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A mixed displacement-pressure formulation for poroelastic materials

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Recently, finite element models based on Biot's displacement (\underline{u}, U) formulation for poroelastic materials have been extensively used to predict the acoustical and structural behavior of multilayer structures. These models while accurate lead to large frequency dependent matrices for three-dimensional problems necessitating important setup time, computer storage and solution time. In this paper, a novel exact mixed displacement pressure (\underline{u}, p) formulation is presented. The formulation derives directly from Biot's poroelasticity equations. It has the form of a classical coupled fluid-structure problem involving the dynamic equations of the skeleton *in vacuo* and the equivalent fluid in the rigid skeleton limit. The governing (\underline{u}, p) equations and their weak integral form are given together with the coupling conditions with acoustic media. The numerical implementation of the presented approach in a finite element code is discussed. Examples are presented to show the accuracy and effectiveness of the presented formulation.

INTRODUCTION

The vibroacoustic performance of finite multilayer systems containing poroelastic materials is of utmost importance for noise control in automobiles, aircraft and several other engineering applications. In the absence of absorbing materials, the vibroacoustic response of complex multilayer structures are classically modeled using the finite element and the boundary element methods. To account for absorbing media, finite element formulations for sound absorbing materials have been developed. They range from simple approaches (impedance techniques: Beranek and Ver, 1992; equivalent fluid models: Craggs, 1978; Panneton *et al.*, 1995) to sophisticated approaches based on the Biot theory (Kang and Bolton, 1995; Johansen *et al.*, 1995; Coyette and Wynendaele, 1995; Panneton and Atalla, 1996, 1997a). For the general case of multilayer systems containing poroelastic media, the use of the sophisticated approaches becomes necessary. These approaches are mainly based on the classical displacement (\underline{u}, U) formulation of Biot's poroelasticity equations (Biot, 1956). However, it has been shown that while accurate the (\underline{u}, U) formulation has the disadvantage of requiring cumbersome calculations for large finite element models and spectral analyses (Panneton, 1996; Panneton and Atalla, 1996, 1997a). To alleviate these difficulties, a mixed formulation using the solid phase displacement and the interstitial fluid pressure as variables seems more suitable for large size finite element models since it requires only four degrees of freedom per node. However, few efforts have been devoted to this idea. Lately, a simplified mixed (\underline{u}, p) formulation was proposed by Göransson (1995). However, in his formulation, Göransson neglected the strain couplings between the fluid and solid phases of the porous material. He solved the one-dimensional wave propagation in porous media using a weighted residual formulation based on Galerkin's method. In geomechanics, several authors have investigated mixed formulations (see Simon and Zienkiewicz,

1986 for references). However, all these formulations are based on assumptions that neglect inertia coupling, which is not acceptable for vibroacoustic applications. Based on these investigations, it is widely believed in acoustics that a (\underline{u}, p) formulation is not possible without limiting assumptions.

In this paper, an exact (\underline{u}, p) formulation recently developed by the authors is presented (Atalla *et al.*, 1996). In this new formulation, the classical Biot-Allard equations are rewritten in terms of the solid phase macroscopic displacement vector and the interstitial fluid phase macroscopic pressure. The new coupled equations have the advantage of recasting the poroelasticity equations in the classical form of a fluid-structure coupled problem wherein the coupling is between an equivalent elastodynamic equation for the skeleton *in vacuo* and an equivalent Helmholtz equation for the fluid in the limit of a rigid skeleton. In the first part of the paper, the classical formulation is recalled. Next, the new formulation is derived followed by its numerical implementation. The coupling conditions between the presented formulation and acoustic formulation are given. Finally, examples demonstrating the accuracy and efficiency of the approach are presented.

I. THEORY

The problem of interest deals with the prediction of the vibroacoustic response (dynamic and acoustic response) of multilayer structures made up of elastic, poroelastic and acoustic media. The poroelastic material may be bonded or unbonded to the structure. The classical assumptions concerning linear acoustic, elastic, and poroelastic wave propagation are assumed (Allard, 1993). Also, the air contained in the porous medium is initially at rest.

A. The Biot's displacement formulation

The modeling of the poroelastic material is based on the Biot's poroelasticity equations (Biot, 1956; Allard, 1993):

$$\operatorname{div} \underline{\underline{\sigma}}^s = \rho_{11} \underline{\underline{\ddot{u}}} + \rho_{12} \underline{\underline{\ddot{U}}} + \tilde{b}(\underline{\underline{\dot{u}}} - \underline{\underline{\dot{U}}}) \quad (1)$$

$$\operatorname{div} \underline{\underline{\sigma}}^f = \rho_{22} \underline{\underline{\ddot{U}}} + \rho_{12} \underline{\underline{\ddot{u}}} - \tilde{b}(\underline{\underline{\dot{u}}} - \underline{\underline{\dot{U}}}),$$

where the tilde symbol indicates that the associated physical property is complex and frequency dependent. In Eq. (1), \underline{u} and \underline{U} denote the solid and fluid macroscopic displacement vectors, respectively. These displacements are average values in the sense of the Biot theory, i.e., the average volume displacements per unit area cross section. Densities ρ_{11} and ρ_{22} are mass coefficients which take into account the fact that the relative flow through the pores is not uniform. They are related to the mass density of the material from which the skeleton is made, ρ_s , and to the mass density of the interstitial fluid, ρ_0 , by :

$$\rho_{11} = (1-h)\rho_s - \rho_{12}, \quad \rho_{22} = h\rho_0 - \rho_{12}, \quad (2)$$

where h is the porosity of the poroelastic material. The coefficient ρ_{12} accounts for the interaction between the inertia forces of the solid and fluid phases; it is function of the tortuosity of the material (Allard, 1993). $\underline{\underline{\sigma}}^s$ and $\underline{\underline{\sigma}}^f$ denote the partial stress tensors associated with the skeleton particle and the macroscopic fluid particle, respectively. Tensor $\underline{\underline{\sigma}}^f$ is related to the averaged pressure prevailing in the saturating fluid by $\underline{\underline{\sigma}}^f = -hp\underline{\underline{1}}$, while tensor $\underline{\underline{\sigma}}^s$ is given by $(1-h)$ times the averaged stress tensor prevailing in the matrix of the poroelastic material. Note that $\underline{\underline{\sigma}}^s$ and $\underline{\underline{\sigma}}^f$ are linearly related to the partial strain tensors $\underline{\underline{\varepsilon}}^s$ and $\underline{\underline{\varepsilon}}^f$ prevailing in the skeleton and the interstitial fluid, respectively. These relations define the stiffness coupling between the solid and the fluid phases and account for dissipation due to thermal exchanges between both phases in the material. Also, they include the structural damping associated with the frame. Finally, \tilde{b} is a frequency dependent viscous damping coefficient accounting for viscous interaction forces. It can be related to the macroscopic flow resistivity of the porous material (Allard, 1993). A detailed discussion about the field quantities occurring in the Biot theory may be found in De Vries (1989) and Geerits (1996, 1997).

