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Localization of two radioactive sources on the plane

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Abstract

It is considered the problem of localization on the plane of two radioactive sources by K detectors. Each detector records a realization of inhomogeneous Poisson process and the intensity function of this process is a sum of a signal arriving from the sources and the constant Poisson noise of known intensity. The time of the beginning of emissions of two sources is known and the main problem is the estimation of the position of the sources. The properties of the MLE and Bayesian estimators are described in the asymptotics of *large signals* in three situations of different regularities of the fronts of the signals: smooth, cusp-type and change-point type.

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Key words: Poisson process, parameter estimation, cusp-type singularity.

1 Introduction

Suppose that there are two radioactive sources and K detectors on the plane. The sources start emission at known time, which can be taken $t = 0$ without loss of generality. The detectors receive Poisson signals with additive noise. The intensity functions of these processes depend on the positions of the detectors (known) and the positions of the sources (unknown) and the main problem is the estimation of the positions of the sources. An example of a possible configuration on the plane of the sources and detectors is given on the Fig. 1.

The problems of localization of the sources with Poisson, Gaussian or more general classes of distributions due to their importance are widely studied in engineering literature, see, e.g., [2], [3], [9], [22], [25], [27], see as well the Handbook [11] and references therein. Mathematical statements are less known. This work is a continuation of the study initiated

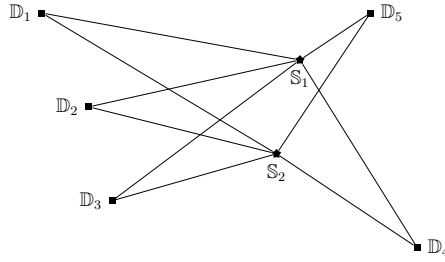


Figure 1: Model of observations. S_1, S_2 are positions of the sources and $D_i, i = 1, \dots, 5$ are positions of the sensors.

in [10] and then developed in [1], [4], [6], [7], [19], where it was always supposed that there is just one source on the plane. In the works [1], [4], [6], [7], [10] it was supposed that the moment of beginning of emission τ_0 is known and the unknown parameter was just a position of the source. The case where τ_0 is unknown too and we have to estimate τ_0 and position of the source was treated in [1] and [4]. In all works the properties of the MLE and BE of the position and position and τ_0 correspondingly are described. Moreover the properties of the least squares estimators are described too. As the types of the data we considered the inhomogeneous Poisson processes and diffusion processes. The properties of the MLE and Bayes estimators are described in the asymptotics of large signals or small noise.

At the present work we suppose that there are two radioactive sources emitting inhomogeneous Poisson processes and the emission starts at the (known) moment $t = 0$. Therefore there are K detectors and we have to estimate the positions of the sources only. The properties of estimators as in all preceding works are described with the help of the Ibragimov-Khasminskii approach (see [14]), which consists in the verification of certain properties of the normalized likelihood ratio process.

The information about the position of sources is in the time of arriving of the signals in the detectors. These times depend on the distances between sources and detectors. The estimators of the positions depend on the estimation of these times and here the form of the fronts of the arriving signals plays important role. We consider three types of fronts: smooth, cusp-type and change-point type. In the smooth case the Fisher information matrix is finite and the estimators are asymptotically normal. The cusp-case corresponds to a front described by a continuous function but the Fisher information does not exist (is infinite). The Bayesian estimators converge to a random vector defined with the help of some functionals of fractional Brownian motion. In the change-point case with discontinuous intensities the limit distribution of bayessian estimators is defined with a help of some Poisson processes. In all cases we discuss the asymptotic efficiency of the proposed estimators.

Special attention is paid to the condition of identifiability, i.e., to the description of the admissible configurations of detectors which allow the consistent estimation of the positions

of the sources. It is shown that if all detectors are on the cross then for any two sources it is possible to define a different couple of sources (twins) which provide the same moments of arriving in all detectors and therefore the consistent estimation is impossible.

2 Main results

2.1 Model of observations

There are K detectors $\mathbb{D}_1, \dots, \mathbb{D}_K$ located on the plane at the (known) points $D_k = (x_k, y_k)$, $k = 1, \dots, K \geq 4$ and two sources \mathbb{S}_1 and \mathbb{S}_2 at the (unknown) points $S_1 = (x_1^\circ, y_1^\circ)$ and $S_2 = (x_2^\circ, y_2^\circ)$. Therefore the unknown parameter is $\vartheta = (x_1^\circ, y_1^\circ, x_2^\circ, y_2^\circ)$, but it will be convenient to write it as $\vartheta = (\vartheta_1, \dots, \vartheta_4) \in \Theta$ and as $\vartheta = (\vartheta^{(1)}, \vartheta^{(2)})$ with obvious notations. The set Θ is an open, bounded, convex subset of \mathcal{R}^4 . We suppose that the positions of the detectors do not coincide and the sources have different positions too. The position of a source does not coincide with the position of a detector.

The both sources start emission at the moment $t = 0$. The k -th detector records a realization $X_k = (X_k(t), 0 \leq t \leq T)$ of the inhomogeneous Poisson process of intensity function

$$\lambda_{k,n}(\vartheta, t) = nS_{1,k}(\vartheta^{(1)}, t) + nS_{2,k}(\vartheta^{(2)}, t) + n\lambda_0, \quad 0 \leq t \leq T,$$

where $n\lambda_0 > 0$ is the intensity of the Poisson noise and $nS_{i,k}(\vartheta^{(i)}, t)$ is the signal recorded from the i -th source, $i = 1, 2$. We suppose that the recorded signals have the following structure

$$S_{i,k}(\vartheta^{(i)}, t) = \psi(t - \tau_k(\vartheta^{(i)})) S_{i,k}(t).$$

Here $S_{i,k}(t) > 0$ is a bounded function and $\tau_k(\vartheta^{(i)})$ is the time of arriving of the signal from the i -th source at the k -th detector, i.e.,

$$\tau_{i,k}(\vartheta) = \nu^{-1} \|D_k - S_i\| = \nu^{-1} (|x_k - x_i^\circ|^2 + |y_k - y_i^\circ|^2)^{1/2}, \quad i = 1, 2,$$

and $\nu > 0$ is the rate of propagation of the signals. The function $\psi(\cdot)$ describes the front of the signals. As in our preceding works (see, e.g. [4]) we take

$$\psi(s) = \left| \frac{s}{\delta} \right|^\kappa \mathbb{I}_{\{0 \leq s \leq \delta\}} + \mathbb{I}_{\{s > \delta\}}, \quad (1)$$

where the parameter $\kappa \geq 0$ describes the regularity of the statistical problem. If $\kappa \geq \frac{1}{2}$, $\delta > 0$, then we have regular statistical experiment, if $\kappa \in (0, \frac{1}{2})$, $\delta > 0$ then we have singularity of cusp type and if $\kappa = 0$, $\delta = 0$, then the intensity is discontinuous function and we have change point model [4]. The examples of these three cases are given on the Fig. 2, where we put $nS_{i,k}(t) \equiv 2$, $n\lambda_0 = 1$ and **a)** $\kappa = 1$, **b)** $\kappa = 1/4$, **c)** $\kappa = 0$.

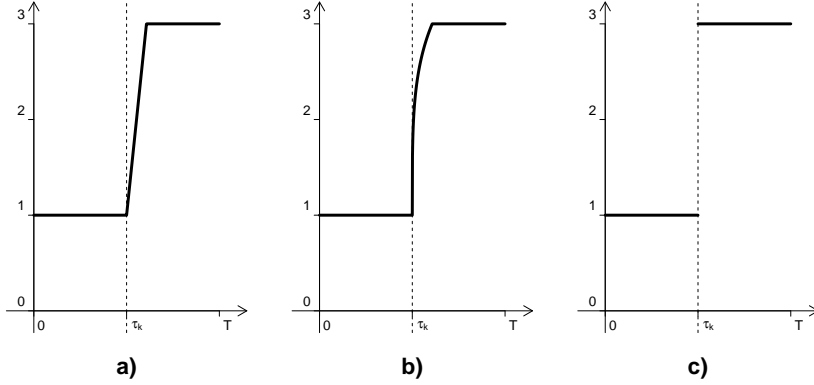


Figure 2: Intensities with three types of fronts of arriving signals

Note that $\psi(s) = 0$ for $s < 0$ and therefore for $t < \tau_k(\vartheta^{(1)}) \wedge \tau_k(\vartheta^{(2)})$ the intensity function $\lambda_{k,n}(\vartheta, t) = n\lambda_0$. The functions $\psi(\cdot)$ in the cases **a)**-**c)** we denote as $\psi_\delta(\cdot)$, $\psi_{\kappa,\delta}(\cdot)$ and $\psi(\cdot)$ respectively.

According to this form of intensity function all information concerning the position of the sources is in the moments of arriving $\tau_k(\vartheta^{(i)})$, $i = 1, 2; k = 1, \dots, K$.

We are interested by the situation where the errors of estimation are small. In the problem of localization it is natural to suppose that the registered intensities take large values. Therefore we study the properties of estimators of the positions in the asymptotics of large intensities, that is why we introduce in the intensity functions the factor n and the asymptotic corresponds to the limit $n \rightarrow \infty$. In the section 3 we explain that the condition $K \geq 4$ is a necessary one for the existence of consistent estimators.

