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### Calculation of the dynamic responses of a railway track on a non-uniform foundation

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Abstract:	<p>The dynamic response of railway track has been investigated by different methods: analytical or numerical models. When rail supports (sleepers) or foundation contain defects, this damaged zone has the stiffness reduced relative to outside the zone and the track becomes consequently non-uniform. The existed analytic models cannot be applied easily for such a type of track, meanwhile the numerical model has an inconvenient of the computing time. This paper presents an analytical model to calculate the dynamic responses of the railway track on a non-uniform foundation. In this model, the rail is considered as an infinite beam resting on equidistant supports and subjected to moving forces. The sleepers responses are stable provided there is a sufficiently large distance between the applied forces and the defective area. By combining this condition and the dynamic equation of the beam, we can deduce a relation between the reaction forces in the supports and the displacement of the beam. The resulting relationship is entirely general for arbitrary combinations of supports and foundation systems. Moreover, it allows us to calculate the dynamic response of the system by including the constitutive laws of the supports. These responses can be obtained with standard numerical methods either directly or with iterative techniques, depending on the complexity of the constitutive relations. The numerical examples show that a defective zone causes an increase in the rail displacement and overloads the supports.</p>

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## Abstract

The dynamic response of railway track has been investigated by different methods: analytical or numerical models. When rail supports (sleepers) or foundation contain defects, this damaged zone has the stiffness reduced relative to outside the zone and the track becomes consequently non-uniform. The existed analytic models cannot be applied easily for such a type of track, meanwhile the numerical model has an inconvenient of the computing time. This paper presents an analytical model to calculate the dynamic responses of the railway track on a non-uniform foundation. In this model, the rail is considered as an infinite beam resting on equidistant supports and subjected to moving forces. The sleepers responses are stable provided there is a sufficiently large distance between the applied forces and the defective area. By combining this condition and the dynamic equation of the beam, we can deduce a relation between the reaction forces in the supports and the displacement of the beam. The resulting relationship is entirely general for arbitrary combinations of supports and foundation systems. Moreover, it allows us to calculate the dynamic response of the system by including the constitutive laws of the supports. These responses can be obtained with standard numerical methods either directly or with iterative techniques, depending on the complexity of the constitutive relations. The numerical examples show that a defective zone causes an increase in the rail displacement and overloads the supports.

## Keywords

Analytic model; Dynamic response; non-uniform foundation; Railway; Periodically supported beam; Structural dynamic

## 1 Introduction

The dynamics of railway tracks has been investigated by analytic models since the 1970s by Mead (1970). In these models, the track is modeled by a beam supported by a visco-elastic foundation or by equidistant supports (so-called the periodically supported beams). The response of the track subjected to an excitation or moving forces has been developed in numerous articles. In 1996, Belotserkovskiy (1996) calculated the oscillations of an infinite periodic beams subjected to a moving concentrated loads. By considering that the rail rests on an identical periodic simple elastic support, the beam responses have been performed with the help of the Fourier's transform. Nordborg (1998a,b) presented a model of the track by using the Green's function. This work has been developed by taking into account the wheel inertia force or the rail/wheel surface roughnesses. Metrikine and Popp (1999); Vostroukhov and Metrikine (2003) calculated the steady state dynamic responses of the high-speed railway track. The dependence of the vertical deflection of the rail on a train speed allows to determine the critical velocity of the moving loads. By considering that the loads are harmonic, Sheng et al. (2005, 2007) proposed a method to calculate an infinite periodic structure by basing on the wave-number approach. The non-linearity of the foundation has been taken into account different researches. Ding et al. (2013) presented a method to calculate the responses of a finite beam subjected to moving loads by studying the convergence of Garlerkin truncation. By using a harmonic balance method, Hoang et al. (2018) solved

the same problem with an iteration procedure. Moreover, Hoang et al. (2016, 2017) demonstrated a relation between the reaction force of the support applied on the beam and its displacement in the frequency domain. This relation can be held for any types of support. Recently, Tran et al. (2019) presented a dynamic model for the railway sleeper resting on a Kelvin-Voigt's foundation in which the sleeper responses agree well with the measurements. Recently, a semi-analytical model for the railway is developed by Tran et al. (2021) to study the influence of the non-homogeneous foundation on the sleeper responses.

However, if the foundation contains a defective zone or the sleepers are damaged, the support constitutive laws are nonidentical. Consequently, the classical models are not applicable obviously. Moreover, the non-uniform supports can occur in numerous problems of railways track. Xia and Zhang (2005); Majka and Hartnett (2008) studied the

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case of railway bridges under high speed moving loads by analytical and numerical models. Varandas et al. (2011); Shahraki et al. (2015); Paixão et al. (2018) focus on the track responses at the transition zones. Meanwhile, Castro Jorge et al. (2015) observe the influence of the defect of foundation and/or sleepers on the track responses with the help of finite element method. Recently, the influence of non-uniform foundations on the railway track response has been studied by considering beams resting on a continuous foundation by Paixão et al. (2014); Shahraki et al. (2015). However, among these researches, the rails are supported periodically and these models do not take into account sleepers spacing.

This article presents a new analytic model for railway tracks with a non-uniform zone. Firstly, in the Section 2 we consider the rail as an Euler-Bernoulli beam which is supported periodically. Each support of a rail is a mass-spring-damper system which can have different dynamic stiffness between the non-uniform and normal zone of the track. The beam is subjected by moving loads which are described with the help of the Dirac's delta distribution. Moreover, as assumption was made that the responses of supports are stable when the distance between the support and the non-uniform zone is large enough. By using the Fourier's transforms and the Dirac comb properties, a relation between the support reaction forces and the beam displacement has been established. Then, by combining this relation with the constitutive laws of the supports, two methods to calculate the dynamical responses of the problem have been developed in the Section 3: direct and iterative method. The solutions of the iterative method are obtained when the convergence of an procedure iteration is meet. The choice of the method depend on the complexity of the track conditions. In the Section 4, the influence of the non-uniform zone on the dynamic response has been studied with numerical examples. Finally, the concluding remarks are drawn in the Section 5.