Using the solid and fluid displacement vectors $(\underline{u}, \underline{U})$ as primary variables and assuming harmonic oscillations $(e^{j\omega t})$, the weak integral form of the poroelasticity equations, Eq. (1), reads (Panneton and Atalla, 1996):

$$\begin{aligned} & \int_{\Omega_p} (\underline{\underline{\sigma}}^s(\underline{u}, \underline{U}) : \underline{\underline{\varepsilon}}^s(\delta \underline{u}) - \rho_{11} \omega^2 \underline{u} \cdot \delta \underline{u} - \rho_{12} \omega^2 \underline{U} \cdot \delta \underline{u} \\ & + j\omega \tilde{b}(\underline{u} - \underline{U}) \cdot \delta \underline{u}) d\Omega \\ & - \int_{\partial\Omega_p} \delta \underline{u} \cdot [\underline{\underline{\sigma}}^s(\underline{u}, \underline{U}) \cdot \underline{n}] dS = 0, \\ & \int_{\Omega_p} (\underline{\underline{\sigma}}^f(\underline{u}, \underline{U}) : \underline{\underline{\varepsilon}}^f(\delta \underline{U}) - \rho_{22} \omega^2 \underline{U} \cdot \delta \underline{U} - \rho_{12} \omega^2 \underline{u} \cdot \delta \underline{U} \\ & - j\omega \tilde{b}(\underline{u} - \underline{U}) \cdot \delta \underline{U}) d\Omega \\ & - \int_{\partial\Omega_p} \delta \underline{U} \cdot [\underline{\underline{\sigma}}^f(\underline{u}, \underline{U}) \cdot \underline{n}] dS = 0, \quad \forall (\delta \underline{u}, \delta \underline{U}), \end{aligned} \quad (3)$$

where $\delta \underline{u}$ and $\delta \underline{U}$ denote admissible variations of \underline{u} and \underline{U} ,

respectively, and where Ω_p and $\partial\Omega_p$ denote the medium domain and its boundary.

For the finite element implementation of Eq. (1), an analogy with the three-dimensional elastic solid elements is used; however, this time, six degrees of freedom per node are used. They account for three displacement components of the solid phase and three displacement components of the fluid phase. However, because of the viscous and thermal dissipation mechanisms, the system's matrices are frequency dependent. In consequence, for large 3D multilayer structures, this formulation has the disadvantage of requiring cumbersome calculations for large finite element models and spectral analyses. To alleviate these difficulties, a mixed displacement pressure formulation is presented.

B. The mixed displacement pressure formulation

In the following, an exact mixed formulation valid for harmonic motion is given. It derives directly from Biot poroelasticity equations, Eq. (1). The developments assume that the porous material properties are homogeneous. Note that a more general time domain formulation valid for anisotropic materials is given by Gorog *et al.* (1997).

For harmonic motion $(e^{j\omega t})$, introducing the effective densities

$$\begin{aligned} \tilde{\rho}_{11} &= \rho_{11} + \frac{\tilde{b}}{j\omega} \\ \tilde{\rho}_{22} &= \rho_{22} + \frac{\tilde{b}}{j\omega} \\ \tilde{\rho}_{12} &= \rho_{12} - \frac{\tilde{b}}{j\omega}, \end{aligned} \quad (4)$$

system (1) is rewritten:

$$\begin{aligned} \omega^2 \tilde{\rho}_{11} \underline{u} + \omega^2 \tilde{\rho}_{12} \underline{U} + \operatorname{div} \underline{\underline{\sigma}}^s &= 0 \\ \omega^2 \tilde{\rho}_{22} \underline{U} + \omega^2 \tilde{\rho}_{12} \underline{u} - h \nabla p &= 0. \end{aligned} \quad (5)$$

Using the second equation in (5), the displacement vector of the fluid phase \underline{U} is expressed in terms of the pressure p in the pores and in terms of the displacement vector of the solid phase particle \underline{u} :

$$\underline{U} = \frac{h}{\tilde{\rho}_{22} \omega^2} \nabla p - \frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} \underline{u}. \quad (6)$$

This last equation will be used to express system (5) in terms of the (\underline{u}, p) variables.

1. The solid phase equation in terms of (\underline{u}, p) variables

Using Eq. (6), the first equation in (5) transforms into:

$$\omega^2 \tilde{\rho}_{11} \underline{u} + h \frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} \nabla p + \operatorname{div} \underline{\underline{\sigma}}^s = 0, \quad (7)$$

where the following effective density has been introduced:

$$\tilde{\rho} = \tilde{\rho}_{11} - \frac{(\tilde{\rho}_{12})^2}{\tilde{\rho}_{22}}. \quad (8)$$

