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# The fundamental solution of a 1D evolution equation with a sign changing diffusion coefficient

Éric Bonnetier\*      Pierre Etoré(★)†      Miguel Martinez‡

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## Abstract

In this work we investigate a 1D evolution equation involving a divergence form operator where the diffusion coefficient inside the divergence is sign changing. Equivalently the evolution equation of interest can be interpreted as behaving locally like a heat equation, and involving a transmission condition at some interface that prescribes in particular a change of sign of the first order space derivatives across the interface. We especially focus on the construction of fundamental solutions for the evolution equation. As the second order operator involved in the evolution equation is not elliptic, this cannot be performed by standard tools for parabolic PDEs. However we manage in a first time to provide a spectral representation of the semigroup associated to the equation, which leads to a first expression of the fundamental solution. In a second time, examining the case when the diffusion coefficient is piecewise constant but remains positive, we do probabilistic computations involving the killed Skew Brownian Motion (SBM), that provide a certain explicit expression of the fundamental solution for the positive case. It turns out that this expression also provides a fundamental solution for the case when the coefficient is sign changing, and can be interpreted as defining a pseudo SBM. This pseudo SBM can be approached by a rescaled pseudo asymmetric random walk. We infer from these different results various approximation schemes that we test numerically.

## Keywords:

Negative Index Materials; Evolution equations; Spectral representation of semigroups; Skew Brownian motion ; Pseudo processes

## 1 Introduction

### 1.1 Mathematical modeling of negative index materials (NIM)

Since the seminal paper of Veselago [20], negative index materials (NIM) have drawn considerable attention, due to the spectacular way in which electromagnetic, acoustic or elastic waves may propagate in such media. Electromagnetic fields for instance may be enhanced by several orders of magnitude due to plasmonic resonance of the fields in the neighborhood of particles with negative permittivity or permeability. Their possibility of localizing and concentrating waves has made NIM's a subject of great interest for many applications, for instance in communication and medical imaging.

From the point of view of mathematical modeling, a large part of the work on NIM's has focused on the so-called electrostatic approximation, for a composite medium made of inclusions (or phases) of NIM's embedded into a homogeneous dielectric matrix phase. This means that the time-harmonic Maxwell system is reduced to a diffusion for one of the components of either the electric or magnetic field, provided the geometry of the device is assumed to have a direction of invariance, and provided

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the typical dimensions of the inhomogeneities are small with respect to the incident wavelength. In the simplest case, the remaining equation takes the form of a transmission equation of type

$$\begin{cases} \operatorname{div}(A(x)\nabla u(x)) = f(x) \\ + \text{boundary conditions} \end{cases} \quad (1)$$

in a domain  $\Omega$ , and where the conductivity  $A(x)$  takes negative values in the NIM phases and a positive value in the surrounding homogeneous dielectric matrix phase.

At first sight, the above PDE looks elliptic. However, because the conductivity changes sign, the associated bilinear form

$$\mathcal{E}(u, v) = \int_{\Omega} A(x)\nabla u(x)\nabla v(x),$$

fails to be coercive in the appropriate Sobolev space  $H$  and it is not possible to invoke the Lax-Milgram Lemma to show existence of solutions to (1).

In many cases though, depending on the geometry of the NIM inhomogeneities, one can show that the form  $\mathcal{E}$  is T-coercive : there exists an invertible operator  $T : H \rightarrow H$  such that  $(u, v) \in H \times H \rightarrow \mathcal{E}(u, Tv)$  is coercive. In this case, applying the Lax-Milgram Lemma to the latter bilinear form yields well-posedness of the PDE, except possibly for some values of the negative conductivities, for which the associated operator is not invertible and may even lose its Fredholm character [1, 2, 17].

If the form  $\mathcal{E}$  is T-coercive, does the PDE have all the properties of an elliptic PDE ?

## 1.2 Investigation of a simple one dimensional situation

In this work, we investigate a simple one-dimensional situation. We consider an evolution equation involving the same divergence form operator as in (1). Namely this equation is:

$$\begin{cases} A(x)\partial_t u(t, x) = \frac{1}{2}\partial_x(A(x)\partial_x u(t, x)), & x \in I, t > 0, \\ u(0, x) = u_0(x) & x \in I, \\ u(t, \pm a) = 0, & t > 0, \end{cases} \quad (2)$$

where  $I$  is the interval  $(-a, a)$  ( $a > 0$  or  $a = +\infty$ ), and where the conductivity  $A$  is defined by

$$A(x) = \begin{cases} k, & x \in I^- := (-a, 0) \\ 1, & x \in I^+ := (0, a). \end{cases}$$

In other words, we assume that a dielectric with unit conductivity fills in the right part of the interval  $I$ , whereas the left part of  $I$  is filled with a general index material of index  $k \in \mathbb{R}^*$  with possibly  $k < 0$ . Note that the time derivative of  $u$  is multiplied by the conductivity  $A(x)$ .

When  $k > 0$ , Eq. (2) is parabolic and this is not a significant change from the usual parabolic setting with no multiplicative variable term in front of the time derivative, as far as existence and uniqueness of solutions are concerned. When  $k < 0$ , considering an equation where the time derivative of  $u$  is multiplied by a function  $B$  that has the same sign as  $A$  seems to be the only way to define a tractable first-order evolution equation associated with the ‘diffusion’ operator  $\partial_x(A(x)\partial_x u)$ , for which solutions can be constructed in the ‘energy’ space  $L^2(0, T, H_0^1(I))$  for any initial datum  $u_0 \in L^2(I)$  [9].

In particular, note that (2) is equivalent to

$$\partial_t u(t, x) = \frac{1}{2}u''(t, x), \quad x \in I^- \cup I^+, \quad t > 0 \quad (3)$$

$$u(0, x) = u_0(x), \quad x \in (-a, a) \quad (4)$$

and the conditions at the interfaces

$$\begin{cases} u(t, a) = u(t, -a) = 0, \\ u(t, 0^-) = u(t, 0^+), \\ k u'(t, 0^-) = u'(t, 0^+). \end{cases} \quad (5)$$

Note that the particular case where  $k = -1$  corresponds to the value of the negative conductivity mentioned before where the associated operator is not invertible and loses its Fredholm character and is discarded from our study.

In this paper, when  $k \in \mathbb{R}^* \setminus \{-1\}$ , we construct a fundamental solution for (3)-(4)-(5) (equivalently (2)) in two manners. First we use spectral analysis computations. Second we interpret the fundamental solution as the transition function of a stochastic process, or rather as we explain below, of a pseudo process.

### 1.3 Probabilistic interpretation

In the case where  $A$  remains strictly positive, it is well known that the fundamental solution of (2) can be interpreted as the transition function of a stochastic process : this can be shown for example by using the stochastic calculus for Dirichlet forms (see for e.g. [7]). In the particular one dimensional case under consideration and when  $k > 0$ , the solution  $u(t, x)$  of problem (3)-(4)-(5), eq. (2) writes, via a Feynman-Kac formula (see e.g. [10]), as the expectation  $\mathbb{E}^x(u_0(\tilde{X}_t))$ , where  $\tilde{X}$  stands here for a skew Brownian motion (SBM) of parameter  $\beta = \frac{1-k}{1+k}$  killed when reaching  $-a$  or  $a$  (we have denoted  $\mathbb{E}^x(\cdot) = \mathbb{E}(\cdot | \tilde{X}_0 = x)$ ; for an account on skew Brownian motion we refer to [8]). Thus the transition probability density function (transition function, in short) of  $\tilde{X}$  provides a fundamental solution of (2).

When  $k$  is possibly negative the skew Brownian motion may not be well defined, however the above construction formally provides a solution to (3)-(4)-(5) that extends the probabilistic representation. With this perspective, our idea is then to consider a *pseudo* expectation involving a *pseudo* skew Brownian motion that has to be properly defined as a *pseudo* process. A pseudo process  $Y = (Y_t)$  is a family of *pseudo-random variable*. By pseudo-random variable we mean a measurable function defined on a space (a *pseudo*-probability space) that is endowed with a signed measure with a total mass equal to 1 (for an account on pseudo random variables we refer to [12, 13] and the references therein). A pseudo process is defined mainly by its transition function, which allows to compute pseudo transition probabilities.

When  $I = \mathbb{R}$  ( $a = +\infty$ ), this program can be successfully performed, exploiting immediately the fact that in the case  $k > 0$  the transition function of the SBM is known ([21]). Then one can check that this transition function provides a fundamental solution of (2) and defines a pseudo SBM, in the general case  $k \in \mathbb{R}^* \setminus \{-1\}$ . Moreover, we prove that the obtained (pseudo) probabilistic representation lends itself to numerical approximation via the convergence of pseudo skew random walks to the pseudo skew Brownian motion in a pseudo weak sense.

The bounded case where  $I = (-a, a)$  ( $a < +\infty$ ) is more involved because it is not so clear to propose a clear statement of what should be the definition of the 'killed' pseudo skew Brownian motion as a pseudo probabilistic process. This is mainly because 'killing' is a trajectorial operation and that the trajectories of a pseudo processes have not a clear meaning. Still, in the same manner as before, we manage to extend the representation of the solution  $u(t, x)$  of problem (3)-(4)-(5) at the level of the fundamental solution, in the case  $k \in \mathbb{R}^* \setminus \{-1\}$ . One preliminary step will be to consider the case  $k > 0$  and to compute the transition function of the killed skew Brownian  $\tilde{X}$ , by probabilistic arguments.

### 1.4 Organization of the paper

This work is organized as follows. In Section 2, we construct  $L^2(0, \infty, H_0^1(I))$  solutions to (2) for any  $k \neq -1$  with the help of a family of eigenfunctions of the bilinear form  $a(u, v) = \int_I A(x) \partial_x u \partial_x v$  which are shown to form a basis of the space  $L^2(I)$ . In Section 3, we concentrate on the case where  $a = +\infty$  and show that the solutions to (2) can be obtained as convolutions of the initial datum with a kernel  $\check{p}(t, x, y)$  which has a (pseudo) probabilistic interpretation, in link with the pseudo skew-Brownian motion, that we introduce. We also show the convergence of pseudo skew random walks to the pseudo skew Brownian motion in pseudo weak sense. In Section 4, we treat the bounded case where  $a < +\infty$ . Finally, in Section 5 we construct several numerical schemes for solving (2) using our results. The numerical comparison of these schemes is reported to Section 6.

## 2 Spectral representation of the semigroup

In this section, we consider the eigenvalue problem : find  $\lambda \in \mathbb{C}$  and  $u \in H_0^1(I)$  such that

$$-(A(x)u'(x)) = \lambda^2 A(x)u(x), \quad \text{in } I, \quad (6)$$

Seeking a solution to (6) in the form

$$u(x) = \begin{cases} a_1 \cos(\lambda x) + b_1 \sin(\lambda x), & x \in I^-, \\ a_2 \cos(\lambda x) + b_2 \sin(\lambda x), & x \in I^+, \end{cases} \quad (7)$$

and expressing the transmission and boundary conditions (5), one is led to solving the linear system

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & k & 0 & -1 \\ \cos(\lambda a) & -\sin(\lambda a) & 0 & 0 \\ 0 & 0 & \cos(\lambda a) & \sin(\lambda a) \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = 0,$$

from which one obtains the dispersion relation

$$\sin(2\lambda a)(k+1) = 0. \quad (8)$$

Assuming  $k \neq -1$ , the associated eigen-elements can be grouped in two families : a set of even functions

$$f_{k,n}(x) = \cos\left(\frac{(2n-1)\pi}{2a}x\right), \quad x \in I,$$

associated to  $\lambda_n = \frac{(2n-1)\pi}{2a}$ ,  $n \geq 1$ , and the set of functions

$$g_{k,n}(x) = \begin{cases} \sin\left(\frac{n\pi}{a}x\right) & \text{if } x \in I^-, \\ k \sin\left(\frac{n\pi}{a}x\right) & \text{if } x \in I^+, \end{cases}$$

associated to  $\mu_n = \frac{n\pi}{a}$ ,  $n \geq 1$ .

When  $k > 0$ , the bilinear form  $u, v \rightarrow \int_I A(x)u(x)v(x) dx$  is a scalar product in  $L^2(I)$ , while the form

$$u, v \in H_0^1(I) \rightarrow a(u, v) = \int_I A(x)u'(x)v'(x) dx,$$

is coercive and symmetric in  $H_0^1(I)$ , and it is well-known that the solutions to (6) form a basis of  $L^2(I)$  and of  $H_0^1(I)$ . The next Proposition shows that this is still the case when  $k < 0$ .

