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Adrien Dubouloz, Frédéric Déglise, Paul Østvaer

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PUNCTURED TUBULAR NEIGHBORHOODS AND STABLE HOMOTOPY AT INFINITY

ADRIEN DUBOULOZ, FRÉDÉRIC DÉGLISE, AND PAUL ARNE ØSTVÆR

ABSTRACT. We initiate a study of punctured tubular neighborhoods and homotopy theory at infinity in motivic settings. We use the six functors formalism to give an intrinsic definition of the stable motivic homotopy type at infinity of an algebraic variety. Our main computational tools include cdh-descent for normal crossing divisors, Euler classes, Gysin maps, and homotopy purity. Under ℓ -adic realization, the motive at infinity recovers a formula for vanishing cycles due to Rapoport-Zink; similar results hold for Steenbrink’s limiting Hodge structures and Wildeshaus’ boundary motives. Under the topological Betti realization, the stable motivic homotopy type at infinity of an algebraic variety recovers the singular complex at infinity of the corresponding topological space. We coin the notion of homotopically smooth morphisms with respect to a motivic ∞ -category and use it to show a generalization to virtual vector bundles of Morel-Voevodsky’s purity theorem, which yields an escalated form of Atiyah duality with compact support. Further, we study a quadratic refinement of intersection degrees, taking values in motivic cohomotopy groups. For relative surfaces, we show the stable motivic homotopy type at infinity witnesses a quadratic version of Mumford’s plumbing construction for smooth complex algebraic surfaces. Our construction and computation of stable motivic links of Du Val singularities on normal surfaces is expressed entirely in terms of Dynkin diagrams. In characteristic $p > 0$, this improves Artin’s analysis on Du Val singularities through étale local fundamental groups. The main results in the paper are also valid for ℓ -adic sheaves, mixed Hodge modules, and more generally motivic ∞ -categories.

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1. INTRODUCTION

1.1. Context and motivation. Topology at infinity is essentially the study of topological properties that persistently occur in complements of compact sets. A space is intuitively simply connected at infinity if one can collapse loops far away from any small subspace. Euclidean space \mathbf{R}^n , $n \geq 3$, is the unique open contractible n -manifold that is simply connected at infinity. For example, the Whitehead manifold is not simply connected at infinity and therefore not homeomorphic to \mathbf{R}^3 . This article describes our first attempt at finding a unified theory of punctured tubular neighborhoods and homotopy at infinity for open manifolds and smooth varieties. Our overriding goal is to develop a study of intrinsic motivic invariants which can distinguish between \mathbf{A}^1 -contractible varieties. For background on motivic homotopy theory and \mathbf{A}^1 -contractible varieties, we refer to the survey [7]. The quest for finding invariants that can help classify smooth varieties over fields up to \mathbf{A}^1 -homotopy can be traced back to work by Asok-Morel [6]. Their ideas on \mathbf{A}^1 -h-cobordisms and \mathbf{A}^1 -surgery theory, with applications towards vector bundles over projective spaces in Asok-Kebekus-Wendt [5], have inspired our search for motivic invariants with a pronounced geometric topological flavor. Another great source of inspiration is Zariski's cancellation problem [50], which remains difficult because of the lack of computable invariants available to distinguish non-isomorphic \mathbf{A}^1 -contractible smooth affine varieties such as the Koras-Russell cubic threefold and \mathbf{A}^3 (see [40], [54]).

Our approach makes extensive use of the six-functor formalism in stable motivic homotopy theory, as developed in [9, 27]; we review and complement this material in Section 4. Let S be a qcqs (quasi-compact quasi-separated) base scheme. Its stable motivic homotopy category $\mathrm{SH}(S)$ is a closed symmetric monoidal ∞ -category, see, e.g., [42, 52, 59, 77]. To any separated S -scheme of finite type $f: X \rightarrow S$ we define $\Pi_S^\infty(X)$, the *stable motivic homotopy type at infinity of X* , by the homotopy exact sequence

$$(1.1.0.a) \quad \Pi_S^\infty(X) \rightarrow f_!f^!(\mathbf{1}_S) \xrightarrow{\alpha_X} f_*f^!(\mathbf{1}_S)$$

Here $\mathbf{1}_S$ is the motivic sphere spectrum over S , $f_!f^!(\mathbf{1}_S) = \Pi_S(X)$ is the stable homotopy type of X and $f_*f^!(\mathbf{1}_S) = \Pi_S^c(X)$ is the properly supported stable homotopy type of X . The canonical morphism α_X is obtained from the six-functor formalism for the stable motivic homotopy category $\mathrm{SH}(S)$, which implies the following fundamental properties.

- If X/S is smooth, then $f_!f^!(\mathbf{1}_S) = \Sigma^\infty X_+$ is the motivic suspension spectrum of X
- If X/S is proper, then α_X is an isomorphism
- The morphism α_X is covariant with respect to proper morphisms and contravariant with respect to étale morphisms

With the intrinsic definition of $\Pi_S^\infty(X)$ in (1.1.0.a) we deduce a number of novel properties in the spirit of proper homotopy theory. Let us fix a compactification \bar{X} of X over S and denote by ∂X

its reduced *boundary*. Then the induced immersions $j : X \rightarrow \bar{X}$, $i : \partial X \rightarrow \bar{X}$ form a diagram of S -schemes

$$(1.1.0.b) \quad \begin{array}{ccc} X & \xrightarrow{j} & \bar{X} & \xleftarrow{i} & \partial X \\ & \searrow f & \downarrow & \swarrow g & \\ & & S & & \end{array}$$

We observe the stable homotopy type at infinity of X is determined by the data in (1.1.0.b) via a canonical equivalence

$$(1.1.0.c) \quad \Pi_S^\infty(X) \simeq g_* i^* j_* f^! (\mathbf{1}_S)$$

This shows that $\Pi_S^\infty(X)$ is independent of the chosen compactification and that our construction has properties analogous to Deligne's vanishing cycle functor for étale sheaves, see [37]. We may reformulate (1.1.0.c) by means of the canonically induced homotopy exact sequence

$$(1.1.0.d) \quad \Pi_S^\infty(X) \rightarrow \Pi_S(\partial X) \oplus \Pi_S(X) \xrightarrow{i_* + j_*} \Pi_S(\bar{X})$$

In the notation in (1.1.0.b), let us assume \bar{X} , ∂X are smooth S -schemes, and write N for the normal bundle of ∂X in \bar{X} . In Section 4.4 we use the Euler class $e(N)$ in $\mathrm{SH}(S)$ to deduce the homotopy exact sequence

$$(1.1.0.e) \quad \Pi_S^\infty(X) \rightarrow \Pi_S(\partial X) \xrightarrow{e(N)} \Sigma^\infty \mathrm{Th}_S(N)$$

It is helpful to think of the passage from (1.1.0.a) to (1.1.0.e) in the language of problem-solving. Our “problem” is to understand $\Pi_S^\infty(X)$ and the “solution” in the smooth case is the Euler class for the normal bundle of the closed immersion $\partial X \hookrightarrow \bar{X}$.

In the following, we further assume \bar{X} is a smooth proper S -scheme and ∂X is a normal crossing divisor on \bar{X} . We may write $\partial X = \cup_{i \in I} \partial_i X$ as the union of its irreducible components $\partial_i X$, so there is a canonical closed immersion $\nu_i : \partial_i X \rightarrow \bar{X}$. For any subset $J \subset I$, we equip $\partial_J X := \cap_{j \in J} \partial_j X$ with its reduced subscheme structure, where \cap is suggestive notation for fiber products over the boundary ∂X . If $J \subset K$, there is a canonical proper morphism $\nu_K^J : \partial_K X \rightarrow \partial_J X$. By means of descent for the cdh-covering

$$\sqcup_{i \in I} \partial_i X \rightarrow \partial X$$

we identify $\Pi_S(\partial X)$ with the colimit¹ of the naturally induced diagram in $\mathrm{SH}(S)$

$$(1.1.0.f) \quad \Pi_S(\partial_I X) \longrightarrow \bigoplus_{\#J=\#I-1} \Pi_S(\partial_J X) \xrightarrow{\cdots} \bigoplus_{\#J=\#I-2} \Pi_S(\partial_J X) \xrightarrow{\cdots} \cdots \xrightarrow{\cdots} \bigoplus_{i \in I} \Pi_S(\partial_i X)$$

The face map on the summand $\Pi_S(\partial_K X)$ is defined by the pushforward maps

$$\sum_{J \subset K, \#J=\#K-1} (\nu_K^J)_*$$

Similarly, we identify $\Sigma^\infty \mathrm{Th}_S(N)$ with the limit of the naturally induced diagram in $\mathrm{SH}(S)$

$$(1.1.0.g) \quad \bigoplus_{i \in I} \Sigma^\infty \mathrm{Th}_S(N_i) \xrightarrow{\cdots} \bigoplus_{\#J=2} \Sigma^\infty \mathrm{Th}_S(N_J) \xrightarrow{\cdots} \bigoplus_{\#J=3} \Sigma^\infty \mathrm{Th}_S(N_J) \xrightarrow{\cdots} \cdots \longrightarrow \Sigma^\infty \mathrm{Th}_S(N_I)$$

Here, N_J is the normal bundle of $\partial_J X$ in \bar{X} , and the coface map on the summand $\Sigma^\infty \mathrm{Th}_S(N_K)$ is defined by the Gysin maps

$$\sum_{J \subset K, \#J=\#K-1} (\nu_K^J)^!$$

¹Limits and colimits in this paper are taken in the sense of ∞ -categories.

Our general computations culminate in Theorem 4.2.1, where we identify $\Pi_S^\infty(X)$ with the homotopy fiber of the map

$$\operatorname{colim}_{n \in (\Delta^{\text{inj}})^{\text{op}}} \left(\bigoplus_{J \subset I, \#J=n+1} \Pi_S(\partial_J X) \right) \xrightarrow{\mu} \lim_{n \in \Delta^{\text{inj}}} \left(\bigoplus_{J \subset I, \#J=m+1} \Sigma^\infty \operatorname{Th}_S(N_J) \right)$$

induced by

$$(\mu_{i,j})_{i,j \in I} : \bigoplus_{i \in I} \Pi_S(\partial_i X) \longrightarrow \bigoplus_{j \in I} \Sigma^\infty \operatorname{Th}_S(N_j)$$

More precisely, $\mu_{i,j}$ is shorthand for the composite map

$$\Pi_S(\partial_i X) \xrightarrow{\nu_{i*}} \Pi_S(\bar{X}) \rightarrow \Sigma^\infty \left(\frac{\bar{X}}{\bar{X} - \partial_j X} \right) \xrightarrow{\simeq} \Sigma^\infty \operatorname{Th}_S(N_j)$$

To refine these techniques, we develop a theory of duality with compact support. We generalize the homotopy purity theorem and give new examples of rigid objects in the process. Our approach is based on the notion of a homotopically smooth morphism. If $f : X \rightarrow S$ is a smoothable lci morphism with virtual bundle τ_f over X , we say that f is *homotopically smooth* (*h-smooth*) if the naturally induced morphism

$$\mathfrak{p}_f : \operatorname{Th}(\tau_f) \rightarrow f^!(\mathbf{1}_S)$$

is an isomorphism (see Definition 2.3.8 for more details). Any closed immersion between smooth varieties over a field is h-smooth. When f is h-smooth and $i : Z \rightarrow X$ is a closed immersion with Z/S h-smooth, Theorem 2.4.3 shows the relative purity isomorphism

$$\Pi_S(X/X - Z, v) \simeq \Pi_S(Z, i^*v + N_i)$$

Here, v is a virtual vector bundle over X and N_i is the (necessarily regular) normal bundle of $i : Z \rightarrow X$. Under the additional assumption that $\Pi_S(X, v)$ is rigid, we show in Section 3.4 the duality with compact support isomorphism

$$\Pi_S(X, v)^\vee \simeq \Pi_S^c(X, -v - \tau_f)$$

This duality isomorphism can be seen as a motivic analog of classical topological results due to Atiyah [8, §3], Milnor-Spanier [69, Lemma 2]. As an application, we identify the stable motivic homotopy type at infinity of hyperplane arrangements in Section 3.5.

We define the punctured tubular neighborhood $\operatorname{TN}_S^\times(X, Z)$ of a closed immersion $i : Z \rightarrow X$ in Section 4. For points on hypersurfaces in affine space, this key invariant specializes in links considered successfully in topology by Milnor and Mumford (see [68], [71]). It turns out that $\operatorname{TN}_S^\times(X, Z)$ is a local invariant in the sense that it only depends on a Nisnevich neighborhood of Z in X , and, moreover, it satisfies a cdh-excision property (see Corollary 4.1.8). The geometric content of our construction is transparently visible in examples, e.g., for an ordinary double point on a threefold (see Example 4.1.10). We invite the interested reader to compare with Levine's notion of motivic punctured tubular neighborhoods in [66].

In the situation with the compactification of a separated morphism of finite type $f : X \rightarrow S$, see (1.1.0.b), Proposition 4.4.2 shows there exists a canonical isomorphism

$$\Pi_S^\infty(X) \simeq \operatorname{TN}_S^\times(\bar{X}, \partial X)$$

which is natural in $(\bar{X}, X, \partial X)$, covariantly functorial for proper maps, and contravariantly functorial for étale maps. Via this isomorphism, we can study stable motivic homotopy types at infinity through the geometric construction of punctured tubular neighborhoods. This perspective helps us clarify a few simple and unifying principles across motivic ∞ -categories. For example, we generalize Wildeshaus' analytic invariance theorem for boundary motives [87, Theorem 5.1]: A closed pair of S -schemes (X, Z) means a closed immersion $Z \not\rightarrow X$ of S -schemes, and a morphism $\phi : (Y, T) \rightarrow (X, Z)$

is an S -morphism $\phi: Y \rightarrow X$ such that $\phi^{-1}(Z) = T$. Suppose $f: T \rightarrow Z$ is an isomorphism that extends to an isomorphism of the respective formal completions $\mathfrak{f}: \hat{Y}_T \rightarrow \hat{X}_Z$. If S is an excellent scheme, Theorem 4.1.14 shows that there exists a canonical isomorphism

$$\mathfrak{f}^*: \mathrm{TN}_S^\times(Y, T) \xrightarrow{\cong} \mathrm{TN}_S^\times(X, Z)$$

In particular, the stable motivic homotopy type at infinity functor satisfies analytical invariance.

In Section 5, we employ punctured tubular neighborhoods to study a theory of motivic plumbing on surfaces; this constitutes a refinement and extension of Mumford's seminal work in [71]. It is also a successful transportation of a construction from surgery theory into motivic homotopy, extending the ideas of [6]. The setting is a closed pair (X, D) , where X/S has relative dimension two and is smooth in a neighborhood of D . We assume D is a divisor on X which is proper and with smooth reduced crossings over S , see Definition 3.3.2. As in [71] we assume the components $(D_i)_{i \in I}$ of D are rational curves. It turns out that $\Pi_S(D)$ is a sum of an Artin object \mathcal{D}^2 depending on the intersections of the (D_i) 's and the "geometric" part $\bigoplus_{i \in I} \mathbf{1}_S(1)[2]$. Theorem 5.2.7 identifies the punctured tubular neighborhood $\mathrm{TN}_S^\times(X, D)$, or equivalently $\Pi_S^\infty(X - D)$ when X/S is proper, with the homotopy fiber of a naturally induced map

$$\begin{pmatrix} a & b' \\ b & \mu \end{pmatrix}: \mathcal{D} \oplus \bigoplus_{i \in I} \mathbf{1}_S(1)[2] \rightarrow \mathcal{D}^\vee(2)[4] \oplus \bigoplus_{j \in I} \mathbf{1}_S(1)[2]$$

We refer to $\mu = (\mu_{ij}): \bigoplus_{i \in I} \mathbf{1}_S(1)[2] \rightarrow \bigoplus_{j \in I} \mathbf{1}_S(1)[2]$ as the "quadratic Mumford matrix" since, over the complex numbers, the above specializes to computations carried out in [71]. Its coefficients take values in the endomorphism ring of the sphere or unit $\mathbf{1}_S$. When $S = \mathrm{Spec}(\mathcal{O})$ is a semi-local essentially smooth scheme over a field, or 2-integers such as $\mathbb{Z}[\frac{1}{2}]$ in a 2-regular number field [16], we interpret μ_{ij} as the class of a quadratic form $(\partial_i X, \partial_j X)_{quad} \in \mathrm{GW}(\mathcal{O})$ in the Grothendieck-Witt ring called the *quadratic degree* of the intersections of the divisors $\partial_i X$ and $\partial_j X$. The close connection with quadratic forms arises since elements of the i th Chow-Witt group are represented by formal sums of subvarieties Z of codimension i equipped with an element of $\mathrm{GW}(k(Z))$. Moreover, the rank of the quadratic degree equals the corresponding Mumford degree.

Further, we specialize our results to motives. When S is a finite field, a global field, or a number ring, we have the motivic t -structure on rational Artin-Tate motives at our disposal (see [65] for the case of fields, and [80] for number rings). We let $\mathrm{DM}^{\mathrm{AT}}(K, \mathbb{Q})$ be the triangulated category of (constructible) rational Artin-Tate motives. From [65] it follows that $\mathrm{DM}^{\mathrm{AT}}(K, \mathbb{Q})$ admits a motivic t -structure, whose heart is the Tannakian category $\mathrm{MM}^{\mathrm{AT}}(K, \mathbb{Q})$ of Artin-Tate motives. In particular, one gets a homological and monoidal functor

$$\underline{\mathrm{H}}_0: \mathrm{DM}^{\mathrm{AT}}(K, \mathbb{Q}) \rightarrow \mathrm{MM}^{\mathrm{AT}}(K, \mathbb{Q})$$

We define the Artin-Tate motive

$$\underline{\mathrm{H}}_i(\mathrm{TN}^\times(X, D)) := \underline{\mathrm{H}}_0(\mathrm{TN}^\times(X, D)[-i])$$

as the i -th (motivic) homology of the punctured tubular neighborhood of (X, D) . When X is in addition proper over K , this is the homology of the boundary motive of $(X - D)$ (see Example 4.3.3 and Proposition 4.4.2), or the *motivic homology at infinity*

$$\underline{\mathrm{H}}_i^\infty(X - D) = \underline{\mathrm{H}}_i(\mathrm{TN}^\times(X, D))$$

²By analogy with the case of motives, it is the smallest ∞ -category containing $\Pi_S(V)$ for V/S finite étale, and stable under suspensions, homotopy (co)fibers.

In Proposition 5.3.2 we show the homology motive $\underline{H}_i(X)$ vanishes for $i \notin [0, 3]$ and there is an exact sequence in the Tannakian category $\text{MM}^{\text{AT}}(S, \mathbb{Q})$ of Artin-Tate motives

$$\begin{aligned} 0 \rightarrow \underline{H}_3(\text{TN}^\times(X, D)) &\rightarrow \bigoplus_{i \in I} \mathbf{1}_S(2) \xrightarrow{\sum_{i < j} p_{ij}^{i!} - p_{ij}^{j!}} \bigoplus_{i < j} M_S(D_{ij})(2) \\ &\rightarrow \underline{H}_2(\text{TN}^\times(X, D)) \rightarrow \bigoplus_{i \in I} \mathbf{1}_S(1) \xrightarrow{\mu} \bigoplus_{j \in I} \mathbf{1}_S(1) \\ &\rightarrow \underline{H}_1(\text{TN}^\times(X, D)) \rightarrow \bigoplus_{i < j} M_S(D_{ij}) \xrightarrow{\sum_{i < j} p_{ij*}^{i!} - p_{ij*}^{j!}} \bigoplus_{i \in I} \mathbf{1}_S \rightarrow \underline{H}_0(\text{TN}^\times(X, D)) \rightarrow 0 \end{aligned}$$

Here μ is the quadratic Mumford matrix and $M_S(D_{ij})$ is the mixed Artin-Tate motive of $D_{ij} = D_i \times_X D_j$. In the above, $\underline{H}_0(\text{TN}^\times(X, D))$ and $\underline{H}_3(\text{TN}^\times(X, D))$ are pure of respective weights 0 and -4 , while $\underline{H}_1(\text{TN}^\times(X, D))$ and $\underline{H}_2(\text{TN}^\times(X, D))$ are mixed of weights $\{0, -2\}$ and $\{-2, -4\}$, respectively (see [58] for the notion of weights). The above applies also in the category of Artin-Tate-Nori motives $\text{MM}^{\text{AT}}(K, \mathbb{Q})$. Similarly to Artin-Tate motives, this involves constructing a homological functor from Voevodsky's category of geometric motives $\text{DM}_{\text{gm}}(K, \mathbb{Q})$ to the Tannakian category $\mathcal{M}(K)$ of Nori motives over K . We study the example of *Ramanujam's surface* Σ [74]. Over the complex numbers, it is a topologically contractible affine algebraic surface which is not homeomorphic to the affine plane. Working over a field k of characteristic different from 2, Example 5.3.4 identifies Σ 's integral motive at infinity $M^\infty(\Sigma)$ with $\mathbf{1}_k \oplus \mathbf{1}_k(2)[3]$.

Our setup provides universal formulas in the various realizations of motives, e.g., ℓ -adic, rigid, syntomic, Galois representations, etc. For example, the computation (3.3.13.a) specializes under ℓ -adic realization to the Rapoport-Zink formula for vanishing cycles [75, Lemma 2.5], and similarly for Steenbrink's limit Hodge structures [83]. We expect that Proposition 3.3.12 yields an explicit formula for Ayoub's nearby cycles in the semi-stable case, cf. [11].

We illustrate the general with concrete examples of \mathbf{A}^1 -equivalent smooth affine surfaces with non-isomorphic stable motivic homotopy types at infinity. For any integer $n > 0$, the Danielewski surface D_n is the closed subscheme of \mathbf{A}^3 cut out by the equation $x^n z = y(y - 1)$, see [29]. We note that D_1 is the Jouanolou device over \mathbf{P}^1 ; in fact, D_n is \mathbf{A}^1 -equivalent to \mathbf{P}^1 [7, §3.4]. Over any field k , one can distinguish between $\Pi_k^\infty(D_m)$ and $\Pi_k^\infty(D_n)$ for $m \neq n$ by viewing Danielewski surfaces as affine modifications of \mathbf{A}^2 . We refer to Section 5.4 for precise statements and further examples, [41] for background on \mathbf{A}^1 -contractibility of affine modifications, and [48] for first homology at infinity of Danielewski surfaces over the complex numbers. The affine modifications give an affirmative answer to Problem 3.4.5 in [7].

At this stage, we should come clean on some technical points concerning fundamental classes and orientations. First, our setup gives a quadratic generalization of Mumford's plumbing construction [71] using Chow-Witt groups. While Mumford uses orientations on the normal bundles of the branches, which are copies of the projective line, much of the subtleties in our setting come from working with twisted Milnor-Witt K -theory sheaves. The latter is needed to compute the quadratic degree maps of the intersections of the branches taking values in the Grothendieck-Witt ring. On the one hand, we develop the idea of parallelization to compute "the fundamental class of the diagonal" in terms of motivic fundamental classes [35]. In another direction closely related to differential geometry and quadratic enumerative geometry, we discuss the foundations for orientations of algebraic vector bundles via quadratic isomorphisms. Making clever choices of orientations is a key point in our computations of quadratic Mumford matrices. In this way, we can compute stable motivic invariants without appealing to SL -orientations (owing to Proposition 6.1.10 and Proposition 6.1.16). Section 6 explains this material, where we also introduce and show some fundamental properties of quadratic Picard groupoids.

Punctured tubular neighborhoods can also be applied to the study of isolated singularities of surfaces, in particular rational double points, also known as Du Val singularities. In characteristic $p > 0$, Artin [3] showed that the étale local fundamental group of such a singularities cannot always distinguish between double and regular points. We show that, with the exception of E_8 -type singularity, the stable motivic link $\mathrm{TN}^\times(\Gamma)$ of a Du Val singularity is different from the stable motivic link of $\mathrm{TN}^\times(\mathbb{A}_k^2, \{0\}) = \mathbf{1}_k \oplus \mathbf{1}_k(2)$ [3]. In particular, $\mathrm{TN}^\times(\Gamma)$ distinguishes Du Val singularities other than E_8 from regular points. For E_8 and the complex numbers, the identification $\mathrm{TN}^\times(E_8) \simeq \mathrm{TN}^\times(\mathbb{A}_k^2, \{0\})$ reflects the fact that the topological link of E_8 is the Poincaré homology 3-sphere $\Sigma(2, 3, 5)$ [73], a compact topological 3-manifold with the same singular homology groups as S^3 , whose fundamental group is isomorphic to the binary dodecahedral group. We refer to Table 1 for a summary of our computation of stable motivic links of Du Val singularities.

A final comment is that defining the stable homotopy type at infinity Π_S^∞ is the first step towards a refined invariant in unstable motivic homotopy theory. The problem of defining unstable motivic homotopy types at infinity witness the tension between unstable and stable motivic homotopy theory. For example, the six functor formalism is not available in the unstable setting. To remedy this, one can take into account all possible smooth compactifications. Nonetheless, some of the techniques developed in this paper will carry over to unstable motivic homotopy categories, e.g., the calculations in Section 3.3 hold in the cdh-topology, and one can expect more developments along these lines.

Remark 1.1.1. This paper’s results hold more generally for any motivic ∞ -category such as triangulated and abelian mixed motives, Artin-Tate motives, étale motives, torsion and ℓ -adic categories, mixed Hodge modules,... in place of SH. If there exists a realization functor that commutes with the six operations, e.g., the Betti or ℓ -adic realizations, then this follows from the universality of SH.

Conventions. Our results are couched in the axiomatic setting of [27], [62] which complements [9]. We fix a *motivic ∞ -category* ([27, Definition 2.4.45]) \mathcal{T} over the category of qcqs schemes, i.e., a *monoidal stable homotopy functor* according to [9]. Our primary example is the motivic stable homotopy category SH. In the language of presentable stable monoidal ∞ -categories [62], SH is the initial motivic ∞ -category. Thus there is a unique morphism of motivic ∞ -categories $\mathrm{SH} \rightarrow \mathcal{T}$. To maintain intuition, we shall refer to the objects of $\mathcal{T}(S)$ as \mathcal{T} -spectra over S . For more details, see Section 1.2.

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1.2. The motivic formalism. Throughout the paper, all schemes are quasi-coherent and quasi-compact, qcqs, and all separated and smooth maps are assumed to be of finite type. The natural framework for this paper is Morel-Voevodsky’s stable homotopy category $\mathrm{SH}(S)$ of the base scheme S . Owing to the works [9, 10], [27], for varying S , these categories satisfy *Grothendieck’s six functors formalism* which we will use extensively. The elimination of the noetherian hypothesis was achieved in [52, Appendix C]. Most of the results in this paper, however, can be stated in the general formalism of Grothendieck’s six functors, as axiomatized in [27]. We will freely use the language, constructions, and notations from *loc. cit.*, together with its natural ∞ -categorical enhancement of [63, 39] (which applies to premotivic model categories). Let us fix a *motivic triangulated category* \mathcal{T} , see [27, Definition 2.4.45], which also admits an ∞ -categorical enhancement (e.g., it arises from a premotivic model category). We note that \mathcal{T} satisfies Grothendieck’s six functors formalism, summarized for example

in [27, 2.4.50]. The added generality of [63] verifies that the pair of adjoint functors (f^*, f_*) , $(p_!, p^!)$ for p separated, and $(\otimes, \underline{\text{Hom}})$ are in fact adjunctions of ∞ -categories. The above applies to the following examples.

- SH – the stable motivic homotopy category, see e.g., [9, 63].
- $\text{DM}_{\mathbb{Q}}$ – rational mixed motives, see [27, Part IV].
- DM – motives defined as modules over Spitzweck’s motivic cohomology ring spectrum relative to \mathbb{Z} , see [82].³
- $\widetilde{\text{DM}}$ – Milnor-Witt motives defined as modules over Milnor-Witt motivic cohomology, if one restricts to base schemes defined over some field k of characteristic not 2; see [15], [14], [43].
- $\text{DM}_{\acute{\text{e}}\text{t}} = \text{DA}_{\acute{\text{e}}\text{t}}$ – étale mixed motives, see [13, 25].
- $D(-_{\acute{\text{e}}\text{t}}, \mathbb{Z}_{\ell})$ – ℓ -adic étale sheaves on $\mathbb{Z}[1/\ell]$ -schemes, ℓ a prime number, see [19], [25, 7.2.18], and on excellent schemes, also its subcategory $D_c^b(-_{\acute{\text{e}}\text{t}}, \mathbb{Z}_{\ell})$ of bounded complexes with constructible cohomology.
- D_B^{σ} – analytical sheaves on k -schemes for a complex embedding $\sigma : k \rightarrow \mathbb{C}$, $D_B^{\sigma}(X)$ is the derived category of sheaves on the analytical site $X^{\sigma}(\mathbb{C})$. This is classical, see also [12]. More generally, given any mixed Weil theory E over a base field k , by restricting to k -schemes, one has the category D_E of modules over the ring spectrum associated with E . See [27, §17.2] for details.
- D_{Hdg}^m – the category of motivic Hodge modules, which corresponds to complexes of Saito’s mixed Hodge modules of geometric origin (obtained by the realization of mixed motives), see [38].

These examples are naturally related via premotivic adjunctions subject to our conventions above:

$$(1.2.0.a) \quad \text{SH} \xrightarrow{\tilde{M}} \widetilde{\text{DM}} \xrightarrow{\pi} \text{DM} \xrightarrow{a^{\acute{\text{e}}\text{t}}} \text{DM}_{\acute{\text{e}}\text{t}} \begin{array}{l} \nearrow^{\rho_B} D_B^{\sigma} \\ \xrightarrow{\rho_{\ell}} D(-_{\acute{\text{e}}\text{t}}, \mathbb{Z}_{\ell}) \\ \searrow_{\rho_{\text{Hdg}}} D_{\text{Hdg}}^m \end{array}$$

- By our definitions of $\widetilde{\text{DM}}$ and DM, the first two functors are induced by taking free modules. See [27, §7.2], [79] for accounts using model categories.⁴
- The functor $a^{\acute{\text{e}}\text{t}}$ changes the topology, see [44], taking into account the Dold-Kan correspondence and the E_{∞} -ring spectra representing motivic cohomology and étale motivic cohomology.
- The functor ρ_B is defined in [12] (see [27] for mixed Weil theories).
- The functors ρ_{ℓ} and ρ_{Hdg} are defined in [25] and [38], respectively.

Formally, being part of a premotivic adjunction, each of the functors in (1.2.0.a) admits a natural right adjoint. Thus, by construction, they commute with f^* , $p_!$, \otimes . Moreover, when restricting to (quasi-)excellent schemes they also commute with the other three operations in Grothendieck’s six functors formalism, see the indicated references. With rational coefficients, both \tilde{M} and $a^{\acute{\text{e}}\text{t}}$ are equivalences (see [33] and [27], respectively). Furthermore, $\text{SH}_{\mathbb{Q}} \rightarrow \text{DM}_{\mathbb{Q}}$ is split with complementary factor *Morel’s minus part* of SH by [27, 16.2]. The reader should feel free to keep in mind a general \mathcal{T} , or specialize to SH and one of the realization functors in (1.2.0.a).

³This viewpoint was advocated in [78, 79]. If one restricts to schemes over a prime field k and inverts the characteristic exponent of k , one can employ cdh-motives as defined in [26] (using cdh-sheaves with transfers).

⁴The construction can be carried out more easily using (monoidal) ∞ -categories developed in [67].

1.3. Conventions on vector bundles and virtual vector bundles. We will follow the following convention for the correspondence between coherent locally free sheaves and vector bundles: the vector bundle $E = \mathbb{V}(\mathcal{E})$ associated with a coherent locally free sheaf of \mathcal{O}_X -modules \mathcal{E} on a scheme X is the relative spectrum of the symmetric algebra $\text{Sym}(\mathcal{E})$. For a vector bundle $p : V \rightarrow X$, we denote by V^\times the complement of the zero section.

Concerning locally free sheaves and corresponding vector bundles associated with differential properties for morphisms of schemes, we adopt the following conventions:

- Given a smooth morphism $f : X \rightarrow S$, let $\Omega_f = \Omega_{X/S}$ be the sheaf of relative Kähler differentials of f and call it the *cotangent sheaf* of f . Its associated vector bundle, the relative spectrum of the symmetric algebra of Ω_f , is the *tangent bundle* $T_f = T_{X/S}$ of f .
- Given a regular closed immersion $i : Z \rightarrow X$, with corresponding ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$, its *conormal sheaf* is the \mathcal{O}_Z -module $\mathcal{C}_i = \mathcal{C}_{Z/X} = \mathcal{I}_Z/\mathcal{I}_Z^2$. Its associated vector bundle is the *normal bundle* $N_{Z/X}$ of Z in X .
- We denote by $\mathcal{E} \otimes \mathcal{F}$ the tensor product of \mathcal{O}_X -modules and by $\mathcal{E}^\vee := \mathcal{H}om_X(\mathcal{E}, \mathcal{O}_X)$ the dual.