Equation (7) is still dependent on the fluid phase displacement vector \underline{U} because of the dependency $\underline{\sigma}^s = \underline{\sigma}^s(\underline{u}, \underline{U})$. Indeed, the stress-strain relations in the Biot theory are given by (Allard, 1993):

$$\begin{aligned}\underline{\sigma}^s(\underline{u}, \underline{U}) &= \tilde{A} \operatorname{div} \underline{u} \underline{1} + 2N \underline{\varepsilon}^s + \tilde{Q} \operatorname{div} \underline{U} \underline{1} \\ -hp \underline{1} &= \tilde{R} \operatorname{div} \underline{U} \underline{1} + \tilde{Q} \operatorname{div} \underline{u} \underline{1},\end{aligned}\quad (9)$$

in which $\underline{1}$ denotes the identity tensor, the elastic coefficients \tilde{A} and N correspond to the Lamé coefficients for elastic solids, \tilde{Q} is a coupling coefficient between the dilatation and stress of the two phases, and \tilde{R} may be interpreted as the bulk modulus of the air occupying a fraction h of a unit volume of aggregate. \tilde{R} is related to the bulk modulus \tilde{K}_f of the air in the pores by the scale factor h : $\tilde{R} = h\tilde{K}_f$. The elastic coefficients \tilde{A} , \tilde{Q} and \tilde{R} are related to K_b , the bulk modulus of the skeleton *in vacuo*, to K_s , the bulk modulus of the elastic solid from which the skeleton is made and to \tilde{K}_f , the bulk modulus of the air in the pores by (Allard, 1993):

$$\tilde{A} = \frac{(1-h)[1-h-K_b/K_s]K_s + h(K_s/\tilde{K}_f)K_b}{1-h-K_b/K_s+hK_s/\tilde{K}_f} - \frac{2}{3}N, \quad (10)$$

$$\tilde{Q} = \frac{[1-h-K_b/K_s]hK_s}{1-h-K_b/K_s+hK_s/\tilde{K}_f}, \quad (11)$$

$$\tilde{R} = \frac{h^2K_s}{1-h-K_b/K_s+hK_s/\tilde{K}_f}, \quad (12)$$

where N is the complex *in vacuo* shear modulus of the skeleton taking into account structural damping. Since \tilde{K}_f takes into account the thermal effects in the pore, it has a frequency dependent complex amplitude, and so do the elasticity tensors. In consequence, in the numerical implementation of the $(\underline{u}, \underline{U})$ formulation, the stiffness matrix associated to the frame is frequency dependent, which limits the computational efficiency of this formulation (the corresponding matrix should be reconstructed at each computational frequency).

To eliminate the dependency $\underline{\sigma}^s = \underline{\sigma}^s(\underline{u}, \underline{U})$, the first and second equation in Eq. (9) are combined to obtain:

$$\underline{\sigma}^s(\underline{u}, \underline{U}) = \left(\tilde{A} - \frac{\tilde{Q}^2}{\tilde{R}} \right) \operatorname{div} \underline{u} \underline{1} + 2N \underline{\varepsilon}^s - h \frac{\tilde{Q}}{\tilde{R}} p \underline{1}. \quad (13)$$

Next, introducing the tensor $\hat{\underline{\sigma}}^s$ defined by

$$\hat{\underline{\sigma}}^s(\underline{u}) = \left(\tilde{A} - \frac{\tilde{Q}^2}{\tilde{R}} \right) \operatorname{div} \underline{u} \underline{1} + 2N \underline{\varepsilon}^s, \quad (14)$$

the solid phase stress tensor writes:

$$\underline{\sigma}^s(\underline{u}, \underline{U}) = \hat{\underline{\sigma}}^s(\underline{u}) - h \frac{\tilde{Q}}{\tilde{R}} p \underline{1}. \quad (15)$$

Since tensor $\hat{\underline{\sigma}}^s$ depends only on the solid phase displacement vector, Eq. (15) may be used to eliminate the dependency $\underline{\sigma}^s = \underline{\sigma}^s(\underline{u}, \underline{U})$ in Eq. (7). Substituting (15) into (7), the

solid phase equation is obtained in terms of the (\underline{u}, p) variables:

$$\operatorname{div} \hat{\underline{\sigma}}^s(\underline{u}) + \tilde{\rho} \omega^2 \underline{u} + \tilde{\gamma} \nabla p = 0, \quad (16)$$

where the following notation is used:

$$\tilde{\gamma} = h \left(\frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} - \frac{\tilde{Q}}{\tilde{R}} \right). \quad (17)$$

To grasp the physical meaning of Eq. (16), note from Eqs. (10) to (12) that the expression (14) for $\hat{\underline{\sigma}}^s$ is equivalent to:

$$\hat{\underline{\sigma}}^s(\underline{u}) = (K_b - \frac{2}{3}N) \operatorname{div} \underline{u} \underline{1} + 2N \underline{\varepsilon}^s, \quad (18)$$

which is the classical stress-strain relation for an elastic solid. Moreover, since K_b and N denote the bulk modulus and the shear modulus of the skeleton *in vacuo*, respectively, tensor $\hat{\underline{\sigma}}^s$ is the stress tensor of the material *in vacuo*. In other words, $\hat{\underline{\sigma}}^s$ represents the stress tensor prevailing in the poroelastic material when the fluid is totally drained off. In consequence, the first two terms of Eq. (16) represent the elastodynamic equation of the material *in vacuo*. One valuable feature is that $\hat{\underline{\sigma}}^s$ is independent of the bulk modulus of the air and hence does not depend on frequency thus increasing the efficiency of the numerical implementation.

2. The fluid phase equation in terms of (\underline{u}, p) variables

Next, to derive the fluid phase equation in terms of (\underline{u}, p) variables, the divergence of Eq. (6) is taken and gives:

$$\operatorname{div} \underline{U} = \frac{h}{\omega^2 \tilde{\rho}_{22}} \Delta p - \frac{\tilde{\rho}_{12}}{\tilde{\rho}_{22}} \operatorname{div} \underline{u}. \quad (19)$$

Combining this equation with the second equation in (9), the fluid phase equation is obtained in terms of the (\underline{u}, p) variables:

$$\Delta p + \frac{\tilde{\rho}_{22}}{\tilde{R}} \omega^2 p + \frac{\tilde{\rho}_{22}}{h^2} \tilde{\gamma} \omega^2 \operatorname{div} \underline{u} = 0, \quad (20)$$

where $\tilde{\gamma}$ is defined in (17).

Note that Eq. (20) is exactly the classical equivalent fluid equation for absorbing media with a source term (Allard, 1993; Panneton *et al.*, 1995). The first two terms of this equation may be obtained directly from Biot's equations in the limit of a rigid skeleton. Those two terms represent the dynamic behavior of the material when its frame is supposed motionless. It is well known that Eq. (20) models correctly the behavior of rigid porous materials and/or situations in which the material is not mechanically excited or directly bonded to a vibrating structure and that for frequencies higher than the decoupling frequency (Panneton and Atalla, 1996).