**Proposition 2.1.** *When  $k < 0$ , the functions  $(f_{k,n}, g_{k,n})_{n \geq 1}$  form a Hilbert basis of  $L^2(I)$  and of  $H_0^1(I)$ .*

*Proof.* Let  $k < 0$ . In view of the above remark, the functions  $(f_{-k,n}, g_{-k,n})_{n \geq 1}$  are a basis of  $L^2(I)$ , and one has the orthogonal decomposition (with respect to the scalar product associated to  $-k$ )

$$L^2(I) = H_f \oplus H_g,$$

where  $H_f$  (resp.  $H_g$ ) is the vector space generated by the  $f_{-k,n}$ 's (resp. by the  $g_{-k,n}$ 's). Consider the mapping  $T : L^2(I) \rightarrow L^2(I)$  defined by

$$Tu(x) = \begin{cases} u(x) & \text{if } u \in H_f, \\ u(x) & \text{if } u \in H_g \text{ and } x \in I^-, \\ -u(x) & \text{if } u \in H_g \text{ and } x \in I^+. \end{cases}$$

As  $T \circ T = I$ ,  $T$  is an isomorphism, thus the basis  $(f_{-k,n}, g_{-k,n})$  is transformed into a basis of  $L^2(I)$ . It is easy to check that  $(Tf_{-k,n}, Tg_{-k,n}) = (f_{k,n}, g_{k,n})$ . The same arguments show that the  $(f_{k,n}, g_{k,n})$ 's form a basis of  $H_0^1(I)$ .  $\square$

From now on, we assume that  $k \in \mathbb{R}^* \setminus \{-1\}$ , and we drop the index  $k$  in the notation of the basis functions  $f_{k,n}$  and  $g_{k,n}$ . Note that the functions  $f_n, g_n, n \geq 1$  satisfy the following relations:

$$\lambda_p^2 \int_I A(x)f_p(x)\varphi(x) = \int_I A(x)f_p'(x)\varphi'(x) = 0,$$

for  $\varphi = f_q, q \neq p$  or  $\varphi = g_q, q \geq 1$ , and similarly

$$\mu_p^2 \int_I A(x)g_p(x)\varphi(x) = \int_I A(x)g_p'(x)\varphi'(x) = 0,$$

for  $\varphi = g_q, q \neq p$  or  $\varphi = f_q, q \geq 1$ . This corresponds to orthogonality or pseudo-orthogonality properties, depending on the sign of  $k$ .

It follows that any function  $u_0 \in L^2(I)$  can be written in the form

$$u_0(x) = \sum_{n \geq 1} a_n f_n(x) + b_n g_n(x), \quad (9)$$

where the coefficients  $a_n$  and  $b_n$  are given by

$$a_n = \frac{\int_I A(x) u_0(x) f_n(x) dx}{\int_I A(x) |f_n(x)|^2 dx}, \quad b_n = \frac{\int_I A(x) u_0(x) g_n(x) dx}{\int_I A(x) |g_n(x)|^2 dx}. \quad (10)$$

In particular, one can check that

$$\int_I A(x) |f_n(x)|^2 dx = \frac{a(k+1)}{2}, \quad \int_I A(x) |g_n(x)|^2 dx = \frac{ak(k+1)}{2}. \quad (11)$$

**Remark 2.2.** Note that if  $k = -1$  it is impossible to provide the forthcoming representations (12) and (14), with the  $a_n$ 's and  $b_n$ 's computed by (10) and (11). Therefore our assumption  $k \in \mathbb{R}^* \setminus \{-1\}$ .

We now focus on the evolution problem (2) assuming that  $u_0 \in L^2(I)$ . Decomposing  $u_0$  on the basis of eigenfunctions as in (9), it is easy to check that

$$u(t, x) = P_t u_0(x) = \sum_{n \geq 1} a_n e^{-\lambda_n^2 t/2} f_n(x) + b_n e^{-\mu_n^2 t/2} g_n(x), \quad (12)$$

is a weak solution to (2) in the sense that  $u \in L^2(0, \infty, H_0^1(I)), \partial_t u \in L^2(0, \infty, H^{-1}(I))$  and

$$\forall v \in H_0^1(I), \quad \int_I 2A(x) \partial_t u(t, x) v(x) + \int_I A(x) \partial_x u(t, x) \partial_x v(x) = 0, \quad (13)$$

and  $u(0, x) = u_0(x), a.e. x \in I$ . With this definition, using the  $f_n, g_n$ 's as test functions, the weak solution to (2) is easily seen to be unique. Note that the family  $(P_t)$  forms a semigroup on  $L^2(I)$ . Finally, the expression (12) can be rewritten as

$$u(t, x) = \int_I u_0(y) \bar{p}(t, x, y) dy,$$

where the kernel has the form

$$\bar{p}(t, x, y) = \sum_{n \geq 1} A(y) \left( \frac{2}{a(k+1)} f_n(y) f_n(x) e^{-\lambda_n^2 t/2} + \frac{2}{ak(k+1)} g_n(y) g_n(x) e^{-\mu_n^2 t/2} \right). \quad (14)$$

We give a probabilistic derivation of the fundamental solution for (2) in the next sections.

### 3 A probabilistic construction of the fundamental solution for the evolution equation on $\mathbb{R}$

In this section, we consider the case when  $I = \mathbb{R}$ , and obtain the expression of the fundamental solution to (2) following a probabilistic construction. More precisely, we consider the Cauchy problem

$$\begin{cases} 2A(x) \partial_t u(t, x) = (A(x) u'(t, x))', & t > 0, x \in \mathbb{R} \\ u(0, x) = u_0(x), \\ u(t, \cdot) \in H^1(\mathbb{R}) \text{ and } Au'(t, \cdot) \in H^1(\mathbb{R}) & t > 0, \end{cases} \quad (15)$$

or equivalently the PDE

$$\partial_t u(t, x) = \frac{1}{2} u''(t, x), \quad x \in (-\infty, 0) \cup (0, \infty), \quad t > 0,$$

with the initial condition  $u(0, x) = u_0(x)$ ,  $x \in \mathbb{R}$  and the radiation and transmission conditions

$$\begin{cases} \lim_{|x| \rightarrow \infty} u(t, x) &= 0, \\ u(t, 0^-) &= u(t, 0^+) \quad ku'(t, 0^-) = u'(t, 0^+). \end{cases} \quad (16)$$

We first introduce a few notations.

**Notations.** We denote by  $g(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|y-x|^2}{2t}\right)$  the density of a  $\mathcal{N}(x, t)$ . We recall that for any  $t > 0$  and any  $x, y \in \mathbb{R}$  one has

$$\partial_t g(t, x, y) = \frac{1}{2} \partial_{xx}^2 g(t, x, y), \quad (17)$$

$$\partial_x g(t, x, y) = \frac{y-x}{t} g(t, x, y) \quad (18)$$

$$\partial_{xx}^2 g(t, x, y) = \left(\frac{y-x}{t}\right)^2 g(t, x, y) - \frac{1}{t} g(t, x, y). \quad (19)$$

Let  $b < c \in \mathbb{R}$ . The transition density of the Brownian motion killed at the points  $b$  or  $c$  is given by

$$p_W^{(b,c)}(t, x, y) := \sum_{n=-\infty}^{\infty} [g(t, x, y - 2n(c-b)) - g(t, x, 2b - y - 2n(c-b))] \quad (20)$$

(see [4], Appendix I, N° 6). Note that one has for any  $y \in (b, c)$ ,

$$\lim_{x \rightarrow b^+} p_W^{(b,c)}(t, x, y) = 0 \quad \text{and} \quad \lim_{x \rightarrow c^-} p_W^{(b,c)}(t, x, y) = 0 \quad (21)$$

### 3.1 The fundamental solution when $k > 0$

We first assume that  $k > 0$ , so that equation (2) is parabolic. It is known that in this case, under mild assumptions on the initial condition  $u_0$  (for example  $u_0$  is continuous and bounded, see [15]) the solution to the above Cauchy problem is given by

$$u(t, x) = \mathbb{E}^x[u_0(X_t)], \quad (22)$$

where  $X$  is the SBM with parameter  $\beta = \frac{1-k}{1+k} \in (-1, 1)$ . We refer to the survey [15] for the definition and main properties of the SBM. In particular  $X$  solves the Stochastic Differential Equation (SDE) with local time

$$dX_t = dW_t + \beta dL_t^0(X), \quad (23)$$

where  $W$  denotes a standard Brownian motion driving the SDE, and where  $L_t^0(X)$  is the symmetric local time at the point zero and at time  $t$  of  $X$ . Note that  $|\beta| < 1$  ensures the existence of  $X$ , see e.g. [15]. The SBM behaves like a Brownian motion, except at the times when it touches zero, at which its dynamics are biased by the term of local time in (23). In particular we have:

**Lemma 3.1.** [Walsh, [21]] *Let  $\beta \in (-1, 1)$  and let  $X$  be the solution to (23). Under  $\mathbb{P}^0$  one has:*

- i) The process  $|X|$  is distributed as a reflecting Brownian motion  $|W|$  (starting from zero).*
  - ii) The processes  $(\text{sign}(X_t))$  and  $(|X_t|)$  are independent.*
- In addition, for any  $t > 0$  one has  $\mathbb{P}^0(X_t > 0) = \frac{1+\beta}{2}$ .*

From this Lemma and the reflection principle for the Brownian motion, Walsh was able to give an explicit expression of the transition probability density  $p(t, x, y)$  of the SBM, in the form [21]

$$p(t, x, y) = \begin{cases} (1 - \beta)g(t, x, y) & \text{if } x \geq 0, y < 0 \\ g(t, x, y) + \beta g(t, x, -y) & \text{if } x > 0, y > 0 \\ (1 + \beta)g(t, x, y) & \text{if } x \leq 0, y > 0 \\ g(t, x, y) - \beta g(t, x, -y) & \text{if } x < 0, y < 0. \end{cases} \quad (24)$$

Note that (24) can also be written as

$$p(t, x, y) = \begin{cases} (1 - \beta)g(t, x, y) & \text{if } x \geq 0, y < 0 \\ -\beta[g(t, x, y) - g(t, x, -y)] + (1 + \beta)g(t, x, y) & \text{if } x > 0, y > 0 \\ (1 + \beta)g(t, x, y) & \text{if } x \leq 0, y > 0 \\ \beta[g(t, x, y) - g(t, x, -y)] + (1 - \beta)g(t, x, y) & \text{if } x < 0, y < 0. \end{cases} \quad (25)$$

where  $g(t, x, y) - g(t, x, -y) =: \check{g}(t, x, y)$ ,  $x, y > 0$  (resp.  $x, y < 0$ ) can be interpreted as the transition function of a Brownian motion on  $(0, \infty)$  (resp.  $(-\infty, 0)$ ) killed at zero (see [4] and Remark 4.2).

Yet another way to rewrite (24) is to consider  $\hat{g}(t, x, y) := g(t, x, y) + g(t, x, -y)$ ,  $x, y > 0$ , the transition function of a reflected Brownian motion on  $[0, \infty)$  (see [4]), which yields

$$p(t, x, y) = \begin{cases} (1 - \alpha)[\hat{g}(t, x, -y) - \check{g}(t, x, -y)] & \text{if } x \geq 0, y < 0 \\ \alpha\hat{g}(t, x, y) + (1 - \alpha)\check{g}(t, x, y) & \text{if } x > 0, y > 0 \\ \alpha[\hat{g}(t, -x, y) - \check{g}(t, -x, y)] & \text{if } x \leq 0, y > 0 \\ (1 - \alpha)\hat{g}(t, -x, -y) + \alpha\check{g}(t, -x, -y) & \text{if } x < 0, y < 0, \end{cases} \quad (26)$$

where we have set

$$\alpha = (1 + \beta)/2 = 1/(1 + k). \quad (27)$$

It follows from (22) that

$$u(t, x) = \mathbb{E}^x[u_0(X_t)] = \int_{\mathbb{R}} u_0(y)p(t, x, y)dy, \quad (28)$$

so that  $p(t, x, y)$  identifies with the fundamental solution of (15).

This fact also follows by mere computation: Indeed, using (17) and (18) one can check that the function  $p$ , defined by (24), satisfies for any  $y \in \mathbb{R}$

$$\partial_t p(t, x, y) = \frac{1}{2} \partial_{xx}^2 p(t, x, y), \quad \forall x \in (-\infty, 0) \cup (0, \infty), \quad (29)$$

together with the transmission conditions

$$p(t, 0-, y) = p(t, 0+, y), \quad k \partial_x p(t, 0-, y) = \partial_x p(t, 0+, y), \quad (30)$$

and the radiation condition

$$\lim_{|x| \rightarrow \infty} p(t, x, y) = 0. \quad (31)$$

Taking derivatives of the integral on the right-hand side of (28) and using (29)-(31) shows that  $p$  is indeed the fundamental solution of (15).

### 3.2 The fundamental solution in the case $k < 0$ and the pseudo SBM

When  $k \in \mathbb{R}_-^* \setminus \{-1\}$ , setting  $\beta = \frac{1-k}{1+k}$ , we may again define a function  $p$  as in (24). It is easy to check that this function solves (29)-(31), so that

$$u(t, x) = \int_{\mathbb{R}} u_0(y)p(t, x, y) dy,$$

is a solution to the Cauchy problem (15). Consequently,  $p(t, x, y)$  is a fundamental solution to (15) also in the case  $k \in \mathbb{R}_-^* \setminus \{-1\}$ . Note that  $\beta$  is not in  $(-1, 1)$ , so that  $p(t, x, y) dy$  is only a signed measure, and (23) does not define a SBM (see e.g. [14]). However, we can still define a function  $p(t, x, y)$  by (24) or (25), to which we can associate a pseudo SBM, as we describe below.

By *pseudo-random variable*, we mean a measurable function defined on a space (so-called pseudo-probability space) endowed with a signed measure with a total mass equal to 1. We observe that in the case  $k \in \mathbb{R}_-^* \setminus \{-1\}$ , the signed asymmetric heat-type kernel  $p(t, x, y)$  defined in (25) is no longer positive everywhere, however it integrates to 1 w.r.t.  $dy$ . In view of (25), this is clearly the case when  $x = 0$ . When  $x > 0$ , we have

$$\begin{aligned} \int_{-\infty}^{+\infty} p(t, x, y) dy &= -\beta \int_0^{+\infty} [g(t, x, y) - g(t, x, -y)] dy + (1 + \beta) \int_0^{+\infty} g(t, x, y) dy + (1 - \beta) \int_{-\infty}^0 g(t, x, y) dy \\ &= \beta \int_0^{+\infty} g(t, x, -y) dy + \int_0^{+\infty} g(t, x, y) dy + (1 - \beta) \int_{-\infty}^0 g(t, x, y) dy \\ &= \int_{-\infty}^{+\infty} g(t, x, y) dy + \beta \int_0^{+\infty} g(t, x, -y) dy - \beta \int_{-\infty}^0 g(t, x, y) dy \\ &= 1 + \beta \left( - \int_0^{-\infty} g(t, x, z) dz - \int_{-\infty}^0 g(t, x, y) dy \right) = 1. \end{aligned}$$



A similar computation applies to the case  $x < 0$ . One may also check that the Chapman-Kolmogorov identity for the family of transition kernels  $(p(t, x, y)dy)_{x \in \mathbb{R}}$  is also preserved in this case  $k < 0, k \neq -1$ .