## 2 Formulations

### 2.1 Constitutive laws of railway supports

A railway track model is considered throughout this section as shown in Fig. 1. In this model, the rail considered is a beam which is subjected to moving forces  $Q_j$  which are characterized by the distance  $D_j$  to the first moving load  $Q_0$ . The sleepers together with the foundation are considered as mass-spring-damper systems which are distributed periodically along the beam as shown in Fig. 1.

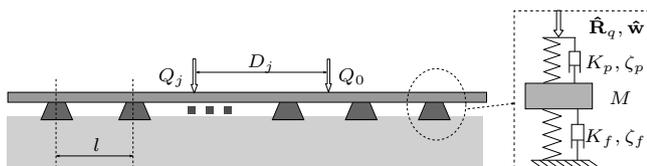


Figure 1. Model of railway track with non-uniform supports

In this figure, each support  $q$  is located by a coordinate  $x = ql$ . The dynamic stiffness  $k_q(\omega)$  of the support  $q$  is defined by:

$$\hat{R}_q(\omega) = -k_q(\omega)\hat{w}_q(\omega) \quad (1)$$

where  $\hat{w}_q(\omega)$ ,  $\hat{R}_q(\omega)$  are respectively the displacement and reaction force of the support  $q$  in the frequency domain. For the mass-spring-damper system as shown in Fig. 1, the dynamic stiffness  $k_q(\omega)$  is calculated as follows:

$$\frac{1}{k_q} = \frac{1}{K_f + i\omega\zeta_f - M\omega^2} + \frac{1}{K_p + i\omega\zeta_p} \quad (2)$$

where  $K$ ,  $\zeta$  and  $M$  are respectively the stiffness, the damping coefficients and the mass of the support. The subscripts ( $\square_f$ ) and ( $\square_p$ ) denote for the foundation and the rail-pad (see Appendix A).

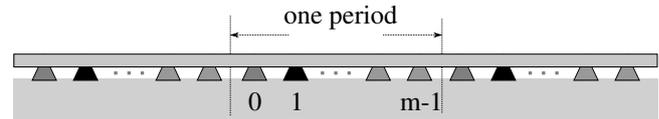


Figure 2. Beam on periodic supports

When the supports are nonidentical, the stiffnesses  $k_q$  on each support are different. The infinite beam leads to an infinite unknown forces  $\hat{R}_q$ . Consequently, the analytic problem can not be solved. In order to reduce the number of unknowns, we consider that the support constitutive laws are periodical along the beam as shown in Fig. 2. This assumption means that two supports at distance  $L = ml$  have the same stiffness  $K_q = K_{q+m}$ . Hence, the track is a periodic structure with period  $L$ . For this reason, we can suppose that the support responses are repeated when the moving forces travel a length equal to one period  $L$ . In other words, for the two supports which has a distance  $L$ , the reaction forces at these positions can be written by the same function with a time delay  $t_L = \frac{L}{v}$  which corresponds to the time for the movement of the forces from one period to another:

$$R_{m+p}(t) = R_p\left(t - \frac{L}{v}\right) \quad (3)$$

where  $R_p(t)$  is the reaction force of the support  $p$ .

It is more convenient for comparing the support responses if they are represented in the same relative time with respect to the moving forces. Because the moving forces take a time  $t = ql/v$  to cover the distance between the support  $q$  and the reference origin ( $x = 0$ ), the support responses without the time delay can be defined as follows:

$$\begin{cases} R_q(t) = R_q\left(t + \frac{ql}{v}\right) \\ w_q(t) = w_q\left(t + \frac{ql}{v}\right) \end{cases} \quad (4)$$

By combining Eq. (3) and (4), we obtain:  $R_{m+p}(t) = R_p(t)$ . Therefore, we have finally  $m$ -unknown forces  $R_p$ .

By substituting the last result into the Eq. (1), the constitutive laws of the supports in one track interval  $L = ml$  can be rewritten in the matrix form as follows:

$$\begin{pmatrix} \hat{R}_0 \\ \hat{R}_1 \\ \vdots \\ \hat{R}_{m-1} \end{pmatrix} = - \begin{pmatrix} k_0 & 0 & \cdots & 0 \\ 0 & k_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{m-1} \end{pmatrix} \begin{pmatrix} \hat{w}_0 \\ \hat{w}_1 \\ \vdots \\ \hat{w}_{m-1} \end{pmatrix} \quad (5)$$

This relation can be rewritten by:

$$\hat{\mathbf{R}} = -\mathbf{K} \hat{\mathbf{w}} \quad (6)$$

where  $\mathbf{K} = \text{diag}(k_0, k_1, \dots, k_{m-1})$ , and  $\hat{\mathbf{w}}$ ,  $\hat{\mathbf{R}}$  are respectively the vectors of displacements  $\hat{w}_p$  and reaction forces  $\hat{R}_p$  in the frequency domain.

For a periodically supported beam that contains a defective zone, we consider that this zone is included in a larger interval with numerous non-defect supports. The number of neighbor supports is big enough so that the responses are unchanged when the moving forces come or leave this interval. These responses are also approximately equal to the steady state results in the case of a non-defect track. Hence, this interval can be considered as one period of periodic supports and we can use again the periodicity condition in Eq. (3).