2.2 Maximum likelihood and Bayesian estimators

As the intensity functions $\lambda_{k,n}(\cdot)$ are bounded and separated from zero the measures $\mathbf{P}_\vartheta^{(n)}$, $\vartheta \in \Theta$ corresponding to the observations X^K with $\vartheta \in \Theta$ in the space of realizations are equivalent and the likelihood ratio function is

$$L(\vartheta, X^K) = \exp \left\{ \sum_{k=1}^K \int_0^T \ln \frac{\lambda_{k,n}(\vartheta)}{n\lambda_0} dX_k(t) - \sum_{k=1}^K \int_0^T [\lambda_{k,n}(\vartheta) - n\lambda_0] dt \right\}, \quad \vartheta \in \Theta \quad (2)$$

The MLE $\hat{\vartheta}_n$ and BE for quadratic loss function $\tilde{\vartheta}_n$ are defined by the usual relations

$$L(\hat{\vartheta}_n, X^K) = \sup_{\vartheta \in \Theta} L(\vartheta, X^K), \quad \tilde{\vartheta}_n = \frac{\int_{\Theta} \vartheta p(\vartheta) L(\vartheta, X^K) d\vartheta}{\int_{\Theta} p(\vartheta) L(\vartheta, X^K) d\vartheta},$$

where $p(\vartheta)$ $\vartheta \in \Theta$ is distribution *a priori* of the (random) parameter ϑ .

2.2.1 Smooth fronts

First we consider the regular case in slightly different situation, which can be seen as a more general. The intensities of the observed processes are supposed to be

$$\lambda_{k,n}(\vartheta, t) = nS_{1,k}(\vartheta^{(1)}, t) + nS_{2,k}(\vartheta^{(2)}, t) + n\lambda_0 = n\lambda_k(\vartheta, t), \quad 0 \leq t \leq T,$$

where $\lambda_k(\vartheta, t)$ is defined by the last equality and $S_{i,k}(\vartheta^{(i)}, t) = S_{i,k}(t - \tau_k(\vartheta^{(i)}))$, $i = 1, 2$; $k = 1, \dots, K$. For the derivatives we have the expressions

$$\frac{\partial S_{1,k}(t - \tau_k(\vartheta^{(1)}))}{\partial \vartheta_1} = \nu^{-1} S'_{1,k}(t - \tau_k(\vartheta^{(1)})) \frac{x_k - x_1^\circ}{\rho_{1,k}} = \nu^{-1} S'_{1,k}(t - \tau_k(\vartheta^{(1)})) \cos(\alpha_{1,k}),$$

where $\rho_{1,k} = \|D_k - S_1\|_2$ and $\cos(\alpha_{1,k}) = (x_k - x_1^\circ) \rho_{1,k}^{-1}$. We denote $\|\cdot\|_2$ and $\|\cdot\|_4$ euclidean norms in \mathcal{R}^2 and \mathcal{R}^4 respectively. Therefore we have as well

$$\frac{\partial S_{1,k}(t - \tau_k(\vartheta^{(1)}))}{\partial \vartheta_2} = \nu^{-1} S'_{1,k}(t - \tau_k(\vartheta^{(1)})) \sin(\alpha_{1,k}).$$

Of course, $\partial S_{1,k}/\partial \vartheta_3 = 0$ and $\partial S_{1,k}/\partial \vartheta_4 = 0$. Recall the notation $\vartheta = (\vartheta^{(1)}, \vartheta^{(2)})^\top$ and $\vartheta^{(1)} = (\vartheta_1, \vartheta_2)^\top$ and $\vartheta^{(2)} = (\vartheta_3, \vartheta_4)^\top$. The similar expressions we have for $\partial S_{2,k}/\partial \vartheta_3$ and $\partial S_{2,k}/\partial \vartheta_4$:

$$\begin{aligned} \frac{\partial S_{2,k}(t - \tau_k(\vartheta^{(2)}))}{\partial \vartheta_3} &= \nu^{-1} S'_{2,k}(t - \tau_k(\vartheta^{(2)})) \cos(\alpha_{2,k}), \\ \frac{\partial S_{2,k}(t - \tau_k(\vartheta^{(2)}))}{\partial \vartheta_4} &= \nu^{-1} S'_{2,k}(t - \tau_k(\vartheta^{(2)})) \sin(\alpha_{2,k}). \end{aligned}$$

Introduce the vectors $m_{1,k} = m_{1,k}(\vartheta^{(1)}) = (\sin(\alpha_{1,k}), \cos(\alpha_{1,k}))^\top$ and $m_{2,k} = m_{2,k}(\vartheta^{(2)}) = (\sin(\alpha_{2,k}), \cos(\alpha_{2,k}))^\top$. We have $\|m_{i,k}\|_2 = 1$, $i = 1, 2$. We will several times use the expansion (below $\varphi_n \rightarrow 0$ and $u^{(1)} = (u_1, u_2)^\top$, $u^{(2)} = (u_3, u_4)^\top$)

$$\tau_k(\vartheta_0^{(i)} + u^{(i)} \varphi_n) = \tau_k(\vartheta_0^{(i)}) - \nu^{-1} \langle m_{i,k}, u^{(i)} \rangle \varphi_n + O(\varphi_n^2). \quad (3)$$

For simplicity of exposition we will use the notations $\tau_{1,k} = \tau_k(\vartheta^{(1)})$ and $\tau_{2,k} = \tau_k(\vartheta^{(2)})$.

The Fisher information matrix is

$$I(\vartheta)_{4 \times 4} = \sum_{k=1}^K \int_0^T \frac{\dot{\lambda}_k(\vartheta, t) \dot{\lambda}_k(\vartheta, t)^\top}{\lambda_k(\vartheta, t)} dt.$$

Here and in the sequel dot means derivative w.r.t. ϑ . The elements of this matrix have the following expressions

$$I(\vartheta)_{11} = \sum_{k=1}^K \int_{\tau_{1,k}} \frac{S'_{1,k}(t - \tau_{1,k})^2 \cos^2(\alpha_{1,k})}{\nu^2 \lambda_k(\vartheta, t)} dt,$$

$$\begin{aligned}
\mathbf{I}(\vartheta)_{12} &= \sum_{k=1}^K \int_{\tau_{1,k}} \frac{S'_{1,k}(t - \tau_{1,k})^2 \cos(\alpha_{1,k}) \sin(\alpha_{1,k})}{\nu^2 \lambda_k(\vartheta, t)} dt \\
\mathbf{I}(\vartheta)_{13} &= \sum_{k=1}^K \int_{\tau_{1,k} \vee \tau_{2,k}} \frac{S'_{1,k}(t - \tau_{1,k}) S'_{2,k}(t - \tau_{2,k}) \cos(\alpha_{1,k}) \cos(\alpha_{2,k})}{\nu^2 \lambda_k(\vartheta, t)} dt, \\
\mathbf{I}(\vartheta)_{14} &= \sum_{k=1}^K \int_{\tau_{1,k} \vee \tau_{2,k}} \frac{S'_{1,k}(t - \tau_{1,k}) S'_{2,k}(t - \tau_{2,k}) \cos(\alpha_{1,k}) \sin(\alpha_{2,k})}{\nu^2 \lambda_k(\vartheta, t)} dt
\end{aligned}$$

The other terms can be written using the same rule.

The regularity conditions are

Conditions \mathcal{R} .

- \mathcal{R}_1 . The functions $S_{i,k}(s) = 0$ for $s \leq 0$ and $S_{i,k}(s) > 0$ for $s > 0$ for all $i = 1, 2; k = 1, \dots, K$.
- \mathcal{R}_2 . The functions $S_{i,k}(\cdot) \in \mathcal{C}^2, i = 1, 2; k = 1, \dots, K$. The set $\Theta \subset \mathcal{R}^4$ is open, bounded and convex.
- \mathcal{R}_3 . The Fisher information matrix is uniformly non degenerate

$$\inf_{\vartheta \in \Theta} \inf_{\|e\|_4=1} e^\top \mathbf{I}(\vartheta) e > 0,$$

where $e \in \mathcal{R}^4$.

- \mathcal{R}_4 . For any $\vartheta_0 = (\vartheta_0^{(1)}, \vartheta_0^{(2)})$ and any $\varepsilon > 0$

$$\begin{aligned}
\inf_{\vartheta \in \Theta} \inf_{\|\vartheta - \vartheta_0\|_4 > \varepsilon} \sum_{k=1}^K \int_0^T [S_{1,k}(t - \tau_{1,k}) + S_{2,k}(t - \tau_{2,k}) \\
- S_{1,k}(t - \tau_{1,k}^\circ) - S_{2,k}(t - \tau_{2,k}^\circ)]^2 dt > 0. \quad (4)
\end{aligned}$$

where $\tau_{i,k}^\circ = \tau_k(\vartheta_0^{(i)}), i = 1, 2$.

It can be shown that if the conditions $\mathcal{R}_1 - \mathcal{R}_3$ are fulfilled then the family of measures $(\mathbf{P}_\vartheta^{(n)}, \vartheta \in \Theta)$ is locally asymptotically normal (LAN) (see Lemma 2.1 in [18]) and therefore we have Hajek-Le Cam's lower bound on the risks of all estimators (see, e.g. [14])

$$\lim_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \sup_{\|\vartheta - \vartheta_0\| \leq \varepsilon} n \mathbf{E}_\vartheta \|\bar{\vartheta}_n - \vartheta\|_4^2 \geq \mathbf{E}_{\vartheta_0} \|\zeta\|_4^2, \quad \zeta \sim \mathcal{N}(0, \mathbf{I}(\vartheta_0)^{-1}) \quad (5)$$

The asymptotically efficient estimator is defined as estimator for which there is equality in (5) for all $\vartheta_0 \in \Theta$.

