\nu)| = e^{\lambda_*} \quad \text{and} \quad \lim_{|\nu| \rightarrow +\infty} \frac{|\rho_+(\nu)|}{|\nu|} = e^{\lambda_*},$$

together with our expression for $\pi^{s,u}(\nu)\mathbf{W}_j(\nu)$ when $\nu \in \mathcal{U}$. \square

Next, we recall that $\sigma(\mathcal{L})$ is tangent to the imaginary axis at the origin. This traduces as $\rho_-(0) = 1$ with $\rho_+(0) = e^{2\lambda_*} > 1$. For $|\nu| \rightarrow 0$, we also note that $\rho_-(\nu)$ has the following expansion

$$\rho_-(\nu) = 1 - \frac{\nu}{c_*} + \frac{\nu^2}{c_*^2 e^{-\lambda_*}} + \mathcal{O}(|\nu|^3). \quad (3.10)$$

It is thus important to examine the behavior of the spatial Green's function close to the origin. Let us remark that the spatial Green's function is well-defined in $B_\delta(0) \cap \mathcal{U}$ for any radius $\delta > 0$ thanks to Lemma 3.1. Our goal is to extend holomorphically $\mathbf{G}_j(\nu)$ to a whole neighborhood of $\nu = 0$ which amounts to passing through the essential spectrum of \mathcal{L} .

Lemma 3.3 (Bounds close to the origin). *There exist some $\epsilon > 0$ and constants $C > 0$ and $\omega > 0$ such that for any $j \in \mathbb{Z}$, the component $\mathbf{G}_j(\nu)$ defined on $B_\epsilon(0) \cap \mathcal{U}$ extends holomorphically to the whole ball $B_\epsilon(0)$ with respect to ν , and the holomorphic extension (still denoted by $\mathbf{G}_j(\nu)$) satisfies the bounds*

$$\forall \nu \in B_\epsilon(0), \quad \forall j \leq 0, \quad |\mathbf{G}_j(\nu)| \leq C e^{-\omega|j|},$$

and

$$\forall \nu \in B_\epsilon(0), \quad \forall j \geq 1, \quad \left| \mathbf{G}_j(\nu) - \frac{1}{c_*} e^{\varpi(\nu)j} \right| \leq C e^{\operatorname{Re}(\varpi(\nu))j} |\nu|,$$

where

$$\varpi(\nu) := \ln(\rho_-(\nu)) = -\frac{\nu}{c_*} + \frac{\cosh(\lambda_*)}{c_*^3} \nu^2 + \mathcal{O}(|\nu|^3), \quad \forall \nu \in B_\epsilon(0), \quad (3.11)$$

is holomorphic in the ball $B_\epsilon(0)$.

Proof. Most ingredients are similar to the ones we already used in the proof of Lemma 3.1. We just need to adapt our notation, since the hyperbolic dichotomy of $\mathbb{A}(\nu)$ does not hold any longer in a whole neighborhood of the origin $\nu = 0$. We readily see that $\rho_-(\nu)$ extends holomorphically to a neighborhood of $\nu = 0$, with corresponding eigenvector

$$\mathbb{E}^0(\nu) := \begin{pmatrix} 1 \\ \rho_-(\nu) \end{pmatrix} \in \mathbb{C}^2,$$

which also depends holomorphically on ν in a neighborhood of $\nu = 0$. Note that $\mathbb{E}^0(\nu)$ contributes to the stable subspace of $\mathbb{A}(\nu)$ for those $\nu \in \mathcal{U}$ close to $\nu = 0$ but the situation is less clear as ν passes the essential spectrum. Nevertheless, we still have the decomposition

$$\mathbb{C}^2 = \mathbb{E}^0(\nu) \oplus \mathbb{E}^u(\nu), \quad \forall \nu \in B_\epsilon(0),$$

for a sufficiently small $\epsilon > 0$. We denote by $\pi^0(\nu)$ and $\pi^u(\nu)$ the holomorphic projectors associated with the above decomposition. We readily see that for each $\nu \in B_\epsilon(0) \cap \mathcal{U}$, we can reuse the expressions derived in Lemma 3.1 to write

$$\pi^0(\nu) \mathbf{W}_j(\nu) = -e^{\lambda_*} \sum_{\ell=-\infty}^{j-1} \delta_{\ell} \rho_-(\nu)^{j-1-\ell} \pi^0(\nu) \mathbf{e}, \quad j \in \mathbb{Z},$$

and

$$\pi^u(\nu) \mathbf{W}_j(\nu) = e^{\lambda_*} \sum_{\ell=0}^{\infty} \delta_{j+\ell} \rho_+(\nu)^{-1-\ell} \pi^u(\nu) \mathbf{e}, \quad j \in \mathbb{Z}.$$

The unstable component $\pi^u(\nu) \mathbf{W}_j(\nu)$ obviously extends holomorphically to the whole ball $B_\epsilon(0)$, as a consequence, we get

$$|\pi^u(\nu) \mathbf{W}_j(\nu)| \leq C e^{-\omega|j|}, \quad \forall \nu \in B_\epsilon(0), \quad \forall j \leq 0,$$

for appropriate constants $C > 0$ and $\omega > 0$. We now focus on the vector $\pi^0(\nu) \mathbf{W}_j(\nu)$ and readily remark that it also extends holomorphically to the whole ball $B_\epsilon(0)$, and we deduce that

$$\forall \nu \in B_\epsilon(0), \quad \mathbf{G}_j(\nu) = \langle \pi^0(\nu) \mathbf{W}_j(\nu), \mathbf{e} \rangle_{\mathbb{C}^2} = \begin{cases} 0 & \text{for } j \leq 0, \\ \Psi(\nu) e^{\varpi(\nu)j} & \text{for } j \geq 1, \end{cases} \quad (3.12)$$

where we have set

$$\Psi(\nu) := -e^{\lambda_*} \frac{\langle \pi^0(\nu) \mathbf{e}, \mathbf{e} \rangle_{\mathbb{C}^2}}{e^{\varpi(\nu)}}, \quad \forall \nu \in B_\epsilon(0). \quad (3.13)$$

Note that eventually upon reducing the size of ϵ , we can always ensure that $\varpi(\nu) = \ln(\rho_-(\nu))$ is holomorphic on $B_\epsilon(0)$, and thus Ψ is also holomorphic on $B_\epsilon(0)$ with $\Psi(0) = \frac{1}{c_*} > 0$. Indeed, the projection $\pi^0(\nu)$ can be expressed as

$$\pi^0(\nu) = \frac{1}{\rho_+(\nu) - \rho_-(\nu)} \left\langle \cdot, \begin{pmatrix} \rho_+(\nu) \\ -1 \end{pmatrix} \right\rangle_{\mathbb{C}^2} \begin{pmatrix} 1 \\ \rho_-(\nu) \end{pmatrix},$$

such that

$$\Psi(0) = -e^{\lambda_*} \langle \pi^0(0) \mathbf{e}, \mathbf{e} \rangle_{\mathbb{C}^2} = \frac{e^{\lambda_*}}{e^{2\lambda_*} - 1} = \frac{1}{c_*} > 0.$$

Here, we denoted by $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ the usual scalar product on \mathbb{C}^2 . □

Lemma 3.4 (Intermediate bounds). *There exist $C > 0$ and $\beta_0 > 0$ such that*

$$\forall \nu \in \mathcal{U} \cap (B_R(0) \setminus B_\epsilon(0)), \quad |\mathbf{G}_j(\nu)| \leq C \begin{cases} e^{-\beta_0|j|}, & \text{for } j \leq 0, \\ 1, & \text{for } j \geq 1. \end{cases}$$

Proof. The set $\mathcal{U} \cap (B_R(0) \setminus B_\epsilon(0))$ is a compact domain of \mathbb{C} . As a consequence, we can obtain uniform constants from Lemma 3.1 by using a simple compactness argument. □

3.2 The temporal Green's function

In this section, we study the temporal Green's function $(\mathcal{G}_j(t))_{j \in \mathbb{Z}}$ solution of (3.3) starting from the initial sequence δ . We use inverse Laplace transform to express $(\mathcal{G}_j(t))_{j \in \mathbb{Z}}$ as the following contour integral

$$\forall t > 0, \quad \forall j \in \mathbb{Z}, \quad \mathcal{G}_j(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\nu t} \mathbf{G}_j(\nu) d\nu,$$

where $\Gamma \subset \mathbb{C}$ is some well-chosen contour in the complex plane which does not intersect the spectrum of \mathcal{L} . For example, one can take the sectorial contour $\Gamma = \{\gamma_0 - \gamma_1|\xi| + i\xi \mid \xi \in \mathbb{R}\}$ for some well chosen $\gamma_{0,1} > 0$. Our aim is to obtain pointwise bounds on $\mathcal{G}_j(t)$ for $t \geq 1$ and we will divide the analysis into several cases.

3.2.1 Exponential pointwise estimates away from $j \sim c_* t$

In the next lemmas, we show that the pointwise temporal Green's function $\mathcal{G}_j(t)$ decays exponentially in time and space whenever $j - c_* t$ is sufficiently large and $t \geq 1$.

Lemma 3.5. *There exists a constant $L > 4R/\kappa$ with $R > 1$ chosen in Lemma 3.2, such that for $|j| > Lt$ and $t \geq 1$, one has the estimate*

$$|\mathcal{G}_j(t)| \leq C e^{-\kappa_1 t - \kappa_2 |j|},$$

for some $C > 0$ and $\kappa_{1,2} > 0$.

Proof. We let $|j| > Lt$ and $t \geq 1$. We recall the estimate of Lemma 3.2

$$\forall \nu \geq R, \quad \forall j \in \mathbb{Z}, \quad |\mathbf{G}_j(\nu)| \leq Ce^{-\kappa|j|}.$$

Let $L > 4R/\kappa$ and we take the contour which is defined as

$$\Gamma = \{2R - \delta|\xi| + \mathbf{i}\xi \mid \xi \in \mathbb{R}\},$$

with $0 < \delta < \sqrt{3}$ small enough such that $|\nu| > R$ for all $\nu \in \Gamma$. As a consequence, we obtain that

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \leq Ce^{2Rt - \kappa|j|} \int_0^{\infty} e^{-\delta\xi t} d\xi \leq C_0 \frac{e^{2Rt - \kappa|j|}}{t}.$$

Since $|j| > Lt$, we have that

$$2Rt - \kappa|j| < \underbrace{\left(2R - \frac{\kappa L}{2}\right)}_{<0} t - \frac{\kappa}{2}|j|,$$

and the result follows. \square

From now on, we will always assume that $|j| \leq Lt$ and $t \geq 1$. We first deal with the case where $j \leq 0$.

Lemma 3.6. *There are some $C > 0$ and $\beta_{1,2} > 0$ such that for any $-Lt \leq j \leq 0$ with $t \geq 1$, one has the bound*

$$|\mathcal{G}_j(t)| \leq Ce^{-\beta_1 t - \beta_2 |j|}.$$

Proof. We consider Γ as the union of Γ_1 and Γ_2 where Γ_1 is a vertical line within the ball $B_{\epsilon}(0)$ and Γ_2 is given as $\Gamma_2 = \{-\delta|\xi| + \mathbf{i}\xi \mid \xi \in \mathbb{R}, |\xi| \geq \xi_*\}$. More precisely, there exists $\eta \in (0, \epsilon)$ small enough such that $-\eta \pm \mathbf{i}\sqrt{\epsilon^2 - \eta^2} \in \mathcal{U}$. We define $\Gamma_1 = \{-\eta + \mathbf{i}\xi \mid \xi \in \mathbb{R}, |\xi| \leq \sqrt{\epsilon^2 - \eta^2}\} \subset B_{\epsilon}(0)$ and $\delta = \frac{\eta}{\sqrt{\epsilon^2 - \eta^2}}$ with $\xi_* = \sqrt{\epsilon^2 - \eta^2}$ in the definition of Γ_2 . We have

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_1} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \leq Ce^{-\eta t - \omega|j|},$$

using the estimate on $\mathbf{G}_j(\nu)$ from Lemma 3.3. We denote $\Gamma_2^{in} = \Gamma_2 \cap B_R(0)$ and $\Gamma_2^{out} = \Gamma_2 \setminus \Gamma_2^{in}$. On Γ_2^{in} , we use Lemma 3.4

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_2^{in}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \leq \frac{C}{t} e^{-\eta t - \beta_0 |j|},$$

and on Γ_2^{out} we use the bound of Lemma 3.2 to get

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_2^{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \leq \frac{C}{t} e^{-\eta t - \beta |j|}.$$

\square

Actually, the above estimate can be easily extended for $j \in \mathbb{Z}$ with $1 \leq j \leq (c_* - \theta)t$ for any $\theta \in (0, c_*)$.

Lemma 3.7. *There are some $C > 0$ and $\beta_{1,2} > 0$ such that for any $1 \leq j \leq (c_* - \theta)t$ with $\theta \in (0, c_*)$ and $t \geq 1$ such that $(c_* - \theta)t \geq 1$, one has the bound*

$$|\mathcal{G}_j(t)| \leq C e^{-\beta_1 t - \beta_2 |j|}.$$

Proof. We consider the same contours as in the previous Lemma. On Γ_2^{out} , we have the exact same estimates since the spatial Green's function enjoys the same pointwise bound. On Γ_2^{in} , we note that we have

$$\left| \frac{1}{2\pi i} \int_{\Gamma_2^{in}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \leq \frac{C}{t} e^{-\eta t},$$

and as $j \leq (c_* - \theta)t$ we have $-\eta t \leq -\frac{\eta}{2}t - \frac{\eta}{2(c_* - \theta)}|j|$. The main difference, is that now, on Γ_1 , we have

$$\forall \nu \in \Gamma_1, \quad |\mathbf{G}_j(\nu)| \leq C e^{\operatorname{Re}(\varpi(\nu))j},$$

from Lemma 3.3. We denote by Γ_1^\pm the portion of Γ_1 where $\operatorname{Re}(\varpi(\nu))$ is respectively positive and negative. On Γ_1^- , we readily obtain

$$\left| \frac{1}{2\pi i} \int_{\Gamma_1^-} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \leq C e^{-\eta t}.$$

Next, from the expansion (3.11), we get that

$$\forall \nu \in B_\epsilon(0), \quad \operatorname{Re}(\varpi(\nu)) = -\frac{\operatorname{Re}(\nu)}{c_*} + \frac{\cosh(\lambda_*)}{c_*^3} (\operatorname{Re}(\nu)^2 - \operatorname{Im}(\nu)^2) + \mathcal{O}(|\nu|^3),$$

such we can solve $\operatorname{Re}(\varpi(\nu)) = 0$ for $\operatorname{Re}(\nu)$ as a function of $\operatorname{Im}(\nu)$ and we get

$$\operatorname{Re}(\varpi(\nu)) = 0 \Leftrightarrow \operatorname{Re}(\nu) = -\frac{\cosh(\lambda_*)}{c_*^2} \operatorname{Im}(\nu)^2 + \mathcal{O}(|\operatorname{Im}(\nu)|^3) \text{ for } \nu \in B_\epsilon(0).$$

This implies that there exists $\beta_0 > 0$ such that

$$|\operatorname{Im}(\nu)| \leq \beta_0 \sqrt{\eta}, \quad \nu \in \Gamma_1^+.$$

We deduce that

$$\operatorname{Re}(\varpi(\nu)) \leq \frac{\eta}{c_*} + \beta_1 \eta^{3/2},$$

for some $\beta_1 > 0$ and any $\nu \in \Gamma_1^+$. As a consequence, we can derive the following bound

$$-\eta t + j \operatorname{Re}(\varpi(\nu)) \leq \eta t \left(-1 + \frac{j}{c_* t} + \beta_1 \frac{\sqrt{\eta} j}{t} \right) \leq \eta t \left(-\frac{\theta}{c_*} + \beta_1 (c_* - \theta) \sqrt{\eta} \right).$$

We can always reduce η such that $\beta_1 (c_* - \theta) \sqrt{\eta} < \frac{\theta}{2c_*}$ and we get

$$\left| \frac{1}{2\pi i} \int_{\Gamma_1^+} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \leq C e^{-\frac{\eta \theta}{2c_*} t}.$$

□

3.2.2 Generalized Gaussian estimate

We are thus now let to study the case where $(c_* - \theta)t \leq j \leq Lt$ with $t \geq 1$ and the main result of this section is a generalized Gaussian estimate for the Green's function which reads as follows. Let us already note that the large constant $L > 0$ from Lemma 3.5 can be fixed large enough such that it further verifies $L > c_*$, which we assume throughout the sequel.

Lemma 3.8 (Generalized Gaussian estimate). *There exist two constants $C > 0$ and $\beta > 0$ such that for any $(c_* - \theta)t \leq j \leq Lt$ with $j \in \mathbb{Z}$, $\theta \in (0, c_*)$ and $t \geq 1$, the temporal Green's function satisfies the estimate*

$$|\mathcal{G}_j(t)| \leq \frac{C}{\sqrt{t}} \exp\left(-\beta \frac{(j - c_*t)^2}{t}\right).$$

Before proceeding with the proof, we need to introduce some notations. First, from Lemma 3.3, one can find $\epsilon_* > 0$ and two constants $0 < \beta_* < \cosh(\lambda_*) < \beta^*$ such that for each $\epsilon \in (0, \epsilon_*)$ there exist constants $C > 0$ and $\omega > 0$ such that

$$\forall \nu \in B_\epsilon(0), \quad |\mathbf{G}_j(\nu)| \leq \begin{cases} Ce^{-\omega|j|}, & j \leq 0, \\ Ce^{\operatorname{Re}(\varpi(\nu))j}, & j \geq 1, \end{cases}$$

together with the bound

$$\operatorname{Re}(\varpi(\nu)) \leq -\frac{1}{c_*} \operatorname{Re}(\nu) + \frac{\beta^*}{c_*^3} \operatorname{Re}(\nu)^2 - \frac{\beta_*}{c_*^3} \operatorname{Im}(\nu)^2, \quad \forall \nu \in B_\epsilon(0), \quad (3.14)$$

where ϖ is holomorphic in $B_\epsilon(0)$ and given in Lemma 3.3. Note that the bound (3.14) and the existence of the constants $0 < \beta_* < \cosh(\lambda_*) < \beta^*$ directly come from the expansion (3.11) of ϖ . Note also that β_*, β^* only depend on $\epsilon_* > 0$ and are uniform in $\epsilon \in (0, \epsilon_*)$. Next, following the strategy developed in [13, 34], we introduce a family of parametrized curves of the form

$$\Gamma_p := \left\{ \operatorname{Re}(\nu) - \frac{\beta^*}{c_*^2} \operatorname{Re}(\nu)^2 + \frac{\beta_*}{c_*^2} \operatorname{Im}(\nu)^2 = \psi(p) \mid -\eta \leq \operatorname{Re}(\nu) \leq p \right\}, \quad (3.15)$$

with $\psi(p) := p - \frac{\beta^*}{c_*^2} p^2$. We readily note that the curve Γ_p intersects the real axis at $\nu = p$. We now explain how we select $\eta > 0$ and $p > -\eta$ in the above definition of Γ_p .