Given any morphism of $f : X \rightarrow S$, we let $\mathcal{L}_f = \mathcal{L}_{X/S}$ be its associated cotangent complex. In general, this is a complex of \mathcal{O}_X -modules. When f is a local complete intersection morphism (lci for short), \mathcal{L}_f is a perfect complex. Moreover, when $f : X \rightarrow S$ is lci smoothable, say $f = p \circ i : X \rightarrow Y \rightarrow S$ where $i : X \rightarrow Y$ is a regular closed immersion and $p : Y \rightarrow S$ is smooth, we have $\mathcal{L}_f = (\mathcal{C}_i \rightarrow i^*\Omega_p)$ where $i^*\Omega_p$ and \mathcal{C}_i are in homological degree 0 and 1, respectively.

We will use Deligne’s category $\underline{\mathbb{K}}(X)$ of virtual coherent locally free sheaves of \mathcal{O}_X -modules on a scheme X (see [36]). Given a locally free sheaf \mathcal{E} on X , we denote by $\langle \mathcal{E} \rangle$ its image in $\underline{\mathbb{K}}(X)$. The correspondence between coherent locally free sheaves and vector bundles extends using the same convention as above to a correspondence between virtual locally free sheaves \mathcal{V} and their associated virtual vector bundles $v = \mathbb{V}(\mathcal{V})$. Henceforth, we will switch freely between (virtual) locally free sheaves and (virtual) vector bundles without frequent mention.

Recall also that $\underline{\mathbb{K}}(X)$ can be described using Thomason’s K-theory space $K(X)$ (the infinite loop space associated with Thomason’s K-theory spectrum, [84, 3.1]) as follows: we view the simplicial set $K(X)$ as an ∞ -category and consider its associated ∞ -groupoid $K(X)^\simeq$ (the sub- ∞ -category generated by 1-morphisms that are equivalences). Then $\underline{\mathbb{K}}(X)$ is the homotopy category associated with $K(X)^\simeq$ — according to [36, 4.12, end of 4.4] and [84, 3.1.1]. This presentation has the advantage of giving an explicit functor

$$D_{\text{perf}}(X) \rightarrow \underline{\mathbb{K}}(X), \mathcal{K} \mapsto \langle \mathcal{K} \rangle$$

by associating to a perfect complex \mathcal{K} of \mathcal{O}_X -modules the corresponding 0-simplex of $K(X)$, which follows from the very construction of Thomason using complicial biWaldhausen categories.

Recall Deligne’s (rank-)determinant functor of Picard categories

$$\underline{\mathbb{K}}(X) \xrightarrow{(\text{rk}, \det)} \underline{\mathbb{Z}}_X \times \underline{\text{Pic}}(X), \mathcal{V} \mapsto (\text{rk } \mathcal{V}, \det \mathcal{V})$$

where $\underline{\text{Pic}}(X)$ denote Deligne’s Picard category of invertible sheaves on X [36], and for a virtual locally free sheaf \mathcal{V} , $\det \mathcal{V}$ is the *determinant* of \mathcal{V} and $\text{rk } \mathcal{V}$ is its virtual rank.

Given an lci morphism $f : X \rightarrow S$, the *virtual tangent bundle* $\tau_f = \tau_{X/S}$ of X/S is the virtual vector bundle on X associated to $\langle \mathcal{L}_f \rangle$. The *canonical sheaf* $\omega_f = \omega_{X/S}$ of X/S is the determinant $\det \langle \mathcal{L}_f \rangle$ of $\langle \mathcal{L}_f \rangle$. For a morphism of schemes $f : X \rightarrow Y$ and a (virtual) locally free sheaf \mathcal{V} on Y , we denote by $f^{-1}\mathcal{V}$ the pullback of \mathcal{V} to X .

2. COMPLEMENTS ON SIX FUNCTORS

2.1. Thom spaces.

2.1.1. The *Thom space* of a vector bundle $p : V \rightarrow X$ with zero section $s : X \rightarrow V$ is the object

$$\text{Th}(V) = \text{Th}_X(V) := p_{\sharp} s_* (\mathbf{1}_X) \in \mathcal{T}(X)$$

Here p_{\sharp} is the left adjoint of p^* . For a coherent locally free sheaf of \mathcal{O}_X -modules \mathcal{E} , we use also sometimes use the notation $\mathrm{Th}(\mathcal{E})$ as a short hand for $\mathrm{Th}(\mathbb{V}(\mathcal{E}))$. The *Tate twist* is a particular case of this notation, namely, we have $\mathbf{1}_X(n) = \mathrm{Th}(\mathcal{O}_X^n)[-2n] = \mathrm{Th}(\mathbf{A}_X^n)[-2n]$. According to the stability property of \mathcal{T} ([27, 2.4.4, 2.4.14]), the object $\mathrm{Th}(V)$ is \otimes -invertible in $\mathcal{T}(X)$ with \otimes -inverse ([27, 2.4.1, 2.4.12])

$$\mathrm{Th}(-V) := s^!p^*(\mathbf{1}_X) = s^!(\mathbf{1}_V)$$

The construction of Thom spaces is functorial in V and, as a consequence of the localization property of \mathcal{T} ([27, 2.4.6, 2.4.10]), it uniquely extends to a monoidal functor (cf. [27, 2.4.15] and [9, 1.5.18])

$$\mathrm{Th} : \underline{\mathbf{K}}(X) \rightarrow \mathcal{T}(X)$$

from Deligne's category $\underline{\mathbf{K}}(X)$ of virtual locally free sheaves on X . For an arbitrary (resp. separated) morphism of schemes $f : Y \rightarrow X$ and a virtual vector bundle v over X , the projection formula and the \otimes -invertibility of $\mathrm{Th}(V)$ imply the exchange isomorphism

$$(2.1.1.a) \quad f^* \mathrm{Th}(v) \xrightarrow{\simeq} \mathrm{Th}(f^{-1}v) \quad (\text{resp. } \mathrm{Th}(f^{-1}v) \otimes f^!(\mathbf{1}_X) \xrightarrow{\simeq} f^! \mathrm{Th}(v))$$

To comply with Morel-Voevodsky's definition, we introduce the following.

Definition 2.1.2. Let $f : X \rightarrow S$ be a smooth morphism and let v a virtual vector bundle over X . The *Thom space of v relative to S* is the object

$$\mathrm{Th}_S(v) = f_{\sharp}(\mathrm{Th}(v)) \in \mathcal{T}(S)$$

Beware that when f is not the identity, the functor Th_S is not monoidal.

In the sequel, when we do not indicate the base of a Thom space, we consider it over the same base scheme as the virtual bundle.

Example 2.1.3. (1) If $\mathcal{T} = \mathrm{SH}$ and $v = \langle V \rangle$ for a vector bundle V/X , then by homotopy purity $\mathrm{Th}_S(v) \simeq \Sigma^\infty(V/V^\times)$.

(2) If $\mathcal{T} = \widetilde{\mathrm{DM}}$, the Thom space $\mathrm{Th}_S(v)$ depends only on the rank and determinant of v (see [33, §7] for a more precise statement).

(3) If \mathcal{T} is oriented in the sense of [27, 2.4.38], e.g., any category under $\widetilde{\mathrm{DM}}$ in (1.2.0.a), then for every virtual vector bundle v of virtual rank n on a smooth S -scheme $p : X \rightarrow S$, there is a canonical *Thom isomorphism* $\mathrm{Th}_S(v) \xrightarrow{\simeq} \mathbf{1}_S(n)[2n]$ compatible with pullbacks and the \otimes -structure on the functor Th . Since Thom spaces are always reduced to Tate twists for oriented theories, this is mainly interesting for generalized theories such as Chow-Witt groups, hermitian K-theory, and stable (co)homotopy.

2.2. Internal theories and functoriality. The six functors formalism encodes the axioms of four (co)homology theories; see e.g., [20] for the combination of cohomology and Borel-Moore homology. Next, we give a systematic definition from the motivic point of view.

Definition 2.2.1. Let $f : X \rightarrow S$ be a separated morphism and let v a virtual vector bundle over X . One associates to X/S and v the following objects of $\mathcal{T}(S)$:

- Homotopy: $\Pi_S(X, v) = f_!(\mathrm{Th}(v) \otimes f^!(\mathbf{1}_S))$
- Cohomotopy: $\mathrm{H}_S(X, v) = f_*(\mathrm{Th}(v) \otimes f^*(\mathbf{1}_S)) \simeq f_*(\mathrm{Th}(v))$
- Borel-Moore (or properly supported) homotopy: $\Pi_S^c(X, v) = f_*(\mathrm{Th}(v) \otimes f^!(\mathbf{1}_S))$
- Properly supported cohomotopy: $\mathrm{H}_S^c(X, v) = f_!(\mathrm{Th}(v) \otimes f^*(\mathbf{1}_S)) \simeq f_!(\mathrm{Th}(v))$

When $v = 0$, we simply write $\Pi_S(X)$, $\mathrm{H}_S(X)$, $\Pi_S^c(X)$, $\mathrm{H}_S^c(X)$.

The natural transformation $\alpha_f : f_! \rightarrow f_*$ yields canonical maps:

$$(2.2.1.a) \quad \alpha_{X/S} : \Pi_S(X, v) \rightarrow \Pi_S^c(X, v)$$

$$(2.2.1.b) \quad \alpha'_{X/S} : \mathrm{H}_S^c(X, v) \rightarrow \mathrm{H}_S(X, v) \text{ ("forgetting proper support")}$$

Both $\alpha_{X/S}$ and $\alpha'_{X/S}$ are isomorphisms whenever X/S is proper.

Remark 2.2.2. If X/S is smooth separated, $\Pi_S(X)$ is called the premotive of X/S in [27]. For all \mathcal{T} , with the exception of $D_c^b(-, \mathbb{Z}_\ell)$, the objects $\Pi_S(X)(n)$ for X/S smooth generate $\mathcal{T}(X)$ under colimits.

Example 2.2.3. Here is a summary comparing our notations with more familiar ones.

- (1) $\mathcal{T} = \text{SH}$ and X/S smooth: $\Pi_S(X) = \Sigma^\infty X_+$ and for a vector bundle V on X , we have $\Pi_S(X, \langle V \rangle) = \Sigma^\infty \text{Th}(V)$.
- (2) $\mathcal{T} = \text{DM}$ and X/S smooth: $\Pi_S(X)$ is Voevodsky's motive $M_S(X)$ of X/S . When X/S is proper and X is regular, $H_S(X) =: h_S(X)$ is the *relative Chow-motive* of X/S . It is a pure motive of weight 0 in the sense of Bondarko. See [60] for the comparison of these objects with Corti-Hanamura's definition.
- (3) $\mathcal{T} = \text{DM}$, k a perfect field, X/k smooth separated: $\Pi_k(X) = M(X) = \underline{C}_* L(X)$, where, with the notations of [86, chap. 5], \underline{C}_* is the Suslin complex functor, and $L(X)$ is the sheaf with transfers represented by X . If k is of characteristic 0, or one works with $\text{DM}[1/p]$ if k has characteristic $p > 0$, then $\Pi_k^c(X) = M^c(X) = \underline{C}_* L^c(X)$ where $L^c(X)$ the sheaf of quasi-finite correspondences (see [86, chap. 5] in characteristic 0 and [25, 8.10] in general).
- (4) $\mathcal{T} = D_c^b(-\text{ét}, \mathbb{Z}_\ell)$ and $f : X \rightarrow S$ any morphism: $H_S(X) = \mathbf{R}f_*(\mathbb{Z}_\ell)$ is the complex computing étale cohomology of X in $D_c^b(S_{\text{ét}}, \mathbb{Z}_\ell)$. In particular, if $S = \text{Spec}(k)$, the complex compute absolute étale cohomology of X after forgetting the action of the absolute Galois group of k . Similarly, $H_S^c(X)$ computes cohomology with compact support.
- (5) $\mathcal{T} = \text{DM}_h$: using the model category of [25], for a smooth S -scheme X , $\Pi_S(X)$ is obtained as the infinite suspension of the h-sheaf represented by X .

Remark 2.2.4. As explained in Section 1.2, the comparison functors from SH to the other motivic categories \mathcal{T} considered in *loc. cit.* commute with the six operations provided that one restricts to excellent base schemes. In particular, the four internal theories considered in SH realize the corresponding theories in \mathcal{T} – of course, this universal property of SH was at the heart of Voevodsky's theory since the beginning. See [39] for a complete account incorporating the six functors. Practically any assertion concerning these internal theories proved in SH is equally valid in \mathcal{T} .

2.2.5. Natural functoriality: For a morphism $f : Y \rightarrow X$ between separated S -schemes, we have the following naturally induced maps (which explain our choice of terminology):

- $f_* : \Pi_S(Y, f^{-1}v) \rightarrow \Pi_S(X, v)$
- $f^* : H_S(X, v) \rightarrow H_S(Y, f^{-1}v)$
- $f_* : \Pi_S^c(Y, f^{-1}v) \rightarrow \Pi_S^c(X, v)$, when f is proper
- $f^* : H_S^c(X, v) \rightarrow H_S^c(Y, f^{-1}v)$, when f is proper

In addition, when f is proper then the comparison maps $\alpha_{X/S}$ and $\alpha'_{X/S}$ (see (2.2.1.a) and (2.2.1.b)) are compatible with f_* and f^* .

Remark 2.2.6. Homotopy twisted by some virtual bundle w on Y , with or without compact support, is not covariant functorial unless w is the pullback of some virtual bundle on X .

Example 2.2.7. Suppose X/S is a separated S -scheme, and let $\nu : X_0 \rightarrow X$ be the immersion on the underlying reduced subscheme (in fact any nil-immersion will work). The localization property for \mathcal{T} implies that (ν^*, ν_*) is an equivalence of categories ([27, 2.3.6]). As $\nu_* = \nu_!$ it follows that $\nu^* = \nu^!$. For any virtual vector bundle v on X and $v_0 = \nu^*(v)$, one deduces the naturally induced isomorphisms

$$\begin{aligned} \nu_* : \Pi_S(X_0, v_0) &\xrightarrow{\cong} \Pi_S(X, v), \quad \nu_* : \Pi_S^c(X_0, v_0) \xrightarrow{\cong} \Pi_S^c(X, v) \\ \nu^* : H_S(X, v) &\xrightarrow{\cong} H_S(X_0, v_0), \quad \nu^* : H_S^c(X, v) \xrightarrow{\cong} H_S^c(X_0, v_0) \end{aligned}$$

In particular, with $v = 0$, we get

$$\Pi_X(X_0) \simeq \Pi_X^c(X_0) \simeq H_X(X_0) \simeq H_X^c(X_0) \simeq \mathbf{1}_X$$

2.2.8. A smooth separated S -scheme $f : X \rightarrow S$ is said to be *stably \mathbf{A}^1 -contractible over S* if the induced map $f_* : \Pi_S(X) \rightarrow \mathbf{1}_S$ is an isomorphism. Note that due to the existence of the conservative family $(s^*)_{s \in S}$ of [27, Prop. 4.3.17], this property is equivalent to ask that for every point $s \in S$, the fiber X_s is stably \mathbf{A}^1 -contractible over $\kappa(s)$.

Lemma 2.2.9. *Let S be a regular scheme and suppose $f : X \rightarrow S$ is stably \mathbf{A}^1 -contractible over S . Then every virtual bundle v over X is constant relative to S , i.e., $v = f^*v_0$ for some virtual vector bundle v_0 over S .*

*Moreover, let T be the tangent bundle of X/S and let v_0 be the virtual vector bundle over S such that $\langle T \rangle = f^*v_0$. Then there is a naturally induced isomorphism*

$$f_*f^!(-) \simeq \mathrm{Th}_S(v_0) \otimes -$$

Proof. The first assertion is a consequence of the representability of K_0 in $\mathrm{SH}(S)$. To prove the assertion, one considers for every object \mathbb{E} of $\mathcal{T}(S)$ the composite of exchange isomorphisms

$$f_*f^!(\mathbb{E}) \stackrel{(a)}{\simeq} f_*(\mathrm{Th}(T) \otimes f^*(\mathbb{E})) = f_*(\mathrm{Th}(f^*v_0) \otimes f^*(\mathbb{E})) \stackrel{(b)}{\simeq} \mathrm{Th}(v_0) \otimes f_*f^*(\mathbb{E}) \stackrel{(c)}{\simeq} \mathrm{Th}(v_0) \otimes \mathbb{E}$$

Here (a) is an instance of the relative purity isomorphism, (b) follows from the fact that $\mathrm{Th}(v_0)$ is \otimes -invertible, and (c) holds because f is a stable \mathbf{A}^1 -weak equivalence and since f is smooth, one has: $f_*f^*(\mathbb{E}) \simeq \underline{\mathrm{Hom}}(\Pi_S(X), \mathbb{E})$.⁵ \square

Definition 2.2.10. Let $f : Y \rightarrow X$ be a morphism of separated S -schemes and let v be a virtual vector bundle over X . We denote the homotopy cofiber of $f_* : \Pi_S(Y, f^{-1}v) \rightarrow \Pi_S(X, v)$ by $\Pi_S(X/Y, v)$ so that there is an homotopy exact sequence

$$\Pi_S(Y, f^{-1}v) \rightarrow \Pi_S(X, v) \rightarrow \Pi_S(X/Y, v)$$

2.3. Virtual fundamental classes, homotopical smoothness and purity.

2.3.1. Exceptional functoriality (Gysin maps): Due to the existence of the fundamental classes introduced in [35] the four theories in Definition 2.2.1 satisfy exceptional functoriality (see [35, 4.3.4] for the general case of a triangulated motivic category).

Let $f : Y \rightarrow X$ be a smoothable lci morphism, i.e., f factors as a regular closed immersion followed by a smooth morphism, with cotangent complex \mathcal{L}_f and associated virtual tangent bundle τ_f . One deduces, from the system of fundamental classes in [35, Theorem 3.3.2], the canonical natural transformation

$$(2.3.1.a) \quad \mathfrak{p}_f(-) : \mathrm{Th}(\tau_f) \otimes f^* \rightarrow f^!$$

By adjunction, one deduces trace and cotrace maps (see §4.3.4 in *loc. cit.*)

$$\mathrm{tr}_f : f_!(\mathrm{Th}(\tau_f) \otimes f^*) \rightarrow \mathrm{Id} \quad \text{and} \quad \mathrm{cotr}_f : \mathrm{Id} \rightarrow f_*(\mathrm{Th}(-\tau_f) \otimes f^!)$$

The latter maps induce the *Gysin maps*:

- $f^! : \Pi_S(X, v) \rightarrow \Pi_S(Y, f^{-1}v - \tau_f)$, when f is proper
- $f_! : \mathrm{H}_S(Y, f^{-1}v + \tau_f) \rightarrow \mathrm{H}_S(X, v)$, when f is proper
- $f^! : \Pi_S^c(X, v) \rightarrow \Pi_S^c(Y, f^{-1}v - \tau_f)$
- $f_! : \mathrm{H}_S^c(Y, f^{-1}v + \tau_f) \rightarrow \mathrm{H}_S^c(X, v)$

Again, assuming f is proper, the comparison maps $\alpha_{X/S}$ and $\alpha'_{X/S}$ are compatible with the above Gysin morphisms in the obvious sense.

2.3.2. Fundamental classes. Characteristic classes are cohomology classes used for classification and computations. It is also possible to define these invariants as cohomotopy classes. Recall also that fundamental classes extend to bivariant homotopy (suitably twisted), see [35] as already mentioned in 2.3.1.

⁵Recall the last isomorphism follows from the axioms of premotivic categories: indeed by the smooth projection formula, $f_*f^*(-) = \Pi_S(X) \otimes -$ and we conclude as f_*f^* is right adjoint to $f_!f^*$.

Example 2.3.3. Euler exact sequence and Euler classes. Let $f : X \rightarrow S$ be a smooth S -scheme and let V be a vector bundle of rank r on X . From the localization triangle associated with the zero section s of V and the homotopy property $\Pi_S(V) \simeq \Pi_S(X)$, one derives the homotopy exact sequence

$$\mathrm{Th}_S(V)[-1] \rightarrow \Pi_S(V^\times) \rightarrow \Pi_S(X) \xrightarrow{s^!} \mathrm{Th}_S(V)$$

Note that, by definition, when $X = S$, then $s^! : \mathbf{1}_X \rightarrow \mathrm{Th}(V)$ is the realization in $\mathcal{S}(X)$ of V 's Euler class $e(V) \in \mathrm{SH}(X)$ defined in [35, Definition 3.1.2]. When $f : X \rightarrow S$ is not the identity, then $s^!$ is the image of the realization of $e(V)$ by $f_!$. This justifies our notation $e_S(V, \mathcal{S}) = s^!$. In particular, note that $e_S(V, \mathcal{S})$ is zero whenever V contains the trivial line bundle \mathbf{A}_X^1 as a direct summand (*loc. cit.*, Corollary 3.1.8).

In the case S is the spectrum of a field, we have the following:

- (1) When $\mathcal{S} = \mathrm{DM}$ or, more generally, when \mathcal{S} is oriented, the *motivic Euler class*

$$e(V) : \mathbf{1}_X \rightarrow \mathrm{Th}(V) \simeq \mathbf{1}_X(n)[2n]$$

corresponds to the top Chern class $c_n(V)$ under the isomorphism $\mathbb{H}_M^{2n,n}(X) \simeq \mathrm{CH}^n(X)$.

- (2) As a map in $\widetilde{\mathrm{DM}}(X)$, the realization of the stable homotopy Euler class $e(V)$ corresponds to Barge-Morel-Fasel's Euler class in the Chow-Witt group $\widetilde{\mathrm{CH}}^n(X, \det V)$ of X twisted by the determinant of V .

For a smoothable lci morphism $f : X \rightarrow S$ with virtual tangent bundle τ_f one has the canonical class

$$\eta_f : \mathrm{Th}(\tau_f) \rightarrow f^!(\mathbf{1}_S)$$

which we will consider as a homotopy class in

$$H_0^{\mathcal{S}}(X/S, \tau_f) := [\mathrm{Th}(\tau_f), f^!(\mathbf{1}_S)] = [f_!(\mathrm{Th}(\tau_f)), \mathbf{1}_S]$$

for the bivariant homology theory (with respect to \mathcal{S}) of X/S and twist τ_f . In fact, this bivariant class is a cohomotopy class; that is, an element of the abelian group

$$H_{\mathcal{S}}^n(X, \tau_f) := [\mathbf{1}_X, \mathrm{Th}(\tau_f)[n]]$$

We impose the following assumptions.

- (1) f is proper.
(2) there exists a virtual bundle v over S and an isomorphism $\epsilon : \tau_f \simeq f^{-1}(v)$. The couple (ϵ, v) , or simply ϵ when v is clear, will be called an *f -parallelization* of τ_f .

In this case, we can consider the composite map

$$H_{\mathcal{S}}^0(X) \xrightarrow{\epsilon_*} H_{\mathcal{S}}^0(X, \tau_f - f^{-1}v) \xrightarrow{f_!} H_{\mathcal{S}}^0(S, -v)$$

Here, the choice of ϵ yields the first map, and the second one is the Gysin map in cohomotopy (see *op. cit.*). The image of the unit element 1 in cohomotopy $H_{\mathcal{S}}^0(X)$ can be deduced from the fundamental class η_f via the composite

$$\mathrm{Th}(v) \xrightarrow{ad_f} f_* f^*(\mathrm{Th}(v)) \simeq f_!(\mathrm{Th}(f^{-1}v)) \xrightarrow{\epsilon_*} f_!(\mathrm{Th}(\tau_f)) \xrightarrow{\eta_f} \mathbf{1}_S$$

Definition 2.3.4. Let $f : X \rightarrow S$ be a proper smoothable lci map with an f -parallelization (ϵ, v) of its virtual tangent bundle. The associated twisted fundamental class is given by

$$\eta_f^\epsilon = f_! \epsilon_*(1) \in H_{\mathcal{S}}^0(S, -v)$$

When $f = i : Z \rightarrow X$ is a regular closed immersion, and we consider an f -parallelization (ϵ, v) of its normal bundle N_i , corresponding to an f -parallelization $\epsilon' : \tau_i = -\langle N_i \rangle \rightarrow -v$, we also define the twisted fundamental class of (Z, ϵ) in X as

$$[Z]_X^\epsilon = f_! \epsilon'_*(1) \in H_{\mathcal{S}}^0(X, v)$$

Example 2.3.5. In our definition, the reader might be surprised by the cohomotopical index 0. The “true” degree is hidden in the twist. In particular, for $\mathcal{T} = \text{DM}$ (resp. $\widetilde{\text{DM}}$), and a rank d virtual bundle v over a smooth k -scheme X , we have

$$H_{\text{DM}}^0(X, v) \simeq \text{CH}^d(X), \quad (\text{resp. } H_{\widetilde{\text{DM}}}^0(X, v) \simeq \widetilde{\text{CH}}^d(X, \det v))$$

The Chow (resp. Chow-Witt) group of X (resp. twisted by the invertible sheaf $\det(v)$). For $\mathcal{T} = \text{SH}$, there is also a canonical isomorphism $H_{\text{SH}}^0(X, v) \simeq \widetilde{\text{CH}}^d(X, \det(v))$, see Proposition 6.2.2 in the Appendix. In the motivic case or any of the oriented triangulated motivic categories of (1.2.0.a), the motivic fundamental class of a closed immersion $i : Z \rightarrow X$ and f -parallelization (ϵ, v) is the usual cycle class of Z in $\text{CH}^d(X)$ (resp. in the relevant cohomology in degree $2d$ and twist d). It is independent of the chosen f -parallelization. This is not the case in the category of Milnor-Witt motives and in SH, as modifying the twist \mathcal{L} in $\widetilde{\text{CH}}^d(X, \mathcal{L})$ can change the group.

Example 2.3.6. Given a regular closed immersion $i : Z \rightarrow X$, a way to obtain an i -parallelization of the normal bundle N_i is to consider an lci morphism $p : X \rightarrow Z'$ such that $p \circ i$ is étale. Indeed, in that case, if τ_p denotes the virtual tangent bundle of p , we get a canonical isomorphism $\epsilon : \langle N_i \rangle \simeq i^{-1}\tau_p$ as the tangent bundle of $p \circ i$ is trivial.

An important example for us comes from the diagonal immersion $\delta : X \rightarrow X \times_S X$ of a smooth S -scheme X . It admits two smooth retractions given by the projections p_j , for $j = 1, 2$. We denote the corresponding twisted fundamental classes by

$$[\Delta_{X/S}]_{X \times X}^j \in H_{\mathcal{T}}^0(X \times_S X, p_j^{-1}\langle T_{X/S} \rangle)$$

2.3.7. Homotopical smoothness and purity.

Definition 2.3.8. (See also [35, Definition 4.3.7]). Let $f : X \rightarrow S$ be a smoothable lci morphism with virtual tangent bundle τ_f . We say that f is *homotopically smooth* (*h-smooth*) with respect to the motivic ∞ -category \mathcal{T} if the natural transformation

$$\mathfrak{p}_f(-) : \text{Th}(\tau_f) \otimes f^* \rightarrow f^!$$

(see (2.3.1.a)) evaluated at the sphere spectrum $\mathbf{1}_S$ is an isomorphism $\mathfrak{p}_f : \text{Th}(\tau_f) \rightarrow f^!(\mathbf{1}_S)$.

2.3.9. One gets the following basic properties of h-smoothness: considering composable lci smoothable morphisms $f, g, h = f \circ g$ (which is also lci smoothable), if f and g (resp. f and h) are h-smooth, then so is h (resp. g). Moreover, if $g^!$ is conservative, g and h being h-smooth implies that f is h-smooth. On the other hand, h-smoothness is not stable under base change.

Example 2.3.10. Here are some examples of h-smooth maps $f : X \rightarrow S$.

- f is smooth
- X, S are smooth over some base B and f is a morphism of B -schemes
- X, S are regular over a field k and \mathcal{T} is continuous, see [33, Appendix A] (all our examples are continuous in this sense)
- (Absolute purity) X and S are regular and $\mathcal{T} = \text{SH}_{\mathbb{Q}}, \text{DM}_{\mathbb{Q}}, \text{DM}_{\text{ét}}, D(-_{\text{ét}}, \mathbb{Z}_{\ell})$

In particular, a closed immersion between smooth varieties over a field is h-smooth. On the other hand, not all regular closed immersions are h-smooth:

Example 2.3.11. Consider the regular closed immersion

$$i : Z = Z_1 \cup_{\{o\}} Z_2 \rightarrow X = \mathbf{A}^2$$

of the union of coordinate axes $Z_j \simeq \mathbf{A}^1$, $j = 1, 2$ in the affine plane \mathbf{A}^2 over a field k . We claim that i is not h-smooth (see Corollary 3.3.7 and Example 3.3.8 for more context).

The normal bundle $N_{Z/X}$ is the trivial line bundle of rank 1. Let $i_0 : \{o\} \rightarrow Z$ be the induced closed immersion and note that the composite immersion $i \circ i_0 : \{o\} \rightarrow X$ is h-smooth, with trivial normal bundle $N_{\{o\}/X}$ of rank 2. Now apply cdh-descent to the canonically induced cdh-distinguished square of closed immersions

$$\begin{array}{ccc} \{o\} & \xrightarrow{i_{0,1}} & Z_1 \\ i_{0,2} \downarrow & \searrow i_0 & \downarrow i_1 \\ Z_2 & \xrightarrow{i_2} & Z \end{array}$$

We obtain the homotopy exact sequence

$$\mathbf{1}_Z \rightarrow i_{1*}\mathbf{1}_{Z_1} \oplus i_{2*}\mathbf{1}_{Z_2} \rightarrow i_{0*}\mathbf{1}_{\{o\}}$$

Applying $i_0^!$ to this sequence and using the base change isomorphisms $i_0^!i_{j,*}(\mathbf{1}_{Z_j}) \simeq i_{0,j}^!(\mathbf{1}_{Z_j})$ and the purity isomorphisms $i_{0,j}^!(\mathbf{1}_{Z_j}) \simeq \mathrm{Th}_{\{o\}}(-N_{\{o\}/Z_j}) \simeq \mathbf{1}_k(-1)[-2]$ for the h-smooth closed immersions $i_{0,j} : \{o\} \rightarrow Z_j$ we get the homotopy exact sequence

$$i_0^!(\mathbf{1}_Z) \rightarrow \mathbf{1}_k(-1)[-2] \oplus \mathbf{1}_k(-1)[-2] \rightarrow \mathbf{1}_k$$

The second map in the above sequence is given by a pair of elements in $\pi_{2r,r}(k)$ for some $r < 0$. Hence it is trivial, and we obtain the isomorphism (see Corollary 3.3.7 for a generalization)

$$i_0^!(\mathbf{1}_Z) \simeq \mathbf{1}_k(-1)[-2] \oplus \mathbf{1}_k(-1)[-2] \oplus \mathbf{1}_k[-1]$$

On the other hand, if i was h-smooth, we would have $i^!(\mathbf{1}_X) \simeq \mathrm{Th}_Z(-N_{Z/X})$. Hence, by applying $i_0^!$ and using (2.1.1.a) and the \otimes -invertibility of $\mathrm{Th}_{\{o\}}(i_0^{-1}N_{Z/X})$, we would obtain isomorphisms

$$i_0^!(\mathbf{1}_Z) \simeq i_0^!i^!(\mathbf{1}_X) \otimes \mathrm{Th}_{\{o\}}(i_0^{-1}N_{Z/X}) \simeq \mathrm{Th}_{\{o\}}(-N_{\{o\}/X}) \otimes \mathrm{Th}_{\{o\}}(i_0^{-1}N_{Z/X}) \simeq \mathbf{1}_k(-1)[-2]$$

The h-smoothness property allows one to compare the four different theories in Definition 2.2.1 and generalizes the smooth case. More precisely, the ‘‘associativity formula’’ for fundamental classes in [35, Theorem 3.3.2] implies the next result.

Proposition 2.3.12. *Let $f : X \rightarrow S$ be an h-smooth morphism with virtual tangent bundle τ_f . Then the purity isomorphism $\mathfrak{p}_f : \mathrm{Th}(\tau_f) \rightarrow f^!(\mathbf{1}_S)$ induces isomorphisms*

$$\begin{aligned} \Pi_S(X, v) &= f_!(\mathrm{Th}(v) \otimes f^!(\mathbf{1}_S)) \xrightarrow{\mathfrak{p}_f^{-1}} f_!(\mathrm{Th}(v) \otimes \mathrm{Th}(\tau_f)) = \mathrm{H}_S^c(X, v + \tau_f) \\ \Pi_S^c(X, v) &= f_*(\mathrm{Th}(v) \otimes f^!(\mathbf{1}_S)) \xrightarrow{\mathfrak{p}_f^{-1}} f_*(\mathrm{Th}(v) \otimes \mathrm{Th}(\tau_f)) = \mathrm{H}_S(X, v + \tau_f) \end{aligned}$$

Moreover, these isomorphisms transform the natural functoriality (resp. Gysin map) in the source to the Gysin map (resp. natural functoriality) on the target.

