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Dynamic response of a cracked multi-span continuous beam subjected to a moving multi-axle vehicle load

Truong-Son Pham¹, Tien Hoang¹, Denis Duhamel¹, Gilles Foret¹, Franziska Schmidt², François-Baptiste Cartiaux³, Véronique Le Corvec³

¹ Laboratoire Navier, UMR 8205, Ecole des Ponts ParisTech, Université Gustave Eiffel, CNRS, 6-8 avenue Blaise Pascal, Champs-sur-Marne, France

² Université Gustave Eiffel, IFSTTAR/MAST, 14-20 boulevard Newton, Champs-sur-Marne, France

³ OSMOS Group, 37 rue la Pérouse, Paris, France

email: truong-son.pham@enpc.fr, tien.hoang@enpc.fr, denis.duhamel@enpc.fr, gilles.foret@enpc.fr, franziska.schmidt@univ-eiffel.fr, cartiaux@osmos-group.com, lecorvec@osmos-group.com

ABSTRACT: An analytical model is developed to calculate the dynamic response of a cracked multi-span continuous beam, subjected to a multiple-axle vehicle load. In this model, the vehicle load is considered as a series of concentrated moving loads at a constant velocity. Meanwhile, the crack is modelled by a spiral spring. Thereafter, the cracked span is considered as a two-span beam, connected by the spring at the position of the crack. By using the Euler-Bernoulli beam theory, the relations between the two successive spans can be obtained from the boundary conditions at the supports and the position of the crack. This technique makes it possible to calculate the eigenmodes of the beam. The response of the beam is finally obtained by the superposition of the modes. This model allows us to quickly analyze the influence of the cracks (depth, location) and of the vehicle (loads, velocity) on the response. Moreover, this model is validated by finite element modelling and some numerical results are given.

KEYWORDS: Multi-span continuous beam; Cracked structure; Moving loads; Transfer Matrix Method.

1 INTRODUCTION

The problems of elastic structures subjected to moving loads are great interest to industry and civil engineering. Indeed, there are many applications in the civil engineering domain, especially in bridge evaluation and assessment. Several studies have been realised in the last century. Fryba [1] presented an analytical solution for the problem of a simply supported beam and a continuous beam subject to a moving load. The results are obtained by combination of displacement functions. Ichikawa *et al.* [2] used the eigenmodes superposition to find the solution of a continuous beam with uniform rectangular cross section subjected to a moving load at constant velocity. Dugush and Eisenberger [3] described the mode shapes of a continuous beam with non-uniform section in terms of infinite polynomial series. In another approach, Foda *et al.* [4] used the Green's function to calculate the dynamic response of a simply supported beam subjected to a point force. Recently, Hoang *et al.* [5] used the Fourier transform and the properties of the Dirac comb to calculate the dynamic response of a periodically supported beam.

The existence of structural damage will alter the response of the system to moving loads. When a structure is cracked, its local flexibility increases. Consequently, its natural frequencies are reduced and its responses are amplified in comparison to an uncracked structure under the same force. The dynamic behaviours of cracked beams have been the subject of many researches [6-9]. In these studies, the crack is modelled by a rotational spring, the rigidity of which depends on the relative depth of the crack (ratio between the depth of the crack and the height of the beam) and possibly on the material constituting the beam. Rizos *et al.* [10] used the strain energy density function to calculate the local flexibility. Chondros *et al.* [11] completed a theory of continuous crack beam vibration using the crack-disturbance function. By comparison with the

numerical results of a beam model subjected to the three-point bending flexural test, the model of rotational spring at crack section is validated for a depth of the crack up to 50 percent of the height of the beam. Rizos' model will be used in this current study due to the results of this comparative study.

The problem of a damaged beam subjected to a moving force was also studied by several approaches. Bilello *et al.* [12] used the mode superposition with the introduction of the transfer matrix, which connects simultaneously the displacement, the slope, the bending moment and the shear force between two beam segments. With approximately the same idea, Lin *et al.* [13] directly computed the response of a cantilever beam using the Duhamel integral. In this model, a transfer matrix at the crack section is introduced.

In this paper, we present an analytical method that allows to quickly calculate the dynamic response of a continuous beam subjected to a system of several concentrated moving loads. The beam is divided into several segments which are assumed to follow the Euler-Bernoulli beam theory. The deformed state of each beam segment is characterized by a vector of coefficients. A transfer matrix is introduced to ensure the compatibility requirements at the interface of two consecutive segments. Hence, the deformed state of a beam segment can be described in terms of the deformed state of its neighbours. The natural frequencies can be obtained quickly by solving an eigenvalue problem. The time-dependent response (displacement, strain) of the beam will be obtained by superposition of modes. In the real life, beam-shaped structures such as bridges are subjected to rolling loads exerted by vehicles. This study allows to develop a method to detect and localize damage from strain measurements. Nowadays, strain measurement by Optical Fiber is one of the most innovative Structural Health Monitoring (SHM) techniques, which allows to monitor the structure in real-time. Besides, the traditional

measurements of acceleration to determine modal parameters of the structure (natural frequencies, mode shapes...) and assess them in the long term.