First, for each $\epsilon \in (0, \epsilon_*)$, we denote by $\nu_\epsilon, \bar{\nu}_\epsilon \in \partial B_\epsilon(0) \cap \sigma(\mathcal{L})$ and let $\operatorname{Re}(\nu_\epsilon) = -\eta_\epsilon < 0$. Then, we fix $\eta > 0$ such that the curve Γ_p with $p = 0$ intersects the line $-\eta + \mathbf{i}\mathbb{R}$ inside the open ball $B_\epsilon(0)$ and readily note that $\eta \in (0, \eta_\epsilon)$. And next, we denote $\iota_\epsilon \in (0, \epsilon)$ the unique real number such that Γ_p with $p = \iota_\epsilon$ intersects the line $-\eta + \mathbf{i}\mathbb{R}$ precisely on the boundary of $B_\epsilon(0)$ with η fixed previously. Finally, we also introduce

$$\zeta := \frac{j - c_*t}{2t} \quad \text{and} \quad \gamma := \frac{j \beta^*}{t c_*^2} > 0,$$

and the specific value of p in the definition of Γ_p is fixed depending on the ratio ζ/γ as follows

$$p := \begin{cases} \frac{\zeta}{\gamma}, & \text{if } -\frac{\eta}{2} \leq \frac{\zeta}{\gamma} \leq \iota_\epsilon, \\ \iota_\epsilon, & \text{if } \frac{\zeta}{\gamma} > \iota_\epsilon, \\ -\frac{\eta}{2}, & \text{if } \frac{\zeta}{\gamma} < -\frac{\eta}{2}. \end{cases}$$

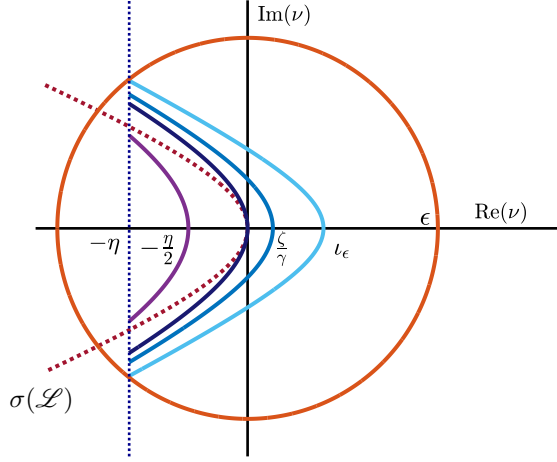


Figure 1: *Illustration of the geometry of the family of parametrized curved Γ_p within the ball $B_\epsilon(0)$ for different values of $p \in [-\eta/2, \nu_\epsilon]$, adapted from [13]. The extremal curves are given when $p = -\eta/2$ to the left (magenta curve) and when $p = \nu_\epsilon$ to the right (light blue curve) where $0 < \nu_\epsilon < \epsilon$ is precisely defined such that Γ_p with $p = \nu_\epsilon$ intersects the line $-\eta + i\mathbb{R}$ on the boundary of $B_\epsilon(0)$. The dashed dark red curve represents the spectrum $\sigma(\mathcal{L})$.*

The geometry of the family of parametrized curves Γ_p is illustrated in Figure 1 for different values of p . Note that with our careful choice of parametrization, we have that Γ_p with $p = 0$ (dark blue curve in Figure 1) lies to the right of the spectral curve with tangency at the origin.

Let us remark that the motivation for introducing such quantities comes from our above estimate on the Green's function. Indeed, for all $\nu \in \Gamma_p \subset B_\epsilon(0)$ and any $j \geq 1$, we have

$$\begin{aligned}
 j\operatorname{Re}(\varpi(\nu)) &\leq j \left(-\frac{1}{c_*} \operatorname{Re}(\nu) + \frac{\beta_*^*}{c_*^3} \operatorname{Re}(\nu)^2 - \frac{\beta_*}{c_*^3} \operatorname{Im}(\nu)^2 \right) = j \left(-\frac{\psi(p)}{c_*} \right) \\
 &= \frac{j}{c_*} \left(-p + \frac{\beta_*^*}{c_*^2} p^2 \right) = -tp + \frac{t}{c_*} (-2\zeta p + \gamma p^2) \\
 &= -tp - \frac{t\zeta^2}{c_*\gamma} + \frac{t}{c_*} \gamma \left(p - \frac{\zeta}{\gamma} \right)^2.
 \end{aligned} \tag{3.16}$$

Our precise choice of p , which depends on the ratio ζ/γ , will always allow us to control the above terms. Furthermore, since we consider here the range $(c_* - \theta)t \leq j \leq Lt$, we readily have that $-\frac{\theta}{2} \leq \zeta \leq \frac{L-c_*}{2}$, and our generalized Gaussian estimate will come from those $\zeta \sim 0$.

Before proceeding with the analysis, we make two final claims that will be useful in the course of the proof of Lemma 3.8.

1. We note that for each $\nu \in \Gamma_p$, with Γ_p defined in (3.15), there exists some constant $c_* > 0$ such that

$$\operatorname{Re}(\nu) \leq p - c_* \operatorname{Im}(\nu)^2, \tag{3.17}$$

since the function ψ , defined as $\psi(p) = p - \frac{\beta_*^*}{c_*^2} p^2$, satisfies $\psi'(0) = 1$, and for each $p \in [-\eta, \epsilon]$, one has $\psi'(t) \leq c_0$ for some $c_0 > 0$.

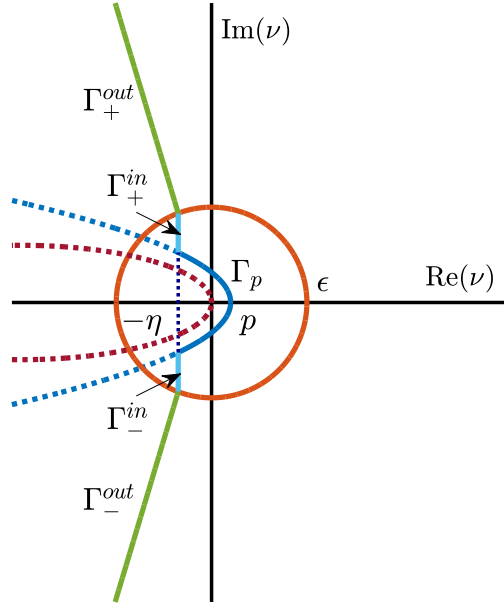


Figure 2: Illustration of the contour used in the case $-\frac{\eta}{2} \leq \frac{\zeta}{\gamma} \leq \iota_\epsilon$ when $(c_* - \theta) \leq j \leq Lt$ with $t \geq 1$. The contour is composed of $\Gamma_-^{out} \cup \Gamma_-^{in} \cup \Gamma_p \cup \Gamma_+^{in} \cup \Gamma_+^{out}$. The contours Γ_\pm^{in} are the portions of the line $-\eta + i\mathbb{R}$ which lie inside $B_\epsilon(0)$ (light blue curves) while the contours Γ_\pm^{out} are half lines lying in the resolvent set \mathcal{U} with $|\text{Im}(\nu)| \geq \sqrt{\epsilon^2 - \eta^2}$ (green curves). Finally, Γ_p is defined in (3.15) and intersects the real axis at p (blue curve).

- Up to further reducing the size of $\epsilon_* > 0$, a direct application of the implicit function theorem demonstrates that there exists a smooth function $\Phi : (-\epsilon_*, \epsilon_*) \times (-\epsilon_*, \epsilon_*) \rightarrow \mathbb{R}$ and some $C > 0$ such that for any $\epsilon \in (0, \epsilon_*)$ and $p \in (-\epsilon, \epsilon)$, the curve Γ_p can be parametrized as

$$\Gamma_p = \{\nu \in B_\epsilon(0) \mid \text{Re}(\nu) = \Phi(\text{Im}(\nu), p)\},$$

with

$$\text{Re}(\nu) = p - \frac{\beta^*}{c_*^2} \text{Im}(\nu)^2 + \mathcal{O}(|\text{Im}(\nu)|^3 + |p|^3), \quad (3.18)$$

together with

$$\left| \frac{\partial \Phi(\text{Im}(\nu), p)}{\partial \text{Im}(\nu)} \right| \leq C, \quad \text{for each } |\text{Im}(\nu)| \leq \epsilon \text{ and } |p| \leq \epsilon.$$

Finally, in what follows, we will use the notation $f \lesssim g$ whenever $f \leq Cg$ for some constant $C > 0$ independent of j and t .

We are now ready to prove Lemma 3.8.

Proof of Lemma 3.8. Throughout, we have $(c_* - \theta) \leq j \leq Lt$ with $t \geq 1$ and for each $\epsilon \in (0, \epsilon_*)$ the constant $\eta \in (0, \eta_\epsilon)$ has been fixed as explained above. We divide the analysis into three cases depending on the ratio ζ/γ .

Case: $-\frac{\eta}{2} \leq \frac{\zeta}{\gamma} \leq \iota_\epsilon$. We will consider a contour Γ which is the union of three contours given by Γ_p with $p = \zeta/\gamma$, together with Γ^{in} and Γ^{out} which are defined as follows. We refer to Figure 2 for a geometrical illustration. The contour Γ^{in} is composed of the two portions of the segment $\text{Re}(\nu) = -\eta$ which lie inside the ball $B_\epsilon(0)$, that is

$$\Gamma^{in} = \Gamma_-^{in} \cup \Gamma_+^{in} = \left\{ -\eta + \mathbf{i}\xi \mid -\sqrt{\epsilon^2 - \eta^2} \leq \xi \leq -\xi_* \right\} \cup \left\{ -\eta + \mathbf{i}\xi \mid \xi_* \leq \xi \leq \sqrt{\epsilon^2 - \eta^2} \right\},$$

with

$$\xi_* := \sqrt{\frac{c_*^2}{\beta_*} (\psi(p) - \psi(-\eta))} > 0.$$

On the other hand, the contour Γ^{out} is taken of the form

$$\Gamma^{out} = \Gamma_-^{out} \cup \Gamma_+^{out} = \left\{ -\delta_0 + \delta_1 \xi + \mathbf{i}\xi \mid \xi \leq -\sqrt{\epsilon^2 - \eta^2} \right\} \cup \left\{ -\delta_0 - \delta_1 \xi + \mathbf{i}\xi \mid \xi \geq \sqrt{\epsilon^2 - \eta^2} \right\},$$

for two constants $\delta_0 > 0$ and $\delta_1 > 0$ chosen such that $\Gamma^{out} \subset \mathcal{U}$ and Γ_\pm^{out} intersect Γ_\pm^{in} exactly on $\partial B_\epsilon(0)$. The later condition implies that $\delta_0 = \eta - \delta_1 \sqrt{\epsilon^2 - \eta^2}$, and then one can take $0 < \delta_1 < \frac{\eta}{\sqrt{\epsilon^2 - \eta^2}}$ as small as required such that both $\delta_0 > 0$ and $\Gamma^{out} \subset \mathcal{U}$ are verified. As a consequence, we have that

$$\mathcal{G}_j(t) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\zeta/\gamma}} e^{\nu t} \mathbf{G}_j(\nu) d\nu + \frac{1}{2\pi\mathbf{i}} \int_{\Gamma^{in}} e^{\nu t} \mathbf{G}_j(\nu) d\nu + \frac{1}{2\pi\mathbf{i}} \int_{\Gamma^{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu,$$

and we have to estimate the above three integrals. We start with the first one, which is the one that will produce the desired Gaussian estimate. For each $\nu \in \Gamma_p \subset B_\epsilon(0)$, we know from (3.16)-(3.17) that

$$\begin{aligned} t\text{Re}(\nu) + j\text{Re}(\varpi(\nu)) &\leq t(\text{Re}(\nu) - p) - \frac{t\xi^2}{c_*\gamma} + \frac{t}{c_*}\gamma \left(p - \frac{\zeta}{\gamma} \right)^2 \\ &\leq -c_*\text{Im}(\nu)^2 t - \frac{t\xi^2}{c_*\gamma} + \frac{t}{c_*}\gamma \left(p - \frac{\zeta}{\gamma} \right)^2 \end{aligned}$$

such that, with $p = \zeta/\gamma$, we obtain that

$$t\text{Re}(\nu) + j\text{Re}(\varpi(\nu)) \leq -c_*\text{Im}(\nu)^2 t - \frac{\zeta^2}{\gamma} \frac{t}{c_*}.$$

As a consequence, we have

$$\begin{aligned} \left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\zeta/\gamma}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| &\lesssim \int_{\Gamma_{\zeta/\gamma}} e^{t\text{Re}(\nu) + j\text{Re}(\varpi(\nu))} |d\nu| \\ &\lesssim e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}} \int_{\Gamma_{\zeta/\gamma}} e^{-c_*\text{Im}(\nu)^2 t} |d\nu| \\ &\lesssim e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}} \int_{-\xi_*}^{\xi_*} e^{-c_*\xi^2 t} d\xi \\ &\lesssim \frac{e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}}}{\sqrt{t}}. \end{aligned}$$

Finally, we remark that

$$-\frac{\zeta^2}{\gamma} \frac{t}{c_*} = -\frac{(j - c_*t)^2}{4t} \frac{1}{\gamma c_*} \leq -\frac{(j - c_*t)^2}{4t} \frac{c_*}{L\beta_*},$$

since $(c_* - \theta) \frac{\beta_*}{c_*^2} \leq \gamma \leq \frac{L\beta_*}{c_*^2}$ as $(c_* - \theta)t \leq j \leq Lt$, and we have obtained the desired Gaussian estimate.

Next, for all $\nu = -\eta + \mathbf{i}\xi \in \Gamma^{in} \subset B_\epsilon(0)$ and $j \geq 1$ we have that

$$\begin{aligned} j\operatorname{Re}(\varpi(\nu)) &\leq j \left(-\frac{1}{c_*} \operatorname{Re}(\nu) + \frac{\beta_*}{c_*^3} \operatorname{Re}(\nu)^2 - \frac{\beta_*}{c_*^3} \operatorname{Im}(\nu)^2 \right) = j \left(-\frac{\psi(-\eta)}{c_*} - \frac{\beta_*}{c_*^3} \xi^2 \right) \\ &= j \left(-\frac{\psi(p)}{c_*} - \frac{\beta_*}{c_*^3} \underbrace{(\xi^2 - \xi_*^2)}_{\geq 0} \right) \leq j \left(-\frac{p}{c_*} + \frac{\beta_*}{c_*^3} p^2 \right), \end{aligned}$$

since $\psi(-\eta) + \frac{\beta_*}{c_*^2} \xi_*^2 = \psi(p)$ by definition of $\xi_* > 0$. Thus, we have

$$\begin{aligned} t\operatorname{Re}(\nu) + j\operatorname{Re}(\varpi(\nu)) &\leq -\eta t + j \left(-\frac{p}{c_*} + \frac{\beta_*}{c_*^3} p^2 \right) \\ &= \frac{t}{c_*} [-(\eta + p)c_* - 2\zeta p + \gamma p^2] \\ &\leq -\frac{\eta}{2} t - \frac{t}{c_*} \frac{\zeta^2}{\gamma}, \end{aligned}$$

since $-\eta/2 \leq \zeta/\gamma = p$ in this case. As a consequence, we obtain an estimate of the form

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma^{in}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \lesssim e^{-\frac{\eta}{2}t - \frac{t}{c_*} \frac{\zeta^2}{\gamma}}.$$

This term can be subsumed into the previous Gaussian estimate.

For the remaining contribution on Γ^{out} , we further split it into two parts $\Gamma^{out} = \Gamma_1^{out} \cup \Gamma_2^{out}$ where $\Gamma_1^{out} := \Gamma^{out} \cap B_R(0)$ and Γ_2^{out} is the complementary part. It follows from Lemma 3.2 and Lemma 3.4 that

$$\begin{aligned} \left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_1^{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| &\lesssim e^{-\delta_0 t}, \\ \left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_2^{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| &\lesssim e^{-\delta_0 t - \kappa j} \underbrace{\int_R^{+\infty} e^{-\delta_1 \xi t} d\xi}_{\leq \frac{C}{t}}. \end{aligned}$$

It only remains to check that a temporal exponential can be subsumed into a Gaussian estimate. Indeed, using the definition of ζ and γ together with the fact we supposed that $-\frac{\eta}{2} \leq \frac{\zeta}{\gamma} \leq \iota_\epsilon$, we can always find a constant $\kappa_0 > 0$ such that

$$\kappa_0 \frac{(j - c_*t)^2}{t} \leq t.$$

Case: $\frac{\zeta}{\gamma} > \iota_\epsilon$. The contour Γ is decomposed into $\Gamma_p \cup \Gamma^{out}$ with

$$\Gamma^{out} = \Gamma_-^{out} \cup \Gamma_+^{out} = \left\{ -\delta_0 + \delta_1 \xi + \mathbf{i} \xi \mid \xi \leq -\sqrt{\epsilon^2 - \eta^2} \right\} \cup \left\{ -\delta_0 - \delta_1 \xi + \mathbf{i} \xi \mid \xi \geq \sqrt{\epsilon^2 - \eta^2} \right\},$$

for two constants $\delta_0 > 0$ and $\delta_1 > 0$ chosen as in the previous case. This time for Γ_p we select $p = \iota_\epsilon$. Note that for each $\nu \in \Gamma_{\iota_\epsilon} \subset B_\epsilon(0)$, we have from (3.16)-(3.17)

$$\begin{aligned} t\operatorname{Re}(\nu) + j\operatorname{Re}(\varpi(\nu)) &\leq -c_* \operatorname{Im}(\nu)^2 t + \frac{t}{c_*} (-2\zeta p + \gamma p^2) \\ &= -c_* \operatorname{Im}(\nu)^2 t + \frac{t}{c_*} (-2\zeta \iota_\epsilon + \gamma \iota_\epsilon^2). \end{aligned}$$

But as $\frac{\zeta}{\gamma} > \iota_\epsilon$, we get in particular that $\zeta > \gamma \iota_\epsilon > 0$ such that

$$-2\zeta \iota_\epsilon + \gamma \iota_\epsilon^2 < -\gamma \iota_\epsilon^2 < 0,$$

and thus

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\iota_\epsilon}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \lesssim \frac{e^{-\frac{t}{c_*} \gamma \iota_\epsilon^2}}{\sqrt{t}}.$$

Now, since $\frac{\zeta}{\gamma} > \iota_\epsilon$, we get that $2\gamma \iota_\epsilon t < 2\zeta t = j - c_* t \leq (L - c_*)t$, and thus there exists some constant $\kappa_1 > 0$ such that

$$\frac{(j - c_* t)^2}{t^2} \leq \kappa_1,$$

which shows that the above exponential decaying in time bound can be subsumed into a Gaussian estimate. The estimate on Γ^{out} is similar as in the previous case and we get

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma^{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \leq \left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_+^{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| + \left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_-^{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \lesssim e^{-\delta_0 t},$$

which can once again be subsumed into a Gaussian estimate.