2.4. Closed pairs. A closed S -pair is a pair (X, Z) consisting of a separated S -scheme $f : X \rightarrow Z$ and a closed subscheme $i : Z \hookrightarrow X$ of X . For such a pair we denote by $j : X - Z \rightarrow X$ the complementary open immersion, so that we have a commutative diagram

$$(2.4.0.a) \quad \begin{array}{ccccc} Z & \xrightarrow{i} & X & \xleftarrow{j} & X - Z \\ & \searrow p & \downarrow f & \swarrow q & \\ & & S & & \end{array}$$

A morphism $(\Phi, \varphi) : (Y, T) \rightarrow (X, Z)$ of closed S -pairs is a topologically cartesian commutative diagram

$$(2.4.0.b) \quad \begin{array}{ccc} T & \longrightarrow & Y \\ \varphi \downarrow & & \downarrow \Phi \\ Z & \longrightarrow & X \end{array}$$

Here, the horizontal maps are closed immersions. Note that $\Pi_S(X/X - Z)$ is functorial for morphisms of closed S -pairs. A morphism of closed S -pairs (Φ, φ) is said to be cartesian if (2.4.0.b) is cartesian. It is said to be Nisnevich-excisive (resp. cdh-excisive) if (2.4.0.b) is Nisnevich-distinguished (resp. cdh-distinguished) in the sense of [85]. An excisive morphism of closed S -pairs induces an isomorphism in $\mathcal{T}(S)$. Indeed, this follows from Nisnevich excision, which is implied by the localization property in [27, 3.3.4].

Definition 2.4.1. A closed S -pair (X, Z) is *weakly smooth* (resp. *weakly h-smooth*) if there exists a Nisnevich neighborhood V of Z in X such that V and Z are smooth (resp. h-smooth, see Definition 2.3.8) over S .

We note that for closed S -pairs as in Definition 2.4.1, the closed immersion $i : Z \rightarrow X$ is necessarily regular with normal bundle $N_{Z/X}$.

2.4.2. Suppose (X, Z) is a closed S -pair with the property that X is h-smooth over S in some Nisnevich neighborhood of its closed subscheme Z . Then, although the cotangent complex $\mathcal{L}_{X/S}$ might not be a perfect complex on X , it restricts by assumption to a perfect complex on a suitable Nisnevich neighborhood of Z in X . Thus, one can canonically define $i^{-1}\tau_{X/S}$ as a virtual vector bundle on Z (by choosing an appropriate Nisnevich neighborhood and showing that it is independent of the choice).

We extend the Morel-Voevodsky homotopy purity theorem as follows, see also Theorem 3.4.3 for a refinement when Z has smooth crossing singularities.

Theorem 2.4.3. *Let (X, Z) be a closed S -pair and let v be virtual vector bundle on X . Then the following hold:*

(1) *If X is h-smooth over S in a Nisnevich neighborhood of Z , then there are canonical purity isomorphisms*

$$(2.4.3.a) \quad \begin{aligned} \Pi_S(X/X - Z, v) &\simeq H_S^c(Z, i^{-1}v + i^{-1}\tau_{X/S}) \\ H_S(X/X - Z, v) &\simeq \Pi_S^c(Z, i^{-1}v - i^{-1}\tau_{X/S}) \end{aligned}$$

(2) *If moreover (X, Z) is weakly h-smooth, then there are canonical purity isomorphisms*

$$(2.4.3.b) \quad \begin{aligned} \Pi_S(X/X - Z, v) &\simeq \Pi_S(Z, i^{-1}v + \langle N_{Z/X} \rangle) \\ H_S(X/X - Z, v) &\simeq H_S(Z, i^{-1}v - \langle N_{Z/X} \rangle) \end{aligned}$$

which are natural for morphisms of weakly h-smooth closed S -pairs.

Proof. By Nisnevich excision for closed S -pairs, we are reduced to the case where $f : X \rightarrow S$ is h-smooth, with virtual tangent bundle τ_f . The fact that the two isomorphisms do not depend on the choice of a Nisnevich neighborhood follows by the functoriality of the excision isomorphism. With the notation (2.4.0.a), by inserting $\mathrm{Th}(v) \otimes f^!(\mathbf{1}_S)$ in the localization exact homotopy sequence

$$j_!j^! \rightarrow \mathrm{Id} \rightarrow i_*i^*$$

and applying $f_!$ we get the exact homotopy

$$\Pi_S(X - Z, j^{-1}v) \rightarrow \Pi_S(X, v) \rightarrow p_!(\mathrm{Th}(i^{-1}v) \otimes i^*f^!(\mathbf{1}_S))$$

Here we used the identifications

$$\begin{aligned} f_!j_!j^!(\mathrm{Th}(v) \otimes f^!(\mathbf{1}_S)) &\simeq q_!(\mathrm{Th}(j^{-1}v) \otimes q^!(\mathbf{1}_S)) = \Pi_S(X - Z, j^{-1}v) \\ f_!i_*i^*(\mathrm{Th}(v) \otimes f^!(\mathbf{1}_S)) &\simeq f_!i_!(\mathrm{Th}(i^{-1}v) \otimes i^*f^!(\mathbf{1}_S)) = p_!(\mathrm{Th}(i^{-1}v) \otimes i^*f^!(\mathbf{1}_S)) \end{aligned}$$

In particular, there is an isomorphism

$$\Pi_S(X/X - Z, v) \simeq p_!(\mathrm{Th}(i^{-1}v) \otimes i^*f^!(\mathbf{1}_S))$$

The purity isomorphism then yields the desired isomorphism

$$\begin{aligned} \Pi_S(X/X - Z, v) &\simeq p_!(\mathrm{Th}(i^{-1}v) \otimes i^* f^!(\mathbf{1}_S)) \xrightarrow{\mathfrak{p}_f^{-1}} p_!(\mathrm{Th}(i^{-1}v) \otimes i^*(\mathrm{Th}(\tau_f) \otimes f^*(\mathbf{1}_S))) \\ &= p_!(\mathrm{Th}(i^{-1}v + i^{-1}\tau_f)) = \mathrm{H}_S^c(Z, i^{-1}v + i^{-1}\tau_f) \end{aligned}$$

In the case where Z/S is h-smooth, with virtual tangent bundle τ_p , the purity isomorphism \mathfrak{p}_p in Proposition 2.3.12 yields in turn an isomorphism

$$\mathrm{H}_S^c(Z, i^{-1}v + i^{-1}\tau_f) \cong \Pi_S(Z, i^{-1}v + i^{-1}\tau_f - \tau_p) = \Pi_S(Z, i^{-1}v + \langle N_{Z/X} \rangle)$$

The second isomorphism in Theorem 2.4.3 is now a direct consequence of the h-smoothness property of Z/S .

The dual statements for $\mathrm{H}_S(X/X - Z, v)$ follow from similar arguments applied to the dual localization homotopy exact sequence $i_! v^! \rightarrow Id \rightarrow j_* j^*$. \square

2.5. Computations of weak duals.

2.5.1. Recall [30, 5.2] that an object M of a monoidal category with unit $\mathbf{1}$ is said to be *rigid* (or *strongly dualizable*) with dual M^\vee if there exists pairing and co-pairing maps

$$\mu : M \otimes M^\vee \rightarrow \mathbf{1}, \epsilon : \mathbf{1} \rightarrow M^\vee \otimes M$$

satisfying relations that express the functors $M \otimes -$ and $- \otimes M^\vee$ as both left and right adjoints. In a general symmetric monoidal category, if an object M is rigid, then $\underline{\mathrm{Hom}}(M, \mathbf{1})$ is a (strong) dual of M , and the duality pairing is given by the evaluation map $M \otimes \underline{\mathrm{Hom}}(M, \mathbf{1}) \rightarrow \mathbf{1}$. This justifies the terminology *weak dual* of M for the object $\underline{\mathrm{Hom}}(M, \mathbf{1})$. Next, we highlight some weaker results which will be useful in the remaining. We first pin down a notion that appears to be missing in previous works on the six functors formalism.

Definition 2.5.2. A separated morphism $f : X \rightarrow S$ is called *pre- \mathcal{T} -dualizing* if the map

$$(2.5.2.a) \quad \mathbf{1}_X \rightarrow \underline{\mathrm{Hom}}(f^!(\mathbf{1}_S), f^!(\mathbf{1}_S))$$

obtained by adjunction from the identity of $f^!(\mathbf{1}_S)$ is an isomorphism in $\mathcal{T}(X)$.

Example 2.5.3. According to Definition 2.3.8, any h-smooth morphism is pre-dualizing.

Remark 2.5.4. The notion of a pre-dualizing morphism is closely linked with Grothendieck-Verdier duality, as shown in [27, 4.4.11]. In fact, if $f^!(\mathbf{1}_S)$ is a dualizing object ([27, Definition 4.4.4]), then f is pre-dualizing. Thus, it follows from [9] that f is pre-SH-dualizing as soon as its target is smooth over a field of characteristic 0. In many cases, if the target of f is regular, then f is pre-dualizing: see [57] for $D(-\acute{\mathrm{e}}\mathrm{t}, \mathbb{Z}_\ell)$, [27] for DM, [26] for DM $_{\acute{\mathrm{e}}\mathrm{t}}$, and [33] for SH $_{\mathbb{Q}}$.

The following proposition provides formulas for some weak duals, hence for potential strong duals when they exist.

Proposition 2.5.5. *Let $f : X \rightarrow S$ be a separated S -scheme and let v be a virtual vector bundle over X . Then the following hold:*

- (1) *There exists a canonical isomorphism*

$$\underline{\mathrm{Hom}}(\mathrm{H}_S^c(X, v), \mathbf{1}_S) \xrightarrow{\cong} \Pi_S^c(X, -v)$$

which is functorial in X , for both the natural functoriality for proper maps (2.2.5) and for the Gysin morphisms for smoothable lci morphisms (2.3.1).

- (2) *If moreover f is pre-dualizing, then there exists an isomorphism*

$$\underline{\mathrm{Hom}}(\Pi_S(X, v), \mathbf{1}_S) \xrightarrow{\cong} \mathrm{H}_S(X, -v)$$

which is again functorial for the natural functorialities and Gysin maps.

- (3) If moreover f is h -smooth, with virtual tangent bundle τ_f , then the purity isomorphism \mathfrak{p}_f induces canonical isomorphisms

$$\underline{\mathrm{Hom}}(\Pi_S(X, v), \mathbf{1}_S) \simeq \Pi_S^c(X, -v - \tau_f) \quad \text{and} \quad \underline{\mathrm{Hom}}(\mathrm{H}_S^c(X, v), \mathbf{1}_S) \simeq \mathrm{H}_S(X, -v + \tau_f)$$

which are natural with respect to the natural functorialities and the Gysin maps, both restricted to proper morphisms.

Proof. To prove the isomorphism in (1) we use

$$\begin{aligned} \underline{\mathrm{Hom}}(\mathrm{H}_S^c(X, v), \mathbf{1}_S) &= \underline{\mathrm{Hom}}(f_!(\mathrm{Th}(v)), \mathbf{1}_S) \xrightarrow{(a)} f_* \underline{\mathrm{Hom}}(\mathrm{Th}(v), f^!(\mathbf{1}_S)) \\ &\simeq f_*(\mathrm{Th}(-v) \otimes f^!(\mathbf{1}_S)) = \Pi_S^c(X, -v) \end{aligned}$$

Here, (a) (resp. (b)) follows from the internal interpretation of the fact that $f^!$ is right adjoint to $f_!$ (resp. that $\mathrm{Th}(v)$ is \otimes -invertible).

To deduce (2), we consider the isomorphisms

$$\begin{aligned} \underline{\mathrm{Hom}}(\Pi_S(X, v), \mathbf{1}_S) &= \underline{\mathrm{Hom}}(f_!(\mathrm{Th}(v) \otimes f^!(\mathbf{1}_S)), \mathbf{1}_S) \xrightarrow{(a)} f_* \underline{\mathrm{Hom}}(\mathrm{Th}(v) \otimes f^!(\mathbf{1}_S), f^!(\mathbf{1}_S)) \\ &\simeq f_* \left(\mathrm{Th}(-v) \otimes \underline{\mathrm{Hom}}(f^!(\mathbf{1}_S), f^!(\mathbf{1}_S)) \right) \\ &\simeq f_*(\mathrm{Th}(-v) \otimes \mathbf{1}_X) = \mathrm{H}_S(X, -v) \end{aligned}$$

Here, (a) and (b) are justified as before in (1), and (c) follows from the assumption that f is pre-dualizing. The isomorphisms in (3) are a combination of (1) and (2), and the isomorphisms of Proposition 2.3.12.

Each functoriality statement is clear by construction. \square

Example 2.5.6. Here are known examples to which Proposition 2.5.5 applies to give formulas for strong duals:

- (1) For $\mathcal{T} = \mathrm{SH}(k)$, where k is a field of characteristic 0, according to [76, Theorem 1.4] any constructible spectrum is rigid. It follows from [10] that the six operations preserve constructibility for morphisms of k -schemes finite type.
 - (a) In particular, $\mathrm{H}_k^c(X, v)$ and $\Pi_k^c(X, v)$ are both rigid, and the point (1) above shows that $\Pi_k^c(X, v)$ is dual to $\mathrm{H}_k^c(X, -v)$ (and reciprocally).
 - (b) Similarly, $\Pi_k(X, v)$ and $\mathrm{H}_k(X, v)$ are constructible, and thus rigid. As Remark 2.5.4 shows that X/k is pre-dualizing, point (2) of the above proposition shows that $\Pi_k(X, v)$ is dual to $\mathrm{H}_k(X, -v)$. See Proposition 3.4.1 for a generalization.
 - (c) Finally, if X is smooth, point (3) shows that $\Pi_k(X, v)$ is dual to $\Pi_k^c(X, -v - \langle T_{X/k} \rangle)$, which is the expected generalization of Poincaré duality. This result will be extended in Theorem 3.4.2.
- (2) Using [22, Theorem 2.4.9] (see also [53, Theorem 5.8]), the same results hold in $\mathrm{SH}(k)[1/p]$ if k has positive characteristic p .

Over a base scheme S of positive dimension, the situation is more complicated. When X/S is smooth and proper, Example 2.5.7 shows that $\Pi_S(X, v) = \Pi_S^c(X, v)$ is rigid for any virtual vector bundle v . Theorems 3.4.1, 3.4.3 and Corollary 3.5.4 below give several new examples of rigid relative spectra and motives. In general, neither properness nor smoothness alone ensures rigidity, see Example 2.5.8.

Example 2.5.7. Poincaré duality (see [30, 5.4]). Let $f : X \rightarrow S$ be a smooth proper S -scheme with tangent bundle T . Then, for any virtual bundle v over X , $\Pi_S(X, v)$ is rigid with dual

$$\Pi_S(X, -\langle T \rangle - v) = \mathrm{Th}_S(-v - \langle T \rangle)$$

Note that the given expression of the dual corresponds to that in Proposition 2.5.5(2) via the purity isomorphism $\Pi_S(X, -v - \langle T \rangle) \simeq \mathrm{H}_S^c(X, -v) = \mathrm{H}_S(X, -v)$ of Proposition 2.3.12.

Indeed, letting $\delta : X \rightarrow X \times_S X$ be the diagonal closed immersion, the pairing and co-pairing maps are given by the composite maps

$$\begin{aligned} \Pi_S(X, v) \otimes \Pi_S(X, -v - \langle T \rangle) &\stackrel{(*)}{\simeq} \Pi_S(X \times_S X, -\langle p_1^{-1}T_f \rangle) \xrightarrow{\delta^!} \Pi_S(X) \xrightarrow{f_*} \mathbf{1}_S \\ \mathbf{1}_S \xrightarrow{f^!} \Pi_S(X, -\langle T \rangle) &\xrightarrow{\delta_*} \Pi_S(X \times_S X, -\langle p_1^{-1}T \rangle) \stackrel{(*)}{\simeq} \Pi_S(X, -v - \langle T \rangle) \otimes \Pi_S(X, v) \end{aligned}$$

Here the labels $(*)$'s are instances of the Künneth isomorphism (2.6.1.b) given in the next subsection. The required identities follow from the base change formula for Gysin morphisms in [35, 3.3.2(iii)].

Example 2.5.8. Let $i : Z \rightarrow S$ be a h-smooth closed immersion (e.g., Z and S are smooth over a field k) with nonempty open complement $j : U \rightarrow S$. We claim that $\Pi_S(U) = j_!(\mathbf{1}_U)$ is not rigid. Indeed, assuming the contrary, according to Proposition 2.5.5 its dual would be isomorphic to $j_*(\mathbf{1}_U)$. Since i^* is monoidal, it would follow that $i^*j_!(\mathbf{1}_U)$ is rigid with dual $i^*j_*(\mathbf{1}_U)$. The first spectrum is trivial, whereas purity identifies the second one with an extension of $\mathbf{1}_Z$ by $\mathrm{Th}(N_{Z/S})$, which is thus necessarily a nontrivial spectrum. An identical (dual) argument shows that $\Pi_S(Z)$ is not rigid.

In a similar vein, [72, Remark 8.2] gives the following: let $S = \mathrm{Spec}(R)$ be the spectrum of a discrete valuation R with quotient field K . Then $\Pi_S(\mathrm{Spec}(K))$ is not rigid in $\mathrm{SH}(S)$.

2.6. Künneth isomorphisms. We collect here several variants of Künneth formulas (see also Proposition 3.3.6).

Example 2.6.1. Künneth isomorphisms. Let X, Y be separated S -schemes and v, w be virtual vector bundles over X, Y , respectively. Then one deduces from the projection and base change formulas a canonical isomorphism (obtained from exchange isomorphisms, see [27])

$$(2.6.1.a) \quad H_S^c(X, v) \otimes H_S^c(Y, w) \simeq H_S^c(X \times_S Y, p_1^{-1}v + p_2^{-1}w)$$

If X and Y are in addition smooth over S , then we have the more usual Künneth formula (see [27, 1.1.37])

$$(2.6.1.b) \quad \Pi_S(X, v) \otimes \Pi_S(Y, w) \simeq \Pi_S(X \times_S Y, p_1^{-1}v + p_2^{-1}w)$$

One can also deduce (2.6.1.b) from the previous one by using the relative purity isomorphism. Example 2.6.2 shows the second Künneth formula (2.6.1.b) fails in the non-smooth case.

Example 2.6.2. One can extend the Künneth formula (2.6.1.b) to the non smooth case (see below for example) but one still needs assumptions. Indeed, one cannot replace in general smoothness by h-smoothness. For example, for the zero section $s : X \rightarrow \mathbf{A}_X^n = S$, $n \geq 1$, one has $\Pi_S(X) = s_*(\mathbf{1}_X)(n)[2n]$ and

$$\Pi_S(X) \otimes_S \Pi_S(X) = s_*(\mathbf{1}_X)(n)[2n] \otimes s_*(\mathbf{1}_X)(n)[2n] = s_*(\mathbf{1}_X)(2n)[4n]$$

The latter is different from $\Pi_S(X \times_S X) = \Pi_S(X)$ (in any of our motivic ∞ -categories).

2.6.3. In the following result, we give some new cases of Künneth formulas to compute stable homotopy at infinity (see Propositions 4.3.5 and 4.3.6). To a cartesian square of separated morphisms

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p} & Y \\ q \downarrow & \searrow h & \downarrow g \\ X & \xrightarrow{f} & S \end{array}$$

we associate the following commutative diagram of exchange transformations and the map $\alpha_?$ forgetting proper support

$$(2.6.3.a) \quad \begin{array}{ccc} f_! f^!(\mathbf{1}) \otimes g_! g^!(\mathbf{1}) & \xrightarrow{\alpha_f \otimes \alpha_g} & f_* f^!(\mathbf{1}) \otimes g_* g^!(\mathbf{1}) \\ \sim \downarrow & & \downarrow (1) \\ g_!(g^* f_! f^!(\mathbf{1}) \otimes g^!(\mathbf{1})) & \xrightarrow{\alpha_g(\alpha_f)} & g_*(g^* f_* f^!(\mathbf{1}) \otimes g^!(\mathbf{1})) \\ \sim \downarrow & & \downarrow (2) \\ g_!(p_! q^* f^!(\mathbf{1}) \otimes g^!(\mathbf{1})) & \xrightarrow{\alpha_g(\alpha_p)} & g_*(p_* q^* f^!(\mathbf{1}) \otimes g^!(\mathbf{1})) \\ \sim \downarrow & & \downarrow (3) \\ h_!(q^* f^!(\mathbf{1}) \otimes p^* g^!(\mathbf{1})) & \xrightarrow{\alpha_h} & h_*(q^* f^!(\mathbf{1}) \otimes p^* g^!(\mathbf{1})) \\ (4) \downarrow & & \downarrow (4) \\ h_!(h^!(\mathbf{1})) & \xrightarrow{\alpha_h} & h_*(h^!(\mathbf{1})) \end{array}$$

Here, α_r denotes any map induced by the natural transformation $r_! \rightarrow r_*$.

Theorem 2.6.4. *With the above notation, assume that one of the following conditions is satisfied:*

- i) Y is smooth and proper over S .
- ii) S is the spectrum of a field k of characteristic exponent p and either \mathcal{T} is $\mathbb{Z}[1/p]$ -linear or receives a realization functor from $\mathrm{DM}_{\text{ét}}$ as in (1.2.0.a).
- iii) Y is smooth and stably \mathbf{A}^1 -contractible over S with stably constant tangent bundle T_g (see 2.2.8).

Then all the vertical maps in (2.6.3.a) are isomorphisms, and there is an induced commutative diagram

$$\begin{array}{ccc} \Pi_S(X) \otimes \Pi_S(Y) & \xrightarrow{\alpha_X \otimes \alpha_Y} & \Pi_S^c(X) \otimes \Pi_S^c(Y) \\ \sim \downarrow & & \downarrow \sim \\ \Pi_S(X \times_S Y) & \xrightarrow{\alpha_{XY}} & \Pi_S^c(X \times_S Y) \end{array}$$

Proof. In each case, we have to prove that the morphisms (1) to (4) in (2.6.3.a) are isomorphisms. Case i) is transparent. Next, we consider Case ii). If \mathcal{T} is $\mathbb{Z}[1/p]$ -linear then all the isomorphisms follow from [61, Theorem 2.4.6] with $Y_1 = Y_2 = S$, $X_1 = X$, $X = Y$. More precisely, the composite of (1), (2), and (3) is an isomorphism due to point (2) of 2.4.6, and (4) is an isomorphism by (3) of 2.4.6. If \mathcal{T} receives a functor from $\mathrm{DM}_{\text{ét}}$, one can reduce to the latter case by appealing to [23, Sec. 3.1].

It remains to prove the assertion in Case iii). The isomorphism (4) follows from the fact that g (resp. q) is smooth with tangent bundle T_g (resp. $T_q = p^* T_g$), and from the relative purity isomorphism

$$q^* f^!(\mathbf{1}_S) \otimes p^* g^!(\mathbf{1}_S) \simeq q^* f^!(\mathbf{1}_S) \otimes p^* \mathrm{Th}(T_g) \simeq q^! f^!(\mathbf{1}_S) = h^!(\mathbf{1}_S)$$

Using Lemma 2.2.9 applied respectively to q and g , one deduces

$$h_* h^!(\mathbf{1}_S) = f_* q_* q^! f^!(\mathbf{1}_S) \simeq f_* \mathrm{Th}(f^* v_0) \otimes f^!(\mathbf{1}_S) = \mathrm{Th}(v_0) \otimes f_* f^!(\mathbf{1}_S) \simeq f_* f^!(\mathbf{1}_S) \otimes g_* g^!(\mathbf{1}_S)$$

where v_0 is the virtual vector bundle over S such that $\langle T_g \rangle = g^* v_0$.

It is now a formal, though lengthy, exercise to check that the preceding isomorphism is equal to the composition of the maps (1)-(4). \square

3. CANONICAL RESOLUTIONS OF CROSSING SINGULARITIES

3.1. Ordered Čech semi-simplicial scheme associated to a closed cover.

3.1.1. Let X be a noetherian scheme and consider a finite closed cover of X , i.e., a surjective map

$$p : X_\bullet = \sqcup_{i \in I} X_i \rightarrow X$$

obtained from a finite collection of closed immersions $\nu_i : X_i \rightarrow X$, $i \in I$. We let $\cap = \times_X$ be a shorthand for the fiber product of closed X -schemes. For every nonempty subset $J \subset I$ we set $X_J = \cap_{j \in J} X_j$ and denote by $\nu_J : X_J \rightarrow X$ the canonically induced closed immersion. For every pair

of nonempty subsets $J \subset K$ of I , we let $\nu_K^J : X_K \rightarrow X_J$ be the canonically induced closed immersion so that we have $\nu_K = \nu_J \circ \nu_K^J$.

The Čech simplicial X -scheme $\check{S}_*(X_\bullet/X)$ associated with p takes the form

$$(3.1.1.a) \quad \check{S}_n(X_\bullet/X) := \bigsqcup_{(i_0, \dots, i_n) \in I^{n+1}} X_{i_0} \cap \dots \cap X_{i_n}$$

with degeneracy morphisms $\delta_n^k : \check{S}_n(X_\bullet) \rightarrow \check{S}_{n-1}(X_\bullet)$, $k = 0, \dots, n$, given by the sum of the canonical immersions

$$X_{i_0} \cap \dots \cap X_{i_k} \cap \dots \cap X_{i_n} \rightarrow X_{i_0} \cap \dots \cap \widehat{X_{i_k}} \cap \dots \cap X_{i_n}$$

The choice of a total ordering on I induces a natural bijection between the set of subsets $J \subset I$ of cardinality $\#J = n + 1$ and the set of $(n + 1)$ -tuples $(i_0, \dots, i_n) \in I^{n+1}$ given by mapping a subset J to the unique $(n + 1)$ -tuple $(i_0, \dots, i_n) \in I^{n+1}$ such that $J = \{i_0, \dots, i_n\}$ and $i_0 < \dots < i_n$. In the following we fix such a total ordering and we set

$$(3.1.1.b) \quad \check{S}_n^{\text{ord}}(X_\bullet/X) := \bigsqcup_{\substack{(i_0, \dots, i_n) \in I^{n+1} \\ i_0 < \dots < i_n}} X_{i_0} \cap \dots \cap X_{i_n} = \bigsqcup_{J \subset I, \#J = n+1} X_J$$

There is a canonical embedding $\check{S}_n^{\text{ord}}(X_\bullet/X) \subset \check{S}_*(X_\bullet/X)$ of \mathbb{N} -graded Z -schemes given in degree n by mapping each $X_{j_0} \cap \dots \cap X_{j_n}$ to itself via the identity. The degeneracy morphisms δ_n^k in the simplicial structure on $\check{S}_*(X_\bullet/X)$ preserve $\check{S}_n^{\text{ord}}(X_\bullet/X)$ and induce degeneracy morphisms

$$\delta_n^k = \bigsqcup_{J = \{i_0 < \dots < \widehat{i_k} < \dots < i_n\} \subset K = \{i_0 < \dots < i_n\}} \nu_K^J : \check{S}_n^{\text{ord}}(X_\bullet/X) \rightarrow \check{S}_{n-1}^{\text{ord}}(X_\bullet/X)$$

endowing $\check{S}_n^{\text{ord}}(X_\bullet/X)$ with the structure of a semi-simplicial X -scheme⁶. We refer to the latter as the *ordered Čech semi-simplicial X -scheme* associated to the finite closed cover $p : X_\bullet \rightarrow X$.

Remark 3.1.2. By construction, the ordered Čech semi-simplicial scheme $\check{S}_n^{\text{ord}}(X_\bullet/X)$ is bounded by the cardinality $\#I$ of the index set I in the sense that $\check{S}_n^{\text{ord}}(X_\bullet/X) = \emptyset$ for all $n > \#I$. In particular, it is much smaller than $\check{S}_*(X_\bullet/X)$.

3.2. Ordered hyperdescent for closed covers.

3.2.1. We now use the ∞ -categorical enhancement of the motivic category \mathcal{T} , and in particular the adjunction of ∞ -functors (f^*, f_*) and $(f_!, f^!)$. Let us fix a base scheme S and write Sch_S for the category of separated S -schemes. To any object \mathbb{E} of $\mathcal{T}(S)$, we associate the covariant ∞ -functor

$$\Pi_S(-; \mathbb{E}) : \text{Sch}_S \rightarrow \mathcal{T}(S), (f : X \rightarrow S) \mapsto f_! f^!(\mathbb{E})$$

and, dually, the contravariant ∞ -functor

$$\text{H}_S(-; \mathbb{E}) : \text{Sch}_S^{\text{op}} \rightarrow \mathcal{T}(S), (f : X \rightarrow S) \mapsto f_* f^*(\mathbb{E})$$

3.2.2. Back to the setup in 3.1.1, we assume in addition that $f : X \rightarrow S$ is a separated S -scheme. For every nonempty subset $J \subset I$, we let $f_J : X_J \rightarrow S$ be the composite of the closed immersion $\nu_J : X_J \rightarrow X$ with $f : X \rightarrow S$. To the ordered Čech semi-simplicial X -scheme $\check{S}_n^{\text{ord}}(X_\bullet/X)$ and any object \mathbb{E} of $\mathcal{T}(S)$, we associate the functors

$$\begin{aligned} & ((\Delta^{\text{inj}})^{\text{op}} \rightarrow \text{Sch}_S) \xrightarrow{\Pi_S(-; \mathbb{E})} \mathcal{T}(S) \\ & (\Delta^{\text{inj}} \rightarrow \text{Sch}_S^{\text{op}}) \xrightarrow{\text{H}_S(-; \mathbb{E})} \mathcal{T}(S) \end{aligned}$$

⁶Recall that a semi-simplicial object in a category \mathcal{C} is a contravariant functor from $\Delta^{\text{inj}} \rightarrow \mathcal{C}$, where Δ^{inj} denotes the category of finite ordered sets with injective maps as morphisms.

By using the augmentation map to X , we obtain canonical maps involving the limit and colimit of the preceding functors

$$(3.2.2.a) \quad \Pi_{X_\bullet/X;\mathbb{E}} : \operatorname{colim}_{n \in (\Delta^{\text{inj}})^{\text{op}}} \left(\bigoplus_{J \subset I, \#J=n+1} \Pi_S(X_J; \mathbb{E}) \right) \rightarrow \Pi_S(X; \mathbb{E})$$

$$(3.2.2.b) \quad \mathbb{H}_{X_\bullet/X;\mathbb{E}} : \mathbb{H}_S(X; \mathbb{E}) \rightarrow \lim_{n \in \Delta^{\text{inj}}} \left(\bigoplus_{J \subset I, \#J=n+1} \mathbb{H}_S(X_J; \mathbb{E}) \right)$$

The next theorem interprets the colimit (resp. limit) as the “standard” resolution of homology (resp. cohomology) of X/S with \mathbb{E} -coefficients.

Theorem 3.2.3. *For every finite closed cover $p : X_\bullet \rightarrow X$, the maps $\Pi_{X_\bullet/X;\mathbb{E}}$ and $\mathbb{H}_{X_\bullet/X;\mathbb{E}}$ are both isomorphisms in $\mathcal{T}(S)$.*

Proof. Using Example 2.2.7, we can reduce to the case where X and each X_i are reduced.

Let us consider the case of $\Pi_{X_\bullet/S;\mathbb{E}}$. For every nonempty subset $J \subset I$, there is an isomorphism $f_J! f_J^! \simeq f_J! \nu_J! \nu_J^! f_J^!$. So by replacing \mathbb{E} with $f^!(\mathbb{E})$, we are reduced to the case $S = X$. There is, see for example [33, B.20], a conservative family of functors

$$i_x^! : \mathcal{T}(X) \rightarrow \mathcal{T}(\operatorname{Spec}(\kappa(x))), x \in X$$

Therefore, it suffices to show $i_x^!(\Pi_{X_\bullet/X;\mathbb{E}})$ is an isomorphism for all $x \in X$. Given $J \subset I$, we consider the following cartesian square

$$\begin{array}{ccc} X'_J & \xrightarrow{i'_x} & X_J \\ \nu'_J \downarrow & & \downarrow \nu_J \\ \{x\} & \xrightarrow{i_x} & X \end{array}$$

By proper base change for the proper map ν_J , we have an isomorphism $i_x^! \nu_J! \nu_J^! \simeq \nu'_J! i_x^! \nu_J^!$. Since, on the other hand, we have $\nu'_J! i_x^! \nu_J^! \simeq \nu'_J! \nu_J^! i_x^!$, and because the pullback of the ordered Čech complex $\check{S}_*^{\text{ord}}(X_\bullet/X)$ along $\{x\} \rightarrow X$ corresponds to the ordered Čech complex $\check{S}_*^{\text{ord}}(X_\bullet \times_X \{x\}/\{x\})$, we deduce the isomorphism

$$i_x^!(\Pi_{X_\bullet/X;\mathbb{E}}) \simeq \Pi_{X_\bullet \times_X \{x\}/\{x\}; i_x^! \mathbb{E}}$$

Since X is reduced, we may therefore assume $X = \{x\}$ is the Zariski spectrum of a field. In this case, the X'_i 's are closed reduced subschemes of the reduced scheme $\{x\}$, and thus the closed cover $p' : \sqcup_{i \in I} X'_i \rightarrow \{x\}$ is given by a sum of identity maps. To conclude, one can then observe, for example, the existence of explicit homotopy contraction of the semi-simplicial augmented pointed X -scheme

$$\check{S}_*^{\text{ord}}(X_\bullet/\{x\})_+ \rightarrow \{x\}_+$$

The proof for the map $\mathbb{H}_{X_\bullet/X;\mathbb{E}}$ is entirely analogous, using the conservative family of functors

$$i_x^* : \mathcal{T}(X) \rightarrow \mathcal{T}(\operatorname{Spec}(\kappa(x))), x \in X$$

of [27, Proposition 4.3.17]. □

Remark 3.2.4. In formulas (3.2.2.a) and (3.2.2.b), one can arbitrarily replace the closed subscheme X_J of X by its reduction according to Example 2.2.7. In the followings, we will use that possibility without further warning.