2 FORMULATION OF THE PROBLEM

Consider a continuous beam of N_s spans of total length L . A crack is present on the k^{th} span. The cracked span is divided into two separate beams: k_1 and k_2 , which are connected by a rotational spring with given section flexibility. The continuous beam can be modelled by a system of $N = N_s + 1$ separate beams. The total length of the beam is $L = \sum_{i=1}^{k_1} L_i + \sum_{i=k_2}^{N_s} L_i$, where L_i is the length of the i^{th} beam segment. L_{k_1} is the position of the crack starting from the left support of the k^{th} span. The beam is subjected to a M -axles vehicle load, which enters the beam with a constant velocity v . d_i denotes the distance between the i^{th} and the $(i+1)^{\text{th}}$ axle and $D_{i-1} = \sum_{j=1}^{i-1} d_j$ is the distance between the first axle and the i^{th} axle ($i \geq 2$). If there is only one axle, we take $D_0 = 0$ by convention.

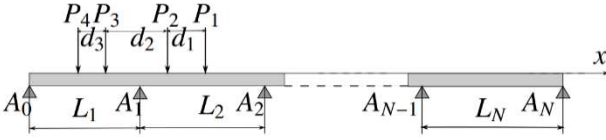


Figure 1. A continuous beam with N -span subjected to M moving loads.

For the formulation of the problem of forced vibration of a cracked continuous beam, the following assumptions are suggested:

- Each separate beam follows the Euler-Bernoulli beam theory and has linear elastic behaviour. The beam has a uniform cross section.
- The moving mass remains in contact with the beam at all times.
- Initially, the mass starts from the left end of the continuous beam and moves at constant speed.

The dynamic equation of the whole beam in the global coordinates system can be expressed as:

$$EI \frac{\partial^4 u}{\partial x^4} + \eta \frac{\partial u}{\partial t} + \rho A \frac{\partial^2 u}{\partial t^2} = f(x, t) \quad (1)$$

where EI is the flexural stiffness, ρ is the mass density and A is the cross-sectional area of the beam, η is the external damping coefficient. Moreover, u is the transverse displacement of the beam, $f(x, t)$ is the distribution of the external load of the moving loads, x is the global coordinate along the beam ($0 \leq x \leq L$), from its left end and t is the time ($0 \leq t \leq (L + D_{M-1})/v$).

The transverse displacement is blocked at each support point, while the bending moment is cancelled at both ends of the beam. These boundary conditions can be written as:

- $u(0) = u(\sum_{i=1}^r L_i) = 0, r = 1, 2, \dots, k, \dots, N_s$
- $M(0) = M(L) = 0 \Rightarrow u''(0) = u''(L) = 0$.

The moving load exerted by each axle is considered as a concentrated load. Using the Dirac distribution $\delta(x)$ to describe

each concentrated force, the expression for the load distribution $f(x, t)$ is given by:

$$\begin{aligned} f(x, t) &= P_1 \delta(x - vt) + P_2 \delta(x - vt + D_1) + \dots \\ &\quad + P_M \delta(x - vt + D_{M-1}) \\ &= \sum_{i=1}^M P_i \delta(x - vt + D_{i-1}) \end{aligned}$$

By separation of the variables, the response of the beam is decoupled into two terms: the time dependent term e^{zt} , ($z \in \mathbb{C}$) and the mode shapes $\phi(x)$:

$$u(x, t) = \phi(x)e^{zt} \quad (2)$$

The homogenous differential equation of the beam is:

$$EI \frac{\partial^4 u}{\partial x^4} + \eta \frac{\partial u}{\partial t} + \rho A \frac{\partial^2 u}{\partial t^2} = 0 \quad (3)$$

Substitute (2) into (3), then arrange the time dependent functions on the left-hand side and the x dependent functions on the right-hand side, we have:

$$\frac{\phi''''(x)}{\phi(x)} = -\frac{\rho A z^2 + \eta z}{EI} = \lambda^4 \quad (4)$$

where λ is a constant to be determined and the prime operator $(\star)'$ denotes the derivative of the function with respect to x . To calculate the natural frequencies, η is first set to zero. Equation (4) becomes:

$$\frac{\phi''''(x)}{\phi(x)} = -\frac{\rho A z^2}{EI} = \lambda^4 \quad (5)$$

2.1 Natural frequencies and mode shapes

From equation (5), the governing equation for the free vibrations of the r^{th} beam segment can be written as:

$$\frac{d^4 \phi_r(x_r)}{dx_r^4} - \lambda^4 \phi_r(x_r) = 0 \quad (6)$$

where $\phi_r(x_r)$ is the mode shape of the r^{th} beam segment, which has the following form:

$$\begin{aligned} \phi_r(x_r) &= A_r \sin \lambda x_r + B_r \cos \lambda x_r \\ &\quad + C_r \sinh \lambda x_r + D_r \cosh \lambda x_r \end{aligned} \quad (7)$$

Hereafter, we use the notation $x_r = x - \sum_{i=1}^{r-1} L_i$, ($0 \leq x_r \leq L_r$) to design the local coordinates of the r^{th} beam segment.