## 2.2 Dynamic equation of the rail

In the model as shown in Fig. 1, the beam is subjected to the support reaction forces  $\{R_q\}$  and to the moving forces  $\{Q_j\}$ . The total force  $F(x, t)$  applied on the rail can be represented with the help of the Dirac's delta distribution as follows:

$$F(x, t) = \sum_{q \in \mathbb{Z}} R_q(t) \delta(x - ql) - \sum_j Q_j \delta(x + D_j - vt) \quad (7)$$

By substituting Eq. (3) into the aforementioned equation, the total force can be expressed by:

$$F(x, t) = \sum_{n \in \mathbb{Z}} \sum_{p=0}^{m-1} r_p \left( t - \frac{nL}{v} \right) \delta(x - pl - nL) - \sum_j Q_j \delta(x + D_j - vt) \quad (8)$$

When the rail is modeled by an Euler-Bernoulli's beam, the dynamic equation subjected to a force  $F(x, t)$  is written as follows:

$$EI \frac{\partial^4 w_r(x, t)}{\partial x^4} + \rho S \frac{\partial^2 w_r(x, t)}{\partial t^2} = F(x, t) \quad (9)$$

where  $w_r(x, t)$  is the beam displacement in time domain. The notations  $\rho$ ,  $E$ ,  $S$  and  $I$  represent the density, the Young's modulus, the section and the inertia moment of the beam, respectively. The combination of the Fourier's transform of the Eq. (9) with respect to  $x$ , and the Dirac comb property (see [Bracewell \(2000\)](#)) leads to the following result (see [Appendix B](#)):

$$\hat{w}_r(x, \omega) e^{i\omega \frac{x}{v}} = \sum_{p=0}^{m-1} \hat{R}_p(\omega) \eta(pl - x) - w_e(\omega) \quad (10)$$

where  $\hat{w}_r$ ,  $\hat{R}_p$  are respectively the rail displacement and the reaction force in the frequency domain. Besides, the two functions  $w_e(\omega)$  and  $\eta(x)$  are determined as follows:

$$w_e(\omega) = \frac{1}{vEI} \sum_j \frac{Q_j e^{-i\frac{\omega}{v} D_j}}{\left(\frac{\omega}{v}\right)^4 - \lambda_e^4} \quad (11)$$

$$\eta(x) = \frac{1}{LEI} \sum_{n \in \mathbb{Z}} \frac{e^{i\frac{2\pi n}{L} x}}{\left(\frac{\omega}{v} + \frac{2\pi n}{L}\right)^4 - \lambda_e^4} \quad (12)$$

Indeed, the function  $\eta(x)$  with  $x \in [0, L]$  is a series which can be reduced to a simple expression (see [Appendix C](#)). Eq. (10) establishes a relation between the beam displacement and the support reaction forces. Particularly, the support responses  $(\hat{w}_q, \hat{R}_q)$  can be calculated by substituting its coordinate  $x = ql$  into the Eq. (10) and we obtain:

$$\hat{w}_q = \hat{w}_r(ql, \omega) e^{i\omega \frac{ql}{v}} = \sum_{p=0}^{m-1} \hat{R}_p \eta_{p-q} - w_e \quad (13)$$

where  $\eta_{q-p}$  is deduced from Eq. (12) with  $x = (p - q)l$ .

It is remarkable that  $\forall q > p$ ,  $\eta_{p-q} = \eta_{m+p-q}$ . Thus, Eq. (13) can be written in matrix form as follows:

$$\begin{pmatrix} \eta_0 & \eta_1 & \cdots & \eta_{m-1} \\ \eta_{m-1} & \eta_0 & \cdots & \eta_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1 & \eta_2 & \cdots & \eta_0 \end{pmatrix} \begin{pmatrix} \hat{R}_0 \\ \hat{R}_1 \\ \vdots \\ \hat{R}_{m-1} \end{pmatrix} = w_e \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \begin{pmatrix} \hat{w}_0 \\ \hat{w}_1 \\ \vdots \\ \hat{w}_{m-1} \end{pmatrix} \quad (14)$$

By taking the notations: matrix  $\mathbf{C}$ , vectors  $\hat{\mathbf{R}}$ ,  $\mathbf{I}$ ,  $\hat{\mathbf{w}}$  corresponding to the terms of the Eq. (14), this result can be expressed as:

$$\mathbf{C} \hat{\mathbf{R}} = w_e \mathbf{I} + \hat{\mathbf{w}} \quad (15)$$

Eq. (15) describes relation between the support reaction forces and displacements in one period of the structure. It is a result of the periodicity condition and the dynamic equation. In addition, this relation does not depend on support behaviors.

*Remarks:* the matrix  $\mathbf{C}$  in Eq. (15) is circular and it can be represented by a convolution product (see [Harris and Stocker \(1998\)](#)):

$$\hat{\mathbf{w}} = \boldsymbol{\eta} * \hat{\mathbf{R}} - w_e \mathbf{I} \quad (16)$$

where  $\boldsymbol{\eta} = (\eta_0 \ \eta_1 \ \cdots \ \eta_{m-1})^T$ . We can represent the aforementioned equation by using the discrete Fourier's transform as follows:

$$\mathcal{F}_q\{\hat{\mathbf{w}}\} = \mathcal{F}_q\{\boldsymbol{\eta}\} \mathcal{F}_q\{\hat{\mathbf{R}}\} - w_e \mathcal{F}_q\{\mathbf{I}\} \quad (17)$$

where  $\mathcal{F}_q\{\square\}$  with  $(0 \leq q \leq m - 1)$  denotes for the component  $q + 1$  of discrete Fourier's transform. In addition, we have:

$$\mathcal{F}_q\{\mathbf{I}\} = m\delta_{0q} \quad \text{and} \quad \mathcal{F}_q\{\boldsymbol{\eta}\} = \chi_q \quad (18)$$

where  $\delta_{0q} = 1$  if  $q = 0$  and  $\delta_{0q} = 0$  otherwise; the function  $\chi_q$  is shown in [Appendix C](#). Thus, Eq. (17) becomes:

$$\mathcal{F}_q\{\hat{\mathbf{R}}\} = \kappa_q \mathcal{F}_q\{\hat{\mathbf{w}}\} + m\mathcal{Q}_e \delta_{0q} \quad (19)$$

where  $\kappa_q = \chi_q^{-1}$  and  $\mathcal{Q}_e = \kappa_0 w_e$ .