Remark 3.2.5. Theorem 3.2.3 does not extend to arbitrary cdh-covers. For instance, it does not work for the proper cdh-cover $\mathbf{P}_k^1 \rightarrow \operatorname{Spec} k$ for apparent reasons: for such a connected cover, one needs the whole Čech complex to get a resolution of the point. Similarly, the ordered Čech complex associated with a nontrivial finite étale cover does not yield a resolution in the étale topology. In the

cdh-topology it is possible to generalize Theorem 3.2.3 by replacing closed covers $p : X_\bullet \rightarrow X$ by proper cdh-covers such that there exists a stratification of X having the property that for every stratum Y , there exists a member of the covering family $X_i \rightarrow X$ for which $X_i \times_X Y \rightarrow Y$ is an isomorphism. The proof of Theorem 3.2.3 carries over to this setting by applying the proper base change theorem, and this generalization allows in particular to incorporate the elementary cdh-covers. A similar consideration applies to Nisnevich covers.

3.3. Schemes and subschemes with crossing singularities.

Notations 3.3.1. Let Z be a separated S -scheme with finitely many irreducible components $Z'_i, i \in I$. For every nonempty subset $J \subset I$, we let $Z'_J = (\bigcap_{j \in J} Z'_j)$ and $Z_J = (Z'_J)_{\text{red}}$. We denote by ν_J the canonically induced closed immersion of Z_J in Z . For every pair of nonempty subsets $J \subset K$ of I , we denote by $\nu_K^J : Z_K \rightarrow Z_J$ the naturally induced closed immersion. For a virtual vector bundle v on Z and a nonempty subset $J \subset I$, we let $v_J = \nu_J^{-1}v$.

For a closed S -pair (X, Z) corresponding to a closed subscheme $i : Z \rightarrow X$ with irreducible components $Z'_i, i \in I$, we extend the above notation by setting

$$\bar{\nu}_J = i \circ \nu_J : Z_J \rightarrow Z \rightarrow X$$

For a virtual vector bundle v on X , we let v_J denote the pullback of v to Z_J by $\bar{\nu}_J$.

We fix the following terminology on normal crossing singularities in the rest of this paper.

Definition 3.3.2. With the notation above, we say that Z has *smooth* (resp. *regular*, *h-smooth*) *reduced crossing* over S if, for any non-empty $J \subset I$, Z_J is a smooth (resp. regular, h-smooth) S -scheme.

With our conventions, the intersection of the irreducible components of Z is allowed to have non-trivial multiplicity. Note that h-smoothness is insensible to reduction; we will simply write *h-smooth crossing*.

Proposition 3.3.3. *Let Z/S be an h-smooth crossing scheme and let v is a virtual vector bundle on Z . Then $\Pi_S(Z, v)$ is isomorphic to the colimit in the underlying ∞ -category of $\mathcal{T}(S)$ of the diagram*

$$(3.3.3.a) \quad \Pi_S(Z_I, v_I) \rightrightarrows \bigoplus_{K \subset I, \#K = \#I - 1} \Pi_S(Z_K, v_K) \rightrightarrows \dots \bigoplus_{J \subset I, \#J = 2} \Pi_S(Z_J, v_J) \rightrightarrows \bigoplus_{i \in I} \Pi_S(Z_i, v_i)$$

with degeneracy maps

$$(\delta_n^k)_* = \sum_{J = \{i_0 < \dots < \widehat{i_k} < \dots < i_n\} \subset K = \{i_0 < \dots < i_n\}} (\nu_K^J)_*$$

and with augmentation map

$$\sum_{i \in I} \nu_{i*} : \bigoplus_{i \in I} \Pi_S(Z_i, v_i) \rightarrow \Pi_S(Z, v)$$

Dually, $H_S(Z, v)$ is isomorphic to the limit of the diagram

$$(3.3.3.b) \quad \bigoplus_{i \in I} H_S(Z_i, v_i) \rightrightarrows \bigoplus_{J \subset I, \#J = 2} H_S(Z_J, v_J) \rightrightarrows \dots \bigoplus_{K \subset I, \#K = \#I - 1} H_S(Z_K, v_K) \rightrightarrows H_S(Z_I, v_I)$$

with co-degeneracy maps

$$(\delta_n^k)^* = \sum_{J = \{i_0 < \dots < \widehat{i_k} < \dots < i_n\} \subset K = \{i_0 < \dots < i_n\}} (\nu_K^J)^*$$

and with co-augmentation map

$$\sum_{i \in I} \nu_i^* : H_S(Z, v) \rightarrow \bigoplus_{i \in I} H_S(Z_i, v_i)$$

Proof. Consider the closed cover $Z_\bullet = \bigsqcup Z'_i \rightarrow Z$ of Z by its irreducible components. Noting that by Example 2.2.7 we have, for every $J \subset I$, canonical isomorphisms $\Pi_S(Z'_J, v'_J) \simeq \Pi_S(Z_J, v_J)$ and $\mathbb{H}(Z'_J, v'_J) \simeq \mathbb{H}(Z_J, v_J)$, the assertion follows by appealing to Theorem 3.2.3 with $S = X = Z$, $X_\bullet = Z_\bullet$ and $\mathbb{E} = \mathrm{Th}(v) \otimes f^!(\mathbf{1}_S)$ (resp. $\mathbb{E} = \mathrm{Th}(v)$) and then applying $f_!$ (resp. f_*) to the obtained resolution. \square

Example 3.3.4. In the case $\mathcal{T} = \mathrm{SH}$, the S -scheme Z in Proposition 3.3.3 defines a sheaf of sets \underline{Z} on Sm_S . We claim the preceding computation yields an isomorphism $\Pi_S(Z) \simeq \Sigma^\infty \underline{Z}_+$ in $\mathrm{SH}(S)$. A proof uses the \mathbf{P}^1 -stable \mathbf{A}^1 -homotopy category $\underline{\mathrm{SH}}_{\mathrm{cdh}}(S)$ over S for the big cdh site; i.e., the site of finite type S -schemes endowed with the cdh-topology in the style of [27, §6.1]. Theorem 3.2.3 holds in $\underline{\mathrm{SH}}_{\mathrm{cdh}}(S)$ due to cdh-descent, so the comparison reduces to the smooth case, which holds by the general properties of an enlargement.

Example 3.3.5. In the cases $\mathcal{T} = \mathrm{DM}_{\acute{\mathrm{e}}\mathrm{t}}, \mathrm{DM}_{\mathbb{Q}}$, Proposition 3.3.3 takes a simpler form for motives. Indeed, one considers the complex of representable h-sheaves

$$(3.3.5.a) \quad \mathbb{Z}_S^h(Z_I) \xrightarrow{d_{c-2}} \dots \xrightarrow{d_1} \bigoplus_{J \subset I, \#J=2} \mathbb{Z}_S^h(Z_J) \xrightarrow{d_0} \bigoplus_{i \in I} \mathbb{Z}_S^h(Z_i)$$

with differentials given by the alternating sums $d_i = \sum_{k=0}^n (-1)^k (\delta_n^k)_*$. This complex defines an object K of $\mathrm{DM}_{\mathrm{h}}^{\mathrm{eff}}(S)$, which computes the homotopy colimit of (3.3.3.a). Proposition 3.3.3 can thus be formulated by saying that the infinite suspension of the complex K is isomorphic to $M_S(Z)$. As in the preceding example, one can compute the motive $M_S(Z)$ as the infinite suspension of the h-sheaf $\mathbb{Z}_S^h(Z)$ represented by Z . This formula is a motivic relative version of the classical computation of the homology of a normal crossing scheme. It actually gives back the known formulas by realization of motives (Betti, étale, etc...).

A dual formula holds for computing the relative Chow motive $h_S(Z) = f_* f^*(\mathbf{1}_S)$. To that end, we consider the isomorphism $\mathrm{h}(Z_\bullet/Z, \mathbf{1}_S)$ of Theorem 3.2.3: $h_S(Z)$ is quasi-isomorphic to the image of the complex (3.3.5.a) under the (derived) internal Hom functor $\mathbf{R}\underline{\mathrm{Hom}}(-, \mathbf{1}_S)$, see also Proposition 3.4.1.

Next, we show a Künneth formula for smooth crossings schemes.

Proposition 3.3.6. *Suppose Z, T are smooth crossings S -schemes, and v, w are virtual bundles over Z and T , respectively. Then the canonical map (2.6.3.a) is an isomorphism*

$$\Pi_S(Z, w) \otimes \Pi_S(T, w) \xrightarrow{\simeq} \Pi_S(Z \times_S T, v \times_S w)$$

Proof. The case where Z/S is smooth and T/S is smooth crossing follows from Proposition 3.3.3 and the fact \otimes commutes with homotopy colimits (as a left adjoint). To treat the case where Z/S has smooth crossings, we can therefore argue by induction on the number of irreducible components of Z . Let Z' be an irreducible component of Z and Z'' the union of the other irreducible components. The cdh-distinguished homotopy exact sequence associated with the cdh-cover (Z', Z'') of Z takes the form

$$(3.3.6.a) \quad \Pi_S(Z' \times_Z Z'') \rightarrow \Pi_S(Z') \oplus \Pi_S(Z'') \rightarrow \Pi_S(Z)$$

By induction, the result holds for Z' (resp. Z'' and $Z' \times_Z Z''$) and T . We conclude by tensoring (3.3.6.a) with $\Pi_S(T)$ and applying descent for the cdh-cover $(Z' \times_S T, Z'' \times_S T)$ of $Z \times_S T$. \square

As another corollary, the following computation explains the defect of absolute purity.

Corollary 3.3.7. *Let $i : Z \rightarrow X$ be a closed immersion such that Z/X has h -smooth crossings. Then $i^!(\mathbf{1}_X)$ is isomorphic to the homotopy colimit of the diagram*

$$\mathrm{Th}_Z(N_I) \rightrightarrows \bigoplus_{K \subset I, \#K=\#I-1} \mathrm{Th}_Z(-N_K) \rightrightarrows \dots \bigoplus_{J \subset I, \#J=2} \mathrm{Th}_Z(-N_J) \rightrightarrows \bigoplus_{i \in I} \mathrm{Th}_Z(-N_i)$$

Here N_J is the normal bundle of Z_J in Z , $\mathrm{Th}_Z(-N_J)$ is the associated Thom space (of the opposite), seen over Z . For any $J \subset K$, we have the Gysin map

$$(\nu_K^J)_! : \mathrm{Th}_Z(-N_J) = \mathrm{H}_Z(Z_J, \langle -N_J \rangle) \rightarrow \mathrm{H}_Z(Z_K, \langle -N_K \rangle) = \mathrm{Th}_Z(-N_K)$$

and the degeneracy maps

$$(\delta_n^k)_* = \sum_{J=\{i_0 < \dots < \widehat{i_k} < \dots < i_n\} \subset K = \{i_0 < \dots < i_n\}} (\nu_K^J)_!$$

Proof. Applying Proposition 3.3.3 to Z/X , $v = 0$, yields a computation of $\Pi_X(Z) = i_! i^!(\mathbf{1}_X)$ as a colimit. We conclude by applying i^* and noticing that $\nu_J^!(\mathbf{1}_X) = \mathrm{Th}(-N_J)$ since $\nu_J : Z_J \rightarrow X$ is h-smooth by assumption (see Definition 2.3.8). \square

Example 3.3.8. Applying Corollary 3.3.7 to a strict normal crossing divisor in a regular scheme, we obtain a homotopy extension of a fundamental computation in étale cohomology. It also explains the failure of absolute purity for snc divisors and, more generally, for regular closed immersions that are h-smooth. The augmentation map

$$(3.3.8.a) \quad \epsilon_i : \bigoplus_{i \in I} \mathrm{Th}_Z(-N_i) \rightarrow i^!(\mathbf{1}_X)$$

coming from the above corollary can be seen as the “best” approximation of the fundamental class associated with i , in the spirit of [35].

3.3.9. Consider a closed S -pair (X, Z) such that Z has h-smooth crossings over S and such that for every nonempty subset $J \subset I$, $\bar{\nu}_J : Z_J \rightarrow X$ is an h-smooth closed immersion (see Notations 3.3.1). This holds, for instance, when X is h-smooth in a Nisnevich neighborhood of Z . In such circumstances, $\bar{\nu}_J$ is, in particular, a regular immersion. We denote its associated normal bundle by N_J . Denote by $j : X - Z \rightarrow X$ the complementary open immersion.

Proposition 3.3.10. *Let (X, Z) be a closed S -pair such that Z has h-smooth crossings over S and such that X is h-smooth over S in a Nisnevich neighborhood of Z . Let v be a virtual vector bundle on X .*

Then the object $\Pi_S(X - Z, j^{-1}v)$ is isomorphic to the limit of the diagram

$$(3.3.10.a) \quad \Pi_S(X, v) \xrightarrow{\epsilon} \bigoplus_{i \in I} \Pi_S(Z_i, v_i + \langle N_i \rangle) \rightrightarrows \bigoplus_{J \subset I, \#J=2} \Pi_S(Z_J, v_J + \langle N_J \rangle) \rightrightarrows \dots \rightrightarrows \Pi_S(Z_I, v_I + \langle N_I \rangle)$$

given by the sums of Gysin maps

$$\epsilon = \sum_{i \in I} \bar{\nu}_i^! \quad (\delta_k^n)_! = \sum_{J=\{i_0 < \dots < \widehat{i_k} < \dots < i_n\} \subset K = \{i_0 < \dots < i_n\}} (\nu_K^J)_!$$

associated to the closed immersions $\bar{\nu}_i : Z_i \rightarrow X$ and $\nu_K^J : Z_K \rightarrow Z_J$.

Dually, the object $\mathrm{H}_S(X - Z, j^{-1}v)$ is isomorphic to the colimit of the diagram

$$(3.3.10.b) \quad \mathrm{H}_S(Z_I, v_I - \langle N_I \rangle) \rightrightarrows \dots \rightrightarrows \bigoplus_{J \subset I, \#J=2} \mathrm{H}_S(Z_J, v_J - \langle N_J \rangle) \rightrightarrows \bigoplus_{i \in I} \mathrm{H}_S(Z_i, v_i - \langle N_i \rangle) \xrightarrow{\epsilon'} \mathrm{H}_S(X, v)$$

given by sums of Gysin maps

$$\epsilon' = \sum_{i \in I} \bar{\nu}_j^! \quad (\delta_k^n)_! = \sum_{J=\{i_0 < \dots < \widehat{i_k} < \dots < i_n\} \subset K = \{i_0 < \dots < i_n\}} (\nu_K^J)_!$$

Proof. With reference to (2.4.0.a), inserting $\mathbb{E} = \mathrm{Th}(v) \otimes f^!(\mathbf{1}_S)$ in the localization homotopy exact sequence $j_! j^! \rightarrow \mathrm{Id} \rightarrow i_* i^*$ and applying $f_!$ yields the homotopy exact sequence

$$\Pi_S(X - Z, j^{-1}v) = f_! j_! j^!(\mathbb{E}) \rightarrow \Pi_S(X, \mathbb{E}) = f_!(\mathbb{E}) \rightarrow f_! i_* i^*(\mathbb{E})$$

By applying Theorem 3.2.3 to the closed cover $\bigsqcup Z'_i \rightarrow Z$ of Z by its irreducible components, and then applying f_i and arguing as in the proof of Proposition 3.3.3, we obtain the isomorphism

$$f_! i_* i^*(\mathbb{E}) \simeq \lim_{n \in \Delta^{\text{inj}}} \left(\bigoplus_{J \subset I, \#J=n+1} f_! \bar{\nu}_{J*} \bar{\nu}_J^*(\mathbb{E}) \right)$$

The object $f_! \bar{\nu}_{J*} \bar{\nu}_J^*(\mathbb{E})$ of $\mathcal{T}(S)$ depends only on a Nisnevich neighborhood of Z in X . Thus, under our hypotheses, we may replace X by an h-smooth Nisnevich neighborhood of Z in X and assume that $f : X \rightarrow S$ itself is h-smooth, say with virtual relative tangent bundle τ_f . We then have the purity isomorphism $\mathbb{E} \simeq \text{Th}(v) \otimes \text{Th}(\tau_f)$. Furthermore, under our assumptions, for every $J \subset I$, $\bar{\nu}_J : Z_J \rightarrow X$ and $f_J = f \circ \bar{\nu}_J : Z_J \rightarrow S$ are h-smooth morphisms. Since $\bar{\nu}_J^{-1} \tau_f = \tau_{f_J} + \langle N_J \rangle$, where τ_{f_J} is the virtual tangent bundle of the h-smooth morphism f_J and $\text{Th}(\tau_{f_J}) \simeq f_J^!(\mathbf{1}_S)$ by purity, we obtain the isomorphisms

$$\begin{aligned} f_! \bar{\nu}_{J*} \bar{\nu}_J^*(\mathbb{E}) &= f_{J!} \text{Th}(\bar{\nu}_J^{-1} \tau_f) \otimes \text{Th}(v_J) \simeq f_{J!} (\text{Th}(\tau_{f_J}) \otimes \text{Th}(N_J) \otimes \text{Th}(v_J)) \\ &\simeq f_{J!} ((\text{Th}(v_J) \otimes f_J^!(\mathbf{1}_S)) \otimes \text{Th}(N_J)) \\ &= \Pi_S(Z_J, v_J + \langle N_J \rangle) \end{aligned}$$

It is then straightforward to check that under these isomorphisms, the maps in the diagram correspond to the announced Gysin maps.

The assertion for $H_S(X - Z, v)$ follows similarly by starting with the dual localization homotopy exact sequence $i_! i^! \rightarrow Id \rightarrow j_* j^*$. We leave further details to the reader. \square

Remark 3.3.11. Let us specialize the preceding result to the cases $\mathcal{T} = \text{DM}, \text{DM}_{\text{ét}}, \text{DM}_{\mathbb{Q}}$, and more specifically $\mathcal{T} = \text{DM}_{\mathbb{Q}}$ when considering Bondarko's weight structure (see [21]). Under the assumption and notations of Proposition 3.3.10, the motive $M_S(X - Z)$ is the limit of the augmented semi-simplicial diagram

$$(3.3.11.a) \quad M_S(X) \xrightarrow{\epsilon} \bigoplus_{i \in I} M_S(Z_i) \langle 1 \rangle \rightrightarrows \bigoplus_{J \subset I, \#J=2} M_S(Z_J) \langle 2 \rangle \dots \rightarrow M_S(Z_I) \langle c \rangle$$

with the same formulas as in (3.3.10.a) for the augmentation ϵ and the coface maps δ_k^n .

In the case where $f : X \rightarrow S$ is smooth and proper, and $Z = D$ is a normal crossing divisor with irreducible components D_i , $i \in I$, the formula for the motive $M_S(X - D)$ of the complement of a normal crossing divisor D of X/S is a relative motivic analog of the de Rham complex with logarithmic poles that Deligne used to define mixed Hodge structures. The motive of the non-proper S -scheme $X - D$ is expressed as the "complex" (3.3.11.a) whose terms $M_S(D_J) \langle \#J \rangle$ are pure of weight 0 for Bondarko's motivic weight structure. In particular, it gives a canonical and functorial weight filtration for the motive $M_S(X - D)$ (recall that a pure object of weight 0 shifted n times has weight n). We view this as a motivic analog of the fact that the weight filtration of the mixed Hodge structure on $X - D$ over $S = \text{Spec}(\mathbb{C})$ arises from the naive filtration of the De Rham complex with logarithmic poles associated with (X, D) .

Dually, we can identify the Chow motive $h_S(X - D)$ with the colimit of the diagram

$$(3.3.11.b) \quad h_S(D_I) \langle -c \rangle \rightarrow \dots \bigoplus_{J \subset I, \#J=2} h_S(D_J) \langle -2 \rangle \rightrightarrows \bigoplus_{i \in I} h_S(D_i) \langle -1 \rangle \xrightarrow{\epsilon'} h_S(X)$$

When $S = \text{Spec}(\mathbb{C})$, it follows from the identification of the orientation of the motivic spectrum representing algebraic De Rham cohomology given in [32, Example 5.4.2(1)] that the De Rham realization of (3.3.11.b), see [24, §3.1], can be canonically identified with the de Rham complex with logarithmic poles associated with (X, D) .

We finally derive the following generalization of a computation due to Rappoport and Zink, see Remark 3.3.13 for details.

Proposition 3.3.12. *Let (X, Z) be a closed S -pair corresponding to a closed immersion $i : Z \rightarrow X$ such that Z has h -smooth crossings over S and such that for every irreducible component Z'_i of Z , the induced closed immersion $\bar{\nu}_i : Z'_i \rightarrow X$ is h -smooth⁷. For every $J \subset I$, let N_J be the normal bundle of the induced regular closed immersion $\bar{\nu}_J : Z_J \rightarrow X$.*

*Then the object $i^*j_*(\mathbf{1}_{X-Z})$ of $\mathcal{T}(Z)$ is isomorphic to the colimit in the underlying ∞ -category of the augmented semi-simplicial diagram of length at most $c + 1$*

$$\mathrm{H}_Z(Z_I, \langle -N_I \rangle) \rightarrow \dots \bigoplus_{J \subset I, \#J=2} \mathrm{H}_Z(Z_J, \langle -N_J \rangle) \rightrightarrows \bigoplus_{i \in I} \mathrm{H}_Z(Z_i, \langle -N_i \rangle) \xrightarrow{\epsilon} \mathbf{1}_Z$$

where the degeneracy maps are given by the formula

$$(\delta_n^k)_! = \sum_{J=\{i_0 < \dots < \widehat{i_k} < \dots < i_n\} \subset K=\{i_0 < \dots < i_n\}} (\nu_K^J)_!$$

using the Gysin maps (see 2.3.1) associated to the regular closed immersions $\nu_K^J : Z_K \rightarrow Z_J$, $J \subset K$, and the augmentation map ϵ is obtained by composing (3.3.8.a) with the canonical map $i^!(\mathbf{1}_X) \rightarrow i^*(\mathbf{1}_X) = \mathbf{1}_Z$.

Dually, the object $i^!j_!(\mathbf{1}_{X-Z})$ in $\mathcal{T}(Z)$ is isomorphic to the limit of the following augmented semi-cosimplicial diagram of length at most $c + 1$

$$\mathbf{1}_Z \xrightarrow{\epsilon'} \bigoplus_{i \in I} \mathrm{H}_Z^c(Z_i, \langle N_i \rangle) \rightrightarrows \bigoplus_{J \subset I, \#J=2} \mathrm{H}_Z^c(Z_J, \langle N_J \rangle) \dots \rightarrow \mathrm{H}_Z^c(Z_I, \langle N_I \rangle)$$

with degeneracy maps

$$(\delta_n^k)' = \sum_{J=\{i_0 < \dots < \widehat{i_k} < \dots < i_n\} \subset K=\{i_0 < \dots < i_n\}} (\nu_K^J)_!$$

Proof. The first assertion immediately follows by applying i^* to the localization triangle

$$i^!i^!(\mathbf{1}_X) \rightarrow \mathbf{1}_X \rightarrow j_*j^*(\mathbf{1}_X) = j_*(\mathbf{1}_{X-Z})$$

and using the computation of Corollary 3.3.7. The other assertion is obtained similarly, starting from the dual localization triangle and applying. \square

Remark 3.3.13. Let \mathcal{T} be a motivic ∞ -category with a realization functor from $\mathrm{DM}_{\text{ét}}$ as in (1.2.0.a). Assume that X is regular and that $Z = D$ is a normal crossing divisor in X with irreducible components D_i , $i \in I$. The above formula shows that the motive $i^*j_*(\mathbf{1}_{X-Z})$ is the colimit in the underlying ∞ -category of the diagram

$$(3.3.13.a) \quad \nu_{I*}(\mathbf{1}_{D_I})(c)[2c] \xrightarrow{d_{c-2}} \dots \xrightarrow{d_1} \bigoplus_{J \subset I, \#J=2} \nu_{J*}(\mathbf{1}_{D_J})(2)[4] \xrightarrow{d_0} \bigoplus_{i \in I} \nu_{i*}(\mathbf{1}_{D_i})(1)[2] \xrightarrow{\epsilon} \mathbf{1}_D$$

Here, $d_n = \sum_k (-1)^k (\delta_n^k)_!$ is the alternate sum of Gysin maps associated with the relevant closed immersions (see 2.3.1, given that $\nu_{J*}(\mathbf{1}_{D_J}) = \mathrm{H}_D(D_J)$). The computation for (3.3.13.a) specializes under ℓ -adic realization to the Rapoport-Zink formula, used to compute vanishing cycles [75, Lemma 2.5]. A similar remark applies to Steenbrink's limit Hodge structure [83], with the caveat that our computation for motives does not account for the action of the monodromy operator.

3.4. Application to strong duality. Next, we deduce some applications of the computations of Section 3.3 towards strong duality results.

Proposition 3.4.1. *Let Z/S be a proper S -scheme with smooth crossings, and let v be a virtual bundle over Z . Then $\Pi_S(Z, v)$ is rigid with dual $\mathrm{H}_S(Z, -v)$ isomorphic to limit of the diagram*

$$\bigoplus_{i \in I} \Pi_S(Z_i, -v_i - \langle T_i \rangle) \rightrightarrows \bigoplus_{J \subset I, \#J=2} \Pi_S(Z_J, -v_J - \langle T_J \rangle) \longrightarrow \dots \longrightarrow \Pi_S(Z_I, -v_I - \langle T_I \rangle)$$

where for every $J \subset I$, T_J denotes the tangent bundle of Z_J/S .

⁷This holds in particular when X is h -smooth in a Nisnevich neighborhood of Z .

Proof. According to (3.3.3.a), $\Pi_S(Z, v)$ is isomorphic to the colimit of the finite diagram

$$\Pi_S(Z_I, v_I) \longrightarrow \dots \longrightarrow \bigoplus_{J \subset I, \#J=2} \Pi_S(Z_J, v_J) \rightrightarrows \bigoplus_{i \in I} \Pi_S(Z_i, v_i)$$

whose components are spectra of smooth proper schemes, hence rigid spectra. This implies $\Pi_S(Z, v)$ is rigid. The fact that its dual is $H_S(Z, -v)$ follows from Proposition 2.5.5(2). On the other hand, by (3.3.3.b), $H_S(Z, -v)$ is isomorphic to the colimit of the diagram

$$H_S(Z_I, -v_I) \longrightarrow \dots \longrightarrow \bigoplus_{J \subset I, \#J=2} H_S(Z_J, -v_J) \rightrightarrows \bigoplus_{i \in I} H_S(Z_i, v_i)$$

whose components are isomorphic to $\Pi_S(Z_J, -v_J - \langle T_J \rangle)$ by combining Example 2.5.7 and Proposition 2.5.5(2). \square

Theorem 3.4.2. *Let (X, Z) be a closed S -pair such that X/S is smooth and proper, with tangent bundle T , and such that Z/S has smooth crossings. Let v be a virtual vector bundle on X .*

Then $\Pi_S(X - Z, j^{-1}v)$ and $H_S(X - Z, j^{-1}v)$ are rigid with duals $\Pi_S^c(X - Z, -j^{-1}(v + \langle T \rangle))$ and $H_S^c(X - Z, -j^{-1}(v - \langle T \rangle))$, respectively.

Proof. One first appeals to Proposition 3.3.10 to conclude that $\Pi_S(X - Z, j^{-1}v)$ (resp. $H_S(X - Z, j^{-1}v)$) is rigid as a limit (resp. colimit) of a finite diagram whose components are rigid spectra due to the assumption that X , and hence all the Z_J , $J \subset I$, are smooth proper S -schemes. The given expressions for the dual then follow from Proposition 2.5.5. \square

Finally, we deduce an improvement of Theorem 2.4.3.

Theorem 3.4.3. *Let (X, Z) be a closed S -pair such that Z/S is proper with smooth crossing over S and such that X is smooth in a Nisnevich neighborhood of Z .*

Then, for every virtual vector bundle v on X , $\Pi_S(X/X - Z, v)$ and $H_S(X/X - Z, v)$ are rigid with duals $\Pi_S(Z, -i^{-1}v - i^{-1}\tau_{X/S})$ and $H_S(Z, -i^{-1}v + i^{-1}\tau_{X/S})$, respectively.

Proof. This is a direct combination of Theorem 2.4.3 and Proposition 3.4.1. \square

3.5. Complements of stably contractible arrangements. To illustrate the preceding results, we determine the stable homotopy types of complements of normal crossing S -schemes with stably \mathbf{A}^1 -contractible components.

3.5.1. A *stably \mathbf{A}^1 -contractible arrangement over S* is a closed S -pair (X, Z) consisting of a smooth stably \mathbf{A}^1 -contractible S -scheme X and a closed subscheme $Z \subsetneq X$ with smooth crossing over S that satisfies the following assumptions (see Notations 3.3.1).

- (1) For any $J \subset I$, every connected component of Z_J is stably \mathbf{A}^1 -contractible over S .
- (2) For any $K \subsetneq J \subset I$, Z_K is nowhere dense in Z_J .

For a subset $J \subset I$, we set $n_J = \#J$, and for any generic point x of Z_J we let c_x denote the codimension of x in X .

Example 3.5.2. A basic example of a stably \mathbf{A}^1 -contractible arrangement consists of an arrangement of affine hyperplanes in affine space \mathbf{A}_S^d over S .

Proposition 3.5.3. *Let S be a smooth stably \mathbf{A}^1 -contractible scheme over a field k and let (X, Z) be stably \mathbf{A}^1 -contractible arrangement over S . Then there exists a canonical isomorphism*

$$\Pi_S(X - Z) \simeq \bigoplus_{J \subset I, x \in Z_J^{(0)}} \mathbf{1}_S(c_x)[2c_x - n_J]$$

In addition, if Z is a normal crossing subscheme of X , then the isomorphism takes the form

$$\Pi_S(X - Z) \simeq \bigoplus_{n=0}^d m(n) \mathbf{1}_S(n)[n]$$

Here d is the relative dimension of X over S and $m(n)$ denotes the sum of the number of connected components of all codimension n subschemes Z_J of X .

Proof. According to Proposition 3.3.10 one obtains that $\Pi_S(X - Z)$ is the homotopy limit of the augmented semi-simplicial diagram

$$(3.5.3.a) \quad \Pi_S(X) \rightarrow \bigoplus_{i \in I} \Pi_S(Z_i, N_i) \rightrightarrows \cdots \rightrightarrows \bigoplus_{J \subset I, \#J=n} \Pi_S(Z_J, N_J) \rightrightarrows \cdots$$

Let x is a generic point of Z_J , for $J \subset I$, and write $Z_J(x)$ for the associated connected component. By assumption, $Z_J(x)$ is smooth and stably \mathbf{A}^1 -contractible over S , hence over k . It follows from Lemma 2.2.9 that the rank c_x vector bundle $N_J|_{Z_J(x)}$ is stably trivial, and hence

$$\Pi_S(Z_J, N_J) \simeq \bigoplus_x \Pi_S(Z_J(x), N_J|_{Z_J(x)}) \simeq \bigoplus_x \mathbf{1}_S(c_x)[2c_x]$$

To deduce the first assertion, it suffices to show that the morphisms in (3.5.3.a) are zero. Recall that these maps are sums of Gysin morphisms $(\nu_K^J)^!$ for $J, K \subset I$, $K = J \cup \{k\}$, $\nu_K^J : Z_K \rightarrow Z_J$. We are reduced to consider maps of the form

$$(3.5.3.b) \quad \mathbf{1}_S(c_x)[2c_x] \rightarrow \mathbf{1}_S(c_y)[2c_y]$$

Here, x (resp. y) is a generic point of Z_J (resp. Z_K). Since Z_K is nowhere dense in Z_J , all such maps belong to some stable cohomotopy group $\pi^{2r,r}(S)$ for $r > 0$. The assumption that S is stably \mathbf{A}^1 -contractible over k implies $\pi^{2r,r}(S) \simeq \pi^{2r,r}(k)$. Morel's \mathbf{A}^1 -connectivity theorem shows the latter group is trivial. It follows that the map (3.5.3.b) is zero.

For the second assertion, it suffices to note that if Z is a normal crossing subscheme, then for any $J \subset I$, Z_J has pure codimension n_J in X . \square

Using Proposition 2.5.5(3), we obtain the following rigidity result.

Corollary 3.5.4. *With the notation and assumptions of Proposition 3.5.3, $\Pi_S(X - Z)$ is rigid with dual*

$$\Pi_S^c(X - Z)(-d)[-2d] \simeq \bigoplus_{K \subset I, x \in Z_K^{(0)}} \mathbf{1}_S(-c_x)[-2c_x + n_K]$$

4. PUNCTURED TUBULAR NEIGHBORHOODS AND STABLE HOMOTOPY AT INFINITY

4.1. Punctured tubular neighborhoods.

Definition 4.1.1. Let (X, Z) be a closed S -pair and let v be a virtual vector bundle on X . The *punctured tubular \mathcal{T} -neighborhood* $\mathrm{TN}_S^\times(X, Z, v)$ of Z in X relative to S twisted by v is the homotopy fiber in $\mathcal{T}(S)$ of the composite

$$\beta_{X,Z} : \Pi_S(Z, i^{-1}v) \rightarrow \Pi_S(X, v) \rightarrow \Pi_S(X/X - Z, v)$$

Here the first map is induced by the immersion $i : Z \rightarrow X$, and the second one is defined in Definition 2.2.10. In the case of a trivial twist, we use the notation $\mathrm{TN}_S^\times(X, Z)$.

