Localization of Two Radioactive Sources on the Plane

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Abstract

The problem of localization on the plane of two radioactive sources by K detectors is considered. Each detector records a realization of an inhomogeneous Poisson process whose intensity function is the sum of signals arriving from the sources and of a constant Poisson noise of known intensity. The time of the beginning of emission of the sources is known, and the main problem is the estimation of the positions of the sources. The properties of the maximum likelihood and Bayesian estimators are described in the asymptotics of large signals in three situations of different regularities of the fronts of the signals: smooth, cusp-type and change-point type.

MSC 2000 Classification: 62M02, 62G10, 62G20.

Key words: *Poisson process, parameter estimation, cusp-type singularity.*

1 Introduction

Suppose that there are two radioactive sources and K detectors on the plane. The sources start emitting at a known time, which can be taken $t = 0$ without loss of generality. The detectors receive Poisson signals with additive noise. The intensity functions of these processes depend on the positions of the detectors (known) and the positions of the sources (unknown), and the main problem is the estimation of the positions of the sources. An example of a possible configuration of the sources and of the detectors on the plane is given in Fig. 1.

Due to their practical importance, the problems of localization of the sources with Poisson, Gaussian or more general classes of distributions are widely studied in engineering

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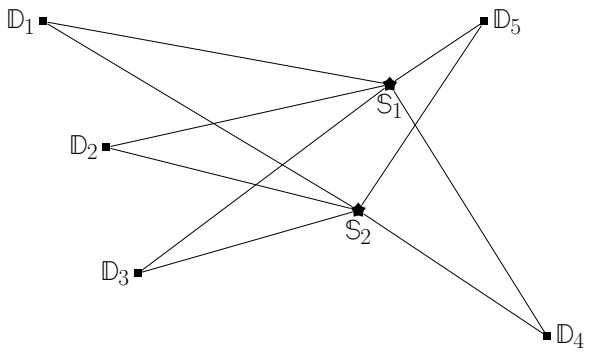

Figure 1: Model of observations: $\mathbb{S}_1$, $\mathbb{S}_2$ are the positions of the sources and $\mathbb{D}_i$, $i = 1, \dots, 5$, are the positions of the sensors

literature, see, e.g., [2,3,8,12,17–20], as well as the Handbook [10] and the references therein. Mathematical statements are less known. This work is a continuation of a study initiated in [9] and then developed in [1,4–6,15], where it was always supposed that there is only one source on the plane. In the works [5,6,9] it was supposed that the moment of the beginning of the emission τ_0 is known and the unknown parameter was just the position of the source. The case where τ_0 is unknown too and we have to estimate both τ_0 and the position of the source was treated in [1, 4]. In all these works the properties of the maximum likelihood estimator (MLE) and of the Bayesian estimators (BEs) of the position (or of τ_0 and of the position) where described. Moreover, the properties of the least squares estimators where equally described. The inhomogeneous Poisson processes and the diffusion processes were considered as models of observations. The properties of the MLE and of the BEs were described in the asymptotics of large signals or in the asymptotics of small noise.

In the present work we suppose that there are K detectors and two radioactive sources emitting signals which can be described as inhomogeneous Poisson processes and the emission starts at the (known) moment $t = 0$. So, we need to estimate the positions of the sources only. As in all preceding works, the properties of the estimators are described with the help of the Ibragimov-Khasminskii approach (see [11]), which consists in the verification of certain properties of the normalized likelihood ratio process.

The information about the positions of the sources is contained in the times of arrival of the signals to the detectors. These times depend on the distances between the sources and the detectors. The estimation of the positions depend on the estimation of these times, and here the form of the fronts of the arriving signals plays an important role. We consider three types of fronts: smooth, cusp-type and change-point type. In the smooth case the Fisher information matrix is finite and the estimators are asymptotically normal. In the cusp-case the fronts are described by a function which is continuous, but the Fisher information does not exist (is infinite). For Poisson processes, statisticaal problems with such type singularities was first considered in [7]. In this case the Bayesian estimators converge to a random vector defined with the help of some functionals of fractional Brownian motions.

In the change-point case with discontinuous intensities the limit distribution of Bayesian estimators is defined with the help of some Poisson processes. In all the three cases we discuss the asymptotic efficiency of the proposed estimators.

Special attention is paid to the condition of identifiability, i.e., to the description of the admissible configurations of detectors which allow the consistent estimation of the positions of the sources. It is shown that if the detectors do not lay on a cross, then it is impossible to find two different pairs of sources which provide the same moments of arrival to the detectors and hence the consistent estimation is possible.

2 Main results

2.1 Model of observations

We suppose that there are K detectors $\mathbb{D}_1, \dots, \mathbb{D}_K$ located on the plane at the (known) points $D_k = (x_k, y_k)^\top$, $k = 1, \dots, K$ ($K \geqslant 4$), and two sources $\mathbb{S}_1$ and $\mathbb{S}_2$ located at the (unknown) points $S_1 = (x_1', y_1')^\top$ and $S_2 = (x_2', y_2')^\top$. Therefore, the unknown parameter is $\vartheta = (x_1', y_1', x_2', y_2')^\top$, but it will be convenient to write it as $\vartheta = (\vartheta_1, \dots, \vartheta_4)^\top \in \Theta$ and as $\vartheta = (\vartheta^{(1)}, \vartheta^{(2)})^\top$ with obvious notations. For the true value of ϑ we will often use the notations $\vartheta_0 = (\vartheta_0^{(1)}, \vartheta_0^{(2)})^\top$, $\vartheta_0^{(1)} = (\vartheta_1^\circ, \vartheta_2^\circ)^\top = (x_1'^\circ, y_1'^\circ)^\top$, $\vartheta_0^{(2)} = (\vartheta_3^\circ, \vartheta_4^\circ)^\top = (x_2'^\circ, y_2'^\circ)^\top$. The set Θ is an open, bounded, convex subset of $\mathcal{R}^4$. We suppose, of course, that the positions of the detectors are all different. We suppose as well that a position of a source does not coincide with a position of a detector.

The sources start emitting at the moment $t = 0$. The k-th detector records a realization $X_k = (X_k(t),\ 0 \leqslant t \leqslant T)$ of an inhomogeneous Poisson process of intensity function

$$\lambda_{k,n}(\vartheta, t) = nS_{1,k}(\vartheta^{(1)}, t) + nS_{2,k}(\vartheta^{(2)}, t) + n\lambda_0, \qquad 0 \leqslant t \leqslant T,$$

where $n\lambda_0 > 0$ is the intensity of the Poisson noise and $nS_{i,k}(\vartheta^{(i)}, t)$ is the signal recorded from the i-th source, $i = 1, 2$. We suppose that the recorded signals have the following structure

$$S_{i,k}(\vartheta^{(i)}, t) = \psi\big(t - \tau_k(\vartheta^{(i)})\big) S_{i,k}(t), \tag{1}$$

where $S_{i,k}(t) > 0$ is a bounded function and $\tau_k(\vartheta^{(i)})$ is the time of the arrival of the signal from the i-th source to the k-th detector, i.e.,

$$\tau_k(\vartheta^{(i)}) = \nu^{-1} \|D_k - S_i\|_2 = \nu^{-1} \big(|x_k - x_i'|^2 + |y_k - y_i'|^2\big)^{1/2}, \qquad i = 1, 2,$$

and $\nu > 0$ is the rate of the propagation of the signals. Here and in the sequel, we denote $\|\cdot\|_2$ and $\|\cdot\|_4$ the Euclidean norms in $\mathcal{R}^2$ and $\mathcal{R}^4$ respectively.

The function $\psi(\cdot)$ describes the fronts of the signals. As in our preceding works (see, e.g., [4]) we take

$$\psi(s) = \Big|\frac{s}{\delta}\Big|^\kappa \mathbb{1}_{\{0 \leqslant s \leqslant \delta\}} + \mathbb{1}_{\{s > \delta\}},$$

where $\delta > 0$ and the parameter $\kappa \geqslant 0$ describes the regularity of the statistical problem. If $\kappa \geqslant \frac{1}{2}$, we have a regular statistical experiment, if $\kappa \in \left(0, \frac{1}{2}\right)$, we have a singularity of cusp type, and if $\kappa = 0$, the intensity is a discontinuous function and we have a change-point model. The examples of these three cases are given in Fig. 2, where we put $nS_{i,k}(t) \equiv 2$, $n\lambda_0 = 1$ and **a)** $\kappa = 1$, **b)** $\kappa = 1/4$, **c)** $\kappa = 0$.

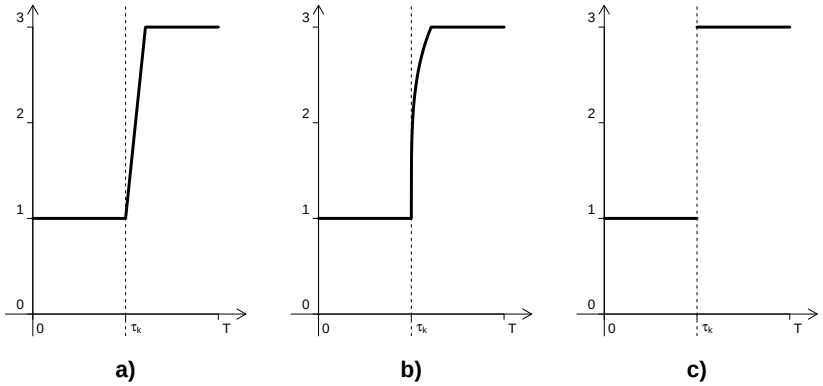

Figure 2: Intensities with three types of fronts of arriving signals

Note that $\psi(s) = 0$ for $s < 0$, and therefore for $t < \tau_k\big(\vartheta^{(1)}\big) \wedge \tau_k\big(\vartheta^{(2)}\big)$ the intensity function is $\lambda_{k,n}(\vartheta, t) = n\lambda_0$. According to the form of the intensity function, all the information concerning the positions of the sources is clearly contained in the moments of arrival $\tau_k(\vartheta^{(i)})$, $i = 1, 2$, $k = 1, \ldots, K$.

We are interested in the situation where the errors of estimation are small. In the problem of localization it is natural to suppose that the registered intensities take large values. Therefore we study the properties of the estimators of the positions in the asymptotics of large intensities, that is why we introduce in the intensity functions the factor n and the asymptotics corresponds to the limit $n \to \infty$. In Section 3 we explain that the condition $K \geqslant 4$ is necessary for the existence of consistent estimators.

2.2 Maximum likelihood and Bayesian estimators

As the intensity functions $\lambda_{k,n}(\cdot)$ are bounded and separated from zero, the measures $\mathbf{P}_\vartheta^{(n)}$, $\vartheta \in \Theta$, induced by the observations $X^K = (X_1, \ldots, X_K)$ on the space of the realizations are equivalent, and the likelihood ratio function is

$$L\big(\vartheta, X^K\big) = \exp\left\{\sum_{k=1}^K \int_0^T \ln \frac{\lambda_{k,n}(\vartheta, t)}{n\lambda_0}\, \mathrm{d}X_k(t) - \sum_{k=1}^K \int_0^T [\lambda_{k,n}(\vartheta, t) - n\lambda_0]\mathrm{d}t\right\}, \quad \vartheta \in \Theta$$

The MLE $\hat{\vartheta}_n$ and the BEs for quadratic loss function $\tilde{\vartheta}_n$ are defined by the usual

relations

$$L\big(\hat{\vartheta}_n, X^K\big) = \sup_{\vartheta\in\Theta} L\big(\vartheta, X^K\big), \qquad\qquad \tilde{\vartheta}_n = \frac{\int_\Theta \vartheta\, p(\vartheta) L\big(\vartheta, X^K\big)\,\mathrm{d}\vartheta}{\int_\Theta p(\vartheta) L\big(\vartheta, X^K\big)\,\mathrm{d}\vartheta},$$

where $p(\vartheta)$, $\vartheta\in\Theta$, is a strictly positive and continuous *a priori* density of the (random) parameter ϑ.