Hence, in accordance to the usual Markov rules, we may define the pseudo-skew Brownian motion as the pseudo-Markov process  $(X_t)_{t \geq 0}$  associated with the signed asymmetric heat-type kernel  $p(t, x, y)dy$  defined in (25) – which is the fundamental solution to (15) also in the case  $k \in \mathbb{R}^+ \setminus \{-1\}$  – by : for  $t > 0$  and  $0 = t_0 < t_1 < \dots < t_m$  and  $x = x_0, x_1, \dots, x_m, y \in \mathbb{R}$ ,

$$\mathbb{P}^x (X_t \in dy) = p(t, x, y)dy$$

and

$$\mathbb{P}^x (X_{t_1} \in dx_1, \dots, X_{t_m} \in dx_m) = \prod_{i=1}^m p(t_i - t_{i-1}, x_{i-1}, x_i)dx_i.$$

Note that since pseudo-Markov processes are defined in terms of a signed measure, it is not clear how one could generalize the definition of the skew Brownian motion over all  $t \geq 0$  in this context. In particular, from a strict probabilistic point of view, the notion of trajectory for pseudo Markov processes indexed by continuous time does not have a clear meaning.

For more results on pseudo-Markov processes, we refer to e.g. [12], [13].

We now give a definition, inspired from [13], for the convergence of a family of pseudo-processes  $((Y_t^\varepsilon)_{t \geq 0})_{\varepsilon > 0}$  towards a pseudo-process  $(Y_t)_{t \geq 0}$ .

**Definition 3.2.** Let  $((Y_t^\varepsilon)_{t \geq 0})_{\varepsilon > 0}$  denote a family of pseudo-processes and  $(Y_t)_{t \geq 0}$  a pseudo-process.

We write

$$(Y_t^\varepsilon)_{t \geq 0} \xrightarrow[\varepsilon \searrow 0]{\text{pseudo-w}} (Y_t)_{t \geq 0}$$

if

$$\forall \ell \in \mathbb{N}^*, \forall t_1, \dots, t_\ell \geq 0, \forall u_1, \dots, u_\ell \in \mathbb{R}, \quad \mathbb{E} \left[ \exp \left( i \sum_{j=1}^{\ell} u_j Y_{t_j}^\varepsilon \right) \right] \xrightarrow[\varepsilon \searrow 0]{} \mathbb{E} \left[ \exp \left( i \sum_{j=1}^{\ell} u_j Y_{t_j} \right) \right].$$

### 3.3 Convergence of the scaled pseudo asymmetric random walk to the pseudo SBM

In this section, we assume that  $k \in \mathbb{R}^+ \setminus \{-1\}$ , and define  $\alpha$  as in (27). If  $k > 0$  then  $\alpha \in (0, 1)$  and we deal with true processes and random walks. If  $k \in \mathbb{R}^+ \setminus \{-1\}$  then  $\alpha \in \mathbb{R} \setminus (0, 1)$  and we deal with pseudo processes and pseudo random walks. Our computations englobe both cases, but the results are new for the case  $\alpha \in \mathbb{R} \setminus (0, 1)$  (for the case  $\alpha \in (0, 1)$  see e.g. [8]).

For any  $t > 0, x \geq 0$  and  $u \in \mathbb{R}$ , we introduce

$$\hat{\psi}(t; x, u) := \mathbb{E}^x \left( e^{iu|W_t|} \right) \quad \check{\psi}(t; x, u) := \mathbb{E}^x \left( e^{iu|\check{W}_t|} \right)$$

the characteristic functions of a Brownian motion on  $[0, \infty)$  reflected at 0 and respectively, killed at 0 (at some time  $t$ ).

For any  $x > 0$ , we use (26) and check at once that

$$\begin{aligned} & \int_{-\infty}^{+\infty} e^{iuy} p(t, x, y) dy \\ &= \alpha \int_0^{+\infty} e^{iuy} \hat{g}(t, x, y) dy + (1 - \alpha) \int_0^{+\infty} e^{iuy} \check{g}(t, x, y) dy \\ & \quad + (1 - \alpha) \left( \int_{-\infty}^0 e^{iuy} \hat{g}(t, x, -y) dy - \int_{-\infty}^0 e^{iuy} \check{g}(t, x, -y) dy \right) \\ &= \alpha \int_0^{+\infty} e^{iuy} \hat{g}(t, x, y) dy + (1 - \alpha) \int_0^{+\infty} e^{iuy} \check{g}(t, x, y) dy \\ & \quad + (1 - \alpha) \left( \int_0^{+\infty} e^{-iuy} \hat{g}(t, x, y) dy - \int_0^{+\infty} e^{-iuy} \check{g}(t, x, y) dy \right) \\ &= \alpha \hat{\psi}(t; x, u) + (1 - \alpha) \hat{\psi}(t; x, -u) + (1 - \alpha) (\check{\psi}(t; x, u) - \check{\psi}(t; x, -u)) = H_{\alpha, \alpha}(t; x, u) \end{aligned} \quad (32)$$

where we define for  $u, \gamma, \gamma' \in \mathbb{R}$  and  $x \geq 0$ ,

$$H_{\gamma, \gamma'}(t; x, u) := \gamma \hat{\psi}(t; x, u) + (1 - \gamma) \hat{\psi}(t; x, -u) + (1 - \gamma') (\check{\psi}(t; x, u) - \check{\psi}(t; x, -u)). \quad (33)$$

If  $x < 0$ , we similarly have

$$\begin{aligned} & \int_{-\infty}^{+\infty} e^{iuy} p(t, x, y) dy \\ &= \alpha \left( \int_0^{+\infty} e^{iuy} \hat{g}(t, |x|, y) dy - \int_0^{+\infty} e^{iuy} \check{g}(t, |x|, y) dy \right) \\ & \quad + (1 - \alpha) \int_{-\infty}^0 e^{iuy} \hat{g}(t, |x|, -y) dy + \alpha \int_{-\infty}^0 e^{iuy} \check{g}(t, |x|, -y) dy \\ &= \alpha \hat{\psi}(t; |x|, u) + (1 - \alpha) \hat{\psi}(t; |x|, -u) + \alpha (\check{\psi}(t; |x|, -u) - \check{\psi}(t; |x|, u)) = H_{\alpha, 1-\alpha}(t; |x|, -u), \end{aligned} \quad (34)$$

while if  $x = 0$ , we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{iuy} p(t, 0, y) dy &= (1 + \beta) \int_0^{+\infty} e^{iuy} g(t, 0, y) dy + (1 - \beta) \int_{-\infty}^0 e^{iuy} g(t, 0, y) dy \\ &= \alpha \int_0^{+\infty} e^{iuy} \hat{g}(t, 0, y) dy + (1 - \beta) \int_0^{+\infty} e^{-iuy} g(t, 0, y) dy \\ &= \alpha \hat{\psi}(t; 0, u) + (1 - \alpha) \hat{\psi}(t; 0, -u). \end{aligned} \quad (35)$$

Fix for a moment  $m_0 \in \mathbb{Z}$ . Constructed on some pseudo probability space  $(\Omega, \mathcal{F}, \mathbb{P}^{m_0})$  we consider  $(S_n)_{n \geq 0}$  the pseudo asymmetric random walk on the integers starting  $\mathbb{P}^{m_0}$ -a.s. from  $S_0 = m_0$  with pseudo transition probabilities given by

$$\begin{aligned} \mathbb{P}^{m_0}(S_{n+1} = S_n + 1 | S_0, \dots, S_n) &= \begin{cases} \alpha & \text{if } S_n = 0 \\ 1/2 & \text{otherwise.} \end{cases} \\ \mathbb{P}^{m_0}(S_{n+1} = S_n - 1 | S_0, \dots, S_n) &= \begin{cases} 1 - \alpha & \text{if } S_n = 0 \\ 1/2 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that the pseudo random sequence  $(S_n)_{n \geq 0}$  constructed likewise is a pseudo Markovian process.

The following reflection principle holds :

**Lemma 3.3.** *For all  $j \in \mathbb{N}^*$  and  $m \in \mathbb{Z}^*$ ,*

$$\mathbb{P}^0(S_j = m) = \begin{cases} \alpha \mathbb{P}^0(|S_j| = m) & \text{if } m > 0 \\ (1 - \alpha) \mathbb{P}^0(|S_j| = |m|) & \text{if } m < 0. \end{cases} \quad (36)$$

Moreover, for any integer  $m_0 > 0$

$$\mathbb{P}^{m_0}(S_j = m) = \begin{cases} \alpha \mathbb{P}^{m_0}(|S_j| = m) + (1 - \alpha) \mathbb{P}^{m_0}(|S_j| = m; \forall n \in \llbracket 1, j \rrbracket, |S_n| \neq 0) & \text{if } m > 0 \\ (1 - \alpha) (\mathbb{P}^{m_0}(|S_j| = |m|) - \mathbb{P}^{m_0}(|S_j| = |m|; \forall n \in \llbracket 1, j \rrbracket, |S_n| \neq 0)) & \text{if } m < 0. \end{cases} \quad (37)$$

and for any integer  $m_0 < 0$ ,

$$\mathbb{P}^{m_0}(S_j = m) = \begin{cases} (1 - \alpha) \mathbb{P}^{|m_0|}(|S_j| = m) + \alpha \mathbb{P}^{|m_0|}(|S_j| = |m|; \forall n \in \llbracket 1, j \rrbracket, |S_n| \neq 0) & \text{if } m < 0 \\ \alpha (\mathbb{P}^{|m_0|}(|S_j| = |m|) - \mathbb{P}^{|m_0|}(|S_j| = |m|; \forall n \in \llbracket 1, j \rrbracket, |S_n| \neq 0)) & \text{if } m > 0. \end{cases} \quad (38)$$

**Remark 3.4.** Note the analogy between the result of Lemma 3.3 and formula (26).

*Proof.* We only treat the case where the random walk starts from 0. The other cases, however tedious, can be analysed in the same fashion.

Let  $n \in \mathbb{N}^*$  and  $m \in \mathbb{Z}^*$  fixed.

Let us introduce  $G_n = \sup(j \leq n : S_j = 0) = \sup(j \leq n : |S_j| = 0)$ .

It is easy to check that the pseudo-path  $(S_{G_n+j})_{j \in \{0, \dots, n-G_n\}}$  has the same pseudo-law under the pseudo conditionnal probability  $\mathbb{P}^0(\cdot | S_{G_n+1} = 1)$  as the pseudo-path  $(|S_{G_n+j}|)_{j \in \{0, \dots, n-G_n\}}$  under the original pseudo probability  $\mathbb{P}^0$ . Observe also that  $\mathbb{P}^0(S_{G_n+1} = 1) = \alpha$ .

If  $m > 0$ , then a.s. on the set  $\{S_n = m\}$ , the value of  $G_n$  cannot be equal to  $n$ , so that  $G_n \leq n-1$  a.s. conditionnaly on this set. Hence, for  $m > 0$  we have

$$\begin{aligned} \mathbb{P}^0(S_n = m) &= \mathbb{P}^0(\{S_n = m\} \cap \{S_{G_n+1} = 1\}) \\ &= \mathbb{P}^0(S_{G_n+(n-G_n)} = m \mid S_{G_n+1} = 1) \mathbb{P}(S_{G_n+1} = 1) \\ &= \mathbb{P}^0(|S_{G_n+(n-G_n)}| = m) \mathbb{P}^0(S_{G_n+1} = 1) \\ &= \alpha \mathbb{P}^0(|S_n| = m). \end{aligned}$$

The case  $m < 0$  is proved in a similar way. We have  $\mathbb{P}^0(S_{G_n+1} = 1) = k/(1+k)$  and it is easily checked that the pseudo-path  $(S_{G_n+j})_{j \in \{0, \dots, n-G_n\}}$  has the same pseudo-law under the conditionnal probability  $\mathbb{P}^0(\cdot | S_{G_n+1} = -1)$  as the pseudo-path  $(-|S_{G_n+j}|)_{j \in \{0, \dots, n-G_n\}}$  under  $\mathbb{P}^0$ .  $\square$

The previous reflection principle is the key argument to show the following result.

**Proposition 3.5.** *The rescaled asymmetric random walk converges in the sense of Definition 3.2 to the pseudo SBM :*

$$\widehat{X}^n := \left( n^{-1/2} S_{\lfloor nt \rfloor} \right)_{t \geq 0} \xrightarrow{\text{pseudo-w}} (X_t)_{t \geq 0} \quad \text{as } n \rightarrow +\infty.$$

*Proof.* Our main concern is to ensure that all arguments for the convergence in the classical case hold in our pseudo probability context. In particular we are not allowed to use a Skorokhod embedding as this is often used for the convergence of random walks.

Taking into account these issues, we only give the main arguments of the proof and leave the computational details to the reader.

Let us fix  $x \geq 0$ ,  $t \geq 0$ ,  $u \in \mathbb{R}$  and set  $m_0 = \lfloor \sqrt{nx} \rfloor$ . We denote

$$\hat{\psi}_0(j; m_0, u) := \mathbb{E}^{m_0}(\exp(iu |S_j|)) \quad \check{\psi}_0(j; m_0, u) := \mathbb{E}^{m_0}(\exp(iu |S_j^\dagger|))$$

the characteristic functions of the symmetric random walk on  $\mathbb{N}$  reflected at 0 and (resp.) killed at 0 (taken at time  $j$ ).

