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Strategy in Geoguessr:

Games of competitive Euclidean geometry

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Abstract

We model a problem featured in the multiplayer version of the online browser game Geoguessr (<http://www.geoguessr.com>). Multiple players compete to find a point in a map, knowing only how much closer or further the opponent's best guess is to the target. We develop strategies and compare them using computer power. We also explore several variations (turn-based versus one-round, 1 dimension, multiple dimensions...).

Keywords: Euclidean geometry, Voronoi game, computer science, game theory, recreational

1 Introduction

In the browser game Geoguessr [16] you find yourself lost in a city somewhere around the globe (using Google Street View [7]). Your goal is to find your location in a map as accurately as possible. Recently, a multiplayer mode was introduced, in which you race against other players to make the closest guess in a limited time and with only a few tries. You only know limited information about your opponents' guesses. It is often the case that one can quickly identify the country the images are from (maybe a flag or a sign gave it up), but guessing closer than the other players is a difficult task.

We model and study three simplified versions of the game. These are instances of the Voronoi game, a facility locating problem introduced in [1]. In the Voronoi game, players compete by placing facilities to serve a distribution of users. Their goal is to maximize the number of users served by their facilities, where a user is served by the closest facility available.

We will assume that the player has correctly identified the country \mathcal{C} , which we will see as a region of \mathbb{R}^2 in our first model¹. Hence the *target* (the user, in the Voronoi

game) is a point x in \mathcal{C} . After any *guess* (facility placement, in the Voronoi game) is made by any player, the game Geoguessr ranks every player by how close their *best* guess is, and displays said ranking indicating how much further is your *best* guess compared to the *best* guess of the player immediately above you in the ranking, as well as how much closer your best guess is compared to the best guess of the player immediately below. As an example, the available information to player A after three guesses are made is displayed in Figure 1. Note that you do not know any additional information — the other players' guesses, which is your best guess so far, the distance from your guesses to the target, etc.

A similar instance of the Voronoi game was studied in [6]. They introduce the so called blind Voronoi game, in which the location of the users is not known. The main difference with our version is the available information after the placement of a facility: every player in the blind Voronoi game knows the location of all facilities placed by any player, the facilities that currently serve each user, and the distance from said facilities to said users.

We will restrict ourselves to games with two players A and B . First, in section 2, we explore a one round version of the game, in which player B makes all their guesses before A starts guessing. We construct a winning strategy for A in at most 5 guesses. In section 2.1, we compare this strategy with mixed strategies that involve randomization, and find that there are strategies that win quicker, in average.

We then consider a turn-based version of the game in section 3, in which two players take turns making one guess per turn. No pure strategy was found, and hence mixed strategies were compared.

In section 4 we explore both versions of this game on a line. The one round version turns out to be trivial (player A can win in 2 guesses). A modification of the turn-based game is more interesting. It can be identified as the game “higher or lower” played in one of the several rounds of the show *The Price is Right* [15]. There is a strategy for both players in which the probability of

¹ This is a simplification, and one would have to take the curvature of the Earth into consideration in the actual game, specially in bigger countries such as Russia. A brief note about this will be said in the end.

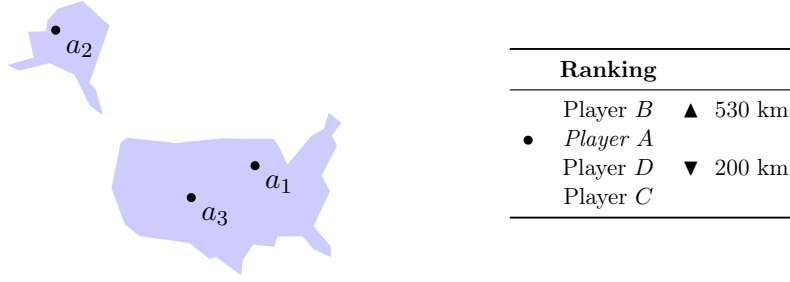


Fig. 1: Pictured, all the available information the player A disposes of after having made three guesses a_1 , a_2 and a_3 . The best guess made by player B is 530 km closer to the target x than the best guess made by A (which could be a_1 , a_2 or a_3). Similarly, the best guess made by player A is 200 km closer to x than the best guess made by player D . Player C 's best guess is worst than player D 's.

winning is close to $\frac{1}{2}$. Finally, we provide some comments about higher dimensional analogues in section 5.