2.2 The configuration of the beam

In this section, the transfer matrices are now introduced in order to ensure the conditions of continuity at the crack ends and at the intermediate support points. The deformed state of a beam segment is characterized by the coefficients A_r, B_r, C_r, D_r . We denote $X_r = [A_r, B_r, C_r, D_r]^T$ the vector which contains the shape coefficients of the mode for the r^{th} beam segment. At the interface of two consecutive segments (i.e. at the crack point and at the intermediate points), the 4×4 transfer matrix $\mathbf{M}_{4 \times 4}$ connects these two segments so that $X_r = \mathbf{M}_{4 \times 4} X_{r-1}$.

The crack

We assume that a crack is present on the k^{th} span. This span is divided into 2 beam segments of length L_{k_1} and L_{k_2} . The crack section is modelled like a rotational spring, at which the

discontinuity of the slope is proportional to the bending moment at this section: $\Delta u'(x) = cM(x)$. Note that $M(x) = EIu''(x)$. This relation can be written as:

$$u'_{k_2}(0, t) - u'_{k_1}(L_{k_1}, t) = Ku''_{k_2}(0, t) \quad (8)$$

For an orthogonal cross section of width b and height h , from the strain energy density function in fracture mechanics, Rizos *et al.* proposed the expression of local flexibility c [13]:

$$c = \frac{6h(1 - \nu^2)}{EI} J_1(a/h) \quad (9)$$

where ν is the Poisson's coefficient of the material of the beam, EI is its bending stiffness, a is the depth of the uniform lateral crack. $J_1(a_1) = J_1(a/h)$ is the dimensionless local compliance function whose expression has the form:

$$\begin{aligned} J_1(a_1) = & 1.8624a_1^2 - 3.95a_1^3 + 16.375a_1^4 \\ & - 37.226a_1^5 + 76.81a_1^6 - 126.9a_1^7 \\ & + 172a_1^8 - 143.97a_1^9 + 66.56a_1^{10} \end{aligned} \quad (10)$$

We obtain finally: $K = EIC = 6h(1 - \nu^2)J_1(a_1)$.

Across the crack, the transverse displacement, the bending moment and the shear force are unchanged. The continuity conditions at the crack position are:

$$\begin{aligned} u_{k_1}(L_{k_1}, t) &= u_{k_2}(0, t) \\ u''_{k_1}(L_{k_1}, t) &= u''_{k_2}(0, t) \\ u'''_{k_1}(L_{k_1}, t) &= u'''_{k_2}(0, t) \end{aligned} \quad (11)$$

From relations (8) and (11), we deduce:

$$X_{k_2} = \mathbf{T}(K, L_{k_1})X_{k_1} = \mathbf{T}_{k_1}X_{k_1} \quad (12)$$

where $\mathbf{T}(K, L_{k_1})$ is a 4×4 matrix which is known as the transfer matrix at the section of the crack.

The intermediate supports

As the intermediate supports are assumed to be infinitely rigid, the transverse displacements are blocked. Meanwhile, the slope of the beam and the bending moment are conserved across the support. Let us consider the $(r - 1)$ th support which connects the $(r - 1)$ th and the r th span. We have the following relations:

$$\begin{aligned} u_r(0, t) &= 0 \\ u_r(L_r, t) &= 0 \\ u'_r(0, t) &= u'_{r-1}(L_{r-1}, t) \\ u''_r(0, t) &= u''_{r-1}(L_{r-1}, t) \end{aligned} \quad (13)$$

From these above relations, we obtain the transfer matrix at the intermediate support $\mathbf{S}(L_{r-1}, L_r)$ such that: $X_r = \mathbf{S}_{4 \times 4}X_{r-1}$.

The left support of the cracked span

Since the crack is present on the k th span, this span is divided into two beam segments: k_1 and k_2 . The span which lies on the left of this cracked span is now called $k - 1$.

The right end of the first beam segment of the crack span is no longer blocked, so that the second condition of the system (13) is not satisfied. Indeed, the right end of the second beam segment (k_2) is found at another intermediate support, which leads to the condition: $u_{k_2}(L_{k_2}, t) = 0$.

This is equivalent to:

$$\mathbf{b}_{k_2}^T X_{k_2} = \mathbf{0} \quad (14)$$

with: $\mathbf{b}_{k_2} = [\sin \lambda L_{k_2} \quad \cos \lambda L_{k_2} \quad \sinh \lambda L_{k_2} \quad \cosh \lambda L_{k_2}]^T$.

By taking into account the relation (12), the equation (14) can be rewritten as:

$$\mathbf{b}_{k_2}^T \mathbf{T}_{k_1} X_{k_1} = \mathbf{0} \quad (15)$$

The other relations remain the same as in the system (13), except the second condition. We obtain finally a 4×4 matrix such that:

$$X_{k_1} = \mathbf{G}(L_{k-1}, \mathbf{T}_{k_1}, \mathbf{b}_{k_2})X_{k-1} \quad (16)$$

The matrix \mathbf{G} is called the transfer matrix at the left support of the cracked span.