The result of Lemma 3.3 implies that

$$\begin{aligned} \mathbb{E}^{m_0} \left( \exp \left( iu n^{-1/2} S_{\lfloor nt \rfloor} \right) \right) &= \alpha \hat{\psi}_0(\lfloor nt \rfloor; m_0, n^{-1/2} u) + (1-\alpha) \hat{\psi}_0(\lfloor nt \rfloor; m_0, -n^{-1/2} u) \\ &\quad + (1-\alpha) \left( \check{\psi}_0(\lfloor nt \rfloor; m_0, n^{-1/2} u) - \check{\psi}_0(\lfloor nt \rfloor; m_0, -n^{-1/2} u) \right) \\ &= \bar{H}_{\alpha, \alpha} \left( \lfloor nt \rfloor; m_0, n^{-1/2} u \right) \end{aligned} \quad (39)$$

where similarly to (33) we define

$$\bar{H}_{\gamma, \gamma'}(j; m_0, v) = \gamma \hat{\psi}_0(j; m_0, v) + (1-\gamma) \hat{\psi}_0(j; m_0, -v) + (1-\gamma') \left( \check{\psi}_0(j; m_0, v) - \check{\psi}_0(j; m_0, -v) \right).$$

From (32) we infer that

$$\mathbb{E}^x(\exp(iu X_t)) = \alpha \hat{\psi}(t; x, u) + (1-\alpha) \hat{\psi}(t; x, -u) + (1-\alpha) \left( \check{\psi}(t; x, u) - \check{\psi}(t; x, -u) \right). \quad (40)$$

We note that  $(|X_t|)$  has the same (pseudo) law under  $\mathbb{P}$  as a standard half-normal distribution on  $[0, \infty)$  and  $(|S_n|)_{n \geq 0}$  has the (pseudo) law under  $\mathbb{P}$  as a standard symmetric reflected random walk. Thus, standard results on random walks ensure that  $\hat{\psi}_0(\lfloor nt \rfloor; m_0, n^{-1/2} u)$  (resp.  $\check{\psi}_0(\lfloor nt \rfloor; m_0, n^{-1/2} u)$ ) converges simply to  $\hat{\psi}(t; x, u)$  (resp. to  $\check{\psi}(t; x, u)$ ) as  $n$  tends to infinity. Combining with (39) and (40) shows that

$$\mathbb{E}^{m_0} \left[ \exp \left( iu n^{-1/2} S_{\lfloor nt \rfloor} \right) \right] \xrightarrow{n \rightarrow +\infty} \mathbb{E}^x [\exp(iu X_t)].$$

The same arguments show that the convergence also holds for  $x < 0$  and  $x = 0$ .

Now let  $x \geq 0$ ,  $0 \leq t_1 < t_2$ ,  $u_1, u_2 \in \mathbb{R}$  and set  $m_0 = \lfloor \sqrt{nx} \rfloor$ . Using the Markov property of the pseudo random walk, we have

$$\begin{aligned} & \mathbb{E}^{m_0} \left[ \exp \left( iu_1 n^{-1/2} S_{\lfloor nt_1 \rfloor} + iu_2 n^{-1/2} S_{\lfloor nt_2 \rfloor} \right) \right] \\ &= \mathbb{E}^{m_0} \left[ \exp \left( i(u_1 + u_2) n^{-1/2} S_{\lfloor nt_1 \rfloor} + iu_2 n^{-1/2} (S_{\lfloor nt_2 \rfloor} - S_{\lfloor nt_1 \rfloor}) \right) \right] \\ &= \mathbb{E}^{m_0} \left[ \exp \left( i(u_1 + u_2) n^{-1/2} S_{\lfloor nt_1 \rfloor} \right) \mathbb{E}^{S_{\lfloor nt_1 \rfloor}} \left[ \exp \left( iu_2 n^{-1/2} S_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor} \right) \right] \right] \end{aligned}$$

which in view of (36) and (37) gives

$$\begin{aligned} & \mathbb{E}^{m_0} \left[ \exp \left( iu_1 n^{-1/2} S_{\lfloor nt_1 \rfloor} + iu_2 n^{-1/2} S_{\lfloor nt_2 \rfloor} \right) \right] \\ &= \alpha \mathbb{E}^{m_0} \left[ \exp \left( i(u_1 + u_2) n^{-1/2} |S_{\lfloor nt_1 \rfloor}| \right) \bar{H}_{\alpha, \alpha} \left( \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor; |S_{\lfloor nt_1 \rfloor}|, u_2 \right) \right] \\ &+ (1 - \alpha) \mathbb{E}^{m_0} \left[ \exp \left( -i(u_1 + u_2) n^{-1/2} |S_{\lfloor nt_1 \rfloor}| \right) \bar{H}_{\alpha, 1-\alpha} \left( \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor; |S_{\lfloor nt_1 \rfloor}|, -u_2 \right) \right] \\ &+ (1 - \alpha) \mathbb{E}^{m_0} \left[ \exp \left( i(u_1 + u_2) n^{-1/2} |S_{\lfloor nt_1 \rfloor}^\dagger| \right) \bar{H}_{\alpha, \alpha} \left( \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor; |S_{\lfloor nt_1 \rfloor}^\dagger|, u_2 \right) \right] \\ &- (1 - \alpha) \mathbb{E}^{m_0} \left[ \exp \left( -i(u_1 + u_2) n^{-1/2} |S_{\lfloor nt_1 \rfloor}^\dagger| \right) \bar{H}_{\alpha, 1-\alpha} \left( \lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor; |S_{\lfloor nt_1 \rfloor}^\dagger|, -u_2 \right) \right], \quad (41) \end{aligned}$$

where we have denoted  $S^\dagger$  the pseudo random walk on  $\mathbb{N}$  killed at zero.

But as already mentionned,  $(|S_n|)_{n \geq 0}$  has the (pseudo)law under  $\mathbb{P}$  of a standard symmetric reflected random walk. Therefore, we can use standard convergence results on random walks : since the convergence of  $x \mapsto \bar{H}_{\gamma, \gamma'} \left( \lfloor nt \rfloor; \lfloor \sqrt{nx} \rfloor, n^{-1/2} u \right)$  and  $x \mapsto \bar{H}_{\gamma, \gamma'} \left( \lfloor nt + 1 \rfloor; \lfloor \sqrt{nx} \rfloor, n^{-1/2} u \right)$  to  $x \mapsto H_{\gamma, \gamma'}(t; x, u)$  are uniform, we deduce that  $\mathbb{E}^{m_0} \left[ \exp \left( iu_1 n^{-1/2} S_{\lfloor nt_1 \rfloor} + iu_2 n^{-1/2} S_{\lfloor nt_2 \rfloor} \right) \right]$  converges as  $n$  tends to infinity to

$$\begin{aligned} \Psi_\alpha(t_1, t_2; x, u_1, u_2) &= \alpha \mathbb{E}^x \left[ \exp \left( i(u_1 + u_2) |X_{t_1}| \right) H_{\alpha, \alpha} \left( t_2 - t_1; |X_{t_2 - t_1}|, u_2 \right) \right] \\ &+ (1 - \alpha) \mathbb{E}^x \left[ \exp \left( -i(u_1 + u_2) |X_{t_1}| \right) H_{\alpha, 1-\alpha} \left( t_2 - t_1; |X_{t_1}|, -u_2 \right) \right] \\ &+ (1 - \alpha) \mathbb{E}^x \left[ \exp \left( i(u_1 + u_2) |X_{t_1}^\dagger| \right) H_{\alpha, \alpha} \left( t_2 - t_1; |X_{t_1}^\dagger|, u_2 \right) \right] \\ &- (1 - \alpha) \mathbb{E}^x \left[ \exp \left( -i(u_1 + u_2) |X_{t_1}^\dagger| \right) H_{\alpha, 1-\alpha} \left( t_2 - t_1; |X_{t_1}^\dagger|, -u_2 \right) \right], \end{aligned}$$

where  $X^\dagger$  denoted the pseudo SBM on  $(0, \infty)$  killed at zero.

It remains to check that  $\Psi_\alpha(t_1, t_2; x, u_1, u_2) = \mathbb{E}^x \left[ \exp \left( i(u_1 X_{t_1} + u_2 X_{t_2}) \right) \right]$  but this is easily checked by the use of the Markov property which derives from the Chapman-Kolmogorov identity. Hence,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[ \exp \left( i \left( u_1 n^{-1/2} S_{\lfloor nt_1 \rfloor} + u_2 n^{-1/2} S_{\lfloor nt_2 \rfloor} \right) \right) \right] = \mathbb{E} \left[ \exp \left( i(u_1 X_{t_1} + u_2 X_{t_2}) \right) \right].$$

Finally, we may extend the previous limiting result by induction to prove that for any  $\ell \in \mathbb{N}^*$ ,  $u_1, \dots, u_\ell \in \mathbb{R}$  and times  $0 \leq t_1 < \dots < t_\ell$ ,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[ \exp \left( i \sum_{j=1}^{\ell} u_j n^{-1/2} S_{\lfloor nt_j \rfloor} \right) \right] = \mathbb{E} \left[ \exp \left( i \sum_{j=1}^{\ell} u_j X_{t_j} \right) \right].$$

The proof is completed.  $\square$

## 4 The evolution equation on a finite interval

In this section we assume  $0 < a < \infty$ .

Again, we first address the case  $k > 0$ , in which we compute the transition function of the skew Brownian motion killed at the end-points of  $I = (-a, a)$ , then we treat the general case  $k \in \mathbb{R}^* \setminus \{-1\}$ .

#### 4.1 The transition function of the SBM killed at $-a$ or $a$ (case $k > 0$ )

In the same manner as for (22), it can be shown that the solution to the parabolic problem (2) on  $(-a, a) \times \mathbb{R}$  can be represented as

$$u(t, x) = \mathbb{E}^x[u_0(\tilde{X}_t)] \quad (42)$$

where  $\tilde{X}$  is the SBM of parameter  $\beta = \frac{1-k}{1+k}$ , killed at  $-a$  or  $a$ . This process behaves like the SBM  $X$  as long as it does not exit from  $(-a, a)$ . When it touches  $-a$  or  $a$  it is sent at a cemetery point  $\partial$ . By convention, for any function  $f$  one has  $f(\partial) = 0$ , which ensures that the homogeneous Dirichlet boundary condition in (5) is satisfied.

Let us assume that  $x \in (-a, a)$  and let  $\tilde{T}_{(-a, a)} = \inf\{t \geq 0 : \tilde{X}_t \notin (-a, a)\}$ . We compute

$$\begin{aligned} \mathbb{E}^x[u_0(\tilde{X}_t)] &= \mathbb{E}^x[u_0(\tilde{X}_t); \tilde{T}_{(-a, a)} \leq t] + \mathbb{E}^x[u_0(\tilde{X}_t); \tilde{T}_{(-a, a)} > t] \\ &= \mathbb{E}^x[u_0(\partial); \tilde{T}_{(-a, a)} \leq t] + \mathbb{E}^x[u_0(\tilde{X}_t); \tilde{T}_{(-a, a)} > t] \\ &= \mathbb{E}^x[u_0(X_t); T_{(-a, a)} > t], \end{aligned} \quad (43)$$

where  $X$  is the SBM considered in Section 3.2 and  $T_{(-a, a)} = \inf\{t \geq 0 : X_t \notin (-a, a)\}$ .

We first derive the expression of the kernel  $\check{p}(t, x, y)$  that satisfies  $\mathbb{P}^x(X_t \in dy; T_{(-a, a)} > t) = \check{p}(t, x, y)dy$ , i.e. the fundamental solution to (2) :

**Proposition 4.1.** *Let  $\beta \in (-1, 1)$ . Let  $X$  be the solution of (23), i.e. the SBM of parameter  $\beta$ . Let  $a > 0$ .*

*For any  $x, y \in (-a, a)$ , any  $t > 0$ , one has  $\mathbb{P}^x(X_t \in dy; T_{(-a, a)} > t) = \check{p}(t, x, y)dy$  with*

$$\check{p}(t, x, y) = \begin{cases} (1 - \beta)p_W^{(-a, a)}(t, x, y) & \text{if } x \geq 0, y < 0 \\ -\beta p_W^{(0, a)}(t, x, y) + (1 + \beta)p_W^{(-a, a)}(t, x, y) & \text{if } x > 0, y > 0 \\ (1 + \beta)p_W^{(-a, a)}(t, x, y) & \text{if } x \leq 0, y > 0 \\ \beta p_W^{(-a, 0)}(t, x, y) + (1 - \beta)p_W^{(-a, a)}(t, x, y) & \text{if } x < 0, y < 0. \end{cases} \quad (44)$$

**Remark 4.2.** Note the consistence of (25) with (44), as  $a \rightarrow \infty$  in (44).

**Remark 4.3.** Note that as

$$\int_{-a}^a \check{p}(t, x, y)dy = \mathbb{P}^x(T_{(-a, a)} > t),$$

$\check{p}(t, x, \cdot)$  does not necessarily integrate to 1. The definition of a genuine family of transition probability measures from  $\check{p}(t, x, y)dy$  would require combining this density measure with a Dirac measure that charges  $\partial$  with probability  $\mathbb{P}^x(T_{(-a, a)} \leq t)$ , see [19], p84. Still by a slight abuse of language we call this kernel the transition function of  $\tilde{X}$ .