Case: $\frac{\zeta}{\gamma} < -\frac{\eta}{2}$. Once again we divide the contour Γ into $\Gamma_p \cup \Gamma^{in} \cup \Gamma^{out}$ with $p = -\frac{\eta}{2}$, and Γ^{in} and Γ^{out} defined as previously. This time, for all $\nu \in \Gamma_{-\frac{\eta}{2}} \subset B_\epsilon(0)$, we have

$$\begin{aligned} t\operatorname{Re}(\nu) + j\operatorname{Re}(\varpi(\nu)) &\leq -c_* \operatorname{Im}(\nu)^2 t + \frac{t}{c_*} (-2\zeta p + \gamma p^2) \\ &\leq -c_* \operatorname{Im}(\nu)^2 t + \frac{t}{c_*} \left(\eta \zeta + \gamma \left(\frac{\eta}{2} \right)^2 \right). \end{aligned}$$

Note that this time $\zeta < -\eta\gamma/2$, and we have

$$\eta \zeta + \gamma \left(\frac{\eta}{2} \right)^2 < -\gamma \frac{\eta^2}{4},$$

thus

$$\left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{-\frac{\eta}{2}}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \lesssim \frac{e^{-\frac{t}{c_*} \gamma \frac{\eta^2}{4}}}{\sqrt{t}}.$$

And once again we can conclude by noticing that $-\theta t \leq j - c_* t \leq -\eta \gamma t$ due to $\zeta < -\eta \gamma / 2$, and the above exponential decaying in time bound can be subsumed into a Gaussian estimate.

Next, for all $\nu \in \Gamma^{in} \subset B_\epsilon(0)$, we have

$$\begin{aligned} t\operatorname{Re}(\nu) + j\operatorname{Re}(\varpi(\nu)) &\leq \frac{t}{c_*} [-(\eta + p)c_* - 2\zeta p + \gamma p^2] \\ &= \frac{t}{c_*} \left[-\frac{\eta}{2}c_* + \zeta\eta + \gamma \left(\frac{\eta}{2}\right)^2 \right] \\ &< -\frac{\eta}{2}t - \frac{t}{c_*}\gamma\frac{\eta^2}{4}, \end{aligned}$$

and we obtain an estimate of the form

$$\left| \frac{1}{2\pi i} \int_{\Gamma^{in}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \lesssim e^{-\frac{\eta}{2}t - \frac{t}{c_*}\gamma\frac{\eta^2}{4}}.$$

This term can be subsumed into a Gaussian estimate. Finally, the contribution along Γ^{out} can be handled as previously. This concludes the proof of the Lemma. \square

As a conclusion, summarizing all the above lemmas, we have obtained the following intermediate result.

Proposition 3.9 (Pointwise estimates). *There are constants $\theta_* > 0$, $\beta_i > 0$ for $i = 1, 2, 3$ and $C > 0$ such that the temporal Green's function $\mathcal{G}(t)$ satisfies the following pointwise estimates:*

- for $|j - c_* t| > \theta_* t$ and $t \geq 1$, one has

$$|\mathcal{G}_j(t)| \leq C \exp(-\beta_1 t - \beta_2 |j|);$$

- for $|j - c_* t| \leq \theta_* t$ and $t \geq 1$, one has

$$|\mathcal{G}_j(t)| \leq \frac{C}{\sqrt{t}} \exp\left(-\beta_3 \frac{|j - c_* t|^2}{t}\right).$$

The above proposition ensures in particular that there exists a constant $C > 0$, such that

$$\sqrt{t} \|\mathcal{G}(t)\|_{\ell^\infty(\mathbb{Z})} + \|\mathcal{G}(t)\|_{\ell^1(\mathbb{Z})} \leq C, \quad t \geq 1.$$

This in turn implies that any solution of the linear problem (3.2) starting from a compactly supported initial condition satisfies

$$|w_j(t)| = \left| \sum_{\ell \in \mathbb{Z}} \mathcal{G}_{j-\ell}(t) w_\ell^0 \right| \leq \frac{C}{\sqrt{t}} \|w^0\|_{\ell^1(\mathbb{Z})}, \quad t \geq 1, \quad j \in \mathbb{Z}. \quad (3.19)$$

It turns out that such an estimate will not be enough in the forthcoming analysis leading to the proof of Theorem 1. It is the purpose of the next section to refine our pointwise estimates for $|j - c_* t| \leq \theta_* t$ and t large enough.

4 Refined pointwise estimates in the sub-linear regime

From the study conducted in the previous section and Lemma 3.8, we expect that $\mathcal{G}_j(t)$ behaves like a Gaussian profile around $j \sim c_*t$ for all $j \in \mathbb{Z} \cap [(c_* - \theta_*)t, (c_* + \theta_*)t]$. It is actually possible to refine our analysis and prove that $\mathcal{G}_j(t)$ can be decomposed as a universal Gaussian profile plus some reminder term which can be bounded and also satisfies a Gaussian estimate. To do so, we work in the asymptotic regime where the time t is large and $|j - c_*t| \leq \vartheta t^\alpha$ for any $\vartheta > 0$ and $\alpha \in (0, 1)$. Before stating our main result, we introduce the normalized Gaussian profile:

$$\mathcal{G}(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R},$$

and we define the following odd cubic polynomial function $\mathcal{P} : \mathbb{R} \mapsto \mathbb{R}$

$$\mathcal{P}(x) := \frac{c_*}{24 \cosh(\lambda_*)^2} (-3x + x^3). \quad (4.1)$$

Proposition 4.1 (Refined asymptotic). *For any $\vartheta > 0$ and $\alpha \in (0, 1)$, there is some $T_0 > 1$ such that for each $j \in \mathbb{Z}$ with $|j - c_*t| \leq \vartheta t^\alpha$ and $t \geq T_0$, one can decompose the temporal Green's function as follows*

$$\mathcal{G}_j(t) = \mathcal{H}_j(t) + \mathcal{R}_j(t), \quad (4.2)$$

where the principal part $\mathcal{H}_j(t)$ is defined as

$$\mathcal{H}_j(t) := \left[\frac{1}{\sqrt{2 \cosh(\lambda_*)t}} + \frac{1}{t} \mathcal{P} \left(\frac{j - c_*t}{\sqrt{2 \cosh(\lambda_*)t}} \right) \right] \mathcal{G} \left(\frac{j - c_*t}{\sqrt{2 \cosh(\lambda_*)t}} \right),$$

and with a Gaussian bound on the remainder term $\mathcal{R}_j(t)$

$$|\mathcal{R}_j(t)| \leq \frac{C}{t^{3/2}} \exp \left(-\beta \frac{(j - c_*t)^2}{t} \right),$$

for some uniform constants $C > 0$ and $\beta > 0$.

We illustrate in Figure 3 our main result of Proposition 4.1. We numerically solved (3.2) starting from the initial sequence $\boldsymbol{\delta}$ to approximate the temporal Green's functions $(\mathcal{G}_j(t))_{j \in \mathbb{Z}}$ for $t \geq 1$. We used a Runge-Kutta method of order 4 to discretize in time the evolution equation (3.2) with fixed time step $\Delta t > 0$, and the spatial domain was taken to $\llbracket -L, L \rrbracket^1$ for some given integer $L \geq 1$. And we denote by $\widetilde{\mathcal{G}}_j^n$ the approximation of $\mathcal{G}_j(t)$ at time $t = t_n = n\Delta t$ for each $j \in \llbracket -L, L \rrbracket$ and $n = 1, \dots, N$. Let \mathcal{E}_n be defined as

$$\mathcal{E}_n := \sup_{j \in \llbracket -L, L \rrbracket} \left| \widetilde{\mathcal{G}}_j^n - \mathcal{H}_j(t_n) \right|, \quad n = 1, \dots, N. \quad (4.3)$$

Then \mathcal{E}_n represents a numerical approximation of $\sup_{j \in \mathbb{Z}} |\mathcal{R}_j(t_n)|$ which is of the order $\mathcal{O}(t_n^{-3/2})$ by Proposition 4.1. In the left panel of Figure 3, we show that $\ln(\mathcal{E}_n)$ is a linear function of $\ln(t_n)$

¹Here, we used the notation $\llbracket A, B \rrbracket$ to denote the set of all integers between $A \in \mathbb{Z}$ and $B \in \mathbb{Z}$ with $A \leq B$.

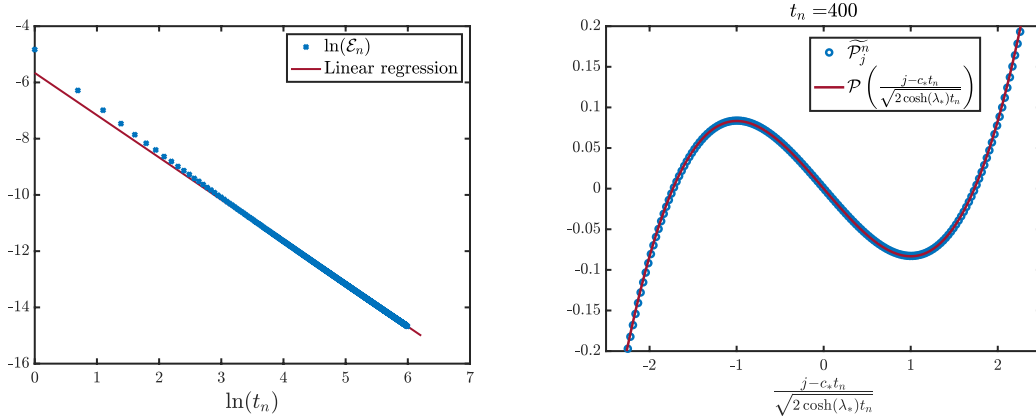


Figure 3: Numerical illustrations of Proposition 4.1. Left. We plot $\ln(\mathcal{E}_n)$ (blue crosses), where \mathcal{E}_n is defined in (4.3), as a function of $\ln(t_n)$ for each $n = 1, \dots, N$. We compare $\ln(\mathcal{E}_n)$ to its linear regression (dark red curve). The slope is found to be $\simeq -1.5033$ which gives a 0.2% error compared to the theoretical value $-\frac{3}{2}$. Right. We compare $\widetilde{\mathcal{P}}_j^n$ (blue circles), defined in (4.4), to $\mathcal{P}\left(\frac{j - c_* t}{\sqrt{2 \cosh(\lambda_*) t}}\right)$ (dark red curve) at time $t = 400$. In the computations, we used $f'(0) = 1$ giving $(c_*, \lambda_*) \simeq (2.0734, 0.9071)$ together with $L = 1000$, $\Delta t = 0.01$ and $N = 40001$.

whose slope is found to be $\simeq -1.5033$ which compares very well with the theoretical value $-\frac{3}{2}$. In the right panel, we recover the universal nature of the profile \mathcal{P} . More precisely, we compare $\widetilde{\mathcal{P}}_j^n$, defined as

$$\widetilde{\mathcal{P}}_j^n := t_n \left[\widetilde{\mathcal{G}}_j^n - \frac{1}{\sqrt{2 \cosh(\lambda_*) t_n}} \mathcal{G}\left(\frac{j - c_* t_n}{\sqrt{2 \cosh(\lambda_*) t_n}}\right) \right] \mathcal{G}\left(\frac{j - c_* t_n}{\sqrt{2 \cosh(\lambda_*) t_n}}\right)^{-1}, \quad j \in \llbracket -L, L \rrbracket, \quad (4.4)$$

to $\mathcal{P}\left(\frac{j - c_* t_n}{\sqrt{2 \cosh(\lambda_*) t_n}}\right)$ as a function of $\frac{j - c_* t_n}{\sqrt{2 \cosh(\lambda_*) t_n}}$ at fixed t_n and observe a very good match confirming numerically the universal nature of our formula (4.1) for \mathcal{P} .

4.1 Proof of Proposition 4.1

We first explain the strategy of the proof. We are going to write $\mathcal{G}_j(t)$ as

$$\mathcal{G}_j(t) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{in}} e^{\nu t} \mathbf{G}_j(\nu) d\nu + \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu,$$

where Γ_{in} is a contour within the ball $B_\epsilon(0)$ and $\Gamma_{out} \subset \{\nu \in \mathbb{C} \mid \operatorname{Re}(\nu) \leq -\eta\}$ is some contour of the form $\{-\delta_0 - \delta_1|\xi| + \mathbf{i}\xi \mid \xi \in \mathbb{R}, |\xi| \geq \sqrt{\epsilon^2 - \eta^2}\}$ for well-chosen $\delta_{0,1} > 0$ and where $\eta \in (0, \epsilon)$ can be chosen as in the proof of Lemma 3.8. Throughout, we assume $\Gamma_{in} \cap \Gamma_{out} = \{\nu_0, \bar{\nu}_0\}$ with $|\nu_0| = \epsilon$ and $\operatorname{Re}(\nu_0) = -\eta$. As a consequence, for t large enough, we have $|j - c_* t| \leq \vartheta t^\alpha \leq \theta_* t$ and the proof of Lemma 3.8 gives

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \leq \frac{C}{t} e^{-\frac{\eta}{2} t} \leq \frac{\tilde{C}}{t^{3/2}} \exp\left(-\beta \frac{(j - c_* t)^2}{t}\right).$$

As a consequence, we are led to prove that

$$\left| \frac{1}{2\pi i} \int_{\Gamma_{in}} e^{\nu t} \mathbf{G}_j(\nu) d\nu - \mathcal{H}_j(t) \right| \leq \frac{C}{t^{3/2}} \exp\left(-\beta \frac{(j - c_* t)^2}{t}\right),$$

whenever $|j - c_* t| \leq \vartheta t^\alpha$ and t large enough. For all $\nu \in B_\epsilon(0)$, we recall from (3.12) that for $j \geq 1$ one has

$$\mathbf{G}_j(\nu) = \Psi(\nu) e^{\varpi(\nu)j},$$

where Ψ is defined in (3.13) and is holomorphic on $B_\epsilon(0)$ and $\Psi(0) = \frac{1}{c_*} > 0$. In fact, simple computation gives

$$\Psi(\nu) = -e^{\lambda_*} \frac{\langle \pi^0(\nu) \mathbf{e}, \mathbf{e} \rangle}{e^{\varpi(\nu)}} = \frac{e^{\lambda_*}}{\rho_+(\nu) - \rho_-(\nu)} = \frac{1}{\sqrt{(\nu + e^{\lambda_*} + e^{-\lambda_*})^2 - 4}},$$

such that

$$\Psi(\nu) = \frac{1}{c_*} - \frac{2 \cosh(\lambda_*)}{c_*^3} \nu + \nu^2 \tilde{\Psi}(\nu), \quad \forall \nu \in B_\epsilon(0),$$

and $\tilde{\Psi}$ is holomorphic on $B_\epsilon(0)$ with $|\tilde{\Psi}(\nu)| \leq C$ for each $\nu \in B_\epsilon(0)$. Next, we are going to need an asymptotic expansion of $\varpi(\nu)$ up to quartic order, that is,

$$\varpi(\nu) = -\frac{\nu}{c_*} + \frac{\cosh(\lambda_*)}{c_*^3} \nu^2 + \frac{\Lambda_*}{c_*^5} \nu^3 + \nu^4 \Omega(\nu), \quad \forall \nu \in B_\epsilon(0), \quad (4.5)$$

where Ω is holomorphic on $B_\epsilon(0)$ with $|\Omega(\nu)| \leq C$ for each $\nu \in B_\epsilon(0)$ and

$$\Lambda_* := -2 - \frac{c_*^2}{3} < 0. \quad (4.6)$$

For future reference, we also denote by $\varphi(\nu)$ the principal part of $\varpi(\nu)$, that is

$$\varphi(\nu) := -\frac{\nu}{c_*} + \frac{\cosh(\lambda_*)}{c_*^3} \nu^2, \quad \forall \nu \in B_\epsilon(0).$$

Finally, we remark that $\beta^*, \beta_* > 0$ in (3.14) can always be taken larger and smaller respectively such that we can also ensure that

$$\operatorname{Re}(\varpi(\nu)) + \frac{|\Lambda_*|}{c_*^5} |\nu|^3 + |\nu|^4 |\Omega(\nu)| \leq -\frac{\operatorname{Re}(\nu)}{c_*} + \frac{\beta^*}{c_*^3} \operatorname{Re}(\nu)^2 - \frac{\beta_*}{c_*^3} \operatorname{Im}(\nu)^2, \quad \nu \in B_\epsilon(0). \quad (4.7)$$

With all these notations in hands, we will decompose $\mathbf{G}_j(\nu)$ as follows

$$\begin{aligned} \mathbf{G}_j(\nu) &= \left(\frac{1}{c_*} - \frac{2 \cosh(\lambda_*)}{c_*^3} \nu + j \frac{\Lambda_*}{c_*^6} \nu^3 \right) e^{\varphi(\nu)j} + \nu^2 \tilde{\Psi}(\nu) e^{\varpi(\nu)j} - 2 \frac{\cosh(\lambda_*)}{c_*^3} \nu \left(e^{\varpi(\nu)j} - e^{\varphi(\nu)j} \right) \\ &\quad + \frac{1}{c_*} \left[e^{j \frac{\Lambda_*}{c_*^5} \nu^3 + j \nu^4 \Omega(\nu)} - \left(1 + j \frac{\Lambda_*}{c_*^5} \nu^3 + j \nu^4 \Omega(\nu) \right) \right] e^{\varphi(\nu)j} + \frac{j}{c_*} \nu^4 \Omega(\nu) e^{\varphi(\nu)j}, \end{aligned}$$

valid for each $\nu \in B_\epsilon(0)$ and $j \geq 1$.

We now introduce the following integrals which will contribute as error terms

$$\begin{aligned}
\mathcal{E}_j^1(t) &:= \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{in}} e^{\nu t} \nu^2 \tilde{\Psi}(\nu) e^{\varpi(\nu)j} d\nu, \\
\mathcal{E}_j^2(t) &:= \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{in}} e^{\nu t} \nu \left(e^{\varpi(\nu)j} - e^{\varphi(\nu)j} \right) d\nu, \\
\mathcal{E}_j^3(t) &:= \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{in}} e^{\nu t} \left[e^{j \frac{\Lambda_*}{c_*^5} \nu^3 + j \nu^4 \Omega(\nu)} - \left(1 + j \frac{\Lambda_*}{c_*^5} \nu^3 + j \nu^4 \Omega(\nu) \right) \right] e^{\varphi(\nu)j} d\nu, \\
\mathcal{E}_j^4(t) &:= \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{in}} e^{\nu t} \nu^4 \Omega(\nu) e^{\varphi(\nu)j} d\nu.
\end{aligned}$$

More precisely, we have the following lemma.

Lemma 4.2. *For any $\vartheta > 0$ and $\alpha \in (0, 1)$, there is some $T_0 > 1$ such that for each $j \in \mathbb{Z}$ with $|j - c_* t| \leq \vartheta t^\alpha$ and $t \geq T_0$ one has*

$$\left| \mathcal{E}_j^k(t) \right| \leq \frac{C}{t^{3/2}} \exp\left(-\beta \frac{(j - c_* t)^2}{t}\right), \quad k = 1, \dots, 4,$$

for some uniform constants $C > 0$ and $\beta > 0$.