Eq. (19) presents a characteristic relation of a periodically supported beam subjected to moving loads. This equation

links the discrete Fourier transforms of the support responses. When all supports are identical, the vectors of reactions forces  $\hat{\mathbf{R}}$  and displacements  $\hat{\mathbf{w}}$  are proportional to  $\mathbf{I}$  and this equation can be reduced to:

$$\hat{R}_0 = \kappa_0 \hat{w}_0 + Q_e \quad (20)$$

This result is already established for periodically supported beams with identical supports by Hoang et al. (2017).

### 3 Calculation of the track responses

In this section, two methods will be presented to calculate the track responses: direct and iterative methods, depending on the complexity of the constitutive relations.

#### 3.1 Direct method

In the previous section, the relations between the reaction forces  $\hat{\mathbf{R}}$  and the displacements  $\hat{\mathbf{w}}$  has been established in the two Eqs. (6) and (15). By combining these two equations, we obtain the following result:

$$(\mathbf{C} \mathbf{K} + \mathbf{I}) \hat{\mathbf{w}} = -w_e \mathbf{I} \quad (21)$$

The displacements of the supports are obtained from Eq. (21):

$$\hat{\mathbf{w}} = -w_e \tilde{\mathbf{K}}^{-1} \mathbf{I} \quad (22)$$

where:  $\tilde{\mathbf{K}} = (\mathbf{C} \mathbf{K} + \mathbf{I})$ . The matrix  $\mathbf{K}$  is a dimensionless with size  $m \times m$  depending on the frequency  $\omega$ . Therefore, the supports responses can obtain by inverting this matrix for all frequencies. When  $m$  is small, this mathematical operation is simple and fast. Otherwise, an iterative method will be presented below to compute the responses.

#### 3.2 Iterative method

Now, we note that:

$$\tilde{\mathbf{R}} = \hat{\mathbf{R}} + K_{nd} \hat{\mathbf{w}} \quad (23)$$

where  $K_{nd}$  is the dynamic stiffness of a non-defect support. By substituting Eq. (23) into Eqs. (6) and (19), we obtain a following mathematical system:

$$\begin{cases} \tilde{\mathbf{R}} = -(\mathbf{K} - K_{nd} \mathbf{I}) \hat{\mathbf{w}} \\ \mathcal{F}_q\{\hat{\mathbf{w}}\} = \frac{1}{\kappa_q + K_{nd}} [\mathcal{F}_q\{\tilde{\mathbf{R}}\} - m Q_e \delta_{1q}] \end{cases} \quad (24)$$

An iterative schema is built for solving the aforementioned equation. From an initial value which is given for the displacement  $\hat{\mathbf{w}}$ , the force  $\tilde{\mathbf{R}}$  can be calculated by using the first equation of Eq. (24). Then, the displacement is recalculated by the second equation with the help of the fast Fourier transform (FFT). The procedure is repeated until the convergence of the track responses is satisfied. The initial values can be chosen as the response of a periodically supported beam given by:

$$\hat{w}_{initial} = \frac{Q_e}{\kappa_0 + K_{nd}} \mathbf{I} \quad (25)$$

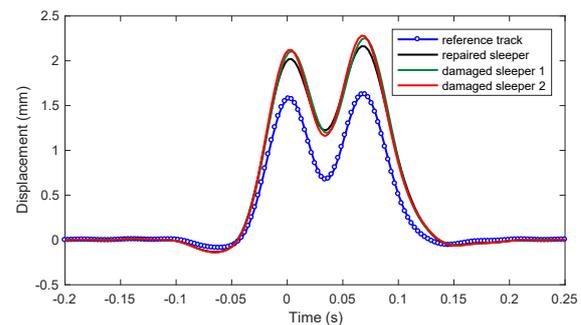
Eqs. (24) and (25) completely define a recurrent sequence of an iteration procedure.

*Remarks:* For a support system with a defective zone,  $K_q = K_{nd}$  for all supports  $q$  outside of this area. Thus, the vector  $\tilde{\mathbf{R}}$  in Eq. (24) has zeros for all components outside of the defective zone.

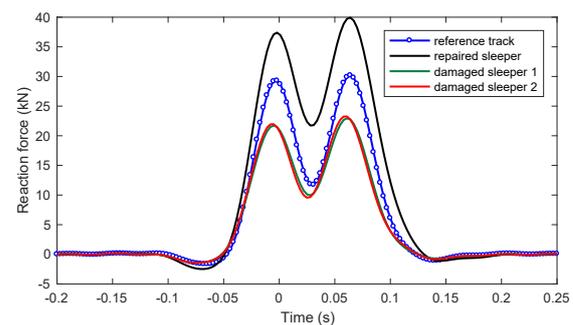
## 4 Numerical examples

### 4.1 Repaired railway track by periodical replacements

In a non-ballasted railway track, the rails are supported by mono-blocks. In order to quickly repair damaged sleepers without substantial traffic interruptions, we can replace one in three damaged blocks. This process creates periodic supports (as shown in Fig. 2) whose one period contains three blocks where a repaired block is located between two damaged ones. We consider such a repaired railway track with parameters given in Table 1. The dynamic stiffness of supports in one period are calculated by Eq. (2). We consider that the damaged sleeper has a smaller stiffness than the repaired one. Fig. 3 shows the responses of the three blocks in one period by using the direct method by considering only a half of track (one rail). In this figure, the reference track corresponds to the responses of a railway track with all repaired blocks. The reference time  $t = 0$  correspond to the moment when the moving force passe on the block position.