2.2.1 Smooth fronts

First we consider the regular case in a slightly different setup, which can be seen as more general. The intensities of the observed processes are supposed to be

$$\lambda_{k,n}(\vartheta,t) = nS_{1,k}(\vartheta^{(1)},t) + nS_{2,k}(\vartheta^{(2)},t) + n\lambda_0 = n\lambda_k(\vartheta,t), \qquad 0\leqslant t\leqslant T,$$

where $\lambda_k(\vartheta,t)$ is defined by the last equality and $S_{i,k}(\vartheta^{(i)},t) = s_{i,k}\big(t-\tau_k(\vartheta^{(i)})\big)$, $i=1,2$, $k=1,\dots,K$.

For the derivatives we have the expressions

$$\frac{\partial s_{1,k}\big(t-\tau_k(\vartheta^{(1)})\big)}{\partial\vartheta_1} = \nu^{-1}s'_{1,k}\big(t-\tau_k(\vartheta^{(1)})\big)\,\frac{x_k-x'_1}{\rho_{1,k}} = \nu^{-1}s'_{1,k}\big(t-\tau_k(\vartheta^{(1)})\big)\,\cos(\alpha_{1,k}),$$

where $\rho_{1,k} = \|D_k - S_1\|_2$ and $\cos(\alpha_{1,k}) = (x_k-x'_1)\rho_{1,k}^{-1}$. Similarly, we obtain

$$\frac{\partial s_{1,k}\big(t-\tau_k(\vartheta^{(1)})\big)}{\partial\vartheta_2} = \nu^{-1}s'_{1,k}\big(t-\tau_k(\vartheta^{(1)})\big)\,\sin(\alpha_{1,k})$$

and, of course, we have $\partial s_{1,k}/\partial\vartheta_3 = \partial s_{1,k}/\partial\vartheta_4 = 0$. Let us recall here that we use the notations $\vartheta = (\vartheta^{(1)},\vartheta^{(2)})^\top$, $\vartheta^{(1)} = (\vartheta_1,\vartheta_2)^\top$ and $\vartheta^{(2)} = (\vartheta_3,\vartheta_4)^\top$. For $\partial s_{2,k}/\partial\vartheta_3$ and $\partial s_{2,k}/\partial\vartheta_4$ we obtain similar expressions:

$$\frac{\partial s_{2,k}\big(t-\tau_k(\vartheta^{(2)})\big)}{\partial\vartheta_3} = \nu^{-1}s'_{2,k}\big(t-\tau_k(\vartheta^{(2)})\big)\,\cos(\alpha_{2,k}),$$
$$\frac{\partial s_{2,k}\big(t-\tau_k(\vartheta^{(2)})\big)}{\partial\vartheta_4} = \nu^{-1}s'_{2,k}\big(t-\tau_k(\vartheta^{(2)})\big)\,\sin(\alpha_{2,k}).$$

Let us now introduce the two vectors $m_{1,k} = m_{1,k}(\vartheta^{(1)}) = \big(\cos(\alpha_{1,k}),\sin(\alpha_{1,k})\big)^\top$ and $m_{2,k} = m_{2,k}(\vartheta^{(2)}) = \big(\cos(\alpha_{2,k}),\sin(\alpha_{2,k})\big)^\top$. We have $\|m_{i,k}\|_2 = 1$, $i=1,2$. We will several times use the expansion (below $u^{(1)} = (u_1,u_2)^\top$, $u^{(2)} = (u_3,u_4)^\top$ and $\varphi_n\to 0$)

$$\tau_k(\vartheta_0^{(i)} + u^{(i)}\varphi_n) = \tau_k(\vartheta_0^{(i)}) - \nu^{-1}\langle m^\circ_{i,k}, u^{(1)}\rangle\,\varphi_n + O(\varphi_n^2),$$

where $m^\circ_{i,k} = m_{i,k}(\vartheta_0^{(i)}) = \big(\cos(\alpha^\circ_{1,k}),\sin(\alpha^\circ_{1,k})\big)^\top$.

For simplicity of exposition we will sometimes use the notations $\tau_{1,k} = \tau_k(\vartheta^{(1)})$ and $\tau_{2,k} = \tau_k(\vartheta^{(2)})$.

The Fisher information matrix is

$$\mathrm{I}(\vartheta)_{4\times 4} = \sum_{k=1}^{K} \int_0^T \frac{\dot\lambda_k(\vartheta,t)\,\dot\lambda_k(\vartheta,t)^\top}{\lambda_k(\vartheta,t)}\,\mathrm{d}t.$$

Here and in the sequel dot means derivative w.r.t. ϑ. The elements of this matrix have the following expressions

$$\begin{aligned}
\mathrm{I}(\vartheta)_{11} &= \sum_{k=1}^{K} \int_{\tau_{1,k}} \frac{s'^2_{1,k}(t-\tau_{1,k})\cos^2(\alpha_{1,k})}{\nu^2\lambda_k(\vartheta,t)}\,\mathrm{d}t,\\
\mathrm{I}(\vartheta)_{12} &= \sum_{k=1}^{K} \int_{\tau_{1,k}} \frac{s'^2_{1,k}(t-\tau_{1,k})\cos(\alpha_{1,k})\sin(\alpha_{1,k})}{\nu^2\lambda_k(\vartheta,t)}\,\mathrm{d}t,\\
\mathrm{I}(\vartheta)_{13} &= \sum_{k=1}^{K} \int_{\tau_{1,k}\vee\tau_{2,k}} \frac{s'_{1,k}(t-\tau_{1,k})\,s'_{2,k}(t-\tau_{2,k})\cos(\alpha_{1,k})\cos(\alpha_{2,k})}{\nu^2\lambda_k(\vartheta,t)}\,\mathrm{d}t,\\
\mathrm{I}(\vartheta)_{14} &= \sum_{k=1}^{K} \int_{\tau_{1,k}\vee\tau_{2,k}} \frac{s'_{1,k}(t-\tau_{1,k})\,s'_{2,k}(t-\tau_{2,k})\cos(\alpha_{1,k})\sin(\alpha_{2,k})}{\nu^2\lambda_k(\vartheta,t)}\,\mathrm{d}t.
\end{aligned}$$

The other terms can be written in a similar way.

The regularity conditions are:

Conditions $\mathscr{R}$.

$\mathscr{R}_1$. *For all $i = 1,2$ and $k = 1,\dots,K$, the functions $s_{i,k}(t) = 0$ for $t \leqslant 0$ and $s_{i,k}(t) > 0$ for $t > 0$.*

$\mathscr{R}_2$. *The functions $s_{i,k}(\cdot) \in \mathcal{C}^2$, $i = 1,2$, $k = 1,\dots,K$. The set $\Theta \subset \mathcal{R}^4$ is open, bounded and convex.*

$\mathscr{R}_3$. *The Fisher information matrix is uniformly non degenerate*

$$\inf_{\vartheta\in\Theta} \inf_{\|e\|_4=1} e^\top \mathrm{I}(\vartheta)\, e > 0,$$

where $e \in \mathcal{R}^4$.

$\mathscr{R}_4$. *(**Identifiability**) For any $\varepsilon > 0$, we have*

$$\inf_{\vartheta_0\in\Theta} \inf_{\|\vartheta-\vartheta_0\|_4>\varepsilon} \sum_{k=1}^{K} \int_0^T \left[S_{1,k}(\vartheta^{(1)},t) + S_{2,k}(\vartheta^{(2)},t) - S_{1,k}(\vartheta_0^{(1)},t) - S_{2,k}(\vartheta_0^{(2)},t)\right]^2 \mathrm{d}t > 0.$$

Let us note that in the setup of this section, the identifiability condition rewrites as follows:

$$\inf_{\vartheta_0\in\Theta}\inf_{\|\vartheta-\vartheta_0\|_4>\varepsilon}\sum_{k=1}^{K}\int_0^T\left[s_{1,k}(t-\tau_{1,k})+s_{2,k}(t-\tau_{2,k})-s_{1,k}(t-\tau^\circ_{1,k})-s_{2,k}(t-\tau^\circ_{2,k})\right]^2\mathrm{d}t>0, \quad (2)$$

where $\tau^\circ_{i,k}=\tau_k(\vartheta_0^{(i)})$, $i=1,2$.

It can be shown that if the conditions $\mathscr{R}_1$–$\mathscr{R}_3$ are fulfilled, the family of measures $\big(\mathbf{P}_\vartheta^{(n)},\vartheta\in\Theta\big)$ is locally asymptotically normal (LAN) (see Lemma 2.1 of [14]), and therefore we have the Hajek-Le Cam's lower bound on the risks of an arbitrary estimator $\bar\vartheta_n$ (see, e.g., [11])

$$\lim_{\varepsilon\to0}\varliminf_{n\to\infty}\sup_{\|\vartheta-\vartheta_0\|_4\leqslant\varepsilon}n\,\mathbf{E}_\vartheta\|\bar\vartheta_n-\vartheta\|_4^2\geqslant\mathbf{E}_{\vartheta_0}\|\zeta\|_4^2,\qquad\zeta\sim\mathcal{N}\big(0,\mathrm{I}(\vartheta_0)^{-1}\big). \quad (3)$$

The asymptotically efficient estimator is defined as an estimator for which there is equality in the inequality (3) for all $\vartheta_0\in\Theta$.

Theorem 1. *Let the conditions $\mathscr{R}$ be fulfilled. Then, uniformly on compacts $\mathbb{K}\subset\Theta$, the MLE $\hat\vartheta_n$ and the BEs $\tilde\vartheta_n$ are consistent and asymptotically normal:*

$$\sqrt{n}\,\big(\hat\vartheta_n-\vartheta_0\big)\Longrightarrow\zeta,\qquad\qquad\sqrt{n}\,\big(\tilde\vartheta_n-\vartheta_0\big)\Longrightarrow\zeta,$$

the polynomial moments converge: for any $p>0$, it holds

$$n^{p/2}\,\mathbf{E}_{\vartheta_0}\|\hat\vartheta_n-\vartheta_0\|_4^p\to\mathbf{E}_{\vartheta_0}\|\zeta\|_4^p,\qquad n^{p/2}\,\mathbf{E}_{\vartheta_0}\|\tilde\vartheta_n-\vartheta_0\|_4^p\to\mathbf{E}_{\vartheta_0}\|\zeta\|_4^p,$$

and both the MLE and the BEs are asymptotically efficient.

Proof. This theorem is a particular case of Theorems 2.4 and 2.5 of [14] (see as well [13]). Note that the model of observations with large intensity asymptotics is equivalent to the model of $n\to\infty$ independent identically distributed inhomogeneous Poisson processes. To verify the condition **B4** of the Theorem 2.4, for $\|\vartheta-\vartheta_0\|_4\leqslant\varepsilon$ and sufficiently small $\varepsilon>0$ we can write

$$\begin{aligned}\sum_{k=1}^{K}\int_0^T\Big[\sqrt{\lambda_k(\vartheta,t)}-\sqrt{\lambda_k(\vartheta_0,t)}\Big]^2\mathrm{d}t&=\frac14\,(\vartheta-\vartheta_0)^\top\mathrm{I}(\vartheta_0)\,(\vartheta-\vartheta_0)\,\big(1+O(\varepsilon)\big)\\&\geqslant\frac18\,(\vartheta-\vartheta_0)^\top\mathrm{I}(\vartheta_0)\,(\vartheta-\vartheta_0)=\frac18\,\|\vartheta-\vartheta_0\|_4^2\,e^\top\mathrm{I}(\vartheta_0)\,e\geqslant\kappa_1\|\vartheta-\vartheta_0\|_4^2.\end{aligned} \quad (4)$$

Here we used the condition $\mathscr{R}_3$.