It is easy to check that $\mathrm{TN}_S^\times(X, Z)$ is functorial for morphisms of closed pairs. Moreover, the functor TN_S^\times sends excisive morphisms to isomorphisms. In particular, the punctured tubular neighborhood only depends on a Nisnevich neighborhood of Z in X . In Corollary 4.1.8 below, we will get an even more useful cdh-excision property.

Remark 4.1.2. Our definition is motivated and inspired by the notion of the *link* of a point on a hypersurface due to Brauner, Zariski, Milnor, Mumford (see [68], [71]). Following Mumford, *loc. cit.*, a suitable pointed tubular neighborhood can compute the link. More specifically, we formally view $\beta_{X,Z}$ as a tubular neighborhood of Z in X , and its homotopy cofiber amounts to the pointed tubular neighborhood by analogy with the Gysin exact sequence (see the next example).

Pushing this analogy, one can show that the complex realization of our definition, when Z is a point on a complex hypersurface in affine space, is precisely the link as discussed above. This fact will be transparently visible in our examples.

Example 4.1.3. Let (V, X) be the closed S -pair corresponding to the zero section $s : X \rightarrow V$ of a vector bundle V on a separated S -scheme X . Then, by definition, one obtains the homotopy exact sequence (see 2.3.3 for notation)

$$\mathrm{TN}_S^\times(V, X) \rightarrow \Pi_S(X) \xrightarrow{e_S(V)} \mathrm{Th}_S(V)$$

In particular, $\mathrm{TN}_S^\times(V, X) = \Pi_S(V^\times)$, where V^\times denotes the complement of the image of s . Hence $\mathrm{TN}_S^\times(V, X)$ is the extension of $\Pi_S(X)$ by $\mathrm{Th}_S(V)[-1]$ classified by the Euler class $e_S(V)$. The vanishing of $e_S(V)$ is, by definition, equivalent to the existence of a splitting

$$\mathrm{TN}_S^\times(V, X) = \Pi_S(X) \oplus \mathrm{Th}_S(V)[-1]$$

Remark 4.1.4. Assume that S is the spectrum of a perfect field k of characteristic exponent p . Example 4.1.3 implies that for the closed S -pair (V, X) corresponding to the zero section $s : X \rightarrow V$ of a vector bundle V of rank r on a separated S -scheme X , $\mathrm{TN}_S^\times(V, X)$ is a strictly finer invariant than its motivic realization. Indeed, the realization in $\mathrm{DM}(k)[1/p]$ of $\mathrm{TN}_S^\times(V, X)$ is the extension of $M(X)$ by $M(X)(r)[2r-1]$ classified by the map $\tilde{c}_r(V) : M(X) \rightarrow M(X)(r)[2r-1]$ induced by multiplication with the top Chern class $c_r(V) \in \mathrm{CH}^r(X) \simeq \mathrm{Hom}(M(X), \mathbf{1}(r)[2r])$. In particular, the sequence splits if $c_r(V) = 0$.

However, the vanishing of the homotopy Euler class $e(V)$, which implies the vanishing of the Euler class in Chow-Witt groups, is a strictly stronger condition than the vanishing of the top Chern class $\tilde{c}_r(V)$. For the smooth affine quadric 5-fold $X : x_1y_1 + x_2y_2 + x_3y_3 = 1$ in \mathbf{A}^6 , the kernel of the surjection $(x_1, x_2, x_3) : k[Q]^3 \rightarrow k[Q]$ defines a nontrivial and stably trivial vector bundle V of rank 2 on X . While V 's Chern classes are trivial, V 's Euler class in $\widetilde{\mathrm{CH}}^2(X) = \mathrm{K}_{-1}^{\mathrm{MW}}(k)$ equals η , see the case $n = 2$ in [4, Lemma 3.5].

Example 4.1.3 admits the following generalization.

Proposition 4.1.5. *Let (X, Z) be a weakly h -smooth closed S -pair (see Definition 2.4.1) with normal bundle $N_{Z/X}$. Then there exists a homotopy exact sequence*

$$\mathrm{TN}_S^\times(X, Z) \longrightarrow \Pi_S(Z) \xrightarrow{e_S(N_{Z/X})} \mathrm{Th}_S(N_{Z/X})$$

In other words, $\mathrm{TN}_S^\times(X, Z) = \Pi_S(N_{Z/X}^\times)$. Moreover, if the Euler class of $N_{Z/X}$ vanishes, then

$$\mathrm{TN}_S^\times(X, Z) = \Pi_S(Z) \oplus \mathrm{Th}_S(N_{Z/X})[-1]$$

Proof. By excision, one can assume that both X and Z are h -smooth over S . By appealing to the purity isomorphism of Theorem 2.4.3, one deduces the commutative diagram

$$\begin{array}{ccccc} \Pi_S(Z) & \xrightarrow{\beta_{X,Z}} & \Pi_S(X/X-Z) & \xrightarrow{\sim} & \Pi_S(Z, N_{Z/X}) \\ & \searrow i_* & \nearrow & \nearrow (1) & \\ & & \Pi_S(X) & \xrightarrow{i^!} & \end{array}$$

Indeed, the commutativity of part (1) follows from the definitions of the Gysin map, the purity isomorphism, and the associativity formula for fundamental classes in [35, Theorem 3.3.2]. Then the homotopy exact sequence follows from the excess intersection formula of [35, Proposition 3.3.4]. The remaining assertions follow as in the previous example. \square

The following result expresses a motivic version of a classical computation of topological punctured tubular neighborhoods, which is a consequence of the octahedron axiom.

Proposition 4.1.6. *Let (X, Z) be a closed S -pair and let v be a virtual vector bundle on X . Then the columns and rows of the following diagram are homotopy exact*

$$(4.1.6.a) \quad \begin{array}{ccccc} 0 & \longrightarrow & \Pi_S(X - Z, j^{-1}v) & \xlongequal{\quad} & \Pi_S(X - Z, j^{-1}v) \\ \downarrow & & \downarrow j_* & & \downarrow \alpha_{X,Z} \\ \Pi_S(Z, i^{-1}v) & \xrightarrow{i_*} & \Pi_S(X, v) & \longrightarrow & \Pi_S(X/Z, v) \\ \parallel & & \downarrow & & \downarrow \\ \Pi_S(Z, i^{-1}v) & \xrightarrow{\beta_{X,Z}} & \Pi_S(X/X - Z, v) & \longrightarrow & \mathrm{TN}_S^\times(X, Z, v)[1] \end{array}$$

(1) is above the top row, (2) is below the middle row.

Proof. Indeed, the middle column (resp. row) follows from Definition 2.2.10, the commutativity of (1) follows from the definition, and that of (2) from the definition of $\beta_{X,Z}$. The lower-right corner of the diagram is just the formulation of the octahedron axiom. \square

Remark 4.1.7. In more classical terms for cohomology with coefficients in a ring spectrum \mathbb{E} , one obtains long exact sequences involving the punctured tubular neighborhood

$$\begin{aligned} \dots &\rightarrow \mathbb{E}_Z^{n,i}(X) \rightarrow \mathbb{E}^{n,i}(Z) \rightarrow \mathbb{E}^{n,i}(\mathrm{TN}_S^\times(X, Z)) \rightarrow \mathbb{E}_Z^{n+1,i}(X) \rightarrow \dots \\ \dots &\rightarrow \mathbb{E}^{n,i}(X, Z) \rightarrow \mathbb{E}^{n,i}(X - Z) \rightarrow \mathbb{E}^{n,i}(\mathrm{TN}_S^\times(X, Z)) \rightarrow \mathbb{E}^{n+1,i}(X, Z) \rightarrow \dots \end{aligned}$$

Here $\mathbb{E}_Z^{**}(X)$ (resp. $\mathbb{E}^{**}(X, Z)$) is the cohomology with support (resp. relative cohomology).

One gets the following practical way of computing punctured tubular neighborhoods by using resolution of singularities:

Corollary 4.1.8. *Let $f : (Y, T) \rightarrow (X, Z)$ be a cdh-excisive morphism of closed S -pairs and let v be a virtual vector bundle on X . Then the induced map*

$$\mathrm{TN}_S^\times(Y, T, f^{-1}v) \rightarrow \mathrm{TN}_S^\times(X, Z, v)$$

is an isomorphism.

Proof. Indeed, according to Proposition 4.1.6, one obtains a commutative diagram whose rows are homotopy exact sequences

$$\begin{array}{ccccc} \mathrm{TN}_S^\times(Y, T, f^{-1}v) & \longrightarrow & \Pi_S(Y - T, f^{-1}(v)|_{Y-T}) & \longrightarrow & \Pi_S(Y/T, f^{-1}v) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{TN}_S^\times(X, Z, v) & \longrightarrow & \Pi_S(X - Z, v|_{X-Z}) & \longrightarrow & \Pi_S(X/Z, v) \end{array}$$

By assumption, the middle vertical map, induced by the restriction of f , is an isomorphism. Moreover, the right-most vertical map is an equivalence according to the cdh-descent property of \mathcal{T} (see [27, 3.3.10]). \square

In particular, one can use any suitable resolution of singularities of a pair (X, Z) to compute the punctured tubular neighborhood of (X, Z) . More precisely, if we can find a cdh-excisive morphism $(Y, T) \rightarrow (X, Z)$ such that (Y, T) is smooth over the base S , then applying Proposition 4.1.5 and Corollary 4.1.8, we get $\mathrm{TN}_S^\times(X, Z) \simeq \Pi_S(N_{T/Y}^\times)$. We obtain several examples from singularity theory in this way — S can be any base, the spectrum of a field k or even of \mathbb{Z} .

Example 4.1.9. Let $\mathbb{P} = \mathbb{P}_S^1$ be the projective line and $\mathcal{O}(-1) = \mathbb{V}(\mathcal{O}_{\mathbb{P}}(1))$ be its tautological line bundle. Consider the relative quadratic cone $X = V(xy - z^2)$ in \mathbf{A}_S^3 . Then by blowing-up the ordinary double point at the origin o_S , one gets a resolution $Y \rightarrow X$ whose exceptional divisor is \mathbb{P} , with normal bundle $\mathcal{O}(-2) = \mathcal{O}(-1)^{\otimes 2}$. Therefore, we have

$$\mathrm{TN}_S^\times(V(xy - z^2), o_S) \simeq \Pi_S(\mathcal{O}(-2)^\times)$$

For $S = \text{Spec}(\mathbb{C})$, the underlying topological manifold of the complex realization of $\mathcal{O}(-2)^\times$ is homotopy equivalent to the total space of unit tangent bundle UTS^2 of the sphere $S^2 = \mathbb{C}\mathbb{P}^1$. As a topological manifold, UTS^2 is homeomorphic to $\mathbb{R}\mathbb{P}^3 \cong \text{SO}(3)$. Our computation thus recovers the stable homotopy type of the link of the germ of complex of hypersurface singularity

$$(V = \{u^2 + v^2 - z^2 = 0\}, 0) \subset (\mathbb{C}^3, 0)$$

defined in [68, Chapter 2] as the intersection of V with a real 5-sphere $S_\varepsilon^5 \subset \mathbb{C}^3 = \mathbb{R}^6$ of sufficiently small radius $\varepsilon > 0$ centered at origin. Our computation also accounts for the real case: the underlying topological manifold of the real realization of $\mathcal{O}(-2)^\times$ is homotopy equivalent to the unit tangent bundle of the circle $S^1 = \mathbb{R}\mathbb{P}^1$, hence to disjoint copies of S^1 . The latter equals the link of the real germ of isolated singularity $(V = \{u^2 - v^2 - z^2 = 0\}, 0) \subset (\mathbb{R}^3, 0)$.

Example 4.1.10. Next we consider an ordinary double point in a 3-fold: say $X = V(xt - yz)$ in \mathbf{A}_S^4 , which is singular at the origin o_S . A resolution of the singularity is given by the blow-up $\tilde{X} \rightarrow X$ of o_S with exceptional divisor $\mathbb{P} \times \mathbb{P}$, whose normal bundle is $\mathcal{O}_{\mathbb{P} \times \mathbb{P}}(-1, -1) = p_1^* \mathcal{O}(-1) \otimes p_2^* \mathcal{O}(-1)$. Another resolution $X^- \rightarrow X$ is given the blow-up of X with center at the the Weil non-Cartier divisor $V(x, y)$. The exceptional locus of $X^- \rightarrow X$ is isomorphic to \mathbb{P} and its normal bundle in X^- is equal to $\mathcal{O}_{\mathbb{P}}(-1) \oplus \mathcal{O}_{\mathbb{P}}(-1)$. This yields two models of the punctured tubular neighborhood

$$\text{TN}_S^\times(V(xt - yz), o_S) \simeq \Pi_S([\mathcal{O}_{\mathbb{P} \times \mathbb{P}}(-1, -1)]^\times) \simeq \Pi_S([\mathcal{O}_{\mathbb{P}}(-1) \oplus \mathcal{O}_{\mathbb{P}}(-1)]^\times)$$

The S -schemes $[\mathcal{O}_{\mathbb{P} \times \mathbb{P}}(-1, -1)]^\times$ and $[\mathcal{O}_{\mathbb{P}}(-1) \oplus \mathcal{O}_{\mathbb{P}}(-1)]^\times$ are actually both isomorphic to $V - \{o_S\}$. For $S = \text{Spec}(\mathbb{C})$ the underlying topological manifolds of the complex realizations of these schemes are homotopy equivalent to the S^1 -bundle over $S^2 \times S^2$ with Euler class $(1, 1) \in H^2(S^2 \times S^2, \mathbb{Z}) \cong \mathbb{Z}^2$ and to the trivial S^3 -bundle over S^2 , respectively. Again, our descriptions recover the (stable) homotopy of the link of the germ of complex of hypersurface singularity

$$(V = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\}, 0) \subset (\mathbb{C}^4, 0),$$

this link being homotopy equivalent the unit tangent bundle $UTS^3 \cong S^2 \times S^3$.

Remark 4.1.11. The reader will find in Theorem 4.2.1 a way of computing punctured tubular neighborhoods when dealing with resolution of singularities whose exceptional locus is snc. This was our main motivation for Section 3.

4.1.12. One can further interpret Proposition 4.1.6 in terms of the six functors formalism. For the closed S -pair (X, Z) , consider the commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{i} & X & \xleftarrow{j} & X - Z \\ & \searrow p & \downarrow f & \swarrow q & \\ & & S & & \end{array}$$

of (2.4.0.a). By crossing the two classical localization triangles one gets, as a functorial enhancement of (4.1.6.a), the following commutative diagram of natural transformations of $\mathcal{T}(X)$

$$(4.1.12.a) \quad \begin{array}{ccccc} 0 & \longrightarrow & j_! j^! & \xlongequal{\quad} & j_! j^! \\ \downarrow & & \text{ad}'_{j_! j^!} \downarrow & & \downarrow \alpha_j \\ i_! i^! & \longrightarrow & Id & \xrightarrow{\text{ad}_{j^*, j_*}} & j_* j^* \\ \parallel & & \downarrow & & \downarrow \\ i_! i^! & \xrightarrow{\beta_i} & i_* i^* & \longrightarrow & i_* i^* j_* j^* \end{array}$$

Each arrow in (4.1.12.a) is a unit or counit for one of the adjunctions (k^*, k_*) or $(k_!, k^!)$, $k = i, j$. The second and third rows (resp. columns) are the classical localization triangle, expressed in terms of natural transformation. In particular, each row and column of (4.1.12.a) is exact homotopy (concretely: gives a homotopy exact sequence in $\mathcal{T}(X)$ when evaluated at any object).

Note, moreover, that α_j is given by the map $j_! \rightarrow j_*$ "forgetting the support" where the map β_i corresponds to a natural transformation $\beta_i : i^! \rightarrow i^*$ which is specific to the case of (closed) immersions.⁸ Note, finally, the classical identification of functors $i^! j_! = i^* j_*[-1]$ obtained by applying the localization triangles (middle row of the previous diagram) post-composed with $j_! j^!$ to get the homotopy exact sequence

$$i^! j_! j^! \rightarrow j_! j^! \rightarrow j_* j^* j_! j^! = j_* j^*$$

Since the last arrow identifies with α_j , one gets $i^! j_! j^! = i_* i^* j_* j^*$, which gives the result since $i_* = i_!$ (resp. $j^* = j^!$) is right invertible.

We thus obtain the following expression of the punctured tubular neighborhood.

Proposition 4.1.13. *There is a canonical isomorphism*

$$\mathrm{TN}_S^\times(X, Z) \simeq p_! i^! j_! q^!(\mathbf{1}_S) = p_! i^* j_* q^!(\mathbf{1}_S)[-1]$$

This relation explains the close connection between punctured tubular neighborhoods and nearby cycles. In this line of thought, we extend [87, Theorem 5.1] and [32, 1.4.6] to our context.

Theorem 4.1.14. *Let S be an excellent scheme, and let $(X, Z), (Y, T)$ be closed S -pairs. Assume that there exists an isomorphism $f : T \rightarrow Z$, which extends to an isomorphism of the respective formal completions $\mathfrak{f} : \hat{Y}_T \rightarrow \hat{X}_Z$. Then there exists a canonical isomorphism*

$$\mathfrak{f}^* : \mathrm{TN}_S^\times(Y, T) \xrightarrow{\cong} \mathrm{TN}_S^\times(X, Z)$$

which is compatible with composition in \mathfrak{f} .

Proof. We can assume that $Z = T$ and that Z is reduced. It suffices to build an isomorphism

$$\tilde{\mathfrak{f}}^* : \mathrm{TN}_S^\times(Y, T) \rightarrow \mathrm{TN}_S^\times(X, Z)$$

and a commutative diagram

$$\begin{array}{ccc} & & \Pi_S(Y/Y - Z) \\ & \nearrow & \downarrow \tilde{\mathfrak{f}}^* \\ Z & & \Pi_S(X/X - Z) \end{array}$$

We can apply the strategy of the proof of [32, Theorem 1.4.6] using Artin's approximation theorem at points of Z (here, we use the assumption that S is excellent) and Zariski hypercovers to globalize the situation. Here we do not need to extend our motivic category to diagrams of base schemes. The proof proceeds with the simplicial schemes corresponding to Zariski hypercoverings directly within the ∞ -category $\mathcal{T}(S)$. \square

4.2. Punctured tubular neighborhood of subschemes with crossing singularities. Based on Theorem 3.2.3, we now state our main tool for computing punctured tubular neighborhoods of h -smooth crossing subschemes (Definition 3.3.2). We adopt the notation of 3.3.1 and 3.3.9.

Theorem 4.2.1. *Let (X, Z) be a closed S -pair such that Z/S has h -smooth crossings over S and X/S is h -smooth in a Nisnevich neighborhood of Z and let v be a virtual vector bundle on X .*

Then $\mathrm{TN}_S^\times(X, Z, v)$ is canonically isomorphic to the homotopy fiber of the map

$$\mathrm{colim}_{n \in (\Delta^{\mathrm{inj}})^{\mathrm{op}}} \left(\bigoplus_{J \subset I, \#J=n+1} \Pi_S(Z_J, v_J) \right) \xrightarrow{\partial} \lim_{n \in \Delta^{\mathrm{inj}}} \left(\bigoplus_{J \subset I, \#J=m+1} \Pi_S(Z_J, v_J + \langle N_J \rangle) \right)$$

Here the direct images define the face maps

$$(\delta_n^k)_* = \sum_{K=\{i_0 < \dots < i_n\}, J=\{i_0 < \dots < i_k < \dots < i_n\}} (\nu_K^J)_*$$

⁸It can also be derived from the exchange transformation $i^! Id^* \rightarrow i^* Id^!$.

in the source, and the Gysin maps define the coface maps

$$(\tilde{\delta}_l^m)^! = \sum_{K=\{i_0<\dots<i_m\}, J=\{i_0<\dots<i_{l-1}<\dots<i_m\}} (\nu_K^J)^!$$

in the target. Moreover, ∂ is induced by the canonical map in degree zero

$$(4.2.1.a) \quad (\delta_{ij} = \bar{\nu}_j^! \bar{\nu}_{i*})_{i,j \in I} : \bigoplus_{i \in I} \Pi_S(Z_i, v_i) \longrightarrow \bigoplus_{j \in I} \Pi_S(Z_j, v_j + \langle N_j \rangle)$$

Finally, using the Euler class $e(N_i) : \mathbf{1}_{Z_i} \rightarrow \mathrm{Th}(N_i)$ (see paragraph 2.3.3) of the normal bundle N_i , one can compute the diagonal coefficients of this matrix as

$$\delta_{ii} = p_{i!}(e(N_i) \otimes \mathrm{Th}(\tau_i + v_i))$$

where $p_i : Z_i \rightarrow S$ is the (h -smooth) projection, with virtual tangent bundle τ_i .

Proof. According to Definition 4.1.1, we have to compute the homotopy fiber of the map

$$\beta_{X,Z} : \Pi_S(Z, v) \rightarrow \Pi_S(X/X - Z, v)$$

Proposition 3.3.3 identifies $\beta_{X,Z}$'s source with the desired colimit whereas Proposition 3.3.10 identifies its target with the desired limit. The computation of the (co)face maps and of ∂ follows from these two propositions. The final remark follows from the definition of $\tilde{\mu}_{ii}$, the excess intersection formula [35, Proposition 3.2.8], and $p_i^!(\mathbf{1}_S) \simeq \mathrm{Th}(\tau_i)$ since p_i is h -smooth by assumption. \square

One can suggestively summarize the computation in Theorem 4.2.1 with the diagram

$$\left(\begin{array}{ccc} \bigoplus_{i_1 < i_2} \Pi_S(Z_{i_1 i_2}) & & \\ \downarrow \downarrow & & \\ \bigoplus_{i \in I} \Pi_S(Z_i) & \xrightarrow{\partial} & \bigoplus_{j \in I} \Pi_S(Z_j, \langle N_j \rangle) \\ & & \downarrow \downarrow \\ & & \bigoplus_{j_1 < j_2} \Pi_S(Z_{j_1 j_2}, \langle N_{j_1 j_2} \rangle) \\ & & \vdots \end{array} \right)$$

Typically, a punctured tubular neighborhood computation will consist of determining the homotopy colimit (resp. limit) of the left (resp. right) column and then determining the map ∂ . For a closed S -pair (X, Z) such that X is smooth over S in a Nisnevich neighborhood of Z , $\tau_{X/S}$ is a well-defined virtual vector bundle on a suitable Nisnevich neighborhood of Z , and its restriction $i^{-1}\tau_{X/S}$ to Z is a well-defined virtual vector bundle on Z , see 2.4.2. Since the twisted punctured tubular neighborhood of Z in X depends only on a Nisnevich neighborhood of Z in X , the object $\mathrm{TN}_S^\times(X, Z, -v - \tau_{X/S})$ is well-defined for every virtual vector bundle v on (a Nisnevich neighborhood of Z in) X . One derives from Theorem 3.4.3 the following strong duality result.

Theorem 4.2.2. *Let (X, Z) be a closed S -pair such that X is smooth in a Nisnevich neighborhood of Z and such Z/S is proper with smooth crossings over S . Then for every virtual vector bundle v on X , $\mathrm{TN}_S^\times(X, Z, v)$ is rigid with dual $\mathrm{TN}_S^\times(X, Z, -v - \tau_{X/S})[-1]$.*

In particular, under the stated hypothesis, the punctured tubular neighborhood $\mathrm{TN}^\times(X, Z)$ is auto-dual, up to twist and shift.

4.3. Stable homotopy at infinity and boundary motives. As explained in the next examples, the following definition is rooted in both classical topology, see [56], and in Wildeshaus' theory of boundary motives [88].

Definition 4.3.1. The *homotopy at infinity* of a separated S -scheme X/S is the homotopy fiber computed in $\mathcal{T}(S)$ of the map $\alpha_{X/S} : \Pi_S(X) \rightarrow \Pi_S^c(X)$ in (2.2.1.a) so that there is a homotopy exact sequence

$$\Pi_S^\infty(X) \longrightarrow \Pi_S(X) \xrightarrow{\alpha_{X/S}} \Pi_S^c(X)$$

Owing to (1.2.0.a), the main case is $\mathcal{T} = \text{SH}$. We refer to the spectrum $\Pi_S^\infty(X)$ in $\text{SH}(S)$ as the *stable homotopy at infinity* of X relative to S .

Example 4.3.2. Let $p : V \rightarrow S$ be a vector bundle and consider the closed pair (V, S) given by the zero section $s : S \rightarrow V$. Then, using purity isomorphisms, one gets the commutative diagram

$$\begin{array}{ccccc} & \Pi_S(V) & \xrightarrow{\alpha_{V/S}} & \Pi_S^c(V) & \xlongequal{\quad} & p_*p^!(\mathbf{1}_S) \\ & \swarrow p_* & & \downarrow \text{can} & \sim \downarrow \mathfrak{p}_p & \\ \mathbf{1}_S & & & & & p_*(\text{Th}_V(p^{-1}V)) \xlongequal{\quad} p_*p^*(\text{Th}_S(V)) \\ & \searrow e(V) & & \downarrow \mathfrak{p}_{V,S} & & \\ & \Pi_S(V/V - Z) & \xrightarrow{\quad} & \text{Th}_S(V) & \xrightarrow{\quad} & \end{array}$$

The fact that p_* and the unit ad_p are isomorphisms follows from \mathbf{A}^1 -homotopy invariance. The purity isomorphism \mathfrak{p}_p exists since p is smooth, and $\mathfrak{p}_{V,S}$ is the (tautological) purity isomorphism. The right-hand side commutes by applying [35, Lemma 3.3.1] to $f = p, i = s, i' = Id_V$, while commutativity of the left-hand side follows by definition of the Euler class $e(V)$ (2.3.3). We deduce the homotopy exact sequence

$$\Pi_S^\infty(V) \rightarrow \mathbf{1}_S \xrightarrow{e(V)} \text{Th}(V)$$

In other words, $\Pi_S^\infty(V) = \Pi_S(V^\times)$ and, if $e(V) = 0$ then $\Pi_S^\infty(V) = \mathbf{1}_S \oplus \text{Th}(V)[-1]$

It follows from the discussion in Section 1.2 that $\Pi_S^\infty(X)$ realizes to the analogous definition for the other motivic ∞ -categories of (1.2.0.a).

Example 4.3.3. Motivic realization. Let S be the spectrum of a perfect field k of characteristic exponent p and let X be a separated k -scheme. Then the motivic realization functor (see also [53], [79] in this case)

$$(4.3.3.a) \quad \text{SH}(k) \rightarrow \text{DM}(k)[1/p]$$

sends $\Pi_k(X)$ to Voevodsky's homological motive $M(X)$ of X ([25, §8.7]), and it sends $\Pi_k^c(X)$ to $M^c(X)$, Voevodsky's homological motive of X with compact support ([25, Proposition 8.10]). It follows that the motivic realization functor sends $\Pi_k^\infty(X)$ to the *boundary motive* $\partial M(X)$ of X (see Wildeshaus [87]). In particular, the Betti or ℓ -adic cohomology of $\Pi_k^\infty(X)$ coincides with the so-called *interior cohomology* of X . We generalize the above discussion to arbitrary base schemes in Section 5.

Example 4.3.4. Betti realization. Let S be the spectrum of a field k which admits a complex embedding σ , and consider the Betti realization functor (see Section 1.2)

$$(4.3.4.a) \quad \text{SH}(k) \rightarrow D_B^\sigma(k) = D(\mathbb{Z})$$

Owing to Ayoub's enhancement of (4.3.4.a) to an arbitrary base scheme using the technique of analytical sheaves [12], one derives that for any separated k -scheme X , the spectrum $\Pi_k(X)$ realizes to the singular chain complex $S_*(X^\sigma)$ of the analytification X^σ of X , and the spectrum $\Pi_k^c(X)$ realizes to the Borel-Moore singular chain complex $S_*^{BM}(X^\sigma)$. As X^σ is locally contractible and σ -compact, the latter is quasi-isomorphic to the complex $S_*^{lf}(W)$ of locally finite singular chains ([56, Chapter 3]). Thus the stable homotopy type at infinity $\Pi_S^\infty(X)$ realizes to the singular complex at infinity $S_*^\infty(X^\sigma)$ (see Definition [56]), defined by the distinguished triangle of chain complexes of abelian groups

$$(4.3.4.b) \quad S_*^\infty(X^\sigma) \rightarrow S_*(X^\sigma) \xrightarrow{\alpha_{X^\sigma}} S_*^{lf}(X^\sigma) \rightarrow S_*^\infty(X^\sigma)[1]$$

As a corollary of Theorem 2.6.4, we get the following computations:

Proposition 4.3.5. *In the setting of Theorem 2.6.4 assume that either i) or ii) holds and that Y/S is proper. Then there is a canonical isomorphism*

$$\Pi_S^\infty(X \times_S Y) \simeq \Pi_S^\infty(X) \otimes \Pi_S(Y)$$

Proposition 4.3.6. *In the setting of Theorem 2.6.4 assume that $g : Y \rightarrow S$ is smooth and stably \mathbf{A}^1 -contractible over S with relative tangent bundle T_g stably constant over S and let v_0 be a virtual vector bundle over S such that $\langle T_g \rangle = g^*v_0$ in $K_0(Y)$. Then there exists a homotopy exact sequence*

$$\Pi_S^\infty(X \times_S Y) \longrightarrow \Pi_S(X) \xrightarrow{\alpha_X \otimes \alpha_Y} \Pi_S^c(X) \otimes \text{Th}(v_0)$$

In particular, if T_g is the pullback of a vector bundle V over S with a trivial Euler class, then

$$\Pi_S^\infty(X \times_S Y) = \Pi_S(X) \oplus \Pi_S^c(X) \otimes \text{Th}_S(V)[-1]$$

Note that the splitting uses Example 4.3.2.

Example 4.3.7. Let X be a smooth stably \mathbf{A}^1 -contractible variety of dimension d over a field k . Then, Proposition 4.3.6 implies that

$$\Pi_k^\infty(X) = \mathbf{1}_k \oplus \mathbf{1}_k(d)[2d-1] = \Pi_k^\infty(\mathbf{A}_k^d)$$

In other words, stable homotopy at infinity cannot distinguish between X and affine space \mathbf{A}_k^d , as one would expect from topology (see [7]). A theory of unstable motivic homotopy at infinity, however, is expected to provide a finer invariant.

Similarly, the situation for smooth morphisms $f : X \rightarrow S$ with stably \mathbf{A}^1 -contractible fibers over a general base S is entirely described by their stable tangent bundles. In particular, if T_f is constant over S , equal to f^*V for some vector bundle V on S , then the stable homotopy type at infinity of X is the same as that of the vector bundle V . It is thus essentially described by the Euler class of V as explained in Example 4.3.2.

Remark 4.3.8. In general, one can interpret $\Pi_S^\infty(X)$ as an extension of $\Pi_S(X)$ by $\Pi_S^c(X)$. This viewpoint is prominent in Wildeshaus' work on boundary motives; a motivic realization, where weight considerations are at stake. In topology, it is well-known that forming a product with Euclidean space \mathbf{R}^n kills the fundamental group at infinity. In our stable context, taking a product with affine space \mathbf{A}^n , or more generally, any smooth stably \mathbf{A}^1 -contractible S -scheme $f : Y \rightarrow S$ of relative dimension n with a trivial relative tangent bundle splits the extension in the sense that

$$\Pi_S^\infty(X \times Y) \simeq \Pi_S(X) \oplus \Pi_S^c(X)(n)[2n-1]$$

As an application of the results and techniques above, we can now wholly determine the homotopy at infinity of complements of stably \mathbf{A}^1 -contractible arrangements in smooth stably \mathbf{A}^1 -contractible schemes over a field (see 3.5.1).

Proposition 4.3.9. *Let S be a smooth stably \mathbf{A}^1 -contractible scheme over a field k and let (X, Z) be a stably \mathbf{A}^1 -contractible arrangement over S such that Z is a normal crossing closed subscheme of X . Then there exists a canonical isomorphism*

$$\Pi_S^\infty(X - Z) \simeq \bigoplus_{i=0}^d m(i) \mathbf{1}_S(i)[i] \oplus \bigoplus_{j=0}^d m(j) \mathbf{1}_S(d-j)[2d-j-1]$$

where d is the dimension of X over S and where $m(n)$ denotes the sum of the number of connected components of all codimension n subschemes Z_j of X .