The transfer matrices

As a reminder, the transfer matrices are used to assure the conditions of continuity and the compatibility requirements at the crack cross section (the matrix \mathbf{T}) and at the intermediate supports (the matrices \mathbf{S} , \mathbf{G}). It is important to note that the matrices \mathbf{S} , \mathbf{G} are singular. This remark can be explained by the fact that between the intermediate supports, the beam segment is blocked at its ends. Consequently, at least one component of the coefficient vector can be expressed in terms of the others. The expressions: $X_r = \mathbf{S}_{4 \times 4}X_{r-1}$ and $X_{k_1} = \mathbf{G}_{4 \times 4}X_{k-1}$ allow to calculate the mode shape coefficients of a span knowing those of the previous span. The full expressions of these transfer matrices are presented in the **Appendix**.

Boundary conditions matrix at the right end of the beam

The beam is simply supported at its right end: $x_N = L_N$. This condition imposes the nullity of the displacement and of the bending moment at this point. Note that the condition $u_N(L_N, t) = 0$ is used to establish the transfer matrices (\mathbf{T} , \mathbf{G} , \mathbf{S}). For this support, there is only one boundary condition concerning the nullity of the bending moment:

$$u''_N(L_N, t) = 0 \quad (17)$$

The condition (17) can be expressed in the following matrix form:

$$\begin{aligned} [-\sin \lambda L_N \quad -\cos \lambda L_N \quad \sinh \lambda L_N \quad \cosh \lambda L_N] X_N \\ = \mathbf{B}(L_N)X_N = 0 \end{aligned} \quad (18)$$

Where $\mathbf{B}(L_N)$ is a 1×4 matrix which describes the boundary condition of the right end of the beam.

2.3 Eigenvalues Problem

For all intermediate supports, except the support that lies on the left of the cracked span, we always have:

$$X_r = \mathbf{S}X_{r-1} \quad (19)$$

For the cracked span, we have these relations:

$$\begin{aligned} X_{k_1} &= \mathbf{G}X_{k-1}, & k &\geq 2 \\ X_{k_2} &= \mathbf{T}X_{k_1}, & k &= 1 \end{aligned} \quad (20)$$

By arrangement of these matrices, we have:

$$\begin{aligned} (\mathbf{B}_N \cdot \mathbf{S}_{N-1,N} \cdot \mathbf{S}_{N-2,N-1} \cdots \mathbf{S}_{k_2,k+1} \cdot \mathbf{T}_{k_1} \\ \cdot \mathbf{G}_{k-1,k_1} \cdot \mathbf{S}_{k-2,k-1} \cdots \mathbf{S}_{1,2}) X_1 = 0 \end{aligned} \quad (21)$$

The matrix product on the left-hand side of equation (21) forms a 1×4 matrix. The resulting matrix is now denoted by \mathbf{R} . Equation (21) can be written shorter as:

$$\mathbf{R}X_1 = 0 \quad (22)$$

with $r = [r_1 \ r_2 \ r_3 \ r_4]$, $r_i \in \mathbb{R}$, $i = 1, 2, 3, 4$.

Since the beam is simply supported at its two ends, at $x = 0$, the bending moment and the transverse displacement are always equal to zero. We have:

$$\begin{cases} u_1(0, t) = 0 \\ u_1''(0, t) = 0 \end{cases} \Rightarrow \begin{cases} B_1 = 0 \\ D_1 = 0 \end{cases} \quad (23)$$

Since $X_1 = [A_1, B_1, C_1, D_1]^T$, we can rewrite the coefficient vector X_1 as $X_1 = [A_1, 0, \gamma(\lambda)A_1, 0]^T$, where $\gamma(\lambda)$ is a function that expresses C_1 in term of A_1 : $C_1 = \gamma(\lambda)A_1$. When the crack is present at the first span, we introduce the vector $\mathbf{b}_{k_2} = [\sin \lambda L_{k_2} \ \cos \lambda L_{k_2} \ \sinh \lambda L_{k_2} \ \cosh \lambda L_{k_2}]^T$ and the transfer matrix at the crack $\mathbf{T}_{k_1} = [\mathbf{t}_1 | \mathbf{t}_2 | \mathbf{t}_3 | \mathbf{t}_4]$.

$$\gamma(\lambda) = \begin{cases} -\frac{\sin \lambda L_1}{\sinh \lambda L_1}, & k \geq 2 \\ -\frac{\mathbf{b}_{k_2}^T \mathbf{t}_1}{\mathbf{b}_{k_2}^T \mathbf{t}_3}, & k = 1 \end{cases} \quad (24)$$

The existence of non-trivial solutions of Equation (22) requires:

$$f(\lambda) = r_1(\lambda) + \gamma(\lambda)r_3(\lambda) = 0 \quad (25)$$

The natural frequencies of the continuous beam are solutions of Equation (25).