Proof. Recalling our notation

$$\zeta = \frac{j - c_* t}{2t}, \quad \text{and} \quad \gamma = \frac{j \beta^*}{t c_*^2},$$

we can always ensure that $\zeta/\gamma \in (-\eta/2, \iota_\epsilon)$ since we consider t large enough and $|j - c_* t| \leq \vartheta t^\alpha$ with $\alpha \in (0, 1)$. Here $\iota_\epsilon > 0$ is the unique real number such that Γ_p defined in (3.15) with $p = \iota_\epsilon$ intersects the line $-\eta + \mathbf{i}\mathbb{R}$ on the boundary of $B_\epsilon(0)$ with $\eta \in (0, \eta_\epsilon)$ and where $\eta_\epsilon > 0$ is defined as in the proof of Lemma 3.8. Then, in each $\mathcal{E}_j^k(t)$ with $k = 1, \dots, 4$, we decompose Γ_{in} as $\Gamma_{in} = \Gamma_{\zeta/\gamma} \cup \Gamma_1$ with

$$\Gamma_1 = \left\{ -\eta + \mathbf{i}\xi \mid -\sqrt{\epsilon^2 - \eta^2} \leq \xi \leq -\xi_* \text{ and } \xi_* \leq \xi \leq \sqrt{\epsilon^2 - \eta^2} \right\},$$

and

$$\xi_* = \sqrt{\frac{c_*^2}{\beta_*} (\psi(\zeta/\gamma) - \psi(-\eta))} > 0.$$

Study of $\mathcal{E}_j^1(t)$. First, we can bound $\mathcal{E}_j^1(t)$ by

$$\begin{aligned}
|\mathcal{E}_j^1(t)| &\lesssim \int_{\Gamma_{\zeta/\gamma}} e^{\operatorname{Re}(\nu)t} |\nu|^2 e^{\operatorname{Re}(\varpi(\nu)j)} |d\nu| + \int_{\Gamma_1} e^{\operatorname{Re}(\nu)t} |\nu|^2 e^{\operatorname{Re}(\varpi(\nu)j)} |d\nu| \\
&= \int_{\Gamma_{\zeta/\gamma}} e^{\operatorname{Re}(\nu)t} \left(\frac{\zeta^2}{\gamma^2} + |\operatorname{Im}(\nu)|^2 \right) e^{\operatorname{Re}(\varpi(\nu)j)} |d\nu| + \int_{\Gamma_1} e^{\operatorname{Re}(\nu)t} (\eta^2 + |\operatorname{Im}(\nu)|^2) e^{\operatorname{Re}(\varpi(\nu)j)} |d\nu|.
\end{aligned}$$

Using similar computations as in the proof of Lemma 3.8, we find

$$\begin{aligned} \frac{\zeta^2}{\gamma^2} \int_{\Gamma_{\zeta/\gamma}} e^{\operatorname{Re}(\nu)t} e^{\operatorname{Re}(\varpi(\nu))j} |d\nu| &\lesssim \frac{1}{\gamma^2} \left(\frac{|j - c_*t|}{\sqrt{t}} \right)^2 \frac{e^{-\beta_0 \frac{|j - c_*t|^2}{t}}}{t^{3/2}} \lesssim \frac{e^{-\beta_0 \frac{|j - c_*t|^2}{2t}}}{t^{3/2}}, \\ \int_{\Gamma_{\zeta/\gamma}} |\operatorname{Im}(\nu)|^2 e^{\operatorname{Re}(\nu)t} e^{\operatorname{Re}(\varpi(\nu))j} |d\nu| &\lesssim e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}} \underbrace{\int_{\Gamma_{\zeta/\gamma}} |\operatorname{Im}(\nu)|^2 e^{-c_* \operatorname{Im}(\nu)^2 t} |d\nu|}_{\leq \frac{C}{t^{3/2}}} \lesssim \frac{e^{-\beta_0 \frac{|j - c_*t|^2}{t}}}{t^{3/2}}. \end{aligned}$$

Let us note that in the first estimate, we used the fact that for each integer $m \geq 1$ one can always find some constant $C_m > 0$ such that $x^m e^{-\beta_0 x^2} \leq C_m e^{-\frac{\beta_0}{2} x^2}$ for any $x \geq 0$.

Finally, the remaining contribution in $\mathcal{E}_j^1(t)$ along Γ_1 gives

$$\int_{\Gamma_1} e^{\operatorname{Re}(\nu)t} (\eta^2 + |\operatorname{Im}(\nu)|^2) e^{\operatorname{Re}(\varpi(\nu))j} |d\nu| \lesssim e^{-\frac{\eta}{2}t - \frac{\zeta^2}{\gamma} \frac{t}{c_*}} \lesssim \frac{e^{-\beta_0 \frac{|j - c_*t|^2}{t}}}{t^{3/2}}.$$

Study of $\mathcal{E}_j^2(t)$. To handle the next error term $\mathcal{E}_j^2(t)$, we will use the fact that

$$\left| e^{\varpi(\nu)j} - e^{\varphi(\nu)j} \right| \leq Cj|\nu|^3 e^{j \left(\operatorname{Re}(\varpi(\nu)) + \frac{|\Lambda_*|}{c_*^3} |\nu|^3 + |\nu|^4 |\Omega(\nu)| \right)}, \quad \nu \in B_\epsilon(0),$$

together with our estimates (4.7) and (3.17) to get

$$\begin{aligned} \left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\zeta/\gamma}} e^{\nu t} \nu \left(e^{\varpi(\nu)j} - e^{\varphi(\nu)j} \right) d\nu \right| &\lesssim j \int_{\Gamma_{\zeta/\gamma}} \left(\frac{|\zeta|^4}{\gamma^4} + |\operatorname{Im}(\nu)|^4 \right) e^{t \operatorname{Re}(\nu) + j \left(\operatorname{Re}(\varpi(\nu)) + \frac{|\Lambda_*|}{c_*^3} |\nu|^3 + |\nu|^4 |\Omega(\nu)| \right)} |d\nu| \\ &\lesssim j e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}} \int_{\Gamma_{\zeta/\gamma}} \left(\frac{|\zeta|^4}{\gamma^4} + |\operatorname{Im}(\nu)|^4 \right) e^{-c_* \operatorname{Im}(\nu)^2 t} |d\nu| \\ &\lesssim j \left(\frac{|\zeta|^4}{\sqrt{t}} + \frac{1}{t^{9/2}} \right) e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}}. \end{aligned}$$

Next, we remark that

$$j \left(\frac{|\zeta|^4}{\sqrt{t}} + \frac{1}{t^{9/2}} \right) = \frac{j}{t} \left(\left(\frac{|j - c_*t|}{2\sqrt{t}} \right)^4 \frac{1}{t^{3/2}} + \frac{1}{t^{7/2}} \right),$$

which then implies that

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\zeta/\gamma}} e^{\nu t} \nu \left(e^{\varpi(\nu)j} - e^{\varphi(\nu)j} \right) d\nu \right| \lesssim \frac{e^{-\beta_0 \frac{|j - c_*t|^2}{t}}}{t^{3/2}}.$$

Similarly, the contribution along Γ_1 gives a Gaussian estimate with an exponential in time decaying factor leading to

$$\left| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_1} e^{\nu t} \nu \left(e^{\varpi(\nu)j} - e^{\varphi(\nu)j} \right) d\nu \right| \lesssim j e^{-\frac{\eta}{2}t - \frac{\zeta^2}{\gamma} \frac{t}{c_*}} \lesssim \frac{e^{-\beta_0 \frac{|j - c_*t|^2}{t}}}{t^{3/2}},$$

where we simply used that $|\nu|^4 \leq \epsilon^2$ since $\Gamma_1 \subset B_\epsilon(0)$ and that

$$j e^{-\frac{\eta}{2}t} = \frac{j}{t} t e^{-\frac{\eta}{2}t} \lesssim \frac{1}{t^{3/2}},$$

for t large enough and $|j - c_*t| \leq \vartheta t^\alpha$.

Study of $\mathcal{E}_j^4(t)$. Before proceeding with $\mathcal{E}_j^3(t)$, we first treat $\mathcal{E}_j^4(t)$ since the computations are very close to the previous case. Indeed, for each $\nu \in \Gamma_{in}$, we use that

$$\left| e^{\nu t} \nu^4 \Omega(\nu) e^{\varphi(\nu)j} \right| \leq C |\nu|^4 e^{\operatorname{Re}(\nu)t} e^{\operatorname{Re}(\varphi(\nu))j},$$

such that we deduce

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\Gamma_{\zeta/\gamma}} e^{\nu t} \nu^4 \Omega(\nu) e^{\varphi(\nu)j} d\nu \right| &\lesssim \int_{\Gamma_{\zeta/\gamma}} \left(\frac{|\zeta|^4}{\gamma^4} + |\operatorname{Im}(\nu)|^4 \right) e^{t\operatorname{Re}(\nu) + j\operatorname{Re}(\varphi(\nu))} |d\nu| \\ &\lesssim e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}} \int_{\Gamma_{\zeta/\gamma}} \left(\frac{|\zeta|^4}{\gamma^4} + |\operatorname{Im}(\nu)|^4 \right) e^{-c_* \operatorname{Im}(\nu)^2 t} |d\nu| \\ &\lesssim \left(\frac{|\zeta|^4}{\sqrt{t}} + \frac{1}{t^{9/2}} \right) e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}} \lesssim \frac{e^{-\beta_0 \frac{|j-c_*t|^2}{t}}}{t^{3/2}}. \end{aligned}$$

The contribution along Γ_1 is treated as usual.

Study of $\mathcal{E}_j^3(t)$. Finally, for the last term $\mathcal{E}_j^3(t)$, we use that

$$\left| e^{j \frac{\Lambda_*}{c_*^5} \nu^3 + j \nu^4 \Omega(\nu)} - \left(1 + j \frac{\Lambda_*}{c_*^5} \nu^3 + j \nu^4 \Omega(\nu) \right) \right| \leq C j^2 |\nu|^6 e^{j \frac{|\Lambda_*|}{c_*^5} |\nu|^3 + j |\nu|^4 |\Omega(\nu)|}, \quad \nu \in B_\epsilon(0),$$

together with

$$\operatorname{Re}(\varphi(\nu)) + \frac{|\Lambda_*|}{c_*^5} |\nu|^3 + |\nu|^4 |\Omega(\nu)| \leq -\frac{\operatorname{Re}(\nu)}{c_*} + \frac{\beta_*}{c_*^3} \operatorname{Re}(\nu)^2 - \frac{\beta_*}{c_*^3} \operatorname{Im}(\nu)^2, \quad \nu \in B_\epsilon(0),$$

to obtain that

$$\begin{aligned} &\left| \frac{1}{2\pi i} \int_{\Gamma_{\zeta/\gamma}} e^{\nu t} \left[e^{j \frac{\Lambda_*}{c_*^5} \nu^3 + j \nu^4 \Omega(\nu)} - \left(1 + j \frac{\Lambda_*}{c_*^5} \nu^3 + j \nu^4 \Omega(\nu) \right) \right] e^{\varphi(\nu)j} d\nu \right| \\ &\lesssim j^2 \int_{\Gamma_{\zeta/\gamma}} \left(\frac{|\zeta|^6}{\gamma^6} + |\operatorname{Im}(\nu)|^6 \right) e^{t\operatorname{Re}(\nu) + j \left(\operatorname{Re}(\varphi(\nu)) + \frac{|\Lambda_*|}{c_*^5} |\nu|^3 + |\nu|^4 |\Omega(\nu)| \right)} |d\nu| \\ &\lesssim j^2 e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}} \int_{\Gamma_{\zeta/\gamma}} \left(\frac{|\zeta|^6}{\gamma^6} + |\operatorname{Im}(\nu)|^6 \right) e^{-c_* \operatorname{Im}(\nu)^2 t} |d\nu| \\ &\lesssim j^2 \left(\frac{|\zeta|^6}{\sqrt{t}} + \frac{1}{t^{13/2}} \right) e^{-\frac{\zeta^2}{\gamma} \frac{t}{c_*}}. \end{aligned}$$

Finally, we note that

$$j^2 \frac{|\zeta|^6}{\sqrt{t}} = \left(\frac{j}{t} \right)^2 \left(\frac{|j - c_*t|}{2\sqrt{t}} \right)^6 \frac{1}{t^{3/2}},$$

such that we eventually get

$$|\mathcal{E}_j^3(t)| \lesssim \frac{e^{-\beta_0 \frac{|j-c_*t|^2}{t}}}{t^{3/2}},$$

by, once again, noticing that the contribution along Γ_1 can be subsumed into the one obtained along $\Gamma_{\zeta/\gamma}$. \square

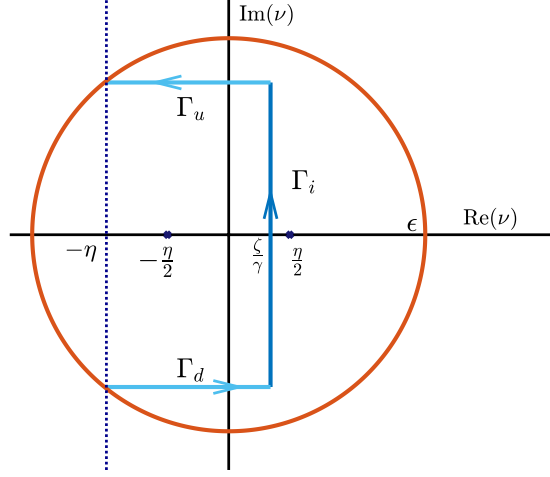


Figure 4: Illustration of the geometry of the contour $\Gamma_{in} = \Gamma_d \cup \Gamma_u \cup \Gamma_i \subset B_\epsilon(0)$ used in the computation of the integrals $\mathcal{I}_j^k(t)$ for $k = 1, 2, 3$.

Next, we move on with the analysis of the leading order terms. We define

$$\begin{aligned}\mathcal{I}_j^1(t) &:= \frac{1}{c_*} \frac{1}{2\pi i} \int_{\Gamma_{in}} e^{\nu t} e^{\varphi(\nu)j} d\nu, \\ \mathcal{I}_j^2(t) &:= -\frac{2 \cosh(\lambda_*)}{c_*^3} \frac{1}{2\pi i} \int_{\Gamma_{in}} e^{\nu t} \nu e^{\varphi(\nu)j} d\nu, \\ \mathcal{I}_j^3(t) &:= \frac{\Lambda_*}{c_*^6} \frac{1}{2\pi i} \int_{\Gamma_{in}} e^{\nu t} j \nu^3 e^{\varphi(\nu)j} d\nu.\end{aligned}$$

In order to evaluate the above three integrals, we introduce some new contours

$$\Gamma_d := \left\{ z - \mathbf{i}\sqrt{\epsilon^2 - \eta^2} \mid -\eta \leq z \leq \frac{\zeta}{\gamma} \right\}, \quad \Gamma_u := \left\{ z + \mathbf{i}\sqrt{\epsilon^2 - \eta^2} \mid -\eta \leq z \leq \frac{\zeta}{\gamma} \right\}$$

and

$$\Gamma_i := \left\{ \frac{\zeta}{\gamma} + \mathbf{i}\xi \mid |\xi| \leq \sqrt{\epsilon^2 - \eta^2} \right\},$$

with this time $\zeta = \frac{j - c_* t}{2t}$ and $\gamma = \frac{j \cosh(\lambda_*)}{t c_*^2}$. We choose t large enough such that

$$-\frac{\eta}{2} \leq \frac{\zeta}{\gamma} \leq \frac{\eta}{2},$$

which is always possible since $|\zeta| \leq \frac{\vartheta}{2t^{1-\alpha}}$ and t is taken large enough. In the following, we will write $\Gamma_{in} = \Gamma_d \cup \Gamma_u \cup \Gamma_i \subset B_\epsilon(0)$ and we refer to Figure 4 for an illustration.

Lemma 4.3. *For any $\vartheta > 0$ and $\alpha \in (0, 1)$, there is some $T_0 > 1$ such that for each $j \in \mathbb{Z}$ with $|j - c_* t| \leq \vartheta t^\alpha$ and $t \geq T_0$ one can decompose $\mathcal{I}_j^1(t)$ as*

$$\begin{aligned}\mathcal{I}_j^1(t) &= \left(\frac{1}{\sqrt{2t \cosh(\lambda_*)}} + \frac{1}{2tc_*} \left(-\left(\frac{j - c_* t}{\sqrt{2 \cosh(\lambda_*)t}} \right) + \left(\frac{j - c_* t}{\sqrt{2 \cosh(\lambda_*)t}} \right)^3 \right) \right) \mathcal{G} \left(\frac{j - c_* t}{\sqrt{2 \cosh(\lambda_*)t}} \right) \\ &\quad + \widetilde{\mathcal{R}}_j^1(t),\end{aligned}$$

with

$$\left| \widetilde{\mathcal{R}}_j^1(t) \right| \leq \frac{C}{t^{3/2}} \exp\left(-\beta \frac{(j - c_* t)^2}{t}\right),$$

for some uniform constants $C > 0$ and $\beta > 0$.

Proof. We first compute

$$\begin{aligned} \frac{1}{c_*} \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_i} e^{\nu t} e^{\varphi(\nu)j} d\nu &= \frac{1}{c_*} \frac{1}{2\pi} \exp\left(-\frac{(j - c_* t)^2}{4tc_*\gamma}\right) \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} e^{-\frac{t\gamma}{c_*}\xi^2} d\xi \\ &= \frac{1}{2\pi\sqrt{t\gamma c_*}} \exp\left(-\frac{(j - c_* t)^2}{4tc_*\gamma}\right) \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} \sqrt{\frac{t\gamma}{c_*}} e^{-z^2} dz. \end{aligned}$$

Since $|j - c_* t| \leq \vartheta t^\alpha$, we have that

$$\frac{1}{\sqrt{t\gamma c_*}} = \frac{1}{\sqrt{t \cosh(\lambda_*)}} \left(1 - \frac{1}{2} \frac{j - c_* t}{tc_*} + \mathcal{O}\left(\left(\frac{j - c_* t}{t}\right)^2\right) \right),$$

together with

$$\left| e^{-\frac{(j - c_* t)^2}{4tc_*\gamma}} - e^{-\frac{(j - c_* t)^2}{4t \cosh(\lambda_*)}} - \frac{(j - c_* t)^3}{4t^2 \cosh(\lambda_*)c_*} e^{-\frac{(j - c_* t)^2}{4t \cosh(\lambda_*)}} \right| \lesssim \frac{1}{t} e^{-\beta \frac{(j - c_* t)^2}{t}},$$

for some $\beta > 0$ and t large enough. Next we remark that

$$\int_{\mathbb{R}} e^{-z^2} dz - \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} \sqrt{\frac{t\gamma}{c_*}} e^{-z^2} dz = 2 \int_{\sqrt{\epsilon^2 - \eta^2}}^{+\infty} \sqrt{\frac{t\gamma}{c_*}} e^{-z^2} dz,$$

for which we can use estimate of the complementary error function $\operatorname{erfc}(x) := 2 \int_x^{+\infty} e^{-z^2} dz$ which satisfies for $x > 1$

$$\operatorname{erfc}(x) \leq \frac{e^{-x^2}}{x}.$$

As consequence, for t large enough such that $\sqrt{\epsilon^2 - \eta^2} \sqrt{\frac{t\gamma}{c_*}} > 1$, one has

$$\left| \int_{\mathbb{R}} e^{-z^2} dz - \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} \sqrt{\frac{t\gamma}{c_*}} e^{-z^2} dz \right| \lesssim \frac{1}{\sqrt{t}} e^{-\beta t},$$

for some $\beta > 0$.