NOTATION. Let $\mathbb{D}(a, r)$ be the disc of center a and radius r , let $\partial\mathbb{D}(a, r) = C(a, r)$ be its circumference and let $\overset{\circ}{\mathbb{D}}(a, r) = B(a, r)$ be its interior (a ball). The metric on \mathcal{C} is the usual Euclidean metric inherited from its ambient space.

2 One round version of the game

Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a compact subspace (the country), and let $x \in \mathcal{C}$ be the target. In this version of the game, we will be playing as player A and competing against another player B who has already made all their guesses. Let $b \in \mathcal{C}$ be their best guess. Our goal is to find a point of \mathcal{C} which is closer to x than b in the least amount of guesses possible. The space of legal guesses will be $\mathcal{S} = \mathbb{R}^2$. At any given moment, we know two things:

- (i) if any of our guesses a_1, \dots, a_n is closer to x than b , and
- (ii) how much closer to x (in distance units) is our best guess compared to b , or how much closer is b compared to our best guess.

Equivalently, let $S = \{a_1, \dots, a_k\}$ be a set of guesses, and let

$$f(S) = \min_i \{d(a_i, x)\} - d(b, x). \quad (1)$$

Then the available information at any given moment is $f(S)$. By convention, let $f(\{\}) = \infty$.

We will say that the game is in a winning state for A if $f(S) \leq 0$. Our goal is to design a strategy to find a set S such that $f(S) \leq 0$, and such that $\#S$ is as small as possible.

Problem 1. Let \mathcal{C} , \mathcal{S} and f be given. A target x was selected from \mathcal{C} , and a guess $b \in \mathcal{C}$ was made by an adversary. After each of our guesses a_k , we can retrieve

the value of $f(\{a_1, \dots, a_k\})$. The goal is to make a guess a_n such that $f(\{a_1, \dots, a_n\}) \leq 0$. Design an algorithm that returns a set S of guesses $\{a_1, \dots, a_n\} \subseteq \mathcal{S}$ such that $f(S) \leq 0$ and minimizes n .

We will now provide a constructive proof of the following theorem.

Theorem 2.1. Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a compact set, let $\mathcal{S} = \mathbb{R}^2$ and let f be defined as in (1). There is a solution to Problem 1 with $n \leq 5$.

Proof. The key fact is that, if a_i is your best guess so far, then $C(a_i, f(\{a_1, \dots, a_i\}))$ is tangent to $C(x, d(b, x))$, by definition of f . We will exploit this to develop our strategy. Also, note that after k guesses, our best guess is indexed by the smallest i such that $f(\{a_1, \dots, a_i\}) = f(\{a_1, \dots, a_k\})$.

Let a_1 be given. We will later discuss the optimal placement of a_1 . If $f(\{a_1\}) \leq 0$, we are finished. If not, then choose a_2, a_3 and a_4 equally distributed in $C(a_1, f(\{a_1\}))$. Similarly, if $f(\{a_1, \dots, a_i\}) \leq 0$ for any $i = 2, 3, 4$, we are finished. We will hereafter always suppose this is never the case.

CLAIM 1. At least one of a_2, a_3, a_4 is closer to x than a_1 .

Let $V_i(\mathbb{R}^2, \{a_1, \dots, a_k\})$ be the Voronoi cell of \mathbb{R}^2 with respect to $\{a_1, \dots, a_k\}$ that contains a_i . In other words, let $V_i(\mathbb{R}^2, \{a_1, \dots, a_k\}) = \{p \in \mathbb{R}^2 : d(a_i, p) \leq d(a_j, p) \forall j \neq i\}$.

