Equilibrium perturbations for stochastic interacting systems

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EQUILIBRIUM PERTURBATIONS FOR STOCHASTIC INTERACTING SYSTEMS

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Abstract. We consider the equilibrium perturbations for two stochastic systems: the d-dimensional generalized exclusion process and the one-dimensional chain of anharmonic oscillators. We add a perturbation of order $N^{-\alpha}$ to the equilibrium profile, and speed up the process by $N^{1+\kappa}$ for parameters $0 < \kappa \leq \alpha$. Under some additional constraints on κ and α, we show the perturbed quantities evolve according to the Burgers equation in the exclusion process, and to two decoupled Burgers equations in the anharmonic chain, both in the smooth regime.

1. Introduction

One of the main aims in the theory of hydrodynamic limit is to derive partial differential equations from microscopic systems. For asymmetric systems with one or several conservation laws, the conserved quantity/quantities usually evolve macroscopically according to the hyperbolic system under the Euler scaling (time accelerated by N and space divided by N), *cf.* [18, 16, 12] for example. To refine the hydrodynamic limit, when the system has only one conservation law, in the seminal paper [5], Esposito, Marra and Yau perturb the asymmetric simple exclusion processes around the equilibrium point with order N^{-1} at the initial time in dimensions $d \geq 3$, and show that the evolution of the perturbed quantity, namely the *equilibrium perturbation*, is governed by the viscous Burgers equation under the diffusive scaling (time accelerated by N^2 and space divided by N). This is closely related to the understanding of Navier–Stokes corrections and is called the incompressible limit in the literature, *cf.* [12, Section 7.7] for details. Later, such kind of result is extended by Seppäläinen [19] to dimension $d = 1$. He considers the one-dimensional Hammersley's model, adds a perturbation of order $N^{-\alpha}$ to the equilibrium, and shows that the perturbation macroscopically obeys the invisid Burgers equation in the time scale $N^{1+\alpha}t$ if $0 < \alpha < 1/2$. The proof uses combinatorial properties of the Hammersley's model and coupling techniques, and the result holds even beyond the appearance of shocks. Independently, Tóth and Valkó [20] obtain the same result for a large class of one-dimensional interacting particle systems, which is called *deposition* models in their paper, but only for $0 < \alpha < 1/5$. Tóth and Valkó employ Yau's relative entropy method [25], thus the result holds only in the smooth regime of the solutions. Very recently, Jara, Landim and Tsunoda [11] consider equilibrium perturbations in weakly asymmetric exclusion processes and derive viscous Burgers equation under the diffusive scaling. Equilibrium perturbations have also been considered for systems with two conservation laws. In [21], Tóth and Valkó prove for a very rich class of systems that the small perturbations are universally driven by a two-by-two system. In [22], Valkó shows that small perturbations around a hyperbolic equilibrium point evolve according to two decoupled Burgers equations. For system with three or more conservation laws, the problem remains generally open.

Key words and phrases. Equilibrium perturbation; exclusion process; oscillator chain; hydrodynamic limit.

1.1. **Nonlinear geometric optics for conservation laws.** To illustrate the idea, for $n \geq 1$, we start from an n-system of conservation laws

$$\partial_t \mathbf{f}(t,v) + \big(m \cdot \nabla_v\big) J\big(\mathbf{f}(t,v)\big) = 0, \quad \mathbf{f}(0,\cdot) = \mathbf{f}^{\mathrm{ini}}, \tag{1.1}$$

where $v \in \mathbb{R}^d$, $\mathbf{f}^{\mathrm{ini}} \in C^1(\mathbb{R}^d;\mathbb{R}^n)$, $m \in \mathbb{R}^d$, $J : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth, nice function and

$$\big(m \cdot \nabla_v\big) J\big(\mathbf{f}(v)\big) := \left(m_1 \frac{\partial}{\partial v_1} + ... + m_d \frac{\partial}{\partial v_d}\right) J\big(\mathbf{f}(v)\big) \in \mathbb{R}^n.$$

To avoid the discussion on boundary terms, we adopt the periodic condition $\mathbf{f}(t, v + e_i) = \mathbf{f}(t,v)$ for each $i = 1$, ..., d. It is equivalent to say that the space variable $v \in \mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$.

The stationary solutions to (1.1) are given by constant functions. We are interested in the behavior of the solution when the initial condition is slightly perturbed from the stationary. More precisely, fix a vector $\mathbf{w}_* \in \mathbb{R}^n$ and consider the perturbed system

$$\partial_t \mathbf{f}^\varepsilon(t,v) + \big(m \cdot \nabla_v\big) J\big(\mathbf{f}^\varepsilon(t,v)\big) = 0, \quad \mathbf{f}^\varepsilon(0,\cdot) = \mathbf{w}_* + \varepsilon \mathbf{w}^{\mathrm{ini}},$$

where $\varepsilon > 0$ and $\mathbf{w}^{\mathrm{ini}} \in C^1(\mathbb{T}^d;\mathbb{R}^n)$. Suppose that $\mathbf{f}^\varepsilon$ decomposes as

$$\mathbf{f}^\varepsilon(t,v) = \mathbf{w}_* + \varepsilon \mathbf{w}(t,v) + \mathbf{w}^\varepsilon(t,v), \quad |\mathbf{w}^\varepsilon(t,v)| = o(\varepsilon).$$

By expanding J at $\mathbf{w}_*$, we obtain the linearized equation of the first order component:

$$\frac{\partial \mathbf{w}}{\partial t} + A \sum_{i=1}^d m_i \frac{\partial \mathbf{w}}{\partial v_i} = 0, \quad \mathbf{w}(0,\cdot) = \mathbf{w}^{\mathrm{ini}}, \tag{1.2}$$

where $A = DJ(\mathbf{w}_*)$ is the Jacobian matrix. Assume that (1.1) is strictly hyperbolic at $\mathbf{w}_*$: A has n distinct, non-zero eigenvalues λ_1, ..., λ_n, with the corresponding left (right) eigenvectors denoted by $\mathbf{l}_1$, ..., $\mathbf{l}_n$ ($\mathbf{r}_1$, ..., $\mathbf{r}_n$):

$$\mathbf{l}_j'(\lambda_j \mathrm{Id} - A) = (\lambda_j \mathrm{Id} - A)\mathbf{r}_j = 0, \quad \mathbf{l}_j' \mathbf{r}_k = \mathbf{1}_{\{j=k\}}.$$

From (1.2), $(d/dt)\mathbf{l}_j'\mathbf{w} = 0$ along each characteristic lines $(t, v + \lambda_j t m)$. Hence,

$$\mathbf{w}(t,v) = \sum_{j=1}^n \mathbf{l}_j' \mathbf{w}^{\mathrm{ini}}\big(\phi_j(t,v)\big)\mathbf{r}_j, \quad \phi_j := v - \lambda_j t m.$$

To observe non-trivial evolution along these lines, we investigate at longer time $\varepsilon^{-1}t$. Assume that the solution decomposes further to the second order as

$$\mathbf{f}^\varepsilon(t,x) = \mathbf{w}_* + \varepsilon \sum_{j=1}^n \sigma_j(\varepsilon t, \phi_j)\mathbf{r}_j + \varepsilon^2 \sum_{j=1}^n \tilde{\sigma}_j(\varepsilon t, \boldsymbol{\phi})\mathbf{r}_j + o(\varepsilon^2),$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n) \in \mathbb{R}^{d\times n}$, $\sigma_j = \sigma_j(s,u)$ and $\tilde{\sigma}_j = \tilde{\sigma}_j(s, u_1, \ldots, u_n)$ are C^1 functions for $j = 1$, ..., n. Inserting the expansion into (1.1) and noting that $A\mathbf{r}_j = \lambda_j \mathbf{r}_j$, we see that the terms of order ε cancel autonomously, while the higher order terms read

$$\begin{aligned}\varepsilon^2 \sum_{j=1}^n \partial_s \sigma_j(\varepsilon t, \phi_j)\mathbf{r}_j + \varepsilon^2 \sum_{j,j'} \big((\lambda_j - \lambda_{j'}) m \cdot \nabla_{u_{j'}}\big)\tilde{\sigma}_j(\varepsilon t, \boldsymbol{\phi})\mathbf{r}_j \\ + \varepsilon^2 \sum_{j,j'} \sigma_j(\varepsilon t, \phi_j)\big(m \cdot \nabla_u\big)\sigma_{j'}(\varepsilon t, \phi_{j'}) H(\mathbf{r}_j, \mathbf{r}_{j'}) = o(\varepsilon^2).\end{aligned}$$

Since $\mathbf{l}_k'\mathbf{r}_j = \mathbf{1}_{\{k=j\}}$, the above holds true if and only if for each $k = 1, ..., n$,

$$\begin{aligned}\partial_s\sigma_k(s,u_k) + \sum_{j=1}^n \big((\lambda_k-\lambda_j)m\cdot\nabla_{u_j}\big)\tilde\sigma_k(s,u_1,\dots,u_n)\\ + \sum_{j,j'}\mathbf{l}_k'H(\mathbf{r}_j,\mathbf{r}_{j'})\sigma_j(s,u_j)\big(m\cdot\nabla_u\big)\sigma_{j'}(s,u_{j'}) = 0.\end{aligned} \tag{1.3}$$

We restrict our discussion to the *non-resonant* situation, in which the wave of each frequency λ_k performs self-consistent time evolution governed by the Burgers equation

$$\partial_s\sigma_k + \mathbf{l}_k'H(\mathbf{r}_k,\mathbf{r}_k)\big(m\cdot\nabla_u\big)\big(2^{-1}\sigma_k^2\big) = 0, \quad \forall\, k = 1,\dots,n. \tag{1.4}$$

Comparing (1.3) and (1.4), $\tilde\sigma_k$ then has to solve the equation

$$\begin{aligned}&\sum_{j=1}^n \big((\lambda_j-\lambda_k)m\cdot\nabla_{u_j}\big)\tilde\sigma_k(s,u_1,\dots,u_n)\\ &= \sum_{(j,j')\neq(k,k)}\mathbf{l}_k'H(\mathbf{r}_j,\mathbf{r}_{j'})\sigma_j(s,u_j)\big(m\cdot\nabla_u\big)\sigma_{j'}(s,u_{j'}).\end{aligned}$$

By superposition, we only need to construct $\tilde\sigma_k$ as

$$\tilde\sigma_k(s,u_1,\dots,u_n) := \sum_{j\neq k}\frac{\mathbf{l}_k'H(\mathbf{r}_j,\mathbf{r}_j)\sigma_j^2(s,u_j)}{2(\lambda_j-\lambda_k)} + \sum_{j\neq j'}\tilde\sigma_{k,j,j'}(s,u_j,u_{j'}),$$

where the functions $\tilde\sigma_{k,j,j'} = \tilde\sigma_{k,j,j'}(s,u,u')$ satisfies that

$$\begin{aligned}&\sum_{j=1}^n\big((\lambda_j-\lambda_k)m\cdot\nabla_u\big)\sum_{j\neq j'}\tilde\sigma_{k,j,j'}(s,u_j,u_{j'})\\ &= \sum_{j\neq j'}\mathbf{l}_k'H(\mathbf{r}_j,\mathbf{r}_{j'})\sigma_j(s,u_j)\big(m\cdot\nabla_u\big)\sigma_{j'}(s,u_{j'}).\end{aligned} \tag{1.5}$$

Hence, we formally obtain a *sufficient condition* for the non-resonant case:

$$\begin{aligned}&\big((\lambda_j-\lambda_k)m\cdot\nabla_u + (\lambda_{j'}-\lambda_k)m\cdot\nabla_{u'}\big)\tilde\sigma_{k,j,j'}(s,u,u')\\ &= \mathbf{l}_k'H(\mathbf{r}_j,\mathbf{r}_{j'})\sigma_j(s,u)\big(m\cdot\nabla_u\big)\sigma_{j'}(s,u'),\end{aligned} \tag{1.6}$$

is solvable for all k, j, $j' = 1$, ..., n such that $j \neq j'$. Notice that the formal calculation above apparently relies on the smoothness of σ_k, so we need to assume that $t < T_{\text{shock}}$, where T_{shock} is the time when the shock first appears in the entropy solution to (1.4).

Observe that the non-resonant condition holds autonomously if $n = 1$. If $n = 2$, $d = 1$, without loss of generality we can set $m = 1$ and solve (1.6) explicitly as

$$\begin{aligned}&\tilde\sigma_{1,1,2} = c_1\sigma_1(s,u)\sigma_2(s,u'), \quad \tilde\sigma_{1,2,1} = c_1\Sigma_2(s,u)\partial_u\sigma_1(s,u'),\\ &\tilde\sigma_{2,1,2} = c_2\Sigma_1(s,u)\partial_u\sigma_2(s,u'), \quad \tilde\sigma_{2,2,1} = c_2\sigma_2(s,u)\sigma_1(s,u'),\end{aligned}$$

where $c_1 = -(\lambda_1-\lambda_2)^{-1}\mathbf{l}_1'H(\mathbf{r}_1,\mathbf{r}_2)$, $c_2 = (\lambda_1-\lambda_2)^{-1}\mathbf{l}_2'H(\mathbf{r}_1,\mathbf{r}_2)$ and Σ_1, Σ_2 are primitives of σ_1, σ_2, respectively, i.e., $\Sigma_1 = \int_0^{\cdot}\sigma_1 du$, $\Sigma_2 = \int_0^{\cdot}\sigma_2 du$. To make Σ_j well-defined on $\mathbb{T}$, we shall assume in addition that $\int_{\mathbb{T}}\sigma_j(0,u)du = 0$ for both $j = 1$, 2, or equivalently

$$\int_{\mathbb{T}}\mathbf{w}^{\text{ini}}(u)du = \int_{\mathbb{T}}\big[\sigma_1(0,u)\mathbf{r}_1 + \sigma_2(0,u)\mathbf{r}_2\big]du = 0. \tag{1.7}$$

The non-resonant expansion is not in general valid for the system with $n \geq 3$. We refer the readers to [13, 4] for rigorous justification and detailed discussion for the resonant case.

