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Stochastic homogenization : a short proof of the annealed Calderón-Zygmund estimate.

Marc Josien

June 3, 2022

Abstract

A building block for many field theories in continuum physics are second-order elliptic operators in divergence form, as given through a coefficient field which may be assimilated to a metric tensor field on \mathbb{R}^d . The mapping properties of these linear operators are a crucial ingredient for analysis. In this paper, we focus on Calderón-Zygmund estimates, that is, on the boundedness of the corresponding Helmholtz projection in $L^p(\mathbb{R}^d)$ -spaces. Even when the coefficient field is uniformly smooth, this estimate may fail for p not close to 2.

We seek an intrinsic criterion on the validity of the Calderón-Zygmund estimate in the whole range of $p \in (1, \infty)$; intrinsic in the sense that it is formulated in terms of the scalar and vector potentials of the harmonic coordinates. We seek genericity in form of a statistical statement, and thus consider general ensembles of coefficient fields. Our criterion comes in form of finite stochastic moments for the potentials, or rather their corrections from being affine. In line with this, the Calderón-Zygmund estimates we obtain are *annealed* as opposed to *quenched*, meaning that there is an inner norm in form of a stochastic moment next to the (outer) L^p -norm in space.

This result grows out of recent progress in quantitative stochastic homogenization; it is ultimately inspired by the classical large-scale regularity theory of Avellaneda and Lin [2]. More specifically, we provide an easier version of the proof given by us in [18], albeit under stronger assumptions. Annealed Calderon-Zygmund estimates were first established in Duerinckx and Otto in [11], and are a very convenient tool for error estimates in stochastic homogenization.

Keywords: Stochastic homogenization, Calderón-Zygmund estimates

AMS classification: 35B27, 35R60, 35J15, 35B45, 42B15

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1 Introduction

Various problems in continuum physics involve elliptic operators in divergence form $-\nabla \cdot a \nabla$. In many contexts (*e.g.* electrostatics, elasticity...), it is relevant to assume that the coefficient field a , which may be seen as a metric tensor field on \mathbb{R}^d , is heterogeneous and random. The large-scale mapping properties of these linear operators are a crucial ingredient for analysis. However, in this task, the main difficulty comes from the fact that a is heterogeneous: this prevents to benefit from most classical elliptic regularity properties, which rely on regularity assumptions. Indeed, for heterogeneous coefficient fields, these are not available on large scales. (The surviving large-scale estimates for heterogeneous coefficient fields are perturbations of the energy estimate, namely variants of the so-called Meyers' estimates [22].)

Here, we focus on a specific mapping property of the operator $\nabla(-\nabla \cdot a \nabla)^{-1} \nabla \cdot$, namely the so-called *annealed Calderón-Zygmund (CZ)* estimates, which are richer and tighter than the classical energy estimate. The wording *annealed CZ* estimate reflects a way of norming the space of random fields f on \mathbb{R}^d by intertwining spatial Lebesgue L^p -norms and local L^r -moments of f for $p, r \in (1, \infty)$. On the one hand, the term *CZ* refers to the well-known L^p -theory for elliptic PDEs due to Calderón and Zygmund. On the other hand, the term *annealed* comes from statistical physics, and ultimately originates from metallurgy. It opposes to the term *quenched*, which precisely describes a *pathwise* way of norming a fixed realization of a random field.

Our result grows out of recent progress in quantitative stochastic homogenization, which is inspired by the classical large-scale regularity theory of Avellaneda and Lin [2]. Hence, our criterion on the validity of the annealed CZ estimates unsurprisingly involves the so-called *correctors* that describe the deviation of the scalar and vector potentials of the a -harmonic coordinates from affine functions: We assume that the stochastic moments of these correctors are finite. This is an intrinsic property of the coefficient field a , the statistics of which we do not fix (unlike most articles in homogenization).

This article is a companion to the more expanded and detailed paper [18], which has the double objective of providing a new proof of the annealed CZ estimates first shown in [11] and exemplifying their use in the context of quantitative stochastic homogenization. The new approach of [18] relies on functional analytics and has the advantage of being less intrusive on the level of the PDE w. r. t. [1, 3, 11, 16, 20] briefly discussed below, which all use large-scale Lipschitz regularity as a crucial input. Here, the aim is to provide a streamlined and simplified version of the proof in [18]. The heuristics are borrowed from [18] but we employ a particular framework where we may use slightly different arguments, which are hopefully more efficient and intuitive while avoiding some technicalities.

CZ estimates in homogenization have been discussed for the last three decades. Indeed, (deterministic) CZ (or L^p) estimates have been first obtained in the context of periodic homogenization in [3] by analyzing the singular integral involving the Green function. Also, they have been deduced from Lipschitz regularity in [20] by using a CZ decomposition originating from [8]. Our result applies in these contexts, where no randomness is involved. Very recently, CZ estimates have been established in the context of stochastic homogenization, either in a quenched form or in an annealed form, *cf.* [1, 11, 16] and [11], respectively.

Annealed CZ estimates play a double role in stochastic homogenization. On the one hand, they are interesting *per se* because they provide natural estimates on the solution of the heterogeneous problem in norms intertwining moments in space and probability. On the other hand, they are very convenient tools in quantitative stochastic homogenization, which aims at obtaining a precise description of the *oscillations* and the *fluctuations* of the solution of the heterogeneous problem: This corresponds to approximating accurately the solution of the heterogeneous problem in strong norms and in weak norms (or on average), respectively. Indeed, as illustrated in [18], we may build a self-contained quantitative homogenization theory allowing for a description of the oscillations and the fluctuations just by appealing to functional calculus (*i.e.* spectral gap estimates, *cf.* [15]) and to these estimates. Hence, we may avoid any quenched result related to the large-scale Lipschitz estimates [16]. Since the article [18] has also the nature of a review article, we refer to it for a detailed overview of recent developments in quantitative homogenization as well as for a thorough justification of the usefulness of the annealed CZ estimates.

1.1 Our framework

We introduce a framework in which our result holds. We purposely choose not to specify the construction of the coefficient field a (which may be related to a Gaussian field with integrable correlation as exemplified in Section 3.2) but rather assume that it enjoys desirable properties on the level of extended correctors (ϕ_i, σ_i) –which play a crucial

role in homogenization. These appear through the following decomposition ¹ of the coefficient field a :

$$ae_i = \bar{a}e_i - a\nabla\phi_i + \nabla \cdot \sigma_i. \quad (1)$$

Here $(e_i)_{i \in \{1, \dots, d\}}$ is the canonical basis of \mathbb{R}^d and:

- \bar{a} is a constant coefficient, the so-called *homogenized* coefficient. The homogenized coefficient is of paramount use in applications, *e.g.* in mechanics, for it encodes the large-scale *homogeneous* behavior $\nabla(-\nabla \cdot \bar{a}\nabla)^{-1}\nabla \cdot$ of the *heterogeneous* operator $\nabla(-\nabla \cdot a\nabla)^{-1}\nabla \cdot$;

- the functions ϕ_i are the correctors, which correct the \bar{a} -harmonic coordinates x_i to a -harmonic coordinates in the sense of

$$-\nabla \cdot a\nabla(\phi_i + x_i) = 0; \quad (2)$$

- the skew-symmetric fields σ_i are the flux-correctors (they indeed correct the flux $a(e_i + \nabla\phi_i)$). Notice that the flux correctors σ_i satisfying (1) are not unique. In general, the following gauge is employed: $-\Delta\sigma_{ijk} = \partial_j(e_k \cdot a(e_i + \nabla\phi_i)) - \partial_k(e_j \cdot a(e_i + \nabla\phi_i))$.

Decomposition (1) may be seen as an a -Helmholtz decomposition of $(a - \bar{a})e_i$ since $a\nabla\phi_i$ is the product of a and a curl-free vector field (that is, a gradient in \mathbb{R}^d) and since $\nabla \cdot \sigma_i$ is divergence-free.

Let $d \geq 1$. Assume that we are given $\lambda > 0$, $\alpha \in (0, 1)$ and a nondecreasing function $\kappa : [1, \infty) \rightarrow [1, \infty)$. We consider an ensemble $\langle \cdot \rangle$ of coefficient fields $a : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ that satisfy the following assumptions:

A.1 a is a symmetric uniformly λ -elliptic and bounded coefficient field, namely there holds

$$\xi \cdot a(x)\xi \geq \lambda|\xi|^2 \quad \text{and} \quad \xi \cdot a^{-1}(x)\xi \geq |\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^d; \quad (3)$$

A.2 a admits the decomposition (1) where \bar{a} is constant, and where the extended correctors (ϕ_i, σ_i) are uniformly bounded with overwhelming probability in the sense of²

$$\sup_{x \in \mathbb{R}^d} |(\phi_i, \sigma_i)(x)|^r \leq \kappa(r) \quad \text{for all } r \in [1, \infty) \text{ and } i \in \{1, \dots, d\}; \quad (4)$$

A.3 a is uniformly Hölder continuous in the sense of

$$[a]_{C^{0,\alpha}(\mathbb{R}^d)} := \sup_{x, x' \in \mathbb{R}^d} \frac{|a(x) - a(x')|}{|x - x'|^\alpha} \leq 1. \quad (5)$$

In the sequel, we do not give the precise dependence of our estimates with respect to the given data d, λ, α and κ . On the contrary, most of the estimates will involve constants depending on $\gamma := (d, \lambda, \alpha, \kappa)$. To emphasize this dependence, we make use of the notation “ $\lesssim_{\beta_1, \dots, \beta_n}$ ” which reads “ $\leq C$ for a (finite) constant C depending on the tuple $(\beta_1, \dots, \beta_n)$ of previously defined parameters”³. Likewise, the notations $\ll_{\beta_1, \dots, \beta_n} 1$ (and $\gg_{\beta_1, \dots, \beta_n} 1$) read “sufficiently small (sufficiently large) with respect to a constant depending on the parameters β_1, \dots, β_n ”. For simplicity, the subscripts might be omitted in the course of the proofs.

Remark 1. Assumption **A.2** is the only assumption related to the homogenization theory – in practice, verifying this assumption requires to appeal to homogenization results, *e.g.* [16]. As is well-known, controlling the growth of the correctors is the royal road to quantitative homogenization (either stochastic or not, *cf.* *e.g.* [12, 16, 19] and [2, 4, 6], respectively). In particular, notice that stationarity of the ensemble $\langle \cdot \rangle$ is not a necessary assumption. Hence, media perturbed by a local defect enter this framework (see [6, 7]).

¹By default, the equations hold in \mathbb{R}^d and we write \int for $\int_{\mathbb{R}^d}$.

²For conciseness, in the context of estimates, we may write (f, g) for the vector field obtained by concatenating the coordinates of the vectors fields f and g .