Once the eigenvalues are found, the mode shapes will be normalized such that: $\int_0^L \phi_m(x)\phi_n(x) = \delta_{mn}$, where δ_{mn} is the Kronecker delta. Since the initial continuous beam is divided into N beam segments, the mode shapes of the beam are obtained by summation of the mode shapes over all segments:

$$\phi_n(x) = \sum_{r=1}^N \phi_{n,r}(x - l_{r-1})[H(x - l_{r-1}) - (x - l_r)] \quad (26)$$

where $\phi_{n,r}$ is the mode shape of the r^{th} beam segment for the n^{th} mode, $l_k = \sum_{i=1}^k L_i$ the distance from the left end of the beam to the right end of the k^{th} beam segment ($l_0 = 0$ by convention) and H denotes the Heaviside step function:

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

2.4 Forced response of the beam under moving loads

The governing equation of the beam is:

$$\begin{aligned} EI \frac{\partial^4 u}{\partial x^4} + \eta \frac{\partial u}{\partial t} + \rho A \frac{\partial^2 u}{\partial t^2} &= f(x, t) \\ &= \sum_{i=1}^M P_i \delta(x - vt + D_{i-1}) \end{aligned} \quad (27)$$

The solution of Equation (27) can be expressed as a linear combination of orthonormal modes $\phi_n(x)$, with $q_n(t)$ the generalized coordinates of the n^{th} mode:

$$u(x, t) = \sum_{n=1}^{\infty} \phi_n(x) q_n(t) \quad (28)$$

The damping coefficient is assumed to be proportional to the mass: $\eta = 2\rho A\omega_n\zeta_n$, where ζ_n is the modal damping ratio for the mode n . Substitute expression (28) into Equation (27) and note that $\omega_n^2 = \lambda^4(EI/\rho A)$, then:

$$\begin{aligned} \sum_{n=1}^{\infty} [\omega_n^2 q_n(t) + 2\omega_n\zeta_n \dot{q}_n(t) + \ddot{q}_n(t)] \phi_n(x) \\ = \sum_{i=1}^M P_i \delta(x - vt + D_{i-1}) \end{aligned} \quad (29)$$

with $p_i = P_i/(\rho A)$. By multiplying both sides of (29) with $\phi_m(x)$ and integrating from 0 to L :

$$\begin{aligned} \sum_{n=1}^{\infty} [\omega_n^2 q_n(t) + 2\omega_n\zeta_n \dot{q}_n(t) + \ddot{q}_n(t)] \int_0^L \phi_n(x) \phi_m(x) dx \\ = \sum_{i=1}^M P_i \int_0^L \delta(x - vt + D_{i-1}) \phi_m(x) dx \end{aligned} \quad (30)$$

Since the mode shapes are orthonormal, Equation (30) can be written as:

$$\ddot{q}_n(t) + 2\omega_n\zeta_n \dot{q}_n(t) + \omega_n^2 q_n(t) = \sum_{i=1}^M p_i \phi_n(vt - D_{i-1}) \quad (31)$$

The general solution of Equation (31) is the sum of its homogeneous solution and of a particular solution. We first look for the particular solution.

For a given linear operator $\mathcal{L}[\star]$ (in this problem, $\mathcal{L}[\star] = d^2/dt^2 + 2\omega_n\zeta_n d/dt + \omega_n^2$), if $q_p(t)$ is a particular solution of the non-homogeneous differential equation: $\mathcal{L}[q(t)] = y(t)$, then $kq_p(t - \tau)$ is also a particular solution of equation $\mathcal{L}[q(t)] = ky(t - \tau)$ for any $k \in \mathbb{R}$. This proposition can be shown by the technique of variable changes.

In Equation (31), we only keep one point force of unit magnitude $p_1 = 1$. Let us consider the following non-homogeneous differential equation:

$$\ddot{q}_{n,r}(t_r) + 2\omega_n\zeta_n \dot{q}_{n,r}(t_r) + \omega_n^2 q_{n,r}(t_r) = \phi_{n,r}(vt), \quad 0 \leq t_r \leq L_r/v \quad (32)$$

We assume that a particular solution of this equation has the same form as the mode shape $\phi_{n,r}$:

$$\begin{aligned} q_{n,r}^p(t_r) &= \bar{A}_{n,r} \sin(\lambda_n vt_r) + \bar{B}_{n,r} \cos(\lambda_n vt_r) \\ &\quad + \bar{C}_{n,r} \sinh(\lambda_n vt_r) + \bar{D}_{n,r} \cosh(\lambda_n vt_r) \\ &= [\sin \Omega_n t_r \ \cos \Omega_n t_r \ \sinh \Omega_n t_r \ \cosh \Omega_n t_r] \bar{X}_{n,r} \end{aligned} \quad (33)$$

where $\Omega_n = \lambda_n v$ and $\bar{X}_{n,r} = [\bar{A}_{n,r}, \bar{B}_{n,r}, \bar{C}_{n,r}, \bar{D}_{n,r}]^T$ is the coefficient vector of the particular solution of the r^{th} beam segment for the n^{th} mode. By using the method of undetermined coefficients, $\bar{X}_{n,r}$ can be expressed in terms of $X_{n,r} = [A_{n,r}, B_{n,r}, C_{n,r}, D_{n,r}]^T$ as the following matrix form:

$$\bar{X}_{n,r} = \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & d & c \end{bmatrix} X_{n,r} = \mathbf{Q}X_{n,r} \quad (34)$$

with:

$$\begin{aligned} a &= \frac{\Delta_{\omega,n}}{\Delta_{\omega,n}^2 + (2\omega_n\zeta_n\Omega_n)^2}, & b &= \frac{2\omega_n\zeta_n\Omega_n}{\Delta_{\omega,n}^2 + (2\omega_n\zeta_n\Omega_n)^2} \\ c &= \frac{S_{\omega,n}}{S_{\omega,n}^2 - (2\omega_n\zeta_n\Omega_n)^2}, & d &= \frac{-2\omega_n\zeta_n\Omega_n}{S_{\omega,n}^2 - (2\omega_n\zeta_n\Omega_n)^2} \\ \Delta_{\omega,n} &= \omega_n^2 - \Omega_n^2, & S_{\omega,n} &= \omega_n^2 + \Omega_n^2 \end{aligned}$$