Proof. Indeed, applying Proposition 3.5.3 and Corollary 3.5.4, we deduce the homotopy exact sequence

$$\Pi_S^\infty(X) \longrightarrow \bigoplus_{i=0}^d m(i) \mathbf{1}_S(i)[i] \longrightarrow \bigoplus_{j=0}^d m(j) \mathbf{1}_S(d-j)[2d-j]$$

To conclude, it suffices to prove that the second map is zero. Since S is stably \mathbf{A}^1 -contractible over the field k , it is given by a sum of elements of the groups $\pi^{2d-i-j, d-i-j}(k)$. Since $d > 0$, these groups are all trivial by Morel's stable \mathbf{A}^1 -connectivity theorem. \square

4.4. Stable homotopy type at infinity via punctured tubular neighborhoods.

4.4.1. Recall that a *compactification* of a separated morphism of finite type $f: X \rightarrow S$ consists of an open immersion $j: X \hookrightarrow \bar{X}$ into a proper S -scheme $\bar{f}: \bar{X} \rightarrow S$. The closed subscheme $\partial X = (\bar{X} - X)_{red}$ of \bar{X} is called the *boundary* of the compactification j . We denote by $i: \partial X \hookrightarrow \bar{X}$ the corresponding closed immersion and set $\partial \bar{f} = \bar{f} \circ i: \partial X \rightarrow S$ in the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & \bar{X} & \xleftarrow{i} & \partial X \\ & \searrow f & \downarrow \bar{f} & \swarrow \partial \bar{f} & \\ & & S & & \end{array}$$

The following result gives our main tool for computing stable homotopy types at infinity. For specializations to topology and motives, see [56] and [88, Theorem 1.6], respectively.

Proposition 4.4.2. *Let $(\bar{X}, \partial X)$ be the closed S -pair associated with a compactification of a separated S -scheme of finite type. Then there exists a canonical isomorphism*

$$\Pi_S^\infty(X) \simeq \mathrm{TN}_S^\times(\bar{X}, \partial X)$$

which is natural in $(\bar{X}, X, \partial X)$, covariantly functorial with respect to proper maps, and contravariantly functorial with respect to étale maps.

Proof. Given the six functors formalism, this is a direct application of Proposition 4.1.6. More precisely, with the notation of 4.4.1, one reduces to the commutative diagram

$$\begin{array}{ccc} f_! f^!(\mathbf{1}_S) & \xrightarrow{\alpha_f} & f_* f^!(\mathbf{1}_S) \\ \sim \downarrow & \xrightarrow{ad'_{j_!, j^!}} & \downarrow \sim \\ \bar{f}_* j_! j^! \bar{f}^!(\mathbf{1}_S) & \xrightarrow{\quad} & \bar{f}_* \bar{f}^!(\mathbf{1}_S) \xrightarrow{ad_{j_*, j^*}} \bar{f}_* j_* j^* \bar{f}^!(\mathbf{1}_S) \end{array}$$

and exactness of the rows and columns of (4.1.12.a). \square

Remark 4.4.3. The above result has the following geometric interpretations. First, using the notations of Proposition 4.1.6 for the closed S -pair $(\bar{X}, \partial X)$ and that of Definition 4.3.1, the commutative diagram in the proof of Proposition 4.4.2 can be recast as

$$\begin{array}{ccc} \Pi_S(X) & \xrightarrow{\alpha_X} & \Pi_S^c(X) \\ \parallel & & \downarrow \sim \\ \Pi_S(\bar{X} - \partial X) & \xrightarrow{\alpha_{\bar{X}, \partial X}} & \Pi_S(\bar{X}/\partial X) \end{array}$$

In particular, considering the Borel-Moore homotopy $\Pi_S^c(X)$ of X naturally leads to considering the object $\bar{X}/\partial X$ obtained by identifying the boundary ∂X of any compactification \bar{X} with a point. The latter can be viewed as a motivic model for the one-point compactification in topology.

Second, $\Pi_S^\infty(X)$ can be canonically identified with the homotopy fiber of the canonical map

$$(4.4.3.a) \quad \Pi_S(\partial X) \oplus \Pi_S(X) \xrightarrow{i_* + j_*} \Pi_S(\bar{X})$$

Under motivic realization, (4.4.3.a) becomes the formula for the boundary motive given in [87, Proposition 2.4].

A reformulation of Proposition 4.4.2 yields the following invariance result for the punctured tubular neighborhood of a closed subscheme Z of a proper S -scheme X :

Corollary 4.4.4. *Let (X, Z) be a closed S -pair such that X/S is proper. Then the punctured tubular neighborhood $\mathrm{TN}_S^\times(X, Z)$ is isomorphic to $\Pi_S^\infty(X - Z)$, and therefore it depends only on the open subscheme $X - Z$.*

By combining Proposition 4.1.5 and Proposition 4.4.2, we obtain the following result.

Corollary 4.4.5. *Let $(\bar{X}, \partial X)$ be the closed S -pair associated to a compactification of a separated S -scheme X . Assume that $(\bar{X}, \partial X)$ is weakly h -smooth with normal bundle $N = N_{\partial X/\bar{X}}$. Then there is a homotopy exact sequence*

$$(4.4.5.a) \quad \Pi_S^\infty(X) \longrightarrow \Pi_S(\partial X) \xrightarrow{e(N)} \mathrm{Th}_S(N)$$

where $e(N)$ is induced by the Euler class of N (see 2.3.3). In particular, $\Pi_S^\infty(X) = \Pi_S(N^\times)$, and when $e(N)$ vanishes, there is a splitting $\Pi_S^\infty(X) = \Pi_S(\partial X) \oplus \mathrm{Th}_S(N)[-1]$.

Remark 4.4.6. Assume that S is the spectrum of a perfect field k of characteristic exponent p . Then the realization in $\mathrm{DM}(k)[1/p]$ of the homotopy exact sequence (4.4.5.a) is the homotopy exact sequence

$$\partial M(X) \longrightarrow M(\partial X) \xrightarrow{\tilde{c}_r(N)} M(\partial X)(r)[2r]$$

where $\partial M(X)$ is the boundary motive of X in Example 4.3.3, r is the rank of the normal bundle N of ∂X in \bar{X} and the map $\tilde{c}_r(N)$ is induced by multiplication with the top Chern class $c_r(N) \in \mathrm{CH}^r(\partial X) \simeq \mathrm{Hom}(M(\partial X), \mathbf{1}(r)[2r])$. Corollary 4.4.5 implies that $\Pi_k^\infty(X)$ is a strictly finer invariant than $\partial M(X)$, see Remark 4.1.4.

4.5. Interpretation in terms of fundamental classes. In what follows, we observe connections between stable homotopy at infinity and more generally punctured tubular neighborhoods and certain fundamental classes.

Proposition 4.5.1. *Let $f : X \rightarrow S$ be a smooth morphism with relative tangent bundle T_f . Then the map $\alpha'_{X/S}$ obtained by adjunction from the composite*

$$\Pi_S(X) \xrightarrow{\alpha_{X/S}} \Pi_S^c(X) \simeq \underline{\mathrm{Hom}}(\Pi_S(X, -T_f), \mathbf{1}_S),$$

where the isomorphism uses Proposition 2.5.5(4), fits into the commutative diagram

$$\begin{array}{ccc} \Pi_S(X) \otimes \Pi_S(X, -T_f) & \xrightarrow{\alpha'_{X/S}} & \mathbf{1}_S \\ \simeq \downarrow & & \uparrow f_* \\ \Pi_S(X \times_S X, -p_j^{-1}T_f) & \xrightarrow{\delta^!} & \Pi_S(X) \end{array}$$

The left vertical map is the Künneth isomorphism (2.6.1.b) and $\delta^!$ is the Gysin map (2.3.1) associated with the diagonal immersion $\delta : X \rightarrow X \times_S X$.

In other words, the map $\alpha_{X/S}$, whose homotopy cofiber is the stable homotopy at infinity of X/S , can be computed under the canonical isomorphisms

$$\begin{aligned} [\Pi_S(X), \Pi_S^c(X)] &\simeq [\Pi_S(X) \otimes \Pi_S(X, -T_f), \mathbf{1}_S] \\ &\simeq [\Pi_S(X \times_S X), \mathrm{Th}(p_j^{-1}T_f)] = H_{\mathcal{D}}^0(X \times_S X, p_j^{-1}(T_f)) \end{aligned}$$

as the twisted fundamental class $[\Delta_{X/S}]_{X \times_S X}^j$ of the diagonal, with respect to the δ -parallelization corresponding to the smooth retraction p_j of δ , see Example 2.3.6.

Proof. For notational convenience, let $p_1 : X \times_S X \rightarrow X$ be the projection on the first factor. The associativity formula in [35, Theorem 3.3.2] shows the equality of fundamental classes $\eta_\delta \cdot \eta_{p_1} = 1$. The assumption that f is smooth implies the cartesian square

$$\begin{array}{ccc} X \times_S X & \xrightarrow{p_1} & X \\ p_2 \downarrow & \Delta & \downarrow f \\ X & \xrightarrow{f} & S \end{array}$$

is Tor-independent. Thus the transversal base change formula in [35, Theorem 3.3.2] implies the equality $\Delta^*(\eta_f) = \eta_{p_1}$ from which the commutativity of the square follows. \square

Remark 4.5.2. Computing fundamental classes of the diagonal is a famous problem, at the center of the Chow-Künneth conjecture, for example. The previous proposition shows the link between determining the stable homotopy type at infinity, or the boundary motive, of X/S and computing the (twisted) fundamental class of its diagonal. The main difference with the Chow-Künneth conjecture is that we are interested mainly in the non-proper case.

Similarly, one gets the following link between punctured tubular neighborhoods and another fundamental class.

Proposition 4.5.3. *Let (X, Z) be a closed S -pair such that X/S is smooth with relative tangent bundle $T_{X/S}$ and such that Z/S is proper and has smooth crossings (see Definition 3.3.2). Then the map $\beta'_{X,Z}$ obtained by adjunction from*

$$\Pi_S(Z) \xrightarrow{\beta_{X,Z}} \Pi_S(X/X - Z) \simeq \Pi_S(Z, -\langle i^{-1}T_{X/S} \rangle)^\vee,$$

where the isomorphism follows from Theorem 4.2.2, fits into the commutative diagram

$$\begin{array}{ccc} \Pi_S(Z) \otimes \Pi_S(Z, -\langle i^{-1}T_{X/S} \rangle) & \xrightarrow{\beta'_{X,Z}} & \mathbf{1}_S \\ \text{Id} \otimes i_* \downarrow & & \uparrow q_* \\ \Pi_S(Z) \otimes \Pi_S(X, -\langle T_{X/S} \rangle) & \xrightarrow{(*)} \Pi_S(Z \times_S X, -\langle p_j^{-1}T_{X/S} \rangle) \xrightarrow{\gamma_i^\dagger} & \Pi_S(Z) \end{array}$$

where γ_i^\dagger is the Gysin morphism associated to the graph immersion $\gamma_i = \text{Id} \times i : Z \rightarrow Z \times_S X$.

In other words, the map $\beta_{X,Z}$, whose cone is the punctured tubular neighborhood $\text{TN}_S^\times(X, Z)$ of the pair (X, Z) , can be computed under the canonical isomorphisms

$$\begin{aligned} [\Pi_S(Z), \Pi_S(X/X - Z)] &\simeq [\Pi_S(Z), \Pi_S(Z, -\langle i^{-1}T_{X/S} \rangle)^\vee] \simeq [\Pi_S(Z) \otimes \Pi_S(Z, -\langle i^{-1}T_{X/S} \rangle), \mathbf{1}_S] \\ &\simeq [\Pi_S(Z \times_S X, -\langle p_j^{-1}T_{X/S} \rangle), \mathbf{1}_S] \simeq H_{\mathcal{F}}^0(Z \times_S X, p_j^{-1}T_{X/S}) \end{aligned}$$

as the twisted fundamental class $[\Gamma_i]_{Z \times_S X}^{\text{can}}$ of the graph γ_i of the closed immersion $i : Z \rightarrow X$, with obvious γ_i -parallelization $N_{\gamma_i} \simeq \gamma_i^{-1}(p_j^{-1}T_{X/S})$.

Proof. First, let us note that γ_i is a section of the smooth separated morphism $Z \times_S X \rightarrow Z$. So it is a regular closed immersion whose normal bundle is isomorphic to the relative tangent bundle $p_j^*T_{X/S}$ of $Z \times_S X$ over Z . This justifies the existence of the Gysin map γ_i^\dagger using 2.3.1. Secondly, the isomorphism $(*)$ follows from the Künneth isomorphism of Proposition 3.3.6. A routine check using the definitions of the maps shows that the diagram commutes. \square

4.5.4. Pushing the idea from the preceding result, one obtains a method of computation for the decomposition of punctured tubular neighborhoods obtained in Theorem 4.2.1. We use the notations of *op. cit.*: (X, Z) is a closed S -pair, $Z = \cup_{i \in I} Z_i$. Furthermore, we make the following assumptions.

- (1) X/S is smooth with relative tangent bundle $T_{X/S}$.
- (2) Z/S is proper and has smooth crossings.

In fact, as Z_i/S is smooth and proper, one deduces from Example 2.5.7 that $\Pi_S(Z_i, N_i)$ is rigid with dual $\Pi_S(Z_i, -\langle T_{X/S}^i \rangle)$, where we denote by $T_{X/S}^i$ the restriction of $T_{X/S}$ to Z_i and use the isomorphism of virtual vector bundles $\langle T_{X/S}^i \rangle = \langle N_i \rangle + \langle T_{Z_i/S} \rangle$. Combined with the Künneth formula (2.6.1.b), one gets a canonical isomorphism

$$(4.5.4.a) \quad \varphi : [\Pi_S(Z_i), \Pi_S(Z_j, N_j)] \xrightarrow{\cong} H_{\mathcal{F}}^0(Z_i \times_S Z_j, p_2^{-1}T_{X/S}^j)$$

Proposition 4.5.5. *Consider the above assumptions and the cartesian square of closed immersions*

$$(4.5.5.a) \quad \begin{array}{ccc} Z'_{ij} & \longrightarrow & X \\ \nu'_{ij} \downarrow & & \downarrow \delta \\ Z_i \times_S Z_j & \xrightarrow{\bar{\nu}_i \times_S \bar{\nu}_j} & X \times_S X \end{array}$$

Let $\delta_{ij} : \Pi_S(Z_i) \rightarrow \Pi_S(Z_j, N_j)$ be the map appearing in Theorem 4.2.1.

(1) Through the isomorphism (4.5.4.a), we have

$$\delta_{ij} = (\bar{\nu}_i \times_S \bar{\nu}_j)^*([\Delta_{X/S}]^2_{X \times X})$$

where the right-hand side is the second twisted fundamental class of the diagonal of X/S (see Example 2.3.6).

(2) If $i = j$, ν'_{ii} is the diagonal δ_i of Z_i/S . We consider the map

$$H^0_{\mathcal{T}}(X, N_i) \xrightarrow{\epsilon_{2*}} H^0_{\mathcal{T}}(Z_i, N_{\delta_i} + \delta_i^{-1} p_2^{-1} T_{X/S}) \xrightarrow{\delta_{i!}} H^0_{\mathcal{T}}(Z_i \times_S Z_i, p_2^{-1} T_{X/S})$$

where the first map is induced by the canonical isomorphism of virtual bundles

$$\epsilon_2 : \langle N_i \rangle \simeq \langle N_i \rangle - \langle T_{X/S}^i \rangle + \langle \delta_i^{-1} p_2^{-1} T_{X/S} \rangle \simeq \langle N_{\delta_i} \rangle + \langle \delta_i^{-1} p_2^{-1} T_{X/S} \rangle$$

over Z_i and $\delta_{i!}$ is the Gysin map in cohomotopy (see 2.3.1). Let also $e(N_i)$ be the Euler class of the normal bundle N_i of Z_i/X (see Example 2.3.3). Then through the isomorphism (4.5.4.a), we have

$$\delta_{ii} = \delta_{i!}(\epsilon_{2*} e(N_i))$$

(3) Assume furthermore that (4.5.5.a) is transversal: ν'_{ij} is regular with normal bundle isomorphic to the restriction of T_X to Z_{ij} , i.e., it is of proper codimension. Then δ_{ij} can be computed through the isomorphism (4.5.4.a) as

$$\delta_{ij} = [Z'_{ij}]^2_{Z_i \times Z_j}$$

Here, $[Z'_{ij}]^2_{Z_i \times Z_j} \in H^0_{\mathcal{T}}(Z_i \times_S Z_j, p_2^{-1} \langle T_X^j \rangle)$ is the twisted fundamental class of ν'_{ij} with respect to the obvious ν'_{ij} -parallelization.

Proof. The first statement follows from the definition of the explicit duality pairing given in Example 2.5.7, and the properties of fundamental classes. For compatibility with composition and transversal base change formula for closed immersions, see [35, Lemma 3.2.13, Ex. 3.2.9(i)]. The second (resp. third) computation follows from the first one and the excess intersection (resp. transversal base change) formula for the above cartesian square. \square

Example 4.5.6. When \mathcal{T} is an oriented motivic category, i.e., one of the categories under DM in (1.2.0.a), and we assume that the second condition of the proposition holds, then $\delta_{ij} = [Z'_{ij}]_{Z_i \times Z_j}$ is the image of the usual cycle class of the natural diagonal immersion of Z'_{ij} by the cycle class map

$$\mathrm{CH}^d(Z_i \times_S Z_j) \rightarrow H^{2d,d}_{\mathcal{T}}(Z_i \times_S Z_j)$$

where d is the dimension of X/S . In particular, we get $\delta_{ij} = \delta_{ji}$ after making the identification $\mathrm{CH}^d(Z_i \times_S Z_j) = \mathrm{CH}^d(Z_j \times_S Z_i)$. That is, the matrix in Theorem 4.2.1 is *symmetric*. In the non-oriented case, this will no longer be true in general, as we will illustrate in the forthcoming section.

5. MOTIVIC PLUMBING

In this section, we explain in which sense our punctured tubular neighborhood gives rise to a motivic version of Mumford's plumbing construction from [71] and show how to extend some of the computations of *loc. cit.* in the \mathbf{A}^1 -homotopical context.

5.1. Assumptions and notation. We impose the following assumptions.

(M1) (X, D) is a closed S -pair, S an arbitrary base scheme.⁹

⁹On the stable homotopy case we will often assume that S is semi-local. In our examples S will be a field.

- (M2) X/S has relative dimension 2 and is smooth in a neighborhood of D .
- (M3) D is a divisor on X , proper and with smooth reduced crossings over S .¹⁰
- (M4) The components $(D_i)_{i \in I}$ of D are rational curves. For $i \in I$, we fix an isomorphism $\alpha_i : D_i \rightarrow \mathbf{P}_S^1$. This determines 3 distinct rational points on D_i , say $0_i, 1_i, \infty_i$. We assume that the point ∞_i on D_i is not a point on another irreducible component D_j of D . We let $\omega_i = \det(\Omega_{D_i/S}) \cong \alpha_i^* \mathcal{O}_{\mathbf{P}_S^1}(-2)$ be the canonical sheaf of D_i and we denote by T_i its associated line bundle.

Later on, we will add one of two other assumptions (M5a)/(M5b) to this list (see Lemma 5.2.5). In the above situation, we follow the conventions of the previous section:

- Let $\Omega_{X/S}|_D$ be the restriction to D of the relative cotangent sheaf of X/S (computed in a neighborhood of D) and denote by $T_X|_D$ the corresponding vector bundle of rank 2 on D . We let $\omega_X|_D = \det(\Omega_{X/S}|_D)$.
- For every $i \in I$, we denote by $\nu_i : D_i \rightarrow D$ and by $\bar{\nu}_i : D_i \rightarrow X$ the natural closed immersions and by $p_i : D_i \rightarrow S$ the projection. We let $\mathcal{C}_i = \mathcal{C}_{D_i/X}$ be the conormal sheaf of D_i in X , and we denote by N_i the associated line bundle.
- We fix an arbitrary order on I . For $i < j$ in I , we put $D'_{ij} = D_i \times_X D_j$, and let D_{ij} be its reduction. Under our assumptions, the projection $p_{ij} : D_{ij} \rightarrow S$ is a finite étale morphism. There are closed immersions $\nu_{ij}^l : D_{ij} \rightarrow D_l, l = i, j$. We set $\mathcal{C}_{ij} = \mathcal{C}_{D_{ij}/X}$ and $\mathcal{C}_{ij}^l = \mathcal{C}_{D_{ij}/D_l}$ the conormal sheaves of D_{ij} in X and D_l respectively, $l = i, j$, and we denote by N_{ij} and N_{ij}^l the associated vector bundles of respective rank 2 and 1 on D_{ij} .

For $i < j$, there are canonical isomorphisms

$$(5.1.0.a) \quad \mathcal{C}_{ij} \simeq \Omega_X|_{D_{ij}} \quad \mathcal{C}_{ij}^l \simeq \omega_l|_{D_{ij}}, \quad l = i, j$$

$$(5.1.0.b) \quad \omega_X|_{D_i} \simeq \mathcal{C}_i \otimes \omega_i, \quad \det(\mathcal{C}_{ij}) \simeq \mathcal{C}_{ij}^l \otimes \mathcal{C}_l|_{D_{ij}} \quad l = i, j$$

These define canonical isomorphisms of virtual vector bundles $T_X|_{D_i} \simeq T_i + N_i$ and

$$T_X|_{D_{ij}} \simeq N_l|_{D_{ij}} + N_{ij}^l \simeq N_l|_{D_{ij}} + T_l|_{D_{ij}} \quad l = i, j$$

To get more precise results, we will also introduce at some point (see (Proposition 5.2.8 and the subsequent results) the following strengthening of assumption (M3):

- (M3+) S is the spectrum of a field k , the irreducible components D_i of D intersect transversely (at all points D_{ij}^λ), and the residue field κ_{ij}^λ of D_{ij}^λ is finite separable over k .

The transversality assumption implies that for every $i < j$, we have $D_{ij} = D_i \times_X D_j = \bigsqcup_\lambda D_{ij}^\lambda$. Furthermore, it guarantees for all $i \neq j$ and all λ the existence of a canonical isomorphism $\mathcal{C}_i|_{D_{ij}^\lambda} \simeq \mathcal{C}_{ij}^j|_{D_{ij}^\lambda}$ and hence of a canonical isomorphism

$$(5.1.0.c) \quad \mathcal{C}_i|_{D_{ij}^\lambda} \otimes \omega_j^\vee|_{D_{ij}^\lambda} \simeq \mathcal{C}_{ij}^j|_{D_{ij}^\lambda} \otimes \omega_j^\vee|_{D_{ij}^\lambda} \stackrel{(5.1.0.a)}{\simeq} \omega_j|_{D_{ij}^\lambda} \otimes \omega_j^\vee|_{D_{ij}^\lambda} \simeq \kappa_{ij}^\lambda$$

5.2. Punctured tubular neighborhoods and quadratic Mumford matrices. Under a reasonable orientability assumption, the computations in this subsection apply to any motivic ∞ -category \mathcal{T} . We will later focus on the (universal) cases of SH and $\mathrm{DM}_{\mathbb{Q}}$.

5.2.1. Under the assumptions and notation in Section 5.1, we will use the formula given in Theorem 4.2.1 in two steps. First, since D is an h-smooth crossing S -scheme, Proposition 3.3.3 identifies $\Pi_S(D)$ with the homotopy cofiber of the canonical map

$$(5.2.1.a) \quad d_1 = \sum_{i < j} \nu_{ij*}^i - \nu_{ij*}^j : \bigoplus_{i < j} \Pi_S(D_{ij}) \rightarrow \bigoplus_{i \in I} \Pi_S(D_i)$$

¹⁰See Definition 3.3.2.

The isomorphisms α_i of 5.1(M4) determine an isomorphism

$$(5.2.1.b) \quad \gamma_1 = \sum_{i \in I} \alpha_{i*} : \bigoplus_{i \in I} \Pi_S(D_i) \rightarrow \bigoplus_{i \in I} \mathbf{1}_S \oplus \bigoplus_{i \in I} \mathbf{1}_S(1)[2]$$

The following lemma is immediate.

Lemma 5.2.2. *Under the above assumptions and notation, the composite $\gamma_1 \circ d_1$ factors through the inclusion of the direct summand $\bigoplus_{i \in I} \mathbf{1}_S$ on the right-hand-side of (5.2.1.b). Moreover, the factorization coincides with the map*

$$(5.2.2.a) \quad q_1 = \sum_{i < j} p_{ij*}^i - p_{ij*}^j : \bigoplus_{i < j} \Pi_S(D_{ij}) \rightarrow \bigoplus_{i \in I} \mathbf{1}_S$$

where p_{ij*}^l is the composition of the inclusion of the l -factor of the left hand-side with p_{ij*} .

In particular, there is a canonical isomorphism

$$(5.2.2.b) \quad \Pi_S(D) \simeq \mathcal{D} \oplus \bigoplus_{i \in I} \mathbf{1}_S(1)[2]$$

where \mathcal{D} is the homotopy cofiber of q_1 .

Remark 5.2.3. One can interpret the preceding lemma by saying that $\Pi_S(D)$ is a sum of the combinatorial part \mathcal{D} depending on the combinatorics of the intersection of the irreducible components of D , which is a smooth Artin object¹¹ and the “geometric” part $\bigoplus_{i \in I} \mathbf{1}_S(1)[2]$.

5.2.4. The description of the target $\Pi_S(X/X - D)$ of the map $\beta_{X,D}$ in Definition 4.1.1 is more involved, in particular in the non-oriented case.

Lemma 5.2.5. *Consider the assumptions of Section 5.1. Assume one of the following condition holds:*

(M5a) *The motivic ∞ -category \mathcal{T} is oriented.*

(M5b) *The motivic category is $\mathcal{T} = \widehat{\text{DM}}$, or $\mathcal{T} = \text{SH}$ and $K_0(S)$ is infinite cyclic. Furthermore, we assume given for every $i \in I$ two orientation classes $\epsilon_i \in \mathcal{O}r_{D_i}(\mathcal{C}_i)$ and $\tau_i \in \mathcal{O}r_{D_i}(\omega_i)$ (see 6.1.5) such that for every $i < j$, the equality*

$$(\epsilon_i \otimes \tau_i)|_{D_{ij}} = (\epsilon_j \otimes \tau_j)|_{D_{ij}}$$

holds in $\mathcal{O}r_{D_{ij}}(\omega_X|_{D_{ij}}) \stackrel{(5.1.0.b)}{\simeq} \mathcal{O}r_{D_{ij}}((\mathcal{C}_i \otimes \omega_i)|_{D_{ij}})$.

Then there exists an isomorphism

$$\epsilon_* : \Pi_S(X/X - D) \rightarrow \Pi_S(D)^\vee(2)[4] = \underline{\text{Hom}}(\Pi_S(D), \mathbf{1}_S)(2)[4]$$

in $\mathcal{T}(S)$, canonical in case (M5a) and depending canonically on the orientations classes $(\epsilon_i)_{i \in I}$ and $(\tau_i)_{i \in I}$ in case (M5b).

Combining the isomorphism ϵ_* with Lemma 5.2.2, we get the isomorphism

$$(5.2.5.a) \quad \Pi_S(X/X - D) \simeq \mathcal{D}^\vee(2)[4] \oplus \bigoplus_{j \in I} \mathbf{1}_S(1)[2]$$

Note moreover that \mathcal{D}^\vee is still an Artin object as in Remark 5.2.3.

Remark 5.2.6. (1) Assumption (M5b) implies that the conormal sheaves \mathcal{C}_i of all the divisors D_i in X must have even degree.

(2) When $\omega_X|_D$ is orientable, say with a given orientation class $\varphi \in \mathcal{O}r_D(\omega_X|_D)$, one way of fulfilling condition (M5b) is to choose for τ_i the orientation given by the canonical isomorphism $c_i : \omega_i \rightarrow (\mathcal{O}_{D_i}(-1))^{\otimes 2}$ and for ϵ_i the image of the orientation class $\varphi|_{D_i} \otimes \tau_i^\vee \in \mathcal{O}r_{D_i}(\omega_X|_{D_i} \otimes \omega_i^\vee)$ under the isomorphism $\mathcal{O}r_{D_i}(\omega_X|_{D_i} \otimes \omega_i^\vee) \stackrel{(5.1.0.b)}{\simeq} \mathcal{O}r_{D_i}(\mathcal{C}_i)$.

¹¹By analogy with the case of motives, it is the smallest ∞ -category containing $\Pi_S(V)$ for V/S finite étale, and stable under suspensions, homotopy (co)fibers.

Proof. In the case of assumption (M5a), the lemma follows from Theorem 3.4.3 by canonically trivializing twists of vector bundles.

Next we treat the more involved case of (M5b). Consider the Gysin morphisms

$$\begin{aligned} (\nu_{ij}^l)^\dagger : \Pi_S(D_l, -\langle T_l \rangle)(2)[4] &\stackrel{(5.1.0.a)}{\simeq} \Pi_S(D_l, -\langle N_{ij}^l \rangle)(2)[4] \rightarrow \Pi_S(D_{ij})(2)[4] \\ (\nu_{ij}^l)^\dagger : \Pi_S(D_l, N_l) &\rightarrow \Pi_S(D_{ij}, (\nu_{ij}^l)^{-1}N_l - \langle N_{ij}^l \rangle) \stackrel{(5.1.0.b)}{\simeq} \Pi_S(D_{ij}, N_{ij}) \end{aligned}$$

Then for every $l = i, j$, we have a commutative diagram of isomorphisms

$$\begin{array}{ccc} \Pi_S(D_l)^\vee(2)[4] & \xrightarrow{(\nu_{ij*}^l)^\vee} & \Pi_S(D_{ij})^\vee(2)[4] \\ \theta_l \downarrow & & \downarrow \theta_{ij} \\ \Pi_S(D_l, -\langle T_l \rangle)(2)[4] & \xrightarrow{(\nu_{ij}^l)^\dagger} & \Pi_S(D_{ij})(2)[4] \\ \eta_*^{(l)} \downarrow & & \downarrow \epsilon_*^{(ij,l)} \\ \Pi_S(D_l, N_l) & \xrightarrow{(\nu_{ij}^l)^\dagger} & \Pi_S(D_{ij}, N_{ij}) \end{array}$$

The top square is the canonical commutative diagram coming from the explicit duality pairing constructed in Example 2.5.7. The vertical isomorphism of the bottom square is defined as follows: The isomorphism $\epsilon_*^{(ij,l)}$ is obtained either from the **SL**-orientation of \widetilde{DM} or in case $\mathcal{T} = \text{SH}$ from Proposition 6.1.16 by using the orientation $(\epsilon_l \otimes \tau_l)|_{D_{ij}}$ of $\det \mathcal{C}_{ij} \stackrel{(5.1.0.a)}{\simeq} \omega_X|_{D_{ij}}$. The isomorphism $\eta_*^{(l)}$ is obtained in the same way by using the isomorphisms

$$\Pi_S(D_l, N_l) \simeq \Pi_S(D_l)(1)[2] \quad \text{and} \quad \Pi_S(D_l, -\langle T_l \rangle) \simeq \Pi_S(D_l)(-1)[-2]$$

associated to the chosen orientations ϵ_l and τ_l , respectively. The assumption that the orientations $(\epsilon_i \otimes \tau_i)|_{D_{ij}}$ and $(\epsilon_j \otimes \tau_j)|_{D_{ij}}$ coincide in $\mathcal{O}r_{D_{ij}}(\omega_X|_{D_{ij}})$ ensures that $\epsilon_*^{(ij,i)} = \epsilon_*^{(ij,j)}$. It follows that the diagram

$$\begin{array}{ccc} \bigoplus_{i \in I} \Pi_S(D_i)^\vee(2)[4] & \xrightarrow{\sum_{i < j} (\nu_{ij*}^i)^\vee - (\nu_{ij*}^j)^\vee} & \bigoplus_{i < j} \Pi_S(D_{ij})^\vee(2)[4] \\ \bigoplus \eta_*^{(i)} \theta_i \downarrow & & \downarrow \bigoplus \epsilon_*^{(ij,i)} \theta_{ij} = \bigoplus \epsilon_*^{(ij,j)} \theta_{ij} \\ \bigoplus_{i \in I} \Pi_S(D_l, N_l) & \xrightarrow{\sum_{i < j} (\nu_{ij}^i)^\dagger - (\nu_{ij}^j)^\dagger} & \bigoplus_{i < j} \Pi_S(D_{ij}, N_{ij}) \end{array}$$

is commutative. This provides the desired canonical isomorphism between the homotopy fiber $\Pi_S(D)^\vee(2)[4]$ of top line (see (5.2.1.a)) and the homotopy fiber $\Pi_S(X/X - D)$ of the bottom line (see Proposition 3.3.10). \square

Using the above two lemmas, we can refine Theorem 4.2.1 as follows:

Theorem 5.2.7. *Assume conditions (M1)-(M4) of Section 5.1 as well as one of the conditions (M5a) or (M5b) of Lemma 5.2.5. Then the punctured tubular neighborhood $\text{TN}_S^\times(X, D)$ in $\mathcal{T}(S)$, or equivalently (Proposition 4.4.2) the homotopy at infinity $\Pi_S^\infty(X - D)$ when X/S is in addition proper, is isomorphic to the homotopy fiber of the map*

$$\beta = \begin{pmatrix} a & b' \\ b & \mu \end{pmatrix} : \mathcal{D} \oplus \bigoplus_{i \in I} \mathbf{1}_S(1)[2] \rightarrow \mathcal{D}^\vee(2)[4] \oplus \bigoplus_{j \in I} \mathbf{1}_S(1)[2]$$

Here $\mu : \bigoplus_{i \in I} \mathbf{1}_S(1)[2] \rightarrow \bigoplus_{j \in I} \mathbf{1}_S(1)[2]$ is given by a square matrix $(\mu_{ij})_{i,j \in I^2}$ with coefficients in the endomorphism ring $\text{End}_{\mathcal{T}}(\mathbf{1}_S)$.