³The cautious reader might be surprised that our estimates may depend on the *function* κ , although we did not introduce any topology for this function. However, in our applications, we only use the notation “ $\lesssim_{\gamma, \beta_1, \dots, \beta_n}$ ” in cases where $\beta_1, \dots, \beta_n \in \mathbb{R}$ and the induced constant C in the estimate depends only on κ through the evaluation of $\kappa(C')$, for a sufficiently large constant C' depending on $(d, \lambda, \alpha, \beta_1, \dots, \beta_n)$ (recall that κ is a nondecreasing function).

1.2 Result

The purpose of this article is to provide a simple proof of the annealed CZ estimates. We define the annealed CZ norms by

$$\|h\|_{p,r} := \left(\int \langle |h|^r \rangle^{\frac{p}{r}} \right)^{\frac{1}{p}} \quad (6)$$

and establish

Theorem 1.1. *Let the random fields ∇u and f be square-integrable and related by*

$$-\nabla \cdot a \nabla u = \nabla \cdot f. \quad (7)$$

*Then, for any ensemble $\langle \cdot \rangle$ of coefficient fields a satisfying Assumptions **A.1**, **A.2**, and **A.3**, there holds*

$$\|\nabla u\|_{p,r} \lesssim_{\gamma,p,r,r'} \|f\|_{p,r'} \quad \text{for all exponents } 1 \leq r < r' \leq \infty \text{ and } 1 < p < \infty. \quad (8)$$

We emphasize the special order of the norms in (6): we strive for $L^p_{\mathbb{R}^d}$ estimates for *vector-valued* functions in the space $L^r_{\langle \cdot \rangle}$ (in the parlance of [21]) and *not* to control the moments of a quenched CZ estimate, which would write in the functional spaces $L^r_{\langle \cdot \rangle}(L^p_{\mathbb{R}^d})$.

The advantage of Theorem 1.1 is that there is no restriction on the exponents $(p, r) \in (1, \infty)^2$. In this sense, this results is the natural extension of the classical CZ estimates for homogeneous coefficient, which hold for any $p \in (1, \infty)$, in opposition to the perturbative Meyers estimate⁴, which requires p being close to 2. This is highly desirable when studying the fluctuations⁵ of the solution to (7), *cf.* [10].

As already noticed in [11] and explained in [18, Sec. 7.1], the loss in stochastic integrability is unavoidable when the exponents (p, r) are far from $(2, 2)$. This is due to the randomness of the coefficient field a , which appears here through the fact that the extended correctors (ϕ_i, σ_i) are not $\langle \cdot \rangle$ -almost surely bounded but only satisfy (4). Indeed, since the medium is random, if p is far from 2, then there generically exists a r. h. s. f of (7) such that the constant C in the CZ estimate $\int |\nabla u|^p \leq C \int |f|^p$ is arbitrarily large. In particular, this justifies that (8) cannot hold for $p = r = r'$.

Remark 2. We study the CZ estimates on the level of the a -Helmholtz decomposition, and not on the (somewhat more common) level of the operator $\nabla^2(-\nabla \cdot a \nabla \cdot)^{-1}$. These first ones are indeed useful because, on the one hand, many physical problems take the form (7) and, on the other hand, the quantity ∇u is oscillating on scale 1, so that there is no hope to get a strong estimate on $\nabla^2 u$ as if the heterogeneous operator $(-\nabla \cdot a \nabla \cdot)^{-1}$ were replaced by the homogeneous operator $(-\nabla \cdot \bar{a} \nabla \cdot)^{-1}$.

1.3 Strategy

As in classical in homogenization (see *e.g.* [2]), the large-scale regularity theory for the heterogeneous operator $\nabla \cdot a \nabla$ is inherited from the constant-coefficient regularity theory for its homogenized counterpart $\nabla \cdot \bar{a} \nabla$. The two-scale expansion bridges the gap between the two operators (see (14) and (15) below for the two-scale expansion). In our proof, we employ extensively functional analysis –in particular real and complex interpolation. Remarkably enough, in opposition to many articles in stochastic homogenization, *e.g.* [3, 11, 16, 20], this does not require large-scale Lipschitz regularity theory, *cf.* [16, Th. 1]. Indeed, here, we make use of a single large-scale regularity result for the heterogeneous operator: the energy estimate.

Our goal is to establish (8). If a were constant, then we would directly obtain Theorem 1.1 in form of

Lemma 1.2 (See Lemma 7.4 of [18]). *Assume that the constant coefficient \bar{a} is elliptic and bounded in the sense of (3). Then, for any square-integrable random functions \bar{v} and g and square-integrable random field f related through*

$$\left(\frac{1}{T} - \nabla \cdot \bar{a} \nabla \right) \bar{v} = \frac{1}{T} g + f, \quad (9)$$

there holds

$$\left\| \left(\frac{\bar{v}}{\sqrt{T}}, \nabla \bar{v} \right) \right\|_{p,r} \lesssim_{d,\lambda,p,r} \left\| \left(\frac{g}{\sqrt{T}}, f \right) \right\|_{p,r} \quad \text{for all } p, r \in (1, \infty). \quad (10)$$

⁴In this regard, the philosophy and the proof of this result are very different from the annealed CZ estimates that can be obtained by the perturbative Meyers strategy and which only holds for $(p, r = r')$ sufficiently close to $(2, 2)$ (see [18, Prop. 7.1(i)]).

⁵There, the particular case $p = 4$ plays a crucial role.

(Lemma 1.2 corresponds to [18, Lem. 7.4], in which we provide a proof based on a straightforward application of the Mihlin theorem [21, Th. 1.1] for vector-valued functions in UMD spaces.) Notice that the constant in (10) does not depend on the massive parameter T , the appearance of which is explained below, so that the statement is preserved for $T = \infty$ and $g = 0$. The above result conveniently applies to the homogenized problem, *cf.* (13) below. The philosophy of our proof is to transfer this result to the level of the a -Helmholtz projection.

A first natural but intractable idea is to directly compare (7) with its homogenized counterpart: $-\nabla \cdot \bar{a} \nabla \bar{v} = \nabla \cdot f$, for which estimate (17) holds. However, the function f enjoys no additional regularity which would help to control the error between u and its two-scale expansion.

A more elaborate idea found out in [18] is to split ∇u into two contributions: a *high-pass* part and a *low-pass* part. The high-pass part $u_>$ is defined by means of the *massive* a -Helmholtz projection $\nabla(\frac{1}{T} - \nabla \cdot a \nabla)^{-1} \nabla \cdot$ as

$$\left(\frac{1}{T} - \nabla \cdot a \nabla\right) u_> = \nabla \cdot f \quad (11)$$

for arbitrary $T \in [1, \infty)$, and the remaining low-pass part $u_< := u - u_>$ is thus characterized by

$$-\nabla \cdot a \nabla u_< = \frac{1}{T} u_>. \quad (12)$$

The term *massive* comes from quantum field theory and is motivated by the interpretation of $\frac{1}{T} - \nabla \cdot a \nabla$ as the generator of a diffusion coupled to desorption at exponential rate $\frac{1}{T}$. This massive term is well-known to provide an infra-red cut-off: more precisely, up to exponentially small tails, the operator $\nabla(\frac{1}{T} - \nabla \cdot a \nabla)^{-1} \nabla \cdot$ has locality properties on scale \sqrt{T} (as can be guessed from the expression of the fundamental solution of $\frac{1}{T} - \Delta$).

As suggested by the terminology, the high-pass and the low-pass parts enjoy locality and regularity properties, respectively. Indeed, on the one hand, the high-pass part $\nabla u_>$ benefits from the locality properties on scale \sqrt{T} of the massive a -Helmholtz projection. On the other hand, since the r. h. s. of (12) is regular, we may advantageously approximate $u_<$ by its homogenized counterpart \bar{u} which solves

$$-\nabla \cdot \bar{a} \nabla \bar{u} = \frac{1}{T} u_>. \quad (13)$$

Indeed, approximating $u_<$ by its two-scale expansion $(1 + \phi_i \partial_i) \bar{u}$, we classically have that the associated error

$$w := u_< - (1 + \phi_i \partial_i) \bar{u} \quad (14)$$

satisfies the following intertwining relation (*cf.* (1), (12), and (13))

$$-\nabla \cdot a \nabla w = \nabla \cdot ((\phi_i a - \sigma_i) \nabla \partial_i \bar{u}). \quad (15)$$

As a consequence, formally, we may control ∇w by means of $\nabla^2 \bar{u}$, which is itself well-behaved. This leads us to the following \sqrt{T} -dependent splitting of ∇u into high-pass and low-pass parts:

$$\nabla u = (\nabla u_> + \partial_i \bar{u} (e_i + \nabla \phi_i)) + (\nabla u_< - \partial_i \bar{u} (e_i + \nabla \phi_i)). \quad (16)$$

For any exponents $p \in (1, \infty)$ and $1 \leq r < r'$, our goal is to control quantitatively the smallest constant $C_{p,r,r'} \geq 1$ in the annealed CZ estimate

$$\|\nabla u\|_{p,r} \leq C_{p,r,r'} \|f\|_{p,r'}, \quad (17)$$

where the random square integrable fields ∇u and f are related by the a -Helmholtz projection (7). (As discussed in Remark 3 below, we shall make the a priori qualitative assumption that $C_{p,r,r'} < \infty$.) Through the splitting (16), we relate (17) to the corresponding estimate on the level of the massive a -Helmholtz projection. The latter is expressed by defining the smallest constant $C_{p,r,r'}(T) \geq 1$ such that

$$\|\nabla u_>\|_{p,r} \leq C_{p,r,r'}(T) \|f\|_{p,r'}, \quad (18)$$

for any square-integrable random fields $u_>$ and f satisfying (11). Therefore, we aim for establishing simultaneously that $C_{p,r,r'} \lesssim 1$ and $C_{p,r,r'}(T) \lesssim 1$ for all $(p, r, r') \in (1, \infty)^3$ such that $r < r'$.

Our starting point is the classical weighted energy estimate (see [18, Proof of Lem. 7.5, Step 2] for a proof):

Lemma 1.3 (See [18], proof of Lemma 7.5, Step 2). *Let $T \geq 1$. There holds*

$$C_{2,2,2}(T) \lesssim_{d,\lambda} 1. \quad (19)$$

Moreover, if the square-integrable random fields $u_{>}$ and f are related through (11), there holds

$$\left\| \omega_T \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{2,2} \lesssim_{d,\lambda} \|\omega_T f\|_{2,2}, \quad (20)$$

where the weight is of exponential form

$$\omega_T(x) := \exp \left(- \frac{|x|}{C\sqrt{T}} \right) \quad \text{for } C \gg_{d,\lambda} 1. \quad (21)$$

Lemma 1.3 relies only on the ellipticity property (3) of the coefficient field a , and is therefore unrelated to homogenization. (In this regard, we underline that the energy estimate is the only large-scale regularity ingredient we use for the a -Helmholtz projection.) It plays a double role. First, it establishes the result (8) for the case $p = r = r' = 2$ (letting $T \uparrow \infty$), which is used later as an anchor in our iterative procedure. Second, it encloses the locality property on scale \sqrt{T} of solutions to the massive equation by means of the *pivotal* estimate (20). Estimate (20) is a key ingredient to get suboptimal estimates on the massive operator, which are independent of homogenization but, on the other hand, involve the cut-off scale \sqrt{T} :

Lemma 1.4. *Let $T \geq 1$. There holds*

$$C_{p,r,r}(T) \lesssim_{\gamma,p,r} \sqrt{T}^d \quad \text{provided } p \in (1, \infty) \text{ and } 1 \leq r < \infty. \quad (22)$$

Lemma 1.4 and its proof are close to [18, Lem. 7.5] and are displayed in Section 2.4 for self-consistency⁶. It relies on (20), where we replace the $\|\cdot\|_{2,2}$ norms in by generic $\|\cdot\|_{p,r}$ norms by using the locality property of the exponential weights. In this procedure, we lose the volume factor \sqrt{T}^d associated with the cut-off scale. However, this main argument only applies to the large scales; we tackle the small scales by an independent ingredient: we may gain in local integrability thanks to the (deterministic) elliptic small-scale CZ estimates in which no loss arises, namely Lemma 2.5.