1.2. **Equilibrium perturbation.** The goal of the present paper is to derive the non-resonant system (1.4) of Burgers equations as a decent scaling limit for some stochastic interacting system. Formally speaking, for a scaling parameter $N \in \mathbb{N}_+$, we study a particle system $\zeta(t) = \{\zeta_x(t)\}$, where the microscopic position x belongs to the periodic lattice $\mathbb{T}_N^d := \mathbb{Z}^d/(N\mathbb{Z}^d)$. Suppose that the system has n conserved quantities $\mathbf{g} = (g_1, \dots, g_n)$ such that, under the *Euler* space-time rescaling $(Nt, [Nv])$ their empirical densities evolve with the macroscopic conservative system (*hydrodynamic limit*):

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \mathbf{g}\big(\zeta_x(Nt)\big)\delta_{\frac{x}{N}}(dv) \Rightarrow \mathbf{f}(t,v)dv, \quad N \to \infty,$$

where $\mathbf{f}$ is the solution to (1.1). To do the perturbation, fix $\alpha > 0$, $\mathbf{w}_* \in \mathbb{R}^n$, $\mathbf{w}^{\mathrm{ini}} \in C^1(\mathbb{T}^d; \mathbb{R}^n)$ and start the dynamics from some initial distribution such that

$$\frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \mathbf{g}\big(\zeta_x(0)\big)\delta_{\frac{x}{N}}(dv) \Rightarrow \big(\mathbf{w}_* + N^{-\alpha}\mathbf{w}^{\mathrm{ini}}(v)\big)dv, \quad N \to \infty.$$

Let $\{\sigma_k; k = 1, \dots, n\}$ be the smooth solutions to (1.4). In the non-resonant situation, the arguments in the previous part suggests the formal asymptotic formula

$$\frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \mathbf{g}\big(\zeta_x(Nt)\big)\delta_{\frac{x}{N}}(dv) \approx \left[\mathbf{w}_* + N^{-\alpha}\sum_{j=1}^n \sigma_j(N^{-\alpha}t, v - \lambda_j tm)\mathbf{r}_j\right] dv.$$

Choosing $\kappa \le \alpha$ and using the variables $s = N^{-\kappa}t$, $u = v - \lambda_j tm$, one obtains that

$$\frac{1}{N^{d-\alpha}} \sum_{x\in\mathbb{T}_N^d} \mathbf{l}_k'\big[\mathbf{g}\big(\zeta_x(N^{1+\kappa}s)\big) - \mathbf{w}_*\big]\delta_{\frac{x}{N}-N^\kappa\lambda_k sm}(du) \Rightarrow \sigma_k^{(\alpha,\kappa)}(s,u)du, \tag{1.8}$$

as $N \to \infty$ for each $k = 1$, ..., n, where

$$\sigma_k^{(\alpha,\kappa)}(s,u) = \begin{cases} \sigma_k(s,u), & \text{if } \kappa = \alpha, \\ \sigma_k^{\mathrm{ini}}(u) := \mathbf{l}_k'\mathbf{w}^{\mathrm{ini}}(u), & \text{if } \kappa < \alpha. \end{cases}$$

Remark 1.1 (Fluctuation)**.** *The convergence in* (1.8) *is available only if* $\alpha < d/2$*. To see that, let us assume that the equilibrium states of the dynamics are given by the family of canonical Gibbs measures* $\otimes_x \nu_{\mathbf{w}}(d\zeta_x)$*, where* $\mathbf{w}$ *is the corresponding equilibrium value of the conserved vector* $\mathbf{g}$*. Starting the dynamics from equilibrium initial state with* $\mathbf{w} = \mathbf{w}^*$*. If* $\mathbf{g}$ *possesses finite variance, then the central limit theorem for i.i.d. random variables yields that for each time* $s > 0$*,*

$$\frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} \mathbf{l}_k'\big[\mathbf{g}\big(\zeta(s)\big) - \mathbf{w}^*\big]\delta_{\frac{x}{N}}(du)$$

converges weakly, as $N \to \infty$*, to some Gaussian random field. The macroscopic time evolutions of these fields are called the equilibrium fluctuations. For the equilibrium fluctuations for the models studied in this paper, see e.g.* [3, 7, 17].

In this paper, we rigorously prove (1.8) for two microscopic models: (i) the generalized exclusion in any dimension, where the total number of particles is the only conservation law ($n = 1$) and (ii) the nonlinear Hamiltonian dynamics in one dimension, disturbed by noises that preserve only the total momentum and volume stretch ($n = 2$).

1.2.1. *Generalized exclusion process.* In Section 2 and 3, we study the d-dimensional *generalized exclusion process* with at most $K > 0$ particles per site. By [18, 12], the density of the particles under

the Euler scaling evolves according to the hyperbolic equation

$$\partial_t \varrho + m \cdot \nabla J(\varrho) = 0, \quad \varrho(0,\cdot) = \varrho^{\mathrm{ini}}, \tag{1.9}$$

where $J(\varrho) = \varrho(K-\varrho)$ is the macroscopic flux at density ϱ, and $m \neq 0$ is the mean of the underlying transition kernel. Fix $\varrho_* \in (0,K)$. We start the process from a perturbation of the constant profile

$$\varrho_N^{\mathrm{ini}}(u) := \varrho_* + N^{-\alpha}\rho^{\mathrm{ini}}(u), \tag{1.10}$$

where $\rho^{\mathrm{ini}} \in C^\infty(\mathbb{T}^d)$. We speed up the process by $N^{1+\kappa}$, $\kappa \leq \alpha$, and choose the reference measure associated to the *smooth* profile

$$\varrho_* + N^{-\alpha}\rho(N^{\kappa-\alpha}t, u - N^\kappa \lambda t m),$$

where $\lambda := J'(\varrho_*) = K - 2\varrho_*$ and ρ is the classical solution to the Burgers equation

$$\partial_s \rho(s,u) - m \cdot \nabla\big(\rho^2(s,u)\big) = 0, \quad \rho(0,u) = \rho^{\mathrm{ini}}(u), \tag{1.11}$$

up to the first shock appears. Then, in Theorem 2.1, we show that under some restrictions on κ and α, the relative entropy is of order $o(N^{d-2\alpha})$. In particular, we partially extend the results in [5, 20] to $0 < \alpha < 1/3$ if $d = 1$, and to $0 < \alpha < 1$ if $d \geq 2$. As a result, in Theorem 2.3, we show the perturbed quantity evolves according to (1.11).

The proof relies on the refined relative entropy method recently introduced by Jara and Menezes [10, 9], also used in many other contexts, e.g., [6, 8, 11]. Compared to the technique in [20], our proof does not involve the spectral gap estimate or logarithmic Sobolev inequality for the underlying microscopic dynamics, which is known as a hard problem for general interacting particle systems.

1.2.2. *Anharmonic chain of oscillators.* In Section 4 and 5, we study the one-dimensional *chain of (unpinned) anharmonic oscillators* with conservative noise. The noise is modeled by Langevin thermostats acting at each position and fixing the temperature to $T = \beta^{-1}$. The only two conserved quantities of the dynamics are the momentum $\mathfrak{p}$ and the length $\mathfrak{r}$, hence it corresponds to the case $n = 2$, $d = 1$. The hydrodynamic equation is given by the following p-system

$$\partial_t \mathfrak{p} = \partial_v \boldsymbol{\tau}(\mathfrak{r}), \quad \partial_t \mathfrak{r} = \partial_v \mathfrak{p}, \quad (\mathfrak{p},\mathfrak{r})(0,\cdot) = (\mathfrak{p},\mathfrak{r})^{\mathrm{ini}},$$

where $\boldsymbol{\tau} = \boldsymbol{\tau}_\beta$ is the equilibrium tension. It is proved in [2] together with the energy conservation law in the smooth regime and in [14] after the appearance of shocks.

Define $J(\mathfrak{p},\mathfrak{r}) := (-\boldsymbol{\tau}(\mathfrak{r}), -\mathfrak{p})$ and fix $(\mathfrak{p}_*,\mathfrak{r}_*) \in \mathbb{R}^2$ such that $\boldsymbol{\tau}'(\mathfrak{r}) \neq 0$. Denote by $A = DJ(\mathfrak{p}_*,\mathfrak{r}_*)$ and note that $A\boldsymbol{v}_- = \sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}\boldsymbol{v}_-$, $A\boldsymbol{v}_+ = -\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}\boldsymbol{v}_+$ for $\boldsymbol{v}_\pm = (\pm\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}, 1)'$. Similarly to the exclusion process, we start the dynamics from a perturbed profile

$$(\mathfrak{p}_*,\mathfrak{r}_*)' + N^{-\alpha}\sigma_-^{\mathrm{ini}}\boldsymbol{v}_- + N^{-\alpha}\sigma_+^{\mathrm{ini}}\boldsymbol{v}_+,$$

where $\sigma_\pm^{\mathrm{ini}} \in C^1(\mathbb{T})$. We also speed up the time to be $N^{1+\kappa}t$ for some $\kappa \leq \alpha$. Choose the reference measure associated to the slowly varying parameters

$$(\mathfrak{p}_*,\mathfrak{r}_*)' + N^{-\alpha}\sum_{j=\pm}\sigma_j\big(N^{\kappa-\alpha}t, v + jN^\kappa\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}t\big)\boldsymbol{v}_j,$$

where (σ_-,σ_+) is the smooth solution to

$$\begin{aligned}
&\partial_s\sigma_-(s,u) + \frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{4\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}}\partial_u\big(\sigma_-^2(s,u)\big) = 0, \quad \sigma_-(0,u) = \sigma_-^{\mathrm{ini}}(u),\\
&\partial_s\sigma_+(s,u) - \frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{4\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}}\partial_u\big(\sigma_+^2(s,u)\big) = 0, \quad \sigma_+(0,u) = \sigma_+^{\mathrm{ini}}(u).
\end{aligned} \tag{1.12}$$

We prove in Theorem 4.1 that in the smooth regime, the relative entropy grows with the order $o(N^{1-2\alpha})$. As a consequence, we prove in Theorem 4.2 that the macroscopic perturbation is governed by the decoupled system (1.12).

The proof of the relative entropy estimate is based on the equivalence of ensembles for inhomogeneous Gibbs states and the uniform gradient estimate for the Poisson equation associated to the generator of the stochastic dynamics, *cf.* [24, Section 8 & 9]. It is worth pointing out that, unlike the exclusion, the equilibrium perturbation for Hamiltonian dynamics has not been investigated before, mainly due to the technical difficulties in obtaining fine estimate for the relative entropy.

2. Asymmetric generalized exclusion process

In this and the next sections, we consider the asymmetric generalized exclusion process. Fix a positive integer $K > 0$, which denotes the maximum number of particles at each site. The state space of the generalized exclusion process is $\Omega_N^d := \{0, 1, \dots, K\}^{\mathbb{T}_N^d}$. For a configuration $\eta \in \Omega_N^d$, η_x denotes the number of particles at site $x \in \mathbb{T}_N^d$. Denote by $\{e_i\}_{1\le i\le d}$ the canonical basis of $\mathbb{T}_N^d$ and let $e_i = -e_{i-d}$ for $i = d+1$, ..., $2d$. Given the jump rates $\{p_i \ge 0; 1 \le i \le 2d\}$, consider the generator $\mathcal{L}_N$ which is given for any function $f : \Omega_N^d \to \mathbb{R}$ by

$$\mathcal{L}_N f(\eta) = \sum_{i=1}^{2d} \sum_{x\in\mathbb{T}_N^d} c_{x,i}(\eta)\big[f(\eta^{x,x+e_i}) - f(\eta)\big],$$

where the jump rate is given by $c_{x,i}(\eta) := p_i\eta_x(K - \eta_{x+e_i})$ and $\eta^{x,y}$ is the configuration obtained from η after a particle jumps from x to y,

$$(\eta^{x,y})_z = \begin{cases} \eta_x - 1, & z = x, \\ \eta_y + 1, & z = y, \\ \eta_z, & \text{otherwise.} \end{cases}$$

Assume that $p_i + p_{i+d} > 0$ for each $1 \le i \le d$. Denote $m = (m_i)_{1\le i\le d} \in \mathbb{R}^d$ where $m_i = p_i - p_{i+d}$. Also assume that m is a nonzero vector, hence $\mathcal{L}_N$ is asymmetric.

The generator $\mathcal{L}_N$ has a family of product invariant measures indexed by the particle density. For $\varrho \in [0, K]$, let ν_ϱ^1 be the binomial measure $B(K, \varrho K^{-1})$:

$$\nu_\varrho^1(k) := \binom{K}{k}\left(\frac{\varrho}{K}\right)^k\left(1 - \frac{\varrho}{K}\right)^{K-k}, \quad \forall\, k = 0, \dots, K.$$

Denote by ν_ϱ^N the product measure on Ω_N^d such that $\nu_\varrho^N(\eta_x) = \nu_\varrho^1(\eta_x)$ for each $x \in \mathbb{T}_N^d$. It is not hard to check that the family $\{\nu_\varrho^N; 0 \le \varrho \le K\}$ is invariant under $\mathcal{L}_N$. Observe that the average number of particles per site under ν_ϱ^N is $\varrho = E_{\nu_\varrho^1}[\eta_x]$.

Let $\mu_{N,0}$ be associated to the profile ϱ_N^{ini} in (1.10) with some $\alpha > 0$. For $0 < \kappa \le \alpha$, denote by $\{\eta = \eta(t); t \ge 0\}$ the Markov process generated by $N^{1+\kappa}\mathcal{L}_N$ starting from $\mu_{N,0}$. Define

$$\varrho_N(t, u) := \varrho_* + N^{-\alpha}\rho(N^{\kappa-\alpha}t, u - N^\kappa\lambda t m), \tag{2.1}$$

where $\rho(t, u)$ is the smooth solution to (1.11). Note that for N sufficiently large, $\varrho_N \in (0, K)$. Denote by $\mu_{N,t}$ the distribution of $\eta(t)$ and by $\nu_{N,t}$ the slowly varying product measure

$$\nu_{N,t}(d\eta) = \bigotimes_{x\in\mathbb{T}_N^d} \nu_{\varrho_{N,x}}^1(d\eta_x), \quad \varrho_{N,x} := \varrho_N\left(t, \frac{x}{N}\right). \tag{2.2}$$

Let $f_{N,t}(\eta) := \nu_{N,t}(\eta)^{-1}\mu_{N,t}(\eta)$ be the Radon–Nikodym derivative. For a probability measure μ on Ω_N^d and some μ–density f, *i.e.* $f \geq 0$ and $\int_{\Omega_N^d} f d\mu = 1$, define the relative entropy as

$$H(f;\mu) := \int_{\Omega_N^d} f \log f d\mu = \sum_{\eta\in\Omega_N^d} f(\eta)\log f(\eta)\mu(\eta).$$

To shorten the notations, denote

$$H_N(t) := H(f_{N,t};\nu_{N,t}), \quad \forall\, t\in[0,T].$$

It is well known that the the entropy solution to the Burgers equation (1.11) may develop shocks in a finite time interval even if the initial density is smooth, *cf.* [15] for example. Since our proof depends on the relative entropy method, which requires the reference density profile to be smooth, throughout the article, we assume the initial density profile ρ^{ini} is smooth, and when $\kappa = \alpha$, fix a time horizon $T > 0$ such that the solution to the Burgers equation (1.11) is smooth during the time interval $[0,T]$.

The followings are the main results of this part.

Theorem 2.1. *Suppose that $H_N(0) = o(N^{d-2\alpha})$.*

(i) *If $d = 1$, $\alpha \in (0,1/2)$, then for any $t\in[0,T]$, $H_N(t) = o(N^{1-2\alpha})$ for $\kappa \in (0,\alpha]\cap(0,1-2\alpha)$;*
ii) *If $d \geq 2$, $\alpha > 0$, then for any $t\in[0,T]$, $H_N(t) = o(N^{d-2\alpha})$ for $\kappa\in(0,\alpha]\cap(0,1)$.*

Remark 2.2. *It is believed that in dimension $d = 1$, the above result extends to $\kappa\in(0,\alpha]$ even beyond the appearance of shocks, cf.* [20].

As a direct consequence of the above theorem, we have the following law of large numbers for the perturbed quantities.

Theorem 2.3. *Suppose the assumptions in Theorem 2.1 hold and further that $\alpha < d/2$. Then, for any $\varphi\in C(\mathbb{T}^d)$ and any $\varepsilon > 0$,*

$$\lim_{N\to\infty}\mu_{N,t}\left\{\left|\frac{1}{N^{d-\alpha}}\sum_{x\in\mathbb{T}_N^d}(\eta_x-\varrho_*)\varphi\left(\frac{x}{N}-N^\kappa\lambda tm\right) - \int_{\mathbb{T}^d}\rho^{(\alpha,\kappa)}(t,u)\varphi(u)du\right| > \varepsilon\right\} = 0,$$

with the function $\rho^{(\alpha,\kappa)}$ defined as

$$\rho^{(\alpha,\kappa)}(t,u) = \begin{cases}\rho^{\mathrm{ini}}(u), & \kappa<\alpha,\\ \rho(t,u), & \kappa=\alpha,\end{cases}$$

where $\rho(t,u)$ is the entropy solution to (1.11).