In the case (M5b), we call μ the *quadratic Mumford matrix* of (X, D) . The next proposition provides an explicit description of this matrix.

Proposition 5.2.8. *We assume conditions (M1)-(M4) of Section 5.1.*

- In case (M5a), and $\mathcal{T} = \text{DM}, \text{DM}_{\text{ét}}, \text{D}_B^\sigma, D(-\text{ét}, \mathbb{Z}_\ell), \text{D}_{\text{Hdg}}^m$ (see diagram (1.2.0.a)) we have $\text{End}_{\mathcal{T}}(k) = \mathbb{Z}$ and for every i, j ,

$$\mu_{ij} = \deg([D_i] \cdot [D_j]) = (D_i, D_j)$$

is the usual intersection number of the (effective Cartier) divisors D_i and D_j .

- In case (M5b), $\text{End}_{\mathcal{T}}(\mathbf{1}_{\mathbb{Z}[\frac{1}{2}]}) \otimes \mathbb{Z}_{(2)} = \text{GW}(\mathbb{Z}[\frac{1}{2}]) \otimes \mathbb{Z}_{(2)}^{12}$ and $\text{End}_{\mathcal{T}}(\mathbf{1}_k) = \text{GW}(k)$ for any field k . Then, for every i , we have

$$\mu_{ii} = \widetilde{\deg}_{\tau'_i} e(N_i, \epsilon_i)$$

where $e(N_i, \epsilon_i) \in \widetilde{\text{CH}}^1(D_i)$ is the Euler class of the oriented bundle (N_i, ϵ_i) , and $\widetilde{\deg}_{\tau'_i} : \widetilde{\text{CH}}^1(X) \rightarrow \text{GW}(k)$ is the quadratic τ'_i -degree associated with the quadratic isomorphism $\tau'_i = \tau_i^{-1} : \mathcal{O}_{D_i} \xrightarrow{\sim} \omega_i$ (see (6.2.3.a) and Remark 5.2.9(2)).

Furthermore, under the additional assumption (M3+) and for every $i \neq j$, we have

$$\mu_{ij} = \sum_{\lambda} \langle \text{Tr}_{\kappa_{ij}^\lambda/k}(u_{ij}^\lambda \cdot xy) \rangle$$

Here $\text{Tr}_{\kappa_{ij}^\lambda/k}$ is the trace form (in variables x, y), and $u_{ij}^\lambda \in \kappa_{ij}^\lambda$ is a unit whose quadratic class is the image of the orientation class

$$\epsilon_{ij}^\lambda = (\epsilon_i \otimes \tau_j^\vee)|_{D_{ij}^\lambda} \in \mathcal{O}r_{D_{ij}^\lambda}((\mathcal{C}_i \otimes \omega_j^\vee)|_{D_{ij}^\lambda})$$

under the isomorphism $\mathcal{O}r_{D_{ij}^\lambda}((\mathcal{C}_i \otimes \omega_j^\vee)|_{D_{ij}^\lambda}) \stackrel{(5.1.0.c)}{\simeq} \mathcal{O}r_{D_{ij}^\lambda}(\kappa_{ij}^\lambda) \stackrel{\text{Remark 6.1.7}}{\simeq} Q(\kappa_{ij}^\lambda)$.

Proof. By construction, for every $(i, j) \in I^2$, the coefficient μ_{ij} is computed as the composite map

$$\begin{aligned} \mathbf{1}_S(1)[2] &\xrightarrow{p_i^!} \Pi_S(D_i, -T_i)(1)[2] \xrightarrow{(1)} \Pi_S(D_i) \xrightarrow{(\bar{v}_i)^*} \Pi_S(X) \\ &\xrightarrow{(\bar{v}_j)^!} \Pi_S(D_j, N_j) \xrightarrow{(2)} \Pi_S(D_j)(1)[2] \xrightarrow{(p_j)^*} \mathbf{1}_S(1)[2] \end{aligned}$$

Here $p_i : D_i \rightarrow \text{Spec}(k)$ (resp. $\bar{v}_i : D_i \rightarrow X$) is the projection map (resp. inclusion), the map (1) (resp. (2)) is the isomorphism from Lemma 5.2.2 (Lemma 5.2.5). The case (M5a) follows readily. In the case (M5b), (1) corresponds to the isomorphism induced by the orientation τ_i and (2) from the orientation ϵ_j . The assertion then follows readily from the definitions. \square

Remark 5.2.9. (1) The assumption made in Lemma 5.2.5 case (M5b) that $(\epsilon_i \otimes \tau_i)|_{D_{ij}}$ and $(\epsilon_j \otimes \tau_j)|_{D_{ij}}$ are equivalent orientations the sheaf $\omega_X|_{D_{ij}^\lambda}$ implies under assumption (M3+) that for all $i \neq j$ and all λ the orientation classes ϵ_{ij}^λ and ϵ_{ji}^λ of κ_{ij}^λ are equal, hence that the matrix μ is symmetric.

- (2) The element $\widetilde{\deg}_{\tau_i} e(N_i, \epsilon_i) \in \text{GW}(k)$ coincides with the Euler number $n^{\text{GS}}(N_i, \sigma_0, \rho_i)$ of the zero section σ_0 of N_i with respect to the relative orientation class $\tau_i^{-1} \circ \epsilon_i : \mathcal{C}_i \xrightarrow{\sim} \omega_i$ in $\mathcal{O}r_{D_i}(\mathcal{C}_i \otimes \omega_i^\vee)$ (see Example 6.1.8 for explanations) of \mathcal{C}_i considered by Bachmann-Wickelgren in [17]. One can check that in our setting, this element is actually independent on the chosen orientations, equal to $\frac{1}{2}(D_i, D_i)h$, where $h = \langle 1 \rangle + \langle -1 \rangle \in \text{GW}(k)$ is the class of the hyperbolic plane and where $(D_i, D_i) = \deg(\mathcal{C}_i^\vee) \in 2\mathbb{Z}$ is the usual self-intersection number of D_i^{13} . In contrast, the coefficients $\mu_{ij}, i \neq j$ of the matrix μ do depend by construction on the choice of the orientations ϵ_i and τ_i made in assumption (M5b) of Lemma 5.2.5.

¹²See [16, §5] for more general results.

¹³ \mathcal{C}_i has even degree on account of being orientable, see Remark 5.2.6(1).

5.3. Abelian mixed Artin-Tate motives.

5.3.1. In the following, we apply Theorem 5.2.7 to rational abelian mixed Artin-Tate motives. That is, we use $\mathcal{T} = \mathrm{DM}_{\mathbb{Q}}$, and restrict for simplicity to the case $S = \mathrm{Spec}(K)$ for some field K .

Let (X, D) be a K -pair satisfying the assumptions (M1)-(M4). Theorem 5.2.7 implies that the motive $M(\mathrm{TN}^{\times}(X, D))$ over K is Artin-Tate: it is in the smallest thick triangulated subcategory $\mathrm{DM}^{\mathrm{AT}}(K, \mathbb{Q})$ of $\mathrm{DM}(K, \mathbb{Q})$ which contains motives of the form $M(L)(n)$, where L/K is a finite separable extension of K .

To state the next result, we moreover consider one of the following settings:

- (1) Assume K is a field of Kronecker index at most one;¹⁴ for example, a number field, a finite field or a finitely generated field of transcendence degree 1 over a finite field. We let $\mathrm{DM}^{\mathrm{AT}}(K, \mathbb{Q})$ be the triangulated category of (constructible) Artin-Tate motives over \mathbb{Q} . From [65] it follows that $\mathrm{DM}^{\mathrm{AT}}(K, \mathbb{Q})$ admits a motivic t-structure (uniquely characterized), whose heart is the Tannakian category $\mathrm{MM}^{\mathrm{AT}}(K, \mathbb{Q})$ of abelian Artin-Tate motives. In particular, one gets a homological and monoidal functor

$$\underline{H}_0 : \mathrm{DM}^{\mathrm{AT}}(K, \mathbb{Q}) \rightarrow \mathrm{MM}^{\mathrm{AT}}(K, \mathbb{Q})$$

- (2) Assume K is a field of characteristic 0 with a fixed complex embedding. Then we can consider the Tannakian category $\mathcal{M}(K)$ of Nori motives over K , as defined in [55], together with its canonical (universal) homological functor

$$\underline{H}_0 : \mathrm{DM}_{gm}(K, \mathbb{Q}) \rightarrow \mathcal{M}(K)$$

In that case, we define the category of Artin-Tate-Nori motives $\mathrm{MM}^{\mathrm{AT}}(K, \mathbb{Q})$ as the smallest thick abelian subcategory of $\mathcal{M}(K)$ which contains $\underline{H}_0 \mathrm{DM}^{\mathrm{AT}}(K, \mathbb{Q})$. As in the previous case, we get a homological functor.

Under these assumptions, we define the Artin-Tate-Nori motive

$$\underline{H}_i(\mathrm{TN}^{\times}(X, D)) := \underline{H}_0(\mathrm{TN}^{\times}(X, D)[-i])$$

as the i -th (motivic) homology of the punctured tubular neighborhood of (X, D) . When X is in addition proper over K , this is the homology of the boundary motive of $(X - D)$ (see Example 4.3.3 and Proposition 4.4.2), or the *motivic homology at infinity*

$$\underline{H}_i^{\infty}(X - D) = \underline{H}_i(\mathrm{TN}^{\times}(X, D))$$

Proposition 5.3.2. *Under the above assumptions, the homology motive $\underline{H}_i(X)$ vanishes for $i \notin [0, 3]$ and there is an exact sequence*

$$\begin{aligned} 0 \rightarrow \underline{H}_3(\mathrm{TN}^{\times}(X, D)) &\rightarrow \bigoplus_{i \in I} \mathbf{1}_S(2) \xrightarrow{\sum_{i < j} p_{ij}^{i!} - p_{ij}^{j!}} \bigoplus_{i < j} M_S(D_{ij})(2) \\ &\rightarrow \underline{H}_2(\mathrm{TN}^{\times}(X, D)) \rightarrow \bigoplus_{i \in I} \mathbf{1}_S(1) \xrightarrow{\mu} \bigoplus_{j \in I} \mathbf{1}_S(1) \\ &\rightarrow \underline{H}_1(\mathrm{TN}^{\times}(X, D)) \rightarrow \bigoplus_{i < j} M_S(D_{ij}) \xrightarrow{\sum_{i < j} p_{ij*}^i - p_{ij*}^j} \bigoplus_{i \in I} \mathbf{1}_S \rightarrow \underline{H}_0(\mathrm{TN}^{\times}(X, D)) \rightarrow 0 \end{aligned}$$

Here, μ is the Mumford matrix, and $M_S(D_{ij}) = \underline{H}_0(M_S(D_{ij}))$ is seen as an abelian Artin-Tate motive, or as an Artin-Tate-Nori motive.

Note in particular that $\underline{H}_0(\mathrm{TN}^{\times}(X, D))$ and $\underline{H}_3(\mathrm{TN}^{\times}(X, D))$ are pure of respective weights 0 and -4 , while $\underline{H}_1(\mathrm{TN}^{\times}(X, D))$ and $\underline{H}_2(\mathrm{TN}^{\times}(X, D))$ are in general mixed of weights $\{0, -2\}$ and $\{-2, -4\}$, respectively (see [58] for the notion of weights on Artin-Tate-Nori motives).

¹⁴Recall the Kronecker index of a field F , of transcendence degree d over its prime subfield and characteristic p , is either $d + 1$ if $p = 0$ or d if $p > 0$.

Proof. Theorem 5.2.7 provides a distinguished triangle computing $M(\mathrm{TN}^\times(X, D))$. The above long exact sequence follows by using the long exact sequence associated with the homological functor \underline{H}_0 . \square

Remark 5.3.3. One obtains similar exact sequences of mixed motives over more general bases S using:

- (1) [80]: when $S \subset \mathrm{Spec} \mathcal{O}_K$, \mathcal{O}_K a number ring;
- (2) [58]: S a smooth K -scheme, for a field K with a complex embedding $K \subset \mathbb{C}$.

Indeed, the indicated references provide us with a suitable category of Artin-Tate(-Nori) motives, and one can make precisely the same calculation (taking into account the dimension of S as we use perverse motivic t-structures).

Example 5.3.4. To illustrate Theorem 5.2.7, Proposition 5.2.8, we compute Wildeshaus' *boundary motive*, or equivalently the motive at infinity (Example 4.3.3), of *Ramanujam's surface* Σ over a field k of characteristic different from 2. We work in $\mathcal{T} = \mathrm{DM}$, the integral category of motives.

First we recall the construction of Σ . Given a cuspidal cubic $C \subset \mathbb{P}_k^2$ and a smooth k -rational conic $Q \subset \mathbb{P}_k^2$ intersecting C with multiplicity 5 in a k -rational point p , let Σ be the complement of the proper transforms of C and Q in the blow-up $\sigma : \mathbb{F}_1 \rightarrow \mathbb{P}_k^2$ of the remaining k -rational intersection point q of C and Q (see [51] for Hirzebruch surfaces \mathbb{F}_n , $n \geq 0$). Over the complex numbers, the underlying analytic space of Σ is a topologically contractible open smooth manifold non-homeomorphic to \mathbb{R}^4 whose topological fundamental group at infinity $\pi_1^\infty(\Sigma)$ is infinite with trivial abelianization, see [74].

A compactification $X = \bar{\Sigma}$ of Σ with smooth crossing boundary $D = \partial\Sigma$ is obtained from \mathbb{F}_1 by blowing-up the singular point of C , with exceptional divisor $E \simeq \mathbb{P}_k^1$. The irreducible components of D are then E and the proper transforms of Q and C , with respective self-intersections $E^2 = -1$, $Q^2 = 4$ and $C^2 = 3$. Furthermore, Q and C intersects with multiplicity 5 at the unique point p and E and Q intersects with multiplicity 2 at a unique k -rational point.

Next we apply Theorem 5.2.7 to the pair (X, D) . One first obtains that the *Artin part* $\mathcal{D} = \mathbf{1}_k$, and that the maps a, b, b' are all zero for degree reasons (see also the proof of Proposition 5.4.2). Then from Proposition 5.2.8, the map $\mu : (\mathbf{1}_k(1)[2])^{\oplus 3} \rightarrow (\mathbf{1}_k(1)[2])^{\oplus 3}$ is given by the integer valued intersection matrix

$$\begin{pmatrix} 4 & 5 & 2 \\ 5 & 3 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

Its Smith normal form is the diagonal matrix $\Delta(1, 1, 1)$ in $M_{3,3}(\mathbb{Z})$. Theorem 5.2.7 implies the boundary motive of Σ is isomorphic to homotopy fiber of the trivial map $\mathbf{1}_k \rightarrow \mathbf{1}_k(2)[4]$. In summary, we obtain

$$\partial M(\Sigma) = M^\infty(\Sigma) \simeq \mathbf{1}_k \oplus \mathbf{1}_k(2)[3]$$

5.4. Punctured tubular neighborhoods of orientable trees of rational curves.

5.4.1. Consider the assumptions (M1)-(M4), (M3+) of Section 5.1, in the special case where D is an orientable tree of smooth k -rational curves on a smooth surface X/k over a field k , that is:

- (1) D is a smooth normal crossing divisor on X with irreducible components $D_i \simeq \mathbb{P}_k^1$, $i \in I$, such that for every $i \neq j$, D_{ij} is either empty or consists of a single k -rational point.
- (2) For every $i \in I$, the conormal sheaf \mathcal{C}_i of D_i is X is orientable, hence isomorphic to $\mathcal{O}_{D_i}(2n_i)$ for some $n_i \in \mathbb{Z}$.
- (3) The incidence complex Γ of D is a tree.

Recall that $h = \langle 1 \rangle + \langle -1 \rangle = 1 + \langle -1 \rangle \in \mathrm{GW}(k)$ denotes the class of the hyperbolic plane. As an application of the general computation of Proposition 5.2.8, we get:

Proposition 5.4.2. *Under the above assumptions, there is a choice of orientations $(\epsilon_i)_{i \in I}$ fulfilling condition (M5b) of Lemma 5.2.5 which guarantees that the punctured tubular neighborhood $\mathrm{TN}_S^\times(X, D)$ in $\mathrm{SH}(k)$ is*

isomorphic to

$$\mathbf{1}_k \oplus \text{hofib}(\mu) \oplus \mathbf{1}_k(2)[3]$$

where the quadratic Mumford matrix $\mu : \bigoplus_{i \in I} \mathbf{1}_k(1)[2] \rightarrow \bigoplus_{i \in I} \mathbf{1}_k(1)[2]$ is obtained from the integer-valued Mumford matrix $(D_i, D_j)_{i,j}$ by replacing each diagonal entry $(D_i, D_i) = -2n_i$ by $-n_i h$.

Proof. First, assume any choice of orientation ϵ_i fulfilling condition (M5b). Denote by $J \subset I \times I$ the subset consisting of pairs $i < j$ such that $D_{ij} \neq \emptyset$. Since Γ is a tree, we have $\sharp J = \sharp I - 1$ and $\bigoplus_{i < j} \Pi_k(D_{ij}) = \bigoplus_{(i,j) \in J} \mathbf{1}_k$. The map q_1 in (5.2.2.a) is given by a matrix in $M_{\sharp J, \sharp I}(\mathbb{Z})$, whose Smith normal form is the diagonal matrix

$$\begin{pmatrix} \text{id}_{\sharp J} \\ 0 \end{pmatrix}$$

The homotopy cofiber \mathcal{D} of q_1 is thus equal to that of the trivial map $0 \rightarrow \mathbf{1}_S$, hence to $\mathbf{1}_S$. This implies that $\mathcal{D}^\vee = \mathbf{1}_S$. By Morel's \mathbb{A}^1 -connectivity theorem, $\text{Hom}_{\text{SH}(k)}(\mathbf{1}_k, \mathbf{1}_k(i)[2i]) = 0$ for all $i > 0$. Thus Theorem 5.2.7 implies that $\text{TN}_S^\times(X, D)$ is the homotopy fiber of the map

$$\beta = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} : \mathbf{1}_S \oplus \bigoplus_{i \in I} \mathbf{1}_S(1)[2] \rightarrow \mathbf{1}_S(2)[4] \oplus \bigoplus_{j \in I} \mathbf{1}_S(1)[2]$$

By Proposition 5.2.8 and Remark 5.2.9(2), the diagonal entries of μ are equal to the Euler classes $e(\mathcal{C}_i^\vee) = e(\mathcal{O}_{\mathbb{P}_k^1}(-2n_i)) = -n_i h \in \text{GW}(k)$.

To finish the proof, we will now show that, up to modifying the τ_i , there always exists a choice of orientations ϵ_i fulfilling condition (M5b) and such that the orientations ϵ_{ij} of $\kappa(D_{ij}) = k$ appearing in Proposition 5.2.8 are all equivalent to the canonical orientation of k defined as the inverse of the multiplication homomorphism $m : k \otimes k \rightarrow k, a \otimes b \mapsto ab$. In turn, this shows our remaining assertion about the coefficients away from the diagonal in the matrix μ .

This can be seen as follows. Let D_0 be any irreducible component of D , which we view as the root of the incidence tree Γ of D and denote by D_1, \dots, D_s be the irreducible components of D which interest D_0 . Let $\epsilon_0 : \mathcal{C}_0 \rightarrow \mathcal{L}_0^{\otimes 2}$ and $\tau_0 : \omega_0 \rightarrow \mathcal{M}_0^{\otimes 2}$ be any fixed choice of orientations. For any orientation $\tau_j : \omega_j \rightarrow \mathcal{M}_j^{\otimes 2}, j = 1, \dots, s$ the orientation

$$\epsilon_{j,0} = \epsilon_j \otimes {}^t \tau_0^{-1} : k \simeq \mathcal{C}_j|_{D_{0,j}} \otimes \omega_0^\vee|_{D_{0,j}} \rightarrow (\mathcal{L}_j|_{D_{0,j}} \otimes \mathcal{M}_0^\vee|_{D_{0,j}})^{\otimes 2}$$

is equivalent to $u_{0,j} \cdot m^{-1} : k \rightarrow k \otimes k$, defined by $1 \mapsto u_{0,j} 1 \otimes 1$ for some element $u_{0,j} \in k^*$ uniquely determined up to multiplication by an element of $(k^*)^2$. If $u_{0,j} \neq 1$ then by replacing τ_j by $\tau_j \circ (\times u_{0,j}^{-1})$ we obtain an orientation whose associated unit $u_{0,j}$ of k as in Proposition 5.2.8 is equal to 1. The same argument implies the existence of orientations $\epsilon_j : \mathcal{C}_j \rightarrow \mathcal{L}_j^{\otimes 2}, j = 1, \dots, s$, such that for all $j = 1, \dots, s$, the unit $u_{j,0}$ of k associated to the orientation

$$\epsilon_{0,j} = \epsilon_0 \otimes {}^t \tau_j^{-1} : k \simeq \mathcal{C}_0|_{D_{0,j}} \otimes \omega_j^\vee|_{D_{0,j}} \rightarrow (\mathcal{L}_0|_{D_{0,j}} \otimes \mathcal{M}_j^\vee|_{D_{0,j}})^{\otimes 2}$$

is equal to 1.

Since the incidence complex Γ of D is a tree, the incidence complex of $\bigcup_{i \neq 0} D_i$ is a union of trees Γ_j with the components $D_j, j = 1, \dots, s$ as their respective roots, with the just constructed orientations $\tau_j : \omega_j \rightarrow \mathcal{M}_j^{\otimes 2}$ and $\epsilon_j : \mathcal{C}_j \rightarrow \mathcal{L}_j^{\otimes 2}$. By repeating for each of these trees the same argument as above, we obtain by induction the existence of a collection of orientations τ_i and ϵ_i for which the units $u_{ij} \in k = \kappa_{ij}$ associated to the orientations ϵ_{ij} and ϵ_{ji} are all equal to 1.

We claim the constructed collection of orientations has the property that for every $i \neq j$, the orientations $(\epsilon_i \otimes \tau_i)|_{D_{ij}}$ and $(\epsilon_j \otimes \tau_j)|_{D_{ij}}$ of $\omega_X|_{D_{ij}}$ appearing in Lemma 5.2.5 are equivalent. Indeed, for

$$o_{ij} = (\tau_i \otimes \tau_j)|_{D_{ij}} : \mathcal{F}_{ij} = \omega_i|_{D_{ij}} \otimes \omega_j|_{D_{ij}} \rightarrow (\mathcal{M}_i|_{D_{ij}} \otimes \mathcal{M}_j|_{D_{ij}})^{\otimes 2}$$

the orientations $(\epsilon_i \otimes \tau_i)|_{D_{ij}}$ and $(\epsilon_j \otimes \tau_j)|_{D_{ij}}$ are obtained by tensoring the equivalent ones ϵ_{ij} and ϵ_{ji} with o_{ij} using the canonical isomorphisms

$$\mathcal{C}_i|_{D_{ij}} \otimes \omega_i|_{D_{ij}} \simeq \mathcal{C}_i|_{D_{ij}} \otimes \omega_j^\vee|_{D_{ij}} \otimes \mathcal{F}_{ij} \quad \text{and} \quad \mathcal{C}_j|_{D_{ij}} \otimes \omega_j|_{D_{ij}} \simeq \mathcal{C}_j|_{D_{ij}} \otimes \omega_i^\vee|_{D_{ij}} \otimes \mathcal{F}_{ij}$$

The assertion follows now from the application of the formula in Proposition 5.2.8 which gives

$$\mu_{ij} = \langle \text{Tr}_{\kappa_{ij}/k}(u_{ij} \cdot xy) \rangle = \langle 1 \rangle = 1 = (D_i, D_j) \in \text{GW}(k)$$

□

In the next subsection, we illustrate our techniques by explicitly computing the punctured tubular neighborhoods of Du Val singularities on normal surfaces and the stable homotopy types at infinity of Danielewski hypersurfaces, a family of smooth affine surfaces of historical interest in the context of the Zariski cancellation problem.

Example 1: Stable motivic links of Du Val singularities on normal surfaces. Let X_0 be a geometrically integral normal surface essentially of finite type over a field k with an isolated k -rational double point x , also called a Du Val singularity. Recall from [2], [3] that among many equivalent characterizations, this means that letting $\pi : X \rightarrow X_0$ be the minimal desingularization of X_0 and $\pi_{\bar{k}} : X_{\bar{k}} \rightarrow X_{0,\bar{k}}$ be the base change to an algebraic closure \bar{k} of k , the following holds:

- (1) $\pi_{\bar{k}}^{-1}(x_{\bar{k}})$ is a smooth normal crossing divisor whose irreducible components are proper \bar{k} -rational curves E_i intersecting each other transversely at \bar{k} -rational points only.
- (2) The curves E_i have self-intersection number -2 and the intersection matrix $(E_i, E_j)_{i,j}$ is negative definite.

The incidence graph of the divisor $E = \pi_{\bar{k}}^{-1}(x_{\bar{k}})$ is one of the classical Dynkin diagram of type A_n , $n \geq 1$, D_n , $n \geq 4$, E_6 , E_7 and E_8 depicted in the left column of Table 1. If \bar{k} has characteristic different from 2, 3 and 5, the completion of the local ring $\mathcal{O}_{X_{0,\bar{k}},x_{\bar{k}}}$ is isomorphic to $\bar{k}[[x, y, z]]/(f)$ where f is one of the polynomials listed in the second column of Table 1, in particular the analytic local isomorphism type of the singularity depends only on the Dynkin diagram.¹⁵ Over a non-closed field, Du Val singularities A_n , D_n and E_6 can in general have non-trivial k -forms depending on the action of the Galois group $\text{Gal}(\bar{k}/k)$ on the irreducible components of E . We now assume in addition that all the irreducible components of E are defined over the base field k and isomorphic to \mathbb{P}_k^1 .¹⁶ For such singularities, the closed pair (X, E) satisfies the assumptions in 5.4.1, and the punctured tubular neighborhood $\text{TN}_S^\times(X_0, x)$ of x in X_0 is a natural invariant of the Nisnevich germ of x in X_0 which, by Corollary 4.1.8, can be computed as the punctured tubular neighborhood $\text{TN}_S^\times(X, E)$. Applying Proposition 5.4.2, we obtain the following

Proposition 5.4.3. *With the assumption above, the punctured tubular neighborhood $\text{TN}_k^\times(X_0, x)$ is isomorphic to*

$$\mathbf{1}_k \oplus \text{hofib}(\mu(\Gamma)) \oplus \mathbf{1}_k(2)[3]$$

Here $\mu(\Gamma)$ is the square matrix with entries in $\text{GW}(k)$ obtained from the integer valued intersection matrix $(E_i, E_j)_{i,j}$ associated to the Dynkin diagram $\Gamma = A_n, D_n, E_6, E_7, E_8$ by replacing each diagonal entry -2 by $-h$.

The above proposition implies that the *stable motivic link* $\text{TN}^\times(\Gamma) := \text{TN}_k^\times(X_0, x)$ of the Du Val singularity germ (X, x_0) depends only on the Dynkin diagram Γ . We summarize these links in Table 1.

5.4.4. Let us explain how to compute with Smith normal forms the part $\text{hofib}(\mu(\Gamma))$ of $\text{TN}^\times(\Gamma)$, the stable homotopy punctured tubular neighborhood associated with du Val singularities in Table 1. A priori, this is non-standard since we are considering a matrix $\mu(\Gamma)$ with coefficients in the non-principal (even non-reduced!) ring

$$\mathbb{Z}_\epsilon := \mathbf{G}_m(\mathbb{Z}) = \mathbb{Z}[\epsilon]/(\epsilon^2 - 1).$$

¹⁵In characteristics 2, 3 and 5, there are finitely many additional “normal forms”, see [3] for the complete list.

¹⁶Over a field of characteristic zero, this amounts to restricting to “split” Du Val singularities $A_n^-, D_n^-, E_6^-, E_7^-$ and E_8^- , see [64].



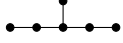

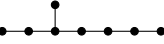
Dynkin diagram	Normal form over k	$\mathrm{TN}^\times(\Gamma)$
A_n^- 	$x^2 - y^2 - z^{n+1} = 0$	$\begin{cases} \mathbf{1}_k \oplus \mathrm{hofib}(-mh) \oplus \mathbf{1}_k(2)[3] & n = 2m - 1 \\ \mathbf{1}_k \oplus \mathrm{hofib}(\frac{n}{2}h + 1) \oplus \mathbf{1}_k(2)[3] & n \equiv 0 [4] \\ \mathbf{1}_k \oplus \mathrm{hofib}((\frac{n}{2} + 1)h - 1) \oplus \mathbf{1}_k(2)[3] & n \equiv 2 [4] \end{cases}$
D_n^- 	$x^2 + y^2z - z^{n-1} = 0$	$\begin{cases} \mathbf{1}_k \oplus \mathrm{hofib}(-h) \oplus \mathbf{1}_k(2)[3] & n = 2m \\ \mathbf{1}_k \oplus \mathrm{hofib}(-2h) \oplus \mathbf{1}_k(2)[3] & n = 2m + 1 \end{cases}$
E_6^- 	$x^2 + y^3 - z^4 = 0$	$\mathbf{1}_k \oplus \mathrm{hofib}(2h - 1) \oplus \mathbf{1}_k(2)[3]$
E_7 	$x^2 + y^3 + yz^3 = 0$	$\mathbf{1}_k \oplus \mathrm{hofib}(-h) \oplus \mathbf{1}_k(2)[3]$
E_8 	$x^2 + y^3 + z^5 = 0$	$\mathbf{1}_k \oplus \mathbf{1}_k(2)[3]$

TABLE 1. Stable motivic links of classical split forms of Du Val Singularities

However, one can consider the two quotient rings $\mathbb{Z}_\pm = \mathbb{Z}_\epsilon / (\epsilon \pm 1)$, both isomorphic to \mathbb{Z} and the canonical injective map $\pi : \mathbb{Z}_\epsilon \rightarrow \mathbb{Z}_+ \times \mathbb{Z}_-$ with image given by pairs (n, m) such that $n \equiv m \pmod{2}$ (see [28, 3.1.1, 3.1.2]). We begin with the matrix $\mu(\Gamma)$ having coefficients in \mathbb{Z}_ϵ , and compute the Smith normal form $\mu(\Gamma)_\pm = S_p m D_\pm T_\pm$ of the matrix obtained by mapping to the principal ring \mathbb{Z}_\pm (i.e., setting $\epsilon = \pm 1$). If the invertible matrices (S_+, S_-) , (T_+, T_-) , as well as the diagonal matrix (D_+, D_-) , are in the image of π (coefficients by coefficients), one can define unique lifts (S, T, D) with coefficients in \mathbb{Z}_ϵ , such that S and T are invertible and the relation $\mu(\Gamma) = S D T$ holds true. In this situation, we deduce the desired Smith normal form and in $\mathrm{SH}(k)$ we obtain an isomorphism

$$\mathrm{hofib}(\mu(\Gamma)) \simeq \mathrm{hofib}(D)$$

Remark 5.4.5. We note that except for the E_8 case, the stable motivic link $\mathrm{TN}^\times(\Gamma)$ of a Du Val singularity is different from the stable motivic link of $\mathrm{TN}^\times(\mathbb{A}_k^2, \{0\}) = \mathbf{1}_k \oplus \mathbf{1}_k(2)[3]$ of a regular point on a surface. In particular, $\mathrm{TN}^\times(\Gamma)$ distinguishes Du Val singularities other than E_8 from regular points. This is in contrast with the étale local fundamental groups of these singularities, which, in characteristic $p > 0$, do not distinguish a double point of the form A_{p^e} from a regular point, see [3]. For E_8 and the complex numbers, we can interpret the equality $\mathrm{TN}^\times(E_8) = \mathrm{TN}^\times(\mathbb{A}_k^2, \{0\})$ as a reminder that the topological link of E_8 is the Poincaré homology 3-sphere $\Sigma(2, 3, 5)$. It is a compact topological 3-manifold with the same singular homology groups as S^3 , whose fundamental group is isomorphic to the binary dodecahedral group.

Example 2: Danielewski hypersurfaces. For a field k and $n \geq 1$, the Danielewski hypersurface D_n is the smooth affine surface D_n in \mathbb{A}_k^3 cut out by the equation $x^n z = y(y - 1)$. Owing to [29], D_n becomes a Zariski locally trivial \mathbf{G}_a -bundle over the affine line with two origins $\check{\mathbf{A}}_k^1$ (using the factorization of the surjective projection $\pi_n = \mathrm{pr}_x : D_n \rightarrow \mathbf{A}_k^1$). Thus D_n is \mathbf{A}^1 -equivalent to $\check{\mathbf{A}}_k^1$ and \mathbf{P}_k^1 . The threefolds $D_n \times \mathbf{A}_k^1$ are isomorphic, but the surfaces D_n are pairwise non-isomorphic. Over \mathbb{C} , Danielewski [29], Fieseler [48] established this by showing the underlying complex analytic manifolds have non-isomorphic first singular homology groups at infinity. Our methods provide a base field independent argument that distinguishes between the D_n 's via their stable homotopy types at infinity.