Up to now, all the arguments only rest on ellipticity, boundedness, and local regularity of a , namely Assumptions **A.1** and **A.3**. Then comes the core of the proof, which is devoted to a buckling argument and where the homogenization theory – and thus Assumption **A.2** – plays a crucial role through our splitting (16). First, under a mild quantification of the control on the constants $C_{p,r,r'}(T)$ for the massive a -Helmholtz projection in a given range of exponents, an inner buckling argument provides an optimal estimate on the level of the massless a -Helmholtz projection in a slightly smaller range of exponents, cf. Lemma 2.1. Second, this estimate is upgraded to an optimal estimate for the massive a -Helmholtz projection, cf. Lemma 2.2.

Last, we conclude by an iterative argument which we briefly sketch: By complex interpolation between the optimal energy estimate and the suboptimal estimates of Lemma 1.4, we may ascertain a sufficiently well controlled growth of the constants $C_{p,r,r'}(T)$ for the massive operator in a range of exponents (p, r, r') not far from $(2, 2, 2)$ (the latter exponents correspond to the energy estimate). Next, we apply the two-step buckling argument described above to get optimal estimates on $C_{p,r,r'}$ and $C_{p,r,r'}(T)$ in this zone. Iterating the argument, we gradually enlarge the zone in which optimal estimates are proved and finally reach any 3-tuple of exponents $(p, r, r') \in (1, \infty)^3$ with $r < r'$ in a finite number of steps.

The main subtlety of the proof is that the annealed CZ estimates for both the massive and the massless the a -Helmholtz projection generically suffer from a loss in the stochastic exponent. Such a phenomenon is due to the fact that the correctors (ϕ_i, σ_i) are not almost-surely bounded but only controlled by (4) (see the discussion in Section 1.2). In practice, it prevents us from utilizing a direct buckling argument. On the contrary, we have to employ a real interpolation argument. Indeed, the loss in stochastic integrability of the low-pass part is counterbalanced by the gain in stochastic integrability of the high-pass part. There, the massive parameter \sqrt{T} plays the role of a potentiometer: When it grows, it damps the low-pass part and favors the high-pass part. (We refer to Step 3 of the proof of Lemma 2.1 for more details.) Notice that this additional difficulty is inherent to stochastic homogenization and would not appear if we restricted our proof to deterministic cases (*e.g.* periodic, quasi-periodic, periodic with defects...).

⁶The main difference w. r. t. [18, Lem. 7.5] is that we make use of the easier estimate (5) instead of (73), and we slightly change the order of the arguments in a more efficient way, using only local CZ estimates in Step 3.

Remark 3 (Finiteness of $C_{p,r,q}$). The cautious reader may fear that there is a circular argument since we assume that the massless constants $C_{p,r,r'}$ is finite in order to derive a bound on it. However, we underline that the final bound we obtain on $C_{p,r,r'}$ only requires its apriori finiteness (but does not depend on the precise value of the apriori bound). Even though we will not justify here that $C_{p,r,r'} < \infty$, there are two ways to circumvent this difficulty:

- Assume that the random medium is periodic of period L , so that periodic homogenization provides $C_{p,r,r'} < \infty$. Then, our argument shows that the final estimate on $C_{p,r,r'}$ is independent of L ; hence, we may retrieve the desired result without periodization in the limit $L \uparrow \infty$. In this task, the bounds (4) on the periodic correctors shall obviously be independent of L (this is shown in a particular context in [9]).
- First prove Theorem 1.1 on the level of the massive operators (since, by Lemma 1.4, the constants $C_{p,r,r'}(T)$ are always finite) and then take the limit $T \uparrow \infty$. This is the philosophy of [18].

1.4 Organization of the paper

Section 2 contains the proof of Theorem 1.1. First, in Section 2.1, we show how all the ingredients fit together in order to get Theorem 1.1. Then we show the intermediate results: In Section 2.2, we obtain an optimal estimate for the massless a -Helmholtz projection from slightly suboptimal estimates on the massive a -Helmholtz projection, that is Lemma 2.1. In Section 2.3, this estimate is upgraded in Lemma 2.2 to an optimal estimate for the massive a -Helmholtz projection. Last, in Section 2.4, we establish the suboptimal massive estimates, namely Lemma 1.4. We conclude this section by emphasizing in Section 2.5 on the main similarities and differences between this proof and [18]. Finally, in Section 3, we briefly discuss our Assumptions **A.1**, **A.2**, and **A.3**, and we provide a concrete example of application of our theorem, namely a coefficient field built by convolving a bounded function applied to a Gaussian field. The Appendix is devoted to the proof of a technical result based on classical local CZ estimates.

2 Details of proof

2.1 Proof of Theorem 1.1

Theorem 1.1 requires two upcoming intermediate results. The most important intermediate result allows for jumping from slightly suboptimal massive estimates to massless estimates. It replaces and simplifies [18, Proof of Prop. 7.3(ii), Parts 1 & 2].

Lemma 2.1. *Let $r < \tilde{r} < \tilde{\tilde{r}} < r' < r'' < r'''$ be such that for some $\theta \in (0, 1]$ we have*

$$\frac{1}{r'} < \frac{1}{s} := \theta \frac{1}{\tilde{\tilde{r}}} + (1 - \theta) \frac{1}{r''}. \quad (23)$$

Suppose that for some constant $\Lambda \in [1, \infty)$ there holds

$$C_{p,r'',r'''}(T) + C_{p,\tilde{r},\tilde{\tilde{r}}}(T) \leq \Lambda \sqrt{T}^{1-\theta} \quad \text{for all } T \in [1, \infty). \quad (24)$$

Then we have

$$C_{p,r,r'} \lesssim_{\gamma,p,r,\tilde{r},\tilde{\tilde{r}},r',r'',r'''} \Lambda^{\frac{1}{\theta}}. \quad (25)$$

The proof of Lemma 2.1 crucially relies on the introduced splitting (16) as well as on the annealed CZ estimates for the homogeneous problem, namely Lemma 1.2.

The second result is somehow the ‘‘converse statement’’ corresponding to Lemma 2.1 and allows for jumping from massless estimates (and slightly suboptimal massive estimates) to optimal massive estimates. Indeed, it yields an optimal estimate on the massive constant $C_{p,r,r'''}(T)$ under the mild assumption that $C_{p,r'',r'''}(T) \lesssim \sqrt{T}$. It has no equivalent in [18].

Lemma 2.2. *For $r < r' < r'' < r''' < r''''$ we have*

$$C_{p,r,r''''}(T) \lesssim_{\gamma,p,r,r',r'',r''',r''''} C_{p,r',r''}(1 + \frac{C_{p,r''',r''''}(T)}{\sqrt{T}}). \quad (26)$$

The proof of Lemma 2.2 relies on the same ingredients as for Lemma 2.1; however, it requires an additional splitting into high-pass and low-pass parts on the level of the homogenized solution \bar{u} (see (45) and (46) below).

Then, establishing Theorem 1.1 is equivalent to showing iteratively in n the following:

Statement \mathcal{P}_n . For all exponents (p, r, r') belonging to the set \mathcal{E}_n (see Figure 1) defined by

$$\left(1 - \frac{1}{2d}\right)^n \leq \frac{2}{p}, \frac{2}{r} \leq 2 - \left(1 - \frac{1}{2d}\right)^n \quad \text{and} \quad r < r',$$

there holds

$$C_{p,r,r'} \lesssim_{\gamma,p,r,r'} 1 \quad \text{and} \quad C_{p,r,r'}(T) \lesssim_{\gamma,p,r,r'} 1. \quad (27)$$

We may take the limit $T \uparrow \infty$ in estimate (19) of Lemma 1.3 and obtain $C_{2,2,2} \lesssim_{d,\lambda} 1$. Hence, by Jensen's inequality (see (57) below) the Statement \mathcal{P}_0 holds.

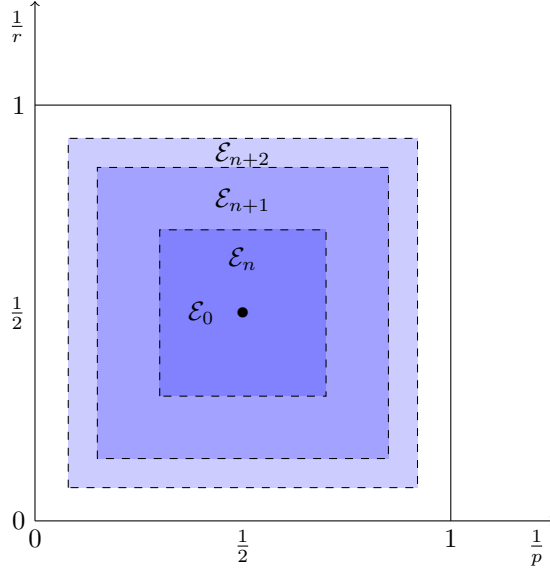


Figure 1: Representation of indexes p and r for $(p, r, r') \in \mathcal{E}_n$.

Induction step. We now assume that the Statement \mathcal{P}_n is satisfied and establish \mathcal{P}_{n+1} . The strategy is the following: First we employ complex interpolation between \mathcal{P}_n and Lemma 1.4 in order to get a larger range of exponents (q, s, s') in which we control the massive constants $C_{q,s,s'}(T)$ in a mildly suboptimal way. Then, we turn this suboptimal estimate into an estimate on the massless constants $C_{q,s,s'}$ by means of Lemma 2.1. Thanks to Lemma 2.2, the latter is upgraded to an optimal estimate on the massive constants $C_{q,s,s'}(T)$ in the same range of exponents, so that we finally get \mathcal{P}_{n+1} .