Remark 2.4. *If $K = 1$, the above results could be interpreted in terms of* second class *particles. The dynamic is defined as follows: there are two kinds of particles, called first and second class particles, in the system. On top of the exclusion rule, the first class particles have priorities to jump over the second class ones. Precisely speaking, if a first class particle jumps to a site occupied by a second class one, then the jump is performed and the two particles exchange their positions; while if a second class particle jumps to a site occupied by a first class one, then the jump is suppressed. At the initial time, independently at each site x, put one first class particle with probability ϱ_*, and one second class particle with probability $N^{-\alpha}\rho^{\mathrm{ini}}(x/N)$. Then, at the macroscopic time t, the density profile of the second class particles, along the characteristic line of the PDE* (1.9), *is described by the function $\rho^{\alpha,\kappa}$.*

Indeed, denote by $\eta_x^1(t)$ (resp. $\eta_x^2(t)$) the number of the first (resp. the second) class particles at site x at time t, then $\eta_x(t) = \eta_x^1(t) + \eta_x^2(t)$. Since the process of the first class particle is in equilibrium,

the distribution of $\eta^1(t)$ *is given by* $\nu^N_{\varrho_*}$ *for any time* $t \geq 0$. *Therefore, for any* $\varphi \in C(\mathbb{T}^d)$,

$$\mathrm{Var}\Big(\frac{1}{N^{d-\alpha}} \sum_{x\in\mathbb{T}^d_N} (\eta^1_x(t) - \varrho_*)\varphi\big(\tfrac{x-\lambda t N^{1+\kappa} m}{N}\big)\Big) = \mathcal{O}\big(N^{2\alpha-d}\big),$$

which converges to zero as $N \to \infty$ *since* $\alpha < d/2$. *As a result, for any* $\varepsilon > 0$,

$$\lim_{N\to\infty} \mu_{N,t}\Big\{\Big|\frac{1}{N^{d-\alpha}} \sum_{x\in\mathbb{T}^d_N} (\eta^1_x - \varrho_*)\varphi\big(\tfrac{x-\lambda t N^{1+\kappa} m}{N}\big)\Big| > \varepsilon\Big\} = 0.$$

Together with Corollary 2.3, we have

$$\lim_{N\to\infty} \mu_{N,t}\Big\{\Big|\frac{1}{N^{d-\alpha}} \sum_{x\in\mathbb{T}^d_N} \eta^2_x\varphi\big(\tfrac{x-\lambda t N^{1+\kappa} m}{N}\big) - \int_{\mathbb{T}^d} \rho^{\alpha,\kappa}(t,u)\varphi(u)\,du\Big| > \varepsilon\Big\} = 0.$$

3. Relative entropy for the generalized exclusion

In this section, we prove Theorems 2.1 and 2.3. Recall the profile ϱ_N defined in (2.1). To make notations simple, in the following calculations, we denote

$$\varrho_x = \varrho_{N,x}(t) := \varrho_N\left(t, \frac{x}{N}\right), \quad \forall\, x \in \mathbb{T}^d_N.$$

For a probability measure μ on Ω^d_N and a μ–density f, define the Dirichlet form

$$D_N(f;\mu) := \sum_{i=1}^{2d} \sum_{x\in\mathbb{T}^d_N} \sum_{\eta\in\Omega^d_N} c_{x,i}(\eta)\left(\sqrt{f(\eta^{x,x+e_i})} - \sqrt{f(\eta)}\right)^2 \mu(\eta). \tag{3.1}$$

We claim that for any $\nu_{N,t}$–density f, there exists $\delta_0 = \delta_0(\varrho_*)$ such that

$$D_N(f;\nu_{N,t}) \geq \delta_0 \sum_{i=1}^{2d} \sum_{x\in\mathbb{T}^d_N} \sum_{\eta\in\Omega^d_N} \eta_x(K-\eta_{x+e_i})\left(\sqrt{f(\eta^{x,x+e_i})} - \sqrt{f(\eta)}\right)^2 \nu_{N,t}(\eta). \tag{3.2}$$

Indeed, the claim holds obviously if $p_i > 0$ for all $1 \leq i \leq 2d$. Suppose that $p_i = 0$ for some $1 \leq i \leq d$, then $p_{i+d} \neq 0$ due to our assumption. Recalling that $e_{i+d} = -e_i$,

$$\begin{aligned}\sum_\eta \eta_x(K-\eta_{x-e_i})\left(\sqrt{f(\eta^{x,x-e_i})} - \sqrt{f(\eta)}\right)^2 \nu_{N,t}(\eta)\\ = \sum_\eta (\eta_x+1)(K-\eta_{x-e_i}+1)\left(\sqrt{f(\eta)} - \sqrt{f(\eta^{x-e_i,x})}\right)^2 \nu_{N,t}(\eta^{x-e_i,x}),\end{aligned}$$

for each $x \in \mathbb{T}^d_N$. Observe from the definition of $\nu_{N,t}$ in (2.2) that

$$\frac{\nu_{N,t}(\eta^{x-e_i,x})}{\nu_{N,t}(\eta)} = \frac{\theta_x}{\theta_{x-e_i}} \frac{\eta_{x-e_i}(K-\eta_x)}{(K-\eta_{x-e_i}+1)(\eta_x+1)},$$

where

$$\theta_x := \frac{\varrho_x}{K-\varrho_x}.$$

Since $|\varrho_x - \varrho_*| \leq N^{-\alpha}||\rho||_\infty$ and $\varrho_* \in (0,K)$, hence there exists some $C = C(\varrho_*) > 0$, such that $C^{-1} < \theta_x < C$ for all x and sufficiently large N. Therefore,

$$\sum_\eta \eta_x(K-\eta_{x-e_i})\left(\sqrt{f(\eta^{x,x-e_i})} - \sqrt{f(\eta)}\right)^2 \nu_{N,t}(\eta)$$

$$\geq C' \sum_{\eta} \eta_{x-e_i}(K-\eta_x)\left(\sqrt{f(\eta)}-\sqrt{f(\eta^{x-e_i,x})}\right)^2 \nu_{N,t}(\eta),$$

which is enough to conclude the claim.

Recall that $f_{N,t} = \mu_{N,t}/\nu_{N,t}$ and let $\psi_{N,t} := (\nu^N_{\varrho_*})^{-1}\nu_{N,t}$ be the derivative of $\nu_{N,t}$ with respect to the stationary measure $\nu^N_{\varrho_*}$. By Yau's relative entropy inequality (see Lemma A.1),

$$\begin{aligned}\frac{d}{dt}H_N(t) \leq & -N^{1+\kappa}D_N(f_{N,t};\nu_{N,t}) \\ & + \sum_{\eta\in\Omega_N^d}\left[N^{1+\kappa}\mathcal{L}^*_{N,t}\mathbf{1}(\eta) - \frac{d}{dt}\log\psi_{N,t}(\eta)\right]\mu_{N,t}(\eta),\end{aligned}$$

where $\mathcal{L}^*_{N,t}$ is the adjoint of $\mathcal{L}_N$ with respect to $\nu_{N,t}$. The main idea is to write the right-hand side of the above inequality as $CH_N(t) + o(N^{d-2\alpha})$ for some finite constant C independent of N, and the result then follows from Grönwall's inequality.

Since $c_{x,i}(\eta)$ is the jump rate from $\eta\in\Omega_N^d$ to $\eta^{x,x+e_i}$, Lemma A.1 yields that

$$\begin{aligned}\mathcal{L}^*_{N,t}\mathbf{1}(\eta) &= \sum_{i=1}^{2d}\sum_{x\in\mathbb{T}_N^d}\left\{\frac{\nu_{N,t}(\eta^{x+e_i,x})}{\nu_{N,t}(\eta)}c_{x,i}(\eta^{x+e_i,x}) - c_{x,i}(\eta)\right\} \\ &= \sum_{i=1}^{2d}\sum_{x\in\mathbb{T}_N^d} p_i\left\{\frac{\varrho_x(K-\varrho_{x+e_i})}{\varrho_{x+e_i}(K-\varrho_x)}\eta_{x+e_i}(K-\eta_x) - \eta_x(K-\eta_{x+e_i})\right\} \\ &= \sum_{i=1}^{2d}\sum_{x\in\mathbb{T}_N^d} p_i\varrho_x(K-\varrho_{x+e_i})\left\{\frac{\eta_{x+e_i}(K-\eta_x)}{\varrho_{x+e_i}(K-\varrho_x)} - \frac{\eta_x(K-\eta_{x+e_i})}{\varrho_x(K-\varrho_{x+e_i})}\right\}.\end{aligned}$$

For any configuration $\eta\in\Omega_N^d$ and any site $x\in\mathbb{T}_N^d$, denote

$$\omega_x = \omega_x(\eta) := \frac{\eta_x-\varrho_x}{\varrho_x(K-\varrho_x)}. \tag{3.3}$$

It could be checked directly that for $x\neq y$,

$$\frac{\eta_x(K-\eta_y)}{\varrho_x(K-\varrho_y)} - \frac{\eta_y(K-\eta_x)}{\varrho_y(K-\varrho_x)} = K\big[\omega_x-\omega_y+(\varrho_x-\varrho_y)\omega_x\omega_y\big]. \tag{3.4}$$

Recall that $e_{i+d} = -e_i$ for $1\leq i\leq d$, and (3.4) permits us to write $\mathcal{L}^*_{N,t}\mathbf{1}(\eta)$ as

$$\mathcal{L}^*_{N,t}\mathbf{1}(\eta) = \sum_{i=1}^{d}\sum_{x\in\mathbb{T}_N^d} b_N(x,x+e_i)\big\{(\omega_{x+e_i}-\omega_x)+(\varrho_{x+e_i}-\varrho_x)\omega_x\omega_{x+e_i}\big\},$$

where for $1\leq i\leq d$,

$$b_N(x,x+e_i) := K\big[p_i\varrho_x(K-\varrho_{x+e_i}) - p_{i+d}\varrho_{x+e_i}(K-\varrho_x)\big].$$

Now we calculate $\frac{d}{dt}\log\psi_{N,t}(\eta)$. Since $\nu_{N,t}$ and $\nu^N_{\varrho_*}$ are both product measures,

$$\log\psi_{N,t}(\eta) = \sum_{x\in\mathbb{T}_N^d}\left[\eta_x\log\left(\frac{\theta_x}{\varrho_*}\right) - K\log\left(\frac{1+\theta_x}{1+\varrho_*}\right)\right].$$

The time derivative then reads

$$\frac{d}{dt}\log\psi_{N,t}(\eta) = \sum_{x\in\mathbb{T}_N^d}\left\{\eta_x\frac{\partial_t\theta_x}{\theta_x} - K\frac{\partial_t\theta_x}{1+\theta_x}\right\}$$

$$= \sum_{x\in\mathbb{T}_N^d} K\Big\{\eta_x \frac{\partial_t \varrho_x}{\varrho_x(K-\varrho_x)} - \frac{\partial_t \varrho_x}{K-\varrho_x}\Big\} = \sum_{x\in\mathbb{T}_N^d} K\omega_x \partial_t \varrho_x.$$

To sum up, we have shown that

$$\frac{d}{dt}H_N(t) \le -N^{1+\kappa} D_N(f_{N,t};\nu_{N,t}) + \mathcal{R}_{N,t} + \mathcal{E}_{N,t}, \tag{3.5}$$

where

$$\mathcal{R}_{N,t} = \sum_{\eta\in\Omega_N^d}\sum_{i=1}^d\sum_{x\in\mathbb{T}_N^d} a_{i,x}^N \omega_x\omega_{x+e_i}\mu_{N,t}(\eta), \tag{3.6}$$

$$a_{i,x}^N = N^{1+\kappa} b_N(x,x+e_i)(\varrho_{x+e_i}-\varrho_x), \tag{3.7}$$

and

$$\mathcal{E}_{N,t} = \sum_{\eta\in\Omega_N^d}\Big\{\sum_{i=1}^d\sum_{x\in\mathbb{T}_N^d} N^{1+\kappa} b_N(x,x+e_i)(\omega_{x+e_i}-\omega_x) - \sum_{x\in\mathbb{T}_N^d} K\omega_x\partial_t\varrho_x\Big\}\mu_{N,t}(\eta).$$

3.1. **Proofs of Theorems 2.1 and Corollary 2.3.** In this subsection, we prove Theorems 2.1 and 2.3. We first deal with the error term $\mathcal{E}_{N,t}$.

Lemma 3.1. *There exists a constant C independent of N such that*

$$\mathcal{E}_{N,t} \le H_N(t) + CN^{d+2\kappa-2\alpha-2}.$$

Proof. For a sequence $\{a_x; x\in\mathbb{T}_N^d\}$ and $1\le i\le d$, let $\nabla_i a_x := a_{x+e_i}-a_x$ and $\nabla_i^* a_x := a_{x-e_i}-a_x$. Using the summation by parts formula,

$$\mathcal{E}_{N,t} = \sum_{\eta\in\Omega_N^d}\mu_{N,t}(\eta)\sum_{x\in\mathbb{T}_N^d}\omega_x\left[N^{1+\kappa}\sum_{i=1}^d\nabla_i^*\big(b_N(x,x+e_i)\big) - K\partial_t\varrho_x\right].$$

By the definition of b_N and Taylor's expansion,

$$\begin{aligned}\nabla_i^*\big(b_N(x,x+e_i)\big) = N^{-1-2\alpha}Km_i\partial_{u_i}\rho^2\Big(N^{\kappa-\alpha}t,\frac{x}{N}-N^\kappa\lambda tm\Big)\\ - N^{-1-\alpha}K\lambda m_i\partial_{u_i}\rho\Big(N^{\kappa-\alpha}t,\frac{x}{N}-N^\kappa\lambda tm\Big)+\epsilon_{N,i}(x),\end{aligned}$$

where $\epsilon_{N,i}(x)=\mathcal{O}(N^{-2-\alpha})$. Meanwhile, by the definition of ϱ_N in (2.1),

$$\partial_t\varrho_x = \big[N^{\kappa-2\alpha}\partial_s - N^{\kappa-\alpha}\lambda(m\cdot\nabla_u)\big]\rho\Big(N^{\kappa-\alpha}t,\frac{x}{N}-N^\kappa\lambda tm\Big).$$

Since ρ solves the Burgers equation (1.11), we can rewrite the $\mathcal{E}_{N,t}$ as

$$\mathcal{E}_{N,t} = N^{1+\kappa}\sum_{\eta\in\Omega_N^d}\sum_{i=1}^d\sum_{x\in\mathbb{T}_N^d}\omega_x\epsilon_{N,i}(x)\,\mu_{N,t}(\eta).$$

It is easy to see $|\mathcal{E}_{N,t}|\le CN^{d-1+\kappa-\alpha}$. To get a better bound, by entropy inequality,

$$\begin{aligned}\mathcal{E}_{N,t} &\le H_N(t) + \log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{N^{1+\kappa}\sum_{i=1}^d\sum_{x\in\mathbb{T}_N^d}\omega_x\epsilon_{N,i}(x)\Big\}\nu_{N,t}(\eta)\Big]\\ &= H_N(t) + \sum_{x\in\mathbb{T}_N^d}\log\Big[\sum_{\eta\in\Omega_N^d}\exp\big\{N^{1+\kappa}\omega_x\sum_{i=1}^d\epsilon_{N,i}(x)\big\}\nu_{N,t}(\eta)\Big].\end{aligned} \tag{3.8}$$

Note that ω_x is bounded and has zero mean with respect to the measure $\nu_{N,t}$. Using the basic inequality

$$e^a \le 1 + a + (1/2)a^2 e^{|a|}, \quad \log(1+a) \le a,$$

we have

$$\mathcal{E}_{N,t} \le H_N(t) + \mathcal{O}(N^{d-2+2\kappa-2\alpha}). \tag{3.9}$$

This concludes the proof of the lemma. □

The following result bounds the term $\mathcal{R}_{N,t}$, whose proof is postponed to Subsection 3.2.