*Proof of Proposition 4.1.* We use Lemma 3.1 and adapt the arguments in [21]. We set  $T_0 = \inf\{t \geq 0 : X_t = 0\}$  and remark that  $T_{(-a, a)} = \inf\{t \geq 0 : X_t \notin (-a, a)\} = \inf\{t \geq 0 : |X_t| = a\}$ . We denote  $\tau_0 = \inf\{t \geq 0 : W_t = 0\}$  and  $\tau_{(-a, a)} = \inf\{t \geq 0 : |W_t| = a\}$ . In the computations below, by a slight abuse of notation we may denote by  $\mathbb{P}^x$  either  $\mathbb{P}(\cdot | X_0 = x)$  or  $\mathbb{P}(\cdot | W_0 = x)$ . This will be clear from the context.

Let  $t > 0$ . We first treat the case  $x > 0, y < 0$ . Using the fact that  $\mathbb{P}^x(X_t \in dy; T_0 > t) = 0$ , and using the strong Markov property of  $X$ , it follows that

$$\begin{aligned} &\mathbb{P}^x(X_t \in dy; T_{(-a, a)} > t) \\ &= \mathbb{P}^x(X_t \in dy; T_0 \leq t; T_{(-a, a)} > t) + \mathbb{P}^x(X_t \in dy; T_0 > t; T_{(-a, a)} > t) \\ &= \mathbb{E}^x[\mathbf{1}_{T_0 \leq t} \mathbb{P}^x(X_t \in dy; T_{(-a, a)} > t | \mathcal{F}_{T_0})] \\ &= \int_0^t \mathbb{P}^0(\tilde{X}_{t-u} \in dy; \tilde{T}_{(-a, a)} > t - u) f_{T_0}^x(u) du, \end{aligned}$$

where  $\tilde{X}$  is another SBM of parameter  $\beta$ , starting from zero under  $\mathbb{P}^0$ ,  $\tilde{T}_{(-a, a)} = \inf\{t \geq 0 : |\tilde{X}_t| = a\}$  and  $f_{T_0}^x(u)du$  is the law of  $T_0$  under  $\mathbb{P}^x$ .

From Lemma 3.1 we have

$$\begin{aligned}
& \mathbb{P}^0(\tilde{X}_{t-u} \in dy; \tilde{T}_{(-a,a)} > t-u) \\
&= \mathbb{P}^0(|\tilde{X}_{t-u}| \in -dy; \tilde{T}_{(-a,a)} > t-u; \tilde{X}_{t-u} < 0) \\
&= \frac{1-\beta}{2} \mathbb{P}^0(|\tilde{W}_{t-u}| \in -dy; \tilde{\tau}_{(-a,a)} > t-u) \\
&= \frac{1-\beta}{2} \left( \mathbb{P}^0(\tilde{W}_{t-u} \in -dy; \tilde{\tau}_{(-a,a)} > t-u) + \mathbb{P}^0(-\tilde{W}_{t-u} \in -dy; \tilde{\tau}_{(-a,a)} > t-u) \right) \\
&= (1-\beta) \mathbb{P}^0(\tilde{W}_{t-u} \in dy; \tilde{\tau}_{(-a,a)} > t-u)
\end{aligned}$$

where  $\tilde{W}$  is a Brownian motion starting from zero under  $\mathbb{P}^0$  and  $\tilde{\tau}_{(-a,a)} = \inf\{t \geq 0 : |\tilde{W}_t| = a\}$ .

Thus denoting by  $f_{\tau_0}^x(u)du$  the law of  $\tau_0$  under  $\mathbb{P}^x$ , noticing that  $f_{T_0}^x(u) = f_{\tau_0}^x(u)$ , and using this time the strong Markov property of  $W$ , we calculate

$$\begin{aligned}
& \mathbb{P}^x(X_t \in dy; T_{(-a,a)} > t) \\
&= (1-\beta) \int_0^t \mathbb{P}^0(\tilde{W}_{t-u} \in dy; \tilde{\tau}_{(-a,a)} > t-u) f_{\tau_0}^x(u) du, \\
&= (1-\beta) \mathbb{P}^x(W_t \in dy; \tau_0 \leq t; \tau_{(-a,a)} > t) \\
&= (1-\beta) \mathbb{P}^x(W_t \in dy; \tau_{(-a,a)} > t) \\
&= (1-\beta) p_W^{(-a,a)}(t, x, y) dy,
\end{aligned}$$

and obtain the first line of (44).

Next, we treat the case  $x > 0$  and  $y > 0$ . We have

$$\begin{aligned}
& \mathbb{P}^x(X_t \in dy; T_{(-a,a)} > t) \\
&= \mathbb{P}^x(X_t \in dy; T_0 \leq t; T_{(-a,a)} > t) + \mathbb{P}^x(X_t \in dy; T_0 > t; T_{(-a,a)} > t)
\end{aligned}$$

As  $0 < x, y < a$ , the term  $\mathbb{P}^x(X_t \in dy; T_0 > t; T_{(-a,a)} > t)$  corresponds to the transition of a Brownian motion killed at 0 or  $a$ , or in other words,

$$\mathbb{P}^x(X_t \in dy; T_0 > t; T_{(-a,a)} > t) = p_W^{(0,a)}(t, x, y) dy.$$

As for the term  $\mathbb{P}^x(X_t \in dy; T_0 \leq t; T_{(-a,a)} > t)$ , we use similar computations as in the case  $x > 0$ ,  $y < 0$ . Since  $X_t$  is positive this time, we obtain

$$\mathbb{P}^x(X_t \in dy; T_0 \leq t; T_{(-a,a)} > t) = (1+\beta) \mathbb{P}^x(W_t \in dy; \tau_0 \leq t; \tau_{(-a,a)} > t).$$

Notice also that

$$\begin{aligned}
& \mathbb{P}^x(W_t \in dy; \tau_0 \leq t; \tau_{(-a,a)} > t) \\
&= \mathbb{P}^x(W_t \in dy; \tau_{(-a,a)} > t) - \mathbb{P}^x(W_t \in dy; \tau_0 > t; \tau_{(-a,a)} > t) \\
&= p_W^{(-a,a)}(t, x, y) dy - p_W^{(0,a)}(t, x, y) dy.
\end{aligned}$$

Putting all the pieces together yields

$$\begin{aligned}
\mathbb{P}^x(X_t \in dy; T_{(-a,a)} > t) &= \left[ p_W^{(0,a)}(t, x, y) + (1+\beta) \{ p_W^{(-a,a)}(t, x, y) - p_W^{(0,a)}(t, x, y) \} \right] dy \\
&= \left[ -\beta p_W^{(0,a)}(t, x, y) + (1+\beta) p_W^{(-a,a)}(t, x, y) \right] dy,
\end{aligned}$$

and thus, the second line of (44). The remaining cases can be treated in a similar manner.  $\square$

## 4.2 The fundamental solution to (2) in the general case $k \in \mathbb{R}^* \setminus \{-1\}$

When  $k > 0$ , the transition function yields a fundamental solution to the evolution equation. When  $k < 0, k \neq -1$ , as in the case when  $I = \mathbb{R}$ , we check that its expression also yields a fundamental solution.

**Lemma 4.4.** *Let  $k \in \mathbb{R}^* \setminus \{-1\}$  and define  $\beta = \frac{1-k}{1+k}$ . The kernel  $\check{p}(t, x, y)$  defined by (44) is the fundamental solution to (3)-(5), or to (2). Equivalently it satisfies for any  $y \in (-a, a)$ ,*

$$\partial_t \check{p}(t, x, y) = \frac{1}{2} \partial_{xx}^2 \check{p}(t, x, y), \quad \forall x \in (-a, 0) \cup (0, a), \quad (45)$$

and

$$\check{p}(t, 0-, y) = \check{p}(t, 0+, y), \quad (46)$$

$$k \partial_x \check{p}(t, 0-, y) = \partial_x \check{p}(t, 0+, y), \quad (47)$$

and

$$\lim_{x \rightarrow \pm a} \check{p}(t, x, y) = 0. \quad (48)$$

*Proof.* If a kernel  $q(t, x, y)$  is a fundamental solution to (3)-(5), then by definition, the function

$$(t, x) \mapsto u(t, x) = \int_{-a}^a u_0(y) q(t, x, y) dy \quad (49)$$

solves (3)-(5) for any  $u_0 \in C_c^\infty(-a, a)$ . As

$$\begin{aligned} \partial_t u(t, x) &= \int_{-a}^a u_0(y) \partial_t q(t, x, y) dy, & \partial_x u(t, x) &= \int_{-a}^a u_0(y) \partial_x q(t, x, y) dy, \\ \text{and } \partial_{xx}^2 u(t, x) &= \int_{-a}^a u_0(y) \partial_{xx}^2 q(t, x, y) dy, \end{aligned}$$

letting  $u_0$  vary in  $C_c^\infty(-a, a)$ , it is easy to see that  $q(t, x, y)$  satisfies (45)-(48). Conversely, if (45)-(48) hold for  $q(t, x, y)$ , one may check that the function defined by (49) solves (3)-(5). Thus the equivalence.

Consider first  $k > 0$  and  $\beta = \frac{1-k}{1+k}$ . From (42), (43) and Proposition 4.1, it is clear that the kernel  $\check{p}(t, x, y)$  defined by (44) is the fundamental solution to (3)-(5), and therefore satisfies (45)-(48).

Considering (45) and for example the first line of (44) it is clear that for  $x < 0$  and  $y > 0$  one has

$$\partial_t p_W^{(-a, a)}(t, x, y) = \frac{1}{2} \partial_{xx}^2 p_W^{(-a, a)}(t, x, y) \quad (50)$$

(of course this fact can formally be seen from (17) and (20); see also Remark 4.5 below).

Consider now  $k \in \mathbb{R}^* \setminus \{-1\}$ , set  $\beta = \frac{1-k}{1+k}$ , and define the kernel  $\check{p}(t, x, y)$  by (44). From the first line of (44), and (50) it is clear that (45) holds for  $x < 0$  and  $y > 0$ . The other cases can be treated similarly, so that  $\check{p}(t, x, y)$  solves (45) for any  $y \in (-a, a)$ ,  $x \in (-a, 0) \cup (0, a)$ .

Concerning the transmission conditions, we first note that when  $y > 0$ , Eq. (47) can be rewritten

$$(1 + \beta) \partial_x \check{p}(t, 0+, y) = (1 - \beta) \partial_x \check{p}(t, 0-, y) \quad (51)$$

or equivalently, using the definition of  $\check{p}(t, x, y)$  in (44) (with  $y < 0$ )

$$(1 - \beta^2) \partial_x p_W^{(-a, a)}(t, 0-, y) = (-\beta - \beta^2) \partial_x p_W^{(0, a)}(t, 0+, y) + (1 + 2\beta + \beta^2) \partial_x p_W^{(-a, a)}(t, 0+, y),$$

which in turn, due to the continuity of  $\partial_x p_W^{(-a, a)}(t, 0, y)$  at  $x = 0$ , is equivalent to

$$\partial_x p_W^{(0, a)}(t, 0+, y) = 2 \partial_x p_W^{(-a, a)}(t, 0, y). \quad (52)$$

Note that condition (52) only bears on the properties of  $p_W^{(-a, a)}$  and  $p_W^{(0, a)}$ , and is equivalent to (47) whatever the sign of  $k$ , provided  $k \neq -1$ . We know from the case  $k > 0$  that (52) holds, thus (47) must hold for all  $k \neq -1$ . The same argument applies to  $y < 0$ . Equation (46) is a consequence of (21) and of the continuity at  $x = 0$  of the kernel  $p_W^{(-a, a)}(t, x, y)$ .

Finally, the Dirichlet condition (48) follows from (21) and the form of  $\check{p}(t, x, y)$ . We conclude that the latter is indeed the fundamental solution of (3)-(5) when  $k < 0, k \neq -1$ .  $\square$

**Remark 4.5.** In fact it is possible to start from (44), and to use (20) and (17)-(18) in order to formally check (45)-(47), for any  $k \in \mathbb{R}^* \setminus \{-1\}$ . We especially want to explain how one can derive the transmission condition in (47).

As we have already seen in the proof of Lemma 4.4, it is enough to check (52). On one hand one has

$$\begin{aligned} & \partial_x p_W^{(0,a)}(t, 0+, y) \\ &= \sum_{n=-\infty}^{\infty} \left\{ \frac{y-2na}{t} g(t, 0, y-2na) - \frac{-y-2na}{t} g(t, 0, -y-2na) \right\} \\ &= \sum_{n=-\infty}^{\infty} \left\{ \frac{y-2na}{t} g(t, 0, y-2na) + \frac{y+2na}{t} g(t, 0, y+2na) \right\} \\ &= 2 \sum_{n=-\infty}^{\infty} \frac{y+2na}{t} g(t, 0, y+2na) \end{aligned}$$

On the other hand one has

$$\begin{aligned} & \partial_x p_W^{(-a,a)}(t, 0, y) \\ &= \sum_{n=-\infty}^{\infty} \left\{ \frac{y-4na}{t} g(t, 0, y-4na) - \frac{2a-y-4na}{t} g(t, 0, 2a-y-4na) \right\} \\ &= \sum_{n=-\infty}^{\infty} \left\{ \frac{y-4na}{t} g(t, 0, y-4na) + \frac{y+4na-2a}{t} g(t, 0, y+4na-2a) \right\} \\ &= \sum_{n=-\infty}^{\infty} \left\{ \frac{y-2(2n)a}{t} g(t, 0, y-2(2n)a) + \frac{y+2(2n-1)a}{t} g(t, 0, y+2(2n-1)a) \right\} \\ &= \sum_{n=-\infty}^{\infty} \frac{y+2na}{t} g(t, 0, y+2na). \end{aligned}$$

Therefore (52).

Note that in the case  $k \in \mathbb{R}_-^* \setminus \{-1\}$  the kernel  $\check{p}(t, x, y)$  defined by (44) is not positive: indeed in that case  $\beta > 1$  and it suffices to examine the first line of (44).