Assume that the exponents $(p, r, r') \in \mathcal{E}_n$, and $q, s, s', \tilde{r}, \tilde{p} \in (1, \infty)$ satisfy

$$\frac{1}{s'} = \frac{\theta}{r'} + \frac{1-\theta}{\tilde{r}} < \frac{1}{s} = \frac{\theta}{r} + \frac{1-\theta}{\tilde{r}} \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{\tilde{p}} \quad \text{for } \theta \in (0, 1). \quad (28)$$

By complex interpolation between the second inequality of (27) and (22) for $(p, r) \rightsquigarrow (\tilde{p}, \tilde{r})$, we obtain that

$$C_{q,s,s'}(T) \lesssim \sqrt{T}^{d(1-\theta)}.$$

Imposing that $d(1-\theta) \leq \frac{3}{4}$ and recalling the constraints (28) on p, r', r, \tilde{p} and \tilde{r} , we get that

$$C_{q,s,s'}(T) \lesssim \sqrt{T}^{\frac{3}{4}} \quad (29)$$

holds for all (q, s, s') such that

$$\begin{cases} s < s', \\ \frac{2}{q}, \frac{2}{s} > \left(1 - \frac{3}{4d}\right)\left(1 - \frac{1}{2d}\right)^n, \\ \frac{2}{q}, \frac{2}{s} < \left(1 - \frac{3}{4d}\right)\left(2 - \left(1 - \frac{1}{2d}\right)^n\right) + 2\frac{3}{4d} = 2 - \left(1 - \frac{3}{4d}\right)\left(1 - \frac{1}{2d}\right)^n. \end{cases} \quad (30)$$

By (29), we may apply Lemma 2.1 and obtain that for all exponents (q, s, s') satisfying (30) there holds $C_{q,s,s'} \lesssim 1$. Thanks to Lemma 2.2, this in turn yields

$$C_{q,s,s'}(T) \stackrel{(29)}{\lesssim} \left(1 + \frac{\sqrt{T}^{\frac{3}{4}}}{\sqrt{T}}\right) \lesssim 1$$

in the same range of exponents (30). We may restrict this range to \mathcal{E}_{n+1} , so that we have established \mathcal{P}_{n+1} .

By induction, this shows that \mathcal{P}_n holds for all n and completes the proof of Theorem 1.1. \square

2.2 From suboptimal massive estimates to massless estimates, *i.e.* proof of Lemma 2.1

Before proceeding with the proof of Lemma 2.1, we state two elementary results (the proofs of which are given at the end of the section): The first one is a uniform estimate on the gradients of the extended correctors

Lemma 2.3. *There holds:*

$$\sup_{x \in \mathbb{R}^d} \langle |(\nabla \phi_i(x), \nabla \sigma_i(x))|^r \rangle^{\frac{1}{r}} \lesssim_{\gamma,r} 1 \quad \text{for all } r \in [1, \infty) \text{ and } i \in \{1, \dots, d\}. \quad (31)$$

The second one extends the use of the constant $C_{p,r,r'}(T)$ for the massive a -Helmholtz projection (defined by (18)) to nondivergence-form r. h. s. and allows for controlling $\frac{u_{>}}{\sqrt{T}}$ as well, thus recovering a statement analogous to (10) in the constant-coefficient case:

Lemma 2.4. *Assume that the ensemble $\langle \cdot \rangle$ satisfies Assumption A.1. Fix $T > 0$ and $(p, r, r') \in (1, \infty)^3$ such that $r < r'$ and assume that $C_{p,r,r'}(T) < \infty$. If the square integrable functions $u_{>}$ and g and field f are related through*

$$\left(\frac{1}{T} - \nabla \cdot a \nabla\right) u_{>} = \frac{1}{T} g + \nabla \cdot f, \quad (32)$$

then there holds

$$\left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>}\right) \right\|_{p,r} \lesssim_d C_{p,r,r'}(T) \left\| \left(\frac{g}{\sqrt{T}}, f\right) \right\|_{p,r'}. \quad (33)$$

Also, we have $C_{p^*,(r')^*,r^*}(T) \lesssim_d C_{p,r,r'}(T)$ for the dual exponents $\frac{1}{p^*} = 1 - \frac{1}{p}$, $\frac{1}{r^*} = 1 - \frac{1}{r}$, and $\frac{1}{(r')^*} = 1 - \frac{1}{r'}$.

Proof of Lemma 2.1. Recall the splitting (16). The first (high-pass) part is estimated in Step 1, the second (low-pass) part in Step 2.

Step 1: Estimate on the high-pass part. We claim that the first part in (16) is estimated as follows

$$\|\nabla u_{>} + \partial_i \bar{u}(e_i + \nabla \phi_i)\|_{p,r} \lesssim C_{p,\tilde{r},\tilde{r}'}(T) \|f\|_{p,\tilde{r}}. \quad (34)$$

Indeed, by the triangle inequality, Jensen's inequality in probability (namely (57)) and (31) combined with the Hölder inequality, there holds

$$\|\nabla u_{>} + \partial_i \bar{u}(e_i + \nabla \phi_i)\|_{p,r} \lesssim \|\nabla u_{>}\|_{p,r} + \|\partial_i \bar{u}(e_i + \nabla \phi_i)\|_{p,r} \stackrel{(57),(31)}{\lesssim} \|\nabla u_{>}\|_{p,\tilde{r}} + \|\nabla \bar{u}\|_{p,\tilde{r}}. \quad (35)$$

By (11), we may rewrite the constant-coefficient equation (13) satisfied by \bar{u} as $-\nabla \cdot \bar{a} \nabla \bar{u} = \nabla \cdot (a \nabla u_{>} + f)$, on which we may use Lemma 1.2, then recall (18) and make use of (57), to the effect of

$$\|\nabla \bar{u}\|_{p, \tilde{r}} \stackrel{(10)}{\lesssim} \|\nabla u_{>}\|_{p, \tilde{r}} + \|f\|_{p, \tilde{r}} \stackrel{(18)}{\lesssim} C_{p, \tilde{r}, \tilde{r}}(T) \|f\|_{p, \tilde{r}} + \|f\|_{p, \tilde{r}} \stackrel{(57)}{\lesssim} C_{p, \tilde{r}, \tilde{r}}(T) \|f\|_{p, \tilde{r}}. \quad (36)$$

Inserting (36) into (35) yields (34).

Step 2: Estimate on the low-pass part. We claim that the second part in (16) is estimated as follows

$$\|\nabla u_{<} - \partial_i \bar{u}(e_i + \nabla \phi_i)\|_{p, r} \lesssim C_{p, r, r'} \frac{1}{\sqrt{T}} C_{p, r'', r'''}(T) \|f\|_{p, r''}. \quad (37)$$

We indeed reformulate the l. h. s. of (37) as

$$\nabla u_{<} - \partial_i \bar{u}(e_i + \nabla \phi_i) = \nabla w + \phi_i \nabla \partial_i \bar{u}, \quad (38)$$

where we introduce the error in the two-scale expansion (14), which satisfies (15). Appealing to the definition (17) of the massless CZ constant, and using the Hölder inequality combined with the corrector estimates (4), we obtain

$$\|\nabla w\|_{p, r} \stackrel{(17)}{\lesssim} C_{p, r, r'} \| |(\phi_i, \sigma_i)| \nabla^2 \bar{u} \|_{p, r'} \stackrel{(4)}{\lesssim} C_{p, r, r'} \|\nabla^2 \bar{u}\|_{p, r''}. \quad (39)$$

In view of (38) and once more (4), this may be upgraded by the triangle inequality to

$$\|\nabla u_{<} - \partial_i \bar{u}(e_i + \nabla \phi_i)\|_{p, r} \lesssim C_{p, r, r'} \|\nabla^2 \bar{u}\|_{p, r''}. \quad (40)$$

We now appeal to the estimate (10) for the constant-coefficient equation (13) rewritten as $\nabla \cdot \bar{a} \partial_i \nabla \bar{u} = \frac{1}{T} \nabla \cdot (u_{>} e_i)$, and to the property (33) of $C_{p, r'', r'''}(T)$ to obtain

$$\|\nabla^2 \bar{u}\|_{p, r''} \stackrel{(10)}{\lesssim} \frac{1}{T} \|u_{<}\|_{p, r''} \stackrel{(33)}{\lesssim} \frac{1}{\sqrt{T}} C_{p, r'', r'''}(T) \|f\|_{p, r''}. \quad (41)$$

Inserting this into (40) entails (37).

Step 3: Conclusion via Interpolation. By the triangle inequality applied to (16), (34) and (37) combine to

$$\|\nabla u\|_{p, r} \lesssim C_{p, \tilde{r}, \tilde{r}}(T) \|f\|_{p, \tilde{r}} + C_{p, r, r'} \frac{1}{\sqrt{T}} C_{p, r'', r'''}(T) \|f\|_{p, r''},$$

into which we insert our assumption (24), to the effect of

$$\|\nabla u\|_{p, r} \lesssim \Lambda (\sqrt{T}^{1-\theta} \|f\|_{p, \tilde{r}} + C_{p, r, r'} \frac{1}{\sqrt{T}^\theta} \|f\|_{p, r''}) \quad \text{for all } T \in [1, \infty). \quad (42)$$

Notice that, if all the stochastic exponents in (42) were equal to r , then it would directly produce

$$C_{p, r, r} \lesssim \Lambda (\sqrt{T}^{1-\theta} + C_{p, r, r} \frac{1}{\sqrt{T}^\theta}),$$

in which the second r. h. s. term could be absorbed into the l. h. s. for $\sqrt{T} \gg \Lambda^{\frac{1}{\theta}}$, so that (25) would be directly obtained. However, the different stochastic exponents in (42) prevent us from using a direct buckling argument. To overcome this, we use below the J -method of real interpolation (see [5, Sec. 3.2 p. 42]).