Using the above computations, we see that

$$\frac{1}{c_*} \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_i} e^{\nu t} e^{\varphi(\nu)j} d\nu = \frac{1}{\sqrt{4\pi t\gamma c_*}} e^{-\frac{(j - c_* t)^2}{4tc_*\gamma}} - \underbrace{\frac{1}{\pi\sqrt{t\gamma c_*}} e^{-\frac{(j - c_* t)^2}{4tc_*\gamma}} \int_{\sqrt{\epsilon^2 - \eta^2}}^{+\infty} \sqrt{\frac{t\gamma}{c_*}} e^{-z^2} dz}_{:= \mathcal{R}_j^1(t)},$$

with

$$\left| \mathcal{R}_j^1(t) \right| \lesssim \frac{1}{t^{3/2}} e^{-\beta \frac{(j - c_* t)^2}{t}}.$$

Next, we also see that if we define

$$\mathcal{R}_j^2(t) := \frac{1}{\sqrt{4\pi t \gamma c_*}} e^{-\frac{(j-c_*t)^2}{4tc_*\gamma}} - \frac{1}{\sqrt{4\pi t \cosh(\lambda_*)}} \left(1 - \frac{1}{2} \frac{j-c_*t}{tc_*} + \frac{(j-c_*t)^3}{4t^2 \cosh(\lambda_*) c_*} \right) e^{-\frac{(j-c_*t)^2}{4t \cosh(\lambda_*)}},$$

then for t large enough we have that

$$|\mathcal{R}_j^2(t)| \lesssim \frac{1}{t^{3/2}} e^{-\beta \frac{(j-c_*t)^2}{t}},$$

and

$$\frac{1}{c_*} \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_i} e^{\nu t} e^{\varphi(\nu)j} d\nu = \frac{1}{\sqrt{4\pi t \cosh(\lambda_*)}} \left(1 - \frac{1}{2} \frac{j-c_*t}{tc_*} + \frac{(j-c_*t)^3}{4t^2 \cosh(\lambda_*) c_*} \right) e^{-\frac{(j-c_*t)^2}{4t \cosh(\lambda_*)}} + \mathcal{R}_j^1(t) + \mathcal{R}_j^2(t).$$

Finally, we rewrite the above leading order terms as

$$\begin{aligned} \mathcal{A}_j(t) &:= \frac{1}{\sqrt{4\pi t \cosh(\lambda_*)}} \exp\left(-\frac{(j-c_*t)^2}{4t \cosh(\lambda_*)}\right) = \frac{1}{\sqrt{2t \cosh(\lambda_*)}} \mathcal{G}\left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)}t}\right), \\ \mathcal{B}_j(t) &:= -\frac{1}{2} \frac{1}{\sqrt{4\pi t \cosh(\lambda_*)}} \frac{j-c_*t}{tc_*} e^{-\frac{(j-c_*t)^2}{4t \cosh(\lambda_*)}} + \frac{1}{\sqrt{4\pi t \cosh(\lambda_*)}} \frac{(j-c_*t)^3}{4t^2 \cosh(\lambda_*) c_*} e^{-\frac{(j-c_*t)^2}{4t \cosh(\lambda_*)}} \\ &= \frac{1}{2tc_*} \left(-\left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)}t}\right) + \left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)}t}\right)^3 \right) \mathcal{G}\left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)}t}\right), \end{aligned}$$

such that we have the decomposition

$$\mathcal{I}_j^1(t) = \mathcal{A}_j(t) + \mathcal{B}_j(t) + \mathcal{R}_j^1(t) + \mathcal{R}_j^2(t) + \frac{1}{c_*} \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_u \cup \Gamma_d} e^{\nu t} e^{\varphi(\nu)j} d\nu.$$

To conclude the study of $\mathcal{I}_j^1(t)$, it remains to evaluate the integrals on $\Gamma_{d,u}$. We only treat the case Γ_d since the other integral can be handled similarly. We need to bound

$$\frac{1}{2\pi \mathbf{i}} \int_{\Gamma_d} e^{\nu t} e^{\varphi(\nu)j} d\nu = \frac{1}{2\pi \mathbf{i}} \int_{-\eta}^{\xi} e^{t(z - \mathbf{i}\sqrt{\epsilon^2 - \eta^2})} e^{\varphi(z - \mathbf{i}\sqrt{\epsilon^2 - \eta^2})j} dz$$

and upon setting $\xi_\epsilon := \sqrt{\epsilon^2 - \eta^2}$ we have

$$t(z - \mathbf{i}\xi_\epsilon) + \varphi(z - \mathbf{i}\xi_\epsilon)j = \mathbf{i}\xi_\epsilon \left(-t + \frac{j}{c_*} \right) - j \frac{\cosh(\lambda_*)}{c_*^3} \xi_\epsilon^2 + \frac{t}{c_*} (-2\zeta z + \gamma z^2) - 2\mathbf{i}j\xi_\epsilon \frac{\cosh(\lambda_*)}{c_*^3} z.$$

As a consequence,

$$\begin{aligned} \left| \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_d} e^{\nu t} e^{\varphi(\nu)j} d\nu \right| &\lesssim \exp\left(-\frac{t}{c_*} \gamma \xi_\epsilon^2 - \frac{t}{c_*} \frac{\zeta^2}{\gamma}\right) \int_{-\eta}^{\xi} e^{\frac{t}{c_*} \gamma (z - \frac{\zeta}{\gamma})^2} dz \\ &\lesssim \exp\left(-\frac{t}{c_*} \gamma \xi_\epsilon^2 - \frac{t}{c_*} \frac{\zeta^2}{\gamma} + \frac{t}{c_*} \gamma \left(\eta + \frac{\zeta}{\gamma}\right)^2\right). \end{aligned}$$

Finally, we note that

$$-\xi_\epsilon^2 + \left(\eta + \frac{\zeta}{\gamma}\right)^2 = -\epsilon^2 + \eta^2 + \left(\eta + \frac{\zeta}{\gamma}\right)^2 \leq -\epsilon^2 + \eta^2 + \left(\frac{3}{2}\eta\right)^2 < 0,$$

since for fixed $\epsilon > 0$, we can always choose $\eta > 0$ small enough such that the above strict inequality holds. Thus we get

$$\left| \frac{1}{2\pi i} \int_{\Gamma_d} e^{\nu t} e^{\varphi(\nu)j} d\nu \right| \lesssim \frac{1}{t^{3/2}} e^{-\beta \frac{(j-c_*t)^2}{t}}.$$

□

We now turn our attention to $\mathcal{I}_j^2(t)$.

Lemma 4.4. *For any $\vartheta > 0$ and $\alpha \in (0, 1)$, there is some $T_0 > 1$ such that for each $j \in \mathbb{Z}$ with $|j - c_*t| \leq \vartheta t^\alpha$ and $t \geq T_0$ one can decompose $\mathcal{I}_j^2(t)$ as*

$$\mathcal{I}_j^2(t) = -\frac{1}{tc_*} \left(\frac{j - c_*t}{\sqrt{2} \cosh(\lambda_*)t} \right) \mathcal{G} \left(\frac{j - c_*t}{\sqrt{2} \cosh(\lambda_*)t} \right) + \widetilde{\mathcal{R}}_j^2(t),$$

with

$$\left| \widetilde{\mathcal{R}}_j^2(t) \right| \leq \frac{C}{t^{3/2}} \exp \left(-\beta \frac{(j - c_*t)^2}{t} \right),$$

for some uniform constants $C > 0$ and $\beta > 0$.

Proof. We first compute

$$\begin{aligned} \frac{1}{c_*} \frac{1}{2\pi i} \int_{\Gamma_i} e^{\nu t} \nu e^{\varphi(\nu)j} d\nu &= \frac{1}{c_*} \frac{1}{2\pi} \exp \left(-\frac{(j - c_*t)^2}{4tc_*\gamma} \right) \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} \left(\frac{\zeta}{\gamma} + i\xi \right) e^{-\frac{t\gamma}{c_*} \xi^2} d\xi \\ &= \frac{1}{2\pi \sqrt{t\gamma} c_*} \frac{\zeta}{\gamma} \exp \left(-\frac{(j - c_*t)^2}{4tc_*\gamma} \right) \int_{-\sqrt{\epsilon^2 - \eta^2} \frac{\sqrt{t\gamma}}{c_*}}^{\sqrt{\epsilon^2 - \eta^2} \frac{\sqrt{t\gamma}}{c_*}} e^{-z^2} dz \\ &= \frac{\zeta}{\gamma} (\mathcal{A}_j(t) + \mathcal{B}_j(t)) + \frac{\zeta}{\gamma} (\mathcal{R}_j^1(t) + \mathcal{R}_j^2(t)). \end{aligned}$$

For t large enough, it is enough to remark that

$$\frac{\zeta}{\gamma} = c_* \frac{j - c_*t}{2t \cosh(\lambda_*)} \left(1 + \mathcal{O} \left(\frac{j - c_*t}{tc_*} \right) \right),$$

such that

$$\frac{1}{c_*} \frac{1}{2\pi i} \int_{\Gamma_i} e^{\nu t} \nu e^{\varphi(\nu)j} d\nu = \frac{c_*}{2t \cosh(\lambda_*)} \left(\frac{j - c_*t}{\sqrt{2} \cosh(\lambda_*)t} \right) \mathcal{G} \left(\frac{j - c_*t}{\sqrt{2} \cosh(\lambda_*)t} \right) + \mathcal{R}_j^3(t),$$

where

$$\mathcal{R}_j^3(t) := \left(\frac{\zeta}{\gamma} - c_* \frac{j - c_*t}{2t \cosh(\lambda_*)} \right) \mathcal{A}_j(t) + \frac{\zeta}{\gamma} \mathcal{B}_j(t) + \frac{\zeta}{\gamma} (\mathcal{R}_j^1(t) + \mathcal{R}_j^2(t))$$

with

$$\left| \mathcal{R}_j^3(t) \right| \lesssim \frac{1}{t^{3/2}} e^{-\beta \frac{(j-c_*t)^2}{t}}.$$

Similarly to the previous case, the integrals on $\Gamma_{d,u}$ produce terms which can be subsumed into the above Gaussian estimate. □

Finally, we study the last integral $\mathcal{I}_j^3(t)$.

Lemma 4.5. *For any $\vartheta > 0$ and $\alpha \in (0, 1)$, there is some $T_0 > 1$ such that for each $j \in \mathbb{Z}$ with $|j - c_*t| \leq \vartheta t^\alpha$ and $t \geq T_0$ one can decompose $\mathcal{I}_j^3(t)$ as*

$$\mathcal{I}_j^3(t) = \frac{\Lambda_*}{4tc_* \cosh(\lambda_*)^2} \left(-3 \left(\frac{j - c_*t}{\sqrt{2} \cosh(\lambda_*)t} \right) + \left(\frac{j - c_*t}{\sqrt{2} \cosh(\lambda_*)t} \right)^3 \right) \mathcal{G} \left(\frac{j - c_*t}{\sqrt{2} \cosh(\lambda_*)t} \right) + \widetilde{\mathcal{R}}_j^3(t),$$

with

$$\left| \widetilde{\mathcal{R}}_j^3(t) \right| \leq \frac{C}{t^{3/2}} \exp \left(-\beta \frac{(j - c_*t)^2}{t} \right),$$

for some uniform constants $C > 0$ and $\beta > 0$.

Proof. As usual, the main contribution will come from the integration along Γ_i and we have

$$\begin{aligned} \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_i} e^{\nu t} j \nu^3 e^{\varphi(\nu)j} d\nu &= \frac{j}{2\pi} \exp \left(-\frac{(j - c_*t)^2}{4tc_*\gamma} \right) \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} \left(\frac{\zeta}{\gamma} + \mathbf{i}\xi \right)^3 e^{-\frac{t\gamma}{c_*}\xi^2} d\xi \\ &= \frac{j}{2\pi} \exp \left(-\frac{(j - c_*t)^2}{4tc_*\gamma} \right) \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} \left[\left(\frac{\zeta}{\gamma} \right)^3 + 3\mathbf{i} \left(\frac{\zeta}{\gamma} \right)^2 \xi - 3\frac{\zeta}{\gamma} \xi^2 - \mathbf{i}\xi^3 \right] e^{-\frac{t\gamma}{c_*}\xi^2} d\xi \\ &= \frac{j}{2\pi} \exp \left(-\frac{(j - c_*t)^2}{4tc_*\gamma} \right) \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} \left[\left(\frac{\zeta}{\gamma} \right)^3 - 3\frac{\zeta}{\gamma} \xi^2 \right] e^{-\frac{t\gamma}{c_*}\xi^2} d\xi. \end{aligned}$$

This time, for t large enough, we have

$$j \left(\frac{\zeta}{\gamma} \right)^3 = c_*t \left(1 + \frac{j - c_*t}{c_*t} \right) \left(c_* \frac{j - c_*t}{2t \cosh(\lambda_*)} \right)^3 \left(1 + \mathcal{O} \left(\frac{j - c_*t}{tc_*} \right) \right),$$

which gives that

$$\begin{aligned} \frac{j}{2\pi} \left(\frac{\zeta}{\gamma} \right)^3 \exp \left(-\frac{(j - c_*t)^2}{4tc_*\gamma} \right) \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} e^{-\frac{t\gamma}{c_*}\xi^2} d\xi &= \frac{c_*^5}{4t \cosh(\lambda_*)^2} \left(\frac{j - c_*t}{\sqrt{2} \cosh(\lambda_*)t} \right)^3 \mathcal{G} \left(\frac{j - c_*t}{\sqrt{2} \cosh(\lambda_*)t} \right) \\ &\quad + \mathcal{R}_j^4(t), \end{aligned}$$

with

$$\left| \mathcal{R}_j^4(t) \right| \lesssim \frac{1}{t^{3/2}} e^{-\beta \frac{(j - c_*t)^2}{t}}.$$

Regarding the last term, we note first that

$$\begin{aligned} \int_{-\sqrt{\epsilon^2 - \eta^2}}^{\sqrt{\epsilon^2 - \eta^2}} \xi^2 e^{-\frac{t\gamma}{c_*}\xi^2} d\xi &= \frac{1}{\left(\frac{t\gamma}{c_*} \right)^{3/2}} \int_{-\sqrt{\epsilon^2 - \eta^2} \sqrt{\frac{t\gamma}{c_*}}}^{\sqrt{\epsilon^2 - \eta^2} \sqrt{\frac{t\gamma}{c_*}}} z^2 e^{-z^2} dz \\ &= -\frac{1}{\left(\frac{t\gamma}{c_*} \right)^{3/2}} \sqrt{\epsilon^2 - \eta^2} \sqrt{\frac{t\gamma}{c_*}} e^{-\frac{t\gamma}{c_*}(\epsilon^2 - \eta^2)} + \frac{1}{2 \left(\frac{t\gamma}{c_*} \right)^{3/2}} \int_{-\sqrt{\epsilon^2 - \eta^2} \sqrt{\frac{t\gamma}{c_*}}}^{\sqrt{\epsilon^2 - \eta^2} \sqrt{\frac{t\gamma}{c_*}}} e^{-z^2} dz. \end{aligned}$$

As a consequence, we have that

$$-3 \frac{\zeta}{\gamma} \frac{j}{2\pi} e^{-\frac{(j-c_*t)^2}{4tc_*\gamma}} \int_{-\sqrt{\epsilon^2-\eta^2}}^{\sqrt{\epsilon^2-\eta^2}} \xi^2 e^{-\frac{t\gamma}{c_*}\xi^2} d\xi = -3 \frac{\zeta}{\gamma} \frac{j}{4\pi \left(\frac{t\gamma}{c_*}\right)^{3/2}} e^{-\frac{(j-c_*t)^2}{4tc_*\gamma}} \int_{-\sqrt{\epsilon^2-\eta^2}\sqrt{\frac{t\gamma}{c_*}}}^{\sqrt{\epsilon^2-\eta^2}\sqrt{\frac{t\gamma}{c_*}}} e^{-z^2} dz + \mathcal{R}_j^5(t),$$

with

$$|\mathcal{R}_j^5(t)| \lesssim \frac{1}{t^{3/2}} e^{-\beta \frac{(j-c_*t)^2}{t}}.$$

Finally, we remark that

$$j \frac{\zeta}{\gamma} \frac{1}{\left(\frac{t\gamma}{c_*}\right)^{3/2}} = \frac{c_*^5}{2t \cosh(\lambda_*)^{5/2}} \frac{j-c_*t}{\sqrt{t}} \left(1 + \mathcal{O}\left(\frac{j-c_*t}{tc_*}\right)\right),$$

for t large enough such that we readily obtain that

$$-3 \frac{\zeta}{\gamma} \frac{j}{4\pi \left(\frac{t\gamma}{c_*}\right)^{3/2}} e^{-\frac{(j-c_*t)^2}{4tc_*\gamma}} \int_{-\sqrt{\epsilon^2-\eta^2}\sqrt{\frac{t\gamma}{c_*}}}^{\sqrt{\epsilon^2-\eta^2}\sqrt{\frac{t\gamma}{c_*}}} e^{-z^2} dz = -\frac{3c_*^5}{4t \cosh(\lambda_*)^2} \frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)t}} \mathcal{G}\left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) + \mathcal{R}_j^6(t),$$

with

$$|\mathcal{R}_j^6(t)| \lesssim \frac{1}{t^{3/2}} e^{-\beta \frac{(j-c_*t)^2}{t}}.$$

Similarly to the previous case, the integrals on $\Gamma_{d,u}$ produce terms which can be subsumed into the above Gaussian estimate. \square

Conclusion. We can now put together the results that we have obtained. In a first step, we managed to decompose $\mathcal{G}_j(t)$ as

$$\mathcal{G}_j(t) = \mathcal{I}_j^1(t) + \mathcal{I}_j^2(t) + \mathcal{I}_j^3(t) + \mathcal{E}_j^1(t) + \mathcal{E}_j^2(t) + \mathcal{E}_j^3(t) + \mathcal{E}_j^4(t) + \frac{1}{2\pi i} \int_{\Gamma_{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu,$$

and proved that

$$|\mathcal{E}_j^1(t)| + |\mathcal{E}_j^2(t)| + |\mathcal{E}_j^3(t)| + |\mathcal{E}_j^4(t)| + \left| \frac{1}{2\pi i} \int_{\Gamma_{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu \right| \lesssim \frac{1}{t^{3/2}} e^{-\beta \frac{(j-c_*t)^2}{t}}$$

Next, we see that if we set

$$\begin{aligned} \mathcal{H}_j(t) := & \frac{1}{\sqrt{2t \cosh(\lambda_*)}} \mathcal{G}\left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) - 3 \left(\frac{1}{2tc_*} + \frac{\Lambda_*}{4tc_* \cosh(\lambda)^2}\right) \left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \mathcal{G}\left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \\ & + \left(\frac{1}{2tc_*} + \frac{\Lambda_*}{4tc_* \cosh(\lambda)^2}\right) \left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right)^3 \mathcal{G}\left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right), \end{aligned}$$

then using Lemma 4.3, Lemma 4.4 and Lemma 4.5

$$\mathcal{I}_j^1(t) + \mathcal{I}_j^2(t) + \mathcal{I}_j^3(t) = \mathcal{H}_j(t) + \widetilde{\mathcal{R}}_j^1(t) + \widetilde{\mathcal{R}}_j^2(t) + \widetilde{\mathcal{R}}_j^3(t),$$

with

$$\left| \widetilde{\mathcal{R}}_j^1(t) \right| + \left| \widetilde{\mathcal{R}}_j^2(t) \right| + \left| \widetilde{\mathcal{R}}_j^3(t) \right| \lesssim \frac{1}{t^{3/2}} e^{-\beta \frac{(j-c_*t)^2}{t}}.$$

As a consequence, we have that

$$\mathcal{R}_j(t) := \mathcal{G}_j(t) - \mathcal{H}_j(t) = \widetilde{\mathcal{R}}_j^1(t) + \widetilde{\mathcal{R}}_j^2(t) + \widetilde{\mathcal{R}}_j^3(t) + \mathcal{E}_j^1(t) + \mathcal{E}_j^2(t) + \mathcal{E}_j^3(t) + \mathcal{E}_j^4(t) + \frac{1}{2\pi i} \int_{\Gamma_{out}} e^{\nu t} \mathbf{G}_j(\nu) d\nu$$

satisfies

$$|\mathcal{R}_j(t)| \leq \frac{C}{t^{3/2}} \exp\left(-\beta \frac{(j-c_*t)^2}{t}\right),$$

for some uniform constants $C > 0$ and $\beta > 0$. Finally, we see that $\mathcal{H}_j(t)$ can be factored into the following condensed formula

$$\mathcal{H}_j(t) = \left[\frac{1}{\sqrt{2t \cosh(\lambda_*)}} + \frac{1}{t} \mathcal{P} \left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)t}} \right) \right] \mathcal{G} \left(\frac{j-c_*t}{\sqrt{2 \cosh(\lambda_*)t}} \right), \quad t > 0, \quad j \in \mathbb{Z}.$$

where the polynomial function \mathcal{P} is given in (4.1). Indeed, with the expression of Λ_* in (4.6), we readily see that

$$\frac{1}{2c_*} + \frac{\Lambda_*}{4c_* \cosh(\lambda_*)^2} = \frac{c_*}{24 \cosh(\lambda_*)^2}.$$

This concludes the proof of Proposition 4.1.