For $\|\vartheta-\vartheta_0\|_4\geqslant\varepsilon$, we have

$$\begin{aligned}\sum_{k=1}^{K}\int_0^T\Big[\sqrt{\lambda_k(\vartheta,t)}-\sqrt{\lambda_k(\vartheta_0,t)}\Big]^2\mathrm{d}t&=\sum_{k=1}^{K}\int_0^T\frac{\big[\lambda_k(\vartheta,t)-\lambda_k(\vartheta_0,t)\big]^2}{\Big[\sqrt{\lambda_k(\vartheta,t)}+\sqrt{\lambda_k(\vartheta_0,t)}\Big]^2}\,\mathrm{d}t\\&\geqslant C\sum_{k=1}^{K}\int_0^T\big[\lambda_k(\vartheta,t)-\lambda_k(\vartheta_0,t)\big]^2\,\mathrm{d}t\end{aligned}$$

$$\geqslant C\sum_{k=1}^{K}\int_0^T\left[s_{1,k}(t-\tau_{1,k})+s_{2,k}(t-\tau_{2,k})-s_{1,k}(t-\tau^\circ_{1,k})-s_{1,k}(t-\tau^\circ_{1,k})\right]^2\mathrm{d}t$$
$$\geqslant Cg(\varepsilon).$$

Here we used the boundedness of the functions $\lambda_k(\vartheta,t)$ and denoted $g(\varepsilon)>0$ the left hand side of (2). Let us denote $D(\Theta)=\sup_{\vartheta,\tilde\vartheta\in\Theta}\|\vartheta-\tilde\vartheta\|_4$. Then we have

$$\sum_{k=1}^{K}\int_0^T\left[\sqrt{\lambda_k(\vartheta,t)}-\sqrt{\lambda_k(\vartheta_0,t)}\right]^2\mathrm{d}t\geqslant Cg(\varepsilon)\geqslant Cg(\varepsilon)\,\frac{\|\vartheta-\vartheta_0\|_4^2}{D(\Theta)^2}\geqslant\kappa_2\|\vartheta-\vartheta_0\|_4^2. \tag{5}$$

The estimates (4) and (5) can be joined in

$$\sum_{k=1}^{K}\int_0^T\left[\sqrt{\lambda_k(\vartheta,t)}-\sqrt{\lambda_k(\vartheta_0,t)}\right]^2\mathrm{d}t\geqslant\kappa\|\vartheta-\vartheta_0\|_4^2, \tag{6}$$

where $\kappa=\kappa_1\wedge\kappa_2$. Now **B4** follows from (6). □

2.2.2 Cusp type fronts

Let us now turn to the intensity function with cusp-type singularity. Suppose that the observed Poisson processes $X^K=(X_k(t),\ 0\leqslant t\leqslant T,\ k=1,\dots,K)$ have intensity functions

$$\lambda_{k,n}(\vartheta,t)=n\psi_{\kappa,\delta}(t-\tau_{1,k})S_{1,k}(t)+n\psi_{\kappa,\delta}(t-\tau_{2,k})S_{2,k}(t)+n\lambda_0=n\lambda_k(\vartheta,t), \tag{7}$$

where $\kappa\in\left(0,\frac12\right)$ and

$$\psi_{\kappa,\delta}(t-\tau_{i,k})=\left|\frac{t-\tau_{i,k}}{\delta}\right|^\kappa\mathbb{1}_{\{0\leqslant t-\tau_{i,k}\leqslant\delta\}}+\mathbb{1}_{\{t-\tau_{i,k}>\delta\}},\qquad i=1,2,\quad k=1,\dots,K. \tag{8}$$

Note that the intensity function of the Poisson process recorded by one detector has two cusp-type singularities. An example of such an intensity function is given in Fig. 3.

Recall that

$$\begin{aligned}&\nu\left[\tau_k(\vartheta_u^{(1)})-\tau_k(\vartheta_0^{(1)})\right]\\&\quad=\left[(x_k-x_1'^\circ-u_1\varphi_n)^2+(y_k-y_1'^\circ-u_2\varphi_n)^2\right]^{1/2}-\left[(x_k-x_1'^\circ)^2+(y_k-y_1'^\circ)^2\right]^{1/2}\\&\quad=\left(\rho_{1,k}-u_1\varphi_n\cos(\alpha^\circ_{1,k})-u_2\varphi_n\sin(\alpha^\circ_{1,k})\right)\left(1+O(\varphi_n)\right)-\rho_{1,k}\\&\quad=-\langle u^{(1)},m^\circ_{1,k}\rangle\,\varphi_n\left(1+O(\varphi_n)\right)\end{aligned}$$

and $\nu\left[\tau_k(\vartheta_u^{(2)})-\tau_k(\vartheta_0^{(2)})\right]=-\langle u^{(2)},m^\circ_{2,k}\rangle\,\varphi_n\left(1+O(\varphi_n)\right)$.

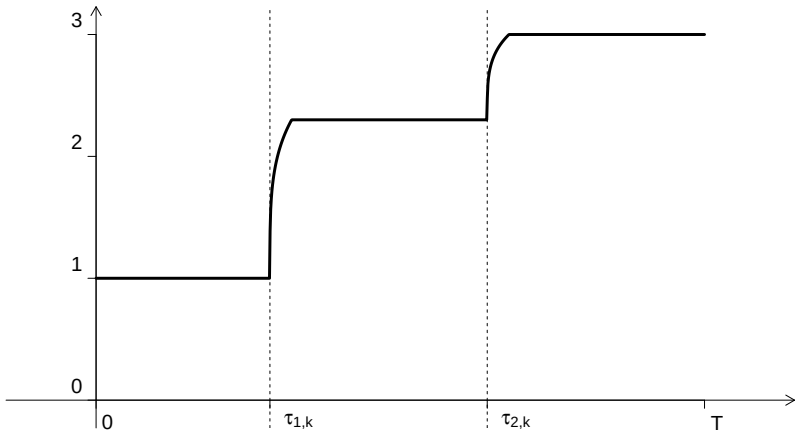

Figure 3: Intensity with two cusp type singularities

Introduce the notations

$$\mathbb{B}_{i,k} = \big(u^{(i)} \,:\, \langle m^{\circ}_{i,k}, u^{(i)}\rangle < 0\big), \qquad \mathbb{B}^{c}_{i,k} = \big(u^{(i)} :\, \langle m^{\circ}_{i,k}, u^{(i)}\rangle \geqslant 0\big), \quad i = 1, 2,$$

$$I_{i,k}(u^{(i)}) = \hat{\Gamma}_{i,k} \int_{\mathcal{R}} \Big[\big| v + \nu^{-1} \langle u^{(i)}, m^{\circ}_{i,k}\rangle \big|^{\kappa} \mathbb{1}_{\{v \geqslant -\nu^{-1}\langle u^{(i)}, m^{\circ}_{i,k}\rangle\}} - v^{\kappa} \mathbb{1}_{\{v>0\}} \Big] \, \mathrm{d}W_{i,k}(v),$$

$$\Gamma^2_{1,k} = \frac{S^2_{1,k}(\tau^{\circ}_{i,k})}{\delta^{2\kappa}\, \lambda_k(\vartheta_0, \tau^{\circ}_{i,k})\, \nu^{2\kappa+1}}\,, \qquad \Gamma^2_{2,k} = \frac{S^2_{2,k}(\tau^{\circ}_{i,k})}{\delta^{2\kappa}\, \lambda_k(\vartheta_0, \tau^{\circ}_{i,k})\, \nu^{2\kappa+1}}\,, \tag{9}$$

$$\hat{\Gamma}_{1,k} = \frac{S_{1,k}(\tau^{\circ}_{i,k})}{\delta^{\kappa}\sqrt{\lambda_k(\vartheta_0, \tau^{\circ}_{i,k})}}\,, \qquad \hat{\Gamma}_{2,k} = \frac{S_{2,k}(\tau^{\circ}_{i,k})}{\delta^{\kappa}\sqrt{\lambda_k(\vartheta_0, \tau^{\circ}_{i,k})}}\,,$$

$$Q^2_{\kappa} = \int_{\mathcal{R}} \big[|v-1|^{\kappa} \mathbb{1}_{\{v \geqslant 1\}} - v^{\kappa} \mathbb{1}_{\{v>0\}} \big]^2 \, \mathrm{d}v, \tag{10}$$

$$Z_{(k)}(u) = \exp\left(\sum_{i=1}^{2} \left[\hat{\Gamma}_{i,k}\, I_{i,k}(u^{(i)}) - \frac{\Gamma^2_{i,k} Q^2_{\kappa}}{2} \, \big| \langle u^{(i)}, m^{\circ}_{i,k}\rangle \big|^{2\kappa+1} \right] \right),$$

$$Z(u) = \prod_{k=1}^{K} Z_{(k)}(u), \qquad \xi = \frac{\int_{\mathcal{R}^4} u Z(u)\, \mathrm{d}u}{\int_{\mathcal{R}^4} Z(u)\, \mathrm{d}u}\,.$$

Here $W_{i,k}(v)$, $v \in \mathcal{R}$, $i = 1, 2$ are two-sided Wiener processes, i.e., $W_{i,k}(v) = W^{+}_{i,k}(v)$, $v \geqslant 0$, and $W_{i,k}(v) = W^{-}_{i,k}(-v)$, $v \leqslant 0$, where $W^{\pm}_{i,k}(\cdot)$ are independent Wiener processes.

Conditions $\mathscr{C}$.

$\mathscr{C}_1$. *The intensities of the observed processes are given by* (7)–(8)*, where the functions* $S_{i,k}(y) \in \mathcal{C}^1$ *and are positive.*

$\mathscr{C}_2$. *The configuration of the detectors and the set* Θ *are such that all signals from the both sources arrive at the detectors during the period* $[0, T]$.

$\mathscr{C}_3$. *The condition $\mathscr{R}_4$ is fulfilled.*

We have the following lower bound on the risks of an arbitrary estimator $\bar{\vartheta}_n$ of the positions of sources:

$$\lim_{\varepsilon\to 0}\varliminf_{n\to\infty}\sup_{\|\vartheta-\vartheta_0\|_4\leqslant\varepsilon} n^{\frac{2}{2\kappa+1}}\,\mathbf{E}_\vartheta\|\bar{\vartheta}_n-\vartheta\|_4^2\geqslant\mathbf{E}_{\vartheta_0}\|\xi\|_4^2.$$

This bound is a particular case of a more general bound given in Theorem 1.9.1 of [11].

Theorem 2. *Let the conditions $\mathscr{C}$ be fulfilled. Then, uniformly on compacts $\mathbb{K}\subset\Theta$, the BEs $\tilde{\vartheta}_n$ are consistent, converge in distribution:*

$$n^{\frac{1}{2\kappa+1}}\big(\tilde{\vartheta}_n-\vartheta_0\big)\Longrightarrow\xi,$$

the polynomial moments converge: for any $p>0$, it holds

$$\lim_{n\to\infty} n^{\frac{p}{2\kappa+1}}\,\mathbf{E}_\vartheta\|\tilde{\vartheta}_n-\vartheta_0\|_4^p=\mathbf{E}_{\vartheta_0}\|\xi\|_4^p,$$

and the BEs are asymptotically efficient.

Proof. Let us study the normalized likelihood ratio

$$Z_n(u)=Z_n^{(\vartheta_0)}(u)=\frac{L\big(\vartheta_0+u\varphi_n,X^K\big)}{L\big(\vartheta_0,X^K\big)},\qquad u\in\mathbb{U}_n=\big(u:\vartheta_0+u\varphi_n\in\Theta\big),$$

where $\varphi_n=n^{-\frac{1}{2\kappa+1}}$. The properties of $Z_n(\cdot)$ which we need are described in the following three lemmas.