3. Poroelasticity equations in terms of (\underline{u}, p) variables

Grouping Eqs. (16) and (20), the Biot poroelasticity equations in terms of (\underline{u}, p) variables are given by:

$$\operatorname{div} \hat{\underline{\underline{\sigma}}}^s(\underline{u}) + \omega^2 \tilde{\rho} \underline{u} + \tilde{\gamma} \nabla p = 0 \quad (21)$$

$$\Delta p + \omega^2 \frac{\tilde{\rho}_{22}}{\tilde{R}} p - \omega^2 \frac{\tilde{\rho}_{22}}{h^2} \tilde{\gamma} \operatorname{div} \underline{u} = 0.$$

This system exhibits the classical form of a fluid-structure coupled equation. However, the coupling is of a volume nature since the poroelastic material is a superposition in space and time of the elastic and fluid phases. The first two terms of the structure equation represent the dynamic behavior of the material *in vacuo* while the first two terms of the fluid equation represent the dynamic behavior of the fluid when the frame is supposed motionless. The third terms in both equations couple the dynamics of the two phases.

The next section discusses the implementation of this formulation in a finite element code. It will be shown that this formulation is symmetric. One obvious advantage of the presented formulation is that four degrees of freedom are needed rather than 6 for the classical $(\underline{u}, \underline{U})$ formulation.

C. The weak integral formulation

Let δu and δp be two admissible variations of the displacement field u and the pressure field p , respectively. Then, using Galerkin's procedure (Reddy, 1991), the following symmetric (\underline{u}, p) coupled weak integral formulation is obtained after few algebraic manipulations:

$$\begin{aligned} & \int_{\Omega_p} \hat{\underline{\underline{\sigma}}}^s(\underline{u}) : \underline{\underline{\varepsilon}}^s(\delta \underline{u}) d\Omega - \omega^2 \int_{\partial\Omega_p} \tilde{\rho} \underline{u} \cdot \delta \underline{u} d\Omega \\ & - \int_{\Omega_p} \tilde{\gamma} \nabla p \cdot \delta \underline{u} d\Omega - \int_{\partial\Omega_p} [\hat{\underline{\underline{\sigma}}}^s \cdot \underline{n}] \cdot \delta \underline{u} dS = 0, \\ & \int_{\Omega_p} \left[\frac{h^2}{\omega^2 \tilde{\rho}_{22}} \nabla p \cdot \nabla \delta p - \frac{h^2}{\tilde{R}} p \delta p \right] d\Omega - \int_{\Omega_p} \tilde{\gamma} \nabla \delta p \cdot \underline{u} d\Omega \\ & + \int_{\partial\Omega_p} \left[\tilde{\gamma} u_n - \frac{h^2}{\tilde{\rho}_{22} \omega^2} \frac{\partial p}{\partial n} \right] \delta p dS = 0, \quad \forall (\delta \underline{u}, \delta p), \end{aligned} \quad (22)$$

where Ω_p and $\partial\Omega_p$ denote the poroelastic domain and its boundary. Note from Eq. (22) the volume nature of the symmetric coupling between the two phases.

Equation (22) exhibits several advantages of the presented formulation over the $(\underline{u}, \underline{U})$ formulation. First, the stiffness matrix associated with the solid phase is frequency independent. Second, it accounts naturally for poroelastic-poroelastic interfaces; no essential boundary conditions are needed. Indeed, let indices 1 and 2 denote the two interfacing poroelastic media, respectively. The continuity equations are given by (Panneton and Atalla, 1996):

$$\begin{aligned} \underline{u}_1 &= \underline{u}_2 \\ h_1(u_{1n} - U_{1n}) &= h_2(u_{2n} - U_{2n}) \\ p_1 &= p_2 \\ \hat{\underline{\underline{\sigma}}}_1^s \cdot \underline{n} &= \hat{\underline{\underline{\sigma}}}_2^s \cdot \underline{n}, \end{aligned} \quad (23)$$

where the first condition ensures the continuity of the solid phase displacement vector. The second equation ensures the continuity of the relative mass flux across the boundary. The

two last equations ensure the continuity of the total normal stresses and the interstitial fluid pressure, respectively. Note that the total stress tensor $\underline{\underline{\sigma}}^t$ is the total stress tensor given by:

$$\underline{\underline{\sigma}}^t(\underline{u}, \underline{U}) = \underline{\underline{\sigma}}^s(\underline{u}, \underline{U}) + \underline{\underline{\sigma}}^f(\underline{u}, \underline{U}) = \underline{\underline{\sigma}}^s(\underline{u}, \underline{U}) - h p \underline{\underline{1}}, \quad (24)$$

which may be written in terms of the in-vacuo stress tensor $\hat{\underline{\underline{\sigma}}}^s$ using Eq. (15):

$$\underline{\underline{\sigma}}^t(\underline{u}, \underline{U}) = \hat{\underline{\underline{\sigma}}}^s(\underline{u}, \underline{U}) - h \left[1 + \frac{\tilde{Q}}{\tilde{R}} \right] p \underline{\underline{1}}. \quad (25)$$

Using the following two relations, derived easily from Eqs. (6), (16) and (17):

$$\begin{aligned} (1-h) \underline{u} \cdot \underline{n} + h \underline{U} \cdot \underline{n} &= \left[1 - h - h \frac{\tilde{Q}}{\tilde{R}} \right] \underline{u} \cdot \underline{n} \\ &\quad - \left[\tilde{\gamma} \underline{u} \cdot \underline{n} - \frac{h^2}{\tilde{\rho}_{22} \omega^2} \frac{\partial p}{\partial n} \right], \end{aligned} \quad (26)$$

$$h(\underline{u} - \underline{U}) \cdot \underline{n} = \left[\tilde{\gamma} \underline{u} \cdot \underline{n} - \frac{h^2}{\tilde{\rho}_{22} \omega^2} \frac{\partial p}{\partial n} \right] + h \left[1 + \frac{\tilde{Q}}{\tilde{R}} \right] \underline{u} \cdot \underline{n}, \quad (27)$$

the coupling conditions given by Eq. (23) may be written in terms of the (\underline{u}, p) variables as follows