Proof of claim. The Voronoi cell $V_1(\mathbb{R}^2, \{a_1, a_2, a_3, a_4\})$ is contained in $\mathbb{D}(a_1, f(\{a_1\})) \not\ni x$. Since $\bigcup_{i=1}^4 V_i(\mathbb{R}^2, \{a_1, a_2, a_3, a_4\}) = \mathbb{R}^2 \ni x$, the claim falls. \square

Let a_{i_0} be the best guess so far; $i_0 \in \{2, 3, 4\}$. At this point, a possible next step would be to choose a_5, a_6 and a_7 evenly distributed in $C(a_{i_0}, f(\{a_1, \dots, a_{i_0}\}))$. Then, by Claim 1, one of these would be closer to x than both a_1 and a_{i_0} .

Instead, we skip one of these guesses by letting $\{a_5, a_6\} = C(a_1, f(\{a_1\})) \cap C(a_{i_0}, f(\{a_1, \dots, a_{i_0}\}))$. The conclusion still holds:

CLAIM 2. At least one of a_5, a_6 is closer to x than a_{i_0} .

Proof of claim. We know $0 \leq f(\{a_1, \dots, a_{i_0}\}) \leq f(\{a_1\})$. If any of the extremes is attained, the lemma holds (see Figure 2a and 2c). A continuity argument proves the claim. \square

Let a_{i_1} be the best guess so far, $i_1 \in \{5, 6\}$. We have successfully found three balls that are tangent to $B(x, d(b, x))$ and that do not intersect it, namely, $B(a_1, f(\{a_1\}))$, $B(a_{i_0}, f(\{a_1, \dots, a_{i_0}\}))$ and $B(a_{i_1}, f(\{a_1, \dots, a_{i_1}\}))$.

CLAIM 3 (Problem of Apollonius). There is a unique ball $B(x, r)$ which is tangent to three given balls $B(c, R)$, $B(c', R')$, $B(c'', R'')$ and that do not intersect them. One can furthermore construct it (in particular finding x).

Proof of claim. The Problem of Apollonius has been widely studied and many proofs have been known for centuries. See [3, 9] for short historical surveys. We will consider it proved. With an algebraic approach, x and r are the solution to the following system.

$$\begin{cases} d(x, c) = R + r \\ d(x, c') = R' + r \\ d(x, c'') = R'' + r \end{cases} ; \quad r \geq 0. \quad (2)$$

The readers wanting to implement the algorithm hereby presented may choose their preferred methods for solving systems of nonlinear equations. \square

We can thus choose $a_7 = x$ as our final guess — $f(\{a_1, \dots, a_7\}) \leq 0$ gives a winning state for A .

So far, we have constructed a solution to Problem 1 with $n \leq 7$, for any starting guess a_1 . We will now discuss the placement of a_1 .

Since \mathcal{C} is a compact subspace of \mathbb{R}^2 , we can bound it by a large enough regular hexagon. Let a_1 be one of its vertices. When choosing a_2, a_3 and a_4 , let a_2 fall in the radius of the hexagon that goes through a_1 . Then, the Voronoi cell $V_2(\mathbb{R}^2, \{a_1, \dots, a_4\})$ will contain the target x , since $V_1(\mathbb{R}^2, \{a_1, \dots, a_4\}) \cup V_2(\mathbb{R}^2, \{a_1, \dots, a_4\})$ will contain the whole hexagon. See Figure 3.

Using this method we can therefore skip guesses a_3 and a_4 , bringing the maximum amount of guesses necessary down to 5, as desired. \blacksquare

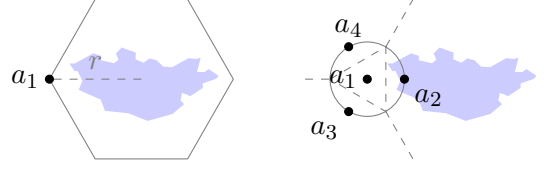


Fig. 3: On the left, a hexagon bounding the country \mathcal{C} . The first guess a_1 is a vertex of the hexagon. Dashed, the radius r of the hexagon through a_1 . On the right, the first four guesses have been made, with a_2 in $C(a_1, f(\{a_1\})) \cap r$. Dashed, the boundaries of the Voronoi cells $V_i(\mathbb{R}^2, \{a_1, \dots, a_4\})$.

See Figure 4 for an illustration of the strategy, which will be hereafter referred to as the **Apollonius** strategy.