For an undamped system (i.e. $\zeta_n = 0$), the matrix \mathbf{Q} is diagonal. The general coordinates of the continuous beam are obtained by summation over its segments:

$$q_n^p(t) = \sum_{r=1}^N q_{n,r}^p(t - t^{r-1}) [H(t - t^{r-1}) - H(t - t^r)] \quad (35)$$

in which $t^r = l_r / v = (\sum_{i=1}^r L_i) / v$ is the moment when the unit force enters the r^{th} span.

The right-hand side of Equation (31) is a linear combination of values of the reduced force p_i with the value of the function $\phi_n(vt - D_{i-1}) = \phi_n(v(t - \tau_{i-1}))$, where $\tau_{i-1} = D_{i-1}/v$ is the instant when the i^{th} axle enters on the bridge ($i = 1, 2, \dots, M$). By superposition of these particular solutions for each axle, we finally obtain the form of particular solution for a system of M concentrated forces:

$$Q_n^p(t) = \sum_{i=1}^M p_i q_n^p(t - \tau_{i-1}) \quad (36)$$

The homogeneous solution of Equation (31) which is corresponding to the n^{th} mode is:

$$Q_n^h(t) = e^{-\omega_n \zeta_n t} [\alpha_n \sin(\omega_n^d t) + \beta_n \cos(\omega_n^d t)] \quad (37)$$

with $\omega_n^d = \omega_n \sqrt{1 - \zeta_n^2}$ the damped angular frequency of the system for the n^{th} natural mode.

Hence, the general solution of a harmonic oscillator subjected to a system of M concentrated forces (equation (31)) is:

$$q_n(t) = Q_n^p(t) + Q_n^h(t) \quad (38)$$

The coefficients α_n and β_n are determined from the conditions of continuity of the generalized coordinates $q_n(t)$ and from the generalized velocity $\dot{q}_n(t)$ when an axle of the vehicle enters a beam segment.

The force response of the continuous beam is the linear combination of the mode shapes with the corresponding generalized coordinates for all the natural modes of the structure. In fact, we only keep a certain number of the first modes so that the total effective modal mass is at least 90% of the mass of the structure.

3 NUMERICAL RESULTS AND DISCUSSION

In this section, a two-spans of equal length, continuous Euler-Bernoulli beam is studied. The total length of the beam is $L = 20 \text{ m}$. The cross section is square of length $b = h = 10 \text{ cm}$. The constitutive material is steel, which has the modulus of elasticity $E = 210 \text{ GPa}$, the Poisson's ratio $\nu = 0.3$ and the density $\rho = 7800 \text{ kg/m}^3$. The beam is subjected to a 2-axle vehicle whose loads are $P_1 = 1 \text{ kN}$ and $P_2 = 2 \text{ kN}$. The distance between these loads is $d = 4 \text{ m}$ (Figure 2). At $t = 0$, the vehicle enters the beam with constant speed $v = 10 \text{ m/s}$. A crack with variable depth is present at the midpoint of the first span. Moreover, different values of modal damping ratio will be considered.

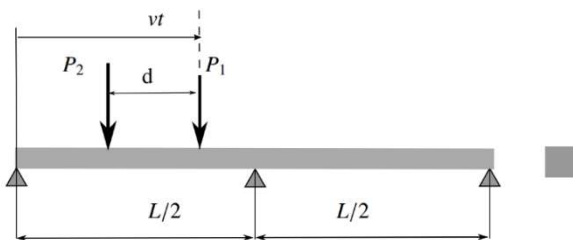


Figure 2. Two-axle vehicle passing a continuous beam with 3 supports at constant speed.

In order to validate the presented method, some results from the analytical model are presented and will be compared with the numerical results of the same problem, modelled in an Open Source FEM software, the *code_Aster*. The FEM model has 100 linear beam elements, which follows the Euler-Bernoulli beam theory. The transient response is calculated on a generalized basis, using Newmark's scheme for temporal integration with $\beta = 1/4$ and $\gamma = 1/2$. The maximum frequency taken into account is up to 100 Hz. In case of cracked beam, a discrete element that having the same rigidity in bending as the crack is introduced.

The present analytical model takes the first 12 frequencies, so that the cumulative effective mass factor is 90,4%. Figure 3 shows the deflection at midspan of each span of the beam in comparison with the results from the FEM model (blue and red lines), when the damping ratio is 2%. The deflection obtained by this model is coherent with the numerical results.

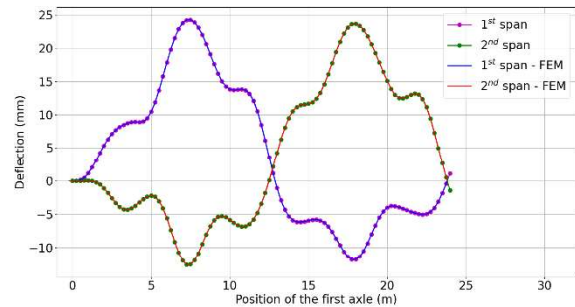


Figure 3. Deflection at midspan with $\zeta = 2\%$.