Proposition 3.2. *For $\ell \ge 1$, let*

$$g_d(\ell) = \begin{cases} \ell, & d = 1, \\ \log \ell, & d = 2, \\ 1, & d \ge 3. \end{cases} \tag{3.10}$$

Then, there exists a constant C independent of N such that

$$\mathcal{R}_{N,t} \le N^{1+\kappa} D_N(f_{N,t}; \nu_{N,t}) + C\left[N^{\kappa-\alpha} + N^{\kappa-2\alpha-1}\ell^d g_d(\ell)\right]\left[H_N(t) + \frac{N^d}{\ell^d}\right] + C\big[H_N(t) + N^{d+2\kappa-4\alpha-2}\ell^d g_d(\ell)\big].$$

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. In Proposition 3.2, take

$$\ell = \ell(N) = \begin{cases} N^{(1+2\alpha-\kappa)/2}, & d = 1, \\ N^{(1+2\alpha-\kappa)/2}/\sqrt{\log N}, & d = 2, \\ N^{(1+2\alpha-\kappa)/d}, & d \ge 3. \end{cases}$$

Therefore, in dimension $d = 1$, if $\kappa < 1 - 2\alpha$ and $\kappa \le \alpha$, then

$$\frac{d}{dt}H_N(t) \le C(1 + N^{\kappa-\alpha})H_N(t) + C\Big(N^{(1-2\alpha+\kappa)/2} + N^{\kappa-2\alpha} + N^{2\kappa-2\alpha-1}\Big) \le CH_N(t) + o(N^{1-2\alpha}).$$

In dimension $d = 2$, if $\kappa < 1$ and $\kappa \le \alpha$, then

$$\frac{d}{dt}H_N(t) \le C(1 + N^{\kappa-\alpha})H_N(t) + C\Big(N^{1-2\alpha+\kappa}\log N + N^{2\kappa-2\alpha}\Big) \le CH_N(t) + o(N^{2-2\alpha}).$$

In dimensions $d \ge 3$, if $\kappa < 1$ and $\kappa \le \alpha$, then

$$\frac{d}{dt}H_N(t) \le C(1 + N^{\kappa-\alpha})H_N(t) + C\Big(N^{d-2\alpha+\kappa-1} + N^{d+2\kappa-2\alpha-2}\Big) \le CH_N(t) + o(N^{d-2\alpha}).$$

We conclude the proof by using Grönwall's inequality. □

The proof of Theorem 2.3 is a direct application of the entropy inequality, *cf.* [12, Corollary 6.1.3] for example. We sketch it below for completeness.

Proof of Theorem 2.3. It suffices to prove

$$\lim_{N\to\infty} \sum_{\eta\in\Omega_N^d} \Big|\frac{1}{N^{d-\alpha}} \sum_{x\in\mathbb{T}_N^d} (\eta_x - E_{\nu_{N,t}}[\eta_x])\varphi\big(\tfrac{x-\lambda t N^{1+\kappa} m}{N}\big)\Big| \mu_{N,t}(\eta) = 0.$$

By the entropy inequality, for any $\gamma > 0$, the integral above is bounded by

$$\frac{H_N(t)}{\gamma N^{d-2\alpha}} + \frac{1}{\gamma N^{d-2\alpha}} \log\Big[\sum_{\eta\in\Omega_N^d} \exp\Big\{\Big|\gamma N^{-\alpha} \sum_{x\in\mathbb{T}_N^d} (\eta_x - E_{\nu_{N,t}}[\eta_x])\varphi\big(\tfrac{x-\lambda t N^{1+\kappa} m}{N}\big)\Big|\Big\}\nu_{N,t}(\eta)\Big].$$

By Theorem 2.1, the first term above converges to zero as $N \to \infty$. Since

$$e^{|c|} \le e^c + e^{-c}, \quad \log(a+b) \le \log 2 + \max\{\log a, \log b\}$$

for any c and any positive reals a, b, and since $\alpha < d/2$, we could remove the absolute value inside the exponential in the second term above, and rewrite it as

$$\frac{1}{\gamma N^{d-2\alpha}} \sum_{x\in\mathbb{T}_N^d} \log\Big[\sum_{\eta\in\Omega_N^d} \exp\Big\{\gamma N^{-\alpha}(\eta_x - E_{\nu_{N,t}}[\eta_x])\varphi\big(\tfrac{x-\lambda t N^{1+\kappa} m}{N}\big)\Big\}\nu_{N,t}(\eta)\Big].$$

Using the basic inequality $e^a \le 1 + a + (1/2)a^2 e^{|a|}$ and $\log(1+a) \le a$, there exists a finite constant C independent of N such that the above formula is bounded by

$$\frac{C}{\gamma N^{d-2\alpha}} \times N^d \times (\gamma N^{-\alpha})^2 = C\gamma.$$

Since γ could be taken arbitrarily small, the proof is completed. □

3.2. **Proof of Proposition 3.2.** It remains to prove Proposition 3.2. The first step is to properly decompose the term $\mathcal{R}_{N,t}$ defined in Eq. (3.6). For $\ell \ge 1$, let $\mathfrak{p}_\ell(\cdot)$ be the uniform measure on $\Lambda_\ell^d = \{0, 1, \dots, \ell-1\}^d$, *i.e.* $\mathfrak{p}_\ell(x) = \ell^{-d}$ if $x \in \Lambda_\ell^d$ and $= 0$ otherwise. Let $\mathfrak{q}_\ell = \mathfrak{p}_\ell * \mathfrak{p}_\ell$ be the convolution of $\mathfrak{p}_\ell$ with itself,

$$\mathfrak{q}_\ell(z) = \sum_{y\in\mathbb{T}_N^d} \mathfrak{p}_\ell(y)\mathfrak{p}_\ell(z-y), \quad z \in \mathbb{T}_N^d.$$

For $x \in \mathbb{T}_N^d$ and $\ell \ge 1$, the spatial average of ω_x in the box $\Lambda_{2\ell-1}^d$ is defined as

$$\omega_x^\ell = \sum_{z\in\mathbb{T}_N^d} \omega_{x+z}\mathfrak{q}_\ell(z).$$

Define

$$\mathcal{R}_{N,t}^\ell = \sum_{\eta\in\Omega_N^d} \sum_{i=1}^d \sum_{x\in\mathbb{T}_N^d} a_{i,x}^N \omega_x \omega_{x+e_i}^\ell \mu_{N,t}(\eta). \tag{3.11}$$

Using the definition of flows introduced in Subsection A.2 and summation by parts formula,

$$\begin{aligned}
\mathcal{R}_{N,t} - \mathcal{R}_{N,t}^\ell &= \sum_{\eta\in\Omega_N^d} \sum_{i=1}^d \sum_{x,z\in\mathbb{T}_N^d} a_{i,x}^N \omega_x \omega_{x+e_i+z}(\delta_0(z) - \mathfrak{q}_\ell(z))\mu_{N,t}(\eta) \\
&= \sum_{\eta\in\Omega_N^d} \sum_{i,j=1}^d \sum_{x,z\in\mathbb{T}_N^d} a_{i,x}^N \omega_x \omega_{x+e_i+z}(\phi_\ell(z,e_j) - \phi_\ell(z-e_j,e_j))\mu_{N,t}(\eta) \\
&= \sum_{\eta\in\Omega_N^d} \sum_{i,j=1}^d \sum_{x,z\in\mathbb{T}_N^d} a_{i,x}^N \omega_x (\omega_{x+e_i+z} - \omega_{x+e_i+e_j+z})\phi_\ell(z,e_j)\mu_{N,t}(\eta).
\end{aligned}$$

Make the change of variables $x \mapsto x - z - e_i$, and put

$$h_{j,x}^\ell = \sum_{i=1}^d \sum_{z\in\mathbb{T}_N^d} a_{i,x-z-e_i}^N \omega_{x-z-e_i}\phi_\ell(z,e_j), \tag{3.12}$$

then we have

$$\mathcal{R}_{N,t} - \mathcal{R}_{N,t}^\ell = \sum_{\eta\in\Omega_N^d} \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} h_{j,x}^\ell (\omega_x - \omega_{x+e_j})\,\mu_{N,t}(\eta).$$

To further decompose the term on the right-hand side of the last line, we introduce the following integration by parts formula.

Lemma 3.3 (Integration by parts formula)**.** *Let $h : \Omega_N^d \to \mathbb{R}$ be such that h does not depend on the values of η_x or η_z for some $x, z \in \mathbb{T}_N^d$. Then, for any $\nu_{N,t}$–density f,*

$$\sum_{\eta\in\Omega_N^d} h(\eta)\big[\omega_z-\omega_x\big]f(\eta)\nu_{N,t}(\eta) = \frac{1}{K}\sum_{\eta\in\Omega_N^d} h(\eta)s_{x,z}^N(\eta)\big(f(\eta^{x,z})-f(\eta)\big)\,\nu_{N,t}(\eta) \\ -(\varrho_z-\varrho_x)\sum_{\eta\in\Omega_N^d} h(\eta)f(\eta)\omega_z\omega_x\nu_{N,t}(\eta),$$

where

$$s_{x,z}^N(\eta) = \frac{\eta_x(K-\eta_z)}{\varrho_x(K-\varrho_z)}.$$

Proof. For $K = 1$, the result is proved in [10, Lemma E.1], and we extend it to general K. Since h does not depend on the values of η_x and η_z, using the change of variables $\eta \mapsto \eta^{z,x}$, we may rewrite the second term on the right-hand side of the above equation as

$$\frac{1}{K}\sum_{\eta\in\Omega_N^d} h(\eta)\Big(\frac{s_{x,z}^N(\eta^{z,x})\nu_{N,t}(\eta^{z,x})}{\nu_{N,t}(\eta)} - s_{x,z}^N(\eta)\Big)f(\eta)\nu_{N,t}(\eta).$$

Direct calculations show that

$$\frac{s_{x,z}^N(\eta^{z,x})\nu_{N,t}(\eta^{z,x})}{\nu_{N,t}(\eta)} = \frac{\eta_z(K-\eta_x)}{\varrho_z(K-\varrho_x)} = s_{z,x}^N(\eta).$$

We conclude the proof by Eq. (3.4). □

Since ϕ_ℓ is supported in $\Lambda_{2\ell-1}^d$, the value of $h_{j,x}^\ell$ does not depend on those of η_x and η_{x+e_j} for $1 \le j \le d$. Recall $\mu_{N,t}(\eta) = f_{N,t}(\eta)\nu_{N,t}(\eta)$. By Lemma 3.3, we have

$$\mathcal{R}_{N,t} - \mathcal{R}_{N,t}^\ell = \frac{1}{K}\sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d}\sum_{\eta\in\Omega_N^d} h_{j,x}^\ell s_{x+e_j,x}^N\big(f_{N,t}(\eta^{x+e_j,x})-f_{N,t}(\eta)\big)\,\nu_{N,t}(\eta) \\ -\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}(\varrho_x-\varrho_{x+e_j})\sum_{\eta\in\Omega_N^d} h_{j,x}^\ell\omega_x\omega_{x+e_j}f_{N,t}\nu_{N,t}(\eta). \quad (3.13)$$

By Cauchy-Schwarz inequality, for any $\gamma > 0$, the first term on the right hand side of the last equation is bounded by

$$\frac{\gamma}{2}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\sum_{\eta\in\Omega_N^d} s_{x+e_j,x}^N\Big[\sqrt{f_{N,t}(\eta^{x+e_j,x})}-\sqrt{f_{N,t}(\eta)}\Big]^2\nu_{N,t}(\eta) \\ +\frac{1}{2K^2\gamma}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\sum_{\eta\in\Omega_N^d}(h_{j,x}^\ell)^2 s_{x+e_j,x}^N\Big[\sqrt{f_{N,t}(\eta^{x+e_j,x})}+\sqrt{f_{N,t}(\eta)}\Big]^2\nu_{N,t}(\eta). \quad (3.14)$$

Since $\varrho_* \in (0, K)$, for N large enough, there exists a constant $C = C(\varrho_*, K) > 0$ such that $S_{x,z}^N \le C$ for any $x, z \in \mathbb{T}_N^d$. Then, by (3.2), the first term in (3.14) is bounded by $C_1\gamma D_N(f_{N,t};\nu_{N,t})$ for some $C_1 = C_1(\varrho_*, K, \delta_0)$. Since for any $\eta \in \Omega_N^d$ and any $x, z \in \mathbb{T}_N^d$,

$$\frac{\nu_{N,t}(\eta^{x,z})}{\nu_{N,t}(\eta)} = \frac{\eta_x(K-\eta_z)}{(K+1-\eta_z)(\eta_z+1))}\frac{\varrho_z(K-\varrho_x)}{\varrho_x(K-\varrho_z)} \le C := C(\varrho_*, K),$$

by Cauchy-Schwarz inequality and exchange of variables $\eta \mapsto \eta^{x,x+e_j}$, the second term in (3.14) is bounded from above by

$$\frac{C}{\gamma}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}_N^d}\sum_{\eta\in\Omega_N^d}(h_{j,x}^{\ell})^2 f_{N,t}(\eta)\nu_{N,t}(\eta).$$

Take $\gamma = N^{1+\kappa}/C_1$, together with (3.13), then we have shown that

$$\begin{aligned}\mathcal{R}_{N,t} \le N^{1+\kappa}D_N(f_{N,t};\nu_{N,t}) + \mathcal{R}_{N,t}^{\ell} + \frac{C}{N^{1+\kappa}}\sum_{j=1}^{d}\sum_{x\in\mathbb{T}_N^d}\sum_{\eta\in\Omega_N^d}(h_{j,x}^{\ell})^2 f_{N,t}(\eta)\,\nu_{N,t}(\eta)\\ +\sum_{j=1}^{d}\sum_{x\in\mathbb{T}_N^d}(\varrho_{x+e_j}-\varrho_x)\sum_{\eta\in\Omega_N^d}h_{j,x}^{\ell}\omega_x\omega_{x+e_j}f_{N,t}(\eta)\nu_{N,t}(\eta) \quad (3.15)\end{aligned}$$

for some $C = C(\varrho_*, K, \delta_0)$.