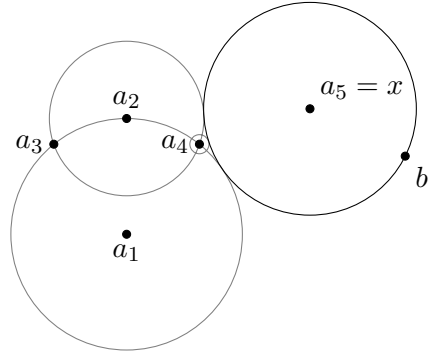


Fig. 4: The **Apollonius** strategy.

Note that there are several choices to be made in an implementation of this strategy. Most notably, if guess a_3 is already better than a_1 and a_2 , one can skip guess a_4 . Therefore, if this choice is made at random, the strategy will end in $n = 4$ steps half the times. We say that the expected performance of this strategy is ≤ 4.5 .

We now turn to the question of finding a strategy with a lower expected performance. For this matter, we will compare several strategies with the help of a computer.

A first naive strategy is to select a point $a_1 \in \mathcal{C}$ at random. If $f(\{a_1\}) \leq 0$ we have finished. If not, choose $a_2 \in \mathcal{C}$ at random and repeat. We will call this the **Random** strategy. It will serve as the base strategy to which compare the rest. Note that this is not the worst strategy, as one can purposely craft worse strategies (e.g., guessing the same point in each turn, or guessing outside \mathcal{C}).

A second strategy will hinge on a fact discussed earlier; x does not belong to $B(a_k, f(\{a_1, \dots, a_k\}))$ for any k . Our strategy will thus draw a_{n+1} at random from $\mathcal{C} - \bigcup_{k=1}^n B(a_k, f(\{a_1, \dots, a_k\}))$. We will call this strategy **RandCircles**.

A second obvious optimization is to keep track of the best guess thus far (just as we did in the proof of Theorem 2.1) and guessing at random from the Voronoi cell

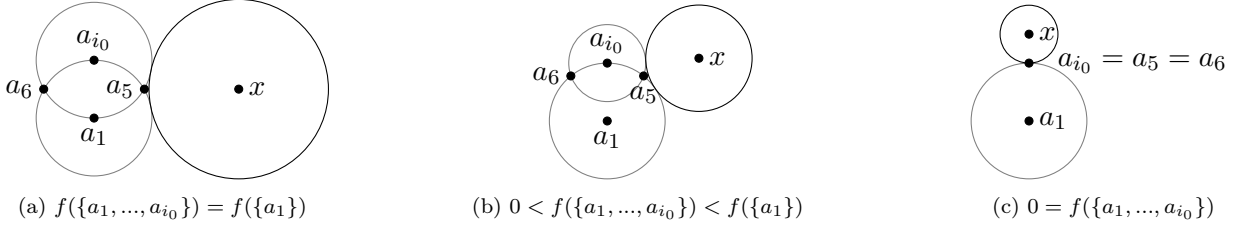


Fig. 2: Geometric configuration of Claim 2 for different values of $f(\{a_1, \dots, a_{i_0}\})$. In each subfigure, three circles are drawn: $C(a_1, f(\{a_1\}))$, $C(a_{i_0}, f(\{a_1, \dots, a_{i_0}\}))$ and $C(x, d(b, x))$.

of the best guess, $V_{i_0}(\mathcal{C}, S)$. We call this strategy **RandVoronoi**.

Simply merging the previous two strategies, we get **RandVorCir**. Now the guesses are chosen at random from $V_{i_0}(\mathcal{C}, S) - \cup_{a_i \in S} B(a_i, f(\{a_1, \dots, a_i\}))$.

Up to this point, every strategy (apart from the **Apollonius** strategy) involves choosing points at random from some region \mathcal{R} in which we have determined that x lies. The next one will be an attempt at avoiding this. In this new strategy, we will guess the point a_i which is, in average, the closest to any given point of \mathcal{R} . That is,

$$a_i = \arg \min \iint_{\mathcal{R}} d(a_i, z) dz. \quad (3)$$

If we take the minimum over the points in \mathbb{R}^2 , the resulting point is the geometric median of \mathcal{R} . If we take the minimum over \mathcal{R} itself, it is the medoid² of \mathcal{R} . They coincide if \mathcal{R} is convex, but we can't make this assumption in general.