Lemma 1. *For any compact $\mathbb{K}\subset\Theta$, the finite-dimensional distributions of $Z_n(\cdot)$ converge, uniformly on $\vartheta_0\in\Theta$, to the finite-dimensional distributions of $Z(\cdot)$.*

Proof. Put $\vartheta_u=\vartheta_0+u\varphi_n$, $\vartheta_u^{(1)}=\vartheta_0^{(1)}+u^{(1)}\varphi_n$ and $\vartheta_u^{(2)}=\vartheta_0^{(2)}+u^{(2)}\varphi_n$. For a fixed u denote $\gamma_n=\sup_{0\leqslant t\leqslant T}\lambda_k(\vartheta_0,t)^{-1}\big|\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_0,t)\big|$ and note that $\gamma_n\to 0$ as $n\to\infty$. Therefore, by Lemma 1.5 of [16], the likelihood ratio admits the representation

$$\begin{aligned}\ln Z_n(u)=&\sum_{k=1}^K\int_0^T\frac{\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_0,t)}{\lambda_k(\vartheta_0,t)}\big[\mathrm{d}X_k(t)-\lambda_k(\vartheta_0,t)\,\mathrm{d}t\big]\big(1+O(\gamma_n)\big)\\&-\frac{n}{2}\sum_{k=1}^K\int_0^T\frac{\big[\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_0,t)\big]^2}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}t\big(1+O(\gamma_n)\big).\end{aligned}$$

Suppose that $\tau_{1,k}^\circ<\tau_{2,k}^\circ$ and set $2\tau=\tau_{1,k}^\circ+\tau_{2,k}^\circ$. Then the second integral can be written as

$$\begin{aligned}&\int_0^T\frac{\big[\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_0,t)\big]^2}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}t\\&\quad=\int_0^\tau\frac{\big[\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_0,t)\big]^2}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}t+\int_\tau^T\frac{\big[\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_0,t)\big]^2}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\tau \frac{\Big(\big[\psi_{\kappa,\delta}\big(t-\tau_k(\vartheta_u^{(1)})\big)-\psi_{\kappa,\delta}(t-\tau_{1,k}^\circ)\big]S_{1,k}(t)+A_k(\vartheta_u,\vartheta_0,t)\Big)^2}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}t \\
&\quad + \int_\tau^T \frac{\Big(B_k(\vartheta_u,\vartheta_0,t)+\big[\psi_{\kappa,\delta}\big(t-\tau_k(\vartheta_u^{(2)})\big)-\psi_{\kappa,\delta}(t-\tau_{2,k}^\circ)\big]S_{2,k}(t)\Big)^2}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}t \\
&= J_{1,k,n}(u^{(1)}) + J_{2,k,n}(u^{(2)}) \qquad (11)
\end{aligned}
$$

with obvious notations. Note that for $t \in [0,\tau]$ the function $A_k(\vartheta,\vartheta_0,t)$ have bounded derivatives w.r.t. ϑ, and that for $t \in [\tau,T]$ the same is true for the function $B_k(\vartheta,\vartheta_0,t)$.

Suppose that $u^{(1)} \in \mathbb{B}_{1,k}$. Then for large n we have $\tau_k(\vartheta_u^{(1)}) > \tau_{1,k}^\circ$, and therefore

$$
\begin{aligned}
J_{1,k,n}(u^{(1)}) &= n\int_{\tau_{1,k}^\circ}^\tau \frac{\Big(\big[\psi_{\kappa,\delta}\big(t-\tau_k(\vartheta_u^{(1)})\big)-\psi_{\kappa,\delta}(t-\tau_{1,k}^\circ)\big]S_{1,k}(t)+A_k(\vartheta_u,\vartheta_0)\Big)^2}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}t \\
&= n\int_{\tau_{1,k}^\circ}^\tau \frac{\Big(\big[\psi_{\kappa,\delta}\big(t-\tau_k(\vartheta_u^{(1)})\big)-\psi_{\kappa,\delta}(t-\tau_{1,k}^\circ)\big]S_{1,k}(t)\Big)^2}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}t + o(1) \\
&= n\int_0^{\tau-\tau_{1,k}^\circ} \frac{\Big(\big[\psi_{\kappa,\delta}\big(s-\tau_k(\vartheta_u^{(1)})+\tau_{1,k}^\circ\big)-\psi_{\kappa,\delta}(s)\big]S_{1,k}(s+\tau_{1,k}^\circ)\Big)^2}{\lambda_k(\vartheta_0,s+\tau_{1,k}^\circ)}\,\mathrm{d}t + o(1) \\
&= n\int_0^{\tau-\tau_{1,k}^\circ} \frac{\Big(\big[\psi_{\kappa,\delta}\big(s+\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\,\varphi_n\big)-\psi_{\kappa,\delta}(s)\big]S_{1,k}(s+\tau_{1,k}^\circ)\Big)^2}{\lambda_k(\vartheta_0,s+\tau_{1,k}^\circ)}\,\mathrm{d}t + o(1) \\
&= n\,\frac{S_{1,k}^2(\tau_{1,k}^\circ)}{\delta^{2\kappa}\,\lambda_k(\vartheta_0,\tau_{1,k}^\circ)}\Bigg[\int_0^{-\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\,\varphi_n} s^{2\kappa}\,\mathrm{d}t \\
&\qquad + \int_{-\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\,\varphi_n}^{\tau-\tau_{1,k}^\circ} \Big[\big|s+\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\,\varphi_n\big|^\kappa - s^\kappa\Big]^2\,\mathrm{d}t\Bigg] + o(1) \\
&= n\varphi_n^{2\kappa+1}\,\Gamma_{1,k}^2\,\big|\langle u^{(1)},m_{1,k}^\circ\rangle\big|^{2\kappa+1}\left[\int_0^1 v^{2\kappa}\,\mathrm{d}v + \int_1^{c/\varphi_n}\big[|v-1|^\kappa - v^\kappa\big]^2\,\mathrm{d}v\right] + o(1) \\
&= \Gamma_{1,k}^2\,\big|\langle u^{(1)},m_{1,k}^\circ\rangle\big|^{2\kappa+1}\int_0^\infty \big[|v-1|^\kappa\,\mathbb{1}_{\{v\geqslant 1\}} - v^\kappa\big]^2\,\mathrm{d}v + o(1) \\
&= \Gamma_{1,k}^2\,\big|\langle u^{(1)},m_{1,k}^\circ\rangle\big|^{2\kappa+1}\,Q_\kappa^2 + o(1), \qquad (12)
\end{aligned}
$$

where we changed the variable $s = -\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\,v$ and used the relation $n\varphi_n^{2\kappa+1}=1$ and the notations (9) and (10).

For the values $u^{(2)} \in \mathbb{B}_{2,k}$ we have a similar expression

$$
J_{2,k,n}(u^{(2)}) = \Gamma_{2,k}^2\,\big|\langle u^{(2)},m_{2,k}^\circ\rangle\big|^{2\kappa+1}\,Q_\kappa^2 + o(1).
$$

If $u^{(1)} \in \mathbb{B}_{1,k}^c$ and $u^{(2)} \in \mathbb{B}_{2,k}^c$, then similar calculations lead to the same integrals.

Hence

$$\int_0^T \frac{\bigl[\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_0,t)\bigr]^2}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}t = \Bigl[\Gamma_{1,k}^2\bigl|\langle u^{(1)},m_{1,k}^\circ\rangle\bigr|^{2\kappa+1}+\Gamma_{2,k}^2\bigl|\langle u^{(2)},m_{2,k}^\circ\rangle\bigr|^{2\kappa+1}\Bigr]Q_\kappa^2+o(1).$$

Let us suppose that $\tau_k(\vartheta_0^{(1)}+u^{(1)}\varphi_n)>\tau_{1,k}^\circ$. Introduce the centered Poisson process $\mathrm{d}\pi_{k,n}(t)=\mathrm{d}X_k(t)-\lambda_{k,n}(\vartheta_0,t)\,\mathrm{d}t$ and consider the stochastic integral

$$\begin{aligned}\int_0^T &\frac{\bigl[\lambda_{k,n}(\vartheta_u,t)-\lambda_{k,n}(\vartheta_0,t)\bigr]}{\lambda_{k,n}(\vartheta_0,t)}\,\mathrm{d}\pi_{k,n}(t)\\ &=\int_{\tau_{1,k}^\circ}^{\tau}\frac{\bigl[\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_0,t)\bigr]}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}\pi_{k,n}(t)+\int_{\tau}^{T}\frac{\bigl[\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_0,t)\bigr]}{\lambda_k(\vartheta_0,t)}\,\mathrm{d}\pi_{k,n}(t)\\ &=I_{1,k,n}(u^{(1)})+I_{2,k,n}(u^{(2)}),\end{aligned}$$

where $2\tau=\tau_k(\vartheta_u^{(1)})+\tau_{1,k}^\circ$.

Using the same relations as above, for $u^{(1)}\in\mathbb{B}_{1,k}$ we can write

$$\begin{aligned}&I_{1,k,n}(u^{(1)})\\ &\quad=\int_0^{\tau-\tau_{1,k}^\circ}\frac{\bigl[\psi_{\kappa,\delta}\bigl(s+\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\,\varphi_n\bigr)-\psi_{\kappa,\delta}(s)\bigr]S_{1,k}(s+\tau_{1,k}^\circ)}{\lambda_k(\vartheta_0,s+\tau_{1,k}^\circ)}\,\mathrm{d}\pi_{k,n}(t)+o(1)\\ &\quad=\frac{S_{1,k}(\tau_{1,k}^\circ)}{\delta^\kappa\,\lambda_k(\vartheta_0,\tau_{1,k}^\circ)}\Biggl[\int_0^{-\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\,\varphi_n}s^\kappa\,\mathrm{d}\pi_{k,n}(s+\tau_{1,k}^\circ)\\ &\qquad+\int_{-\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\,\varphi_n}^{\tau-\tau_{1,k}^\circ}\Bigl[\bigl|s+\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\,\varphi_n\bigr|^\kappa-s^\kappa\Bigr]\mathrm{d}\pi_{k,n}(s+\tau_{1,k}^\circ)\Biggr]+o(1)\\ &\quad=\hat\Gamma_{1,k}\Biggl[\int_0^{c/\varphi_n}\Bigl[\bigl|v+\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\bigr|^\kappa\,\mathbb{1}_{\{v\geqslant-\nu^{-1}\langle u^{(1)},m_{1,k}^\circ\rangle\}}-v^\kappa\Bigr]\mathrm{d}W_{1,k,n}(v)\Biggr]+o(1).\end{aligned}$$

Here we changed the variable $s=v\varphi_n$, used the relation $\sqrt{n}\,\varphi_n^{\kappa+1/2}=1$ and denoted

$$W_{1,k,n}(v)=\frac{\pi_{k,n}(v\varphi_n+\tau_{1,k}^\circ)-\pi_{k,n}(\tau_{1,k}^\circ)}{\sqrt{n\varphi_n\lambda_k(\vartheta_0,\tau_{1,k}^\circ)}}\,.$$

This process has the following first two moments: $\mathbf{E}_{\vartheta_0}W_{1,k,n}(v)=0$,

$$\mathbf{E}_{\vartheta_0}W_{1,k,n}(v)^2=\frac{1}{\varphi_n\,\lambda_k(\vartheta_0,\tau_{1,k}^\circ)}\int_{\tau_{1,k}^\circ}^{\tau_{1,k}^\circ+v\varphi_n}\lambda_k(\vartheta_0,t)\,\mathrm{d}t=v\bigl(1+o(1)\bigr),$$

$$\mathbf{E}_{\vartheta_0}W_{1,k,n}(v_1)W_{1,k,n}(v_2)=\frac{1}{\varphi_n\,\lambda_k(\vartheta_0,\tau_{1,k}^\circ)}\int_{\tau_{1,k}^\circ}^{\tau_{1,k}^\circ+(v_1\wedge v_2)\varphi_n}\lambda_k(\vartheta_0,t)\,\mathrm{d}t=(v_1\wedge v_2)\bigl(1+o(1)\bigr).$$

By the central limit theorem (see, e.g., Theorem 1.2 of [14])

$$
\begin{aligned}
I_{1,k,n}(u^{(1)}) &= \hat{\Gamma}_{1,k} \int_0^{c/\varphi_n} \Big[\big| v + \nu^{-1} \langle u^{(1)}, m^{\circ}_{1,k} \rangle \big|^{\kappa} \mathbb{1}_{\{v \geqslant -\nu^{-1} \langle u^{(1)}, m^{\circ}_{1,k} \rangle\}} - v^{\kappa} \Big] \, \mathrm{d}W_{1,k,n}(v) \\
&\Longrightarrow \hat{\Gamma}_{1,k} \int_0^{\infty} \Big[\big| v + \nu^{-1} \langle u^{(1)}, m^{\circ}_{1,k} \rangle \big|^{\kappa} \mathbb{1}_{\{v \geqslant -\nu^{-1} \langle u^{(1)}, m^{\circ}_{1,k} \rangle\}} - v^{\kappa} \Big] \, \mathrm{d}W^{+}_{1,k}(v) \\
&= \hat{\Gamma}_{1,k} \, I_{1,k}(u^{(1)}),
\end{aligned}
$$

where $W^{+}_{1,k}(v)$, $v \geqslant 0$, is a standard Wiener process. A similar limit for the integral $I_{2,k,n}(u^{(2)})$ can be obtained in the same way.