Theorem 1. *Let the conditions \mathcal{R} be fulfilled, then the MLE and BE are consistent, uniformly on compacts $\mathbb{K} \subset \Theta$ asymptotically normal*

$$\sqrt{n} \left(\hat{\vartheta}_n - \vartheta_0 \right) \Longrightarrow \zeta, \quad \sqrt{n} \left(\tilde{\vartheta}_n - \vartheta_0 \right) \Longrightarrow \zeta,$$

the polynomial moments converge: for any $p > 0$

$$n^{\frac{p}{2}} \mathbf{E}_{\vartheta_0} \|\hat{\vartheta}_n - \vartheta_0\|_4^p \rightarrow \mathbf{E}_{\vartheta_0} \|\zeta\|_4^p, \quad n^{\frac{p}{2}} \mathbf{E}_{\vartheta_0} \|\tilde{\vartheta}_n - \vartheta_0\|_4^p \rightarrow \mathbf{E}_{\vartheta_0} \|\zeta\|_4^p,$$

the both estimators are asymptotically efficient.

Proof. This theorem is a particular case of Theorems 2.4 and 2.5 in [18] (see as well [17]). Note that the model of observations with large intensity asymptotics is equivalent to the model of n independent identically distributed inhomogeneous Poisson process with $n \rightarrow \infty$. To verify the condition **B4** of the Theorem 2.4 we write: for $\|\vartheta - \vartheta_0\|_4 \leq \varepsilon$ and sufficiently small $\varepsilon > 0$

$$\begin{aligned} \sum_{k=1}^K \int_0^T \left[\sqrt{\lambda_k(\vartheta, t)} - \sqrt{\lambda_k(\vartheta_0, t)} \right]^2 dt &= \frac{1}{4} (\vartheta - \vartheta_0)^\top \mathbf{I}(\vartheta_0) (\vartheta - \vartheta_0) (1 + O(\varepsilon)) \\ &\geq \frac{1}{8} (\vartheta - \vartheta_0)^\top \mathbf{I}(\vartheta_0) (\vartheta - \vartheta_0) = \frac{1}{8} \|\vartheta - \vartheta_0\|^2 e^\top \mathbf{I}(\vartheta_0) e \geq \kappa_1 \|\vartheta - \vartheta_0\|_4^2. \end{aligned} \quad (6)$$

Here we used the condition \mathcal{R}_3 .

For $\|\vartheta - \vartheta_0\|_4 \geq \varepsilon$

$$\begin{aligned} \sum_{k=1}^K \int_0^T \left[\sqrt{\lambda_k(\vartheta, t)} - \sqrt{\lambda_k(\vartheta_0, t)} \right]^2 dt &= \sum_{k=1}^K \int_0^T \frac{[\lambda_k(\vartheta, t) - \lambda_k(\vartheta_0, t)]^2}{\left[\sqrt{\lambda_k(\vartheta, t)} + \sqrt{\lambda_k(\vartheta_0, t)} \right]^2} dt \\ &\geq C \sum_{k=1}^K \int_0^T [\lambda_k(\vartheta, t) - \lambda_k(\vartheta_0, t)]^2 dt \\ &\geq C \sum_{k=1}^K \int_0^T [S_{1,k}(t - \tau_{1,k}) + S_{2,k}(t - \tau_{2,k}) - S_{1,k}(t - \tau_{1,k}^\circ) - S_{1,k}(t - \tau_{1,k}^\circ)]^2 dt \\ &\geq Cg(\varepsilon) \end{aligned}$$

Here we used the boundedness of the functions $\lambda_k(\vartheta, t)$ and denoted $g(\varepsilon) > 0$ the left hand side of (4). Let us denote $D(\Theta) = \sup_{\vartheta, \tilde{\vartheta} \in \Theta} \|\vartheta - \tilde{\vartheta}\|_4$. Then we have

$$\sum_{k=1}^K \int_0^T \left[\sqrt{\lambda_k(\vartheta, t)} - \sqrt{\lambda_k(\vartheta_0, t)} \right]^2 dt \geq Cg(\varepsilon) \geq Cg(\varepsilon) \frac{\|\vartheta - \vartheta_0\|_4^2}{D(\Theta)^2} \geq \kappa_2 \|\vartheta - \vartheta_0\|_4^2 \quad (7)$$

The estimates (6) and (7) can be joined in

$$\sum_{k=1}^K \int_0^T \left[\sqrt{\lambda_k(\vartheta, t)} - \sqrt{\lambda_k(\vartheta_0, t)} \right]^2 dt \geq \kappa \|\vartheta - \vartheta_0\|_4^2 \quad (8)$$

where $\kappa = \kappa_1 \wedge \kappa_2$. Now **B4** follows from (8). □

2.2.2 Cusp type fronts

Let us return to the intensity function with cusp type singularity. Suppose that the observed Poisson processes $X^K = (X_k(t), 0 \leq t \leq T, k = 1, \dots, K)$ have intensity functions

$$\lambda_{k,n}(\vartheta, t) = n\psi_{\kappa,\delta}(t - \tau_{1,k})S_{1,k}(t) + n\psi_{\kappa,\delta}(t - \tau_{2,k})S_{2,k}(t) + n\lambda_0 = n\lambda_k(\vartheta, t), \quad (9)$$

where $\kappa \in (0, \frac{1}{2})$ and

$$\psi_{\kappa,\delta}(t - \tau_{i,k}) = \left| \frac{t - \tau_{i,k}}{\delta} \right|^\kappa \mathbb{I}_{\{0 \leq t - \tau_{i,k} \leq \delta\}} + \mathbb{I}_{\{t - \tau_{i,k} > \delta\}}, \quad i = 1, 2, \quad k = 1, \dots, K. \quad (10)$$

An example of intensity function with two cusp type singularities of the Poisson process recorded by one detector is given on Fig. 3

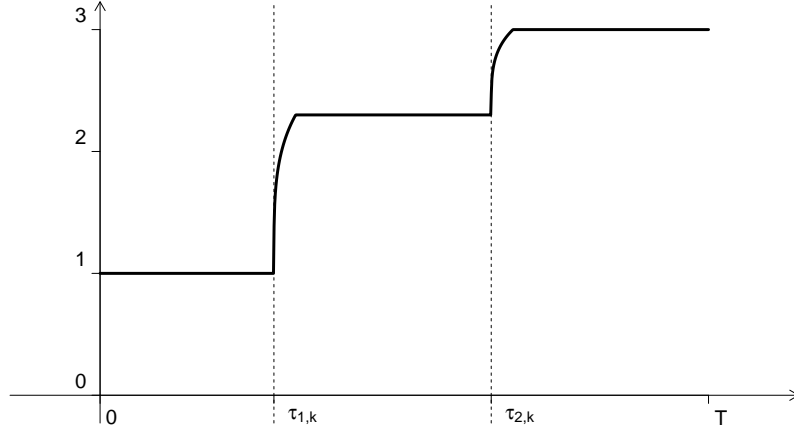


Figure 3: Intensity with two cusp type singularities

Recall that

$$\begin{aligned} & \nu \left[\tau_k(\vartheta_u^{(1)}) - \tau_k(\vartheta_0^{(1)}) \right] \\ &= \left[(x_k - x_1^\circ - u_1\varphi_n)^2 + (y_k - y_1^\circ - u_2\varphi_n)^2 \right]^{1/2} - \left[(x_k - x_1^\circ)^2 + (y_k - y_1^\circ)^2 \right]^{1/2} \\ &= (1 - u_1\varphi_n \cos(\alpha_{1,k}) - u_2\varphi_n \sin(\alpha_{1,k})) (1 + O(\varphi_n)) - \rho_{1,k} \\ &= -\langle u^{(1)}, m_{1,k} \rangle \varphi_n (1 + O(\varphi_n)) \end{aligned}$$

and $\nu \left[\tau_k(\vartheta_u^{(2)}) - \tau_k(\vartheta_0^{(2)}) \right] = -\langle u^{(2)}, m_{2,k} \rangle \varphi_n (1 + O(\varphi_n))$.

Introduce the notations

$$\begin{aligned} \mathbb{B}_{i,k} &= (u^{(i)} : \langle m_{i,k}, u^{(i)} \rangle < 0), & \mathbb{B}_{i,k}^c &= (u^{(i)} : \langle m_{i,k}, u^{(i)} \rangle \geq 0), \quad i = 1, 2, \\ I_{i,k}(u^{(i)}) &= \hat{\Gamma}_{i,k} \int_{\mathcal{R}} \left[|v + \nu^{-1} \langle u^{(i)}, m_{i,k} \rangle|^\kappa \mathbb{I}_{\{v \geq -\nu^{-1} \langle u^{(i)}, m_{i,k} \rangle\}} - v^\kappa \mathbb{I}_{\{v > 0\}} \right] dW_{i,k}(v), \\ \Gamma_{1,k}^2 &= \frac{S_{1,k}(\tau_k(\vartheta_0^{(1)}))^2}{\delta^{2\kappa} \lambda_k(\vartheta_0, \tau_k(\vartheta_0^{(1)})) \nu^{2\kappa+1}}, & \Gamma_{2,k}^2 &= \frac{S_{2,k}(\tau_k(\vartheta_0^{(2)}))^2}{\delta^{2\kappa} \lambda_k(\vartheta_0, \tau_k(\vartheta_0^{(2)})) \nu^{2\kappa+1}}, \end{aligned} \quad (11)$$

$$\hat{\Gamma}_{1,k} = \frac{S_{1,k}(\tau_k(\vartheta_0^{(1)}))}{\delta^\kappa \sqrt{\lambda_k(\vartheta_0, \tau_k(\vartheta_0^{(1)}))}}, \quad \hat{\Gamma}_{2,k} = \frac{S_{2,k}(\tau_k(\vartheta_0^{(2)}))}{\delta^\kappa \sqrt{\lambda_k(\vartheta_0, \tau_k(\vartheta_0^{(2)}))}} \quad (12)$$

$$Q_\kappa^2 = \int_{\mathcal{R}} [|v - 1|^\kappa \mathbb{I}_{\{v \geq 1\}} - v^\kappa \mathbb{I}_{\{v > 0\}}]^2 dv \quad (13)$$

$$Z_{(k)}(u) = \exp \left(\sum_{i=1}^2 \left[\hat{\Gamma}_{i,k} I_{i,k}(u^{(i)}) - \frac{\Gamma_{i,k}^2 Q_\kappa^2}{2} |\langle u^{(i)}, m_{i,k} \rangle|^{2\kappa+1} \right] \right)$$

$$Z(u) = \prod_{k=1}^K Z_{(k)}(u), \quad \xi = \frac{\int_{\mathcal{R}^4} u Z(u) du}{\int_{\mathcal{R}^4} Z(u) du}.$$

Here $W_{i,k}(v)$, $v \in \mathcal{R}$, $i = 1, 2$ are two-sided Wiener processes, i.e., $W_{i,k}(v) = W_{i,k}^+(v)$, $v \geq 0$ and $W_{i,k}(v) = W_{i,k}^-(v)$, $v \leq 0$, where $W_{i,k}^+(\cdot)$ and $W_{i,k}^-(\cdot)$ are independent Wiener processes.