These two points need not be uniquely determined by the previous formula. For instance, suppose that our country is a perfect annulus. Then there is an infinite number of medoids. To solve this, choose one of those medoids at random. Another problem is that, in general, the median does not need to be inside \mathcal{C} . This causes serious problems, and examples of this do not need to be bizarre at all. For instance, let \mathcal{C} be any compact country. Suppose that, after choosing our first point a_1 , which will be the median of \mathcal{C} , the value of $f(\{a_1\})$ is positive and the disc $\mathbb{D}(a_1, f(\{a_1\}))$ is completely contained in \mathcal{C} . The median of the new region $\mathcal{R} = \mathcal{C} - B(a_1, f(\{a_1\}))$ will still be a_1 , hence the algorithm will stabilize in a point which is not a valid solution.

Consequently, we discard the median as a valid strategy. By changing the way we pick a_i in the

RandVorCir from “random” to “medoid” we define the **MedoidVorCir** strategy.

A final strategy we will discuss is **MedVorCirApoll**: start with the **MedoidVorCir** strategy until the best guess is updated three times. Then, the Problem of Apollonius can be exploited just as in the **Apollonius** strategy to find the target in the next guess.

2.1 Comparison of strategies

In the previous section, we discussed seven different strategies to solve Problem 1. We will now compute $N = 1000$ iterations of each strategy and compare the average value of $\#S = n$.

In our implementation, \mathcal{C} will be a square. Its dimensions should not be relevant, as the problem is the same in any scale. We implemented two versions of the **Apollonius** strategy: a first naive version will choose a_1 at random and is guaranteed to end in 7 steps, whilst the second one will choose a_1 as discussed in the end of the proof of Theorem 2.1. We have recorded the results from our comparison in Table 1.

Strategy	Average n	Largest n
Random	7.58	3526
RandCircles	3.02	336
RandVoronoi	2.69	23
RandVorCir	2.26	13
MedVorCir	1.87	10
MedVorCirApoll	1.80	8
Apollonius7	3.81	7
Apollonius5	3.63	5

Tab. 1: Comparison of strategies. Iterations: $N = 1000$. Country shape: square.

The strategies which involve randomness do not perform well when $d(x, b)$ is really small. Most notably, there is one outlier in the data of **Random**: one instance of the algorithm returned $n = 3526$. Without this outlier, the average drops down to 4.00 — which is still the worst performing strategy.

² There are multiple algorithms to approximate the median and medoid of a discrete cluster of points [11, 12]. The problem of finding these points is often called the Fermat-Weber problem [4]. We are interested on a continuous version of this problem, but a discrete implementation over a random sample of our region will suffice.

Another key point is that many of the iterations returned $n = 1$. This is due to choosing x and b at random — if also chosen at random, a_1 is expected to be a winning guess half of the times. This explains why the **Apollo-nius** strategies have such high averages; they are superior to **RandCircles** when $d(x, b)$ is small, but the rigidity of their definition plays against them when $d(x, b)$ is big.

The strategies **Apollonius5** and **Apollonius7** are guaranteed to end very quickly, and we are guaranteed to find x to any desired accuracy. However, they do not perform well in comparison to **MedVorCir**, as this last strategy uses all the information available in each turn to estimate x . The strategy resulting from merging these two, **MedVorCirApoll**, is the best strategy overall among the ones discussed. Again, because it uses all available information in every turn and, if possible, it applies the solution to the Problem of Apollonius to skip unnecessary turns.

3 Turn-based version of the game

In this section, each player will only be able to make one guess per turn. We will therefore drop the assumption that player B has already made all their guesses.