4.2 Sharp asymptotics from odd compactly supported initial conditions

Now, recalling our notation for the linear operator \mathcal{L} given by

$$(\mathcal{L}\mathbf{r})_j = e^{\lambda_*} (r_{j-1} - 2r_j + r_{j+1}) - c_*(r_{j+1} - r_j), \quad j \in \mathbb{Z},$$

we let $w_j(t)$ be the solution of the linear Cauchy problem

$$w'_j(t) - (\mathcal{L}w(t))_j = 0, \quad t > 0, \quad j \in \mathbb{Z}, \quad (4.8)$$

with a nontrivial, odd and compactly supported initial condition w_j^0 for $j \in \mathbb{Z}$ in the sense that

$$w_j^0 = -w_j^0 \geq 0 \quad \forall j \geq 0, \quad w_j^0 = 0 \quad \forall j \geq J_w + 1,$$

for some positive $J_w \geq 1$. Then, $w_j(t)$ can be written in the form

$$w_j(t) = \sum_{\ell=1}^{J_w} (\mathcal{G}_{j-\ell}(t) - \mathcal{G}_{j+\ell}(t)) w_\ell^0, \quad t > 0, \quad j \in \mathbb{Z}.$$

In particular, there exists $j_0 \geq 1$ such that $w_j(t) > 0$ for all $t \geq 0$ and $j - c_*t \geq j_0$.

Now, from Proposition 4.1, for $t \geq T_0$ and each $|j - c_*t| \leq \vartheta t^\alpha$ with $\vartheta > 0$ and $\alpha \in (0, 1)$, we have that the temporal Green's function $\mathcal{G}_{j\pm\ell}(t)$ can be decomposed as

$$\mathcal{G}_{j\pm\ell}(t) = \left[\frac{1}{\sqrt{2 \cosh(\lambda_*)t}} + \frac{1}{t} \mathcal{P} \left(\frac{j \pm \ell - c_*t}{\sqrt{2 \cosh(\lambda_*)t}} \right) \right] \mathcal{G} \left(\frac{j \pm \ell - c_*t}{\sqrt{2 \cosh(\lambda_*)t}} \right) + \mathcal{R}_{j\pm\ell}(t),$$

with

$$|\mathcal{R}_{j\pm\ell}(t)| \leq \frac{C}{t^{3/2}} \exp\left(-\beta \frac{(j \pm \ell - c_*t)^2}{t}\right).$$

Therefore

$$\begin{aligned} w_j(t) &= \frac{1}{\sqrt{2t \cosh(\lambda_*)}} \sum_{\ell=1}^J \left(\mathcal{G}\left(\frac{j-\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) - \mathcal{G}\left(\frac{j+\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \right) w_\ell^0 \\ &+ \frac{1}{t} \sum_{\ell=1}^J \left(\mathcal{P}\left(\frac{j-\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \mathcal{G}\left(\frac{j-\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) - \mathcal{P}\left(\frac{j+\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \mathcal{G}\left(\frac{j+\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \right) w_\ell^0 \\ &+ \sum_{\ell=1}^J (\mathcal{R}_{j-\ell}(t) - \mathcal{R}_{j+\ell}(t)) w_\ell^0. \end{aligned}$$

By Proposition 4.1, we have the bound

$$\left| \sum_{\ell=1}^J (\mathcal{R}_{j-\ell}(t) - \mathcal{R}_{j+\ell}(t)) w_\ell^0 \right| \lesssim \frac{1}{t^{3/2}} \exp\left(-\beta_0 \frac{(j-c_*t)^2}{t}\right) \left(\sum_{\ell=1}^J w_\ell^0 \right),$$

for some $\beta_0 > 0$. Next we remark that for $|j - c_*t| \leq \vartheta t^\alpha$ and as $t \rightarrow +\infty$

$$\begin{aligned} &\exp\left(-\frac{(j-\ell-c_*t)^2}{4t \cosh(\lambda_*)}\right) - \exp\left(-\frac{(j+\ell-c_*t)^2}{4t \cosh(\lambda_*)}\right) \\ &= \exp\left(-\frac{(j-c_*t)^2 + \ell^2}{4t \cosh(\lambda_*)}\right) \left(e^{\frac{\ell(j-c_*t)}{2t \cosh(\lambda_*)}} - e^{-\frac{\ell(j-c_*t)}{2t \cosh(\lambda_*)}} \right) \\ &= \exp\left(-\frac{(j-c_*t)^2 + \ell^2}{4t \cosh(\lambda_*)}\right) \left[\frac{\ell(j-c_*t)}{t \cosh(\lambda_*)} + \mathcal{O}\left(\left(\frac{(j-c_*t)}{t}\right)^3\right) \right]. \end{aligned}$$

As a consequence for $|j - c_*t| \leq \vartheta t^\alpha$, we get

$$\begin{aligned} &\frac{1}{\sqrt{2t \cosh(\lambda_*)}} \sum_{\ell=1}^J \left(\mathcal{G}\left(\frac{j-\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) - \mathcal{G}\left(\frac{j+\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \right) w_\ell^0 \\ &\sim \frac{1}{\sqrt{4\pi t \cosh(\lambda_*)}} \frac{j-c_*t}{t \cosh(\lambda_*)} \exp\left(-\frac{(j-c_*t)^2}{4t \cosh(\lambda_*)}\right) \left(\sum_{\ell=1}^J \ell w_\ell^0 \right), \end{aligned}$$

as $t \rightarrow +\infty$. Now, restricting ourselves to the diffusive regime $\alpha \in (0, 1/2]$, we get that the remaining contributions coming from the difference of the terms in $\mathcal{P}\left(\frac{j\pm\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \mathcal{G}\left(\frac{j\pm\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right)$ are of higher order. Indeed, in that case, we get

$$\begin{aligned} &\sum_{\ell=1}^J \left(\mathcal{P}\left(\frac{j-\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \mathcal{G}\left(\frac{j-\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) - \mathcal{P}\left(\frac{j+\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \mathcal{G}\left(\frac{j+\ell-c_*t}{\sqrt{2 \cosh(\lambda_*)t}}\right) \right) w_\ell^0 \\ &\sim -C_* \frac{j-c_*t}{t^{3/2}} \exp\left(-\frac{(j-c_*t)^2}{4t \cosh(\lambda_*)}\right) \left(\sum_{\ell=1}^J \ell w_\ell^0 \right), \end{aligned}$$

as $t \rightarrow +\infty$ for some universal constant $C_* > 0$. In summary, we have proved the following lemma.

Lemma 4.6. *The solution $(w_j(t))_{j \in \mathbb{Z}}$ of the linear Cauchy problem (4.8) starting from an odd, nontrivial and compactly supported sequence $(w_j^0)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ satisfies, for $|j - c_*t| \leq \vartheta t^\alpha$ with $\alpha \in (0, 1/2]$ and any $\vartheta > 0$, the asymptotic expansion*

$$w_j(t) \sim \frac{1}{\cosh(\lambda_*)^{3/2} \sqrt{4\pi}} \frac{j - c_*t}{t^{3/2}} \left(\sum_{\ell=1}^{\infty} \ell w_\ell^0 \right) \quad \text{as } t \rightarrow +\infty. \quad (4.9)$$

5 Logarithmic delay of the position for the level sets

The aim of this section is to prove Theorem 1 which shows the logarithmic delay on the expansion of the level sets of the solution. To start with, we set

$$v_j(t) := e^{\lambda_*(j - c_*t)} u_j(t), \quad t > 0, \quad j \in \mathbb{Z},$$

such that the sequence $(v_j(t))_{j \in \mathbb{Z}}$ is now solution to the following modified lattice Fisher-KPP equation

$$\begin{cases} v_j'(t) = e^{\lambda_*} (v_{j-1}(t) - 2v_j(t) + v_{j+1}(t)) - c_*(v_{j+1}(t) - v_j(t)) - \mathcal{R}_j(t; v_j(t)), & t > 0, \quad j \in \mathbb{Z}, \\ v_j(0) = v_j^0 = e^{\lambda_* j} u_j^0, & j \in \mathbb{Z}. \end{cases} \quad (5.1)$$

Here, the nonlinear term $\mathcal{R}_j(t; s)$ is defined as

$$\mathcal{R}_j(t; s) := f'(0)s - e^{\lambda_*(j - c_*t)} f(e^{-\lambda_*(j - c_*t)} s), \quad (5.2)$$

for $(t, j) \in \mathbb{R} \times \mathbb{Z}$. Next, we remark that $\mathcal{R}_j(t; s) \geq 0$ for all $s \in \mathbb{R}$ and $(t, j) \in \mathbb{R} \times \mathbb{Z}$. This directly comes from our standing assumptions that $0 < f(u) \leq f'(0)u$ for $u \in (0, 1)$ and the fact that we extended linearly f for $u \in (-\infty, 0) \cup (1, +\infty)$. As a consequence, we readily notice that $\mathcal{R}_j(t; v_j(t)) \geq 0$ for $(t, j) \in (0, +\infty) \times \mathbb{Z}$, since $0 < e^{-\lambda_*(j - c_*t)} v_j(t) = u_j(t) < 1$ in $(0, +\infty) \times \mathbb{Z}$. Finally, throughout the section, for $x \in \mathbb{R}$ the integer part of x will be denoted as $\lfloor x \rfloor \in \mathbb{Z}$.

5.1 Upper and lower bounds for $v_j(t)$

In this section, we provide upper and lower barriers for the function $v_j(t)$ for t sufficiently large and $j \in \mathbb{Z}$ ahead of the position $j - c_*t \sim 0$.

5.1.1 An upper barrier for $v_j(t)$

We start with the construction of a supersolution to (5.1) for all t large enough and $j \in \mathbb{Z}$ ahead of $j - c_*t \sim 0$ by following the strategy we developed in Section 2. More precisely, consider $\delta \in (0, 1/3)$, that will be as small as needed. We will estimate $v_j(t)$ ahead of $j - c_*t \sim -t^\delta$. To do so, we construct a supersolution for (5.1) as follows:

$$\bar{v}_j(t) := \bar{\xi}(t) w_j(t) + \frac{1}{(1+t)^{\frac{3}{2}-\beta}} \cos\left(\frac{j - c_*t}{(1+t)^\alpha}\right) \mathbb{1}_{\{j \in \mathbb{Z} \mid -t^\delta \leq j - c_*t \leq \frac{3\pi}{2}(1+t)^\alpha\}}, \quad (5.3)$$

for t large enough and $j \in \mathbb{Z}$ with $j - c_*t \geq -t^\delta$, where the unknown $\bar{\xi}(t) \in \mathcal{C}^1$ is assumed to be positive and bounded in $(0, +\infty)$, and $\bar{\xi}'(t) \geq 0$ in $(0, +\infty)$. The other parameters $\alpha \in (1/3, 1/2)$ and $\beta > 0$ will be determined in the course of investigation. Let us now check that $\bar{v}_j(t)$ is a supersolution of (5.1) for t large enough and $j \in \mathbb{Z}$ with $j - c_*t \geq -t^\delta$. For notational convenience, we define the following sequence

$$p_j(t) := \frac{1}{(1+t)^{\frac{3}{2}-\beta}} \cos\left(\frac{j-c_*t}{(1+t)^\alpha}\right), \quad t > 0, \quad j \in \mathbb{Z}.$$

The key point of the forthcoming computations will be to verify that the action of the linear operator \mathcal{L} on the above cosine perturbation is well behaved within the range of interest.

First of all, we look at the region $\{j \in \mathbb{Z} \mid j - c_*t \geq \frac{3\pi}{2}(1+t)^\alpha (> j_0)\}$ for all t large enough, in which $\bar{v}_j(t) = \bar{\xi}(t)w_j(t)$. It is easily seen that $\bar{v}'_j(t) - (\mathcal{L}\bar{v}(t))_j \geq 0$ due to the assumption that $\bar{\xi}'(t) \geq 0$ for $t > 0$ and that $w_j(t) > 0$ for all $t \geq 0$ and $j \in \mathbb{Z}$ with $j - c_*t \geq j_0$. It remains to discuss the region $\{j \in \mathbb{Z} \mid -t^\delta \leq j - c_*t \leq \frac{3\pi}{2}(1+t)^\alpha\}$ for all large times. We divide it into three zones:

$$\begin{aligned} R_1 &:= \left\{j \in \mathbb{Z} \mid -t^\delta \leq j - c_*t \leq j_0\right\}, & R_2 &:= \left\{j \in \mathbb{Z} \mid j_0 \leq j - c_*t \leq \frac{\pi}{4}(1+t)^\alpha\right\}, \\ R_3 &:= \left\{j \in \mathbb{Z} \mid \frac{\pi}{4}(1+t)^\alpha \leq j - c_*t \leq \frac{3\pi}{2}(1+t)^\alpha\right\}. \end{aligned}$$

In region R_1 . There holds

$$\frac{-t^\delta}{(1+t)^\alpha} \leq \frac{j-c_*t}{(1+t)^\alpha} \leq \frac{j_0}{(1+t)^\alpha}.$$

Due to $\delta < \alpha$ and due to the asymptotics (4.9) of $w_j(t)$, we get that

$$p_j(t) \sim \frac{1}{(1+t)^{\frac{3}{2}-\beta}} \quad \text{and} \quad -\frac{1}{(1+t)^{\frac{3}{2}-\delta}} \lesssim w_j(t) \lesssim \frac{1}{(1+t)^{\frac{3}{2}}} \quad \text{for } t \text{ large enough.}$$

Therefore the cosine perturbation will enable $\bar{v}_j(t)$ to be positive in this region. To be specific, we first require that $\beta > \delta > 0$, so that the cosine term plays a dominant role here, namely,

$$\bar{v}_j(t) \sim \frac{1}{(1+t)^{\frac{3}{2}-\beta}} > 0 \quad \text{for } t \text{ large enough.} \quad (5.4)$$

Moreover, a straightforward computation gives that, for t large enough,

$$(\bar{\xi}(t)w_j(t))' - (\mathcal{L}[\bar{\xi}(t)w(t)])_j = \bar{\xi}'(t)w_j(t) \gtrsim -\frac{\bar{\xi}'(t)}{(1+t)^{\frac{3}{2}-\delta}},$$

and we also have

$$\begin{aligned}
p'_j(t) - (\mathcal{L}p(t))_j &= \frac{\beta - \frac{3}{2}}{(1+t)^{\frac{3}{2}-\beta+1}} \cos\left(\frac{j-c_*t}{(1+t)^\alpha}\right) + \frac{1}{(1+t)^{\frac{3}{2}-\beta}} \left(\frac{c_*}{(1+t)^\alpha} + \alpha \frac{j-c_*t}{(1+t)^{\alpha+1}} \right) \sin\left(\frac{j-c_*t}{(1+t)^\alpha}\right) \\
&\quad + \frac{4e^{\lambda_*}}{(1+t)^{\frac{3}{2}-\beta}} \sin^2\left(\frac{1/2}{(1+t)^\alpha}\right) \cos\left(\frac{j-c_*t}{(1+t)^\alpha}\right) \\
&\quad - \frac{2c_*}{(1+t)^{\frac{3}{2}-\beta}} \sin\left(\frac{1/2}{(1+t)^\alpha}\right) \sin\left(\frac{j-c_*t+1/2}{(1+t)^\alpha}\right) \\
&\sim \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}}.
\end{aligned}$$

Therefore, in order to ensure that $\bar{v}'_j(t) - (\mathcal{L}\bar{v}(t))_j \geq 0$ for t large enough in this region, it suffices for $\bar{\xi}(t)$ to satisfy

$$\frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} \gg \frac{\bar{\xi}'(t)}{(1+t)^{\frac{3}{2}-\delta}} \quad \text{for } t \text{ large enough.}$$

We can then require that

$$0 \leq \bar{\xi}'(t) \ll \frac{1}{(1+t)^{2\alpha+\delta-\beta}} \quad \text{for } t > 0. \quad (5.5)$$

In region R_2 . There holds

$$\frac{j_0}{(1+t)^\alpha} \leq \frac{j-c_*t}{(1+t)^\alpha} \leq \frac{\pi}{4}.$$

We notice that $w_j(t) > 0$ for all $t > 0$ in this region, so does the cosine perturbation. Since $\bar{\xi}(t)$ is assumed a priori to be positive in $(0, +\infty)$, one has that the function $\bar{v}_j(t) > 0$ for all $t > 0$ in this region. Moreover,

$$(\bar{\xi}(t)w_j(t))' - (\mathcal{L}[\bar{\xi}(t)w(t)])_j = \bar{\xi}'(t)w_j(t) \geq 0$$

for all $t > 0$ in this region, due to our requirement that $\bar{\xi}'(t) \geq 0$ in $(0, +\infty)$. By the same calculation as in region R_1 , we have the following asymptotics in R_2

$$\left| p'_j(t) - (\mathcal{L}p(t))_j \right| \sim \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} > 0 \quad \text{for } t \text{ large enough.}$$

Consequently, there holds $\bar{v}'_j(t) - (\mathcal{L}\bar{v}(t))_j \geq 0$ for t large enough in region R_2 .