$$\begin{aligned} \underline{u}_1 &= \underline{u}_2 \\ \left[\tilde{\gamma}_1 u_{1n} - \frac{h_1^2}{\tilde{\rho}_{22} \omega^2} \frac{\partial p_1}{\partial n} \right] + h \left[1 + \frac{\tilde{Q}_1}{\tilde{R}_1} \right] u_{1n} \\ &= \left[\tilde{\gamma}_2 u_{2n} - \frac{h_2^2}{\tilde{\rho}_{22} \omega^2} \frac{\partial p_2}{\partial n} \right] + h \left[1 + \frac{\tilde{Q}_2}{\tilde{R}_2} \right] u_{2n} \end{aligned} \quad (28)$$

$$p_1 = p_2$$

$$\hat{\underline{\underline{\sigma}}}_1^s \cdot \underline{n} - h \left[1 + \frac{\tilde{Q}_1}{\tilde{R}_1} \right] u_{1n} = \hat{\underline{\underline{\sigma}}}_2^s \cdot \underline{n} - h \left[1 + \frac{\tilde{Q}_2}{\tilde{R}_2} \right] u_{2n}.$$

Using the fourth relation, the first boundary integrals in Eq. (22) related to the interface $\partial\Omega_p$ between the two coupled domains reads:

$$\begin{aligned} I_1 &= - \int_{\partial\Omega_p} h_1 \left(1 + \frac{\tilde{Q}_1}{\tilde{R}_1} \right) p_1 \delta u_{1n} dS \\ &\quad + \int_{\partial\Omega_p} h_2 \left(1 + \frac{\tilde{Q}_2}{\tilde{R}_2} \right) p_2 \delta u_{2n} dS. \end{aligned} \quad (29)$$

Similarly, using the second and fourth relations of Eq. (28), the second boundary integral of Eq. (22) reduces to

$$\begin{aligned} I_2 &= - \int_{\partial\Omega_p} h_1 \left(1 + \frac{\tilde{Q}_1}{\tilde{R}_1} \right) u_{1n} \delta p_1 dS \\ &\quad + \int_{\partial\Omega_p} h_2 \left(1 + \frac{\tilde{Q}_2}{\tilde{R}_2} \right) u_{2n} \delta p_2 dS. \end{aligned} \quad (30)$$

The signs in both equations account for the direction of the normal vector which is assumed to be directed from medium

1 toward medium 2. Equations (29) and (30) show that the two poroelastic media will be coupled through symmetrical coupling terms. This coupling occurs naturally between the solid phase normal displacement of one medium and the interstitial fluid pressure of the other medium. In addition to these surface coupling terms, the kinematic relation $\underline{u}_1 = \underline{u}_2$ and $p_1 = p_2$ will have to be explicitly applied on $\partial\Omega_p$.

For the majority of poroelastic media used in acoustics $K_b/K_s \ll 1$ (Allard, 1993). Thus since [see Eqs. (11) and (12)]:

$$h \frac{\tilde{Q}}{\tilde{R}} = 1 - h - \frac{K_b}{K_s}, \quad (31)$$

we have:

$$h \left(1 + \frac{\tilde{Q}}{\tilde{R}} \right) \cong 1. \quad (32)$$

In this case, both integrals I_1 and I_2 fall to zero and only the kinematic relations have to be applied. These conditions are taken into account through assembling. This is in contrast with the (u, U) formulation which involves the imposition of the second boundary condition in (23) through kinematic relations.

Another advantage of the presented (u, p) formulation over the (u, U) formulation lies in its coupling with air media. Indeed, denoting by p^a the pressure in the air layer, the continuity equations are given by (Panneton and Atalla, 1996):

$$\begin{aligned} \frac{1}{\rho_0 \omega^2} \frac{\partial p^a}{\partial n} &= (1-h)u_n + hU_n \\ \underline{\underline{\sigma}}^t \cdot \underline{n} &= -p^a \underline{1}_n \\ \underline{\underline{\sigma}}^s \cdot \underline{n} &= -(1-h)p^a \underline{1}_n; \quad \underline{\underline{\sigma}}^f \cdot \underline{n} = -hp^a \underline{1}_n. \end{aligned} \quad (33)$$

The first equation ensures the continuity of the normal volume velocity (modified Euler's equation due to the porosity), while the second and third equations ensure the continuity of the normal stresses. Using Eqs. (25) and (26), the coupling conditions in Eq. (33) are proved to be equivalent to:

$$\begin{aligned} \left[1 - h - h \frac{\tilde{Q}}{\tilde{R}} \right] u_n - \left[\tilde{\gamma} u_n - \frac{h^2}{\tilde{\rho}_{22} \omega^2} \frac{\partial p}{\partial n} \right] &= \frac{1}{\rho_0 \omega^2} \frac{\partial p^a}{\partial n} \\ p &= p^a \end{aligned} \quad (34)$$

$$\underline{\underline{\hat{\sigma}}}^s \cdot \underline{n} = \left[h \frac{\tilde{Q}}{\tilde{R}} - (1-h) \right] p \underline{n}.$$

Note that the first and third conditions in (34) lead to symmetric coupling terms of the form [see Eq. (22)]:

$$I_1 = \int_{\partial\Omega_p} \left(1 - h - \frac{\tilde{Q}}{\tilde{R}} \right) p \delta u_n dS \quad (35)$$

and

$$I_2 = \int_{\partial\Omega_p} \left(1 - h - \frac{\tilde{Q}}{\tilde{R}} \right) u_n \delta p dS, \quad (36)$$

where $\partial\Omega_p$ denotes the poroelastic-air interface. Once again for the majority of porous materials used in acoustics, the approximation given by Eq. (32) applies and these coupling terms vanish. Thus, only the continuity of the pressure at the poroelastic-air interface should be accounted for. In this case, contrary to the (\underline{u}, U) formulation, no coupling matrix is needed. This represents another advantage and time saving feature of the presented formulation.

Finally, it is worth mentioning that a detailed description of the boundary, support and loading conditions of the presented (\underline{u}, p) formulation together with corresponding validation examples will be presented in a sequel paper (Debergue *et al.*, 1998).