So even if we complement the density measure  $\check{p}(t, x, y)dy$  with Dirac measures in order to obtain a family of transition pseudo probability measures (see our Remark 4.3), the latter will not define a Markov process. One could however introduce the pseudo-process associated to this family, in the spirit of Section 3.2.

**Remark 4.6.** We end up this section by noticing that as expected, the form of the kernel (44) coincides with (14). Indeed, the transition density of the Brownian motion on an interval  $(a, b)$ , killed at  $a$  or  $b$  has the following spectral representation (see Appendix 1 in [4])

$$p_W^{(a,b)}(t, x, y) = \frac{2}{b-a} \sum_{n \geq 1} \exp\left(-\frac{n^2 \pi^2}{2(b-a)^2} t\right) \sin\left(\frac{n\pi}{b-a}(x-a)\right) \sin\left(\frac{n\pi}{b-a}(y-a)\right).$$

Thus one has

$$p_W^{(-a,a)}(t, x, y) = \frac{1}{a} \sum_{n \geq 1} \exp\left(-\frac{n^2 \pi^2}{8a^2} t\right) \sin\left(\frac{n\pi}{2a}(x-a)\right) \sin\left(\frac{n\pi}{2a}(y-a)\right).$$

Regrouping the terms with odd and even indices, the above expression rewrites, when  $x \geq 0$  and  $y < 0$

$$\begin{aligned} p_W^{(-a,a)}(t, x, y) &= +\frac{1}{a} \sum_{q \geq 1} \exp\left(-\frac{q^2 \pi^2}{2a^2} t\right) \sin\left(\frac{q\pi}{a}(x-a)\right) \sin\left(\frac{q\pi}{a}(y-a)\right) \\ &\quad +\frac{1}{a} \sum_{q \geq 1} \exp\left(-\frac{(2q-1)^2 \pi^2}{8a^2} t\right) \cos\left(\frac{(2q-1)\pi}{2a}(x-a)\right) \cos\left(\frac{(2q-1)\pi}{2a}(y-a)\right) \\ &= \frac{1}{a} \sum_{q \geq 1} \left[ \frac{1}{k} e^{-\mu_q^2 t/2} g_q(x) g_q(y) + e^{-\lambda_q^2 t/2} f_q(x) f_q(y) \right] \end{aligned}$$



Multiplying by  $1 - \beta = \frac{2k}{k+1}$  to retrieve the first line of (44), one easily recovers the expression (14) when  $x \geq 0$  and  $y < 0$ . A similar calculation applies for the other possible choices of  $x$  and  $y$ .

## 5 Various numerical schemes for the PDE (2)

In this section we want to construct several numerical schemes for the approximation of the solution  $u$  of (2), inspired by the theoretical results of the previous sections.

In Section 5.1 we construct a scheme  $\bar{u}_{spec}^N$  inspired by the spectral representation of the semigroup of Section 2.

In Section 5.2 we explain how we can infer a finite difference type scheme  $\bar{u}_{RW}^n$  from the scaled pseudo asymmetric random walk  $\hat{X}^n$  of Section 3.3.

In Section 5.3 we construct a scheme  $\bar{u}_{fund}^{h,N}$ , which is inspired by the fact that the fundamental solution  $\check{p}(t, x, y)$  computed in Section 4.1 involves transition functions of killed Brownian motions, which can be seen as the fundamental solutions of simple heat equations, with homogeneous Dirichlet boundary conditions.

An initial condition  $u_0$  is given (in Section 5.1 it is of class  $L^2(I)$ , in Sections 5.2 and 5.3 we can imagine it is continuous and bounded).

We have  $k \in \mathbb{R}^* \setminus \{-1\}$  and set  $\beta = \frac{1-k}{1+k}$  and  $\alpha = \frac{1+\beta}{2}$ .

In the case  $k \in (0, \infty)$  several numerical schemes are available (one can for example first perform a finite element discretization w.r.t. the space variable, and then a Crank-Nicholson scheme, e.g. [18]), including probabilistic ones (e.g. [5], [16], [6]). Thus, for this well explored case the schemes presented hereafter may provide additional methods (up to our knowledge they have never been proposed in this form).

Their main interest is that they allow to handle the case  $k \in \mathbb{R}_-^* \setminus \{-1\}$ , for which in particular classical probabilistic methods (e.g. [5], [16], [6]) cannot be applied. In the case  $k > 0$  the latter deeply rely on stochastic simulations of the trajectories of the SBM of parameter  $\beta \in (-1, 1)$ . When  $k \in \mathbb{R}_-^* \setminus \{-1\}$  one can define a pseudo-SBM, but it proves difficult to define associated trajectories. Note also that the techniques for proving the convergence of deterministic schemes do not apply in the case  $k \in \mathbb{R}_-^* \setminus \{-1\}$ .

### 5.1 Scheme inspired by the spectral representation of the semi-group

We assume that  $u_0 \in L^2(I)$  and fix a truncation order  $N \in \mathbb{N}^*$ . Then  $\bar{u}_{spec}^N$  is defined by

$$\bar{u}_{spec}^N(t, x) = \sum_{n=1}^N a_n e^{-\lambda_n^2 t/2} f_n(x) + b_n e^{-\mu_n^2 t/2} g_n(x)$$

where the  $\lambda_n$ 's,  $\mu_n$ 's,  $g_n$ 's and  $f_n$ 's are as in Section 2, and the  $a_n$ 's and  $b_n$ 's are defined as in (10). That is to say  $\bar{u}_{spec}^N(t, x)$  is obtained by keeping the first  $N$  terms in the spectral representation (12) of  $u(t, x)$ .

Of course in order to use this method we have to be able to compute the  $a_n$ 's and  $b_n$ 's. In Section 6 we consider two examples where these computations can be done explicitly. If this is not the case we can resort to numerical integration. We sum up hereafter the proposed algorithm (Algorithm 1).

**ALGORITHM 1:** Computation of  $\bar{u}_{spec}^N(t, x)$ .

**Parameters of the method:** A time  $0 \leq t < \infty$  at which we want to compute the approached solution. A truncation order  $N \in \mathbb{N}^*$ . Remember that the  $\lambda_n$ 's,  $\mu_n$ 's,  $g_n$ 's and  $f_n$ 's are as in Section 2.

**Algorithm:** 1) Compute

$$a_n = \frac{\int_I A(x) u_0(x) f_n(x) dx}{\int_I A(x) |f_n(x)|^2 dx}, \quad b_n = \frac{\int_I A(x) u_0(x) g_n(x) dx}{\int_I A(x) |g_n(x)|^2 dx}$$

(either exactly or by numerical integration).

2) Return

$$\bar{u}_{spec}^N(t, x) = \sum_{n=1}^N a_n e^{-\lambda_n^2 t/2} f_n(x) + b_n e^{-\mu_n^2 t/2} g_n(x).$$

As the series in (12) is normally convergent we get immediately the following convergence result.

**Proposition 5.1.** *Let us consider  $u$  the solution of (2). Let  $t \geq 0$  and let us consider for any  $N \in \mathbb{N}^*$  the function  $\bar{u}_{spec}^N(t, \cdot)$  defined by Algorithm 1. We have*

$$\|u(t, \cdot) - \bar{u}_{spec}^N(t, \cdot)\|_{L^2(I)} \xrightarrow{N \rightarrow \infty} 0.$$

## 5.2 Scheme inspired by the scaled pseudo asymmetric random walk

Let  $u_0 : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and bounded and consider first (15). Let us fix  $n \in \mathbb{N}^*$  a discretization order.

From Proposition 3.5,  $u(T, x) = \mathbb{E}^x[u_0(X_T)]$  is approached by  $\mathbb{E}^x[u_0(\hat{X}_T^n)] = \mathbb{E}^x[u_0(n^{-1/2} S_{\lfloor nT \rfloor})]$ . We recall that the expectation symbols have to be understood as pseudo expectations, and that  $X$  is the pseudo SBM of Section 3.2.

Let  $T > 0$  and let us assume that  $nT = N$ , an integer. Denoting  $u_0^n$  the function defined by  $u_0^n(z) = u_0(n^{-1/2} z)$ ,  $z \in \mathbb{Z}$ , one has

$$\mathbb{E}^x[u_0(\hat{X}_T^n)] \approx \mathbb{E}^{\lfloor \sqrt{nx} \rfloor}[u_0^n(S_N)]$$

where the expectation in the right hand side is computed under  $\mathbb{P}^{\lfloor \sqrt{nx} \rfloor}$  s.t.  $\mathbb{P}^{\lfloor \sqrt{nx} \rfloor}(S_0 = \lfloor \sqrt{nx} \rfloor) = 1$ .

Let us denote  $v_N = u_0^n$ . Using the (pseudo) Markov property of the (pseudo) random walk  $S$  we get

$$\mathbb{E}^{\lfloor \sqrt{nx} \rfloor}[u_0^n(S_N)] = \mathbb{E}^{\lfloor \sqrt{nx} \rfloor}[\mathbb{E}[v_N(S_N) | S_0, \dots, S_{N-1}]] = \mathbb{E}^{\lfloor \sqrt{nx} \rfloor}[v_{N-1}(S_{N-1})] \quad (53)$$

where we have denoted  $v_{N-1}(z) = \mathbb{E}[v_N(S_N) | S_{N-1} = z]$ ,  $z \in \mathbb{Z}$ . In fact, defining more generally,

$$v_{m-1}(z) = \mathbb{E}[v_m(S_m) | S_{m-1} = z], \quad z \in \mathbb{Z}, \quad 1 \leq m \leq N,$$

and proceeding to further conditionings in (53) we get

$$\mathbb{E}^{\lfloor \sqrt{nx} \rfloor}[u_0^n(S_N)] = \mathbb{E}^{\lfloor \sqrt{nx} \rfloor}[v_{N-2}(S_{N-2})] = \dots = \mathbb{E}^{\lfloor \sqrt{nx} \rfloor}[v_0(S_0)] = v_0(\lfloor \sqrt{nx} \rfloor).$$

Note that from the (possibly pseudo) transition probabilities of the random walk  $S$  we have for any  $1 \leq m \leq N$ ,

$$v_{m-1}(z) = \frac{1}{2}[v_m(z+1) + v_m(z-1)]\mathbf{1}_{z \neq 0} + [\alpha v_m(z+1) + (1-\alpha)v_m(z-1)]\mathbf{1}_{z=0}, \quad z \in \mathbb{Z}.$$

To sum up, one may approach  $\mathbb{E}^x[u_0(\hat{X}_T^n)]$  by computing  $v_0(\lfloor \sqrt{nx} \rfloor)$  through the dynamical programming procedure

$$\begin{cases} v_N(z) & = u_0^n(z), & \forall z \in \mathbb{Z} \\ v_{m-1}(z) & = \frac{1}{2}[v_m(z+1) + v_m(z-1)]\mathbf{1}_{z \neq 0} \\ & \quad + [\alpha v_m(z+1) + (1-\alpha)v_m(z-1)]\mathbf{1}_{z=0}, & \forall z \in \mathbb{Z}, \forall 1 \leq m \leq N. \end{cases} \quad (54)$$

The algorithm (54) is written in a recursive form. We now rewrite it in an iterative form -using also a new set of notations, in order to stress the fact that it is very similar to an explicit finite difference scheme, with space step  $h = n^{-1/2}$  and time step  $\delta t = n^{-1}$ .

Let us consider the space grid  $\{x_j\}_{j \in \mathbb{Z}}$  defined by  $x_j = j/\sqrt{n}$  for any  $j \in \mathbb{Z}$ , and the scheme  $\{U_j^m\}$ , for  $j \in \mathbb{Z}$ ,  $0 \leq m \leq M$ , defined by

$$U_j^0 = u_0^n(j) = u_0(x_j), \quad j \in \mathbb{Z} \quad (55)$$

and, for  $0 \leq m \leq N - 1$ ,

$$U_j^{m+1} = \frac{1}{2}U_{j+1}^m + \frac{1}{2}U_{j-1}^m \quad \text{for } j \neq 0, \quad (56)$$

$$U_0^{m+1} = \alpha U_1^{m+1} + (1 - \alpha)U_{-1}^{m+1}. \quad (57)$$

It is obvious that (55)-(57) is equivalent to (54), in other words  $U_j^M = v_0(j)$  for any  $j \in \mathbb{Z}$ .

Let us explain briefly why we may interpret (55)-(57) as an explicit finite difference scheme. The simplest way is to examine the case  $k = 1$ . Then  $\alpha = \frac{1}{2}$  and (56)-(57) becomes

$$U_j^{m+1} = \frac{U_{j+1}^m + U_{j-1}^m}{2} \quad \text{for any } j \in \mathbb{Z}. \quad (58)$$

Besides (15) becomes simply the heat equation

$$\partial_t u(t, x) = \frac{1}{2}u''(t, x), \quad x \in \mathbb{R}, \quad t > 0 \quad (59)$$

$$u(0, x) = u_0(x) \quad x \in \mathbb{R}$$

Performing an explicit finite difference scheme with space step  $h$  and time step  $\delta t$  for Eq. (59) amounts to considering a space grid  $\{x_j^h\}_{j \in \mathbb{Z}}$  defined by  $x_j^h = jh$  for any  $j \in \mathbb{Z}$ , and to compute  $\{U_j^m\}$ , for  $j \in \mathbb{Z}$ ,  $0 \leq m \leq M$ , by

$$U_j^0 = u_0(x_j^h) \quad j \in \mathbb{Z}$$

and

$$\frac{U_j^{m+1} - U_j^m}{\delta t} = \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{2h^2} \quad \text{for any } j \in \mathbb{Z}$$

for any  $0 \leq m \leq N - 1$ . Taking  $h = n^{-1/2}$  and  $\delta t = n^{-1}$  (note that this corresponds to touching the bound giving the CFL condition, see e.g. [3]) we get (55) and (58). Therefore the interpretation. In fact it seems that the transition (pseudo) probabilities of the random walk  $S$  suggests how to take into account the transmission condition in (16) in a finite difference scheme for (15), leading to condition (57).