(a) Time domain



(b) Reaction force against the blocks

**Figure 3.** Responses of a periodically repaired track

We note that the displacements of the three blocks are approximately similar and they are larger than for the reference track. Indeed, the stiffness of the damaged blocks are smaller than that of the repaired ones. Consequently, the reference track is more rigid than the repaired track and it is the cause of the displacement increase. The three reaction forces are more varied. The load on the repaired sleeper is more important than the damaged blocks. In conclusion,

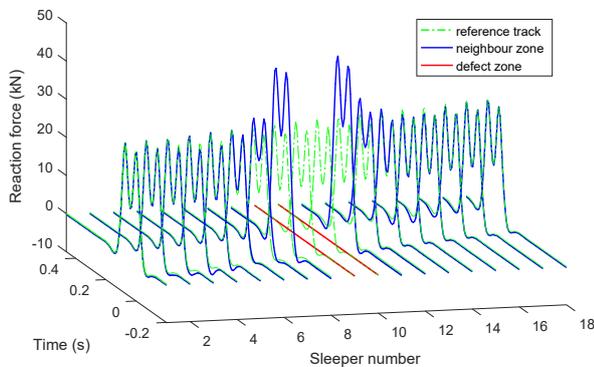
Parameters	Notation	Unit	Value
Rail mass	$\rho S$	kg m <sup>-1</sup>	60
Rail stiffness	$EI$	MN m <sup>2</sup>	6.3
Train speed	$v$	km h <sup>-1</sup>	160
Wheel vertical load	$Q$	kN	100
Block mass	$M$	kg	90
Sleeper spacing	$l$	m	0.6
Axle distance	$D$	m	3
Number of axles	$n$		2
Stiffness of rail-pad	$K_p$	MN m <sup>-1</sup>	200
Damping coefficient of rail-pad	$\zeta_p$	MN s m <sup>-1</sup>	1.0
Stiffness of repaired block	$K_f$	MN m <sup>-1</sup>	20
Damping coefficient of repaired block	$\zeta_f$	MN s m <sup>-1</sup>	0.2
Stiffness of damaged block	$K'_f$	MN m <sup>-1</sup>	10
Damping coefficient of damaged block	$\zeta'_f$	MN s m <sup>-1</sup>	0.2

**Table 1.** Parameters of a railway track

periodic replacement can improve the stiffness of railway tracks but the repaired supports will be overloaded.

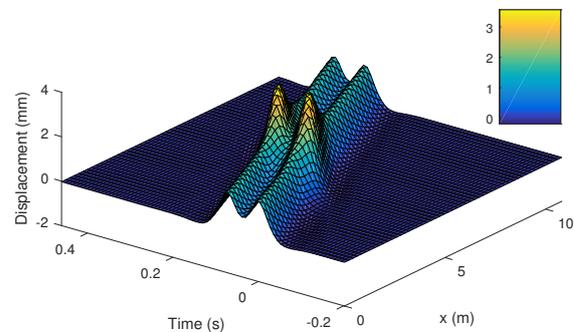
#### 4.2 Railway track with a defective zone

We consider the railway track given in Table 1 with a defective zone containing two hanging blocks which its stiffness are zero. The track responses have been calculated by the iterative method for an interval of 18 supports where the two hanging blocks are located at the center of the interval (block number 9 and 10).



**Figure 4.** Sleeper responses of a damaged track

The support responses in this interval are shown in the Fig. 4. The reaction forces of the hanging blocks (red lines at numbers 9 and 10) are zero because their stiffness is null. Otherwise, the reaction forces of the blocks far from the defective zone (sleepers 1-6 and 13-18) are similar to the responses of the reference track. It means that the influence of the defect block is negligible from the third neighboring block onward. This result agrees well with the periodicity condition that the responses are unchanged outside the interval and they are approximately equal to the reference responses. However, the first and the second neighbor sleepers of the defective area are overloaded. Particularly, the overloaded rate of the first sleeper is 53%. As a consequence, the lifetime of these sleepers would be shortened.



**Figure 5.** Rail displacement of a damaged track

Fig. 5 shows the rail displacement in time domain. The two streaks correspond to the location of two moving forces. In addition, the two maximum peaks correspond to the moment when the moving forces pass through the defective zones. We see that the displacement of the point forces increase from 1.6mm to 3.6mm, which cause a track leveling.

## 5 Conclusions

In this paper, an analytic model has been developed to calculate the response of a non-uniform railway track subjected to moving forces. The rail is modeled as an infinite beam resting on nonidentical periodic supports with the help of the Euler-Bernoulli's beam model. Generally, this model has an infinite unknowns number. However, the unknowns number of the problem is reduced by using a periodicity condition in one interval of the track. Consequently, the dynamic equation of the beam leads to a relation between the beam displacement and the support reaction forces by combining the periodicity condition. This model permits to calculate rapidly dynamic responses of the damaged track. However, it can not take into account the dependent of support as the propagation of vibrations or the shear stress in the foundation. Then, the direct and iterative methods have been developed to compute the dynamical track responses. This model has been applied to the dynamics of railway tracks with sleepers or blocks with defective

support conditions. The numerical results show that the neighbor sleepers are overloaded and the rail displacement increases at the damaged zone.