Suppose that $u^{(2)} \in \mathbb{B}^{c}_{2,k}$, i.e., $\langle u^{(2)}, m^{\circ}_{2,k} \rangle \geqslant 0$. Then

$$
\begin{aligned}
I_{2,k,n}(u^{(2)}) &= \hat{\Gamma}_{2,k} \int_{-\nu^{-1} \langle u^{(2)}, m^{\circ}_{2,k} \rangle}^{c/\varphi_n} \Big[\big| v + \nu^{-1} \langle u^{(2)}, m^{\circ}_{2,k} \rangle \big|^{\kappa} - v^{\kappa} \, \mathbb{1}_{\{v \geqslant 0\}} \Big] \, \mathrm{d}W_{2,k,n}(v) \\
&\Longrightarrow \hat{\Gamma}_{2,k} \int_{-\nu^{-1} \langle u^{(2)}, m^{\circ}_{2,k} \rangle}^{\infty} \Big[\big| v + \nu^{-1} \langle u^{(2)}, m^{\circ}_{2,k} \rangle \big|^{\kappa} - v^{\kappa} \, \mathbb{1}_{\{v \geqslant 0\}} \Big] \, \mathrm{d}W^{+}_{2,k}(v) \\
&= \hat{\Gamma}_{2,k} \, I_{2,k}(u^{(2)}),
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
\ln Z_{(k),n}(u) = \int_0^T & \frac{\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)}{\lambda_k(\vartheta_0, t)} \big[\mathrm{d}X_k(t) - \lambda_k(\vartheta_0, t) \, \mathrm{d}t \big] \\
& - \frac{n}{2} \int_0^T \frac{\big[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t) \big]^2}{\lambda_k(\vartheta_0, t)} \, \mathrm{d}t + o(1) \\
\Longrightarrow \hat{\Gamma}_{1,k} \, I_{1,k}(u^{(1)}) & - \frac{\Gamma^2_{1,k} Q^2_{\kappa}}{2} \big| \langle u^{(1)}, m^{\circ}_{1,k} \rangle \big|^{2\kappa+1} \\
& + \hat{\Gamma}_{2,k} \, I_{2,k}(u^{(2)}) - \frac{\Gamma^2_{2,k} Q^2_{\kappa}}{2} \big| \langle u^{(2)}, m^{\circ}_{2,k} \rangle \big|^{2\kappa+1} \\
= \ln Z_{(k)}(u).
\end{aligned}
$$

The Wiener processes $W_{i,k}(\cdot)$, $i = 1, 2$, $k = 1, \ldots, K$, are independent and this convergence provides the convergence of one-dimensional distributions

$$
Z_n(u) \Longrightarrow Z(u) = \prod_{k=1}^{K} Z_{(k)}(u), \qquad u \in \mathcal{R}^4.
$$

The convergence of finite-dimensional distributions can be proved following the same lines, but we omit it since it is too cumbersome and does not use new ideas or tools. $\square$

Lemma 2. *There exists a constant $C > 0$ such that for any $L > 0$, it holds*

$$
\sup_{|u| < L} \mathbf{E}_{\vartheta_0} \big| Z_n(u)^{1/2} - Z_n(u')^{1/2} \big|^2 \leqslant C \, (1 + L^{1-2\kappa}) \, \|u - u'\|_4^{1+2\kappa}.
$$

Proof. By Lemma 1.5 of [14], we have

$$\begin{aligned}\mathbf{E}_{\vartheta_0}\big|Z_n(u)^{1/2}-Z_n(u')^{1/2}\big|^2 &\leqslant \sum_{k=1}^{K}\int_0^T\left[\sqrt{\lambda_{k,n}(\vartheta_u,t)}-\sqrt{\lambda_{k,n}(\vartheta_{u'},t)}\right]^2\mathrm{d}t\\ &\leqslant Cn\sum_{k=1}^{K}\int_0^T\big[\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_{u'},t)\big]^2\,\mathrm{d}t.\end{aligned}\tag{13}$$

Note that the parts of integrals in (11) containing differentiable functions $A_k(\vartheta_u,\vartheta_0,t)$ and $B_k(\vartheta_u,\vartheta_0,t)$ have estimates like

$$n\int_0^\tau\big[A_k(\vartheta_u,\vartheta_0,t)-A_k(\vartheta_{u'},\vartheta_0,t)\big]^2\,\mathrm{d}t\leqslant C\,n\varphi_n^2\,\|u-u'\|^2=C\,n^{-1+2\kappa}\,\|u-u'\|_4^2.$$

Therefore, following (12), we obtain the relations

$$\begin{aligned}&\sum_{k=1}^{K}\int_0^T\big[\lambda_k(\vartheta_u,t)-\lambda_k(\vartheta_{u'},t)\big]^2\,\mathrm{d}t\\ &\qquad\leqslant C\left(\|u^{(1)}-u'^{(1)}\|_2^{1+2\kappa}+\|u^{(2)}-u'^{(2)}\|_2^{1+2\kappa}\right)+C\,n^{-1+2\kappa}\,\|u-u'\|_4^2\\ &\qquad\leqslant C\,\|u-u'\|_4^{1+2\kappa}+C\,n^{-1+2\kappa}\,\|u-u'\|_4^2\\ &\qquad\leqslant C\,(1+L^{1-2\kappa})\,\|u-u'\|_4^{1+2\kappa}.\end{aligned}$$

□

Lemma 3. *There exist $c_*>0$ such that*

$$\mathbf{P}_{\vartheta_0}\Big(Z_T(u)>e^{-c_*\|u\|_4^{2\kappa+1}}\Big)\leqslant e^{-c_*\|u\|_4^{2\kappa+1}}$$

Proof. By the same Lemma 1.5 of [14], we have

$$\begin{aligned}\mathbf{P}_{\vartheta_0}\Big(Z_T(u)>e^{-c_*\|u\|_4^{2\kappa+1}}\Big)&\leqslant e^{\frac{c_*}{2}\|u\|_4^{2\kappa+1}}\,\mathbf{E}_{\vartheta_0}Z_n(u)^{1/2}\\ &=\exp\left(\frac{c_*}{2}\,\|u\|_4^{2\kappa+1}-\frac{1}{2}\sum_{k=1}^{K}\int_0^T\left(\sqrt{\lambda_{k,n}(\vartheta_u,t)}-\sqrt{\lambda_{k,n}(\vartheta_u,t)}\right)^2\mathrm{d}t\right).\end{aligned}$$

Using calculations similar to those of (11) and (12), we obtain for $\|\vartheta_u-\vartheta_0\|_4\leqslant\varepsilon$ and sufficiently small $\varepsilon>0$ the estimate

$$\sum_{k=1}^{K}\int_0^T\left(\sqrt{\lambda_{k,n}(\vartheta_u,t)}-\sqrt{\lambda_{k,n}(\vartheta_u,t)}\right)^2\mathrm{d}t\geqslant c_1\,\|u\|_4^{2\kappa+1}.$$

Consider now the case $\|\vartheta_u - \vartheta_0\|_4 > \varepsilon$. Let us denote

$$g(\varepsilon) = \inf_{\|\vartheta-\vartheta_0\|_4>\varepsilon} \sum_{k=1}^{K} \int_0^T \left[S_{1,k}(\vartheta^{(1)},t) + S_{2,k}(\vartheta^{(2)},t) - S_{1,k}(\vartheta_0^{(1)},t) - S_{2,k}(\vartheta_0^{(2)},t)\right]^2 \mathrm{d}t$$

and remark that $\varphi_n \|u\|_4 \leqslant D = \sup_{\vartheta,\vartheta'\in\Theta} \|\vartheta - \vartheta'\|_4$. Hence $n > D^{-(2\kappa+1)}\|u\|_4^{2\kappa+1}$. As $g(\varepsilon) > 0$ we can write

$$\begin{aligned}
&\sum_{k=1}^{K} \int_0^T \left(\sqrt{\lambda_{k,n}(\vartheta_u,t)} - \sqrt{\lambda_{k,n}(\vartheta_u,t)}\right)^2 \mathrm{d}t \\
&\qquad\qquad \geqslant c \sum_{k=1}^{K} \int_0^T \left[\lambda_{k,n}(\vartheta_u,t) - \lambda_{k,n}(\vartheta_0,t)\right]^2 \mathrm{d}t \\
&\qquad\qquad \geqslant c\, n\, g(\varepsilon) \geqslant c\, g(\varepsilon) D^{-(2\kappa+1)} \|u\|_4^{2\kappa+1} = c_2 \|u\|_4^{2\kappa+1}.
\end{aligned}$$

Finally, denoting $\bar{c} = c_1 \wedge c_2$ and setting $c_* = \bar{c}/3$, we obtain

$$\frac{c_*}{2}\|u\|_4^{2\kappa+1} - \frac{1}{2}\sum_{k=1}^{K} \int_0^T \left(\sqrt{\lambda_{k,n}(\vartheta_u,t)} - \sqrt{\lambda_{k,n}(\vartheta_u,t)}\right)^2 \mathrm{d}t \leqslant -c_* \|u\|_4^{2\kappa+1}.$$

□

The properties of the process $Z_n(\cdot)$ established in Lemmas 1-3 allow us to cite Theorem 1.10.2 of [11] and, therefore, to obtain the properties of BEs stated in Theorem 2. □

2.2.3 Change point type fronts

Suppose that the intensity functions of the observed inhomogeneous Poisson processes have jumps at the moments of arrival of the signals, i.e., the intensities are

$$\lambda_{k,n}(\vartheta,t) = n\mathbb{1}_{\{t>\tau_k(\vartheta^{(1)})\}} S_{1,k}(t) + n\mathbb{1}_{\{t>\tau_k(\vartheta^{(2)})\}} S_{2,k}(t) + n\lambda_0 = n\lambda_k(\vartheta,t), \tag{14}$$

where $0 \leqslant t \leqslant T$, $k = 1,\ldots,K$. As before, $\vartheta = (\vartheta^{(1)},\vartheta^{(2)})^\top \in \Theta$, where $\vartheta^{(1)} = (\vartheta_1,\vartheta_2)^\top = (x_1',y_1')^\top$ (position of the source $\mathbb{S}_1$) and $\vartheta^{(2)} = (\vartheta_3,\vartheta_4)^\top = (x_2',y_2')^\top$ (position of the source $\mathbb{S}_2$), and

$$\tau_k(\vartheta^{(i)}) = \nu^{-1}\big((x_k - x_i')^2 + (y_k - y_i')^2\big)^{1/2} = \tau_{i,k}.$$

Introduce the notations

$$\ell_{i,k} = \ln\biggl(\frac{\lambda_0}{S_{i,k}(\tau^\circ_{i,k}) + \lambda_0}\biggr), \quad \mathbb{B}_{i,k} = \bigl(u^{(i)} \,:\, \langle u^{(i)}, m^\circ_{i,k}\rangle < 0\bigr),\ i = 1,2,\ k = 1,\dots,K,$$

$$Z_{(k)}(u) = \exp\Biggl(\sum_{i=1}^{2}\Bigl[\ell_{i,k}\, x^+_{i,k}\bigl(-\nu^{-1}\,\langle u^{(i)}, m^\circ_{i,k}\rangle\bigr) - \langle u^{(i)}, m^\circ_{i,k}\rangle\,\nu^{-1}\,S_{i,k}(\tau^\circ_{i,k})\Bigr]\mathbb{1}_{\mathbb{B}_{i,k}}$$

$$- \sum_{i=1}^{2}\Bigl[\ell_{i,k}\, x^-_{i,k}\bigl(-\nu^{-1}\,\langle u^{(i)}, m^\circ_{i,k}\rangle\bigr) - \langle u^{(i)}, m^\circ_{i,k}\rangle\,\nu^{-1}\,S_{i,k}(\tau^\circ_{i,k})\Bigr]\mathbb{1}_{\mathbb{B}^c_{i,k}}\Biggr),$$

$$Z(u) = \prod_{k=1}^{K} Z_{(k)}(u), \qquad \eta = \frac{\int_{\mathcal{R}^4} u\, Z(u)\,\mathrm{d}u}{\int_{\mathcal{R}^4} Z(u)\,\mathrm{d}u}\,.$$

Here

$$x^+_{i,k}\bigl(-\nu^{-1}\,\langle u^{(i)}, m^\circ_{i,k}\rangle\bigr) = y^+_{i,k}(s)_{s=-\nu^{-1}\,\langle u^{(i)}, m^\circ_{i,k}\rangle}, \qquad s \geqslant 0,$$

where $y^+_{i,k}(s)$, $s \geqslant 0$, is a Poisson process with unit intensity. Similarly

$$x^-_{i,k}\bigl(-\nu^{-1}\,\langle u^{(i)}, m^\circ_{i,k}\rangle\bigr) = y^-_{i,k}(-s)_{s=\nu^{-1}\,\langle u^{(i)}, m^\circ_{i,k}\rangle}, \qquad s \leqslant 0,$$

with another Poisson process $y^-_{i,k}(s)$, $s \geqslant 0$ with unit intensity. All Poisson processes $y^\pm_{i,k}(s)$, $s \geqslant 0$, $i = 1,2$, $k = 1,\dots,K$ are independent.