In view of the relation (23), we have indeed the inclusion of the Lorentz spaces $L_{(\cdot)}^{r'} \subset L_{(\cdot)}^{s, 1}$, and $L_{(\cdot)}^{s, 1}$ is obtained by real interpolation of parameters $(\theta, 1)$ between the spaces $L_{(\cdot)}^{\tilde{r}}$ and $L_{(\cdot)}^{r''}$ (see Theorem [5, Th. 5.3.1. p. 113]). Hence, using the J -functional version of real interpolation there exists a family $\{f_L\}_{L \in (0, \infty)}$ of vector fields such that

$$f = \int_0^\infty f_L dL \quad \text{and} \quad \|f\|_{p, r'} \gtrsim \int_0^\infty \left(L^{1-\theta} \|f_L\|_{p, \tilde{r}} + \frac{1}{L^\theta} \|f_L\|_{p, r''} \right) dL. \quad (43)$$

By linearity of the a -Helmholtz projection, this induces a family $\{\nabla u_L\}_{L \in (0, \infty)}$ such that

$$\nabla u = \int_0^\infty \nabla u_L dL \quad \text{and thus} \quad \|\nabla u\|_{p,r} \leq \int_0^\infty \|\nabla u_L\|_{p,r} dL.$$

Applying (42) with $(\nabla u, f)$ replaced by $(\nabla u_L, f_L)$ we obtain

$$\|\nabla u_L\|_{p,r} \lesssim \Lambda C_{p,r,r'}^{1-\theta} \left(\left(\frac{\sqrt{T}}{C_{p,r,r'}} \right)^{1-\theta} \|f_L\|_{p,\tilde{r}} + \left(\frac{C_{p,r,r'}}{\sqrt{T}} \right)^\theta \|f_L\|_{p,r'''} \right).$$

Choosing $\sqrt{T} := 1 + C_{p,r,r'} L$, integrating in L (recall that $C_{p,r,r'} \geq 1$), using Jensen's inequality in probability (57) in conjunction with $\tilde{r} < r'''$, and finally appealing to (43), we get

$$\begin{aligned} \|\nabla u\|_{p,r} &\lesssim \Lambda C_{p,r,r'}^{1-\theta} \int_0^\infty \left(L^{1-\theta} \|f_L\|_{p,\tilde{r}} + L^{-\theta} \|f_L\|_{p,r'''} \right) dL + \Lambda \int_0^1 \|f_L\|_{p,\tilde{r}} dL \\ &\stackrel{(57)}{\lesssim} \Lambda C_{p,r,r'}^{1-\theta} \int_0^\infty \left(L^{1-\theta} \|f_L\|_{p,\tilde{r}} + L^{-\theta} \|f_L\|_{p,r'''} \right) dL + \Lambda \int_0^1 \|f_L\|_{p,r'''} dL \\ &\lesssim \Lambda C_{p,r,r'}^{1-\theta} \int_0^\infty \left(L^{1-\theta} \|f_L\|_{p,\tilde{r}} + L^{-\theta} \|f_L\|_{p,r'''} \right) dL \\ &\stackrel{(43)}{\lesssim} \Lambda C_{p,r,r'}^{1-\theta} \|f\|_{p,r'}. \end{aligned}$$

Last, by definition (17), this entails $C_{p,r,r'} \lesssim \Lambda C_{p,r,r'}^{1-\theta}$. Since $\theta > 0$ and by the qualitative assumption $C_{p,r,r'} < \infty$ (see Remark 3), this yields estimate (25). \square

Proof of Lemma 2.4. We fix g and f and define $u_>$ by (32) and \bar{v} by

$$\left(\frac{1}{T} - \Delta \right) \bar{v} = \frac{1}{T} g + \nabla \cdot f. \quad (44)$$

We first prove (33) and then the duality result.

Step 1: Argument for (33). Subtracting (32) from (44), we get

$$\left(\frac{1}{T} - \Delta \right) (\bar{v} - u_>) = \nabla \cdot (\nabla u_> - a \nabla u_>) \quad \text{and} \quad \left(\frac{1}{T} - \nabla \cdot a \nabla \right) (\bar{v} - u_>) = \nabla \cdot (\nabla \bar{v} - a \nabla \bar{v}).$$

Hence, applying (10) as well as (18) on the first and second above equation, respectively, we get

$$\left\| \frac{\bar{v} - u_>}{\sqrt{T}} \right\|_{p,r} \lesssim \|\nabla u_>\|_{p,r}, \quad \text{and} \quad \|\nabla \bar{v} - \nabla u_>\|_{p,r} \leq C_{p,r,r'}(T) \|\nabla \bar{v}\|_{p,r'}.$$

Whence, applying twice the triangle inequality and the above estimates, there holds

$$\begin{aligned} \left\| \left(\frac{u_>}{\sqrt{T}}, \nabla u_> \right) \right\|_{p,r} &\lesssim \left\| \left(\frac{\bar{v}}{\sqrt{T}}, \nabla \bar{v} \right) \right\|_{p,r} + \|\nabla u_>\|_{p,r} + C_{p,r,r'}(T) \|\nabla \bar{v}\|_{p,r'} \\ &\lesssim \left\| \left(\frac{\bar{v}}{\sqrt{T}}, \nabla \bar{v} \right) \right\|_{p,r} + C_{p,r,r'}(T) \|\nabla \bar{v}\|_{p,r'}. \end{aligned}$$

Anticipating on (57), we also have $\left\| \left(\frac{\bar{v}}{\sqrt{T}}, \nabla \bar{v} \right) \right\|_{p,r} \lesssim \left\| \left(\frac{\bar{v}}{\sqrt{T}}, \nabla \bar{v} \right) \right\|_{p,r'}$. Hence, applying once more (10) for $r \rightsquigarrow r'$ yields (33).

Step 2: Duality argument. Assume now that the square-integrable random fields $u_>$ and f as well as $\tilde{u}_>$ and \tilde{f} are related through (11), respectively, with $\|f\|_{p^*,r^*} < \infty$ and $\|\tilde{f}\|_{p,r'} < \infty$. The variational formulation of (11) yields

$$\frac{1}{T} \int \tilde{u}_> u_> + \int \nabla \tilde{u}_> \cdot a \nabla u_> = - \int \nabla \tilde{u}_> \cdot f = - \int \nabla u_> \cdot \tilde{f}.$$

(Here, we make use of the symmetry of a .) Hence, applying the Hölder inequality and (18), we get

$$\left| \int \nabla u_> \cdot \tilde{f} \right| \lesssim \|\nabla \tilde{u}_>\|_{p,r} \|f\|_{p^*,r^*} \leq C_{p,r,r'}(T) \|\tilde{f}\|_{p,r'} \|f\|_{p^*,r^*}.$$

As a consequence, we obtain by duality that $\|\nabla u_>\|_{p^*,(r')^*} \lesssim C_{p,r,r'}(T) \|f\|_{p^*,r^*}$. This directly entails $C_{p^*,(r')^*,r^*}(T) \lesssim C_{p,r,r'}(T)$. \square

Proof of Lemma 2.3. Since the argument is the same for σ_i and ϕ_i , we only show it for ϕ_i . Using Assumption **A.3**, by the classical Schauder theory applied to (2), we have

$$|\nabla\phi_i(x)| \lesssim \|a\|_{C^{0,\alpha}(\mathbb{B}(x))} + \|\phi_i\|_{L^2(\mathbb{B}(x))} \stackrel{(3),(5)}{\leq} 2 + \|\phi_i\|_{L^2(\mathbb{B}(x))}.$$

Taking the $L^r_{(\cdot)}$ -norm and using (4) implies the desired (31) for ϕ_i . \square

2.3 From massless estimates to optimal massive estimates, *i.e.* proof of Lemma 2.2

Proof of Lemma 2.2. Recall that $C_{p,r,r''''}(T)$ is defined by (18). Hence, we fix f such that $\|f\|_{p,r''''} < \infty$ and define $u_>$ by (11). In order to establish (26), we go beyond the splitting $u = u_> + u_<$ introduced in (12) and also split $\bar{u} = \bar{u}_> + \bar{u}_<$ defined by (13) according to

$$\left(\frac{1}{T} - \nabla \cdot \bar{a}\nabla\right)\bar{u}_> = \frac{1}{T}(\bar{u} - u_<), \quad (45)$$

$$\left(\frac{1}{T} - \nabla \cdot \bar{a}\nabla\right)\bar{u}_< = \frac{1}{T}u. \quad (46)$$

This induces the following decomposition of $\nabla u_>$:

$$\nabla u_> = (\nabla u - \partial_i \bar{u}_<(e_i + \nabla\phi_i)) - (\nabla u_< - \partial_i \bar{u}_<(e_i + \nabla\phi_i)). \quad (47)$$

We tackle separately the first and second contributions in Steps 1 and 2 by showing

$$\|\nabla u - \partial_i \bar{u}_<(e_i + \nabla\phi_i)\|_{p,r} \lesssim C_{p,r',r''} \|f\|_{p,r''''}, \quad (48)$$

$$\|\nabla u_< - \partial_i \bar{u}_<(e_i + \nabla\phi_i)\|_{p,r} \lesssim C_{p,r',r''} \frac{C_{p,r''',r''''}(T)}{\sqrt{T}} \|f\|_{p,r''''}, \quad (49)$$

which, by the triangle inequality, yields (26) in form of

$$\|\nabla u_>\|_{p,r} \lesssim C_{p,r',r''} \left(1 + \frac{C_{p,r''',r''''}(T)}{\sqrt{T}}\right) \|f\|_{p,r''''}. \quad (50)$$

Step 1: Argument for (48). As a consequence of the triangle inequality, Jensen's inequality in probability (57), and the corrector estimate (31) combined with the Hölder inequality, we get

$$\|\nabla u - \partial_i \bar{u}_<(e_i + \nabla\phi_i)\|_{p,r} \stackrel{(57),(31)}{\lesssim} \|\nabla u\|_{p,r'} + \|\nabla \bar{u}_<\|_{p,r'}. \quad (51)$$

Then, thanks to the estimate (10) for the constant coefficient equation (46) in form of $\left(\frac{1}{T} - \nabla \cdot \bar{a}\nabla\right)\partial_i \bar{u}_< = \frac{1}{T}\partial_i u$, we have $\|\nabla \bar{u}_<\|_{p,r'} \lesssim \|\nabla u\|_{p,r'}$. Inserting this into (51) and appealing to (17) (with $(r, r') \rightsquigarrow (r', r'')$) and (57) yields (48).

Step 2: Argument for (49). We rewrite the second r. h. s. of (47) in terms of w defined by (14) as

$$\nabla u_< - \partial_i \bar{u}_<(e_i + \nabla\phi_i) = \nabla w + \phi_i \nabla \partial_i \bar{u} + \partial_i \bar{u}_>(e_i + \nabla\phi_i).$$

Using the triangle inequality and the corrector estimates (4) and (31) in conjunction with the Hölder inequality yields

$$\|\nabla u_< - \partial_i \bar{u}_<(e_i + \nabla\phi_i)\|_{p,r} \stackrel{(4),(31)}{\lesssim} \|\nabla w\|_{p,r} + \|\nabla^2 \bar{u}\|_{p,r'''} + \|\nabla \bar{u}_>\|_{p,r'}. \quad (52)$$

Note that (45) can be reformulated as

$$\left(\frac{1}{T} - \nabla \cdot \bar{a}\nabla\right)\bar{u}_> = -\frac{1}{T}(w + \phi_i \partial_i \bar{u}).$$

Using the constant-coefficient CZ estimate (10) on a differentiated form of this equation and using once more (4) yields

$$\|\nabla \bar{u}_>\|_{p,r'} \stackrel{(10)}{\lesssim} \|\nabla w\|_{p,r'} + \frac{1}{\sqrt{T}} \|\phi_i \partial_i \bar{u}\|_{p,r'} \stackrel{(4)}{\lesssim} \|\nabla w\|_{p,r'} + \frac{1}{\sqrt{T}} \|\nabla \bar{u}\|_{p,r'''}.$$

Inserting this into (52) and appealing to (39) with $(r, r', r'') \rightsquigarrow (r', r'', r''')$ entails

$$\|\nabla u_< - \partial_i \bar{u}_<(e_i + \nabla\phi_i)\|_{p,r} \stackrel{(39)}{\lesssim} C_{p,r',r''} \|\nabla^2 \bar{u}\|_{p,r'''} + \frac{1}{\sqrt{T}} \|\nabla \bar{u}\|_{p,r'''}.$$