## A Appendix A

Let  $w_p(t)$  and  $w_f(t)$  be the variation of displacement of the rail-pad and the foundation at the support  $q$  in the time domain. Under a reaction force  $R_q$ , the equilibrium equation of the sleeper can be written as follows:

$$\begin{aligned} R_q(t) &= K_p w_p(t) + \zeta_p \dot{w}_p(t) \\ &= K_f w_f(t) + \zeta_f \dot{w}_f(t) + M \ddot{w}_f(t) \end{aligned} \quad (26)$$

where  $K_f$ ,  $K_p$ ,  $\zeta_f$ ,  $\zeta_p$  are respectively the stiffness, damping coefficients of the rail-pad and the foundation; and  $M$  is the sleeper mass. Moreover,  $(\square)$  stands for the partial derivative with regard to time  $t$ .

By using the Fourier's transform, the relation between the variation of displacement of the rail-pad  $\hat{w}_p(\omega)$  and the foundation  $\hat{w}_f(\omega)$  in the frequency domain can be obtained:

$$\begin{aligned} \hat{R}_q(\omega) &= (K_p + i\omega\zeta_p) \hat{w}_p(\omega) \\ &= (K_f + i\omega\zeta_f - \omega^2 M) \hat{w}_f(\omega) \end{aligned} \quad (27)$$

By considering a support  $q$  as a spring with a stiffness  $k_q$ , the reaction force applied on the support can be calculated in the frequency domain as follows:

$$\hat{R}_q(\omega) = k_q \hat{w}_q(\omega) = k_q (\hat{w}_p + \hat{w}_f) \quad (28)$$

By substituting the Eq. (28) into the Eq. (27), we can calculate the stiffness equivalent of the support  $q$ :

$$\frac{1}{k_q} = \frac{1}{K_p + i\omega\zeta_p} + \frac{1}{K_f + i\omega\zeta_f - \omega^2 M} \quad (29)$$

## B Appendix B

By performing the Fourier's transform of Eqs. (8) and (9) with respect to  $t$ , we obtain

$$EI \frac{\partial^4 \hat{w}_r(x, \omega)}{\partial x^4} - \rho S \omega^2 \hat{w}_r(x, \omega) = \hat{F}(x, \omega) \quad (30)$$

where:

$$\begin{aligned} \hat{F}(x, \omega) &= \sum_{n \in \mathbb{Z}} \sum_{p=0}^{m-1} \hat{r}_p(\omega) e^{-i\omega \frac{nL}{v}} \delta(x - nL - pl) \\ &\quad - \sum_j \frac{Q_j}{v} e^{-i\omega \frac{x+D_j}{v}} \end{aligned} \quad (31)$$

Then, by performing the Fourier's transform of the aforementioned equation with respect to  $x$ , we obtain

$$\begin{aligned} (EI\lambda^4 - \rho S \omega^2) \tilde{w}_r(\lambda, \omega) \\ = \sum_{n \in \mathbb{Z}} \sum_{p=0}^{m-1} \hat{r}_p(\omega) e^{-i\omega nL - i\lambda(nL+pl)} \\ - 2\pi\delta \left( \lambda + \frac{\omega}{v} \right) \sum_j \frac{Q_j}{v} e^{-i\omega \frac{D_j}{v}} \end{aligned} \quad (32)$$

where  $\tilde{w}_r(\lambda, \omega)$  is the Fourier's transform of  $\hat{w}_r(x, \omega)$  with respect to  $x$ . Moreover, by using the Dirac comb properties [Bracewell \(2000\)](#), the infinite sum in Eq. (32) can be written as follows:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \sum_{p=0}^{m-1} \hat{r}_p(\omega) e^{-i(\lambda + \frac{\omega}{v})nL - i\lambda pl} \\ = \sum_{p=0}^{m-1} \hat{r}_p(\omega) e^{-i\lambda pl} \sum_{n \in \mathbb{Z}} e^{-i(\lambda + \frac{\omega}{v})nL} \\ = \hat{R}(\lambda, \omega) \frac{2\pi}{L} \sum_{n \in \mathbb{Z}} \delta \left( \lambda + \frac{\omega}{v} + \frac{2\pi n}{L} \right) \end{aligned} \quad (33)$$

where  $\hat{R}(\lambda, \omega) = \sum_{p=0}^{m-1} \hat{r}_p(\omega) e^{-i\lambda pl}$ . By substituting Eq. (33) into Eq. (32), we can write

$$\begin{aligned} \tilde{w}_r(\lambda, \omega) &= \frac{2\pi \hat{R}(\lambda, \omega)}{LEI(\lambda^4 - \lambda_e^4)} \sum_{n \in \mathbb{Z}} \delta \left( \lambda + \frac{\omega}{v} + \frac{2\pi n}{L} \right) \\ &\quad - \frac{2\pi\delta \left( \lambda + \frac{\omega}{v} \right)}{vEI(\lambda^4 - \lambda_e^4)} \sum_j Q_j e^{-i\omega \frac{D_j}{v}} \end{aligned} \quad (34)$$

where  $\lambda_e = \sqrt[4]{\frac{\rho S \omega^2}{EI}}$ . Finally,  $\hat{w}_r(x, \omega)$  can be obtained by performing the inverse Fourier's transform of  $\tilde{w}_r(\lambda, \omega)$  with respect to  $\lambda$ :

$$\begin{aligned} \hat{w}_r(x, \omega) &= \sum_{n \in \mathbb{Z}} \sum_{p=0}^{m-1} \frac{\hat{r}_p(\omega)}{LEI} \frac{e^{-i(\frac{2\pi n}{L} + \frac{\omega}{v})(x-pl)}}{\left( \frac{\omega}{v} + \frac{2\pi n}{L} \right)^4 - \lambda_e^4} \\ &\quad - \sum_j \frac{Q_j}{vEI} \frac{e^{-i\omega \frac{x+D_j}{v}}}{\left( \frac{\omega}{v} \right)^4 - \lambda_e^4} \end{aligned} \quad (35)$$