In region R_3 . There holds

$$\frac{\pi}{4} \leq \frac{j-c_*t}{(1+t)^\alpha} \leq \frac{3\pi}{2}.$$

Again, one has

$$(\bar{\xi}(t)w_j(t))' - (\mathcal{L}[\bar{\xi}(t)w(t)])_j = \bar{\xi}'(t)w_j(t) \geq 0$$

for all $t > 0$ in this region. Following an analogous procedure as in preceding cases, it is seen that

$$p'_j(t) - (\mathcal{L}p(t))_j \sim \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} \quad \text{for } t \text{ large enough.}$$

One also observes from (4.9) that

$$w_j(t) \sim \frac{1}{(1+t)^{\frac{3}{2}-\alpha}} \quad \text{for } t \text{ large enough.}$$

However, the cosine perturbation may be negative in this area. Therefore, to ensure that $\bar{v}_j(t) > 0$ and $\bar{v}'_j(t) - (\mathcal{L}\bar{v}(t))_j \geq 0$ for t large enough in this region, we require this time $\alpha > \beta$ and

$$\frac{\bar{\xi}'(t)}{(1+t)^{\frac{3}{2}-\alpha}} \gg \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} \quad \text{for } t \text{ large enough.} \quad (5.6)$$

Conclusion. Gathering (5.5) and (5.6), we should impose as in the continuous case

$$\frac{1}{(1+t)^{3\alpha-\beta}} \ll \bar{\xi}'(t) \ll \frac{1}{(1+t)^{2\alpha+\delta-\beta}} \quad \text{for } t > 0.$$

This is possible so long as $\delta < \alpha$, which is exactly what we have assumed. Let us take

$$\bar{\xi}'(t) \sim \frac{1}{(1+t)^{3\alpha-2\beta}} \quad \text{and} \quad \bar{\xi}(t) = 1 - \frac{1}{(1+t)^{3\alpha-2\beta-1}} \quad \text{for } t \geq 0,$$

which yields that $3\alpha - 2\beta > 2\alpha + \delta - \beta$, i.e., $\beta < \alpha - \delta$. Due to our assumption that the function $\bar{\xi}(t)$ is positive and bounded in $(0, +\infty)$, it suffices to require $3\alpha - 2\beta - 1 > 0$. Hence, we can fix $\delta \in (0, 1/4)$ very small, then there exist $\alpha \in (1/3, 1/2)$ and $\beta > 0$ such that

$$0 < \delta < \beta < \min\left(\alpha - \delta, \frac{3\alpha - 1}{2}\right) < \alpha < \frac{1}{2}. \quad (5.7)$$

As a consequence, one can choose $T_0 > 0$ sufficiently large such that $\bar{v}'_j(t) - (\mathcal{L}\bar{v}(t))_j \geq 0$ for $t \geq T_0$ and $j \in \mathbb{Z}$ with $j - c_*t \geq -t^\delta$, whence $\bar{v}'_j(t) - (\mathcal{L}\bar{v}(t))_j + \mathcal{R}_j(t; \bar{v}_j(t)) \geq 0$ for $t \geq T_0$ and $j \in \mathbb{Z}$ with $j \geq c_*t - t^\delta$, since $\mathcal{R}_j(t; \bar{v}_j(t)) \geq 0$ for $t \geq T_0$ and $j \in \mathbb{Z}$ with $j - c_*t \geq -t^\delta$ by applying (5.2) and $\bar{v}_j(t) > 0$ for $t \geq T_0$ and $j \in \mathbb{Z}$ with $j \geq c_*t - t^\delta$, and such that $c_*T_0 - T_0^\delta > J$, where we recall that $u_j^0 = 0$ for $j \geq J$. Hence, there holds $\bar{v}_j(T_0) > 0 = e^{\lambda_* j} u_j^0 = v_j(0)$ for $j \in \mathbb{Z}$ with $j - c_*T_0 \geq -T_0^\delta$. For $j = \lfloor c_*t - t^\delta \rfloor$, we observe that $v_j(t - T_0) = e^{\lambda_*(j - c_*(t - T_0))} u_j(t - T_0) \leq e^{\lambda_*(c_*T_0 - t^\delta)}$ since $0 \leq u_j(t) \leq 1$ for all $t \geq 0$ and $j \in \mathbb{Z}$, while $\bar{v}_j(t) \sim (1+t)^{-3/2+\beta}$ for $t \geq T_0$ by (5.4), up to increasing T_0 . Then, there is $T_1 > T_0$ sufficiently large such that $(1+t)^{-3/2+\beta} > e^{\lambda_*(c_*T_0 - t^\delta)}$ for all $t \geq T_1$, which will yield that $\bar{v}_j(t) > v_j(t - T_0)$ at $j = \lfloor c_*t - t^\delta \rfloor$ for all $t \geq T_1$. On the other hand, one can choose $K > 0$ large enough such that $K(1+t)^{-3/2+\beta} > e^{\lambda_*(c_*T_0 - t^\delta)}$, that is, $K\bar{v}_j(t) > v_j(t - T_0)$ at $j = \lfloor c_*t - t^\delta \rfloor$ for $t \in [T_0, T_1]$. Therefore, there holds $K\bar{v}_j(t) > v_j(t - T_0)$ for all $t \geq T_0$ and $j \in \mathbb{Z}$ with $j = \lfloor c_*t - t^\delta \rfloor$. We then conclude that $K\bar{v}_j(t)$ is a supersolution of (5.1) for all $t \geq T_0$ and $j \in \mathbb{Z}$ with $j \geq \lfloor c_*t - t^\delta \rfloor$. It follows from the strong maximum principle that $K\bar{v}_j(t) > v_j(t - T_0)$ for all $t > T_0$ and $j \in \mathbb{Z}$ with $j - c_*t > -t^\delta$.

Lemma 5.1. *Let $\delta \in (0, 1/4)$, $\alpha \in (1/3, 1/2)$ and $\beta > 0$ be fixed such that (5.7) holds. Then, there exists $K > 0$ such that the solution $(v_j(t))_{j \in \mathbb{Z}}$ of (5.1) satisfies*

$$v_j(t - T_0) < K\bar{v}_j(t),$$

for all $t > T_0$ and $j \in \mathbb{Z}$ with $j - c_*t > -t^\delta$ where the supersolution $(\bar{v}_j(t))_{j \in \mathbb{Z}}$ is given in (5.3).

5.1.2 Lower barrier for $v_j(t)$

In the special case that f is linear in a small neighborhood of 0, namely, $f(s) = f'(0)s$ for $s \in [0, s_0)$, with $s_0 \in (0, 1)$ small, we would be able to control $v_j(t)$ by some multiple of $w_j(t)$ from below. Nevertheless, the nonlinear term f is not linear in the vicinity of 0 in general, for which we still hope to manage controlling $v_j(t)$ by $w_j(t)$. Accordingly, we need to do it in an area where the nonlinear term is negligible. Let δ , β and α be fixed as in (5.7). The idea is to estimate $v_j(t)$ ahead of $j - c_*t \sim t^\delta$. To do so, let construct a lower barrier as follows:

$$\underline{v}_j(t) := \underline{\xi}(t)w_j(t) - \frac{1}{(1+t)^{\frac{3}{2}-\beta}} \cos\left(\frac{j-c_*t}{(1+t)^\alpha}\right) \mathbb{1}_{\{j \in \mathbb{Z} \mid t^\delta \leq j-c_*t \leq \frac{3\pi}{2}(1+t)^\alpha\}}, \quad (5.8)$$

for t large enough and $j \in \mathbb{Z}$ with $(j_0 <)t^\delta \leq j - c_*t$, where we assume that the unknown $\underline{\xi}(t) \in \mathcal{C}^1$ is positive and bounded away from 0 in $(0, +\infty)$ and satisfies $\underline{\xi}'(t) \leq 0$ in $(0, +\infty)$, which will be made clear in the sequel. Again, we define the sequence $p(t) = (p_j(t))_{j \in \mathbb{Z}}$ as

$$p_j(t) = \frac{1}{(1+t)^{\frac{3}{2}-\beta}} \cos\left(\frac{j-c_*t}{(1+t)^\alpha}\right), \quad t > 0, \quad j \in \mathbb{Z}.$$

Let now verify that $\underline{v}_j(t)$ is a subsolution of (5.1) for t large enough and $j \in \mathbb{Z}$ with $j - c_*t \geq t^\delta$. We first note that, since $f \in \mathcal{C}^2([0, 1])$, there exist $M > 0$ and $s_0 \in (0, 1)$ such that $f(s) - f'(0)s \geq -Ms^2$ for $s \in [0, s_0)$. Gathering this with the linear extension of f on $(-\infty, 0)$, one deduces, for t large enough and $j \in \mathbb{Z}$ with $j - c_*t \geq t^\delta$,

$$\begin{aligned} \mathcal{R}_j(t; s_j(t)) &= f'(0)s_j(t) - e^{\lambda_*(j-c_*t)} f(e^{-\lambda_*(j-c_*t)} s_j(t)) \\ &= e^{\lambda_*(j-c_*t)} \left(f'(0)e^{-\lambda_*(j-c_*t)} s_j(t) - f(e^{-\lambda_*(j-c_*t)} s_j(t)) \right), \\ &\leq M e^{-\lambda_*(j-c_*t)} (s_j(t))^2 \leq M e^{-\lambda_* t^\delta} (s_j(t))^2, \end{aligned}$$

as long as $e^{-\lambda_* t^\delta} s_j(t) \in (0, s_0)$ for t large enough and $j \in \mathbb{Z}$ with $j - c_*t \geq t^\delta$, while $\mathcal{R}_j(t; s_j(t)) = 0$ when $s_j \leq 0$. We shall require $\underline{v}_j(t)$ to satisfy $e^{-\lambda_* t^\delta} \underline{v}_j(t) < s_0$ in $\{j \in \mathbb{Z} \mid j - c_*t \geq t^\delta\}$ for t large enough, which is possible due to the asymptotics of $w_j(t)$ as well as our assumptions on $\underline{\xi}(t)$ and on the parameters.

As proceeded in the previous section, we start with the region $\{j \in \mathbb{Z} \mid j - c_*t \geq \frac{3\pi}{2}(1+t)^\alpha\}$. Here, $\underline{v}_j(t) = \underline{\xi}(t)w_j(t) > 0$, and

$$\begin{aligned} \underline{v}'_j(t) - (\mathcal{L}\underline{v}(t))_j + \mathcal{R}_j(t; \underline{v}_j(t)) &\leq \underline{v}'_j(t) - (\mathcal{L}\underline{v}(t))_j + M e^{-\lambda_* t^\delta} (\underline{v}_j(t))^2 \\ &= \underline{\xi}'(t)w_j(t) + M e^{-\lambda_* t^\delta} \underline{\xi}^2(t)(w_j(t))^2 \\ &= \left(\underline{\xi}'(t) + M \underline{\xi}^2(t) e^{-\lambda_* t^\delta} w_j(t) \right) w_j(t). \end{aligned}$$

Taking into account the boundedness of w_j from (3.19), we see that

$$e^{-\lambda_* t^\delta} w_j(t) \leq C(1+t)^{-2}, \quad (5.9)$$

with some $C > 0$, for all t large enough and $j \in \mathbb{Z}$ with $j - c_*t \geq t^\delta$. It is then sufficient for $\underline{\xi}(t)$ to solve

$$\underline{\xi}'(t) = -CM\underline{\xi}^2(t)(1+t)^{-2} \quad \text{for } t > 0. \quad (5.10)$$

From (5.9) and (5.10), one has that $\underline{v}'_j(t) - (\mathcal{L}\underline{v}(t))_j + \mathcal{R}_j(t; v_j(t)) \leq 0$ for t large enough and $j \in \mathbb{Z}$ with $j - c_*t \geq \frac{3\pi}{2}(1+t)^\alpha$.

Next, we divide the remaining region into two zones:

$$R_1 := \left\{ j \in \mathbb{Z} \mid t^\delta \leq j - c_*t \leq (1+t)^\beta \right\} \quad \text{and} \quad R_2 := \left\{ j \in \mathbb{Z} \mid (1+t)^\beta \leq j - c_*t \leq \frac{3\pi}{2}(1+t)^\alpha \right\}.$$

In region R_1 . Thanks to our choice of δ , β and α , by requiring $\underline{\xi}(0) > 0$ to be sufficiently small, we derive the following estimate

$$\underline{v}_j(t) \lesssim \frac{j - c_*t}{(1+t)^{\frac{3}{2}}} - \frac{1}{(1+t)^{\frac{3}{2}-\beta}} \leq 0 \quad \text{for } t \text{ large enough,} \quad (5.11)$$

which then implies $\mathcal{R}_j(t; \underline{v}_j(t)) = 0$. Since $w_j(t) > 0$ for t large enough in R_1 and since $\underline{\xi}'(t) \leq 0$ in $(0, +\infty)$, we have

$$(\underline{\xi}(t)w_j(t))' - (\mathcal{L}[\underline{\xi}(t)w(t)])_j = \underline{\xi}'(t)w_j(t) \leq 0 \quad \text{for } t \text{ large enough.}$$

Moreover, it follows from a direct calculation that

$$p'_j(t) - (\mathcal{L}p(t))_j \sim \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} \quad \text{for } t \text{ large enough.}$$

We eventually get, in region R_1 ,

$$\underline{v}'_j(t) - (\mathcal{L}\underline{v}(t))_j \sim \underline{\xi}'(t)w_j(t) - \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} < 0 \quad \text{for } t \text{ large enough.}$$

In region R_2 . If $\underline{v}_j(t) \leq 0$, that is, $j - c_*t \sim (1+t)^\beta$, then $\mathcal{R}_j(t; \underline{v}_j(t)) = 0$ and the analysis in the previous case shows that $\underline{v}'_j(t) - (\mathcal{L}\underline{v}(t))_j \leq 0$ for t large enough. Now, it is left to discuss the situation that $\underline{v}_j(t) > 0$ for t large enough in this area R_2 . We note here that

$$0 < \underline{v}_j(t) \lesssim \frac{1}{(1+t)^{\frac{3}{2}-\alpha}} \quad \text{for } t \text{ large enough.}$$

It is obvious to see that $e^{-\lambda_*t^\delta} \frac{1}{(1+t)^{3-2\alpha}} \ll \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}}$ for t large enough, which will imply $e^{-\lambda_*t^\delta} (\underline{v}_j(t))^2 \ll \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}}$ for all t large in this area, whence, for t large enough in R_2 ,

$$\begin{aligned} \underline{v}'_j(t) - (\mathcal{L}\underline{v}(t))_j + \mathcal{R}_j(t; \underline{v}_j(t)) &\leq \underline{v}'_j(t) - (\mathcal{L}\underline{v}(t))_j + Me^{-\lambda_*t^\delta} (\underline{v}_j(t))^2 \\ &\sim \frac{\underline{\xi}'(t)}{(1+t)^{\frac{3}{2}-\beta}} - \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}} \leq 0. \end{aligned}$$

Conclusion. We require that $\underline{\xi}(t)$ is the solution of the ODE (5.10) starting from a sufficiently small initial datum $\underline{\xi}(0) = \underline{\xi}_0 > 0$. Then, $\underline{\xi}(t)$ is positive and uniformly bounded from above and below in $[0, +\infty)$ such that

$$0 < \frac{\underline{\xi}_0}{1 + \underline{\xi}_0 CM} \leq \underline{\xi}(t) \leq \underline{\xi}_0 < +\infty \quad \text{for } t \geq 0.$$

There is $t_0 > 0$ large enough such that $e^{-\lambda_* t^\delta} \frac{1}{(1+t)^{3-2\alpha}} \ll \frac{1}{(1+t)^{\frac{3}{2}-\beta+2\alpha}}$ for $t \geq t_0$ and such that $e^{-\lambda_* t^\delta} \underline{v}_j(t) < s_0$ and $e^{-\lambda_* t^\delta} w_j(t) \leq C(1+t)^{-2}$ for all $t \geq t_0$ and $j \in \mathbb{Z}$ with $j - c_* t \geq t^\delta$. Moreover, it follows from above analysis that $\underline{v}'_j(t) - (\mathcal{L}\underline{v}(t))_j + \mathcal{R}_j(t; \underline{v}_j(t)) \leq 0$ for $t \geq t_0$ and $j \in \mathbb{Z}$ with $j - c_* t \geq t^\delta$. Now let us prove that there is $\kappa > 0$ small enough such that $v_j(t_0+1) > \kappa v_j(t_0)$ for $j \in \mathbb{Z}$ with $j - c_* t_0 \geq t_0^\delta$. As a matter of fact, we notice that the sequence function $z_j(t) := e^{-f'(0)t} w_j(t)$, $j \in \mathbb{Z}$, is the solution of

$$z'_j(t) - (\mathcal{L}z(t))_j + f'(0)z_j(t) = 0 \quad t > 0, j \in \mathbb{Z}, \quad (5.12)$$

with compactly supported initial value $z_j(0) = w_j(0)$, while $v_j(t)$ satisfies (5.12) with “=” replaced by “ \geq ”. Moreover, there exists $\kappa_1 > 0$ small enough such that $v_j(1) = e^{\lambda_*(j-c_*)} u_j(1) > \kappa_1 w_j(0) = \kappa_1 z_j(0)$ in \mathbb{Z} . The strong maximum principle immediately yields that $v_j(t+1) > \kappa_1 e^{-f'(0)t} w_j(t)$ for all $t \geq 0$ and $j \in \mathbb{Z}$. In particular, we have $v_j(t_0+1) > \kappa_1 e^{-f'(0)t_0} w_j(t_0)$ for $j \in \mathbb{Z}$ with $j - c_* t_0 \geq t_0^\delta$. We can then choose $\kappa > 0$ small enough such that $\kappa_1 e^{-f'(0)t_0} w_j(t_0) > \kappa \underline{\xi}_0 w_j(t_0)$ for $j \in \mathbb{Z}$ with $j - c_* t_0 \geq t_0^\delta$ and such that particularly $\kappa_1 e^{-f'(0)t_0} w_j(t_0) > \kappa \left(\underline{\xi}_0 w_j(t_0) + (1+t_0)^{-3/2+\beta} \right)$ for $j \in \mathbb{Z}$ with $c_* t_0 + \frac{\pi}{2}(1+t_0)^\alpha \leq j \leq c_* t_0 + \frac{3\pi}{2}(1+t_0)^\alpha$. This implies that $v_j(t_0+1) > \kappa v_j(t_0)$ for $j \in \mathbb{Z}$ with $j - c_* t_0 \geq t_0^\delta$. For $j = \lfloor c_* t + t^\delta \rfloor + 1$, we have, up to increasing t_0 if necessary, $\kappa v_j(t) < 0$ for all $t \geq t_0$ by (5.11) and $v_j(t+1) > e^{\lambda_* t^\delta} u_j(t+1) > 0$ for all $t \geq t_0$, which implies that $v_j(t+1) > \kappa v_j(t)$ for all $t \geq t_0$. Therefore, $\kappa v_j(t)$ is a subsolution of (5.1) for $t \geq t_0$ and $j \in \mathbb{Z}$ with $j \geq \lfloor c_* t + t^\delta \rfloor + 1$. The strong maximum principle yields that $\kappa v_j(t) < v_j(t+1)$ for $t > t_0$ and $j \in \mathbb{Z}$ with $j - c_* t > t^\delta$.

Lemma 5.2. *Let $\delta \in (0, 1/4)$, $\alpha \in (1/3, 1/2)$ and $\beta > 0$ be fixed such that (5.7) holds. Then, there exists $\kappa > 0$ such that the solution $(v_j(t))_{j \in \mathbb{Z}}$ of (5.1) satisfies*

$$\kappa v_j(t) < v_j(t+1),$$

for all $t > t_0$ and $j \in \mathbb{Z}$ with $j - c_* t > t^\delta$ where the subsolution $(\underline{v}_j(t))_{j \in \mathbb{Z}}$ is given in (5.8).