Next, we shall deal with the last three terms on the right-hand side of the above decomposition (3.15) respectively. We first deal with the term $\mathcal{R}_{N,t}^{\ell}$ defined in (3.11). Observe that

$$\begin{aligned}\sum_{i=1}^{d}\sum_{x\in\mathbb{T}_N^d}a_{i,x}^N\omega_x\omega_{x+e_i}^{\ell} &= \sum_{i=1}^{d}\sum_{x,y,z\in\mathbb{T}_N^d}a_{i,x}^N\omega_x\omega_{x+e_i+z}\mathfrak{p}_\ell(y)\mathfrak{p}_\ell(z-y)\\ &= \sum_{i=1}^{d}\sum_{x\in\mathbb{T}_N^d}\Big(\sum_{y\in\mathbb{T}_N^d}a_{i,x-y}^N\omega_{x-y}\mathfrak{p}_\ell(y)\Big)\Big(\sum_{z\in\mathbb{T}_N^d}\omega_{x+e_i+z}\mathfrak{p}_\ell(z)\Big).\end{aligned}$$

In the last identity, we reindex x by $x-y$ and z by $z+y$. By entropy inequality, for any $\gamma > 0$,

$$\begin{aligned}\mathcal{R}_{N,t}^{\ell} \le \frac{1}{\gamma}\Bigg(H_N(t)\\ +\log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{\gamma\sum_{i=1}^{d}\sum_{x\in\mathbb{T}_N^d}\Big(\sum_{y\in\mathbb{T}_N^d}a_{i,x-y}^N\omega_{x-y}\mathfrak{p}_\ell(y)\Big)\Big(\sum_{z\in\mathbb{T}_N^d}\omega_{x+e_i+z}\mathfrak{p}_\ell(z)\Big)\Big\}\nu_{N,t}(\eta)\Big]\Bigg). \quad (3.16)\end{aligned}$$

Note that for ℓ large enough, the two random variables

$$\begin{aligned}&\Big(\sum_{y\in\mathbb{T}_N^d}a_{i,x'-y}^N\omega_{x'-y}\mathfrak{p}_\ell(y)\Big)\Big(\sum_{z\in\mathbb{T}_N^d}\omega_{x'+e_i+z}\mathfrak{p}_\ell(z)\Big),\\ &\Big(\sum_{y\in\mathbb{T}_N^d}a_{i,x''-y}^N\omega_{x''-y}\mathfrak{p}_\ell(y)\Big)\Big(\sum_{z\in\mathbb{T}_N^d}\omega_{x''+e_i+z}\mathfrak{p}_\ell(z)\Big)\end{aligned}$$

are independent with respect to $\nu_{N,t}$ if $||x'-x''||_\infty > 3\ell$. Denote

$$x = z \pmod{\ell}$$

if $x_i = z_i + \ell k_i$ for some $k_i \in \mathbb{Z}$ and for all $1 \le i \le d$. Note that for any $x \in \mathbb{T}_N^d$, there exists a point $z \in \Lambda_{3\ell-1}^d$ such that $x = z \pmod{3\ell}$, and for fixed z, if $x', x'' = z \pmod{3\ell}$ and $x' \ne x''$, then $||x'-x''||_\infty \ge 3\ell$. By Hölder's inequality and independence, the second term in (3.16) is bounded by

$$\frac{1}{d(3\ell)^d}\sum_{i=1}^{d}\sum_{z\in\Lambda_{3\ell-1}^d}$$

$$\log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{\gamma d(3\ell)^d\sum_{x:x=z\ (\mathrm{mod}\ 3\ell)}\Big(\sum_{y\in\mathbb{T}_N^d}a_{i,x-y}^N\omega_{x-y}\mathfrak{p}_\ell(y)\Big)\Big(\sum_{z\in\mathbb{T}_N^d}\omega_{x+e_i+z}\mathfrak{p}_\ell(z)\Big)\Big\}\nu_{N,t}(\eta)\Big]$$

$$\le\frac{1}{d(3\ell)^d}\sum_{i=1}^d\sum_{x\in\mathbb{T}_N^d}$$

$$\log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{\gamma d(3\ell)^d\Big(\sum_{y\in\mathbb{T}_N^d}a_{i,x-y}^N\omega_{x-y}\mathfrak{p}_\ell(y)\Big)\Big(\sum_{z\in\mathbb{T}_N^d}\omega_{x+e_i+z}\mathfrak{p}_\ell(z)\Big)\Big\}\nu_{N,t}(\eta)\Big]. \quad (3.17)$$

We claim that the for each $x\in\mathbb{T}_N^d$, the random variable

$$\sum_{y\in\mathbb{T}_N^d}a_{i,x-y}^N\omega_{x-y}\mathfrak{p}_\ell(y)$$

is sub-Gaussian of order $C_2N^{2\kappa-2\alpha}\ell^{-d}$ with respect to the measure $\nu_{N,t}$ for some constant $C_2=C_2(\varrho_*,K,\partial_{u_i}\rho)$. Indeed, first note that by the definition of $a_{i,x}^N$ in (3.7), there exists a constant $C=C(\varrho_*,K)$ such that

$$|a_{i,x}^N|\le C||\partial_{u_i}\rho||_\infty N^{\kappa-\alpha}.$$

Since $\omega_x\le C(\varrho_*,K)$ for N large enough, and has mean zero with respect to $\nu_{N,t}$, by Lemma A.5, the random variable $a_{i,x}^N\omega_x$ is sub-Gaussian of order $CN^{2\kappa-2\alpha}$ for some constant $C=C(\varrho_*,K,\partial_{u_i}\rho)$ with respect to $\nu_{N,t}$. Therefore, for any $\theta\in\mathbb{R}$,

$$\log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{\theta\sum_{y\in\mathbb{T}_N^d}a_{i,x-y}^N\omega_{x-y}\mathfrak{p}_\ell(y)\Big\}\nu_{N,t}(\eta)\Big]$$

$$=\sum_{y\in\mathbb{T}_N^d}\log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{\theta a_{i,x-y}^N\omega_{x-y}\mathfrak{p}_\ell(y)\Big\}\nu_{N,t}(\eta)\Big]$$

$$\le\frac{1}{2}\theta^2\sum_{y\in\mathbb{T}_N^d}C^2N^{2\kappa-2\alpha}\mathfrak{p}_\ell(y)^2=\frac{C_2}{2}\theta^2N^{2\kappa-2\alpha}\ell^{-d}.$$

Similarly, one could prove that $\sum_{z\in\mathbb{T}_N^d}\omega_{x+e_i+z}\mathfrak{p}_\ell(z)$ is sub-Gaussian of order $C_2\ell^{-d}$. By Corollary A.4, taking $\gamma=(4C_2d3^d)^{-1}N^{\alpha-\kappa}$, we bound the term in (3.17) by $N^d\log 3/(3\ell)^d$. Therefore, there exists a constant C independent of N such that

$$\mathcal{R}_{N,t}^\ell\le\frac{C}{N^{\alpha-\kappa}}\Big(H_N(t)+\frac{N^d}{\ell^d}\Big). \quad (3.18)$$

The third term in (3.15) is treated in the same way as above. Using Lemma A.2, there exists a constant $C=C(\rho_*,K)$ such that the random variable $h_{j,x}^\ell$ defined in (3.12) is sub-Gaussian of order

$$CN^{2\kappa-2\alpha}\sum_{z\in\mathbb{T}_N^d}\phi_\ell(z,e_j)^2\le C_3N^{2\kappa-2\alpha}g_d(\ell)$$

with respect to the measure $\nu_{N,t}$ for some constant $C_3=C_3(\varrho_*,K,C_0)$. Also note that the two random variables $h_{j,x'}^\ell$ and $h_{j,x''}^\ell$ are independent under the measure $\nu_{N,t}$ if $||x'-x''||_\infty>3\ell$. By entropy inequality and Hölder's inequality, for any $\gamma>0$,

$$\frac{1}{N^{1+\kappa}}\sum_{\eta\in\Omega_N^d}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}(h_{j,x}^\ell)^2f_{N,t}(\eta)\,\nu_{N,t}(\eta)$$

$$\leq \frac{1}{\gamma N^{1+\kappa}}\Big(H_N(t)+\log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{\gamma\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}(h_{j,x}^\ell)^2\Big\}\nu_{N,t}(\eta)\Big]\Big)$$

$$\leq \frac{1}{\gamma N^{1+\kappa}}\Big(H_N(t)+\frac{1}{d(3\ell)^d}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{\gamma d(3\ell)^d(h_{j,x}^\ell)^2\Big\}\nu_{N,t}(\eta)\Big]\Big). \quad (3.19)$$

Take $\gamma=\big[4C_3d3^dN^{2\kappa-2\alpha}\ell^dg_d(\ell)\big]^{-1}$, then by Lemma A.3, there exists a finite constant C independent of N such that the third term in (3.15) is bounded by

$$\frac{C\ell^dg_d(\ell)}{N^{1+2\alpha-\kappa}}\Big(H_N(t)+\frac{N^d}{\ell^d}\Big). \quad (3.20)$$

Now we deal with the last term in (3.15). First note that

$$|\varrho_{x+e_j}-\varrho_x|\leq N^{-1-\alpha}||\partial_{u_j}\rho||_\infty.$$

By entropy inequality and Hölder's inequality, for any $\gamma>0$, we may bound the last term in (3.15) by

$$\frac{1}{\gamma N^{1+\alpha}}\Big(H_N(t)+\log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{\gamma\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}N^{1+\alpha}(\varrho_{x+e_j}-\varrho_x)h_{j,x}^\ell\omega_x\omega_{x+e_j}\Big\}\nu_{N,t}(\eta)\Big]\Big)$$

$$\leq \frac{1}{\gamma N^{1+\alpha}}\Big(H_N(t)$$

$$+\frac{1}{d(3\ell)^d}\sum_{j=1}^d\sum_{x\in\mathbb{T}_N^d}\log\Big[\sum_{\eta\in\Omega_N^d}\exp\Big\{\gamma d(3\ell)^dN^{1+\alpha}(\varrho_{x+e_j}-\varrho_x)h_{j,x}^\ell\omega_x\omega_{x+e_j}\Big\}\nu_{N,t}(\eta)\Big]\Big). \quad (3.21)$$

Since $|\omega_x\omega_{x+e_j}|\leq C:=C(\varrho_*,K)$, and we have already shown that $h_{j,x}^\ell$ is sub-Gaussian of order $C_3N^{2\kappa-2\alpha}g_d(\ell)$, then, for any $\theta\in\mathbb{R}$,

$$\log\Big[\sum_{\eta\in\Omega_N^d}\exp\{\theta h_{k,x}^\ell\omega_x\omega_{x+e_j}\}\nu_{N,t}(\eta)\Big]\leq\log\Big[\sum_{\eta\in\Omega_N^d}\exp\{C|\theta h_{k,x}^\ell|\}\nu_{N,t}(\eta)\Big]$$

$$\leq\log\Big[\sum_{\eta\in\Omega_N^d}\Big(\exp\{C\theta h_{j,x}^\ell\}+\exp\{-C\theta h_{j,x}^\ell\}\Big)\nu_{N,t}(\eta)\Big]\leq\log 2+C\theta^2N^{2\kappa-2\alpha}g_d(\ell) \quad (3.22)$$

for some constant $C=C(\varrho_*,K,C_3)$. Therefore, there exists some constant C independent of N, such that the term on the right hand side of (3.21) is bounded by

$$\frac{C}{\gamma N^{1+\alpha}}\Big[H_N(t)+\frac{N^d}{\ell^d}\times\ell^{2d}g_d(\ell)\gamma^2N^{2\kappa-2\alpha}+\frac{N^d}{\ell^d}\Big].$$

Take $\gamma=N^{-1-\alpha}$, then the last line is bounded by

$$C\Big(H_N(t)+\ell^dg_d(\ell)N^{d+2\kappa-4\alpha-2}+\frac{N^d}{\ell^d}\Big). \quad (3.23)$$

We conclude the proof of Proposition 3.2 by using (3.15), (3.18), (3.20) and (3.23).

4. 1-d chain of anharmonic oscillators

In this and the next sections, we consider a chain of N coupled oscillators in one-dimensional lattice space. All particles have identical mass 1. The momentum and position of the particle $x=1$, ..., N are denoted by $p_x\in\mathbb{R}$ and $q_x\in\mathbb{R}$, respectively. The interaction between two particles $x-1$ and x

is determined by an anharmonic spring with the potential energy $V(q_x - q_{x-1})$, where V is some nice function. The total energy is given by the Hamiltonian

$$\mathcal{H}_N(\eta) := \sum_{x\in\mathbb{T}_N} \frac{p_x^2}{2} + V(q_x - q_{x-1}), \quad \forall\, \eta \in \Omega_N.$$

The corresponding Hamiltonian dynamics then reads

$$\dot{p}_x = -\partial_{q_x}\mathcal{H}_N, \quad \dot{q}_x = \partial p_x \mathcal{H}_N, \quad \forall\, x \in \mathbb{T}_N.$$

Assume that $V \in C^2(\mathbb{R};\mathbb{R})$ and some constant $c > 0$, such that

$$c^{-1} \le V''(r) \le c, \quad \forall\, r \in \mathbb{R}.$$

Define $r_x := q_x - q_{x-1}$ to be the inter-particle distance and require the periodic boundary condition: $(p_{N+1}, r_{N+1}) = (p_1, r_1)$. The configuration space is then $\Omega_N = (\mathbb{R}^2)^{\mathbb{T}_N}$, with its elements denoted by $\eta = \{\eta_x = (p_x, r_x); x \in \mathbb{T}_N\}$.

Observe that the total momentum, the total volume and the Hamiltonian are conserved. Under a generic assumption of *local equilibrium*, Euler equations can be formally obtained as the evolution of these quantities. However, to prove it for the purely deterministic system turns out to be a difficult task. Proper stochastic noise helps us solve this problem. Suppose that at each bond $(x, x+1)$, the deterministic system is contact with a thermal bath at fixed temperature. More precisely, fix some inverse temperature $\beta > 0$ and define

$$\mathcal{Y}_x := \frac{\partial}{\partial r_{x+1}} - \frac{\partial}{\partial r_x}, \quad \mathcal{Y}_x^* = \beta\big(V'(r_{x+1}) - V'(r_x)\big) - \mathcal{Y}_x.$$

For some deterministic parameter $\gamma_N > 0$ that regulates the strength of the heat bath, consider the operator $\mathcal{L}_N$ given by

$$\mathcal{L}_N := \mathcal{A}_N + \gamma_N \mathcal{S}_N, \quad \mathcal{S}_N := -\frac{1}{2}\sum_{x\in\mathbb{T}_N} \mathcal{Y}_x^*\mathcal{Y}_x, \tag{4.1}$$

where $\mathcal{A}_N$ is the Liouville operator given by

$$\mathcal{A}_N := \sum_{x\in\mathbb{T}_N} (p_x - p_{x-1})\frac{\partial}{\partial r_x} + \big(V'(r_{x+1}) - V'(r_x)\big)\frac{\partial}{\partial p_x}.$$

The Markov process generated by $\mathcal{L}_N$ is equivalently expressed by the following system of stochastic differential equations: for each $x \in \mathbb{T}_N$,

$$\begin{cases} dp_x(t) = \big(V'(r_{x+1}) - V'(r_x)\big)dt, \\ dr_x(t) = (p_{x+1} - p_x)dt + \dfrac{\beta\gamma_N}{2}\big(V'(r_{x+1}) + V'(r_{x-1}) - 2V'(r_x)\big)dt \\ \qquad\qquad + \sqrt{\gamma_N}\big(dB_t^{x-1} - dB_t^x\big), \end{cases}$$

where $\{B_\cdot^x; x \ge 1\}$ is an infinite system of independent, standard Brownian motions. Notice that the total momentum $\sum_x p_x$ and the total length $\sum_x r_x$ are the only conserved quantities of the microscopic dynamics. The conservation law of energy is no longer preserved by $\mathcal{S}_N$.