By substituting Eq. (4) into the aforementioned equation, we obtain:

$$\hat{w}_r(x, \omega) e^{i\omega \frac{x}{v}} = \sum_{p=0}^{m-1} \hat{R}_p(\omega) \eta(pl - x) - \sum_j \frac{Q_j}{vEI} \frac{e^{-i\omega \frac{D_j}{v}}}{\left( \frac{\omega}{v} \right)^4 - \lambda_e^4} \quad (36)$$

where  $\eta(x)$  is defined by

$$\eta(x) = \frac{1}{LEI} \sum_{n \in \mathbb{Z}} \frac{e^{i2\pi n \frac{x}{L}}}{\left( \frac{\omega}{v} + \frac{2\pi n}{L} \right)^4 - \lambda_e^4}$$

## C Appendix C

The function  $\eta(x)$  in Eq. (37) can be rewritten as follows

$$\eta(x) = \frac{L}{2\lambda_e^2 EI} \sum_{n \in \mathbb{Z}} \left[ \frac{e^{i2\pi n \frac{x}{L}}}{\left( \frac{\omega L}{v} + 2\pi n \right)^2 - (L\lambda_e)^2} - \frac{e^{i2\pi n \frac{x}{L}}}{\left( \frac{\omega L}{v} + 2\pi n \right)^2 + (L\lambda_e)^2} \right] \quad (37)$$

We will show that each term of the last expression can be deduced as follows

$$\left\{ \begin{array}{l} \sum_{n \in \mathbb{Z}} \frac{e^{i2\pi n \frac{x}{L}}}{\left(\frac{\omega L}{v} + 2\pi n\right)^2 - (L\lambda_e)^2} = \\ \frac{e^{i\omega \frac{x}{v}} \sin \lambda_e(L-x) + e^{-i\omega \frac{L-x}{v}} \sin \lambda_e x}{2L\lambda_e (\cos L\lambda_e - \cos \frac{\omega L}{v})} \\ \sum_{n \in \mathbb{Z}} \frac{e^{i2\pi n \frac{x}{L}}}{\left(\frac{\omega L}{v} + 2\pi n\right)^2 + (L\lambda_e)^2} = \\ \frac{e^{i\omega \frac{x}{v}} \sinh \lambda_e(L-x) + e^{-i\omega \frac{L-x}{v}} \sinh \lambda_e x}{2L\lambda_e (\cosh L\lambda_e - \cos \frac{\omega L}{v})} \end{array} \right. \quad (38)$$

Indeed, the right-hand sides of the two last equations are Fourier series with respect to  $x$ . Therefore, it is sufficient to demonstrate that the Fourier series coefficients of the functions on the left-hand sides correspond to coefficients on the right-hand sides. This demonstration is easy because these functions are trigonometric. Then, by substituting Eqs. (38), (38) into Eq. (37) we obtain the following result

$$\eta(x) = \frac{e^{i\omega \frac{x}{v}}}{4\lambda_e^3 EI} \left[ \frac{\sin \lambda_e(L-x) + e^{-i\omega \frac{L-x}{v}} \sin \lambda_e x}{\cos L\lambda_e - \cos \frac{\omega L}{v}} - \frac{\sinh \lambda_e(L-x) + e^{-i\omega \frac{L-x}{v}} \sinh \lambda_e x}{\cosh L\lambda_e - \cos \frac{\omega L}{v}} \right] \quad (39)$$

Particularly, for  $x = pl$  we can rewrite Eq. (37) as follows

$$\eta_p = \eta(pl) = \frac{1}{LEI} \sum_{n \in \mathbb{Z}} \frac{e^{i2\pi n \frac{pl}{L}}}{\left(\frac{\omega}{v} + \frac{2\pi n}{L}\right)^4 - \lambda_e^4} \quad (40)$$

If we put  $n = rm + q$  (with  $0 \leq q \leq m-1$ ,  $r \in \mathbb{Z}$ ) and  $L = ml$ , the last equation becomes

$$\begin{aligned} \eta_p &= \sum_{q=0}^{m-1} \frac{1}{mlEI} \sum_{r \in \mathbb{Z}} \frac{e^{i2\pi(mr+q)\frac{p}{m}}}{\left(\frac{\omega}{v} + \frac{2\pi(mr+q)}{ml}\right)^4 - \lambda_e^4} \\ &= \frac{1}{m} \sum_{q=0}^{m-1} e^{i2\pi \frac{pq}{m}} \chi_q \end{aligned} \quad (41)$$

where  $\chi_q$  is defined by

$$\chi_q = \frac{1}{lEI} \sum_{r=-\infty}^{\infty} \frac{1}{\left(\frac{\omega}{v} + \frac{2\pi(mr+q)}{ml}\right)^4 - \lambda_e^4} \quad (42)$$

Eq. (41) is the inverse discrete Fourier's transform (see Harris and Stocker (1998)) of vector  $\boldsymbol{\eta} = (\eta_0 \ \eta_2 \ \dots \ \eta_{m-1})^T$ . Therefore,  $\chi_q$  ( $0 \leq q < m$ ) is the discrete Fourier's transform of  $\boldsymbol{\eta}$ .