The players will take turns guessing points $S_A = \{a_1, a_2, \dots\}$ and $S_B = \{b_1, b_2, \dots\}$. In this version, the function f will be replaced by

$$g(S_A, S_B) = \min_i \{d(a_i, x)\} - \min_j \{d(b_j, x)\}. \quad (4)$$

The space of possible guesses is again $\mathcal{S} = \mathbb{R}^2$. The winning condition must be changed, as to make it possible to end the game in a finite number of turns. In similar papers, like [6], the game ends after a fixed amount of turns. For us, however, the game will only end when one of the players makes a guess inside a disc $\mathbb{D}(x, \varepsilon)$ for a certain tolerance ε . This player would then win the game.

Problem 2. *Let \mathcal{C} , \mathcal{S} , g and ε be given. A target x was selected from \mathcal{C} . Two players A and B take turns guessing points $a_1, a_2, \dots \in \mathcal{S}$ and $b_1, b_2, \dots \in \mathcal{S}$ respectively, which we record in the sets S_A, S_B . Neither of the players know what is the other's set of guesses. After each guess, the function $g(S_A, S_B)$ can be retrieved by both players. The goal of each player is to make a guess in $\mathbb{D}(x, \varepsilon)$, where ε is fixed. Design a strategy for both players.*

We will be designing strategies for both players and then comparing them by letting two computers play against each other. As before, a strategy that will serve as a base point for comparison will be the **Random** strategy (which is defined as expected).

More sophisticated strategies involve analyzing the available information in any given state of the game. Suppose

we are playing as player A , without loss of generality. After turn 1, each guess made by any of the players will update the value of $g(S_A, S_B)$. Suppose player A is about to guess.

1. If A is winning, that is, $g(S_A, S_B) \leq 0$, then the new guess a_k is either in $\mathbb{D}(x, \varepsilon)$, ending the game, or not. In the latter case, if $g(S_A \cup \{a_k\}, S_B)$ is less than $g(S_A, S_B)$, then a_k is currently our best guess. Otherwise, the information we get from a_k is that x is not in its Voronoi region $V_k(\mathcal{C}, S_A \cup \{a_k\})$.
2. If B is winning, $g(S_A, S_B) > 0$, and a_k is not in $\mathbb{D}(x, \varepsilon)$, then the information we get from our new guess a_k is similar to the information we got in the one player version of the game. Namely, we can check if our new guess is our best guess so far (hence x is in $V_k(\mathcal{C}, S_A)$). Moreover, if after guessing a_k the player B is still winning, we can be sure that x is not in the disc $\mathbb{D}(a_k, g(S_A \cup \{a_k\}, S_B))$. And if our new guess a_k is closer to x than any guess b_j , that is, $g(S_A \cup \{a_k\}, S_B) \leq 0$, we don't get any additional information (but we give information to our adversary).

Suppose now B is about to guess.

1. If B is winning, i.e. $g(S_A, S_B) > 0$, then B will also be winning after their guess b_k . But if $g(S_A, S_B \cup \{b_k\})$ is greater than $g(S_A, S_B)$, then we know that their newest guess is better than any other one of their guesses.

We can thus update the information we had. For instance, we knew that x was not in $\mathbb{D}(a_i, g(S_A, S_B))$ for any i , and now we can make this disc larger, as x is not in $\mathbb{D}(a_i, g(S_A, S_B \cup \{b_k\}))$ either.

2. If A is winning, $g(S_A, S_B) \leq 0$, and if b_k does not change this fact, then we didn't get any information.

If, however, B is winning after guessing b_k , then we know that x is not in $\mathbb{D}(a_i, g(S_A, S_B \cup \{b_k\}))$ for any i . We can update our information.

Comparing this analysis to the one player version, one can define the strategies **RandVoronoi**, **RandCircles**, **RandVorCir** and **MedVorCir** as expected. The strategies which used the Problem of Apollonius will not be as useful, given that our adversary's best guess is continuously changing. A possible optimization involving said problem would be quite an intricate one — whenever the losing player reduces its distance to the winning player three times whilst not ending the game, and while the winning player does not make a better guess than its best one, then the losing player can find x .