Conditions $\mathscr{D}$.

$\mathscr{D}_1$. *The intensities of the observed processes are* (14), *where the functions $S_{i,k}(y) \in \mathcal{C}^1$ and are positive. The set $\Theta \subset \mathcal{R}^4$ is open, bounded and convex.*

$\mathscr{D}_2$. *The configuration of the detectors and the set Θ are such that the signals from the both sources arrive at the detectors during the period $[0,T]$.*

$\mathscr{D}_3$. *The condition $\mathscr{R}_4$ is fulfilled.*

The following lower bound

$$\lim_{\varepsilon\to 0}\ \varliminf_{n\to\infty}\ \sup_{\|\vartheta-\vartheta_0\|\leqslant\varepsilon} n^2\,\mathbf{E}_\vartheta\|\bar\vartheta_n - \vartheta\|_4^2 \geqslant \mathbf{E}_{\vartheta_0}\|\eta\|_4^2$$

holds. This bound is another particular case of Theorem 1.9.1 of [11].

Theorem 3. *Let the conditions $\mathscr{D}$ be fulfilled. Then, uniformly on compacts $\mathbb{K} \subset \Theta$, the BEs $\tilde\vartheta_n$ are consistent, converge in distribution:*

$$n\,\bigl(\tilde\vartheta_n - \vartheta_0\bigr) \Longrightarrow \eta,$$

the polynomial moments converge: for any $p > 0$, it holds

$$n^p\,\mathbf{E}_{\vartheta_0}\|\tilde\vartheta_n - \vartheta_0\|_4^p \longrightarrow \mathbf{E}_{\vartheta_0}\|\eta\|_4^p,$$

and the BEs are asymptotically efficient.

Proof. The normalized likelihood ratio function in this problem is given by

$$\begin{aligned}\ln Z_n(u) &= \ln Z_n^{(\vartheta_0)}(u)\\ &= \sum_{k=1}^K \int_0^T \ln \frac{\lambda_k(\vartheta_0 + n^{-1}u, t)}{\lambda_k(\vartheta_0, t)}\,\mathrm{d}X_k(t) - n\sum_{k=1}^K \int_0^T \big[\lambda_k(\vartheta_0 + n^{-1}u, t) - \lambda_k(\vartheta_0, t)\big]\,\mathrm{d}t.\end{aligned}$$

We have to check once more the conditions of Theorem 1.10.1 of [11] and to prove three lemmas.

Lemma 4. *For any compact $\mathbb{K} \subset \Theta$, the finite-dimensional distributions of $Z_n(\cdot)$ converge, uniformly on $\vartheta_0 \in \Theta$, to the finite-dimensional distributions of $Z(\cdot)$.*

Proof. Let us set $\vartheta_u = \vartheta_0 + n^{-1}u$ and suppose that $u^{(1)} \in \mathbb{B}_{1,k}$ and $u^{(2)} \in \mathbb{B}_{2,k}$. For n sufficiently large we have $\tau_k(\vartheta_u^{(1)}) > \tau_{1,k}^\circ$ and $\tau_k(\vartheta_u^{(2)}) > \tau_{2,k}^\circ > \tau_k(\vartheta_u^{(1)})$, and so

$$\int_0^T \ln \frac{\lambda_k(\vartheta_u, t)}{\lambda_k(\vartheta_0, t)}\,\mathrm{d}X_k(t) = \int_{\tau_{1,k}^\circ}^{\tau_k(\vartheta_u^{(1)})} \ln \frac{\lambda_0}{S_{1,k}(t) + \lambda_0}\,\mathrm{d}X_k(t) + \int_{\tau_{2,k}^\circ}^{\tau_k(\vartheta_u^{(2)})} \ln \frac{\lambda_0}{S_{2,k}(t) + \lambda_0}\,\mathrm{d}X_k(t).$$

and

$$\int_0^T \big[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)\big]\,\mathrm{d}t = -\int_{\tau_{1,k}(\vartheta_0)}^{\tau_{1,k}(\vartheta_u)} S_{1,k}(t)\,\mathrm{d}t - \int_{\tau_{2,k}(\vartheta_0)}^{\tau_{2,k}(\vartheta_u)} S_{2,k}(t)\,\mathrm{d}t.$$

For a fixed u, using Taylor expansion we can write (recall that $\tau_{i,k}^\circ = \tau_k(\vartheta_0^{(i)})$, $i = 1, 2$)

$$\begin{aligned}\int_{\tau_{1,k}^\circ}^{\tau_k(\vartheta_u^{(1)})} S_{1,k}(t)\,\mathrm{d}t &= \big(\tau_k(\vartheta_u^{(1)}) - \tau_{1,k}^\circ\big)\, S_{1,k}(\tau_{1,k}^\circ) + \frac{1}{n}\int_{\tau_{1,k}^\circ}^{\tau_k(\vartheta_u^{(1)})} S_{1,k}'(\tilde{t})\,\mathrm{d}t\\ &= -(n\nu)^{-1}\,\langle u^{(1)}, m_{1,k}^\circ\rangle\, S_{1,k}(\tau_{1,k}^\circ)\,\big(1 + O(n^{-1})\big)\end{aligned}$$

and

$$\int_{\tau_{2,k}^\circ}^{\tau_k(\vartheta_u^{(2)})} S_{2,k}(t)\,\mathrm{d}t = -(n\nu)^{-1}\,\langle u^{(2)}, m_{2,k}^\circ\rangle\, S_{2,k}(\tau_{2,k}^\circ)\,\big(1 + O(n^{-1})\big).$$

For stochastic integrals, similar calculations provide

$$\int_{\tau_{1,k}^\circ}^{\tau_k(\vartheta_u^{(1)})} \ln \frac{\lambda_0}{S_{1,k}(t) + \lambda_0}\,\mathrm{d}X_k(t) = \ln \frac{\lambda_0}{S_{1,k}(\tau_{1,k}^\circ) + \lambda_0}\,\big[X_k\big(\tau_k(\vartheta_u^{(1)})\big) - X_k(\tau_{1,k}^\circ)\big]\big(1 + O(n^{-1})\big).$$

Further

$$\begin{aligned}X_k\big(\tau_k(\vartheta_u^{(1)})\big) - X_k(\tau_{1,k}^\circ) &= X_k\big(\tau_{1,k}^\circ - n^{-1}\nu^{-1}\langle u^{(1)}, m_{1,k}^\circ\rangle\big) - X_k(\tau_{1,k}^\circ)\\ &\quad + X_k\big(\tau_k(\vartheta_u^{(1)})\big) - X_k\big(\tau_{1,k}^\circ - n^{-1}\nu^{-1}\langle u^{(1)}, m_{1,k}^\circ\rangle\big)\\ &= x_{1,k}^+\big(-\nu^{-1}\langle u^{(1)}, m_{1,k}^\circ\rangle\big) + O(n^{-1/2}).\end{aligned}$$

Here $x^+_{1,k}\bigl(-\nu^{-1}\,\langle u^{(1)}, m^\circ_{1,k}\rangle\,\bigr) = X_k\bigl(\tau^\circ_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m^\circ_{1,k}\rangle\bigr) - X_k(\tau^\circ_{1,k})$ is a Poisson random process and the last estimate was obtained as follows:

$$\begin{aligned}
&\mathbf{E}_{\vartheta_0}\bigl[X_k\bigl(\tau_k(\vartheta^{(1)}_u)\bigr) - X_k\bigl(\tau^\circ_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m^\circ_{1,k}\rangle\bigr)\bigr]^2 \\
&\quad = n\int_{\tau_k(\vartheta^{(1)}_u)}^{\tau^\circ_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m^\circ_{1,k}\rangle} \lambda_k(\vartheta_0, t)\,\mathrm{d}t + \left(n\int_{\tau_k(\vartheta^{(1)}_u)}^{\tau^\circ_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m^\circ_{1,k}\rangle} \lambda_k(\vartheta_0, t)\,\mathrm{d}t\right)^2 \\
&\quad \leqslant \frac{C}{n}\,.
\end{aligned}$$

Hence, if $u^{(1)} \in \mathbb{B}_{1,k}$ and $u^{(2)} \in \mathbb{B}_{2,k}$, it holds

$$\begin{aligned}
\int_0^T \ln\frac{\lambda_k(\vartheta_0 + n^{-1}u, t)}{\lambda_k(\vartheta_0, t)}\,\mathrm{d}X_k(t) &= \ln\left(\frac{\lambda_0}{S_{1,k}(\tau^\circ_{1,k}) + \lambda_0}\right) x^+_{1,k}\bigl(-\nu^{-1}\langle u^{(1)}, m^\circ_{1,k}\rangle\bigr) \\
&\quad + \ln\left(\frac{\lambda_0}{S_{2,k}(\tau^\circ_{2,k}) + \lambda_0}\right) x^+_{2,k}\bigl(-\nu^{-1}\langle u^{(2)}, m^\circ_{2,k}\rangle\bigr) + O(n^{-1/2}).
\end{aligned}$$

If $u^{(1)} \in \mathbb{B}^c_{1,k}$ and $u^{(2)} \in \mathbb{B}^c_{2,k}$, we have

$$\begin{aligned}
\int_0^T \ln\frac{\lambda_k(\vartheta_0 + n^{-1}u, t)}{\lambda_k(\vartheta_0, t)}\,\mathrm{d}X_k(t) &= \ln\left(1 + \frac{S_{1,k}(\tau^\circ_{1,k})}{\lambda_0}\right) x^-_{1,k}\bigl(-\nu^{-1}\langle u^{(1)}, m^\circ_{1,k}\rangle\bigr) \\
&\quad + \ln\left(1 + \frac{S_{2,k}(\tau^\circ_{2,k})}{\lambda_0}\right) x^-_{2,k}\bigl(\nu^{-1}\langle u^{(2)}, m^\circ_{2,k}\rangle\bigr) + O(n^{-1/2}).
\end{aligned}$$

□

Lemma 5. *There exists a constant $C > 0$ such that*

$$\mathbf{E}_{\vartheta_0}\bigl|Z_n(u)^{1/2} - Z_n(u')^{1/2}\bigr|^2 \leqslant C\,\|u - u'\|_4.$$

Proof. We have (see (13))

$$\sup_{\vartheta_0 \in \mathbb{K}} \mathbf{E}_{\vartheta_0}\bigl|Z_n(u)^{1/2} - Z_n(u')^{1/2}\bigr|^2 \leqslant \sum_{k=1}^K n\int_0^T \bigl[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_{u'}, t)\bigr]^2\,\mathrm{d}t.$$

Suppose that $\tau_k(\vartheta^{(1)}_u) > \tau_k(\vartheta^{(1)}_{u'})$ and $\tau_k(\vartheta^{(2)}_u) > \tau_k(\vartheta^{(2)}_{u'}) > \tau_k(\vartheta^{(1)}_u)$. Then

$$\begin{aligned}
&n\int_0^T \Bigl[S_{1,k}(t)\,\mathbb{1}_{\{\tau_k(\vartheta^{(1)}_{u'}) \leqslant t \leqslant \tau_k(\vartheta^{(1)}_u)\}} + S_{2,k}(t)\,\mathbb{1}_{\{\tau_k(\vartheta^{(2)}_{u'}) \leqslant t \leqslant \tau_k(\vartheta^{(2)}_u)\}}\Bigr]^2\,\mathrm{d}t \\
&\qquad = n\int_{\tau_k(\vartheta^{(1)}_{u'})}^{\tau_k(\vartheta^{(1)}_u)} S^2_{1,k}(t)\,\mathrm{d}t + n\int_{\tau_k(\vartheta^{(2)}_{u'})}^{\tau_k(\vartheta^{(2)}_u)} S^2_{2,k}(t)\,\mathrm{d}t \\
&\qquad \leqslant Cn\bigl(\tau_k(\vartheta^{(1)}_u) - \tau_k(\vartheta^{(1)}_{u'})\bigr) + Cn\bigl(\tau_k(\vartheta^{(2)}_u) - \tau_k(\vartheta^{(2)}_{u'})\bigr)
\end{aligned}$$

$$\leqslant C\Big(\big|\langle u'^{(1)} - u^{(1)}, m^{\circ}_{1,k}\rangle\big| + \big|\langle u'^{(2)} - u^{(2)}, m^{\circ}_{2,k}\rangle\big|\Big)$$
$$\leqslant C\big(\|u'^{(1)} - u^{(1)}\|_2 + \|u'^{(2)} - u^{(2)}\|_2\big) \leqslant C\,\|u' - u\|_4.$$