Moreover, we see that the function  $\chi_q$  in Eq. (42) is a particular case of  $\eta(x)$  in Eq. (37) with  $x = 0$ ,  $n = m$  and  $\left(\frac{\omega}{v}\right)$  replaced by  $\left(\frac{\omega}{v} + \frac{2\pi q}{ml}\right)$ . Thus,  $\chi_q$  can be deduced from  $\eta(x)$  by using Eq. (39)

$$\chi_q = \frac{1}{4\lambda_e^3 EI} \left( \frac{\sin l\lambda_e}{\cos l\lambda_e - \cos\left(\frac{\omega l}{v} + \frac{2\pi q}{m}\right)} - \frac{\sinh l\lambda_e}{\cosh l\lambda_e - \cos\left(\frac{\omega l}{v} + \frac{2\pi q}{m}\right)} \right) \quad (43)$$

## References

- Belotserkovskiy PM (1996) On the oscillation of infinite periodic beams subjected to a moving concentrated force. *Journal of Sound and Vibration* 193(3): 705–712.
- Bracewell R (2000) *The Fourier transform and its applications*. McGraw-Hill Higher Education. ISBN 9780073039381.
- Castro Jorge P, Simões F and Pinto da Costa A (2015) Dynamics of beams on non-uniform nonlinear foundations subjected to moving loads. *Computers & Structures* 148: 26–34.
- Ding H, Shi KL, Chen LQ and Yang SP (2013) Dynamic response of an infinite Timoshenko beam on a nonlinear viscoelastic foundation to a moving load. *Nonlinear Dynamics* 73(1-2): 285–298.
- Harris JW and Stocker H (1998) *Handbook of Mathematics and Computational Science*. Springer. ISBN 0387947469.
- Hoang T, Duhamel D and Foret G (2018) Dynamical response of a timoshenko beams on periodical nonlinear supports subjected to moving forces. *Engineering Structures* 176: 673–680. DOI: <https://doi.org/10.1016/j.engstruct.2018.09.028>.
- Hoang T, Duhamel D, Foret G, Yin H, Joyez P and Caby R (2017) Calculation of force distribution for a periodically supported beam subjected to moving loads. *Journal of Sound and Vibration* 388: 327–338. DOI: <http://dx.doi.org/10.1016/j.jsv.2016.10.031>.
- Hoang T, Duhamel D, Foret G, Yin HP and Cumunel G (2016) Response of a periodically supported beam on a nonlinear foundation subjected to moving loads. *Nonlinear Dynamics* 86(2): 953–961. DOI:10.1007/s11071-016-2936-5.
- Majka M and Hartnett M (2008) Effects of speed, load and damping on the dynamic response of railway bridges and vehicles. *Computers & Structures* 86(6): 556–572. DOI:10.1016/j.compstruc.2007.05.002.
- Mead DJ (1970) Free wave propagation in periodically supported, infinite beams. *Journal of Sound and Vibration* 11(2): 181–197.
- Metrikine AV and Popp K (1999) Vibration of a periodically supported beam on an elastic half-space. *European Journal of Mechanics A/Solid* 18: 679–701.
- Nordborg A (1998a) Vertical rail vibrations: Parametric excitation. *Acustica* 84: 289–300.
- Nordborg A (1998b) Vertical rail vibrations: Pointforce excitation. *Acustica* 84: 280–288.
- Paixão A, Fortunato E and Calçada R (2014) Transition zones to railway bridges: Track measurements and numerical modelling. *Engineering Structures* 80: 435–443.
- Paixão A, Varandas JN, Fortunato E and Calçada R (2018) Numerical simulations to improve the use of under sleeper pads at transition zones to railway bridges. *Engineering Structures* 164: 169–182. DOI: <https://doi.org/10.1016/j.engstruct.2018.03.005>.
- Shahraki M, Warnakulasooriya C and Witt KJ (2015) Numerical study of transition zone between ballasted and ballastless railway track. *Transportation Geotechnics* 3: 58–67.
- Sheng X, Jones C and Thompson D (2005) Response of infinite periodic structure to moving or stationary harmonic loads. *Journal of Sound and Vibration* 282: 125–149.
- Sheng X, Li M, Jones CJC and Thompson DJ (2007) Vibration analysis of the continuous beam subjected to a moving mass. *Journal of Sound and Vibration* 303: 873–894.

- 1  
2 Tran LH, Hoang T, Duhamel D, Foret G, Messad S and Loaëc  
3 A (2021) Influence of non-homogeneous foundations on the  
4 dynamic responses of railway sleepers. *International Journal*  
5 *of Structural Stability and Dynamics* 21(01): 2150002. DOI:  
6 10.1142/S0219455421500024.
- 7 Tran LH, Hoang T, Gilles F, Duhamel D, Messad S and Loaëc  
8 A (2019) A fast analytic method to calculate the dynamic  
9 response of railways sleepers. *Journal of Vibration and*  
10 *Acoustics* 141(1). DOI:10.1115/1.4040392.
- 11 Varandas JN, Hölscher P and aG Silva M (2011) Dynamic  
12 behaviour of railway tracks on transitions zones. *Computers &*  
13 *Structures* 89(13-14): 1468–1479. DOI:10.1016/j.compstruc.  
14 2011.02.013.
- 15 Vostroukhov AV and Metrikine A (2003) Periodically supported  
16 beam on a visco-elastic layer as a model for dynamic analysis  
17 of a high-speed railway track. *International Journal of Solids*  
18 *and Structures* 40: 5723–5752.
- 19 Xia H and Zhang N (2005) Dynamic analysis of railway bridge  
20 under high-speed trains. *Computers & Structures* 83(23):  
21 1891–1901. DOI:10.1016/j.compstruc.2005.02.014.  
22  
23  
24  
25  
26  
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