4.1. **Stationary states.** The stationary states of $\mathcal{L}_N$ are given by the family of canonical Gibbs measures indexed by the global momentum $\bar{p} \in \mathbb{R}$ and tension $\tau \in \mathbb{R}$:

$$\nu_{\bar{p},\tau}^N(dp\,dr) = \bigotimes_{x\in\mathbb{T}_N} \sqrt{\frac{\beta}{2\pi}} \exp\left\{-\frac{\beta(p_x - \bar{p})^2}{2}\right\} dp_x \otimes \pi_\tau(dr_x),$$

where the probability measure π_τ is defined as

$$\pi_\tau(dr) := \frac{1}{Z(\tau)} e^{-\beta(V(r)-\tau r)} dr, \quad Z(\tau) := \int_{\mathbb{R}} e^{-\beta(V(r)-\tau r)} dr.$$

Observe that the dependence on β is omitted, since it is fixed hereafter. It is easy to see that $\mathcal{A}_N$, $\mathcal{S}_N$ are respectively anti-symmetric and symmetric with respect to the Gibbs states. Moreover, for all smooth functions f, g on Ω_N,

$$\int_{\Omega_N} f(\mathcal{S}_N g)\, d\nu_{\bar{p},\tau}^N = -\frac{1}{2} \int_{\Omega_N} \sum_{x\in\mathbb{T}_N} (\mathcal{Y}_x f)(\mathcal{Y}_x g) d\nu_{\bar{p},\tau}^N.$$

Define the Gibbs potential $G = G(\tau)$ for $\tau \in \mathbb{R}$ and the free energy $F = F(r)$ for $r \in \mathbb{R}$ by the following Legendre transform

$$G(\tau) := \frac{1}{\beta} \log Z(\tau), \quad F(r) := \sup_{\tau\in\mathbb{R}} \{\tau r - G(\tau)\}.$$

The average length $\bar{r} = \bar{r}(\tau)$ and equilibrium tension $\boldsymbol{\tau} = \boldsymbol{\tau}(r)$ are then given by the convex conjugate variables

$$\bar{r}(\tau) := E_{\pi_\tau}[r] = G'(\tau), \quad \boldsymbol{\tau}(r) := F'(r).$$

4.2. **Equilibrium perturbation.** As illustrated in Section 1, we fix $(\mathfrak{p}_*, \mathfrak{r}_*) \in \mathbb{R}^2$ and consider the distribution $\mu_{N,0}$ associated to the profile $(\mathfrak{p}_N^{\mathrm{ini}}, \mathfrak{r}_N^{\mathrm{ini}})$ given by

$$\begin{pmatrix} \mathfrak{p}_N^{\mathrm{ini}} \\ \mathfrak{r}_N^{\mathrm{ini}} \end{pmatrix} := \begin{pmatrix} \mathfrak{p}_* \\ \mathfrak{r}_* \end{pmatrix} + N^{-\alpha} \sum_{j=\pm} \sigma_j^{\mathrm{ini}} \boldsymbol{v}_j, \quad \boldsymbol{v}_\pm := \begin{pmatrix} \pm\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)} \\ 1 \end{pmatrix},$$

where $\alpha > 0$ and $\sigma_\pm^{\mathrm{ini}} \in C^\infty(\mathbb{T})$. In addition, we require that (*cf.* (1.7))

$$\int_{\mathbb{T}} \sigma_-(u) du = \int_{\mathbb{T}} \sigma_+(u) du = 0.$$

For $0 < \kappa \le \alpha$, denote by $\{\eta(t); t \ge 0\}$ the Markov process generated by $N^{1+\kappa}\mathcal{L}_N$ and the initial distribution $\mu_{N,0}$. As usual, we use the notation $\mu_{N,t}$ for the distribution of $\eta(t)$ on Ω_N.

For $(t,u) \in [0,T] \times \mathbb{T}$, define $(\mathfrak{p},\mathfrak{r}) = (\mathfrak{p},\mathfrak{r})(t,u)$ by

$$\begin{pmatrix} \mathfrak{p}_N \\ \mathfrak{r}_N \end{pmatrix} := \begin{pmatrix} \mathfrak{p}_* \\ \mathfrak{r}_* \end{pmatrix} + N^{-\alpha} \sum_{j=\pm} \sigma_j\big(N^{\kappa-\alpha}t, u + jN^\kappa \sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)} t\big) \boldsymbol{v}_j,$$

where σ_-, σ_+ solve the decoupled system of Burgers equations (1.12). Denote by $\nu_{N,t}$ the slowly varying product measure

$$\nu_{N,t}(d\eta) = \bigotimes_{x\in\mathbb{T}_N^d} \nu^1_{\mathfrak{p}_x^N, \boldsymbol{\tau}_x^N}(d\eta_x), \quad \big(\mathfrak{p}_x^N, \boldsymbol{\tau}_x^N\big) := \big(\mathfrak{p}_N, \boldsymbol{\tau}(\mathfrak{r}_N)\big)\left(t, \frac{x}{N}\right).$$

Let $f_{N,t}$ be the Radon–Nikodym derivative $d\mu_{N,t}/d\nu_{N,t}$ and recall the relative entropy

$$H_N(t) = H(f_{N,t}; \nu_{N,t}) := \int_{\Omega_N} f_{N,t} \log f_{N,t} d\nu_{N,t}. \tag{4.2}$$

Recall that our argument relies on the smoothness of $\sigma_\pm$, hence we require that $t \in [0,T]$ for (i) any $T < T_{\mathrm{shock}}$, the first time when shock appears in the entropy solution to (1.12) if $\kappa = \alpha$ and (ii) any $T > 0$ if $\kappa < \alpha$. The first result is stated below, *cf.* the case $d = 1$ in Theorem 2.1.

Theorem 4.1. *Suppose that $H_N(0) = o(N^{1-2\alpha})$ and $\alpha \in (0, 1/2)$. If $\kappa \in (0,\alpha] \cap (0, (1-2\alpha)/3)$ and $N^{5\kappa+4\alpha-1} \ll \gamma_N \ll N^{1-\kappa}$, then $H_N(t) = o(N^{1-2\alpha})$ for any $t \in [0,T]$.*

Theorem 4.1 is proved in Section 5. With Theorem 4.1 and exactly the same argument used in the proof of Theorem 2.3, we obtain the equilibrium perturbation.

Theorem 4.2. *Under the assumption of Theorem 4.1,*

$$\lim_{N\to\infty}\mu_{N,t}\Big\{\Big|\frac{1}{N^{1-\alpha}}\sum_{x\in\mathbb{T}_N}\Big(\frac{r_x-\mathfrak{r}_*}{2}-\frac{p_x-\mathfrak{p}_*}{2\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}}\Big)\varphi\Big(\frac{x}{N}-N^\kappa\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}t\Big) - \int_{\mathbb{T}_N}\sigma_-^{(\alpha,\kappa)}(t,u)\varphi(u)du\Big|>\varepsilon\Big\}=0,$$

$$\lim_{N\to\infty}\mu_{N,t}\Big\{\Big|\frac{1}{N^{1-\alpha}}\sum_{x\in\mathbb{T}_N}\Big(\frac{r_x-\mathfrak{r}_*}{2}+\frac{p_x-\mathfrak{p}_*}{2\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}}\Big)\varphi\Big(\frac{x}{N}+N^\kappa\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}t\Big) - \int_{\mathbb{T}_N}\sigma_+^{(\alpha,\kappa)}(t,u)\varphi(u)du\Big|>\varepsilon\Big\}=0,$$

for any $t\in[0,T]$, $\varphi\in C(\mathbb{T})$ and $\varepsilon>0$, where

$$\sigma_\pm^{(\alpha,\kappa)}(t,u):=\begin{cases}\sigma_\pm(t,u), & \text{if } 0<\alpha<\frac15,\ \kappa=\alpha,\\ \sigma_\pm^{\mathrm{ini}}(u), & \text{if } 0<\alpha<\frac12,\ 0<\kappa<\min\{\alpha,\frac{1-2\alpha}{3}\}.\end{cases}$$

5. Relative entropy for the oscillator chain

We have seen in (1.6) that, for a system of 2 conservation laws, the non-resonant system of perturbations (1.12) requires proper second order correction terms. Hence, we choose the modified profile $(\tilde{\mathfrak{p}}_N,\tilde{\mathfrak{r}}_N)(t,u)$ given by

$$\begin{pmatrix}\mathfrak{p}_N\\ \mathfrak{r}_N\end{pmatrix}+N^{-2\alpha}\sum_{j=\pm}\tilde\sigma_j\big(N^{\kappa-\alpha}t,u-N^\kappa\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}t,u+N^\kappa\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}t\big)\boldsymbol{\upsilon}_j,$$

where for $(t,u_-,u_+)\in[0,T]\times\mathbb{R}^2$,

$$\begin{aligned}\tilde\sigma_-(t,u_-,u_+)&:=-\frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{8\boldsymbol{\tau}'(\mathfrak{r}_*)}\sigma_+^2(t,u_+)\\ &\quad-\frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{4\boldsymbol{\tau}'(\mathfrak{r}_*)}\big[\partial_u\sigma_-(t,u_-)\Sigma_+(t,u_+)+\sigma_-(t,u_-)\sigma_+(t,u_+)\big],\\ \tilde\sigma_+(t,u_-,u_+)&:=-\frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{8\boldsymbol{\tau}'(\mathfrak{r}_*)}\sigma_-^2(t,u_-)\\ &\quad-\frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{4\boldsymbol{\tau}'(\mathfrak{r}_*)}\big[\sigma_-(t,u_-)\sigma_+(t,u_+)+\Sigma_-(t,u_-)\partial_u\sigma_+(t,u_+)\big].\end{aligned}\tag{5.1}$$

Here $\Sigma_\pm=\int_0^\cdot\sigma_\pm(t,u)du$ are the primitive functions of $\sigma_\pm$.

Let $\tilde\nu_{N,t}$ be the product measure on Ω_N associated to the profile $(\tilde{\mathfrak{p}},\tilde{\mathfrak{r}})(t,\cdot)$. Recall that $\mu_{N,t}$ is the distribution of the dynamics $\eta(t)$. Denote by $\tilde f_{N,t}$ Radon–Nikodym derivative of $\mu_{N,t}$ with respect to $\tilde\nu_{N,t}$ and by $\widetilde H_N(t)$ the corresponding relative entropy. Also recall the relative entropy $H_N(t)$ in (4.2). The following lemma is straightforward.

Lemma 5.1. *For any $t\ge0$, $H_N(t)=o(N^{1-2\alpha})$ if and only if $\widetilde H_N(t)=o(N^{1-2\alpha})$.*

Proof. Assume first that $\widetilde H_N(t)=o(N^{1-2\alpha})$. By the definition of relative entropy,

$$H_N(t)-\widetilde H_N(t)=\int_{\Omega_N}g_{N,t}\,d\mu_{N,t},\quad g_{N,t}:=\log\Big(\frac{d\tilde\nu_{N,t}}{d\nu_{N,t}}\Big).$$

Direct calculations show that $g_{N,t}(\eta)$ is equal to

$$\beta \sum_{x\in\mathbb{T}_N} \Big[(\tilde{\mathfrak{p}}_x^N - \mathfrak{p}_x^N)(p_x - \tilde{\mathfrak{p}}_x^N) + (\tilde{\boldsymbol{\tau}}_x^N - \boldsymbol{\tau}_x^N)(r_x - \tilde{\mathfrak{r}}_x^N)\Big]$$
$$-\beta \sum_{x\in\mathbb{T}_N} \left[\frac{(\tilde{\mathfrak{p}}_x^N - \mathfrak{p}_x^N)^2}{2} + G(\tilde{\boldsymbol{\tau}}_x^N) - G(\boldsymbol{\tau}_x^N) - \tilde{\mathfrak{r}}_x^N(\tilde{\boldsymbol{\tau}}_x^N - \boldsymbol{\tau}_x^N)\right]$$

for each $\eta = (p_x, r_x)_{x\in\mathbb{T}_N} \in \Omega_N$. The second line is bounded from above by $\mathcal{O}(N^{1-4\alpha})$. Meanwhile, by the entropy inequality, the integral of the first line is bounded from above by

$$\widetilde{H}_N(t) + \sum_{x\in\mathbb{T}_N} \log \int_{\Omega_N} \exp\left\{\beta \begin{pmatrix} \tilde{\mathfrak{p}}_x^N - \mathfrak{p}_x^N \\ \tilde{\boldsymbol{\tau}}_x^N - \boldsymbol{\tau}_x^N \end{pmatrix} \cdot \begin{pmatrix} p_x - \tilde{\mathfrak{p}}_x^N \\ r_x - \tilde{\mathfrak{r}}_x^N \end{pmatrix}\right\} d\tilde{\nu}_{N,t}.$$

Observe that the last term is bounded by $\mathcal{O}(N^{1-4\alpha})$ and hence $H_N(t) \le 2\widetilde{H}_N(t) + CN^{1-4\alpha} = o(N^{1-2\alpha})$. The inverse assertion follows similarly. □

Let $\psi_{N,t}$ be the density function of $\tilde{\nu}_{N,t}$ with respect to some fixed reference measure $\nu^N_{\bar{p},\tau}$. Without loss of generality, we can choose $(\bar{p},\tau) = (0,0)$. Define the Dirichlet form (*cf.* (3.1))

$$D_N\big(\tilde{f}_{N,t}; \nu_{N,t}\big) := \int_{\Omega_N} \sum_{x\in\mathbb{T}_N} \Big(\mathcal{Y}_x \sqrt{\tilde{f}_{N,t}}\Big)^2 d\nu_{N,t}.$$

Standard manipulation gives

$$\frac{d}{dt}\widetilde{H}_N(t) = \int_{\Omega_N} \tilde{f}_{N,t}\left(N^{1+\kappa}\mathcal{L}_N \log \tilde{f}_{N,t} - \frac{d}{dt}\log\psi_{N,t}\right) d\tilde{\nu}_{N,t}.$$

Recalling the definition of $\mathcal{L}_N$ in (4.1), we have

$$\begin{aligned}\tilde{f}_{N,t}\mathcal{L}_N \log \tilde{f}_{N,t} &= \mathcal{L}_N \tilde{f}_{N,t} - \frac{\gamma_N}{2}\sum_{x\in\mathbb{T}_N} \tilde{f}_{N,t}^{-1}\big(\mathcal{Y}_x \tilde{f}_{N,t}\big)^2 \\ &= \mathcal{L}_N \tilde{f}_{N,t} - 2\gamma_N \sum_{x\in\mathbb{T}_N}\Big(\mathcal{Y}_x\sqrt{\tilde{f}_{N,t}}\Big)^2.\end{aligned}$$

Integrate it with respect to $\tilde{\nu}_{N,t}$ and notice that

$$\begin{aligned}\int_{\Omega_N} \mathcal{A}_N \tilde{f}_{N,t} d\tilde{\nu}_{N,t} &= -\int_{\Omega_N} \tilde{f}_{N,t}\mathcal{A}_N\psi_{N,t}\frac{d\tilde{\nu}_{N,t}}{\psi_{N,t}} = -\int_{\Omega_N} \tilde{f}_{N,t}\mathcal{A}_N \log\psi_{N,t}\, d\tilde{\nu}_{N,t}, \\ \int_{\Omega_N} \mathcal{S}_N \tilde{f}_{N,t} d\tilde{\nu}_{N,t} &= -\frac{1}{2}\sum_{x\in\mathbb{T}_N}\int_{\Omega_N} \mathcal{Y}_x\tilde{f}_{N,t}\cdot\mathcal{Y}_x\psi_{N,t}\frac{d\tilde{\nu}_{N,t}}{\psi_{N,t}} \\ &\le D_N\big(\tilde{f}_{N,t};\tilde{\nu}_{N,t}\big) + \frac{1}{4}\int_{\Omega_N}\tilde{f}_{N,t}\sum_{x\in\mathbb{T}_N}\big(\mathcal{Y}_x\log\psi_{N,t}\big)^2 d\tilde{\nu}_{N,t},\end{aligned}$$

where the last estimate follows from Cauchy–Schwarz inequality. Hence,

$$\begin{aligned}\frac{d}{dt}\widetilde{H}_N(t) \le -\gamma_N N^{1+\kappa} D\big(\tilde{f}_{N,t};\tilde{\nu}_{N,t}\big) + \int_{\Omega_N}\tilde{f}_{N,t}J_{N,t}\, d\tilde{\nu}_{N,t} \\ + \frac{\gamma_N N^{1+\kappa}}{4}\int_{\Omega_N}\tilde{f}_{N,t}\sum_{x\in\mathbb{T}_N}\big(\mathcal{Y}_x\log\psi_{N,t}\big)^2 d\tilde{\nu}_{N,t},\end{aligned}$$

where $J_{N,t} := (-N^{1+\kappa}\mathcal{A}_N - d/dt)\log\psi_{N,t}$.