Now, we create a crack at the midpoint of the first span. The depth of this crack is equal to half the height of the beam (ratio $a_1 = a/h = 0.5$). Figure 4 shows the deflection at mid-span of the two spans, in case of an undamaged beam (violet and green lines) and a damaged beam (blue and red lines). We can see that the crack has more influence on the deflection of the first span than of the second one. However, this difference is relatively small. At the midpoint of the first span, the difference is about 5%.

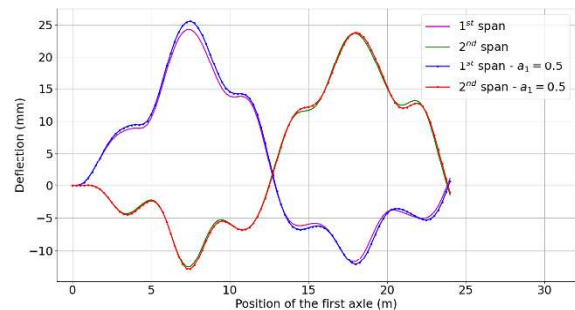


Figure 4. Deflection at midspan with $\zeta = 2\%$.

In case of an undamaged beam, the speed of the vehicle also has an important influence on the amplitude of the response. Figure 5 presents the dependence of the Dynamics

Amplification Factor (DAF) on the dimensionless velocity of the vehicle, which is defined as $\tilde{v} = v/v_{c1}$, where $v_{c1} = \lambda_1 \sqrt{EI/\rho A} = 47 \text{ m/s}$ is the first critical velocity. Remind that $DAF = \max_t u(t) / \max_t u_0(t)$, the ratio of the two largest responses in case of damaged and undamaged structure. We can see that the amplification is greater on the second span than on the first span. At certain speeds, this amplification vanishes. Moreover, the difference of DAF at peaks compared to the undamped cases is proportional to the damped ratio of the system.

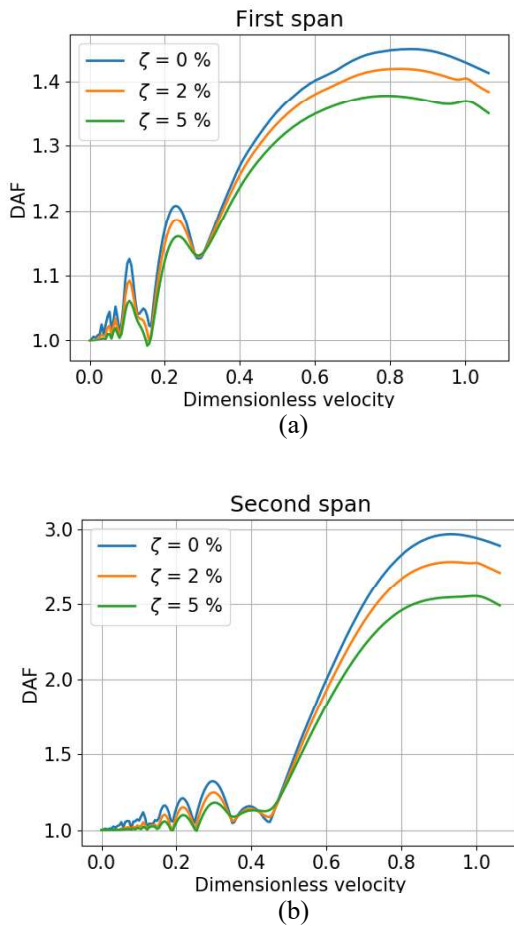


Figure 5. Dynamics amplification factor at different levels of damping

In order to see how the location and the depth of the crack influence the natural frequencies, we carry out some parametric studies. The fundamental frequency of the uncracked beam is $f_0 = 2.35 \text{ Hz}$. Figure 6 shows the variations in the fundamental frequency ($\Delta f_0/f_0 = (f - f_0)/f_0$) depending on the location of the crack with different crack depth $a_1 = a/h$. This relative difference (in percent) is greatest when the crack is near the midpoint of each span. When the crack is at the level of the support, the fundamental frequency does not change.

Figure 7 shows the dependence of the fundamental frequency on the crack depth for the case where the position of the crack has the largest influence on the frequency variation, according to Figure 6 ($x_0 = 4.8 \text{ m}$). When the crack depth is 50% of the beam's height, the change in frequency exceeds 5%.