Last, the constant-coefficient estimates (41) for $(r'', r''') \rightsquigarrow (r''', r''''')$ and (36) for $(\tilde{r}, \tilde{r}) \rightsquigarrow (r''', r''''')$ entail (49). \square

2.4 Suboptimal massive estimates, *i.e.* proof of Lemma 1.4

As previously announced, we control $(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>})$ on the scales smaller than 1 thanks to the local regularity of a . In particular, the local CZ estimates can be used in order to gain in local integrability. This is conveniently expressed by means of a nested version of the norms (6), namely

$$\|h\|_{p,r,q} := \left(\int \left\langle \left(\int_{B_1(x)} |h|^q \right)^{\frac{r}{q}} \right\rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}}, \quad (53)$$

where⁷ $p, r, q \in [1, \infty]$. These norms advantageously separate the small-scale behavior $L_{B_1(x)}^q$ from the large-scale behavior $L_{\mathbb{R}^d}^p$. We make use of the following small-scale regularity result:

Lemma 2.5. *Let $\langle \cdot \rangle$ be an ensemble of coefficient fields a satisfying estimates (3) and (5) for fixed $\lambda > 0$ and $\alpha \in (0, 1)$. If the square integrable random vector field $u_{>}$ and f are related through (11) for $T \geq 1$, then there holds*

$$\left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r} \lesssim_{d,\lambda,\alpha,p,r} \left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,q} + \|f\|_{p,r} \quad \text{for all } p, r, q \in (1, \infty), \quad (54)$$

$$\left\| \left(\frac{u}{\sqrt{T}}, \nabla u \right) \right\|_{p,r,q} \lesssim_{d,\lambda,\alpha,p,r,q} \left\| \left(\frac{u}{\sqrt{T}}, \nabla u \right) \right\|_{p,r,2} + \|f\|_{p,r,q} \quad \text{for all } p, r, q \in (1, \infty). \quad (55)$$

The proof of Lemma 2.5 appeals to the classical CZ and Schauder theories in a slightly intricate way. Since it is essentially a simplification of the proof of [18, Lem. A.2], we postpone it until Appendix A.

Proof of Lemma 1.4. Throughout the proof, the square-integrable random fields $u_{>}$ and f are related by (11).

The proof relies on two ingredients: the locality on scale \sqrt{T} of the massive a -Helmholtz projection and the regularity on scale 1 of the coefficient field due to (5), which are captured through a weighted energy estimate in Lemma 1.3 and local annealed CZ estimates in Lemma 2.5, respectively. These two properties allow for estimates between different spatial $L_{\mathbb{R}^d}^p$ -norms, where locality provides the large-scale cut-off, and regularity the small-scale cut-off. In particular, we will jump between the $L_{\mathbb{R}^d}^2$ -norm, on which scale we have the weighted energy estimate (20), and the $L_{\mathbb{R}^d}^\infty$ -norm, where we may handle the $L_{\langle \cdot \rangle}^r$ -norm.

The proof is divided into 3 steps. In Step 1, we derive some useful properties of the norms (53), and in particular that they decrease with increasing spatial exponent p , see (56). In Step 2, we make use of the locality property on scale \sqrt{T} in order to derive a weak form of (22) on the scale of the norms $\| \cdot \|_{p,r,2}$ defined by (53), see (63) below. In Step 3, finally, we use local regularity (54) and (55) to return to the original norms (6).

Step 1: Nestedness properties of the norms (53). We claim the following properties of the norms (53):

$$\|h\|_{p',r,q} \lesssim_d \|h\|_{p,r,q} \quad \text{provided } p \leq p', \quad (56)$$

$$\|h\|_{p,r} \leq \|h\|_{p,r'} \quad \text{and} \quad \|h\|_{p,r,q} \leq \|h\|_{p,r',q} \quad \text{provided } r \leq r', \quad (57)$$

$$\|h\|_{p,r,q} \leq \|h\|_{p,r,q'} \quad \text{provided } q \leq q', \quad (58)$$

$$\|h\|_{p,r,q} \leq \|h\|_{p,r} \quad \text{provided } q \leq \min\{p, r\}, \quad (59)$$

$$\|h\|_{2,2,2} = \|h\|_{2,2}. \quad (60)$$

We start with the argument for (56). This is due to the norm equivalence

$$\begin{aligned} \|h\|_{\ell_k^p(L_{\langle \cdot \rangle}^r(L^q(B_{1+\sqrt{d}}(k))))} &\lesssim_d \|h\|_{p,r,q} \lesssim_d \|h\|_{\ell_k^p(L_{\langle \cdot \rangle}^r(L^q(B_{1+\sqrt{d}}(k))))} \\ \text{for } \|h\|_{\ell_k^p(L_{\langle \cdot \rangle}^r(L^q(B_{1+\sqrt{d}}(k))))} &:= \left(\sum_{k \in \mathbb{Z}^d} \left\langle \left(\int_{B_{1+\sqrt{d}}(k)} |h|^q \right)^{\frac{r}{q}} \right\rangle^{\frac{p}{r}} \right)^{\frac{1}{p}}, \end{aligned} \quad (61)$$

in conjunction with the embedding of the discrete spaces $\ell^p(\mathbb{Z}^d) \subset \ell^{p'}(\mathbb{Z}^d)$. In turn, the leftmost estimate in (61) follows from the elementary geometric fact that $B_{1+\sqrt{d}}(k)$ can be covered by a d -dependent finite number of balls of radius $\frac{1}{2}$, whereas the rightmost estimate in (61) is obvious by inclusion $B_1(x) \subset B_{1+\sqrt{d}}(k)$ if $x \in k + [0, 1]^d$.

Inequalities (57) and (58) are obtained by Jensen's inequalities in probability and in space, respectively.

⁷The cases p, r or $q = \infty$ requires to interpret (53) by using the norms $L_{\mathbb{R}^d}^\infty$, $L_{\langle \cdot \rangle}^\infty$, or $L_{B_1(x)}^\infty$, respectively.

We turn to (59), which is a consequence of Jensen's inequality: Rewriting definition (53) in form of

$$\|h\|_{p,r,q} = \left(\int \left\langle \left(\int_{B_1} |h(x+z)|^q dz \right)^{\frac{r}{q}} \right\rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}} = \left\| \int_{B_1} |h(\cdot+z)|^q dz \right\|_{L_{\mathbb{R}^d}^{\frac{p}{q}}(L_{(\cdot)}^{\frac{r}{q}})}^{\frac{1}{q}},$$

we learn from the convexity and translation invariance of the involved norms (here we use $q \leq \min\{p, r\}$) that

$$\|h\|_{p,r,q} \leq \left(\int_{B_1} \| |h(\cdot+z)|^q \|_{L_{\mathbb{R}^d}^{\frac{p}{q}}(L_{(\cdot)}^{\frac{r}{q}})} dz \right)^{\frac{1}{q}} \stackrel{(6)}{=} \|h\|_{p,r}.$$

Finally, (60) directly follows from the identity

$$\int h(x) dx = \int \int_{B_1(x)} h(y) dy dx, \quad (62)$$

(Notice that the value 2 of the exponent plays no special role and can be replaced by any other value $p \in [1, \infty]$.)

Step 2: Suboptimal estimates on the scale of norms (53). Now, we are given $p, r \in (1, \infty)$ and establish

$$\left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,2} \lesssim_{\gamma,p,r} \sqrt{T}^d \|f\|_{p,r,2}. \quad (63)$$

By duality (in the same spirit as Lemma 2.4), it is sufficient to show (63) for $r \geq 2$. We choose the weight ω_T as in Lemma 1.3. The two ingredients for (63) are the following norm relations, to be established below:

$$\sup_x \left\langle \int_{B_1(x)} |h|^2 \right\rangle^{\frac{1}{2}} =: \|h\|_{\infty,2,2} \lesssim \|h\|_{2,2,2}, \quad (64)$$

$$\|\omega_T h\|_{2,r,2} \lesssim \sqrt{T}^{\max\{0, d(\frac{1}{2} - \frac{1}{p})\}} \|\omega_{4T} h\|_{p,r,2}, \quad (65)$$

$$\|\omega_{\frac{T}{4}} h\|_{p,r,2} \lesssim \sqrt{T}^{\frac{d}{p}} \|\omega_T h\|_{\infty,r,2}. \quad (66)$$

The merit of passing to the spatial exponent $p = \infty$ is that by duality, if $r > 2$, we have

$$\|h\|_{\infty,r,2} = \sup_{\langle |F|^{\frac{2r}{r-2}} \rangle \leq 1} \|Fh\|_{\infty,2,2}. \quad (67)$$

Estimate (64) is immediate from (56). In the case of $p \geq 2$, (65) follows from appealing to the relation $\omega_T = \omega_{4T}^2$, cf. (21), which allows to use the Hölder inequality, so that it reduces to the obvious $\|\omega_{4T}\|_{\frac{2p}{p-2}, \infty, \infty} \lesssim \sqrt{T}^{d(\frac{1}{2} - \frac{1}{p})}$ (recall $T \geq 1$). In the case of $p \leq 2$, we appeal to (56) and the obvious $\omega_T \leq \omega_{4T}$. For (66), we start from $\omega_{T/4} = \omega_T^2$, use the Hölder inequality, and $\|\omega_T\|_{p, \infty, \infty} \lesssim \sqrt{T}^{\frac{d}{p}}$.

Here comes the argument for (63). Since the case $r = 2$ is simpler (in particular, we do not need to introduce the additional random variable F below), we henceforth assume that $r > 2$. By (66) and (67), we have

$$\left\| \omega_{\frac{T}{4}} \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,2} \stackrel{(66)}{\lesssim} \sqrt{T}^{\frac{d}{p}} \left\| \omega_T \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{\infty,r,2} \stackrel{(67)}{=} \sqrt{T}^{\frac{d}{p}} \sup_{\langle |F|^{\frac{2r}{r-2}} \rangle \leq 1} \left\| \omega_T F \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{\infty,2,2}. \quad (68)$$

Then, (64), (20) (and (60)), and the Hölder inequality yield

$$\left\| \omega_T F \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{\infty,2,2} \stackrel{(64)}{\lesssim} \left\| \omega_T F \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{2,2,2} \stackrel{(20),(60)}{\lesssim} \|\omega_T F f\|_{2,2,2} \leq \|\omega_T f\|_{2,r,2} \langle |F|^{\frac{2r}{r-2}} \rangle^{\frac{r-2}{2r}}. \quad (69)$$

Last, appealing to (65), we obtain

$$\|\omega_T f\|_{2,r,2} \stackrel{(65)}{\lesssim} \sqrt{T}^{\max\{0, d(\frac{1}{2} - \frac{1}{p})\}} \|\omega_{4T} f\|_{p,r,2}. \quad (70)$$

Inserting (70) into (69), which in turn is used in (68), and using $\frac{d}{p} + \max\{0, d(\frac{1}{2} - \frac{1}{p})\} \leq d$ entails

$$\|\omega_{\frac{T}{4}}(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>})\|_{p,r,2} \lesssim \sqrt{T}^d \|\omega_{4T} f\|_{p,r,2}.$$

Since by the definition (53) and the properties of ω_T we have

$$\|\omega_T(\cdot - z)h\|_{p,r,2} \sim \left(\int \left(\omega_T(x - z) \langle \int_{B_1(x)} |h|^2 \rangle^{\frac{r}{2}} \rangle^{\frac{1}{r}} \right)^p dx \right)^{\frac{1}{p}},$$

the desired (63) follows from taking the L^p -norm in the shift z .