With the choice $(\bar{p},\tau)=(0,0)$, $\log\psi_{N,t}=\log(d\tilde{\nu}_{N,t}/\nu^N_{0,0})$ reads

$$\log\psi_{N,t}=\beta\sum_{x\in\mathbb{T}_N}\left[\tilde{\mathfrak{p}}^N_x p_x-\frac{(\tilde{\mathfrak{p}}^N_x)^2}{2}+\tilde{\boldsymbol{\tau}}^N_x r_x-G\big(\tilde{\boldsymbol{\tau}}^N_x\big)+G(0)\right].$$

Elementary computation then shows that

$$\sum_{x\in\mathbb{T}_N}\big(\mathcal{Y}_x\log\psi_{N,t}\big)^2=\beta^2\sum_{x\in\mathbb{T}_N}\big(\tilde{\boldsymbol{\tau}}^N_{x+1}-\tilde{\boldsymbol{\tau}}^N_x\big)^2,$$

$$-\mathcal{A}_N\log\psi_{N,t}=\beta\sum_{x\in\mathbb{T}_N}\begin{pmatrix}\tilde{\boldsymbol{\tau}}^N_{x+1}-\tilde{\boldsymbol{\tau}}^N_x\\ \tilde{\mathfrak{p}}^N_x-\tilde{\mathfrak{p}}^N_{x-1}\end{pmatrix}\cdot\begin{pmatrix}p_x-\tilde{\mathfrak{p}}^N_x\\ V'(r_x)-\tilde{\boldsymbol{\tau}}^N_x\end{pmatrix},$$

$$-\frac{d}{dt}\log\psi_{N,t}=-\beta\sum_{x\in\mathbb{T}_N}\frac{d}{dt}\begin{pmatrix}\tilde{\mathfrak{p}}^N_x\\ \tilde{\mathfrak{r}}^N_x\end{pmatrix}\cdot\begin{pmatrix}p_x-\tilde{\mathfrak{p}}^N_x\\ \boldsymbol{\tau}'(\tilde{\mathfrak{r}}^N_x)(r_x-\tilde{\mathfrak{r}}^N_x)\end{pmatrix}.$$

Therefore, we finally obtain the following inequality, *cf.* (3.5):

$$\frac{d}{dt}\widetilde{H}_N(t)\le-\gamma_N N^{1+\kappa}D\big(\tilde{f}_{N,t};\tilde{\nu}_{N,t}\big)+\beta\big(\mathcal{R}_{N,t}+\mathcal{E}_{N,t}\big),\tag{5.2}$$

where

$$\mathcal{R}_{N,t}:=\int_{\Omega_N}\tilde{f}_{N,t}\sum_{x\in\mathbb{T}_N}\frac{d\tilde{\mathfrak{r}}^N_x}{dt}\big[V'(r_x)-\tilde{\boldsymbol{\tau}}^N_x-\boldsymbol{\tau}'(\tilde{\mathfrak{r}}^N_x)(r_x-\tilde{\mathfrak{r}}^N_x)\big]d\tilde{\nu}_{N,t};$$

$$\mathcal{E}_{N,t}:=\int_{\Omega_N}\tilde{f}_{N,t}\sum_{x\in\mathbb{T}_N}\left[\frac{\gamma_N N^{1+\kappa}\beta}{4}\big(\nabla\tilde{\boldsymbol{\tau}}^N_x\big)^2+\epsilon^N_x\cdot\begin{pmatrix}p_x-\tilde{\mathfrak{p}}^N_x\\ V'(r_x)-\tilde{\boldsymbol{\tau}}^N_x\end{pmatrix}\right]d\tilde{\nu}_{N,t}.$$

Here we use the abbreviation $\nabla f_x=f_{x+1}-f_x$, $\nabla^* f_x=f_{x-1}-f_x$ and define

$$\epsilon^N_x:=N^{1+\kappa}\begin{pmatrix}\nabla\tilde{\boldsymbol{\tau}}^N_x\\ \nabla\tilde{\mathfrak{p}}^N_{x-1}\end{pmatrix}-\frac{d}{dt}\begin{pmatrix}\tilde{\mathfrak{p}}^N_x\\ \tilde{\mathfrak{r}}^N_x\end{pmatrix}.$$

5.1. **Proofs of Theorem 4.1 and 4.2.** Theorem 4.1 follows from the inequality (5.2), Propositions 5.2 and 5.4 below.

Proposition 5.2. *There exists a constant C independent of N, such that*

$$\mathcal{E}_{N,t}\le\widetilde{H}_N(t)+C\big(\gamma_N N^{\kappa-2\alpha}+N^{-1+2\kappa-2\alpha}+N^{1+2\kappa-6\alpha}\big).$$

Proof. Recall that the profile $(\tilde{\mathfrak{p}}_N,\tilde{\mathfrak{r}}_N)$ is explicitly given by

$$\begin{pmatrix}\tilde{\mathfrak{p}}_N\\ \tilde{\mathfrak{r}}_N\end{pmatrix}(t,u)=\begin{pmatrix}\mathfrak{p}_*\\ \mathfrak{r}_*\end{pmatrix}+\sum_{j=\pm}\left[\frac{\sigma_j(s,u_j)}{N^\alpha}+\frac{\tilde{\sigma}_j(s,u_-,u_+)}{N^{2\alpha}}\right]\boldsymbol{v}_j,$$

where $\sigma_\pm=\sigma_\pm(s,u)$, $\tilde{\sigma}_\pm=\tilde{\sigma}(s,u_1,u_2)$ are smooth functions given respectively by (1.12) and (5.1), $s=N^{\kappa-\alpha}t$, $\boldsymbol{v}_-=(-\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)},1)'$, $\boldsymbol{v}_+=(\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)},1)'$ and

$$u_-=u-N^\kappa\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}t,\quad u_+=u+N^\kappa\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}t.$$

For the first term in $\mathcal{E}_{N,t}$, since $\nabla\tilde{\boldsymbol{\tau}}^N_x\le N^{-1}\sup_x|\partial_x\boldsymbol{\tau}(\tilde{\mathfrak{r}}(t,x))|\le CN^{-1-\alpha}$,

$$\frac{\gamma_N N^{1+\kappa}\beta^2}{4}\int_{\Omega_N}\tilde{f}_{N,t}\sum_{x\in\mathbb{T}_N}\big(\nabla\tilde{\boldsymbol{\tau}}^N_x\big)^2d\tilde{\nu}_{N,t}\le C\gamma_N N^{\kappa-2\alpha}.$$

We focus on the second term in $\mathcal{E}_{N,t}$. By Taylor's expansion,

$$\begin{pmatrix}\nabla\tilde{\boldsymbol{\tau}}^N_x\\ \nabla\tilde{\mathfrak{p}}^N_{x-1}\end{pmatrix}=\frac{1}{N}\frac{\partial}{\partial u}\begin{pmatrix}\boldsymbol{\tau}(\tilde{\mathfrak{r}}_N)\\ \tilde{\mathfrak{p}}_N\end{pmatrix}\Big(t,\frac{x}{N}\Big)+\mathcal{O}\big(N^{-2-\alpha}\big).$$

Expanding the function $(\mathfrak{p},\mathfrak{r})\mapsto(\boldsymbol{\tau}(\mathfrak{r}),\mathfrak{p})$ at $(\mathfrak{p}_*,\mathfrak{r}_*)$ up to the second order,

$$\begin{pmatrix}\nabla\tilde{\boldsymbol{\tau}}_x^N\\ \nabla\tilde{\mathfrak{p}}_{x-1}^N\end{pmatrix}=A\sum_{j=\pm}\left[\frac{\partial_u\sigma_j(s,u_j)}{N^{1+\alpha}}+\frac{(\partial_{u_1}+\partial_{u_2})\tilde{\sigma}_j(s,u_-,u_+)}{N^{1+2\alpha}}\right]\boldsymbol{\upsilon}_j \\ +\frac{\partial_u[(\sigma_-(s,u_-)+\sigma_+(s,u_+))^2]}{2N^{1+2\alpha}}\mathbf{b}+\mathcal{O}\big(N^{-2-\alpha}+N^{-1-3\alpha}\big),$$

where $A=\left[\begin{smallmatrix}0&\boldsymbol{\tau}'(\mathfrak{r}_*)\\1&0\end{smallmatrix}\right]$ and $\mathbf{b}=(\boldsymbol{\tau}''(\mathfrak{r}_*),0)'$. Meanwhile,

$$\frac{d}{dt}\begin{pmatrix}\tilde{\mathfrak{p}}_x^N\\ \tilde{\mathfrak{r}}_x^N\end{pmatrix}=\sum_{j=\pm}\left[\frac{\partial_s\sigma_j(s,u_j)}{N^{-\kappa+2\alpha}}+\frac{j\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}\partial_u\sigma_j(s,u_j)}{N^{-\kappa+\alpha}}\right]\boldsymbol{\upsilon}_j \\ +\frac{\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}}{N^{-\kappa+2\alpha}}\sum_{j=\pm}(-\partial_{u_1}+\partial_{u_2})\tilde{\sigma}_j(s,u_-,u_+)\boldsymbol{\upsilon}_j+\mathcal{O}\big(N^{\kappa-3\alpha}\big).$$

Noticing that $A\boldsymbol{\upsilon}_\pm=\pm\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}\boldsymbol{\upsilon}_\pm$ for $j=\pm$,

$$\begin{aligned}\epsilon_x^N=\frac{1}{N^{-\kappa+2\alpha}}\Big\{&\big[-\partial_s\sigma_-(s,u_-)-2\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}\partial_{u_2}\tilde{\sigma}_-(s,u_-,u_+)\big]\boldsymbol{\upsilon}_-\\&+\big[-\partial_s\sigma_+(s,u_+)+2\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}\partial_{u_1}\tilde{\sigma}_+(s,u_-,u_+)\big]\boldsymbol{\upsilon}_+\\&+\frac{\partial_u[(\sigma_-(s,u_-)+\sigma_+(s,u_+))^2]}{2}\mathbf{b}\Big\}+\mathcal{O}\big(N^{-1+\kappa-\alpha}+N^{\kappa-3\alpha}\big).\end{aligned}\tag{5.3}$$

We show that the terms in the first bracket vanishes. From (5.1),

$$\partial_{u_2}\tilde{\sigma}_-(s,u_-,u_+)=-\frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{8\boldsymbol{\tau}'(\mathfrak{r}_*)}\partial_u\big[\sigma_+^2(t,u_+)+2\sigma_-(s,u_-)\sigma_+(s,u_+)\big],$$
$$\partial_{u_1}\tilde{\sigma}_+(s,u_-,u_+)=-\frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{8\boldsymbol{\tau}'(\mathfrak{r}_*)}\partial_u\big[\sigma_-^2(t,u_+)+2\sigma_-(s,u_-)\sigma_+(s,u_+)\big].$$

Therefore, with the identity $\boldsymbol{\tau}''(\mathfrak{r}_*)(\boldsymbol{\upsilon}_--\boldsymbol{\upsilon}_+)=-2\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}\mathbf{b}$,

$$-2\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}\partial_{u_2}\tilde{\sigma}_-(s,u_-,u_+)\boldsymbol{\upsilon}_-+2\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}\partial_{u_1}\tilde{\sigma}_+(s,u_-,u_+)\boldsymbol{\upsilon}_+ \\ =\frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{4\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}}\partial_u\big[\sigma_+^2(s,u_+)\boldsymbol{\upsilon}_--\sigma_-^2(s,u_-)\boldsymbol{\upsilon}_+\big]-\partial_u\big[\sigma_-(s,u_-)\sigma_+(s,u_+)\big]\mathbf{b}.$$

The bracket in (5.3) is then equal to

$$-\partial_s\sigma_-(s,u_-)\boldsymbol{\upsilon}_--\left[\frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{4\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}}\boldsymbol{\upsilon}_+-\frac{1}{2}\mathbf{b}\right]\partial_u\sigma_-^2(s,u_-) \\ -\partial_s\sigma_+(s,u_+)\boldsymbol{\upsilon}_++\left[\frac{\boldsymbol{\tau}''(\mathfrak{r}_*)}{4\sqrt{\boldsymbol{\tau}'(\mathfrak{r}_*)}}\boldsymbol{\upsilon}_-+\frac{1}{2}\mathbf{b}\right]\partial_u\sigma_+^2(s,u_+).$$

Observe from (1.12) that it is identically zero, hence

$$\epsilon_x^N\le C\big(N^{-1+\kappa-\alpha}+N^{\kappa-3\alpha}\big).$$

The conclusion then follows from the relative entropy inequality. □

Corollary 5.3 (Harmonic chain)**.** *When the oscillators perform harmonic interaction, i.e., $V(r)\propto r^2$, $\mathcal{R}_{N,t}$ is identically zero. Hence, Theorems 4.1 and 4.2 hold autonomously.*

Proposition 5.4. *There exists a constant C independent of N, such that*

$$\mathcal{R}_{N,t} \le \frac{\gamma_N N^{1+\kappa}}{\beta} D_N\big(\tilde f_{N,t}; \tilde\nu_{N,t}\big) + C\left(\widetilde H_N(t) + \gamma_N^{-\frac15} N^{\frac45+\kappa-\frac65\alpha}\right).$$

Proof. We follow the proof of [24, Lemma 3.1]. To shorten the notation, let

$$a_x^N := \partial_t \tilde{\mathfrak{r}}_N\left(t, \frac{x}{N}\right), \quad \phi_x := V'(r_x) - \boldsymbol\tau(\tilde{\mathfrak{r}}_x^N) - \boldsymbol\tau'(\tilde{\mathfrak{r}}_x^N)\big(r_x - \tilde{\mathfrak{r}}_x^N\big).$$

Observe that

$$\big|a_x^N\big| \le N^{-\alpha} \sum_{j=\pm} jN^\kappa \partial_u \sigma_j |\boldsymbol\upsilon_j| + \mathcal{O}\big(N^{\kappa-2\alpha}\big) \le CN^{\kappa-\alpha},$$
$$\big|a_x^N - a_y^N\big| \le C|x-y|N^{-1+\kappa-\alpha}.$$

Fix a mesoscopic scale $\ell = \ell(N) \ll N$ and define

$$\phi_x^\ell := \frac1\ell \sum_{y=0}^{\ell-1} \phi_{x+y}, \quad \bar\phi_x^\ell := E_{\tilde\nu_{N,t}}\left[\phi_x^\ell \;\middle|\; \sum_{y=0}^{\ell-1} r_{x+y}\right].$$

First, one can replace ϕ_x by its block average. The error is bounded by

$$\begin{aligned}\mathcal{R}_{N,t} - \int_{\Omega_N} \tilde f_{N,t} \sum_{x\in\mathbb{T}_N} a_x^N \phi_x^\ell \, d\tilde\nu_{N,t} &= \int_{\Omega_N} \tilde f_{N,t} \sum_{x\in\mathbb{T}_N} \phi_x \sum_{y=0}^{\ell-1} \frac{a_x^N - a_{x-y}^N}{\ell} d\tilde\nu_{N,t} \\ &\le \widetilde H_N(t) + C\ell^2 N^{-1+2\kappa-2\alpha}.\end{aligned} \tag{5.4}$$

Next, let $\psi_x^\ell = \psi_x^\ell(r_x, \dots, r_{x+\ell-1})$ solve the Poisson equation

$$\sum_{y=0}^{\ell-2} \mathcal{Y}_{t,x+y}^* \mathcal{Y}_{x+y} \psi_x^\ell = \phi_x^\ell - \bar\phi_x^\ell,$$

where for each x, $\mathcal{Y}_{t,x}^*$ is the adjoint operator of $\mathcal{Y}_x$ with respect to $\tilde\nu_{N,t}$:

$$\mathcal{Y}_{t,x}^* := \beta\big[V'(r_{x+1}) - V'(r_x) - \boldsymbol\tau_{x+1}^N + \boldsymbol\tau_x^N\big] - \mathcal{Y}_x.$$