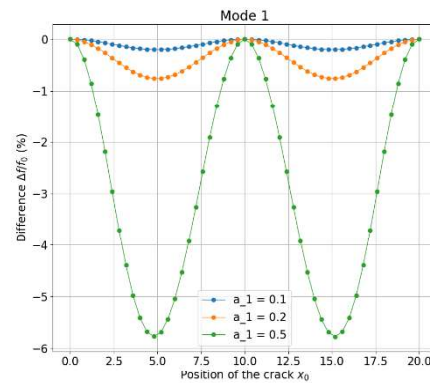


Figure 6. Change of frequency as a function of crack location

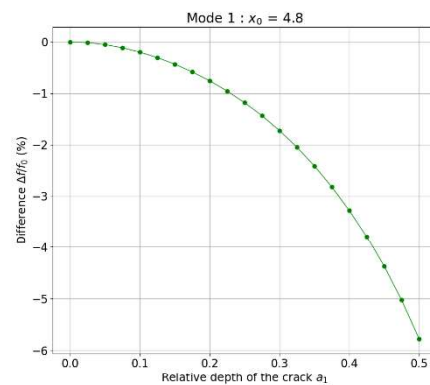


Figure 7. Change of frequency as a function of dimensionless crack depth

4 CONCLUSION

A direct analytical method to derive the eigenmodes for cracked and uncracked beams under point moving loads was presented in this work. The final results are obtained by mode decomposition. The mode shapes and the generalized coordinates are expressed explicitly by analytical functions. This model allows us to quickly calculate the response of a continuous Euler-Bernoulli beam with the presence of damage. Hence, we can perform parametric studies with several parameters (crack location, crack depth, load velocity...) to see how these factors influence the final results. This is useful to highlighting some critical features of the beam and quickly investigating a beam-shaped system.

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Appendix

Transfer matrix at the crack

$$T_{4 \times 4} = \begin{bmatrix} -b.s + c & -s - b.c & b.sh & b.ch \\ s & c & 0 & 0 \\ -b.s & -b.c & b.sh + ch & sh + b.ch \\ 0 & 0 & sh & ch \end{bmatrix}$$

Where:

$$b = \frac{K\lambda}{2}$$

$$s = \sin \lambda L_{k_1}, c = \cos \lambda L_{k_1}$$

$$sh = \sinh \lambda L_{k_1}, ch = \cosh \lambda L_{k_1}$$

Transfer matrix at the intermediate support

At the intermediate support (except the left support of the cracked span), the transfer matrix is an 4×4 matrix whose components are product of two vectors:

$$S_{4 \times 4} = \begin{bmatrix} sn_1^T & sn_2^T & hn_3^T & -hn_2^T \\ sp_1^T & sp_2^T & -hp_1^T & -hp_2^T \\ sm_1^T & sm_2^T & hm_3^T & -hm_2^T \\ -sp_1^T & -sp_2^T & hp_1^T & hp_2^T \end{bmatrix}$$

Where s, h are the vectors which contain the sinusoidal and hyperbolic in terms of the $(r-1)$ th span:

$$s = [\sin \lambda L_{r-1} \quad \cos \lambda L_{r-1}], h = [\sinh \lambda L_{r-1} \quad \cosh \lambda L_{r-1}]$$

The vectors n_i, m_i ($i = 1, 2, 3$) contain the information of the r th span:

$$n_1 = \left[-\frac{1}{2}\kappa \quad \eta \right], n_2 = \left[-\eta \quad -\frac{1}{2}\kappa \right], n_3 = \left[\frac{1}{2}\kappa \quad \eta \right]$$

$$m_1 = \left[-\frac{1}{2}\kappa \quad 1 - \eta \right], m_2 = \left[-(1 - \eta) \quad -\frac{1}{2}\kappa \right],$$

$$m_3 = \left[\frac{1}{2}\kappa \quad 1 - \eta \right]$$

in which:

$$\kappa = \frac{\cos \lambda L_r - \cosh \lambda L_r}{\sin \lambda L_r - \sinh \lambda L_r}$$

$$\eta = -\frac{\sinh \lambda L_r}{\sin \lambda L_r - \sinh \lambda L_r}$$

And p_1, p_2 are the vectors which contain real coefficients:

$$p_1 = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}, p_2 = \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix}$$

Transfer matrix at the left support of the cracked span

The transfer matrix at the left support of the cracked span G has the same structure as the transfer matrix at intermediate support S , but its components change.

$$G_{4 \times 4} = \begin{bmatrix} sn_1^T & sn_2^T & hn_3^T & -hn_2^T \\ sp_1^T & sp_2^T & -hp_1^T & -hp_2^T \\ sm_1^T & sm_2^T & hm_3^T & -hm_2^T \\ -sp_1^T & -sp_2^T & hp_1^T & hp_2^T \end{bmatrix}$$

The vectors s, h now contain the sinusoidal and hyperbolic in terms of the $(k-1)$ th span:

$$s = [\sin \lambda L_{k-1} \quad \cos \lambda L_{k-1}], h = [\sinh \lambda L_{k-1} \quad \cosh \lambda L_{k-1}]$$

The vectors n_i, m_i keep the same structures, which contain the information of the crack span:

$$\kappa = \frac{b^T t_2 - b^T t_4}{b^T t_1 - b^T t_3}$$

$$\eta = \frac{-b^T t_3}{b^T t_1 - b^T t_3}$$

with $b = [\sin \lambda L_{k_2} \quad \cos \lambda L_{k_2} \quad \sinh \lambda L_{k_2} \quad \cosh \lambda L_{k_2}]^T$ and t_i , ($i = 1, 2, 3, 4$) are the columns of the transfer matrix at the crack T_{k_1} .