Step 3: Conclusion, namely proof of (22). Let $p, r \in (1, \infty)$. We first upgrade (63) to

$$\left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,q} \lesssim_{\gamma,p,r,q} \sqrt{T}^d \|f\|_{p,r,q} \quad \text{for all } q \in (1, \infty). \quad (71)$$

Then, selecting $q := \min\{p, r\}$ and applying (54), (71), and (59) (since $T \geq 1$) entails the desired estimate (22) via

$$\left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r} \stackrel{(54)}{\lesssim} \left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,q} + \|f\|_{p,r} \stackrel{(71)}{\lesssim} \sqrt{T}^d \|f\|_{p,r,q} + \|f\|_{p,r} \stackrel{(59)}{\lesssim} \sqrt{T}^d \|f\|_{p,r}.$$

Here comes the argument for (71). By duality (in the same spirit as Lemma 2.4), it suffices to establish it for $q \geq 2$. By (55), (63), and (58), and since $T \geq 1$, we obtain (71) in form of

$$\left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,q} \stackrel{(55)}{\lesssim} \left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,2} + \|f\|_{p,r,q} \stackrel{(63)}{\lesssim} \sqrt{T}^d \|f\|_{p,r,2} + \|f\|_{p,r,q} \stackrel{(58)}{\lesssim} \sqrt{T}^d \|f\|_{p,r,q}.$$

□

2.5 Differences with respect to [18, Prop. 7.1]

We compare here our strategy with [18, Prop. 7.1], which contains a more general (but more complicated) version of the present article. We first underline that [18] and the present article share the same fundamental ideas; in particular, Section 2.4, and 2.1 are simplified versions of arguments from [18], whereas Sections 2.2 and 2.3 substantially differ from this reference. The very interest of our new simplified proof is that it allows for emphasizing on the fundamental ingredients of the proof while suppressing many technicalities.

The main difference with respect to [18] is that we employ homogenization results on the level of the massless operator. Conveniently, this only requires Assumption **A.2** on the correctors, which has been verified in the literature for various frameworks (see, *e.g.* [16]), but it also comes with the disadvantage of being unable to treat the case of sublinear but unbounded correctors (as in the relaxed assumption (72) of Section 3.1). On the contrary, in [18], reasoning only on the level of the massive operators comes with the benefit of having always a locality property on the scale \sqrt{T} , so that (a mild modification of) the case (72) of Section 3.1 may be treated as well. However, this requires to introduce more elaborated objects, namely the *massive correctors* the growth of which has to be controlled by an independent argument, and to appeal to two different scales related to massive parameters $\sqrt{\tau} \ll \sqrt{T}$. In this respect, replacing $(\sqrt{\tau}, \sqrt{T}) \rightsquigarrow (\sqrt{T}, \infty)$ provides a formal correspondence between the two proofs.

As a side-effect, in the proof of Lemma 2.1, we carry out the real interpolation argument on the level of the stochastic exponent related to the r. h. s. of (11). Thus, we need to appeal to the J -method of interpolation, whereas the K -method was used in [18], for the reasoning was conducted on the level of the stochastic exponent related to the l. h. s. of (11).

Last, we emphasize our use of a simple local regularity assumption, namely (5), instead of the more complicated assumption (73) used in [18]. Besides numerous simplifications in the proofs, the only visible effect is that the use of local regularity results causes no loss in stochastic exponent (see *e.g.* (54) and (55)), so that the loss in (8) is only due to stochastic homogenization, *cf.* discussion in Section 1.2.

3 Short discussion on the assumptions and an example of application

3.1 Relaxing the assumptions

We make Assumptions **A.1**, **A.2** and **A.3** in order to simplify the proof of Theorem 1.1 below. However, these may be relaxed as explained below (we also refer the interested reader to [18, Prop. 7.1(ii)]).

In Assumption **A.1**, the symmetry of a is convenient but unnecessary. However, it comes with the advantage that the extended correctors (ϕ_i^*, σ_i^*) associated with the adjoint a^* obviously coincide with (ϕ_i, σ_i) . If a is not symmetric, we have to assume that these satisfy (4) as well. Notice that we treat with *scalar* equations for simplicity, but the result of this paper applies as well to elliptic systems without any modification in the proof.

In Assumption **A.2**, instead of (4), we may only assume that the extended correctors (ϕ_i, σ_i) are quantitatively strictly sublinear with overwhelming probability, in the sense that there exists $\beta \in [0, 1)$ such that

$$\sup_{x_0 \in \mathbb{R}^d} \langle |(\phi_i, \sigma_i)(x_0 + x) - (\phi_i, \sigma_i)(x_0)|^r \rangle^{\frac{1}{r}} \leq \kappa(r)(1 + |x|)^\beta \quad \text{for all } x \in \mathbb{R}^d, r \in [1, \infty) \text{ and } i \in \{1, \dots, d\}. \quad (72)$$

Such a quantified growth is known to be sufficient to get quantitative homogenization properties (see, *e.g.* [16]) and is indeed sufficient in the proofs of [18] for obtaining annealed CZ estimates –with the slight subtlety it has to be transposed on the level of the *massive* correctors.

Assumption **A.3** on local regularity of a is of lesser importance. For example, we may morally only assume that a is Hölder continuous with overwhelming probability in the sense that there holds

$$\langle [a]_{C^{0,\alpha}(\mathbb{B}_1)}^r \rangle^{\frac{1}{r}} \leq \kappa(r) \quad \text{for all } r \in [1, \infty). \quad (73)$$

Assumption (73) may ultimately be removed, *c.f.* [11, Th. 6.1], at the price of downgrading the resulting estimate (8) below to

$$\int \left\langle \left(\int_{\mathbb{B}_1(x)} |\nabla u|^2 \right)^{\frac{r'}{2}} \right\rangle^{\frac{p}{r'}} dx \lesssim_{\gamma,p,r,r'} \int \left\langle \left(\int_{\mathbb{B}_1(x)} |f|^2 \right)^{\frac{r}{2}} \right\rangle^{\frac{p}{r}} dx \quad \text{for all } 1 \leq r' < r \leq \infty \text{ and } 1 < p < \infty,$$

where we denote the ball of center x and radius R by $\mathbb{B}_R(x)$ (if $R = 1$ or $x = 0$, these parameters are omitted) and the averaged integral by $\int_{\mathbb{D}} = \frac{1}{|\mathbb{D}|} \int$.

3.2 A simple example of application

We propose here a possible example of application of Theorem 1.1 close to [18].

Let $\alpha, \beta \in (0, 1)$ and $\lambda > 0$. Assume that we are given an ensemble $\langle \cdot \rangle$ of stationary and centered Gaussian fields g characterized by the covariance function c , the (non-negative) Fourier transform $\mathcal{F}c$ of which is assumed to satisfy an improved integrability condition:

$$\mathcal{F}c(k) \leq (1 + |k|)^{-d-2\beta} \quad \text{for all } k \in \mathbb{R}^d. \quad (74)$$

The coefficient a is defined through the convolution

$$a := \eta * A(g), \quad (75)$$

where $A \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^{d \times d})$ is a Lipschitz application in the set of symmetric, elliptic, and bounded matrices, namely

$$\xi \cdot A(h)\xi \geq \lambda|\xi|^2 \quad \text{and} \quad \xi \cdot A(h)^{-1}\xi \geq |\xi|^2 \quad \text{for all } h \in \mathbb{R}, \xi \in \mathbb{R}^d, \quad (76)$$

and where $\eta \in C_c^{0,\alpha}(\mathbb{B}_1)$ is a Hölder-continuous cut-off function with values in \mathbb{R}_+ satisfying $\int \eta = 1$. We claim that we may apply Theorem 1.1 to such an ensemble $\langle \cdot \rangle$ of coefficient fields. More precisely, we show the following:

Lemma 3.1. *Let $d > 2$. Assume that $\langle \cdot \rangle$ satisfies the above assumptions. Then, up to rescaling $a \rightsquigarrow a(\mu)$ for a constant*

$$\mu \gg_{d,\|\eta\|_{C^{0,\alpha}(\mathbb{B}_1)}} 1, \quad (77)$$

*the ensemble $\langle \cdot \rangle$ also satisfies Assumptions **A.1**, **A.2**, and **A.3** of Section 1.1, for*

$$\kappa(r) \lesssim_{d,\lambda,\alpha,\beta,\|A'\|_{L^\infty(\mathbb{R}^d)},\|\eta\|_{C^{0,\alpha}(\mathbb{B}_1)},r} 1.$$

Proof. Since a self-contained proof may be found in a slightly different context in [15, Prop. 1], we only give here the main ingredients and some additional references.

First, by (76) we easily verify that Assumption **A.1** is satisfied. Second, by definition (75), we have by the properties of convolution

$$[a]_{C^{0,\alpha}(B_1(x))} \leq \|\eta\|_{C^{0,\alpha}(B_1)} \int_{B_2(x)} |A(g(y))| dy \stackrel{(76)}{\lesssim} d, \|\eta\|_{C^{0,\alpha}(B_1)} 1.$$

This shows (5) for $a \rightsquigarrow a(\mu \cdot)$ where μ is given by (77), so that Assumption **A.3** is satisfied for $a \rightsquigarrow a(\mu \cdot)$.