Note that by applying the scheme (55)-(57) we get for any  $j \in \mathbb{Z}$ , and any  $0 \leq m \leq N$  an approximation  $U_j^m$  of  $u(\frac{m}{n}, x_j = \frac{j}{\sqrt{n}})$ .

For computational purposes we have to consider a PDE problem with a bounded space domain, and this is our PDE of interest (2).

Firstly, the domain  $[-a, a]$  is discretized with a grid  $\{x_j\}_{j=-N_a}^{N_a}$  with  $N_a = \sqrt{n} a$  (we assume this quantity is an integer) and  $x_j = j/\sqrt{n}$  for  $-N_a \leq j \leq N_a$ .

Secondly, we have to adapt (55) and (57) to a bounded domain (see Algorithm 2 just hereafter) and thirdly, we have to take into account the Dirichlet boundary conditions by imposing  $U_{-N_a}^m = U_{N_a}^m = 0$  for  $1 \leq m \leq N$ .

**ALGORITHM 2:** Computation of  $\bar{u}_{RW}^n(t, x)$ .

**Parameters of the method:** A time horizon  $0 < T < \infty$  and a discretization order  $n \in \mathbb{N}^*$ .

We set  $N = nT$  and  $N_a = \sqrt{n}a$  and assume this quantities are integers.

We set  $x_j = j/\sqrt{n}$  for  $-N_a \leq j \leq N_a$ .

**Algorithm:** 1) Set  $U_j^0 = u_0(x_j)$  for any  $-N_a + 1 \leq j \leq N_a - 1$ .

2) For  $0 \leq m \leq N - 1$ , compute

$$\begin{aligned} U_j^{m+1} &= \frac{1}{2}U_{j+1}^m + \frac{1}{2}U_{j-1}^m \quad \text{for } -N_a + 1 \leq j \leq N_a - 1 \\ U_0^{m+1} &= \alpha U_1^{m+1} + (1 - \alpha)U_{-1}^{m+1} \end{aligned}$$

with the convention that  $U_{-N_a}^m = U_{N_a}^m = 0$ .

3) Return a piecewise constant function  $\bar{u}_{RW}^n(t, x)$  satisfying

$$\bar{u}_{RW}^n\left(\frac{m}{n}, x_j\right) = U_j^m, \quad \forall -N_a + 1 \leq j \leq N_a - 1, \quad \forall 0 \leq m \leq M,$$

and

$$\bar{u}_{RW}^n\left(\frac{m}{n}, \pm a\right) = 0, \quad \forall 1 \leq m \leq M.$$

It should be possible to adapt the results of Proposition 3.5 to prove convergence of the above scheme. This would require considering the trajectories of killed scaled pseudo asymmetric random walks, which presents difficulties we have decided not to address in the present paper.

Nevertheless, we suspect that the function  $\bar{u}_{RW}^n : [0, T] \times [-a, a] \rightarrow \mathbb{R}$  defined by Algorithm 2 should converge towards  $u$  the solution of (2):

$$\sup_{(t,x) \in [0,T] \times [-a,a]} |u(t, x) - \bar{u}_{RW}^n(t, x)| \xrightarrow[n \rightarrow \infty]{} 0. \quad (60)$$

This will be illustrated by the numerical experiments of Section 6.

### 5.3 Scheme inspired by the expression of the fundamental solution involving the transition function of the killed Brownian motion

Let  $u_0 \in C(I; \mathbb{R})$ . We denote  $u_0^+ = u_0 \mathbf{1}_{\mathbb{R}_+}$  and  $u_0^- = u_0 \mathbf{1}_{\mathbb{R}_-}$ . Let  $x \in (0, a)$ , from Lemma 4.4 and Eq. (44) we have

$$\begin{aligned} u(t, x) &= \int_{-a}^a u_0(y) \check{p}(t, x, y) dy \\ &= (1 - \beta) \int_{-a}^a u_0^-(y) p_W^{(-a,a)}(t, x, y) dy + (1 + \beta) \int_{-a}^a u_0^+(y) p_W^{(-a,a)}(t, x, y) dy \\ &\quad - \beta \int_0^a u_0^+(y) p_W^{(0,a)}(t, x, y) dy \\ &= (1 - \beta)u_1(t, x) + (1 + \beta)u_2(t, x) - \beta u_{3,+}(t, x), \end{aligned} \quad (61)$$

where the functions  $u_1$ ,  $u_2$  and  $u_{3,+}$  are respectively solution of the following heat equations:

$$\begin{cases} \partial_t u_1(t, x) = \frac{1}{2} \partial_{xx}^2 u_1(t, x), & x \in I, t > 0, \\ u_1(0, x) = u_0^-(x) & x \in I, \\ u_1(t, \pm a) = 0, & t > 0, \end{cases} \quad \begin{cases} \partial_t u_2(t, x) = \frac{1}{2} \partial_{xx}^2 u_2(t, x), & x \in I, t > 0, \\ u_2(0, x) = u_0^+(x) & x \in I, \\ u_2(t, \pm a) = 0, & t > 0, \end{cases}$$

and

$$\begin{cases} \partial_t u_{3,+}(t, x) & = \frac{1}{2} \partial_{xx}^2 u_{3,+}(t, x), & x \in (0, a), t > 0, \\ u_{3,+}(0, x) & = u_0^+(x) & x \in (0, a), \\ u_{3,+}(t, 0) = u_{3,+}(t, a) & = 0, & t > 0. \end{cases}$$

Indeed  $p_W^{(-a,a)}(t, x, y)$  (resp.  $p_W^{(0,a)}(t, x, y)$ ) may be viewed as the fundamental solution of the heat equation with half Laplacian on the domain  $(-a, a)$  (resp.  $(0, a)$ ), with homogeneous Dirichlet boundary conditions ([4], Appendix I, N° 6).

In the same manner, for  $x < 0$  we have

$$u(t, x) = (1 - \beta)u_1(t, x) + (1 + \beta)u_2(t, x) + \beta u_{3,-}(t, x), \quad (62)$$

with  $u_1$  and  $u_2$  as before and  $u_{3,-}$  the solution of

$$\begin{cases} \partial_t u_{3,-}(t, x) & = \frac{1}{2} \partial_{xx}^2 u_{3,-}(t, x), & x \in (-a, 0), t > 0, \\ u_{3,-}(0, x) & = u_0^-(x) & x \in (-a, 0), \\ u_{3,-}(t, -a) = u_{3,-}(t, 0) & = 0, & t > 0. \end{cases}$$

Our idea is to perform finite different schemes for  $u_1, u_2, u_{3,\pm}$  and to combine them through (61)(62) in order to get a scheme for the approximation of  $u$ . We sum up the procedure in Algorithm 3 where we use implicit finite different schemes, which are known to be unconditionally stable ([3]). In the present case they are also consistent and therefore convergent by Lax principle.

**ALGORITHM 3:** Computation of  $\bar{u}_{fund}^{h,N}(t, x)$ .

**Parameters of the method:** A time horizon  $0 < T < \infty$ .

A time discretization order  $N$  is given and we set  $\delta t = T/N$ .

A space step  $h$  is given and we set  $N_a = a/h$  (we assume this is an integer).

We set  $x_j = jh$  for  $-N_a \leq j \leq N_a$ .

**Algorithm:** 1) Set  $U_{1,j}^0 = u_0^-(x_j)$  for any  $-N_a + 1 \leq j \leq N_a - 1$ .

Set  $U_{2,j}^0 = u_0^+(x_j)$  for any  $-N_a + 1 \leq j \leq N_a - 1$ .

Set  $U_{3+,j}^0 = u_0^+(x_j)$  for any  $1 \leq j \leq N_a - 1$ .

Set  $U_{3-,j}^0 = u_0^-(x_j)$  for any  $-N_a + 1 \leq j \leq -1$ .

2) For  $0 \leq m \leq N - 1$ , compute the vectors  $U_{1,\pm}^{m+1}, U_{2,\pm}^{m+1}, U_{3\pm}^{m+1}$  by applying the implicit finite difference schemes

$$\begin{aligned} \frac{U_{1,j+1}^{m+1} - U_{1,j}^m}{\delta t} &= \frac{U_{1,j+1}^{m+1} - 2U_{1,j}^{m+1} + U_{1,j-1}^{m+1}}{2h^2} \quad \text{for } -N_a + 1 \leq j \leq N_a - 1 \\ \frac{U_{2,j+1}^{m+1} - U_{2,j}^m}{\delta t} &= \frac{U_{2,j+1}^{m+1} - 2U_{2,j}^{m+1} + U_{2,j-1}^{m+1}}{2h^2} \quad \text{for } -N_a + 1 \leq j \leq N_a - 1 \\ \frac{U_{3+,j+1}^{m+1} - U_{3+,j}^m}{\delta t} &= \frac{U_{3+,j+1}^{m+1} - 2U_{3+,j}^{m+1} + U_{3+,j-1}^{m+1}}{2h^2} \quad \text{for } 1 \leq j \leq N_a - 1 \\ \frac{U_{3-,j+1}^{m+1} - U_{3-,j}^m}{\delta t} &= \frac{U_{3-,j+1}^{m+1} - 2U_{3-,j}^{m+1} + U_{3-,j-1}^{m+1}}{2h^2} \quad \text{for } -N_a + 1 \leq j \leq -1 \end{aligned}$$

with the conventions that  $U_{1,\pm N_a}^{m+1} = U_{2,\pm N_a}^{m+1} = U_{3+,N_a}^{m+1} = U_{3+,0}^{m+1} = U_{3-,-N_a}^{m+1} = U_{3-,0}^{m+1} = 0$ .

3) Return a piecewise constant function  $\bar{u}_{fund}^{h,N}(t, x)$  satisfying

$$\bar{u}_{fund}^{h,N}(m \delta t, x_j) = (1 - \beta)U_{1,j}^m + (1 + \beta)U_{2,j}^m - \beta U_{3+,j}^m, \quad \forall 1 \leq j \leq N_a - 1, \quad \forall 0 \leq m \leq M,$$

$$\bar{u}_{fund}^{h,N}(m \delta t, x_j) = (1 - \beta)U_{1,j}^m + (1 + \beta)U_{2,j}^m + \beta U_{3-,j}^m, \quad \forall -N_a + 1 \leq j \leq -1, \quad \forall 0 \leq m \leq M,$$

$$\bar{u}_{fund}^{h,N}(m \delta t, 0) = (1 - \beta)U_{1,0}^m + (1 + \beta)U_{2,0}^m, \quad \forall 0 \leq m \leq M,$$

and

$$\bar{u}_{fund}^{h,N}(m \delta t, \pm a) = 0, \quad \forall 1 \leq m \leq M.$$

From the convergence of the finite difference schemes we immediately get the following convergence result.

**Proposition 5.2.** *Let us consider  $u$  the solution of (2). Let  $0 < T < \infty$  and let us consider for any  $N \in \mathbb{N}^*$  and any  $h \in (0, a)$  the function  $\bar{u}_{fund}^{h,N} : [0, T] \times [-a, a] \rightarrow \mathbb{R}$  defined by Algorithm 3. We have*

$$\sup_{(t,x) \in [0,T] \times [-a,a]} |u(t,x) - \bar{u}_{fund}^{h,N}(t,x)| \xrightarrow{h \downarrow 0, N \rightarrow \infty} 0.$$

**Remark 5.3.** In fact we could infer several other numerical schemes from Eq. (44). For example, consider  $x > 0$  and  $t > 0$ . Eq. (61) also implies that

$$u(t,x) = (1 - \beta) \mathbb{E}^x[u_0^-(W_t); t < \tau_{(-a,a)}] + (1 + \beta) \mathbb{E}^x[u_0^+(W_t); t < \tau_{(-a,a)}] - \beta \mathbb{E}^x[u_0^+(W_t); t < \tau_{(0,a)}].$$

So we could consider approaching each of the above expectations by Monte Carlo sums involving samples of independent Brownian motions and the corresponding stopping times  $\tau_{(-a,a)}$  or  $\tau_{(0,a)}$  (see e.g. [11]).

However, as the space dimension is one, we know that this Monte Carlo method would be slower than the finite difference approach described in Algorithm 3. Nevertheless such an approach could be interesting if we would address the problem in a space of higher dimension.

## 6 Numerical experiments

**Example 1.** We take  $I = (-1, 1)$  (i.e.  $a = 1$ ) and  $k = -0.5$ . We choose the following initial condition

$$u_0(x) = \frac{10x^3 - 3x^2 - 9x + 4}{2}, \quad \forall x \in I.$$

In order to use Algorithm 1 we have to compute the  $a_n$ 's and  $b_n$ 's, through Eq. (10). Remember that the quantities  $\int_I A(x)|f_n(x)|^2 dx$  and  $\int_I A(x)|g_n(x)|^2 dx$  are given by Eq. (11). Besides one can compute

$$\begin{aligned} & \int_{-1}^1 A(x) f_n(x) u_0(x) dx \\ &= \frac{k}{(2n-1)^4 \pi^4} (-528 (n-1/2) \pi (-1)^n - 480 - 72 (n-1/2)^2 \pi^2) \\ & \quad + \frac{1}{(2n-1)^4 \pi^4} \left( -528 \pi \left( 1/33 (n-1/2)^2 \pi^2 - \frac{9}{11} \right) (n-1/2) (-1)^n + 480 + 72 (n-1/2)^2 \pi^2 \right) \end{aligned}$$

and

$$\int_{-1}^1 A(x) g_n(x) u_0(x) dx = -\frac{k (-1)^n (\pi^2 n^2 - 60)}{\pi^3 n^3}.$$

Recalling that here  $\lambda_n = \frac{(2n-1)\pi}{2}$  and  $\mu_n = n\pi$ ,  $n \geq 1$ , we have everything at hand to perform Algorithm 1.