Conditions \mathcal{C} .

\mathcal{C}_1 . *The intensities of the observed processes are (9), (10), where the functions $S_{i,k}(y) \in \mathcal{C}^1$ and are positive.*

\mathcal{C}_2 . *The configuration of the detectors and the set Θ are such that all signals from the both sources arrive at the detectors during the period $[0, T]$.*

\mathcal{C}_3 . *The condition \mathcal{R}_4 is fulfilled.*

We have the following lower bound on the risks of all estimators of the positions of sources:

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \sup_{\|\vartheta - \vartheta_0\| \leq \varepsilon} n^{\frac{2}{2\kappa+1}} \mathbf{E}_\vartheta \|\bar{\vartheta}_n - \vartheta\|_4^2 \geq \mathbf{E}_{\vartheta_0} \|\xi\|_4^2. \quad (14)$$

This bound is a particular case of more general bound given in [14], Theorem 1.9.1.

Theorem 2. *Let the conditions \mathcal{C} be fulfilled. Then the BE $\tilde{\vartheta}_n$ of the parameter ϑ is consistent, uniformly on compacts $\mathbb{K} \subset \Theta$ converges in distribution*

$$n^{\frac{1}{2\kappa+1}} (\tilde{\vartheta}_n - \vartheta_0) \Longrightarrow \xi,$$

the moments converge: for any $p > 0$

$$\lim_{n \rightarrow \infty} n^{\frac{2p}{2\kappa+1}} \mathbf{E}_{\vartheta} \|\tilde{\vartheta}_n - \vartheta\|_4^p = \mathbf{E}_{\vartheta_0} \|\xi\|_4^p,$$

and BE is asymptotically efficient.

Proof. Let us study the normalized likelihood ratio

$$Z_n(u) = \frac{L(\vartheta_0 + u\varphi_n, X^K)}{L(\vartheta_0, X^K)}, \quad u \in \mathbb{U}_n = (u : \vartheta_0 + u\varphi_n \in \Theta),$$

where $\varphi_n = n^{-\frac{1}{H}}$. The properties of $Z_n(\cdot)$, which we need are described in three lemmas

Lemma 1. *The finite dimensional distributions of $Z_n(\cdot)$ converge to the finite dimensional distributions of $Z(\cdot)$.*

Proof. Put $\vartheta_u = \vartheta_0 + u\varphi_n$, $\vartheta_u^{(1)} = \vartheta_0^{(1)} + u^{(1)}\varphi_n$ and $\vartheta_u^{(2)} = \vartheta_0^{(2)} + u^{(2)}\varphi_n$. For a fixed u denote $\gamma_n = \sup_{0 \leq t \leq T} \lambda_k(\vartheta_0, t)^{-1} |\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)|$ and note that $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore by Lemma 1.5 in [20] the LR admits the representation

$$\begin{aligned} \ln Z_n(u) &= \sum_{k=1}^K \int_0^T \frac{\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)}{\lambda_k(\vartheta_0, t)} [dX_k(t) - \lambda_k(\vartheta_0, t) dt] (1 + O(\gamma_n)) \\ &\quad - \frac{n}{2} \sum_{k=1}^K \int_0^T \frac{[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)]^2}{\lambda_k(\vartheta_0, t)} dt (1 + O(\gamma_n)) \end{aligned}$$

Suppose that $\tau_k(\vartheta_0^{(1)}) \neq \tau_k(\vartheta_0^{(2)})$, $\tau_k(\vartheta_0^{(1)}) < \tau_k(\vartheta_0^{(2)})$ and set $2\tau = \tau_k(\vartheta_0^{(1)}) + \tau_k(\vartheta_0^{(2)})$. Then the ordinary integral we write as follows

$$\begin{aligned} &\int_0^T \frac{[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)]^2}{\lambda_k(\vartheta_0, t)} dt \\ &= \int_0^\tau \frac{[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)]^2}{\lambda_k(\vartheta_0, t)} dt + \int_\tau^T \frac{[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)]^2}{\lambda_k(\vartheta_0, t)} dt \\ &= \int_0^\tau \frac{\left(\left[\psi_{\kappa, \delta} \left(t - \tau_k(\vartheta_u^{(1)}) \right) - \psi_{\kappa, \delta} \left(t - \tau_k(\vartheta_0^{(1)}) \right) \right] S_{1,k}(t) + A_k(\vartheta_u, \vartheta_0) \right)^2}{\lambda_k(\vartheta_0, t)} dt \\ &\quad + \int_\tau^T \frac{\left(B_k(\vartheta_u, \vartheta_0) + \left[\psi_{\kappa, \delta} \left(t - \tau_k(\vartheta_u^{(2)}) \right) - \psi_{\kappa, \delta} \left(t - \tau_k(\vartheta_0^{(2)}) \right) \right] S_{2,k}(t) \right)^2}{\lambda_k(\vartheta_0, t)} dt \\ &= J_{1,k,n}(u^{(1)}) + J_{2,k,n}(u^{(2)}) \end{aligned} \tag{15}$$

with obvious notations. The function $A_k(\vartheta_u, \vartheta_0)$ on the interval $[0, \tau]$ and the function $B_k(\vartheta_u, \vartheta_0)$ on the interval $[\tau, T]$ have bounded derivatives w.r.t. ϑ .

Suppose that $u^{(1)} \in \mathbb{B}_{1,k}$ then for large n we have $\tau_k(\vartheta_u^{(1)}) > \tau_k(\vartheta_0^{(1)}) = \tau_{1,k}$ and therefore

$$\begin{aligned}
J_{1,k,n}(u^{(1)}) &= n \int_{\tau_{1,k}}^{\tau} \frac{\left(\left[\psi_{\kappa,\delta}(t - \tau_k(\vartheta_u^{(1)})) - \psi_{\kappa,\delta}(t - \tau_{1,k}) \right] S_{1,k}(t) + A_k(\vartheta_u, \vartheta_0) \right)^2}{\lambda_k(\vartheta_0, t)} dt \\
&= n \int_{\tau_{1,k}}^{\tau} \frac{\left(\left[\psi_{\kappa,\delta}(t - \tau_k(\vartheta_u^{(1)})) - \psi_{\kappa,\delta}(t - \tau_{1,k}) \right] S_{1,k}(t) \right)^2}{\lambda_k(\vartheta_0, t)} dt + o(1) \\
&= n \int_0^{\tau - \tau_{1,k}} \frac{\left(\left[\psi_{\kappa,\delta}(s - \tau_k(\vartheta_u^{(1)}) + \tau_{1,k}) - \psi_{\kappa,\delta}(s) \right] S_{1,k}(s + \tau_{1,k}) \right)^2}{\lambda_k(\vartheta_0, s + \tau_{1,k})} ds + o(1) \\
&= n \int_0^{\tau - \tau_{1,k}} \frac{\left(\left[\psi_{\kappa,\delta}(s + \nu^{-1}\langle u^{(1)}, m_{1,k} \rangle \varphi_n) - \psi_{\kappa,\delta}(s) \right] S_{1,k}(s + \tau_{1,k}) \right)^2}{\lambda_k(\vartheta_0, s + \tau_{1,k})} ds + o(1) \\
&= n \frac{S_{1,k}(\tau_{1,k})^2}{\delta^{2\kappa} \lambda_k(\vartheta_0, \tau_{1,k})} \left[\int_0^{-\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle \varphi_n} s^{2\kappa} dt \right. \\
&\quad \left. + \int_{-\nu^{-1}\langle u^{(1)}, e_{1,k} \rangle \varphi_n}^{\tau - \tau_{1,k}} \left[|s + \nu^{-1}\langle u^{(1)}, m_{1,k} \rangle \varphi_n|^\kappa - s^\kappa \right]^2 dt \right] + o(1) \\
&= n \varphi_n^{2\kappa+1} \Gamma_{1,k}^2 |\langle u^{(1)}, m_{1,k} \rangle|^{2\kappa+1} \left[\int_0^1 v^{2\kappa} dv + \int_1^{c/\varphi_n} [|v - 1|^\kappa - v^\kappa]^2 dv \right] + o(1) \\
&= \Gamma_{1,k}^2 |\langle u^{(1)}, m_{1,k} \rangle|^{2\kappa+1} \int_0^\infty [|v - 1|^\kappa \mathbb{I}_{\{v \geq 1\}} - v^\kappa]^2 dv + o(1) \\
&= \Gamma_{1,k}^2 |\langle u^{(1)}, m_{1,k} \rangle|^{2\kappa+1} Q_\kappa^2 + o(1), \tag{16}
\end{aligned}$$

where we changed the variable $s = -\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle v$, used relation $n\varphi_n^{2\kappa+1} = 1$ and notation (11),(13).