II. NUMERICAL IMPLEMENTATION

In the presented work, the weak formulation [Eq. (22)] has been discretized using linear volume elements with four degrees of freedom per node: the three solid macroscopic displacements \underline{u} and the pressure p in the interstitial fluid. Two finite volume elements have been implemented: a six node wedge element and an eight node brick element. Accordingly, within a finite element, it is assumed that the solid phase displacement vector and the pressure can be represented in matrix form following

$$\underline{u}^e = [N_s] \{u_n\}^e \quad \text{and} \quad p^e = [N_f] \{p_n\}^e, \quad (37)$$

where $[N_s]$ and $[N_f]$ are the element's shape functions used to approximate the solid phase displacement vector and the interstitial pressure within element "e." $\{u_n\}^e$ and $\{p_n\}^e$ are respectively the nodal displacement and the nodal pressure variables.

Since the solid phase governing equation is similar to a 3D elastodynamic equation, an analogy with solid elements is used. Similarly, since the fluid phase equation is similar to the Helmholtz's equation, an analogy with acoustic finite elements is used.

Substituting Eq. (37) into Eq. (22), the following matrices are obtained:

$$\int_{\Omega_p} \underline{\underline{\hat{\sigma}}}^s(\underline{u}) : \underline{\underline{\epsilon}}^s(\delta \underline{u}) d\Omega \Rightarrow \langle \delta u_n \rangle [K] \{u_n\}, \quad (38)$$

$$\int_{\Omega_p} \tilde{\rho} \underline{u} \cdot \delta \underline{u} d\Omega \Rightarrow \langle \delta u_n \rangle [\tilde{M}] \{u_n\}, \quad (39)$$

$$\int_{\Omega_p} \tilde{\gamma} \underline{\nabla} p \cdot \delta \underline{u} dS \Rightarrow \langle \delta u_n \rangle [\tilde{C}] \{p_n\}, \quad (40)$$

$$\int_{\Omega_p} \frac{h^2}{\tilde{\rho}_{22}} (\underline{\nabla} p \cdot \underline{\nabla} \delta p) d\Omega \Rightarrow \langle \delta p_n \rangle [\tilde{H}] \{p_n\}, \quad (41)$$

$$\int_{\Omega_p} \frac{h^2}{\tilde{R}} p \cdot \delta p d\Omega \Rightarrow \langle \delta p_n \rangle [\tilde{Q}] \{p_n\}, \quad (42)$$

$$\int_{\Omega_p} \tilde{\gamma} \underline{u} \cdot \underline{\nabla} \delta p dS \Rightarrow \langle \delta p_n \rangle [\tilde{C}]^T \{u_n\}, \quad (43)$$

where $\{u_n\}$ and $\{p_n\}$ represent the solid phase and the fluid phase global nodal variables, respectively. $[\tilde{M}]$ and $[K]$ represent equivalent mass and stiffness matrices for the solid

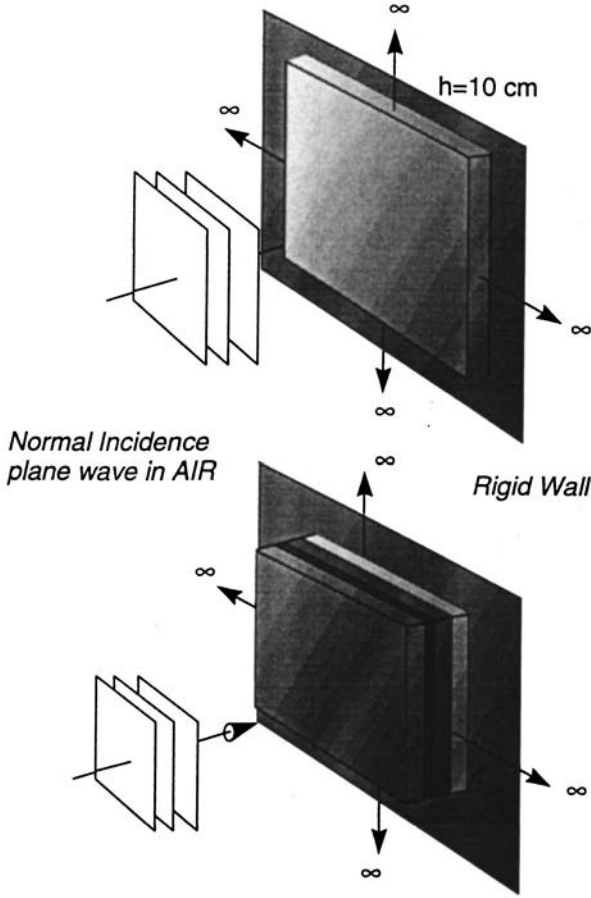


FIG. 1. Geometry of the surface impedance problem. (a) Single poroelastic layer; and (b) multilayered poroelastic material.

phase, respectively. $[\tilde{H}]$ and $[\tilde{Q}]$ represent equivalent kinetic and compression energy matrices for the fluid phase, respectively. Finally, $[\tilde{C}]$ is a volume coupling matrix between the solid phase displacement variables and the fluid phase pressure variable. Note that contrary to a classical fluid-structure coupling problem, the coupling within a poroelastic system is of a volume nature. The numerical evaluation of the different matrices is classical and will not be detailed in this paper. Note finally that the discretization of the interface integrals depends on the boundary conditions of the system (= loading conditions).

Substituting Eqs. (38) to (43) into Eq. (22), the following coupled system is formed:

$$\begin{pmatrix} [K] - \omega^2[\tilde{M}] & -[\tilde{C}] \\ -\omega^2[\tilde{C}]^T & [\tilde{H}] - \omega^2[\tilde{Q}] \end{pmatrix} \begin{Bmatrix} u_n \\ p_n \end{Bmatrix} = \begin{Bmatrix} F_s \\ F_p \end{Bmatrix}, \quad (44)$$

where $\{F_p\}$ denotes the loading vector for the poroelastic medium (these vectors depends on the nature of the excitation; they are not detailed here). Solution of system (44) leads simultaneously to the nodal displacements of the solid phase and to the nodal pressures of the fluid phase. Once more, note that system (44) exhibits the classical form of a fluid structure coupled system. This system is smaller compared to the corresponding system in the classical (u, U) formulation since four degrees of freedom are used rather

than six. Moreover, for a homogeneous poroelastic medium, all the frequency dependency in matrices (38) to (43) show up as multiplication coefficients that may be taken out of the integrals. Thus, the matrices may be calculated and assembled, once for all, for a given mesh. The spectra of the frequency dependent coefficient are accounted for during the formation of the global system (44). This is in contrast with the (u, U) formulation where the frequency dependence of the stiffness matrix is implicit in the stress-strain relations, thus necessitating recalculation of the matrix at each frequency.