We begin by constructing explicit smooth projective completions \bar{D}_n of the surfaces D_n , whose boundaries are strict normal crossing divisors. The morphism $\varphi_n = \mathrm{pr}_{x,y} : D_n \rightarrow \mathbf{A}_k^2$ expresses D_n as the affine modification of \mathbf{A}_k^2 with center at the closed subscheme Z_n with ideal $(x^n, y(y - 1))$ and divisor $D_n = \mathrm{div}(x^n)$, cf. [41]. Furthermore, φ_n decomposes into a sequence of affine modifications

$$(5.4.5.a) \quad \varphi_n = \varphi_1 \circ \psi_2 \cdots \circ \psi_n : D_n \rightarrow D_{n-1} \rightarrow \cdots \rightarrow D_2 \rightarrow D_1 \rightarrow \mathbf{A}_k^2$$

given by $\psi_\ell : D_\ell \rightarrow D_{\ell-1}; (x, y, z) \mapsto (x, y, xz)$, with center at the closed subscheme $Y_{\ell-1} = (x, z)$ and divisor $H_\ell = \text{div}(x)$. That is, $\varphi_1 : D_1 \rightarrow \mathbf{A}_k^2$ is the birational morphism obtained by blowing-up the points $(0, 0), (0, 1)$ in \mathbf{A}_k^2 and removing the proper transform of $\{0\} \times \mathbf{A}_k^1$, and $\psi_\ell : D_\ell \rightarrow D_{\ell-1}$ is the birational morphism obtained by blowing-up the points $(0, 0, 0), (0, 0, 1)$ in $\pi_\ell^{-1}(0)$ and removing the proper transform of $\pi_{\ell-1}^{-1}(0)$.

Now consider the open embedding $\mathbf{A}_k^2 \hookrightarrow \mathbf{P}_k^1 \times \mathbf{P}_k^1; (x, y) \mapsto ([x : 1], [y : 1])$. Then $C_\infty = \mathbf{P}_k^1 \times [1 : 0]$ and $F_\infty = [1 : 0] \times \mathbf{P}_k^1$ are irreducible components of $\mathbf{P}_k^1 \times \mathbf{P}_k^1$ and we set $F_0 = [0 : 1] \times \mathbf{P}_k^1$. Let $\bar{\varphi}_1 : \bar{D}_1 \rightarrow \mathbf{P}_k^1 \times \mathbf{P}_k^1$ be the blow-up of the points $([0 : 1], [0 : 1]), ([0 : 1], [1 : 1])$ in F_0 , with respective exceptional divisors $E_{1,0}, E_{1,1}$. From now on the proper transform of F_0 in \bar{D}_1 is also denoted by F_0 . With these definitions, there is a commutative diagram

$$\begin{array}{ccc} D_1 & \longrightarrow & \bar{D}_1 \\ \varphi_1 \downarrow & & \downarrow \bar{\varphi}_1 \\ \mathbf{A}_k^2 & \longrightarrow & \mathbf{P}_k^1 \times \mathbf{P}_k^1 \end{array}$$

Here, $D_1 \hookrightarrow \bar{D}_1$ is the open immersion given by the complement of the support of the strict normal crossing divisor $\partial D_1 = C_\infty \cup F_\infty \cup F_0$. The closures in \bar{D}_1 of the two irreducible components $\{x = y = 0\}$ and $\{x = y - 1 = 0\}$ of $\pi_1^{-1}(0)$ equal the exceptional divisors $E_{1,0}$ and $E_{1,1}$, respectively. We calculate the self-intersection numbers $C_\infty^2 = F_\infty^2 = 0, F_0^2 = -2$ in \bar{D}_1 ; that is, the usual degrees of the respective normal line bundles of these curves in \bar{D}_1 , see e.g., [49, Chapter 5.6], [81, Chapter IV].

To construct $\bar{D}_n, n \geq 2$, we start with \bar{D}_1 and proceed inductively by performing the same sequence of blow-ups as for the affine modifications $\psi_\ell : D_\ell \rightarrow D_{\ell-1}$ in (5.4.5.a). This yields birational morphisms $\bar{\psi}_\ell : \bar{D}_\ell \rightarrow \bar{D}_{\ell-1}$ consisting of the blow-up of one point on $E_{\ell,0} - E_{\ell-1,0}$ and another point on $E_{\ell,1} - E_{\ell-1,1}$ with respective exceptional divisors $E_{\ell+1,0}$ and $E_{\ell+1,1}$ (by convention $E_{0,0} = E_{0,1} = F_0$). Moreover, D_ℓ embeds into \bar{D}_ℓ as the complement of the support of the strict normal crossing divisor $\partial D_\ell = C_\infty \cup F_\infty \cup F_0 \cup \bigcup_{i=1}^{\ell-1} (E_{i,0} \cup E_{i,1})$ in such a way that the closures of the two irreducible components $\{x = y = 0\}$ and $\{x = y - 1 = 0\}$ of $\pi_\ell^{-1}(0)$ coincide with the divisors $E_{\ell+1,0}$ and $E_{\ell+1,1}$, respectively. By construction, there is a commutative diagram

$$\begin{array}{ccccccc} \bar{D}_\ell & \xrightarrow{\bar{\psi}_\ell} & \bar{D}_{\ell-1} & \longrightarrow & \cdots & \longrightarrow & \bar{D}_2 & \xrightarrow{\bar{\psi}_2} & \bar{D}_1 & \xrightarrow{\bar{\varphi}_1} & \mathbf{P}_k^1 \times \mathbf{P}_k^1 \\ \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\ D_\ell & \xrightarrow{\psi_\ell} & D_{\ell-1} & \longrightarrow & \cdots & \longrightarrow & D_2 & \xrightarrow{\psi_2} & D_1 & \xrightarrow{\varphi_1} & \mathbf{A}_k^2 \end{array}$$

For every $n \geq 2$, we may visualize the boundary divisor ∂D_n as a fork of $2n + 1$ copies of \mathbf{P}_k^1

$$\begin{array}{c} (E_{1,0}, -2) \text{ --- } \cdots \text{ --- } (E_{n-1,0}, -2) \\ \diagdown \quad \diagup \\ (F_\infty, 0) \text{ --- } (C_\infty, 0) \text{ --- } (F_0, -2) \\ \diagup \quad \diagdown \\ (E_{1,1}, -2) \text{ --- } \cdots \text{ --- } (E_{n-1,1}, -2) \end{array}$$

intersecting transversally in k -rational points, with the indicated self-intersection numbers for each irreducible component. We may order the irreducible components of ∂D_n by setting

$$F_\infty < C_\infty < F_0 < E_{1,0} < \cdots < E_{n-1,0} < E_{1,1} < \cdots < E_{n-1,1}$$

Definition 6.1.4. The *quadratic Picard groupoid* $\underline{\text{Pic}}^{\text{or}}(X)$ of a scheme X is the category whose objects are invertible sheaves on X and with morphisms, the quadratic isomorphisms.

Let $\underline{\text{Pic}}(X)$ denotes Deligne's Picard category of invertible sheaves on X (see Section 1.3 for our conventions). There is an obvious functor

$$\rho_X : \underline{\text{Pic}}(X) \rightarrow \underline{\text{Pic}}^{\text{or}}(X)$$

which is the identity on objects and maps an isomorphism $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ to the equivalence class of the quadratic pre-isomorphism $\phi \otimes m^{-1} : \mathcal{L} = \mathcal{L} \otimes \mathcal{O}_X \rightarrow \mathcal{L}' \otimes (\mathcal{O}_X)^{\otimes 2}$. Moreover, one checks the following properties:

- (1) The tensor product of invertible sheaves induces a symmetric monoidal structure on $\underline{\text{Pic}}^{\text{or}}(X)$, such that ρ becomes monoidal. Therefore $\underline{\text{Pic}}^{\text{or}}(X)$ is a Picard groupoid and ρ_X is a natural transformation of Picard groupoids.
- (2) Given a morphism of schemes $f : Y \rightarrow X$, the pullback of invertible sheaves induces a functor $f^* : \underline{\text{Pic}}^{\text{or}}(X) \rightarrow \underline{\text{Pic}}^{\text{or}}(Y)$ such that ρ_X is natural in X .

We henceforth denote by Isom (resp. Isom_Q) the sets of isomorphisms (resp. quadratic isomorphisms) of invertible sheaves.

6.1.5. Orientation classes. The notion of quadratic isomorphisms naturally recovers Barge-Morel's formalism of orientations. Given an invertible sheaf \mathcal{L} over a scheme X , we define the set of *orientation classes* of \mathcal{L} as

$$\mathcal{O}r_X(\mathcal{L}) = \text{Isom}_Q(\mathcal{L}, \mathcal{O}_X) = \{(\epsilon, \mathcal{M}) \mid \epsilon : \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X \otimes \mathcal{M}^{\otimes 2}\} / \sim$$

This assignment is functorial for quadratic isomorphisms. The monoidal structure on $\underline{\text{Pic}}^{\text{or}}(X)$ induces a product

$$\mathcal{O}r_X(\mathcal{L}) \otimes \mathcal{O}r(\mathcal{L}') \rightarrow \mathcal{O}r_X(\mathcal{L} \otimes \mathcal{L}'), (\epsilon, \epsilon') \mapsto \epsilon \cdot \epsilon' = (m^{-1} \otimes \text{id}_{(\mathcal{M} \otimes \mathcal{M}')^{\otimes 2}}) \circ (\epsilon \otimes \epsilon')$$

The composition law

$$\mathcal{O}r_X(\mathcal{O}_X) \otimes \mathcal{O}r(\mathcal{O}_X) \rightarrow \mathcal{O}r_X(\mathcal{O}_X \otimes \mathcal{O}_X) \xrightarrow{m^{-1}} \mathcal{O}r_X(\mathcal{O}_X)$$

defines an abelian group structure on $\mathcal{O}r_X(\mathcal{O}_X)$. Its neutral element is the class of the quadratic pre-isomorphism $m^{-1} : \mathcal{O}_X \rightarrow \mathcal{O}_X^{\otimes 2}$.¹⁷

Moreover, the preceding product induces an action of $\mathcal{O}r_X(\mathcal{O}_X)$ on $\mathcal{O}r_X(\mathcal{L})$. Next we record a fundamental fact about orientations whose proof is elementary and left to the reader.

Theorem 6.1.6. *For any scheme X , there is a short exact sequence of abelian groups*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{G}_m(X)/\mathbf{G}_m(X)^2 & \longrightarrow & \mathcal{O}r_X(\mathcal{O}_X) & \longrightarrow & \text{Pic}(X)_2 \longrightarrow 0 \\ & & u \longmapsto & & m^{-1} \circ (\times u) & & \\ & & & & (\epsilon, \mathcal{M}) \longmapsto & & \mathcal{M} \end{array}$$

where $\text{Pic}(X)_2$ is the 2-torsion subgroup of $\text{Pic}(X)$.

The action of $\mathcal{O}r_X(\mathcal{O}_X)$ on $\mathcal{O}r_X(\mathcal{L})$ is faithful. Moreover, when $\text{Pic}(X)$ has no 2-torsion, the abelian group $\mathcal{O}r_X(\mathcal{O}_X) \simeq \mathbf{G}_m(X)/\mathbf{G}_m(X)^2$ acts fully faithfully on the set $\mathcal{O}r_X(\mathcal{L})$. In particular, two classes of orientations of \mathcal{L} differ by a uniquely defined element of $\mathbf{G}_m(X)/\mathbf{G}_m(X)^2$ (modulo this action).

Remark 6.1.7. To summarize, an invertible sheaf \mathcal{L} on X is orientable if and only if its class in $\text{Pic}(X)$ is 2-divisible. If $\text{Pic}(X)$ has no 2-torsion, then two orientations of \mathcal{L} differs by a unique quadratic class $\bar{\phi} \in \mathbf{G}_m(X)/\mathbf{G}_m(X)^2$ for some global invertible function ϕ on X .

For instance, if $X = \mathbf{P}_k^1$ is the projective line over a field k , an invertible sheaf \mathcal{L} is orientable if and only if it has an even degree; moreover, two orientations of \mathcal{L} differ by a unique quadratic class in $Q(k) = k^*/(k^*)^2$.

¹⁷One can check that the composition of quadratic isomorphisms also induces this group structure.

Example 6.1.8. With reference to Example 6.1.3, the previous definitions and *loc. cit.* readily imply that the set $\mathcal{O}r_X(\mathcal{L} \otimes \omega_X^\vee)$ is in bijection with quadratic isomorphisms $\epsilon : \mathcal{L} \rightarrow \omega_X$ and also with relative orientations of $L = \mathbb{V}(\mathcal{L})$ in the sense of Bachmann-Wickelgren [17].

6.1.9. Recall from [33, 7.13] the monoidal twisted Thom space functor (see also 2.1.1)

$$\mathrm{Tw} := \mathrm{Tw}_X : \underline{\mathrm{Pic}}(X) \rightarrow \mathrm{SH}(X), \mathcal{L} \mapsto \mathrm{Th}(\mathcal{L})(-1)[-2]$$

Proposition 6.1.10. *There exists a canonical monoidal extension $\tilde{\mathrm{Tw}}$ of the twisted Thom space functor and a natural isomorphism*

$$\begin{array}{ccc} \underline{\mathrm{Pic}}(X) & \xrightarrow{\mathrm{Tw}} & \mathrm{SH}(X) \\ & \searrow \rho & \downarrow \sim \\ & & \underline{\mathrm{Pic}}^{\mathrm{or}}(X) \end{array} \quad \begin{array}{c} \\ \\ \nearrow \tilde{\mathrm{Tw}} \end{array}$$

which is the identity on objects.

Proof. To a quadratic isomorphism $\epsilon : \mathcal{L} \rightarrow \mathcal{L}'$, represented by $\epsilon : \mathcal{L} \rightarrow \mathcal{L}' \otimes \mathcal{M}^{\otimes 2}$, we associate an isomorphism in $\mathrm{SH}(X)$

$$\epsilon_* : \mathrm{Tw}(\mathcal{L}) \rightarrow \mathrm{Tw}(\mathcal{L}' \otimes \mathcal{M}^{\otimes 2}) \simeq \mathrm{Tw}(\mathcal{L}') \otimes \mathrm{Tw}(\mathcal{M}^{\otimes 2}) \simeq \mathrm{Tw}(\mathcal{L}')$$

The identification of $\mathrm{Tw}(\mathcal{L}' \otimes \mathcal{M}^{\otimes 2})$ follows from the monoidality of Tw and [33, 7.13]. One checks that the isomorphism in $\mathrm{SH}(X)$ depends only on the equivalence class of ϵ and that it is compatible with the composition defined in (6.1.2.a). It is now straightforward to define the desired natural transformation from Tw to the composition of ρ and $\tilde{\mathrm{Tw}}$. \square

Example 6.1.11. It follows that the Thom space $\mathrm{Th}(\mathcal{L})$ of an invertible sheaf \mathcal{L} depends only on the orientation class of \mathcal{L} . More precisely, every quadratic isomorphism $\epsilon : \mathcal{L} \rightarrow \mathcal{L}'$ induces a (well-defined) isomorphism

$$\epsilon_* : \mathrm{Th}(\mathcal{L}) = \tilde{\mathrm{Tw}}(\mathcal{L})(1)[2] \rightarrow \tilde{\mathrm{Tw}}(\mathcal{L}')(1)[2] \simeq \mathrm{Th}(\mathcal{L}')$$

In particular, one associate to any orientation class $\epsilon \in \mathcal{O}r_X(\mathcal{L})$ of \mathcal{L} a canonical isomorphism

$$\epsilon_* : \Pi_S(X, \mathbb{V}(\mathcal{L})) \rightarrow \Pi_S(X)(1)[2]$$

in $\mathrm{SH}(X)$.

6.1.12. Next, we introduce a quadratic version of Deligne's (rank-)determinant functor of Picard categories

$$\underline{\mathbf{K}}(X) \xrightarrow{(\mathrm{rk}, \det)} \underline{\mathbb{Z}}_X \times \underline{\mathrm{Pic}}(X), \mathcal{V} \mapsto (\mathrm{rk} \mathcal{V}, \det \mathcal{V})$$

Definition 6.1.13. The category $\underline{\mathbf{K}}^{\mathrm{or}}(X)$ of virtual vector bundles over X modulo orientation is the groupoid whose objects are virtual locally free sheaves \mathcal{V} on X and a morphism from \mathcal{V} to \mathcal{V}' is a morphism from $(\mathrm{rk} \mathcal{V}, \det \mathcal{V})$ to $(\mathrm{rk} \mathcal{V}', \det \mathcal{V}')$ in $\underline{\mathbb{Z}}_X \times \underline{\mathrm{Pic}}^{\mathrm{or}}(X)$, i.e., a quadratic isomorphism $\det \mathcal{V} \rightarrow \det \mathcal{V}'$ assuming $\mathrm{rk} \mathcal{V} = \mathrm{rk} \mathcal{V}'$.

We refer to morphisms in $\underline{\mathbf{K}}^{\mathrm{or}}(X)$ as quadratic isomorphisms and use the notation $\mathcal{V} \rightarrow \mathcal{V}'$. By definition, $\underline{\mathbf{K}}^{\mathrm{or}}(X)$ is the essential image of the composite functor

$$\underline{\mathbf{K}}(X) \xrightarrow{(\mathrm{rk}, \det)} \underline{\mathbb{Z}}_X \times \underline{\mathrm{Pic}}(X) \xrightarrow{Id \times \rho} \underline{\mathbb{Z}}_X \times \underline{\mathrm{Pic}}^{\mathrm{or}}(X)$$

In particular, one gets a canonical monoidal structure on $\underline{\mathbf{K}}^{\mathrm{or}}(X)$, so that it becomes a Picard groupoid with a canonical monoidal functor

$$\underline{\mathbf{K}}(X) \xrightarrow{\rho} \underline{\mathbf{K}}^{\mathrm{or}}(X)$$

Remark 6.1.14. We can again follow the lines of 6.1.5 and define an orientation of a virtual vector bundle $v = \mathbb{V}(\mathcal{V})$ as either an orientation of its determinant or a quadratic isomorphism $\mathcal{V} \rightarrow \langle r \rangle$ where $r = \mathrm{rk}(\mathcal{V})$ (as a locally constant function on X). The same applies to vector bundles and recovers the classical definition of Barge and Morel.

Example 6.1.15. Let \mathbb{E} be a ring spectrum over a scheme S which is **SL**-oriented in the sense of Panin-Walter (see [1]). Let X be a separated S -scheme and v a virtual bundle over X . Using Thom isomorphisms attached to the **SL**-orientation of \mathbb{E} , one obtains that the v -twisted \mathbb{E} -cohomology of X

$$\mathbb{E}^n(X, v) := [\Pi_S(X, -v), \mathbb{E}[n]]$$

depends only on the pair $(\mathrm{rk}(v), \det(v))$. Moreover, it follows from Proposition 6.1.10 that $\mathbb{E}^n(X, v)$ is functorial in v with respect to isomorphisms modulo orientation.

Chow-Witt groups provide the most fundamental example for us (the unramified Milnor-Witt sheaf $\underline{\mathbf{K}}_*^{MW}$ represents these groups over fields).

Proposition 6.1.16. *Let X be a scheme such that $K_0(X)$ is infinite cyclic, e.g., a semi-local scheme or a unique factorization domain. Then there exists a canonical monoidal extension $\tilde{\mathrm{Th}}_X$ of the Thom space functor and a natural isomorphism*

$$\begin{array}{ccc} \underline{\mathbf{K}}(X) & \xrightarrow{\mathrm{Th}_X} & \mathrm{SH}(X) \\ & \searrow \rho & \downarrow \sim \\ & & \underline{\mathbf{K}}^{\mathrm{or}}(X) \\ & & \nearrow \tilde{\mathrm{Th}}_X \end{array}$$

which is the identity on objects.

Proof. By assumption, the morphism $\underline{\mathbf{K}}(X) \xrightarrow{(\mathrm{rk}, \det)} \underline{\mathbb{Z}}_X \times \mathrm{Pic}(X)$ is an equivalence of categories. Thus, by definition of $\underline{\mathbf{K}}^{\mathrm{or}}(X)$, Proposition 6.1.10 concludes the proof. \square

Remark 6.1.17. Note that, for X as above, we get an explicit quasi-inverse to (rk, \det) by associating to the class of (r, \mathcal{L}) the class of the locally free sheaf $\mathcal{O}_X^{r-1} \oplus \mathcal{L}$. In particular, one can describe the functor $\tilde{\mathrm{Th}}_X$ as the composite

$$\underline{\mathbf{K}}^{\mathrm{or}}(X) \xrightarrow{\sim} \underline{\mathbb{Z}}_X \times \mathrm{Pic}^{\mathrm{or}}(X) \rightarrow \mathrm{SH}(X), \mathcal{V} \mapsto (r = \mathrm{rk} \mathcal{V}, \det \mathcal{V}) \mapsto \mathrm{Tw}_X(\mathcal{V})(r)[2r]$$

The latter description clarifies the functoriality. In fact, the proposition posits the existence of a canonical isomorphism (functorial for isomorphisms in \mathcal{V})

$$\mathrm{Th}_X(\mathcal{V}) \simeq \mathrm{Tw}_X(\det \mathcal{V})(r)[2r], r = \mathrm{rk} \mathcal{V}$$

6.2. Quadratic 0-cycles and quadratic degrees.

6.2.1. Next, we recall a few definitions of Chow-Witt groups suitable for our needs.¹⁸ We fix a base field k , not necessarily perfect but finitely generated over a perfect field k_0 .¹⁹

Given a finitely generated extension field K/k , we let $K_*^{MW}(K)$ be the Milnor-Witt ring of K (see [70, Def. 3.1]). Given an invertible K -vector space \mathcal{L} , we define the twisted Milnor-Witt ring of K by the formula in [70, Rem. 3.21]

$$(6.2.1.a) \quad K_*^{MW}(K, \mathcal{L}) := K_*^{MW}(K) \otimes_{\mathbb{Z}[K^\times]} \mathbb{Z}[\mathcal{L}^\times]$$

where $\mathcal{L}^\times = \mathcal{L} - \{0\}$, using the action of K^\times on $K_*^{MW}(K)$ via the canonical map $K^\times \rightarrow \mathrm{GW}(K) = K_0^{MW}(K)$.

Let now X be an essentially smooth k -scheme of dimension d and \mathcal{L} an invertible sheaf on X . One defines the group of *quadratic 0-cycles* on X twisted by \mathcal{L} as

$$\tilde{Z}^d(X, \mathcal{L}) := \bigoplus_{x \in X^{(d)}} \mathrm{GW}(\kappa(x), \omega_{x/X}^\vee \otimes_{\kappa(x)} \mathcal{L}|_x)$$

Here $X^{(d)}$ is the set of closed points x of X and $\omega_{x/X}$ is the determinant of the $\kappa(x)$ -vector space $\mathcal{C}_{x/X} = \mathfrak{m}_x / \mathfrak{m}_x^2$. The *support* of a quadratic 0-cycle α is the set of points $x \in X^{(d)}$ whose coefficient in α is non-zero. We will consider it as a finite reduced closed subscheme of X .

¹⁸We focus on zero cycles and emphasize (quadratic) cycles rather than cycle classes.

¹⁹This is to be able to use the written account on Chow-Witt groups. This assumption will be removed in [34].

Owing to [70, Rem. 5.13], [45], or [46, Def. 7.2]²⁰ there is a map

$$\operatorname{div} : \bigoplus_{y \in X^{(d-1)}} K_1^{MW}(\kappa(y), \omega_{y/X}^\vee \otimes \mathcal{L}|_y) \longrightarrow \tilde{Z}^d(X, \mathcal{L})$$

Two quadratic zero-cycles are said to be *rationally equivalent* if their difference is in the image of div . The above defines an additive equivalence relation \sim_{rat} on quadratic 0-dimensional cycles, and the d -th Chow-Witt group of X twisted by \mathcal{L} is the quotient

$$\widetilde{\text{CH}}^d(X, \mathcal{L}) = \tilde{Z}^d(X, \mathcal{L}) / \sim_{\text{rat}} = \operatorname{coKer}(\operatorname{div})$$

This group depends functorially on \mathcal{L} for quadratic isomorphisms.

Proposition 6.2.2. *Let X be an essentially smooth k -scheme of dimension d and let $v = \mathbb{V}(\mathcal{V})$ be a virtual vector bundle of rank d on X . Then there is a canonical isomorphism*

$$H_{\text{SH}}^0(X, v) := [\mathbf{1}_X, \operatorname{Th}(v)] \simeq \widetilde{\text{CH}}^d(X, \det \mathcal{V})$$

Proof. With k being finitely generated over a perfect field k_0 , one can work over k_0 or assume that k is perfect. The coniveau spectral sequence (see [31], §1.1.1 and Def. 1.4) associated with the cohomology theory $H_{\text{SH}}^*(X, v)$ takes the form

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H_{\text{SH}}^{p+q}(\operatorname{Th}(N_x X_{(x)}), v) \Rightarrow H_{\text{SH}}^{p+q}(X, v)$$

Here $X_{(x)} = \operatorname{Spec}(\mathcal{O}_{X,x})$ and $N_x X_{(x)}$ is the normal bundle of x (this relies on Morel-Voevodsky's homotopy purity theorem). The E_1 -term is concentrated in the range $p \in [0, d]$ and by the \mathbf{A}^1 -connectivity theorem, in the range $q \leq 0$. According to Morel's computation of the 0-stable stem and Feld's theory [47], there is an isomorphism between complexes

$$E_1^{*,0} \simeq C^*(X, K_*^{MW}, \omega_{X/k}^\vee \otimes \det \mathcal{V})$$

We conclude by looking at the line $p + q = d$. □

6.2.3. Quadratic Degree. Let X/k be a smooth proper scheme of dimension d with canonical sheaf $\omega_X = \det(\Omega_{X/k})$. In the following we define a notion of quadratic degree of ω_X -twisted quadratic 0-cycles. To begin, note that there is a canonical isomorphism

$$\tilde{Z}^d(X, \omega_X) = \bigoplus_{x \in X^{(d)}} \operatorname{GW}(\kappa(x), \omega_{x/X}^\vee \otimes \omega_X|_x) \simeq \bigoplus_{x \in X^{(d)}} \operatorname{GW}(\kappa(x))$$

This holds because for any closed point $x \in X$, the conormal exact sequence

$$0 \rightarrow \mathcal{C}_{x/X} \rightarrow \Omega_X|_x \rightarrow \Omega_{x/k} \rightarrow 0$$

together with the fact that $\Omega_{x/k} = 0$ (as $\kappa(x)/k$ is étale) provides a canonical isomorphism of $\kappa(x)$ -vector spaces $\omega_{x/X}^\vee \otimes \omega_X|_x = \det(\mathcal{C}_{x/X})^\vee \otimes \omega_X|_x \simeq \kappa(x)$. Thus, an ω_X -twisted quadratic 0-cycle can be identified with a formal sum $\alpha = \sum_{i \in I} \langle \sigma_i \rangle \cdot x_i$, where $x_i \in X$ is a closed point and σ_i is a symmetric bilinear form over $\kappa(x_i)$.

One defines the *quadratic degree* $\widetilde{\operatorname{deg}}$ of a quadratic cycle α as the proper pushforward associated with the projection of X/k (see [15, Chap. 2, §3]). It is defined at the level of cycles and factorizes through rational equivalence. If one assumes that the support of α is étale over k , then it can be computed explicitly as the element

$$\widetilde{\operatorname{deg}}(\alpha) = \sum_{i \in I} \langle \operatorname{Tr}_{\kappa(x_i)/k} \circ \sigma_i \rangle \in \operatorname{GW}(k)$$

where $\operatorname{Tr}_{\kappa(x_i)/k}$ is the trace form for the finite separable (by assumption) extension $\kappa(x_i)/k$ (apply *loc. cit.* Lemma 2.3).

²⁰In Morel's notation, $\tilde{Z}^d(X, \mathcal{L})$ is the d -th term of the Rost-Schmid complex $C_{\text{RS}}^*(X, K_d^{MW}\{\mathcal{L}\})$ while in Feld's notation it is the end of the complex $C_*(X, K_*^{MW}, \omega_{X/k}^\vee \otimes \mathcal{L})$, where $\omega_{X/k} = \det(\Omega_{X/k})$ is the canonical sheaf of X/k .

More generally, let \mathcal{L} be an invertible sheaf over X with a relative orientation (see Example 6.1.8) given by a quadratic isomorphism $\epsilon : \mathcal{L} \rightarrow \omega_X$. We define the *quadratic ϵ -degree* as the composite

$$(6.2.3.a) \quad \widetilde{\text{deg}}_\epsilon : \tilde{Z}^d(X, \mathcal{L}) \xrightarrow{\epsilon_*} \tilde{Z}^d(X, \omega_X) \xrightarrow{\widetilde{\text{deg}}} \text{GW}(k)$$

To compute this degree concretely, assuming that the support of α is étale over k , one first choose any representative $\tilde{\epsilon} : \mathcal{L} \xrightarrow{\epsilon} \omega_X \otimes \mathcal{M}^{\otimes 2}$ of the quadratic isomorphism ϵ . Then, by linearity, one is reduced to quadratic cycles of the form

$$\alpha = \langle \sigma \rangle \otimes (t^* \otimes l).x \in \text{GW}(\kappa(x), \omega_{x/X}^\vee \otimes \mathcal{L}|_x) = \text{GW}(\kappa(x)) \otimes_{\mathbb{Z}[\kappa(x)^\times]} \mathbb{Z}[(\omega_{x/X}^\vee \otimes \mathcal{L}|_x)^\times]$$

where $t = t_1 \wedge \dots \wedge t_d$ belongs to $\omega_{x/X}$ and correspond to a local parametrisation of $x \in X$, and $l \in \mathcal{L}|_x^\times$. Then $\epsilon_x(l) = w_l \otimes (u \otimes u)$ where $w_l \in \omega_X|_x$ and $u \in \mathcal{M}|_x^\times$. Finally, one gets a canonical isomorphism $\omega_X|_x \simeq \omega_{x/X}$ so that t defines a non-zero linear form

$$t^* : \omega_X|_x \simeq \omega_{x/X} \rightarrow \kappa(x)$$

Putting everything together, we get the formula

$$(6.2.3.b) \quad \widetilde{\text{deg}}_\epsilon (\langle \sigma \rangle \otimes (t^* \otimes l).x) = \langle \text{Tr}_{\kappa(x)/k}(t^*(w_l) \cdot \sigma) \rangle \in \text{GW}(k)$$

6.2.4. Oriented degree of oriented cycles We keep the notation and hypotheses of the previous paragraph and assume that ω_X is orientable, with chosen orientation class $\tau \in \mathcal{O}r_X(\omega_X)$. Suppose that Y is a reduced d -codimensional closed subscheme of X , such that Y/k is étale. Let $\omega_{Y/X} = \det \mathcal{C}_{Y/X}$ be the determinant of the conormal sheaf of Y in X and assume given an orientation classe $\epsilon \in \mathcal{O}r_Z(\omega_{Y/X})$. This allows us to define a canonical quadratic 0-cycle $[Y]_X^\epsilon \in \tilde{Z}^d(X)$ associated with (Y, ϵ) , as the image of $\sum_{x \in Y_{(0)}} \langle 1 \rangle .x \in \tilde{Z}^0(Y)$ under the composite map

$$\tilde{Z}^0(Y) \xrightarrow{\epsilon_*^{-1}} \tilde{Z}^0(Y, \omega_{Y/X}) \xrightarrow{i_*} \tilde{Z}^d(X)$$

Note that given our conventions, the map i_* is just the identity. As τ corresponds to a quadratic isomorphism $\omega_X \rightarrow \mathcal{O}_X$, we get the τ -degree map

$$\text{deg}_{\tau^{-1}} : \tilde{Z}^d(X) \xrightarrow{\tau_*^{-1}} \tilde{Z}^d(X, \omega_X) \xrightarrow{\widetilde{\text{deg}}} \text{GW}(k)$$

Then the τ -oriented degree of the ϵ -oriented cycle $[Y]_X^\epsilon$ is given by the class

$$\text{deg}_{\tau^{-1}}([Y]_X^\epsilon) = \sum_{x \in Y_{(0)}} \text{Tr}_{\kappa(x)/k} \circ \langle u_x \rangle \in \text{GW}(k)$$

for quadratic classes $u_x \in Q(\kappa(x))$ computed as in the above paragraph.

The quadratic class u_x can be computed in terms of the chosen orientations using again the canonical isomorphism $\phi : \omega_X|_x \simeq \omega_{x/X}$ (as $\kappa(x)/k$ is separable). Namely, we get two induced orientation classes $\epsilon|_x$ and $\phi_*(\tau|_x)$ of the 1-dimensional $\kappa(x)$ -vector space $\omega_{x/X} = \omega_X|_x$, which, according to Remark 6.1.7, are uniquely linked by a relation of the form

$$\epsilon|_x = u_x \cdot \phi_*(\tau|_x)$$

Equivalently, u_x is the element $\epsilon|_x \otimes (\tau|_x)^\vee$ in

$$\mathcal{O}r_{\kappa(x)}(\omega_{x/X} \otimes \omega_X^\vee|_x) \simeq \mathcal{O}r_{\kappa(x)}(\mathcal{O}_{\kappa(x)}) \simeq Q(\kappa(x))$$

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