Then, for each $x \in \mathbb{T}_N$,

$$\int_{\Omega_N} \tilde f_{N,t} a_x^N \big(\phi_x^\ell - \bar\phi_x^\ell\big) d\tilde\nu_{N,t} = \int_{\Omega_N} a_x^N \sum_{y=0}^{\ell-2} \big(\mathcal{Y}_{x+y}\tilde f_{N,t}\big)\big(\mathcal{Y}_{x+y}\psi_x^\ell\big) d\tilde\nu_{N,t}.$$

Summing up for $x \in \mathbb{T}_N$ and applying Cauchy–Schwarz inequality,

$$\begin{aligned}\int_{\Omega_N} \tilde f_{N,t} \sum_{x\in\mathbb{T}_N} a_x^N \big(\phi_x^\ell - \bar\phi_x^\ell\big) d\tilde\nu_{N,t} &\le \frac{\gamma_N N^{1+\kappa}}{\beta(\ell-1)} \int_{\Omega_N} \sum_{x\in\mathbb{T}_N} \sum_{y=0}^{\ell-2} \frac{(\mathcal{Y}_{x+y}\tilde f_{N,t})^2}{4\tilde f_{N,t}} d\tilde\nu_{N,t} \\ &\quad + \frac{\beta(\ell-1)}{\gamma_N N^{1+\kappa}} \int_{\Omega_N} \sum_{x\in\mathbb{T}_N} (a_x^N)^2 \sum_{y=0}^{\ell-2} \tilde f_{N,t} \big(\mathcal{Y}_{x+y}\psi_x^\ell\big)^2 d\tilde\nu_{N,t}.\end{aligned}$$

The first term in the righthand side gives $\beta^{-1}\gamma_N N^{1+\kappa} D_N(\tilde f_{N,t}; \tilde\nu_{N,t})$. To estimate the second term, note that since V'' is bounded, we can apply the gradient estimate for Poisson equation (see [24, Proposition 9.1], also *cf.* [23, Theorem 1.1]) to obtain a constant C independent of x or ℓ, such that

$$\sum_{y=0}^{\ell-2} \big(\mathcal{Y}_{x+y}\psi_x^\ell\big)^2 \le C\ell^4 \sup_{(r_x,\dots,r_{x+\ell-1})} \sum_{y=0}^{\ell-2} \big(\mathcal{Y}_{x+y}[\phi_x^\ell - \bar\phi_x^\ell]\big)^2.$$

Since $\bar{\phi}_x^\ell$ is a function of $r_x + \ldots + r_{x+\ell-1}$, $\mathcal{Y}_{x+y}\bar{\phi}_x^\ell \equiv 0$. Also,

$$\left|\mathcal{Y}_{x+y}\phi_x^\ell\right| = \frac{1}{\ell}\left|V'(r_{x+y+1}) - V'(r_{x+y}) - \boldsymbol{\tau}'(\mathfrak{r}_{x+y+1}^N) + \boldsymbol{\tau}'(\mathfrak{r}_{x+y}^N)\right| \le \frac{C}{\ell}.$$

Direct computation then shows that

$$\int_{\Omega_N} \sum_{x\in\mathbb{T}_N} (a_x^N)^2 \sum_{y=0}^{\ell-2} \tilde{f}_{N,t}\big(\mathcal{Y}_{x+y}\psi_x^\ell\big)^2 d\tilde{\nu}_{N,t} \le C \sum_{x\in\mathbb{T}_N} (a_x^N)^2 \ell^3 \le C'\ell^3 N^{1+2\kappa-2\alpha}.$$

Therefore, we obtain the estimate

$$\int_{\Omega_N} \tilde{f}_{N,t} \sum_{x\in\mathbb{T}_N} a_x^N\big(\phi_x^\ell - \bar{\phi}_x^\ell\big) d\tilde{\nu}_{N,t} \le \frac{\gamma_N N^{1+\kappa}}{\beta} D_N\big(\tilde{f}_{N,t};\tilde{\nu}_{N,t}\big) + C_\beta \frac{\ell^4 N^{\kappa-2\alpha}}{\gamma_N}. \tag{5.5}$$

For the space variance of $\bar{\phi}_x^\ell$, relative entropy inequality reads

$$\int_{\Omega_N} \tilde{f}_{N,t} \sum_{x\in\mathbb{T}_N} a_x^N \bar{\phi}_x^\ell \, d\tilde{\nu}_{N,t} \le \frac{N^{\kappa-\alpha}}{\delta}\left[\widetilde{H}_N(t) + \frac{1}{\ell}\sum_x \log \int_{\Omega_N} e^{\delta N^{\alpha-\kappa} a_x^N \ell \bar{\phi}_x^\ell} d\tilde{\nu}_{N,t}\right],$$

for any $\delta > 0$. Observe that the extra factor ℓ in the last term above is because that $\bar{\phi}_x^\ell$ is independent of $\bar{\phi}_y^\ell$ for any $|x-y| \ge \ell$, see, e.g., [9, Lemma F.12] and [24, Lemma D.3]. Recall that $N^{\alpha-\kappa}a_x^N$ is bounded. To treat the exponential moment in above, we apply the equivalence of inhomogeneous ensembles [24, Proposition 8.3] to obtain that for $\ell = o(N^{\frac{2}{3}})$ and δ sufficiently small but fixed,

$$\log \int_{\Omega_N} \exp\left\{\delta(N^{\alpha-\kappa}a_x^N)\ell\bar{\phi}_x^\ell\right\} d\tilde{\nu}_{N,t} \le C.$$

Since $N^{\kappa-\alpha} \le \mathcal{O}(1)$,

$$\int_{\Omega_N} \tilde{f}_{N,t} \sum_{x\in\mathbb{T}_N} a_x^N \bar{\phi}_x^\ell \, d\tilde{\nu}_{N,t} \le C\big(\widetilde{H}_N(t) + \ell^{-1}N^{1+\kappa-\alpha}\big). \tag{5.6}$$

Finally, the proof is concluded by choosing $\ell(N) = \gamma_N^{\frac{1}{5}} N^{\frac{1+\alpha}{5}}$ and adding up the estimates (5.4), (5.5) and (5.6). □

Proof of Theorem 4.1. By (5.2), Proposition 5.2 and 5.4,

$$\frac{d}{dt}\widetilde{H}_N(t) \le C\Big(\widetilde{H}_N(t) + \gamma_N N^{\kappa-2\alpha} + N^{-1+2\kappa-2\alpha} + N^{1+2\kappa-6\alpha} + \gamma_N^{-\frac{1}{5}} N^{\frac{4}{5}+\kappa-\frac{6}{5}\alpha}\Big).$$

Since $\kappa \le \alpha < 1/2$, $N^{-1+2\kappa-2\alpha} + N^{1+2\kappa-6\alpha} = o(N^{1-2\alpha})$, so that

$$\frac{d}{dt}\widetilde{H}_N(t) \le C\widetilde{H}_N(t) + C\left[\gamma_N N^{\kappa-1} + \big(\gamma_N^{-1}N^{-1+5\kappa+4\alpha}\big)^{\frac{1}{5}} + o(1)\right] N^{1-2\alpha}.$$

The estimate then follows from the choice of γ_N and Grönwall's inequality. □

Theorem 4.2 follows from the exactly same argument as we used in the proof of Theorem 2.3, hence we omit the proof here.

Appendix A. General tools

In this appendix, we state some model independent tools that is used through the paper.

A.1. **Relative entropy inequality.** In this subsection, we introduce a version of Yau's relative entropy inequality. Let $\{X_t\}_{t\ge0}$ be a continuous-time Markov chain on a finite state space S, whose

infinitesimal generator is defined as

$$Lf(x) = \sum_{y\in S} r(x,y)\big[f(y) - f(x)\big], \quad x \in S.$$

Above, $r(x,y) \geq 0$ is the rate at which the chain jumps from x to y, and $f : S \to \mathbb{R}$ is any function. Define the *carré du champ* operator associated to L as

$$\Gamma f(x) = \sum_{y\in S} r(x,y)\big[f(y) - f(x)\big]^2, \quad x \in S.$$

Denote by μ_t the distribution of the process at time t with initial measure μ_0. Let $\{\nu_t\}_{t\geq 0}$ and ν be a family of measures in S such that ν_t is differentiable in time t, and $\nu_t(x) > 0$, $\nu(x) > 0$ for any $x \in S$ and any $t \geq 0$. Denote by f_t (respectively ψ_t) the Radon-Nikodym derivative of μ_t (respectively ν_t) with respect to ν_t (respectively ν),

$$f_t(x) = \frac{\mu_t(x)}{\nu_t(x)}, \quad \psi_t(x) = \frac{\nu_t(x)}{\nu(x)}, \quad x \in S.$$

Define the relative entropy $H(t)$ as

$$H(t) = H(\mu_t|\nu_t) = \int f_t \log f_t d\nu_t,$$

with the convention $0 \log 0 = 0$.

Lemma A.1 (Yau's relative entropy inequality)**.** *For any $t \geq 0$,*

$$H'(t) \leq -\int \Gamma\sqrt{f_t} d\nu_t + \int \Big(L_t^*\mathbf{1} - \frac{d}{dt}\log\psi_t\Big) d\mu_t,$$

where $\mathbf{1}$ is the constant function identical to one, and L_t^* is the adjoint of L with respect to $L^2(\nu_t)$, which acting on any function $g : S \to \mathbb{R}$ is given by

$$L_t^* g(x) = \sum_{y\in S}\Big\{\frac{\nu_t(y) r(y,x)}{\nu_t(x)} g(y) - r(x,y) g(x)\Big\}, \quad x \in S.$$

We refer the readers to [10, Lemma A.1] for proof of the above lemma. Compared to the classical Yau's relative entropy inequality (*cf.* [12, Lemma 6.1.4]), an extra term $\int \Gamma\sqrt{f_t} d\nu_t$ is subtracted in the above version.

A.2. **Flow lemma.** In this subsection, we state a flow lemma introduced by Jara and Menezes [10, 9]. For two measures $\mathfrak{p}$ and $\mathfrak{q}$ on $\mathbb{Z}^d$, we say a function $\phi : \mathbb{Z}^d \times \{e_i\}_{1\leq i\leq d} \to \mathbb{R}$ is a *flow connecting $\mathfrak{p}$ to $\mathfrak{q}$* if for any $z \in \mathbb{Z}^d$,

$$\mathfrak{p}(z) - \mathfrak{q}(z) = \sum_{i=1}^d (\phi(z, e_i) - \phi(z - e_i, e_i)).$$

The support of the flow ϕ is defined as the set of points $\{z, z + e_i\}$ such that $\phi(z, z + e_i) \neq 0$. Using the summation by parts formula, for any function $f : \mathbb{Z}^d \to \mathbb{R}$ and any flow ϕ connects $\mathfrak{p}$ to $\mathfrak{q}$,

$$\sum_{z\in\mathbb{Z}^d} f(z)(\mathfrak{p}(z) - \mathfrak{q}(z)) = \sum_{i=1}^d \sum_{z\in\mathbb{Z}^d} \phi(z, e_i)(f(z) - f(z + e_i)).$$

For $\ell \geq 1$, let $\mathfrak{p}_\ell(\cdot)$ be the uniform measure on $\Lambda_\ell^d = \{0, 1, \ldots, \ell - 1\}^d$, *i.e.* $\mathfrak{p}_\ell(x) = \ell^{-d}$ if $x \in \Lambda_\ell^d$ and $= 0$ otherwise. Let $\mathfrak{q}_\ell = \mathfrak{p}_\ell * \mathfrak{p}_\ell$ be the convolution of $\mathfrak{p}_\ell$ with itself,

$$\mathfrak{q}_\ell(y) = \sum_{z\in\mathbb{T}_N^d} \mathfrak{p}_\ell(z)\mathfrak{p}_\ell(y - z), \quad y \in \mathbb{T}_N^d.$$

In the sequel, we shall use $\mathfrak{q}_\ell$ to define the spatial average of a random variable over a large box instead of the usual $\mathfrak{p}_\ell$. Note that the support of $\mathfrak{q}_\ell$ is contained in $\Lambda^d_{2\ell-1}$. Let $\delta_0(\cdot)$ be the Dirac measure concentrated at the origin. For $\ell \geq 1$, denote

$$g_d(\ell) = \begin{cases} \ell, & d = 1, \\ \log \ell, & d = 2, \\ 1, & d \geq 3. \end{cases} \tag{A.1}$$

The following lemma states that we could construct a flow, which connects δ_0 to $\mathfrak{q}_\ell$, such that the cost is at most of order $g_d(\ell)$. We refer the readers to [9, 10] for its proof.

Lemma A.2 (Flow lemma)**.** *There exists a finite constant C_0 such that for any $\ell \geq 1$, there exists a flow ϕ_ℓ connecting δ_0 to $\mathfrak{q}_\ell$ with support in $\Lambda^d_{2\ell-1}$ such that*

$$\sum_{i=1}^d \sum_{z\in\mathbb{Z}^d} \phi_\ell(z, e_i)^2 \leq C_0 g_d(\ell), \quad \sum_{i=1}^d \sum_{z\in\mathbb{Z}^d} |\phi_\ell(z, e_i)| \leq C_0 \ell.$$

A.3. **Concentration inequalities.** In this subsection, we focus on properties of sub-Gaussian random variables. We say a real-valued random variable X is sub-Gaussian of order σ^2 if

$$\log E[e^{\theta X}] \leq \frac{1}{2}\sigma^2\theta^2, \quad \forall \theta \in \mathbb{R}.$$

The following lemma controls the expectation of the exponential of X^2, and an elementary proof could be found in [10, Proposition F.7]. Below we shall present a different proof.

Lemma A.3. *If the random variable X is sub-Gaussian of order σ^2, then for any $\gamma \leq (4\sigma^2)^{-1}$,*

$$E[e^{\gamma X^2}] \leq 3.$$

The constant 3 above is not optimal and we only need it to be a constant. By Cauchy-Schwarz inequality, it is easy to have the following result.

Corollary A.4. *Let X_i be sub-Gaussian of order σ_i^2 for $i = 1, 2$. Then, for any $\gamma \leq (4\sigma_1\sigma_2)^{-1}$,*

$$E[e^{\gamma X_1 X_2}] \leq 3.$$

Proof of Lemma A.3. Let $\mathcal{N}$ be a standard normal distribution independent of X, and denote by $\langle\cdot\rangle$ the expectation with respect to $\mathcal{N}$. Then, for any $\theta \in \mathbb{R}$,

$$\log\langle e^{\theta\mathcal{N}}\rangle = \frac{1}{2}\theta^2.$$

Therefore,

$$E[e^{\gamma X^2}] = E[\langle e^{\sqrt{2\gamma}X\mathcal{N}}\rangle] = \langle E[e^{\sqrt{2\gamma}X\mathcal{N}}]\rangle.$$

Since X is sub-Gaussian of order σ^2, the last formula is bounded by

$$\langle e^{\gamma\sigma^2\mathcal{N}^2}\rangle = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 + \gamma\sigma^2 x^2} dx = \frac{1}{\sqrt{1 - 2\gamma\sigma^2}}.$$

If $\gamma \leq (4\sigma^2)^{-1}$, it is easy to see the last formula is bounded by 3. This concludes the proof. □

The following lemma states that centered bounded random variables are sub-Gaussian.

Lemma A.5 (Hoeffding's Lemma, [1, Lemma 2.2.2])**.** *If the random variable $X \in [a, b]$ for some $a < b$, the $X - E[X]$ is sub-Gaussian of order $(b - a)^2$.*

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