5.2 Proof of Theorem 1

Using the upper and lower barriers of v_j , constructed in Lemma 5.1 and Lemma 5.2 respectively, as key ingredients, we are now in position to prove Theorem 1 which gives a refined estimate, up to $\mathcal{O}(1)$ precision, of the level sets for initially localized solutions $u_j(t)$ to (1.1) for large times. Before proceeding with the proof of Theorem 1, we normalize the minimal traveling front profile U_{c_*} , solution of (1.10), according to its asymptotic behavior at $+\infty$. More precisely, using the result of [8], we normalize U_{c_*} so as to satisfy asymptotically

$$\frac{U_{c_*}(x)}{x e^{-\lambda_* x}} \xrightarrow{x \rightarrow +\infty} 1. \quad (5.13)$$

Proof of Theorem 1. The proof is based on the comparison between $v_j(t)$ and a variant of the shifted minimal traveling front with logarithmic correction for all large times in the moving zone $|j - c_*t| \leq t^\eta$ for some small η . We will show, respectively, the upper and lower bounds for the level sets of $u_j(t)$ for large times. Let δ , β and α be chosen as in (5.7), which also implies that $\beta < 1/4$. Fix now

$$\eta = \beta + \varepsilon < \alpha \quad (5.14)$$

for some $\varepsilon > 0$ small enough.

Step 1: Upper bound. Let $K\bar{v}_j(t)$ be the supersolution of (5.1), given in Lemma 5.1, for t large enough and $j \in \mathbb{Z}$ with $j - c_*t \geq -t^\delta$. This implies that there exists some constant $C > 0$ such that

$$v_j(t - T_0) \leq K\bar{v}_j(t) \leq C \frac{j - c_*t}{t^{3/2}} \quad \text{for } \lfloor c_*t + t^\eta \rfloor \leq j \leq \lfloor c_*t + t^\eta \rfloor + 1,$$

and for t large enough. Define now the sequence $\psi_j(t)$ by

$$\psi_j(t) = e^{\lambda_*(j - c_*t)} U_{c_*} \left(j - c_*t + \frac{3}{2\lambda_*} \ln t + b \right),$$

where $b \in \mathbb{R}$ is fixed such that $\psi_j(t) \geq K\bar{v}_j(t + T_0 + 1)$ for t large enough and $\lfloor c_*t + t^\eta \rfloor \leq j \leq \lfloor c_*t + t^\eta \rfloor + 1$. This is always possible thanks to our normalization for the minimal traveling front U_{c_*} in (5.13) which ensures that

$$\psi_j(t) \sim \frac{j - c_*t + \frac{3}{2\lambda_*} \ln t + b}{t^{3/2}} \quad \text{for } \lfloor c_*t + t^\eta \rfloor \leq j \leq \lfloor c_*t + t^\eta \rfloor + 1,$$

and for t large enough. Substituting $\psi_j(t)$ into the equation of $v_j(t)$ leads to

$$\begin{aligned} \psi_j'(t) - e^{\lambda_*} (\psi_{j-1}(t) - 2\psi_j(t) + \psi_{j+1}(t)) + c_* (\psi_{j+1}(t) - \psi_j(t)) + \mathcal{R}_j(t; \psi_j(t)) \\ = \frac{3}{2\lambda_* t} U_{c_*}' \left(j - c_*t + \frac{3}{2\lambda_*} \ln t + b \right) \sim \frac{1}{t^{5/2-\eta}} < O\left(t^{-(1-\eta)}\right), \end{aligned}$$

for t large enough and $j \in \mathbb{Z}$ with $\lfloor c_*t - t^\eta \rfloor \leq j \leq \lfloor c_*t + t^\eta \rfloor + 1$. Now, set $s_j(t) := (v_j(t) - \psi_j(t))^+$, being the positive part of $v_j(t) - \psi_j(t)$. Then, $s_j(t)$ satisfies

$$\begin{cases} s_j'(t) - (\mathcal{L}s(t))_j + \mathcal{Q}_j(t; s_j(t)) < O\left(t^{-(1-\eta)}\right), & \lfloor c_*t - t^\eta \rfloor \leq j \leq \lfloor c_*t + t^\eta \rfloor + 1, \\ s_j(t) = O\left(e^{-\lambda_* t^\eta}\right), & j = \lfloor c_*t - t^\eta \rfloor, \\ s_j(t) = 0, & j = \lfloor c_*t + t^\eta \rfloor + 1, \end{cases} \quad (5.15)$$

for t large enough. Here, $\mathcal{Q}_j(t; s_j(t)) = 0$ if $s_j(t) = 0$, and otherwise,

$$\begin{aligned} \mathcal{Q}_j(t; s_j(t)) &= \mathcal{R}_j(t; v_j(t)) - \mathcal{R}_j(t; \psi_j(t)) \\ &= f'(0)s_j(t) - e^{\lambda_*(j - c_*t)} \left(f(e^{-\lambda_*(j - c_*t)} v_j(t)) - f(e^{-\lambda_*(j - c_*t)} \psi_j(t)) \right) \\ &= f'(0)s_j(t) - b_j(t)s_j(t) \geq 0, \end{aligned}$$

in which b_j is a continuous function, and all $\overline{b_j}$'s are bounded in ℓ^∞ norm by $f'(0)$ since $0 < f(s) \leq f'(0)s$ for $s \in (0, 1)$. We claim that, there holds

$$\lim_{t \rightarrow +\infty} \sup_{j \in \mathbb{Z}, |j - c_* t| \leq t^\eta} s_j(t) = 0. \quad (5.16)$$

We use a comparison argument to verify this. Define

$$\overline{s}_j(t) = \frac{1}{t^\lambda} \cos\left(\frac{j - c_* t}{t^\gamma}\right)$$

where $\lambda > 0$ is to be chosen later and $\gamma > \frac{1}{3}$ close to $\frac{1}{3}$ is such that $2\gamma + \eta < 1$, which is possible so long as $\varepsilon > 0$ is chosen sufficiently small in (5.14) such that $\eta \sim \beta < 1/4$ due to (5.7). It follows that $\overline{s}_j(t) \sim t^{-\lambda}$ for t large enough and $j \in \mathbb{Z}$ with $\lfloor c_* t - t^\eta \rfloor \leq j \leq \lfloor c_* t + t^\eta \rfloor + 1$. It also follows from a direct computation that, for t large enough and $j \in \mathbb{Z}$ with $\lfloor c_* t - t^\eta \rfloor \leq j \leq \lfloor c_* t + t^\eta \rfloor + 1$,

$$\overline{s}'_j(t) - e^{\lambda_*} (\overline{s}_{j-1}(t) - 2\overline{s}_j(t) + \overline{s}_{j+1}(t)) + c_*(\overline{s}_{j+1}(t) - \overline{s}_j(t)) \sim \frac{1}{t^{2\gamma+\lambda}}.$$

We also note that $\mathcal{Q}_j(t; \overline{s}_j(t))$ is nonnegative. Therefore, $\overline{s}_j(t)$ (up to multiplying by a large constant) is a supersolution of (5.15) for t large enough and $j \in \mathbb{Z}$ with $\lfloor c_* t - t^\eta \rfloor \leq j \leq \lfloor c_* t + t^\eta \rfloor + 1$ so long as

$$\frac{1}{t^{2\gamma+\lambda}} > \frac{1}{t^{1-\eta}},$$

that is, $2\gamma + \lambda < 1 - \eta$, which implies $0 < \lambda < 1 - 2\gamma - \eta$. Our claim (5.16) is then reached by noticing that

$$\lim_{t \rightarrow +\infty} \sup_{j \in \mathbb{Z}, |j - c_* t| \leq t^\eta} \overline{s}_j(t) = 0.$$

Eventually, one get $v_j(t) \leq \psi_j(t)$ uniformly in $j \in \mathbb{Z}$ with $|j - c_* t| \leq t^\eta$ for t large enough. This implies

$$u_j(t) \leq U_{c_*} \left(j - c_* t + \frac{3}{2\lambda_*} \ln t + b \right), \quad (5.17)$$

uniformly in $j \in \mathbb{Z}$ with $|j - c_* t| \leq t^\eta$ for t large enough.

Step 2: Lower bound. The proof of this part is similar to Step 1. We sketch it for the sake of completeness. Assume that $\kappa \underline{v}_j(t)$ is the subsolution of (5.1), given in Lemma 5.2, for t large enough and $j \in \mathbb{Z}$ with $j - c_* t \geq t^\delta$. Define the sequence $\varphi_j(t)$ by

$$\varphi_j(t) = e^{\lambda_*(j - c_* t)} U_{c_*} \left(j - c_* t + \frac{3}{2\lambda_*} \ln t + a \right).$$

Here, we fix $a \in \mathbb{R}$ such that $\varphi_j(t) \leq \kappa \underline{v}_j(t)$ for t large enough and $\lfloor c_* t + t^\eta \rfloor \leq j \leq \lfloor c_* t + t^\eta \rfloor + 1$. It is also noticed that necessarily $a > b$. Indeed, from the relation $\varphi_j(t) \leq \kappa \underline{v}_j(t) < v_j(t+1) < K \overline{v}_j(t+T_0+1) \leq \psi_j(t)$ for t large enough, say for all $t > T_0 + t_0$, and $j \in \mathbb{Z}$ with $\lfloor c_* t + t^\eta \rfloor \leq j \leq \lfloor c_* t + t^\eta \rfloor + 1$ implies that

$$U_{c_*} \left(j - c_* t + \frac{3}{2\lambda_*} \ln t + a \right) < U_{c_*} \left(j - c_* t + \frac{3}{2\lambda_*} \ln t + b \right).$$

Substituting $\varphi_j(t)$ into the equation of $v_j(t)$ yields

$$\varphi'_j(t) - (\mathcal{L}\varphi(t))_j + \mathcal{R}_j(t; \varphi_j(t)) \sim \frac{1}{t^{5/2-\eta}} < O\left(t^{-(1-\eta)}\right)$$

for t large enough and $j \in \mathbb{Z}$ with $\lfloor c_*t - t^\eta \rfloor \leq j \leq \lfloor c_*t + t^\eta \rfloor + 1$.

Set $z_j(t) := (v_j(t) - \varphi_j(t))^-$, being the negative part of $v_j(t) - \varphi_j(t)$. Then, $z_j(t)$ satisfies

$$\begin{cases} z'_j(t) - (\mathcal{L}z(t))_j + \mathcal{H}_j(t; z_j(t)) < O\left(t^{-(1-\eta)}\right), & \lfloor c_*t - t^\eta \rfloor \leq j \leq \lfloor c_*t + t^\eta \rfloor + 1, \\ z_j(t) = O\left(e^{-\lambda_*t^\eta}\right), & j = \lfloor c_*t - t^\eta \rfloor, \\ z_j(t) = 0, & j = \lfloor c_*t + t^\eta \rfloor + 1, \end{cases}$$

for t large enough. Here, $\mathcal{H}_j(t; z_j(t)) = 0$ when $z_j(t) = 0$; otherwise,

$$\begin{aligned} \mathcal{H}_j(t; z_j(t)) &= \mathcal{R}_j(t; v_j(t)) - \mathcal{R}_j(t; \varphi_j(t)) \\ &= f'(0)z_j(t) - e^{\lambda_*(j-c_*t)} \left(f(e^{-\lambda_*(j-c_*t)}v_j(t)) - f(e^{-\lambda_*(j-c_*t)}\varphi_j(t)) \right) \\ &= f'(0)z_j(t) - d_j(t)z_j(t) \geq 0, \end{aligned}$$

in which d_j is a continuous function, and all d_j 's are bounded in ℓ^∞ norm by $f'(0)$ since $0 < f(s) \leq f'(0)s$ for $s \in (0, 1)$. Following the proof of (5.16) in Step 1, one can show that, there holds

$$\lim_{t \rightarrow +\infty} \sup_{j \in \mathbb{Z}, |j-c_*t| \leq t^\eta} z_j(t) = 0.$$

It then follows that $v_j(t) \geq \varphi_j(t)$ uniformly in $j \in \mathbb{Z}$ with $|j - c_*t| \leq t^\eta$ for t large enough, whence

$$u_j(t) \geq U_{c_*} \left(j - c_*t + \frac{3}{2\lambda_*} \ln t + a \right), \quad (5.18)$$

uniformly in $j \in \mathbb{Z}$ with $|j - c_*t| \leq t^\eta$ for t large enough.

Step 3: Conclusion. Combining (5.17) and (5.18), one gets

$$U_{c_*} \left(j - c_*t + \frac{3}{2\lambda_*} \ln t + a \right) \leq u_j(t) \leq U_{c_*} \left(j - c_*t + \frac{3}{2\lambda_*} \ln t + b \right)$$

uniformly in $j \in \mathbb{Z}$ with $|j - c_*t| \leq t^\eta$ for t large enough. We eventually conclude that, for every $m \in (0, 1)$, there exist $t_0 > 0$ and $C \in \mathbb{R}$ such that

$$j_m(t) \subset \left\{ j \in \mathbb{Z} \mid c_*t - \frac{3}{2\lambda_*} \ln t - C \leq j \leq c_*t - \frac{3}{2\lambda_*} \ln t + C \right\} \quad \text{for all } t \geq t_0.$$

The proof of Theorem 1 is thereby complete. \square

6 Convergence to the logarithmically shifted critical pulled front

This section is devoted to the proof of Theorem 2. As an immediate conclusion from Theorem 1, we know that the transition zone of $u_j(t)$ between the two equilibria 0 and 1 is located around the position $c_*t - \frac{3}{2\lambda_*} \ln t$ for t large enough. Also, it is crucial to note that the proof of Theorem 1 shows in particular that $u_j(t)$ is indeed sandwiched between two finitely shifted minimal traveling fronts with logarithmic delay at large times in a well chosen moving zone. Therefore, the Liouville type result, Proposition 3.3 of [21], can be directly applied and we then follow the strategy proposed in the continuous case [22] to accomplish our proof.

Proof of Theorem 2. Let $C > 0$ be such that $-C < b < a < C$. Assume by contradiction that (1.11) is not true, then there exist $\varepsilon > 0$ and a sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and

$$\min_{|\zeta| \leq C} \sup_{j \in \mathbb{Z}, j \geq 0} \left| u_j(t_n) - U_{c_*} \left(j - c_*t_n + \frac{3}{2\lambda_*} \ln t_n + \zeta \right) \right| \geq \varepsilon$$

for all $n \in \mathbb{N}$. Since $U_{c_*}(-\infty) = 1$ and $U_{c_*}(+\infty) = 0$, together with properties (1.8) and (1.9), there exists $L > 0$ such that

$$\min_{|\zeta| \leq C} \max_{j \in \mathbb{Z}, |j| \leq L} |u_{j+j_n}(t_n) - U_{c_*}(j + \zeta)| \geq \varepsilon \quad \text{with} \quad j_n = \left\lfloor c_*t_n - \frac{3}{2\lambda_*} \ln t_n \right\rfloor \quad (6.1)$$

for all $n \in \mathbb{N}$. Up to extraction of a subsequence, the functions $(u_n)_j(t) = u_{j+j_n}(t + t_n)$ with $j_n = \left\lfloor c_*t_n - \frac{3}{2\lambda_*} \ln t_n \right\rfloor$ converge, in $C_{loc}^1(\mathbb{R})$ for each $j \in \mathbb{Z}$, to a solution $(u_\infty)_j(t)$ of

$$(u_\infty)'_j(t) = (u_\infty)_{j-1}(t) - 2(u_\infty)_j(t) + (u_\infty)_{j+1}(t) + f((u_\infty)_j(t)) \quad \text{for } (t, j) \in \mathbb{R} \times \mathbb{Z},$$

such that $0 \leq (u_\infty)_j(t) \leq 1$ in $\mathbb{R} \times \mathbb{Z}$. Furthermore, (1.8) and (1.9) imply that

$$\lim_{y \rightarrow +\infty} \inf_{\substack{(t,j) \in \mathbb{R} \times \mathbb{Z}, \\ j \leq c_*t - y}} (u_\infty)_j(t) = 1, \quad \lim_{y \rightarrow +\infty} \sup_{\substack{(t,j) \in \mathbb{R} \times \mathbb{Z}, \\ j \geq c_*t + y}} (u_\infty)_j(t) = 0. \quad (6.2)$$

On the other hand, for each fixed $t \in \mathbb{R}$ and $j \in \mathbb{Z}$, the sequence $j + j_n = j + \left\lfloor c_*t_n - \frac{3}{2\lambda_*} \ln t_n \right\rfloor$ satisfies $|j + j_n - c_*(t + t_n)| \leq (t + t_n)^\eta$ for n large enough with η chosen in (5.14), whence it follows from (5.17) and (5.18) that

$$U_{c_*} \left(j - c_*t + \frac{3}{2\lambda_*} \ln(t + t_n)/t_n + a \right) \leq (u_n)_j(t) \leq U_{c_*} \left(j - c_*t + \frac{3}{2\lambda_*} \ln(t + t_n)/t_n + b \right)$$

for n large enough. Therefore,

$$U_{c_*}(j - c_*t + a) \leq (u_\infty)_j(t) \leq U_{c_*}(j - c_*t + b) \quad \text{for } (t, j) \in \mathbb{R} \times \mathbb{Z}.$$

Applying the Liouville-type result [21, Proposition 3.3] by taking any positive interger N as the period, one then gets the existence of $z \in \mathbb{R}$ such that $b \leq z \leq a$ and

$$(u_\infty)_j(t) = U_{c_*}(j - c_*t + z) \quad \text{for } (t, j) \in \mathbb{R} \times \mathbb{Z}.$$

Since u_n converges, up to extraction of a subsequence, to u_∞ locally uniformly in $\mathbb{R} \times \mathbb{Z}$, it follows in particular that $(u_n)_j(0) - U_{c_*}(j+z) \rightarrow 0$ uniformly in $j \in \mathbb{Z}$ with $|j| \leq L$, that is,

$$\max_{j \in \mathbb{Z}, |j| \leq L} |u_{j+j_n}(t_n) - U_{c_*}(j+z)| \rightarrow 0 \quad \text{with} \quad j_n = \left\lfloor c_* t_n - \frac{3}{2\lambda_*} \ln t_n \right\rfloor.$$

Notice that $z \in [b, a] \subset [-C, C]$, one then gets a contradiction with (6.1). This completes the proof of (1.11).

Fix now any $m \in (0, 1)$ and let $(t_n)_{n \in \mathbb{N}}$ and $(j_n)_{n \in \mathbb{N}}$ be the sequences of positive real numbers and of positive integers, respectively, such that $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and $u_{j_n}(t_n) = m$ for all $n \in \mathbb{N}$. Set

$$\xi_n = j_n - \left\lfloor c_* t_n - \frac{3}{2\lambda_*} \ln t_n \right\rfloor,$$

Theorem 1 implies that the sequence $(\xi_n)_{n \in \mathbb{N}}$ is bounded and then, up to extraction of a subsequence, $\xi_n \rightarrow \xi_0 \in \mathbb{Z}$ as $n \rightarrow +\infty$. From the argument of (1.11), the functions

$$(v_n)_j(t) = u_{j+j_n}(t+t_n) \quad \text{with} \quad j_n = \xi_n + \left\lfloor c_* t_n - \frac{3}{2\lambda_*} \ln t_n \right\rfloor$$

converge, up to another subsequence, locally uniformly in t for all $j \in \mathbb{Z}$, to $(v_\infty)_j(t) = U_{c_*}(j - c_* t + \xi_0 + z_0)$ for some $z_0 \in [-C, C]$. Since $(v_n)_{j=0}(0) = u_{j_n}(t_n) = m$, one then has $\xi_0 + z_0 = U_{c_*}^{-1}(m)$. Therefore, the limit v_∞ is uniquely determined and the whole sequence $(v_n)_{n \in \mathbb{N}}$ then converges to the traveling wave $U_{c_*}(j - c_* t + U_{c_*}^{-1}(m))$. The conclusion of Theorem 2 follows. \square

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