We coded all the strategies and made them play against each other. The results are recorded in Table 2. The tol-

	B.Random	B.RandCircles	B.RandVoronoi	B.RandVorCir	B.MedVorCir
A.Random	-0.02	-0.40	-0.84	-0.84	-0.92
A.RandCircles	0.42	0.12	-0.56	-0.64	-0.78
A.RandVoronoi	0.74	0.68	-0.02	-0.12	-0.34
A.RandVorCir	0.84	0.74	0.06	0.00	-0.30
A.MedVorCir	0.86	0.84	0.30	0.20	0.02

Tab. 2: The expected outcome of a game as described in Problem 2 between Player A 's strategy (rows) and Player B 's strategy (columns), where 1 is a victory for A and -1 a loss.

erance ε was adjusted so that the probability of winning at random in any given turn is 1 in 100.

Not surprisingly, **MedVorCir** is the best performing strategy. However, note that in this version, **RandVoronoi** is already a competitive strategy, winning almost as much as **MedVorCir** against weaker strategies and close to tying against stronger ones. This is what was to be expected — when studying strategies to a given game, one tends to focus on the better ones and under represents the worst ones³.

Another thing to realize is that a bad player gives less information to the opponent. This is not well captured in the table, but it did have an impact on the average number of guesses per player (e.g., a typical match between **A.MedVorCir** and **B.Random** was much longer than a typical match between **A.MedVorCir** and **B.RandVorCir**). The variability in game duration contributes to the asymmetry of the table.

4 One dimensional versions

In one dimension, a simple translation of Problem 1 as it is stated becomes trivial.

Theorem 4.1. *Let $\mathcal{C} \subseteq \mathbb{R}$ be a compact set, let $\mathcal{S} = \mathbb{R}$ and let f be defined as in (1). There is a solution to Problem 1 with $n = 2$.*

Proof. Choose $a_1 = \min \mathcal{C}$. Then, one can choose a_2 to be $a_1 + f(\{a_1\}) = x$. ■

A more interesting turn-based version of the problem can be proposed by changing the definition of f , so that less information is provided to our players. For example, order $S_A \cup S_B$ by the turn order, $\{a_1, b_1, a_2, b_2, \dots\}$, and let g be defined as

$$g(S_A, S_B) = \hat{g}(\{a_1, b_1, \dots, c\}) = \begin{cases} +1 & \text{if } c > x, \\ 0 & \text{if } c = x, \\ -1 & \text{if } c < x, \end{cases} \quad (5)$$

³ For the interested reader, there is a not-too-serious article discussing how this problem affects Elo rating (particularly in chess) and proposes a way to overcome it [10].

and consider Problem 2 with $\mathcal{C} \subseteq \mathbb{Z}$. We can think of ± 1 as meaning “higher” or “lower”. An instance of this game is featured as one of the many rounds in the television show *The Price is Right*⁴ [15]: two players compete against each other to guess x to a given accuracy ($\varepsilon < \$1$), with the only information being if your opponent just guessed too high or too low. In this instance of the game, the set of legal guesses \mathcal{S} is dynamically changed to be the smallest interval in which players know the target is. The winning condition is equivalent to $g(S_A, S_B) = 0$.

Theorem 4.2. *Let $\mathcal{C} \subseteq \mathbb{Z}$ be a discrete interval, let g be defined as in (5), let $\varepsilon < 1$. Consider Problem 2, in which \mathcal{S} is given as a function of the guesses: after each turn, update \mathcal{S} to be*

$$\left(\left(\max_{\substack{c \in \{-\infty\} \cup S_A \cup S_B \\ c < x}} \{c\}, \min_{\substack{c \in \{+\infty\} \cup S_A \cup S_B \\ c > x}} \{c\} \right) \cap \mathcal{C} \right)$$

Then,

- there is a strategy for player A in which the probability of winning is greater than $\frac{1}{2} - \frac{1}{2|\mathcal{C}|}$.
- there is a strategy for player B in which the probability of winning is greater than $\frac{1}{2} - \frac{3}{2|\mathcal{C}|}$.

Proof. We prove the first result, the other one is shown analogously. We assume A plays in the odd turns and B in the even turns.

The probability of winning for A , $\mathbb{P}[A \text{ wins}]$, is equal to the following sum

$$\sum_{i=0}^{\lfloor \frac{|\mathcal{C}|-1}{2} \rfloor} \mathbb{P}[A \text{ wins in the } (2i+1)\text{st turn}].$$

Note that this sum is finite, since only finitely many different guesses can be made in a compact discrete interval.