Moreover, it is worth mentioning that the presented formulation while improving significantly the performances of the numerical solutions of the Biot's poroelasticity equations still suffers from the necessity of using a spectral approach necessitated by the nonlinearity of the frequency dependence of the final matrix system. At this stage of the research, two potential solutions may be mentioned. The first is the use of a set of acceptable approximations that lead to a linearisation of the matrix system (see Panneton and Atalla, 1997b). The second is to use the selective modal reduction technique developed by Sgard *et al.* (1997).

Finally, recall from the previous section that coupling conditions are accounted for naturally in the presented formulation compared to the (u, U) formulation. The above mentioned features of the presented approach lead to important savings in setup and solution time as it will be demonstrated in the following numerical examples.

III. NUMERICAL EXAMPLES

In the following, the presented (u, p) formulation is validated by comparisons with the (u, U) formulation for both two-dimensional and three-dimensional problems. Note that detailed validation examples for the (u, U) formulation and its interface with elastic and poroelastic media have been presented elsewhere (Panneton and Atalla, 1996, 1997a).

A. Surface impedance for a single poroelastic layer

As a first example, the surface impedance of laterally infinite poroelastic materials predicted by the presented (u, p) finite element Biot model of Eq. (21) is compared with the prediction of the (u, U) formulation. The configuration under study is depicted in Fig. 1(a). A laterally infinite glass wool layer, described in Table I, is bonded onto a rigid impervious wall. A normal incidence plane wave of unit amplitude excites the absorbing material. To simulate the laterally infinite extent with the finite element model, only the axial macroscopic displacements are considered, i.e., the lateral displacements are set to zero. For both formulations, the poroelastic domain is meshed using an eight node linear brick element. A mesh of $7 \times 7 \times 5$ elements was used. The normal incidence surface impedance is calculated using the axial nodal solid and fluid components, u_n and U_n respectively, at the input surface for the unit acoustic pressure excitation by the following equation:

$$Z_n = \frac{1}{j\omega(hU_n + (1-h)u_n)}. \quad (45)$$

TABLE I. Physical properties and dimensions of the poroelastic materials.

	k_s	ρ_l (kg/m ³)	σ (Ns/m ⁴)	h	N (kPa)	ν	Λ (m)	Λ' (m)	Thickness (cm)
Glass wool	1.06	130	40 000	0.94	2200 (1+j0.1)	0	0.56×10^{-4}	1.10×10^{-4}	10
Blanket	1.18	41	34 000	0.98	110(1+j0.015)	0.3	0.60×10^{-4}	0.87×10^{-4}	0.4
Screen	2.56	125	320×10^4	0.80	1000 (1+j0.1)	0.3	0.06×10^{-4}	0.24×10^{-4}	0.08
Foam A	2.52	31	87 000	0.97	55 (1+j0.055)	0.3	0.37×10^{-4}	1.19×10^{-4}	0.5
Foam B	1.98	16	65 000	0.99	18 (1+j0.1)	0.3	0.37×10^{-4}	1.21×10^{-4}	1.6
Fiberglass	1.4	30	25 000	0.95	21 (1+j0.05)	0.0	0.93×10^{-4}	0.93×10^{-4}	7.62

For the formulation, the normal displacement of the fluid phase U_n is calculated using Eq. (6). This equation necessitates the evaluation of the nodal values of the pressure gradient. The classical procedure relying on evaluation of the pressure gradients at Gauss integration points followed by a least-square procedure in order to get nodal values is used. Note that the results of the $(\underline{u}, \underline{U})$ finite element Biot model have been validated by comparison with an exact analytical calculation (Allard, 1993). Figure 2 presents the real and imaginary parts of the surface impedance predicted by both models. An excellent agreement is observed. For this simple problem, the (\underline{u}, p) formulation took approximately one-fifth of the time needed for the $(\underline{u}, \underline{U})$ formulation.

B. Surface impedance for a multilayer poroelastic material

To show the efficiency of the approach, the surface impedance of a laterally infinite multilayer poroelastic material is calculated. The configuration under study is depicted in Fig. 1(b). A laterally infinite multilayer poroelastic material is bonded onto a rigid impervious wall. From the front face to the rear face, the multilayer consists of a blanket, a screen, foam A, and foam B. The material properties are listed in Table I. A normal incidence plane wave of unit amplitude excites the absorbing material. To simulate the laterally infi-

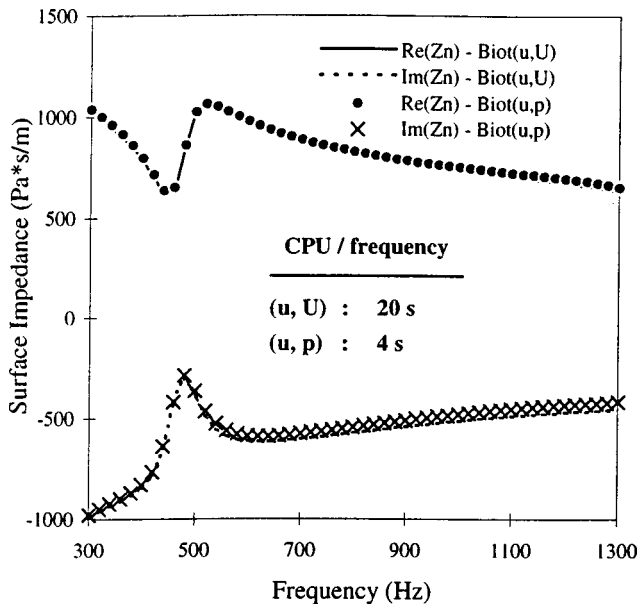


FIG. 2. Comparison of the surface impedance calculated with the $(\underline{u}, \underline{U})$ and the (\underline{u}, p) Biot finite element models for the laterally infinite glass wool layer of Table I.

nite extent with the finite element model, only the axial macroscopic displacements are considered, i.e., the lateral displacements are set to zero. Identical meshes using eight node linear brick elements are used for both formulations. The cross section of the material was meshed using 7×7 elements. The blanket, the screen, foam A and foam B were meshed using five elements. Note that this problem has been validated elsewhere by comparison with an analytical calculation (Allard, 1993). Figure 3 compares the real and imaginary parts of the surface impedance predicted by both models. A perfect agreement is observed. For this problem, the (\underline{u}, p) formulation took approximately one-fourth the time needed for the $(\underline{u}, \underline{U})$ formulation.