□

Lemma 6. *There exists a constant $\kappa > 0$ such that*

$$\mathbf{E}_{\vartheta_0} Z_n(u)^{1/2} \leqslant e^{-\kappa\,\|u\|_4}.$$

Proof. We have

$$\ln \mathbf{E}_{\vartheta_0} Z_n(u)^{1/2} = -\frac{1}{2}\sum_{k=1}^{K}\int_0^T \Big[\sqrt{\lambda_{k,n}(\vartheta_u, t)} - \sqrt{\lambda_{k,n}(\vartheta_0, t)}\Big]^2 \mathrm{d}t$$
$$\leqslant -C\sum_{k=1}^{K}\int_0^T \big[\lambda_{k,n}(\vartheta_u, t) - \lambda_{k,n}(\vartheta_0, t)\big]^2 \mathrm{d}t.$$

Suppose that $\tau_k(\vartheta_u^{(i)}) > \tau^{\circ}_{i,k}$, $\tau^{\circ}_{2,k} > \tau_k(\vartheta_u^{(1)})$ and $\|\vartheta_u - \vartheta_0\|_4 \leqslant \varepsilon$. Then

$$\int_0^T \big[\lambda_{k,n}(\vartheta_u, t) - \lambda_{k,n}(\vartheta_0, t)\big]^2 \mathrm{d}t$$
$$\geqslant c\,n\int_0^T \Big[S_{1,k}(t)\,\mathbb{1}_{\{\tau^{\circ}_{1,k} < t < \tau_k(\vartheta_u^{(1)})\}} + S_{2,k}(t)\,\mathbb{1}_{\{\tau^{\circ}_{2,k} < t < \tau_k(\vartheta_u^{(2)})\}}\Big]^2 \mathrm{d}t$$
$$\geqslant c\,n\left[\int_{\tau^{\circ}_{1,k}}^{\tau_k(\vartheta_u^{(1)})} S^2_{1,k}(t)\,\mathrm{d}t + \int_{\tau^{\circ}_{2,k}}^{\tau_k(\vartheta_u^{(2)})} S^2_{2,k}(t)\,\mathrm{d}t\right]$$
$$\geqslant c\,n\big[\tau_k(\vartheta_u^{(1)}) - \tau^{\circ}_{1,k} + \tau_k(\vartheta_u^{(2)}) - \tau^{\circ}_{2,k}\big]$$
$$\geqslant c\big[-\langle u^{(1)}, m^{\circ}_{1,k}\rangle - \langle u^{(2)}, m^{\circ}_{2,k}\rangle\big]$$
$$\geqslant c\,\frac{-\langle u^{(1)}, m^{\circ}_{1,k}\rangle - \langle u^{(2)}, m^{\circ}_{2,k}\rangle}{\|u\|_4}\,\|u\|_4 = -c\sqrt{2}\,\langle \mathrm{e}, \mathrm{m}_k\rangle\,\|u\|_4.$$

Here $-\langle \mathrm{e}, \mathrm{m}_k\rangle > 0$ and the vectors are

$$\mathrm{e} = \frac{u}{\|u\|_4}, \quad \|\mathrm{e}\|_4 = 1, \quad \mathrm{m}_k = \left(\frac{x_k - x_1'^{\circ}}{\sqrt{2}\rho_k}, \frac{y_k - y_1'^{\circ}}{\sqrt{2}\rho_k}, \frac{x_k - x_2'^{\circ}}{\sqrt{2}\rho_k}, \frac{y_k - y_2'^{\circ}}{\sqrt{2}\rho_k}\right), \quad \|\mathrm{m}_k\|_4 = 1.$$

Note that

$$\inf_{\|\mathrm{e}\|_4 = 1}\sum_{k=1}^{K}\big(-\langle \mathrm{e}, \mathrm{m}_k\rangle\big) = \bar{c} > 0.$$

Indeed, if for some e we have $\bar{c} = 0$, then all scalar products $\langle u^{(i)}, m^{\circ}_{i,k}\rangle = 0$, $i = 1, 2$, $k = 1, \ldots, K$, but such configuration of detectors is impossible.

Let $\|\vartheta_u - \vartheta_0\|_4 > \varepsilon$. Then as in the proof of Lemma 3 we obtain the estimate

$$\sum_{k=1}^{K} \int_0^T \left[\lambda_{k,n}(\vartheta_u, t) - \lambda_{k,n}(\vartheta_0, t)\right]^2 \mathrm{d}t \geqslant n\, g(\varepsilon) \geqslant D^{-1} \|u\|_4 = \tilde{c}\, \|u\|_4.$$

Therefore

$$\sum_{k=1}^{K} \int_0^T \left[\sqrt{\lambda_{k,n}(\vartheta_u, t)} - \sqrt{\lambda_{k,n}(\vartheta_0, t)}\right]^2 \mathrm{d}t \geqslant 2\kappa\, \|u\|_4,$$

with a corresponding $\kappa > 0$. □

Once more, according to Theorem 1.10.2 of [11], the properties of the normalized likelihood ratio $Z_n(\cdot)$ established in Lemmas 4–6 provide us the properties of the BEs stated in Theorem 3. □

3 On identifiability

Recall that we consider the situation where $K \in \mathbb{N}^*$ detectors $\mathbb{D}_1, \ldots, \mathbb{D}_K \in \mathbb{R}^2$ receive signals from two sources $\mathbb{S}_1, \mathbb{S}_2 \in \mathbb{R}^2$. In this section we make no distinction between detectors/sources and their positions, and we suppose that the two signals received by a detector are identical, i.e., we have $S_{1,k}(\cdot) = S_{2,k}(\cdot) = S_k(\cdot)$ in the right hand side of (1), and so the signals recorded by the k-th detector are

$$S_{i,k}(\vartheta^{(i)}, t) = \psi(t - \tau_{i,k})\, S_k(t), \qquad i = 1, 2.$$

Considering the identifiability condition $\mathscr{R}_4$, we can see that if for some $\vartheta \in \Theta$ and some $\varepsilon > 0$ satisfying $\|\vartheta - \vartheta_0\|_4 \geqslant \varepsilon > 0$ we have

$$\sum_{k=1}^{K} \int_0^T \left[\psi(t - \tau_{1,k})\, S_k(t) + \psi(t - \tau_{2,k})\, S_k(t) - \psi(t - \tau^\circ_{1,k})\, S_k(t) - \psi(t - \tau^\circ_{2,k})\, S_k(t)\right]^2 \mathrm{d}t = 0,$$

then, taking into account that $S_k(t) > 0$ and that $\psi(t) = 0$ for $t < 0$ and $\psi(t) \neq 0$ for $t > 0$, we should have (for each k) either

$$\tau_{1,k} = \tau^\circ_{1,k} \quad \text{and} \quad \tau_{2,k} = \tau^\circ_{2,k}$$

or

$$\tau_{1,k} = \tau^\circ_{2,k} \quad \text{and} \quad \tau_{2,k} = \tau^\circ_{1,k}.$$

Of course, in such situation the consistent estimation is impossible, because for two different values of the unknown parameter we have the same statistical model.

We see that the question of identifiability is reduced to the following one: *when having the distances from each detector to the two sources (without knowing which distance corresponds to which source) is it possible to find ϑ?* So, the system will be identifiable if and

only if there exist no two different pairs of sources providing the same K pairs of distances to the detectors.

We introduce the following definition.

Definition 1. *We say that $n \in \mathbb{N}^*$ points $A_1, \ldots, A_n \in \mathbb{R}^2$ "lay on a cross" if there exist a pair of orthogonal lines ℓ_1 and ℓ_2 such that $A_i \in \ell_1 \cup \ell_2$ for all $i = 1, \ldots, n$.*

Now we can state the theorem providing a necessary and sufficient condition for identifiability.

Theorem 4. *A sufficient condition for a system with K detectors to be identifiable is that the detectors do not lay on a cross.*

In absence of restrictions on the positions of the sources (i.e., if we could suppose that $\Theta = \mathcal{R}^4$) this condition would be also necessary.

Before proving this theorem, let us note that at least 4 detectors are necessary for the system to be identifiable. Indeed, any three detectors $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ lay on a cross (take, for example, the line $\overleftrightarrow{\mathbb{D}_1\mathbb{D}_2}$ and the perpendicular to this line passing by $\mathbb{D}_3$).

Note also that 4 detectors do not lay on a cross if (and only if) they are in general linear position (any 3 of them are not aligned) and cannot be split in two pairs such that the lines passing by these pairs are orthogonal.

Finally, note that if 5 (or more) detectors are in general linear position, then they necessarily do not lay on a cross (and hence the system is identifiable). Indeed, if they laid on a cross, at least one of the lines forming the cross would contain at least three of them, and so, they would not be in general linear position.

Proof. Note that if the system is identifiable with $\Theta = \mathcal{R}^4$, it will be also identifiable with any $\Theta \subset \mathcal{R}^4$. So we can suppose $\Theta = \mathcal{R}^4$ and reformulate the theorem using the contrapositions as follows: *the detectors $\mathbb{D}_1, \ldots, \mathbb{D}_K$ lay on a cross if and only if there exist two different pairs of sources providing the same K pairs of distances to them.*

In order to show the necessity, we suppose that the detectors $\mathbb{D}_1, \ldots, \mathbb{D}_K$ lay on a cross formed by a pair of orthogonal line ℓ_1 and ℓ_2, and we need to find two different pairs of sources $\{\mathbb{S}_1, \mathbb{S}_2\}$ and $\{\mathbb{S}'_1, \mathbb{S}'_2\}$ providing the same K pairs of distances to the detectors.

Let us denote O the point of intersection of ℓ_1 and ℓ_2, take an arbitrary point $\mathbb{S}_1$ not belonging to neither ℓ_1, nor ℓ_2, and denote $\mathbb{S}'_1$, $\mathbb{S}'_2$ and $\mathbb{S}_2$ the points symmetric to $\mathbb{S}_1$ with respect to ℓ_1, ℓ_2 and O respectively (see Fig. 4).

Then, clearly, the pairs of sources $\{\mathbb{S}_1, \mathbb{S}_2\}$ and $\{\mathbb{S}'_1, \mathbb{S}'_2\}$ provide the same pair of distances to any point of a cross, and hence the same K pairs of distances to the detectors $\mathbb{D}_1, \ldots, \mathbb{D}_K$.

Now we turn to the proof of the sufficiency. We suppose that two different pairs of sources $\{\mathbb{S}_1, \mathbb{S}_2\}$ and $\{\mathbb{S}'_1, \mathbb{S}'_2\}$ provide the same K pairs of distances to the detectors $\mathbb{D}_1, \ldots, \mathbb{D}_K$, and we need to show that the detectors lay on a cross.

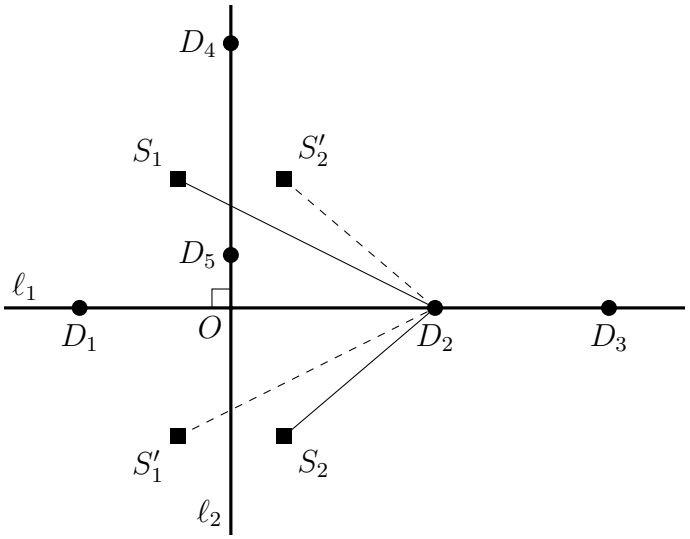

Figure 4: Non identifiability for detectors laying on a cross

We need to distinguish several cases.