It remains to establish that a satisfies Assumption **A.2**. (W. l. o. g. we may assume $\mu = 1$.) As remarked in [15, Prop. 1] in the discrete setting and in [17, Cor. 1] for the continuous setting, it suffices to establish a *spectral gap* inequality to obtain bounded correctors in dimension $d > 2$. Here, we make use of [18, Prop. 4.1]⁸, which requires to establish the following spectral gap inequality:

$$\langle (F - \langle F \rangle)^2 \rangle \lesssim \left\langle \int \left(\int_{B_1(x)} \left| \frac{\partial F}{\partial a(y)} \right| dy \right)^2 dx \right\rangle \quad \text{for any random variable } F \text{ depending on } a. \quad (78)$$

Here appears the Malliavin derivative defined by

$$\lim_{h \rightarrow 0} \frac{F(a + h\tilde{a}) - F(a)}{h} = \int \frac{\partial F(a)}{\partial a(x)} : \tilde{a}(x) dx.$$

Estimate (78) follows from

$$\langle (F - \langle F \rangle)^2 \rangle \lesssim \left\langle \int \left(\int_{B_1(x)} \left| \frac{\partial F}{\partial g(y)} \right| dy \right)^2 dx \right\rangle, \quad (79)$$

which itself is a consequence of (74) (we refer to the argument in [18, Lem. 3.1]). Indeed, by the chain rule we have

$$\frac{\partial F}{\partial g(y)} = \int \frac{\partial F}{\partial a(z)} \frac{\partial a(z)}{\partial g(y)} dz \stackrel{(75)}{=} \int \frac{\partial F}{\partial a(z)} \eta(z - y) A'(g(y)) dz,$$

so that, since η is of compact support in B_1 , there holds

$$\left| \frac{\partial F}{\partial g(x)} \right| \lesssim \|A'\|_{L^\infty(\mathbb{R})} \|\eta\|_{L^\infty(\mathbb{R}^d)} \int_{B_1(y)} \left| \frac{\partial F}{\partial a(z)} \right| dz.$$

Inserting the above estimate into (79) yields (78). □

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⁸An easy inspection of the proof shows that this result holds under Assumptions **A.1** and **A.3** provided the spectral gap inequality (78) is satisfied.

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A Local CZ estimates

Some parts of the proof below are essentially quoted from [18].

Proof of Lemma 2.5. The proof relies on deterministic local CZ estimates enclosed in Lemma A.1 below. We split it into two independent parts.

Part 1: Proof of (54). Let $\rho \geq 0$. We introduce

$$\tilde{a}_{x,\rho}(y) := \begin{cases} a(y) & \text{if } y \in \mathbb{B}_\rho(x), \\ a(x) & \text{otherwise.} \end{cases}$$

Accordingly, we freeze the variable x and define v_x as the solution to

$$\left(\frac{1}{T} - \nabla \cdot \tilde{a}_x \nabla\right)v_x = \nabla \cdot (\mathbf{1}_{\mathbb{B}_\rho(x)} f), \quad (80)$$

where the operator ∇ acts on the variable y . This combines with (11) to

$$\left(\frac{1}{T} - \nabla \cdot a \nabla\right)(v_x - u_{>}) = 0 \quad \text{in } \mathbb{B}_\rho(x). \quad (81)$$

Then, the proof falls in three steps: In Step 1, we establish local and annealed estimates on v_x by means of the Meyers strategy, then in Step 2 we establish local estimates on the locally a -harmonic function $v_x - u_{>}$, and last, we combine these estimates in order to establish (54).

Part 1, Step 1: Estimate on v_x . By (5) we have $\|\tilde{a}_x - a(x)\|_{L^\infty} \leq \rho^\alpha$; therefore, by a perturbative argument *à la Meyers* (see *e.g.* [22] and a more recent presentation in [23, Chap. 2, Th. 2.6.2, p. 122]), we may select $\rho \ll_{d,\lambda,\alpha,p,q,r} 1$ such that

$$\int \left| \left\langle \frac{v_x}{\sqrt{T}}, \nabla v_x \right\rangle \right|^q \lesssim \int_{\mathbb{B}_1(x)} |f|^q, \quad (82)$$

$$\int \left\langle \left| \left\langle \frac{v_x}{\sqrt{T}}, \nabla v_x \right\rangle \right|^r \right\rangle^{\frac{p}{r}} \lesssim \int_{\mathbb{B}_1(x)} \langle |f|^r \rangle^{\frac{p}{r}}. \quad (83)$$

Since the proof is the same for (82) and (83), we only establish (83). To this purpose, we rewrite (80) as

$$\left(\frac{1}{T} - \nabla \cdot a(x) \nabla\right)v_x = \nabla \cdot (\tilde{a}_x - a(x)) \nabla v_x + \nabla \cdot (\mathbf{1}_{\mathbb{B}_1(x)} f).$$

Thus, by Lemma 1.2, it suffices to establish that the operator $\nabla \left(\frac{1}{T} - \nabla \cdot a(x) \nabla\right)^{-1} \nabla \cdot (\tilde{a}_x - a(x))$ is a contraction on $L_{\mathbb{R}^d}^p(L_{(\cdot,\cdot)}^r)$ to obtain (83). Denoting by $\bar{C}_{p,r} < \infty$ the operator norm

$$\bar{C}_{p,r} := \left\| \nabla \left(\frac{1}{T} - \nabla \cdot a(x) \nabla\right)^{-1} \nabla \cdot \right\|_{\mathcal{L}(L_{\mathbb{R}^d}^p(L_{(\cdot,\cdot)}^r))},$$

there obviously holds

$$\left\| \nabla \left(\frac{1}{T} - \nabla \cdot a(x) \nabla\right)^{-1} \nabla \cdot (\tilde{a}_x - a(x)) \right\|_{\mathcal{L}(L_{\mathbb{R}^d}^p(L_{(\cdot,\cdot)}^r))} \leq \bar{C}_{p,r} \rho^\alpha.$$

Thus, for $\rho \ll 1$, the above r. h. s. is strictly smaller than 1, so that we have established the desired contraction property.

Part 1, Step 2: Estimate on $v_x - u_{>}$. By (5), we may apply Lemma A.1 to (81), and more precisely a rescaled version of (88) on $v_x - u_{>}$ in the ball $\mathbb{B}_\rho(x)$, and make use of the triangle inequality, to the effect of

$$\begin{aligned} \left\| \left\langle \frac{v_x - u_{>}}{\sqrt{T}}, \nabla v_x - \nabla u_{>} \right\rangle \right\|_{L^\infty(\frac{1}{2}\mathbb{B}_\rho(x))} &\stackrel{(88)}{\lesssim} \left\| \left\langle \frac{v_x - u_{>}}{\sqrt{T}}, \nabla v_x - \nabla u_{>} \right\rangle \right\|_{L^{\min\{p,r,q\}}(\mathbb{B}_\rho(x))} \\ &\leq \left\| \left\langle \frac{v_x}{\sqrt{T}}, \nabla v_x \right\rangle \right\|_{L^{\min\{p,r,q\}}(\mathbb{B}_1(x))} + \left\| \left\langle \frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right\rangle \right\|_{L^{\min\{p,r,q\}}(\mathbb{B}_1(x))}. \end{aligned}$$

We may estimate the first r. h. s. term by (82), so that, taking the $L_{\mathbb{R}^d}^p(L_{(\cdot)}^r)$ -norm and using (58) and (59) yields

$$\begin{aligned} \left(\int_{\mathbb{R}^d} \left\langle \left\| \left(\frac{v_x - u_{>}}{\sqrt{T}}, \nabla v_x - \nabla u_{>} \right) \right\|_{L^\infty(\frac{1}{2}B_\rho(x))}^r \right\rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}} &\lesssim \|f\|_{p,r,\min\{p,r,q\}} + \left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,\min\{p,r,q\}} \\ &\stackrel{(58),(59)}{\lesssim} \|f\|_{p,r} + \left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,q}. \end{aligned} \quad (84)$$

Part 1, Step 3: Conclusion, namely proof of (54). Covering the balls $B_1(x)$ by smaller balls $\frac{1}{2}B_\rho(x')$, and using the triangle inequality, we may estimate the left-hand side of (54) as follows:

$$\begin{aligned} \left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r} &\lesssim \left(\int \left(\int_{\frac{1}{2}B_\rho(x)} \left\langle \left| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right|^r \right\rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}} \\ &\leq \left(\int \left(\int_{\frac{1}{2}B_\rho(x)} \left\langle \left| \left(\frac{v_x}{\sqrt{T}}, \nabla v_x \right) \right|^r \right\rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}} + \left(\int \left(\int_{\frac{1}{2}B_\rho(x)} \left\langle \left| \left(\frac{u_{>} - v_x}{\sqrt{T}}, \nabla u_{>} - \nabla v_x \right) \right|^r \right\rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}}. \end{aligned}$$

We estimate the first r. h. s. term by means of (83),

$$\left(\int \left(\int_{\frac{1}{2}B_\rho(x)} \left\langle \left| \left(\frac{v_x}{\sqrt{T}}, \nabla v_x \right) \right|^r \right\rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}} \lesssim \left(\int \left(\int_{B_1(x)} \langle |f|^r \rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}} \stackrel{(62)}{\lesssim} \|f\|_{p,r},$$

and the second r. h. s. term by Jensen's inequality and (84), namely

$$\begin{aligned} \left(\int \left(\int_{\frac{1}{2}B_\rho(x)} \left\langle \left| \left(\frac{u_{>} - v_x}{\sqrt{T}}, \nabla u_{>} - \nabla v_x \right) \right|^r \right\rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}} &\lesssim \left(\int \left\langle \left\| \left(\frac{v_x - u_{>}}{\sqrt{T}}, \nabla v_x - \nabla u_{>} \right) \right\|_{L^\infty(\frac{1}{2}B_\rho(x))}^r \right\rangle^{\frac{p}{r}} dx \right)^{\frac{1}{p}} \\ &\stackrel{(84)}{\lesssim} \|f\|_{p,r} + \left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{p,r,q}. \end{aligned}$$

As a consequence, we obtain (54).

Part 2: Proof of (55). We first apply Lemma A.1 on $u_{>}$ in $B_1(x)$, and more precisely (87) for $(p, q) \rightsquigarrow (q, 2)$. Then, taking the $L_{\mathbb{R}^d}^p(L_{(\cdot)}^r)$ -norm yields (55). \square

Lemma A.1. *Assume that coefficient field a satisfies estimates (3) and*

$$[a]_{C^{0,\alpha}(B_2)} < \infty. \quad (85)$$

for $\lambda > 0$ and $\alpha \in (0, 1)$. For $T \geq 1$, we assume that the function $u_{>}$ and the vector field f satisfy the relation

$$\left(\frac{1}{T} - \nabla \cdot a \nabla \right) u_{>} = \nabla \cdot f \quad \text{in } B_2. \quad (86)$$

Then, there holds

$$\left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{L^p(B_1)} \lesssim_{d,\lambda,\alpha,[a]_{C^{0,\alpha}(B_2)},p,q} \left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{L^q(B_2)} + \|f\|_{L^p(B_2)} \quad \text{for all } p, q \in (1, \infty), \quad (87)$$

and, provided $f = 0$,

$$\left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{L^\infty(B_1)} \lesssim_{d,\lambda,\alpha,[a]_{C^{0,\alpha}(B_2)},q} \left\| \left(\frac{u_{>}}{\sqrt{T}}, \nabla u_{>} \right) \right\|_{L^q(B_2)} \quad \text{for all } q \in (1, \infty). \quad (88)$$

Lemma A.1 relies on standard CZ and Schauder theory for (11) (see [13, Th. 7.2 p. 140], and [14, Chap. 8 p. 177-218] or [13, Chap. 5 p. 75-96], respectively). The interested reader may consult the proof in [18, Lem. A.3].