For performing Algorithms 2 and 3 no previous computation is needed.

Figure 1 represents the graphs of  $\bar{u}_{spec}^{200}(T, \cdot)$ ,  $\bar{u}_{RW}^{2.5 \times 10^5}(T, \cdot)$  and  $\bar{u}_{fund}^{2 \times 10^{-3}, 500}(T, \cdot)$  at  $T = 0.4$  with the following choices of parameters:  $N = 200$  for Algorithm 1,  $n = 2.5 \times 10^5$  for Algorithm 2, a space step  $h = 2 \times 10^{-3}$  and a time discretization order  $N = 500$  for Algorithm 3.

We see a very good concordance between the three methods (which actually can be observed for coarser discretizations).

In particular we can numerically check the convergence of the Algorithm 2 announced in Eq. (60). To that aim we consider  $\bar{u}_{spec}^{200}(T, \cdot)$  as the reference solution and report in Table 1 the value of

$$\sup_{x \in [-a,a]} |\bar{u}_{spec}^{200}(T, x) - \bar{u}_{RW}^n(T, x)|$$

for increasing values of  $n$ , up to  $n = 2.5 \times 10^5$ . Convergence is indeed observed.

On Figure 2 we check another interesting phenomenon. We plot the initial condition  $u_0$  and an approximation of  $u(t, \cdot)$  by  $\bar{u}_{fund}^{h,N}(t, \cdot)$  (we keep the same parameters  $h = 2 \times 10^{-3}$  and  $N = 500$  as

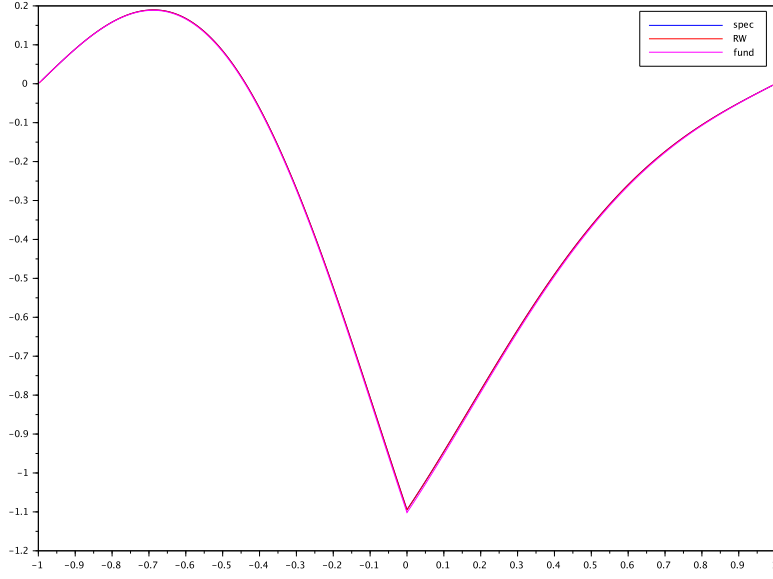


Figure 1: Plot of an approximation of the function  $u(T = 0.4, \cdot)$ , by  $\bar{u}_{spec}^{200}(T, \cdot)$ ,  $\bar{u}_{RW}^{2.5 \times 10^5}(T, \cdot)$  and  $\bar{u}_{fund}^{2 \times 10^{-3}, 500}(T, \cdot)$ , for the initial condition  $u_0(x) = (10x^3 - 3x^2 - 9x + 4)/2$ .

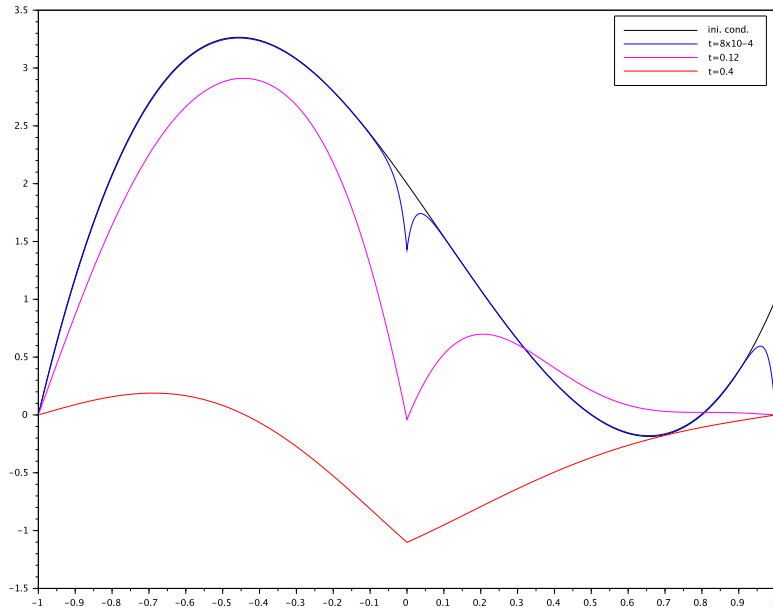


Figure 2: Plot of the initial condition  $u_0$  and of an approximation of the function  $u(t, \cdot)$ , by  $\bar{u}_{fund}^{2 \times 10^{-3}, 500}(t, \cdot)$ , at times  $t = 8 \times 10^{-4}$ ,  $t = 0.12$  and  $t = 0.4$ , for the initial condition  $u_0(x) = (10x^3 - 3x^2 - 9x + 4)/2$ .

n	$\sup_{x \in [-a, a]}  \bar{u}_{spec}^{200}(T, x) - \bar{u}_{RW}^n(T, x) $
100	$4.56 \times 10^{-2}$
625	$9.53 \times 10^{-3}$
$10^4$	$4.4 \times 10^{-4}$
$2.5 \times 10^5$	$1.7 \times 10^{-5}$

Table 1: Approximation error  $\sup_{x \in [-a, a]} |\bar{u}_{spec}^{200}(T, x) - \bar{u}_{RW}^n(T, x)|$  in function of  $n$  (the function  $\bar{u}_{spec}^{200}(T, \cdot)$  is considered as the benchmark).

previously) at times  $t = 8 \times 10^{-4}$ ,  $t = 0.12$  and  $t = 0.4$ . Note that  $t = 8 \times 10^{-4} = T/N$  corresponds to the first time step in the finite difference scheme.

Observe that the transmission condition destroys the regularity of the initial datum. Also observe that the slope of the graph of  $u(t, \cdot)$  is of negative sign at  $0-$ , but twice bigger in absolute value than the positive slope at  $0+$ . This is what we expect as  $k = -0.5$ .

**Example 2.** In this second example we keep  $I = (-1, 1)$  and  $k = -0.5$ , but choose the following initial condition

$$u_0(x) = \mathbf{1}_{x < 0} - 0.5, \quad \forall x \in I. \quad (63)$$

Indeed we want to test the robustness of our numerical schemes to a non smooth initial condition, especially if this initial condition presents a discontinuity at the interface point  $x = 0$ .

Of course for Algorithm 1 the initial condition can be taken in  $L^2(I)$  so one knows that the algorithm converges if we take  $u_0$  defined by (63).

The analysis of probabilistic schemes such as Algorithm 2 usually relies on arguments of convergence in pseudo law, which are usually valid only for smooth functions. We suspect that the smoothing properties of the operator  $\frac{1}{2A} \nabla \cdot (A \nabla)$  could be used to prove convergence of Algorithm 2, even for non smooth initial conditions.

In order to use Algorithm 1 we have to compute the  $a_n$ 's and  $b_n$ 's again, the  $\mu_n$ 's and  $\lambda_n$ 's remaining unchanged. Easy computations show that

$$a_n = -\frac{(-1)^n 2(k-1)}{(2n-1)\pi(k+1)}$$

and

$$b_n = \frac{4}{n\pi(k+1)} \mathbf{1}_{n \text{ is odd}},$$

for  $n \geq 1$ .

We take again an order of truncation  $N = 200$  for Algorithm 1.

Again for the Algorithm 2 we take a discretization order  $n = 2.5 \times 10^5$ , and for the Algorithm 3 we take a space step  $h = 2 \times 10^{-3}$  and a time discretization order  $N = 500$ .

Figure 3 depicts the graphs of  $\bar{u}_{spec}^{200}(T, \cdot)$ ,  $\bar{u}_{RW}^{2.5 \times 10^5}(T, \cdot)$  and  $\bar{u}_{fund}^{2 \times 10^{-3}, 500}(T, \cdot)$  for  $T = 0.4$  and again shows very good agreement between the results of the three schemes.

On Figure 4 we plot the initial condition  $u_0$  and an approximation of  $u(t, \cdot)$  by  $\bar{u}_{fund}^{h, N}(t, \cdot)$  (we keep the same parameters  $h = 2 \times 10^{-3}$  and  $N = 500$  as previously) at times  $t = 0.012$ ,  $t = 0.12$  and  $t = 0.4$ . In particular, the plot illustrates the fact that the solution  $u(t, x)$  does not satisfy a maximum principle, one of the main arguments classically used in the numerical analysis of deterministic finite difference schemes for parabolic equations.

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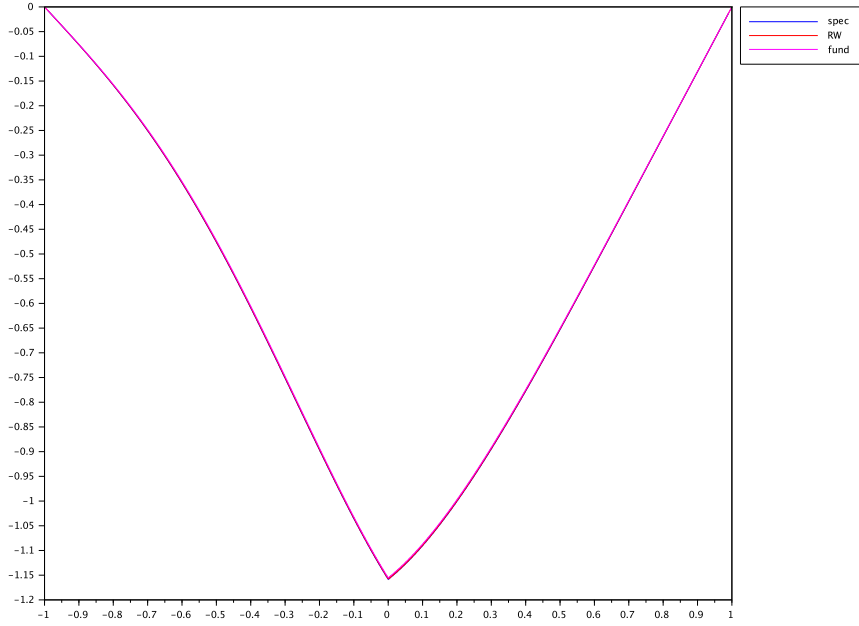


Figure 3: Plot of an approximation of the function  $u(T = 0.4, \cdot)$ , by  $\bar{u}_{spec}^{200}(T, \cdot)$ ,  $\bar{u}_{RW}^{2.5 \times 10^5}(T, \cdot)$  and  $\bar{u}_{fund}^{2 \times 10^{-3}, 500}(T, \cdot)$ , for the initial condition  $u_0(x) = \mathbf{1}_{x < 0} - 0.5$ .

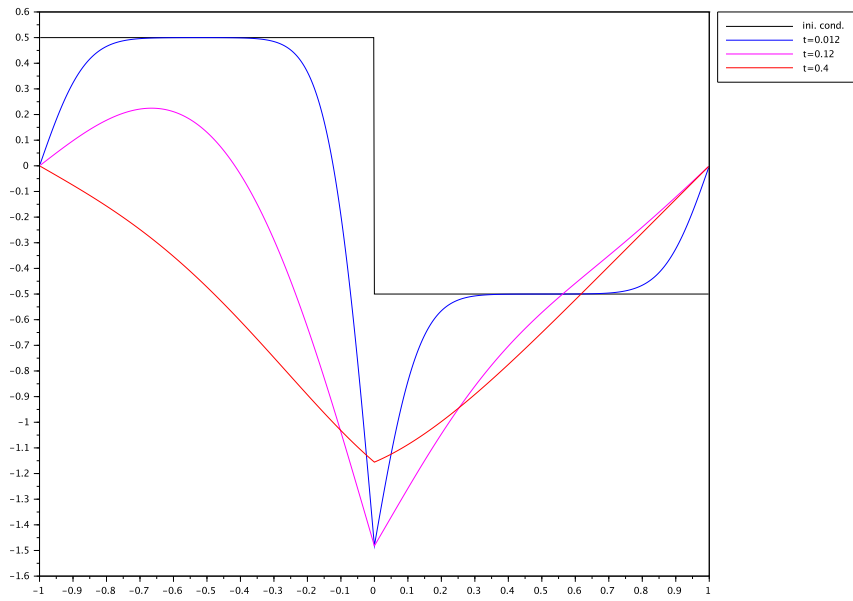


Figure 4: Plot of the initial condition  $u_0$  and of an approximation of the function  $u(t, \cdot)$ , by  $\bar{u}_{fund}^{2 \times 10^{-3}, 500}(t, \cdot)$ , at times  $t = 0.012$ ,  $t = 0.12$  and  $t = 0.4$ , for the initial condition  $u_0(x) = \mathbf{1}_{x < 0} - 0.5$ .

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