C. A rigid cavity with surface absorption

Next, a coupled acoustic-poroelastic problem is considered. The geometry of the problem is depicted in Fig. 4. A wall of a rigid cavity (dimensions: $0.35 \text{ m} \times 0.22 \text{ m} \times 0.01 \text{ m}$) is treated with 10 cm of the fiberglass of Table I. The fiberglass is totally bonded to the rigid wall. The system is excited with a point source positioned at a corner of the rigid cavity. The quadratic pressure in the cavity is calculated using the $(\underline{u}, \underline{U})$ and the (\underline{u}, p) formulations. Identical meshes are used for both formulations. The cavity is discretized using six-node linear pentahedron acoustic elements with one

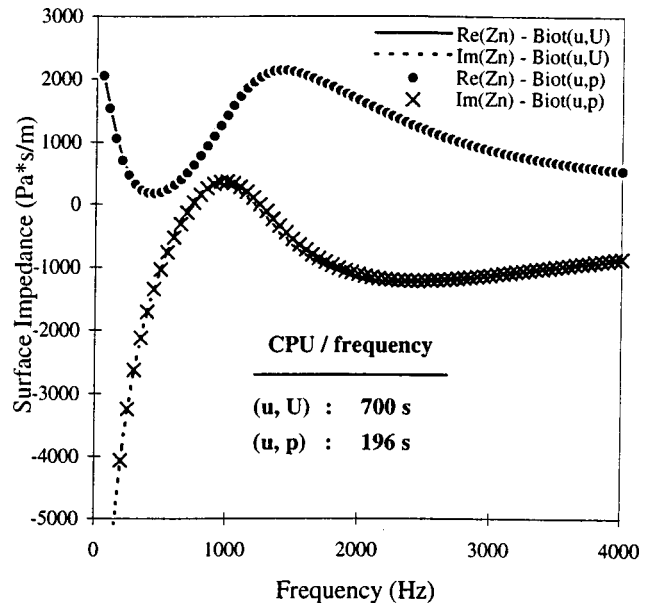


FIG. 3. Comparison of the surface impedance between the $(\underline{u}, \underline{U})$ and the (\underline{u}, p) Biot finite element models for the laterally infinite multilayer.

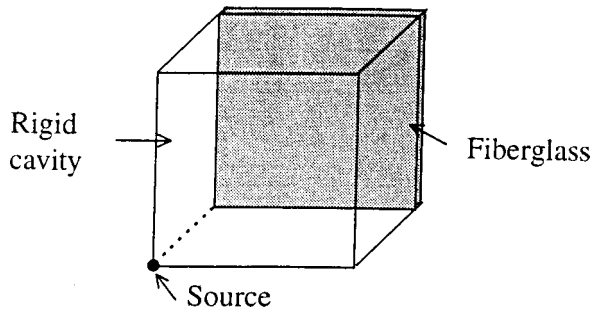


FIG. 4. Geometry of the acoustic-poroelastic problem.

degree-of-freedom per node: the acoustic pressure. A $7 \times 7 \times 7$ mesh is used for the cavity. For the poroelastic domain, a $7 \times 7 \times 3$ mesh is used. Note that this problem has been validated elsewhere by comparison with an impedance model for the poroelastic material (Atalla and Panneton, 1996). The results of the calculation are given in Fig. 5. Once again, excellent agreement is found between the two approaches. The (u, p) formulation took approximately one-twelfth of the time needed with the (u, U) formulation. This is mostly due to the reduction of the number of degrees of freedom for the problem and the more natural way of accounting for the coupling between the poroelastic and the acoustic domain.

IV. CONCLUSION

The presented mixed (u, p) formulation for poroelastic materials has several features:

- (1) It is exact, in the sense that no new assumptions are introduced beyond those governing Biot's poroelastic equations.
- (2) It leads to a classical coupled fluid-structure problem involving the dynamic equations of the skeleton *in vacuo* and the equivalent fluid in the rigid skeleton limit.

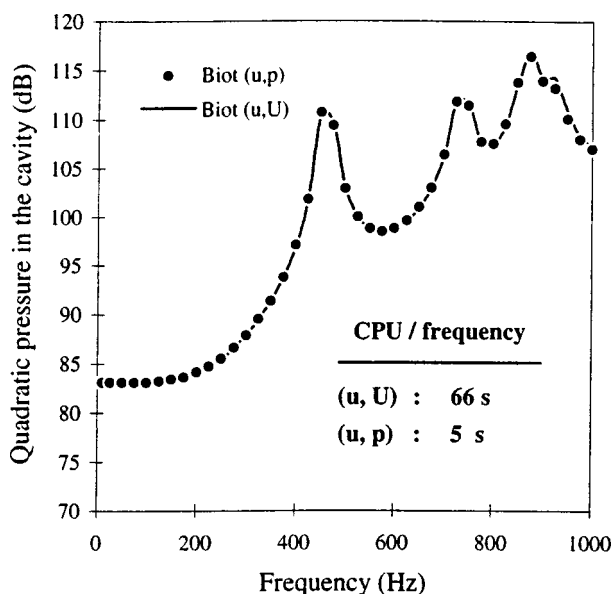


FIG. 5. Comparison of the quadratic velocity at the poroelastic surface calculated with the (u, U) and the (u, p) Biot finite element models for the cavity problem of Fig. 4.

- (3) It handles naturally coupling conditions with acoustic and other pyroelastic media.
- (4) It involves four degrees of freedom per node element compared to six for the classical (u, U) formulation; important savings in computer storage and solution time are achieved.
- (5) The physical nature of its coupled equations and its variables allows for efficient numerical implementation; important savings in setup and solution time are achieved.
- (6) For the presented numerical simulations, the (u, p) was shown to be 5 to 12 times faster than the (u, U) formulation with identical accuracy.

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