For the values $u^{(2)} \in \mathbb{B}_{2,k}$ we have a similar expression

$$J_{2,k,n}(u^{(2)}) = \Gamma_{2,k}^2 |\langle u^{(2)}, e_{2,k} \rangle|^{2\kappa+1} \int_0^\infty [|v - 1|^\kappa \mathbb{I}_{\{v \geq 1\}} - v^\kappa]^2 dv + o(1).$$

If $u^{(1)} \in \mathbb{B}_{1,k}^c$ and $u^{(2)} \in \mathbb{B}_{2,k}^c$, then the similar calculations lead to the same integrals.

Hence

$$\int_0^T \frac{[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)]^2}{\lambda_k(\vartheta_0, t)} dt = \left[\Gamma_{1,k}^2 |\langle u^{(1)}, m_{1,k} \rangle|^{2\kappa+1} + \Gamma_{2,k}^2 |\langle u^{(2)}, m_{2,k} \rangle|^{2\kappa+1} \right] Q_\kappa^2 + o(1).$$

Let us suppose that $\tau_k(\vartheta_0^{(1)} + u^{(1)}\varphi_n) > \tau_k(\vartheta_0^{(1)})$. Introduce the centered Poisson process $d\pi_{k,n}(t) = dX_k(t) - \lambda_{k,n}(\vartheta_0, t) dt$ and consider the stochastic integral

$$\int_0^T \frac{[\lambda_{k,n}(\vartheta_u, t) - \lambda_{k,n}(\vartheta_0, t)]}{\lambda_{k,n}(\vartheta_0, t)} d\pi_{k,n}(t)$$

$$\begin{aligned}
&= \int_{\tau_{1,\kappa}}^{\tau} \frac{[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)]}{\lambda_k(\vartheta_0, t)} d\pi_{k,n}(t) + \int_{\tau}^T \frac{[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)]}{\lambda_k(\vartheta_0, t)} d\pi_{k,n}(t) \\
&= I_{1,k,n}(u^{(1)}) + I_{2,k,n}(u^{(2)}),
\end{aligned}$$

where $2\tau = \tau_k(\vartheta_u^{(1)}) + \tau_k(\vartheta_0^{(1)})$.

Using the same relations as above we can write for $u^{(1)} \in \mathbb{B}_{1,k}$

$$\begin{aligned}
&I_{1,k,n}(u^{(1)}) \\
&= \int_0^{\tau - \tau_{1,k}} \frac{[\psi_{\kappa,\delta}(s + \nu^{-1}\langle u^{(1)}, m_{1,k} \rangle \varphi_n) - \psi_{\kappa,\delta}(s)] S_{1,k}(s + \tau_{1,k})}{\lambda_k(\vartheta_0, s + \tau_{1,k})} d\pi_{k,n}(t) + o(1) \\
&= \frac{S_{1,k}(\tau_{1,k})}{\delta^\kappa \lambda_k(\vartheta_0, \tau_{1,k})} \left[\int_0^{-\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle \varphi_n} s^\kappa d\pi_{k,n}(s + \tau_{1,k}) \right. \\
&\quad \left. + \int_{-\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle \varphi_n}^{\tau - \tau_{1,k}} \left[|s + \nu^{-1}\langle u^{(1)}, m_{1,k} \rangle \varphi_n|^\kappa - s^\kappa \right] d\pi_{k,n}(s + \tau_{1,k}) \right] + o(1) \\
&= \hat{\Gamma}_{1,k} \left[\int_0^{c/\varphi_n} \left[|v + \nu^{-1}\langle u^{(1)}, m_{1,k} \rangle|^\kappa \mathbb{I}_{\{v \geq -\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle\}} - v^\kappa \right] dW_{1,k,n}(v) \right] + o(1)
\end{aligned}$$

Here we changed the variable $s = v\varphi_n$, used the relation $\sqrt{n}\varphi_n^{\kappa+1/2} = 1$ and denoted

$$W_{1,k,n}(v) = \frac{\pi_{k,n}(v\varphi_n + \tau_{1,k}) - \pi_{k,n}(\tau_{1,k})}{\sqrt{n\varphi_n\lambda_k(\vartheta_0, \tau_{1,k})}}.$$

This process has the following first two moments: $\mathbf{E}_{\vartheta_0} W_{1,k,n}(v) = 0$

$$\begin{aligned}
\mathbf{E}_{\vartheta_0} W_{1,k,n}(v)^2 &= \frac{1}{\varphi_n \lambda_k(\vartheta_0, \tau_{1,k})} \int_{\tau_{1,k}}^{\tau_{1,k} + v\varphi_n} \lambda_k(\vartheta_0, t) dt = v(1 + o(1)), \\
\mathbf{E}_{\vartheta_0} W_{1,k,n}(v_1) W_{1,k,n}(v_2) &= \frac{1}{\varphi_n \lambda_k(\vartheta_0, \tau_{1,k})} \int_{\tau_{1,k}}^{\tau_{1,k} + v_1 \wedge v_2 \varphi_n} \lambda_k(\vartheta_0, t) dt = v_1 \wedge v_2 (1 + o(1)).
\end{aligned}$$

By the central limit theorem (see, e.g., Theorem 1.2 in [18])

$$\begin{aligned}
I_{1,k,n}(u^{(1)}) &= \hat{\Gamma}_{1,k} \int_0^{c/\varphi_n} \left[|v + \nu^{-1}\langle u^{(1)}, m_{1,k} \rangle|^\kappa \mathbb{I}_{\{v \geq -\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle\}} - v^\kappa \right] dW_{1,k,n}(v) \\
&\implies \hat{\Gamma}_{1,k} \int_0^\infty \left[|v + \nu^{-1}\langle u^{(1)}, m_{1,k} \rangle|^\kappa \mathbb{I}_{\{v \geq -\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle\}} - v^\kappa \right] dW_{1,k}^+(v) \\
&= \hat{\Gamma}_{1,k} I_{1,k}(u^{(1)}),
\end{aligned}$$

where $W_{1,k}^+(v)$, $v \geq 0$ is a standard Wiener process. The similar limit we have for the integral $I_{2,k,n}(u^{(2)})$.

Suppose that $u^{(1)} \in \mathbb{B}_{1,k}^c$, i.e., $\langle u^{(1)}, m_{1,k} \rangle \geq 0$. Then

$$\begin{aligned} I_{2,k,n}(u^{(1)}) &= \hat{\Gamma}_{2,k} \int_{-\nu^{-1}\langle u^{(2)}, m_{2,k} \rangle}^{c/\varphi_n} \left[|v + \nu^{-1}\langle u^{(1)}, m_{2,k} \rangle|^\kappa - v^\kappa \mathbb{I}_{\{v \geq 0\}} \right] dW_{2,k,n}(v) \\ &\implies \hat{\Gamma}_{2,k} \int_{-\nu^{-1}\langle u^{(2)}, m_{2,k} \rangle}^{\infty} \left[|v + \nu^{-1}\langle u^{(1)}, m_{2,k} \rangle|^\kappa - v^\kappa \mathbb{I}_{\{v \geq 0\}} \right] dW_{2,k}^+(v) \\ &= \hat{\Gamma}_{2,k} I_{2,k}(u^{(2)}), \end{aligned}$$

Therefore we obtained

$$\begin{aligned} \ln Z_{(k),n}(u) &= \int_0^T \frac{\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)}{\lambda_k(\vartheta_0, t)} [dX_k(t) - \lambda_k(\vartheta_0, t) dt] \\ &\quad - \frac{n}{2} \int_0^T \frac{[\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)]^2}{\lambda_k(\vartheta_0, t)} dt + o(1) \\ &\implies \hat{\Gamma}_{1,k} I_{1,k}(u^{(1)}) - \frac{\Gamma_{1,k}^2 Q_\kappa^2}{2} |\langle u^{(1)}, m_{1,k} \rangle|^{2\kappa+1} \\ &\quad + \hat{\Gamma}_{2,k} I_{2,k}(u^{(2)}) - \frac{\Gamma_{2,k}^2 Q_\kappa^2}{2} |\langle u^{(2)}, m_{2,k} \rangle|^{2\kappa+1} \\ &= \ln Z_{(k)}(u). \end{aligned}$$

The Wiener processes $W_{i,k}(\cdot)$ $i = 1, 2; k = 1, \dots, K$ are independent and this convergence provides the convergence of one-dimensional distributions

$$Z_n(u) \implies Z(u) = \prod_{k=1}^K Z_{(k)}(u), \quad u \in \mathcal{R}^4.$$

The convergence of finite-dimensional distributions can be proved following the same lines, but we ommite it because it is too cumbersome and does not use new ideas or tools. \square

Lemma 2. *There exists constant $C > 0$ such that for any $L > 0$*

$$\sup_{|u| < L} \mathbf{E}_{\vartheta_0} \left| Z_n(u)^{1/2} - Z_n(u')^{1/2} \right|^2 \leq C (1 + L^{1-2\kappa}) \|u - u'\|_4^{1+2\kappa} \quad (17)$$

Proof. By Lemma 1.5 in [18] we have

$$\begin{aligned} \mathbf{E}_{\vartheta_0} \left| Z_n(u)^{1/2} - Z_n(u')^{1/2} \right|^2 &\leq \sum_{k=1}^K \int_0^T \left[\sqrt{\lambda_{k,n}(\vartheta_u, t)} - \sqrt{\lambda_{k,n}(\vartheta_{u'}, t)} \right]^2 dt \\ &\leq Cn \sum_{k=1}^K \int_0^T [\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_{u'}, t)]^2 dt. \end{aligned} \quad (18)$$