⁴ Some other games featured in this show have already been subject to game theoretic analysis. See [2].

Let $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \dots$ be the different sets of legal guesses as they update, starting with $\mathcal{S}_0 = \mathcal{C}$. Then,

$$\begin{aligned} \mathbb{P}[A \text{ wins in the } (2i+1)\text{st turn}] &= \mathbb{P}[a_1 \neq x] \cdot \mathbb{P}[b_1 \neq x] \cdots \mathbb{P}[b_i \neq x] \cdot \mathbb{P}[a_{i+1} = x] \\ &= \frac{|\mathcal{S}_0| - 1}{|\mathcal{S}_0|} \cdot \frac{|\mathcal{S}_1| - 1}{|\mathcal{S}_1|} \cdots \frac{|\mathcal{S}_{2i-1}| - 1}{|\mathcal{S}_{2i-1}|} \cdot \frac{1}{|\mathcal{S}_{2i}|}. \end{aligned}$$

The strategy for player A is choosing a_k to be the minimum of the current legal set \mathcal{S} at any turn. This ensures $|\mathcal{S}_{2k} - 1 = |\mathcal{S}_{2k+1}|$. We also know $|\mathcal{S}_{2k-1}| - 1 \geq |\mathcal{S}_{2k}|$. Hence,

$$\begin{aligned} \mathbb{P}[A \text{ wins in the } (2i+1)\text{st turn}] &\geq \frac{|\mathcal{S}_1|}{|\mathcal{S}_0|} \cdot \frac{|\mathcal{S}_2|}{|\mathcal{S}_1|} \cdot \frac{|\mathcal{S}_3|}{|\mathcal{S}_2|} \cdots \frac{|\mathcal{S}_{2i}|}{|\mathcal{S}_{2i-1}|} \cdot \frac{1}{|\mathcal{S}_{2i}|} = \frac{1}{|\mathcal{S}_0|} = \frac{1}{|\mathcal{C}|}. \end{aligned}$$

And therefore,

$$\mathbb{P}[A \text{ wins}] \geq \frac{\lfloor \frac{|\mathcal{C}|-1}{2} \rfloor + 1}{|\mathcal{C}|} \geq \frac{|\mathcal{C}| - 1}{2|\mathcal{C}|} \geq \frac{1}{2} - \frac{1}{2|\mathcal{C}|},$$

as desired. \blacksquare

5 Further comments and higher dimensional variations

Just as we mentioned in the introduction, for the original problem, supposing that the country is a subspace of \mathbb{R}^2 is a simplification that can introduce a substantial error for large enough countries. We can fix this by considering the Problem of Apollonius for spherical regions in \mathbb{S}^2 (see [5]). Following the proof of Theorem 2.1, we see that Claims 1 and 3 therefore hold on a sphere, whilst Claim 2 is not necessarily true. The discussion about the optimal placement of a_1 is also untrue. Recall however that Claim 2 is not crucial to the strategy, but an optimization on n . There is a strategy with $n \leq 8$.

Other generalizations of the Problem of Apollonius involve higher dimensional tangent spheres in \mathbb{R}^m . In general, in good enough conditions⁵, there is a unique $(m-1)$ -sphere tangent and external to $(m+1)$ given $(m-1)$ -spheres. One can get these $m+1$ spheres by choosing $m+1$ points whose convex hull span a regular m -simplex in each step. This way, we are guaranteed to find x in $(m+1) \cdot m + 2$ turns.

This bound is not sharp for $m = 2$, since we could make some optimizations that brought down the number of needed guesses from 8 to 5.

We propose the following question: what is the minimum number of turns that will work in \mathbb{R}^3 ? (at

⁵ The problem has been solved in general [8, 14], but all solutions need not exist [13].

most, $4 \cdot 3 + 2 = 14$.) And in \mathbb{R}^n ?

However, this was not the best strategy in \mathbb{R}^2 . We can define an analogous strategy to **MedVorCirApoll** in n dimensions. The question now is, is this better than **Apollonius** in all dimensions?

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