If in one of the two pairs the sources are located at the same point (say $\mathbb{S}_1 = \mathbb{S}_2$), then at least one of the points $\mathbb{S}_1'$ and $\mathbb{S}_2'$ (say $\mathbb{S}_1'$) must be different from $\mathbb{S}_1$ and $\mathbb{S}_2$ (see the left picture in Fig. 5). Then, if our two pairs of sources provide the same pair of distances to a point D, we should have, in particular, $\rho(D, \mathbb{S}_1) = \rho(D, \mathbb{S}_1')$, and hence $D \in b_{\mathbb{S}_1\mathbb{S}_1'}$. Here and in the sequel, we denote ρ the Euclidean distance, and for any two distinct points A and B, we denote b_{AB} the perpendicular bisector of the segment $\overline{AB}$. Therefore, all the detectors must belong to the line $b_{\mathbb{S}_1\mathbb{S}_1'}$ (and, in particular, they lay on a cross).

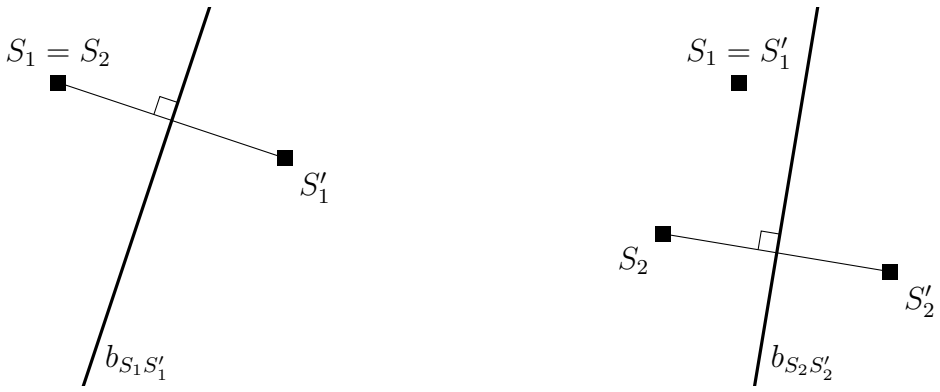

Figure 5: Cases with coinciding sources

So, from now on, we can suppose that $\mathbb{S}_1 \neq \mathbb{S}_2$ and $\mathbb{S}_1' \neq \mathbb{S}_2'$.

Now we consider the case when the two pairs of sources have a point in common (say $\mathbb{S}_1 = \mathbb{S}_1'$). Note that in this case, we must have $\mathbb{S}_2 \neq \mathbb{S}_2'$, since otherwise the pairs of sources will not be different (see the left picture in Fig. 5). Then, our two pairs of sources

provide the same pair of distances to a point D if and only if $\rho(D, \mathbb{S}_2) = \rho(D, \mathbb{S}'_2)$, which is equivalent to $D \in b_{\mathbb{S}_2\mathbb{S}'_2}$. Therefore, all the detectors must belong to the line $b_{\mathbb{S}_2\mathbb{S}'_2}$ (and, in particular, they lay on a cross).

So, from now on, we can suppose that the points $\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}'_1, \mathbb{S}'_2$ are all different. In this case, our two pairs of sources provide the same pair of distances to a point D if and only if D belongs at the same time either to the pair of lines $b_{\mathbb{S}_1\mathbb{S}'_1}$ and $b_{\mathbb{S}_2\mathbb{S}'_2}$, or to the pair of lines $b_{\mathbb{S}_1\mathbb{S}'_2}$ and $b_{\mathbb{S}_2\mathbb{S}'_1}$ $\big($otherwise speaking, if and only if $D \in (b_{\mathbb{S}_1\mathbb{S}'_1} \cap b_{\mathbb{S}_2\mathbb{S}'_2}) \cup (b_{\mathbb{S}_1\mathbb{S}'_2} \cap b_{\mathbb{S}_2\mathbb{S}'_1})\big)$. We need again to distinguish several (sub)cases depending on whether the lines in each of these pairs coincide or not.

First we suppose that $b_{\mathbb{S}_1\mathbb{S}'_1} \neq b_{\mathbb{S}_2\mathbb{S}'_2}$ and $b_{\mathbb{S}_1\mathbb{S}'_2} \neq b_{\mathbb{S}_2\mathbb{S}'_1}$ (see the left picture in Fig. 6). Then each of these pairs of lines meet in at most one point. So, all the detectors must belong to a set consisting of at most two points (and, therefore, there is at most two detectors and they trivially lay on a cross).

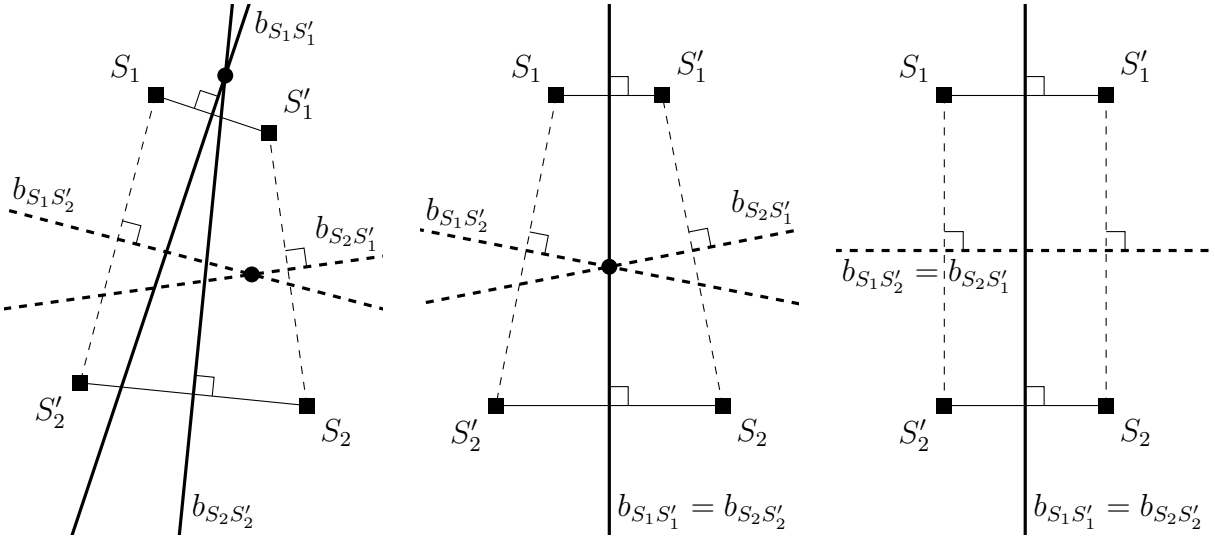

Figure 6: Cases without coinciding sources

Now we suppose that the lines coincide in one of the pairs, and not in the other (say $b_{\mathbb{S}_1\mathbb{S}'_1} = b_{\mathbb{S}_2\mathbb{S}'_2}$ and $b_{\mathbb{S}_1\mathbb{S}'_2} \neq b_{\mathbb{S}_2\mathbb{S}'_1}$). Note that in this case, the points $\mathbb{S}_1$ and $\mathbb{S}'_1$, as well as the points $\mathbb{S}_2$ and $\mathbb{S}'_2$, are symmetric with respect to the line $b_{\mathbb{S}_1\mathbb{S}'_1}$, and hence the lines $b_{\mathbb{S}_1\mathbb{S}'_2}$ and $b_{\mathbb{S}_2\mathbb{S}'_1}$ either do not meet, or meet in a point belonging to $b_{\mathbb{S}_1\mathbb{S}'_1}$ (see the middle picture in Fig. 6). So, finally all the detectors must belong to the line $b_{\mathbb{S}_1\mathbb{S}'_1}$ (and, in particular, they lay on a cross).

It remains to consider the case when $b_{\mathbb{S}_1\mathbb{S}'_1} = b_{\mathbb{S}_2\mathbb{S}'_2}$ and $b_{\mathbb{S}_1\mathbb{S}'_2} = b_{\mathbb{S}_2\mathbb{S}'_1}$ (see the right picture in Fig. 6). In this case, as the segments $\overline{\mathbb{S}_1\mathbb{S}'_1}$ and $\overline{\mathbb{S}_2\mathbb{S}'_2}$ have a common perpendicular bisector, they must, in particular, be parallel (including the case where they lay on a same line). The same reasoning applies to the segments $\overline{\mathbb{S}_1\mathbb{S}'_2}$ and $\overline{\mathbb{S}_2\mathbb{S}'_1}$. Now, the case when the points $\mathbb{S}_1, \mathbb{S}'_1, \mathbb{S}_2, \mathbb{S}'_2$ lay on a same line is impossible, since these points being all different, we

clearly can not have common perpendicular bisectors at the same time for $\overline{\mathbb{S}_1\mathbb{S}'_1}$ and $\overline{\mathbb{S}_2\mathbb{S}'_2}$ and for $\overline{\mathbb{S}_1\mathbb{S}'_2}$ and $\overline{\mathbb{S}_2\mathbb{S}_1}$. Thus, $\mathbb{S}_1\mathbb{S}'_1\mathbb{S}_2\mathbb{S}'_2$ is a parallelogram. Moreover, as its opposite sides have common perpendicular bisectors, $\mathbb{S}_1\mathbb{S}'_1\mathbb{S}_2\mathbb{S}'_2$ is necessarily a rectangle. But in this case, the perpendicular bisectors $b_{\mathbb{S}_1\mathbb{S}'_1}$ and $b_{\mathbb{S}_1\mathbb{S}'_2}$ are orthogonal. Hence, as all the detectors belong to $b_{\mathbb{S}_1\mathbb{S}'_1} \cup b_{\mathbb{S}_1\mathbb{S}'_2}$, they lay again on a cross. $\square$

4 Discussions

The considered models in the cases of cusp and change-point singularities can be easily generalized to the models with signals $s_{i,k}\big(t-\tau_k(\vartheta)\big)$. The proofs will be more cumbersome but the rates and the limit distributions of the studied estimators will be the same.

Remark that the same mathematical models are used in the applications related to the detection of weak optical signals from two sources.

Of course, it will be interesting to see the conditions of identifiability in the situation where the beginning of the emissions of these sources are unknown and have to be estimated together with the positions. In the case of one source such problem was discussed in the work [4].

All useful information about the position of sources, according to the statements of this work, is contained in the times of arrival of the signals $\tau_{1,k}(\vartheta)$, $\tau_{2,k}(\vartheta)$, $k=1,\ldots,K$. It is possible to study another statement of the problem supposing that these moments are estimated separately by observations $X_k=(X_k(t),\ 0\leqslant t\leqslant T)$ for each k, say by $\hat{\tau}_{i,k,n}=\hat{\tau}_{i,k,n}(X_k)$, and then the positions of the sources are estimated on the base of the obtained estimators $\hat{\tau}_{i,k,n}$, $i=1,2$, $k=1,\ldots,K$. Such approach was considered in the works [4, 6] (see as well [16]).

Note that in the works on Poisson source localization, the intensities of Poisson processes are sometimes taken in the form

$$\lambda_k(\vartheta,t)=F\big(\rho_k(\vartheta)\big)\,S_k(t)+\lambda_0,\qquad 0\leqslant t\leqslant T,$$

where $F(\cdot)$ is a known strictly decreasing function of the distance ρ_k between the source and the k-th detector. The developed in the present work approach (smooth case) can be applied for such models too, because here all the useful information is contained in the distances $\rho_k(\vartheta)$.

Acknowledgments. This research was financially supported by the Ministry of Education and Science of Russian Federation (project No. FSWF-2023-0012) for sections 2.1 and 2.2, and by the Russian Science Foundation (project No. 20-61-47043) for sections 2.3 and 2.4.

Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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