Note that the parts of integrals in (15) containing differentiable functions $A_k(\vartheta_u, t)$ and $B_k(\vartheta_u, t)$ have estimates like

$$n \int_0^\tau [A_k(\vartheta_u, t) - A_k(\vartheta_{u'}, t)]^2 dt \leq Cn\varphi_n^2 \|u - u'\|^2 = Cn^{-1+2\kappa} \|u - u'\|_4^2.$$

Therefore, following (16) we obtain the relations

$$\begin{aligned} \sum_{k=1}^K \int_0^T [\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_{u'}, t)]^2 dt &\leq C \left(\|u^{(1)} - u'^{(1)}\|_2^{1+2\kappa} + \|u^{(2)} - u'^{(2)}\|_2^{1+2\kappa} \right) + Cn^{-1+2\kappa} \|u - u'\|^2 \\ &\leq C \left(\|u - u'\|_4^{1+2\kappa} + Cn^{-1+2\kappa} \|u - u'\|^2 \right) \\ &\leq C \left(1 + L^{1-2\kappa} \right) \|u - u'\|_4^{1+2\kappa} \end{aligned}$$

□

Lemma 3. *There exist $c_* > 0$ such that*

$$\mathbf{P}_{\vartheta_0} \left(Z_T(u) > e^{-c_* \|u\|_4^{2\kappa+1}} \right) \leq C e^{-c_* \|u\|_4^{2\kappa+1}} \quad (19)$$

Proof. By the same Lemma 1.5 in [18] we have

$$\begin{aligned} \mathbf{P}_{\vartheta_0} \left(Z_T(u) > e^{-c_* \|u\|_4^{2\kappa+1}} \right) &\leq e^{\frac{c_*}{2} \|u\|_4^{2\kappa+1}} \mathbf{E}_{\vartheta_0} Z_n(u)^{1/2} \\ &= \exp \left(\frac{c_*}{2} \|u\|_4^{2\kappa+1} - \frac{1}{2} \sum_{k=1}^K \int_0^T \left(\sqrt{\lambda_{k,n}(\vartheta_u, t)} - \sqrt{\lambda_{k,n}(\vartheta_0, t)} \right)^2 dt \right). \end{aligned}$$

Using the calculations similar to that of (15) and (16) we obtain for $\|\vartheta_u - \vartheta_0\| \leq \varepsilon$ and sufficiently small $\varepsilon > 0$ the estimate

$$\sum_{k=1}^K \int_0^T \left(\sqrt{\lambda_{k,n}(\vartheta_u, t)} - \sqrt{\lambda_{k,n}(\vartheta_0, t)} \right)^2 dt \geq c_1 \|u\|_4^{2\kappa+1}.$$

Consider the case $\|\vartheta_u - \vartheta_0\|_4 > \varepsilon$. Let us denote

$$g(\varepsilon) = \inf_{\|\vartheta - \vartheta_0\|_4 > \varepsilon} \sum_{k=1}^K \int_0^T [S_{1,k}(t - \tau_{1,k}) + S_{2,k}(t - \tau_{2,k}) - S_{1,k}(t - \tau_{1,k}^\circ) - S_{2,k}(t - \tau_{2,k}^\circ)]^2 dt$$

and remark that $\varphi_n \|u\|_4 \leq D = \sup_{\vartheta, \vartheta' \in \Theta} \|\vartheta - \vartheta'\|_4$. Hence $n > D^{-(2\kappa+1)} \|u\|_4^{2\kappa+1}$. As $g(\varepsilon) > 0$ we can write

$$\sum_{k=1}^K \int_0^T \left(\sqrt{\lambda_{k,n}(\vartheta_u, t)} - \sqrt{\lambda_{k,n}(\vartheta_0, t)} \right)^2 dt$$

$$\begin{aligned}
&\geq c \sum_{k=1}^K \int_0^T (\lambda_{k,n}(\vartheta_u, t) - \lambda_{k,n}(\vartheta_0, t))^2 dt \\
&\geq c n g(\varepsilon) \geq c g(\varepsilon) D^{-(2\kappa+1)} \|u\|_4^{2\kappa+1} = c_2 \|u\|_4^{2\kappa+1}.
\end{aligned}$$

Let us denote $\bar{c} = c_1 \wedge c_2$ and set $c_* = \bar{c}/3$. Then we obtain

$$\frac{c_*}{2} \|u\|_4^{2\kappa+1} - \frac{1}{2} \sum_{k=1}^K \int_0^T \left(\sqrt{\lambda_{k,n}(\vartheta_u, t)} - \sqrt{\lambda_{k,n}(\vartheta_0, t)} \right)^2 dt \leq -c_* \|u\|_4^{2\kappa+1}.$$

□

The properties of the process $Z_n(\cdot)$ established in Lemmas 1-3 allow us to cite Theorem 1.10.2 in [14] and therefore to obtain the properties of BE mentioned in Theorem 2.

□

2.2.3 Change point type fronts.

Suppose that the intensity functions of observed inhomogeneous Poisson processes have jumps at the moments of arriving of the signals, i.e. the intensities are

$$\lambda_{k,n}(\vartheta, t) = n \mathbb{1}_{\{t > \tau_k(\vartheta^{(1)})\}} S_{1,k}(t) + n \mathbb{1}_{\{t > \tau_k(\vartheta^{(2)})\}} S_{2,k}(t) + n \lambda_0 = n \lambda_k(\vartheta, t), \quad (20)$$

where $0 \leq t \leq T$, $k = 1, \dots, K$. As before $\vartheta = (\vartheta^{(1)}, \vartheta^{(2)})^\top \in \Theta$, where $\vartheta^{(1)} = (\vartheta_1, \vartheta_2)^\top = (x_1^\circ, y_1^\circ)^\top$ (position of the source \mathbb{S}_1) and $\vartheta^{(2)} = (\vartheta_3, \vartheta_4)^\top = (x_2^\circ, y_2^\circ)^\top$ (position of the source \mathbb{S}_2),

$$\tau_k(\vartheta^{(i)}) = \nu^{-1} [(x_k - x_i^\circ)^2 + (y_k - y_i^\circ)^2]^{1/2} = \tau_{i,k}$$

Introduce the notations

$$\begin{aligned}
\ell_{i,k} &= \ln \left(\frac{\lambda_0}{S_{i,k}(\tau_{i,k}) + \lambda_0} \right), \quad \mathbb{B}_{i,k} = (u^{(i)} : \langle u^{(i)}, m_{i,k} \rangle < 0), \quad i = 1, 2; \quad k = 1, \dots, K, \\
Z_{(k)}(u) &= \exp \left(\sum_{i=1}^2 [\ell_{i,k} x_{i,k}^+ (-\nu^{-1} \langle u^{(i)}, m_{i,k} \rangle) - \langle u^{(i)}, m_{i,k} \rangle \nu^{-1} S_{i,k}(\tau_{i,k})] \mathbb{1}_{\mathbb{B}_{i,k}} \right. \\
&\quad \left. - \sum_{i=1}^2 [\ell_{i,k} x_{i,k}^- (-\nu^{-1} \langle u^{(i)}, m_{i,k} \rangle) - \langle u^{(i)}, m_{i,k} \rangle \nu^{-1} S_{i,k}(\tau_{i,k})] \mathbb{1}_{\mathbb{B}_{i,k}^c} \right), \\
Z(u) &= \prod_{k=1}^K Z_{(k)}(u), \quad \eta = \frac{\int_{\mathcal{R}^4} u Z(u) du}{\int_{\mathcal{R}^4} Z(u) du},
\end{aligned}$$

Here $x_{i,k}^+(-\nu^{-1}\langle u^{(i)}, m_{i,k} \rangle) = y_{i,k}^+(s)_{s=-\nu^{-1}\langle u^{(i)}, m_{i,k} \rangle}$, $s \geq 0$, where $y_{i,k}^+(s)$, $s \geq 0$ is a Poisson process with unit intensity function. Similarly

$$x_{i,k}^-(-\nu^{-1}\langle u^{(i)}, m_{i,k} \rangle) = y_{i,k}^-(-s)_{s=\nu^{-1}\langle u^{(i)}, m_{i,k} \rangle}, s \leq 0$$

with another Poisson process $y_{i,k}^-(s)$, $s \geq 0$ with unit intensity. All Poisson processes $y_{i,k}^\pm(s)$, $s \geq 0$, $i = 1, 2$; $k = 1, \dots, K$ are independent.

Conditions \mathcal{D} .

\mathcal{D}_1 . The intensities of the observed processes are (20), where the functions $S_{i,k}(y) \in \mathcal{C}^1$ and are positive. The set $\Theta \subset \mathcal{R}^4$ is open, bounded and convex.

\mathcal{D}_2 . The configuration of the detectors and the set Θ are such that all signals from the both sources arrive at the detectors during the period $[0, T]$.

\mathcal{D}_3 . The condition \mathcal{R}_4 is fulfilled.

The following lower bound

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \sup_{\|\vartheta - \vartheta_0\| \leq \varepsilon} n^2 \mathbf{E}_\vartheta \|\tilde{\vartheta}_n - \vartheta\|_4^2 \geq \mathbf{E}_{\vartheta_0} \|\eta\|_4^2$$

holds. This bound is another particular case of Theorem 1.9.1 given in [14].

Theorem 3. Let the conditions \mathcal{D} be fulfilled, then the BE $\tilde{\vartheta}_n$ of positions \mathcal{S}_1 and \mathcal{S}_2 is consistent, converges in distribution

$$n(\tilde{\vartheta}_n - \vartheta_0) \Longrightarrow \eta,$$

the moments converge: for any $p > 0$

$$n^p \mathbf{E}_{\vartheta_0} \|\tilde{\vartheta}_n - \vartheta_0\|_4^p \longrightarrow \mathbf{E}_{\vartheta_0} \|\eta\|_4^p,$$

the BE is asymptotically efficient.

Proof. The normalized LR function in this problem is

$$\begin{aligned} \ln Z_n(u) &= \sum_{k=1}^K \int_0^T \ln \frac{\lambda_k(\vartheta_0 + n^{-1}u, t)}{\lambda_k(\vartheta_0, t)} dX_k(t) \\ &\quad - n \sum_{k=1}^K \int_0^T [\lambda_k(\vartheta_0 + n^{-1}u, t) - \lambda_k(\vartheta_0, t)] dt. \end{aligned}$$

We have to check once more the conditions of Theorem 1.10.1 in [14] and to prove three lemmas.

Lemma 4. *The finite dimensional distributions of $Z_n(\cdot)$ uniformly on compacts $\mathbb{K} \subset \Theta$ converge to the finite dimensional distributions of $Z(\cdot)$.*

Proof. Let us set $\vartheta_u = \vartheta_0 + n^{-1}u$ and suppose that $u^{(1)} \in \mathbb{B}_{1,k}$, $u^{(2)} \in \mathbb{B}_{2,k}$. For n sufficiently large we have $\tau_{1,k}(\vartheta_u) > \tau_{1,k}(\vartheta_0)$, $\tau_{2,k}(\vartheta_u) > \tau_{2,k}(\vartheta_0) > \tau_{1,k}(\vartheta_u)$, then

$$\begin{aligned} \int_0^T \ln \frac{\lambda_k(\vartheta_u, t)}{\lambda_k(\vartheta_0, t)} dX_k(t) \\ = \int_{\tau_{1,k}(\vartheta_0)}^{\tau_{1,k}(\vartheta_u)} \ln \frac{\lambda_0}{S_{1,k}(t) + \lambda_0} dX_k(t) + \int_{\tau_{2,k}(\vartheta_0)}^{\tau_{2,k}(\vartheta_u)} \ln \frac{\lambda_0}{S_{2,k}(t) + \lambda_0} dX_k(t). \end{aligned} \quad (21)$$

and

$$\int_0^T [\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_0, t)] dt = - \int_{\tau_{1,k}(\vartheta_0)}^{\tau_{1,k}(\vartheta_u)} S_{1,k}(t) dt - \int_{\tau_{2,k}(\vartheta_0)}^{\tau_{2,k}(\vartheta_u)} S_{2,k}(t) dt \quad (22)$$

For a fixed u using Taylor expansion we can write (recall that $\tau_{i,k} = \tau_{i,k}(\vartheta_0^{(i)})$, $i = 1, 2$)

$$\begin{aligned} \int_{\tau_{1,k}}^{\tau_{1,k}(\vartheta_u)} S_{1,k}(t) dt &= (\tau_{1,k}(\vartheta_u) - \tau_{1,k}) S_{1,k}(\tau_{1,k}) + \frac{1}{n} \int_{\tau_{1,k}(\vartheta_0)}^{\tau_{1,k}(\vartheta_u)} S'_{1,k}(\tilde{t}) dt \\ &= -(n\nu)^{-1} \langle u^{(1)}, m_{1,k} \rangle S_{1,k}(\tau_{1,k}) (1 + O(n^{-1})) \end{aligned}$$

and

$$\int_{\tau_{2,k}}^{\tau_{2,k}(\vartheta_u)} S_{2,k}(t) dt = -(n\nu)^{-1} \langle u^{(2)}, m_{2,k} \rangle S_{2,k}(\tau_{2,k}) (1 + O(n^{-1})).$$

For stochastic integrals the similar expansions provide

$$\begin{aligned} \int_{\tau_{1,k}}^{\tau_{1,k}(\vartheta_u)} \ln \frac{\lambda_0}{S_{1,k}(t) + \lambda_0} dX_k(t) \\ = \ln \frac{\lambda_0}{S_{1,k}(\tau_{1,k}) + \lambda_0} [X_k(\tau_{1,k}(\vartheta_u)) - X_k(\tau_{1,k})] (1 + O(n^{-1})) \end{aligned}$$

Further

$$\begin{aligned} X_k(\tau_{1,k}(\vartheta_u)) - X_k(\tau_{1,k}) &= X_k(\tau_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle) - X_k(\tau_{1,k}) \\ &\quad + X_k(\tau_{1,k}(\vartheta_u)) - X_k(\tau_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle) \\ &= x_{1,k}^+(-\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle) + O(n^{-1/2}). \end{aligned}$$

Here $x_{1,k}^+(-\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle) = X_k(\tau_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle) - X_k(\tau_{1,k})$ is Poisson random field and the last estimate was obtained as follows

$$\mathbf{E}_{\vartheta_0} (X_k(\tau_{1,k}(\vartheta_u)) - X_k(\tau_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle))^2$$

$$\begin{aligned}
&= n \int_{\tau_{1,k}(\vartheta_u)}^{\tau_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle} \lambda_k(\vartheta_0, t) dt + \left(n \int_{\tau_{1,k}(\vartheta_u)}^{\tau_{1,k} - n^{-1}\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle} \lambda_k(\vartheta_0, t) dt \right)^2 \\
&\leq \frac{C}{n}.
\end{aligned}$$

Hence if $u^{(1)} \in \mathbb{B}_{1,k}$ and $u^{(2)} \in \mathbb{B}_{2,k}$

$$\begin{aligned}
\int_0^T \ln \frac{\lambda_k(\vartheta_0 + n^{-1}u, t)}{\lambda_k(\vartheta_0, t)} dX_k(t) &= \ln \left(\frac{\lambda_0}{S_{1,k}(\tau_{1,k}) + \lambda_0} \right) x_{1,k}^+(-\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle) \\
&\quad + \ln \left(\frac{\lambda_0}{S_{2,k}(\tau_{2,k}) + \lambda_0} \right) x_{2,k}^+(-\nu^{-1}\langle u^{(2)}, m_{2,k} \rangle) + O(n^{-1/2}).
\end{aligned}$$

If $u^{(1)} \in \mathbb{B}_{1,k}^c$ and $u^{(2)} \in \mathbb{B}_{2,k}^c$

$$\begin{aligned}
\int_0^T \ln \frac{\lambda_k(\vartheta_0 + n^{-1}u, t)}{\lambda_k(\vartheta_0, t)} dX_k(t) &= \ln \left(1 + \frac{S_{1,k}(\tau_{1,k})}{\lambda_0} \right) x_{1,k}^-(-\nu^{-1}\langle u^{(1)}, m_{1,k} \rangle) \\
&\quad + \ln \left(1 + \frac{S_{2,k}(\tau_{2,k})}{\lambda_0} \right) x_{2,k}^-(\nu^{-1}\langle u^{(2)}, m_{2,k} \rangle) + O(n^{-1/2}).
\end{aligned}$$

□

Lemma 5. *There exists a constant $C > 0$ such that*

$$\mathbf{E}_{\vartheta_0} \left| Z_n(u)^{1/2} - Z_n(u')^{1/2} \right|^2 \leq C \|u - u'\|_4.$$

Proof. We have (see (18))

$$\sup_{\vartheta_0 \in \mathbb{K}} \mathbf{E}_{\vartheta_0} \left| Z_n(u)^{1/2} - Z_n(u')^{1/2} \right|^2 \leq \sum_{k=1}^K n \int_0^T [\lambda_k(\vartheta_u, t) - \lambda_k(\vartheta_{u'}, t)]^2 dt.$$

Suppose that $\tau_{1,k}(\vartheta_u) > \tau_{1,k}(\vartheta_{u'})$, $\tau_{2,k}(\vartheta_u) > \tau_{2,k}(\vartheta_{u'}) > \tau_{1,k}(\vartheta_u)$. Then

$$\begin{aligned}
&n \int_0^T \left[S_{1,k}(t) \mathbb{I}_{\{\tau_{1,k}(\vartheta_{u'}) \leq t \leq \tau_{1,k}(\vartheta_u)\}} + S_{2,k}(t) \mathbb{I}_{\{\tau_{2,k}(\vartheta_{u'}) \leq t \leq \tau_{2,k}(\vartheta_u)\}} \right]^2 dt \\
&= n \int_{\tau_{1,k}(\vartheta_{u'})}^{\tau_{1,k}(\vartheta_u)} S_{1,k}(t)^2 dt + n \int_{\tau_{2,k}(\vartheta_{u'})}^{\tau_{2,k}(\vartheta_u)} S_{2,k}(t)^2 dt \\
&\leq Cn(\tau_{1,k}(\vartheta_u) - \tau_{1,k}(\vartheta_{u'})) + Cn(\tau_{2,k}(\vartheta_u) - \tau_{2,k}(\vartheta_{u'})) \\
&\leq C \left(|\langle u'^{(1)} - u^{(1)}, m_{1,k} \rangle| + |\langle u'^{(2)} - u^{(2)}, m_{2,k} \rangle| \right) \\
&\leq C \left(\|u'^{(1)} - u^{(1)}\|_2 + \|u'^{(2)} - u^{(2)}\|_2 \right) \leq C \|u' - u\|_4.
\end{aligned}$$

□

Lemma 6. *There exists a constant $\kappa > 0$ such that*

$$\mathbf{E}_{\vartheta_0} Z_n(u)^{1/2} \leq e^{-\kappa \|u\|_4}.$$

Proof. We have

$$\begin{aligned} \ln \mathbf{E}_{\vartheta_0} Z_n(u)^{1/2} &= -\frac{1}{2} \sum_{k=1}^K \int_0^T \left[\sqrt{\lambda_{k,n}(\vartheta_u, t)} - \sqrt{\lambda_{k,n}(\vartheta_0, t)} \right]^2 dt \\ &\leq -C \sum_{k=1}^K \int_0^T [\lambda_{k,n}(\vartheta_u, t) - \lambda_{k,n}(\vartheta_0, t)]^2 dt. \end{aligned}$$

Suppose that $\tau_{i,k}(\vartheta_u) > \tau_{i,k}(\vartheta_0) = \tau_{i,k}$, $\tau_{2,k} > \tau_{1,k}(\vartheta_u)$ and $\|\vartheta_u - \vartheta_0\| \leq \varepsilon$. Then

$$\begin{aligned} &\int_0^T [\lambda_{k,n}(\vartheta_u, t) - \lambda_{k,n}(\vartheta_0, t)]^2 dt \\ &\geq cn \int_0^T \left[S_{1,k}(t) \mathbb{I}_{\{\tau_{1,k} < t < \tau_{1,k}(\vartheta_u)\}} + S_{2,k}(t) \mathbb{I}_{\{\tau_{2,k} < t < \tau_{2,k}(\vartheta_u)\}} \right]^2 dt \\ &\geq cn \left[\int_{\tau_{1,k}}^{\tau_{1,k}(\vartheta_u)} S_{1,k}(t)^2 dt + \int_{\tau_{2,k}}^{\tau_{2,k}(\vartheta_u)} S_{2,k}(t)^2 dt \right] \\ &\geq cn [\tau_{1,k}(\vartheta_u) - \tau_{1,k} + \tau_{2,k}(\vartheta_u) - \tau_{2,k}] \\ &\geq c [-\langle u^{(1)}, m_{1,k} \rangle - \langle u^{(2)}, m_{2,k} \rangle] \\ &\geq c \frac{[-\langle u^{(1)}, m_{1,k} \rangle - \langle u^{(2)}, m_{2,k} \rangle]}{\|u\|_4} \|u\|_4 = -c\sqrt{2} \langle e, m_k \rangle \|u\|_4. \end{aligned}$$

Here $-\langle e, m_k \rangle > 0$ and the vectors are

$$e = \frac{u}{\|u\|_4}, \quad \|e\| = 1, \quad m_k = \left(\frac{x_k - x_1^\circ}{\sqrt{2}\rho_k}, \frac{y_k - y_1^\circ}{\sqrt{2}\rho_k}, \frac{x_k - x_2^\circ}{\sqrt{2}\rho_k}, \frac{y_k - y_2^\circ}{\sqrt{2}\rho_k} \right), \quad \|m_k\|_4 = 1.$$

Note that

$$\inf_{\|e\|=1} \sum_{k=1}^K (-\langle e, m_k \rangle) = \bar{c} > 0.$$

Indeed, if for some e we have $\bar{c} = 0$ then all scalar products $\langle u^{(i)}, m_{i,k} \rangle = 0$, $i = 1, 2; k = 1, \dots, K$ but such configuration of detectors is impossible.

Let $\|\vartheta_u - \vartheta_0\|_4 > \varepsilon$, then as in the proof of Lemma 3 we obtain the estimate

$$\sum_{k=1}^K \int_0^T [\lambda_{k,n}(\vartheta_u, t) - \lambda_{k,n}(\vartheta_0, t)]^2 dt \geq g(\varepsilon) n \geq D^{-1} \|u\|_4 = \bar{c} \|u\|_4.$$

Therefore

$$\sum_{k=1}^K \int_0^T \left[\sqrt{\lambda_{k,n}(\vartheta_u, t)} - \sqrt{\lambda_{k,n}(\vartheta_0, t)} \right]^2 dt \geq 2\kappa \|u\|_4,$$

with the corresponding $\kappa > 0$.

□

Once more the properties of the normalized LR $Z_n(\cdot)$ established in the Lemmas 4-6 according to Theorem 1.10.2 in [14] provide us the mentioned in Theorem 3 properties of BE.

□

3 On identifiability

We consider the situation where $K \in \mathbb{N}^*$ detectors $\mathbb{D}_1, \dots, \mathbb{D}_K \in \mathbb{R}^2$ receive signals from two sources $\mathbb{S}_1, \mathbb{S}_2 \in \mathbb{R}^2$. The two signals received by a detector are identical. We suppose that the positions (location points) of the detectors are all different, and we make no distinction between detectors/sources and their positions.

Suppose that the arrival times are registered without errors, and hence (given that the emission times and the signal propagation speed are known) provide us the distances from each detector to the two sources (without knowing which distance corresponds to which source). So, the system will be *identifiable* (without any further restriction on the positions of the sources and/or the emission times), if there exist no two different pairs of sources providing the same K pairs of distances to the detectors.

We introduce the following definition.

Definition 1. We say that $n \in \mathbb{N}^*$ points $A_1, \dots, A_n \in \mathbb{R}^2$ “lay on a cross”, if there exist a pair of orthogonal lines ℓ_1 and ℓ_2 such that $A_i \in \ell_1 \cup \ell_2$ for all $i = 1, \dots, n$.

Now we can formulate the theorem providing a necessary and sufficient condition for identifiability.

Theorem 4. A system with $K \in \mathbb{N}^*$ detectors will be identifiable if and only if the detectors do not lay on a cross.

Before proving this theorem, let us note that at least 4 detectors are necessary for the system to be identifiable. Indeed, any three detectors $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ lay on a cross (take, for example, the line $\overleftrightarrow{\mathbb{D}_1\mathbb{D}_2}$ and the perpendicular to this line passing by \mathbb{D}_3).

Note also that 4 detectors do not lay on a cross if (and only if) they are in general linear position (any 3 of them are not aligned) and cannot be split in two pairs such that the lines passing by these pairs are orthogonal.

Finally, note that if 5 (or more) detectors are in general linear position, then they necessarily do not lay on a cross (and hence the system is identifiable). Indeed, if they laid on a cross, at least one of the lines forming the cross would contain at least three of them, and so, they would not be in general linear position.

Proof. First we reformulate the theorem in an equivalent way using the contra-positions: *the detectors $\mathbb{D}_1, \dots, \mathbb{D}_K$ lay on a cross if and only if there exist two different pairs of sources providing the same K pairs of distances to them.*

In order to show the necessity, we suppose that the detectors $\mathbb{D}_1, \dots, \mathbb{D}_K$ lay on a cross formed by a pair of orthogonal line ℓ_1 and ℓ_2 , and we need to find two different pairs of sources $\{\mathbb{S}_1, \mathbb{S}_2\}$ and $\{\mathbb{S}'_1, \mathbb{S}'_2\}$ providing the same K pairs of distances to the detectors.

Let us denote O the point of intersection of ℓ_1 and ℓ_2 , take an arbitrary point \mathbb{S}_1 not belonging to neither ℓ_1 , nor ℓ_2 , and denote \mathbb{S}'_1 , \mathbb{S}'_2 and \mathbb{S}_2 the points symmetric to \mathbb{S}_1 with respect to ℓ_1 , ℓ_2 and O respectively (see Figure 4).

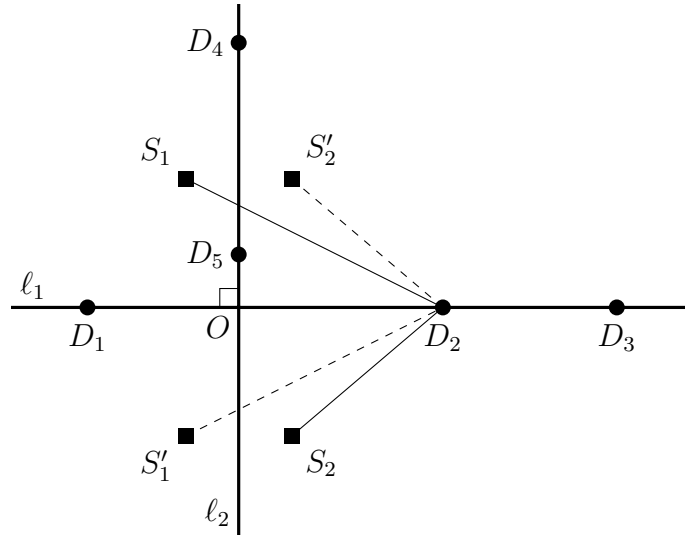


Figure 4: Non identifiability for detectors laying on a cross

Then, clearly, the pairs of sources $\{\mathbb{S}_1, \mathbb{S}_2\}$ and $\{\mathbb{S}'_1, \mathbb{S}'_2\}$ provide the same pair of distances to any point of a cross, and hence the same K pairs of distances to the detectors $\mathbb{D}_1, \dots, \mathbb{D}_K$.

Now we turn to the proof of the sufficiency. We suppose that two different pairs of sources $\{\mathbb{S}_1, \mathbb{S}_2\}$ and $\{\mathbb{S}'_1, \mathbb{S}'_2\}$ provide the same K pairs of distances to the detectors $\mathbb{D}_1, \dots, \mathbb{D}_K$, and we need to show that the detectors lay on a cross.

We need to distinguish several cases.

If in one of the two pairs the sources are located at the same point (say $\mathbb{S}_1 = \mathbb{S}_2$), then at least one of the points \mathbb{S}'_1 and \mathbb{S}'_2 (say \mathbb{S}'_1) must be different from \mathbb{S}_1 and \mathbb{S}_2 (see the left

picture in Figure 5). Then, if our two pairs of sources provide the same pair of distances to a point D , we should have, in particular, $\rho(D, S_1) = \rho(D, S'_1)$, and hence $D \in b_{S_1 S'_1}$. Here and in the sequel, we denote ρ the Euclidean distance, and for any two distinct points A and B , we denote b_{AB} the perpendicular bisector of the segment \overline{AB} . Therefore, all the detectors must belong to the line $b_{S_1 S'_1}$ (and, in particular, they lay on a cross).

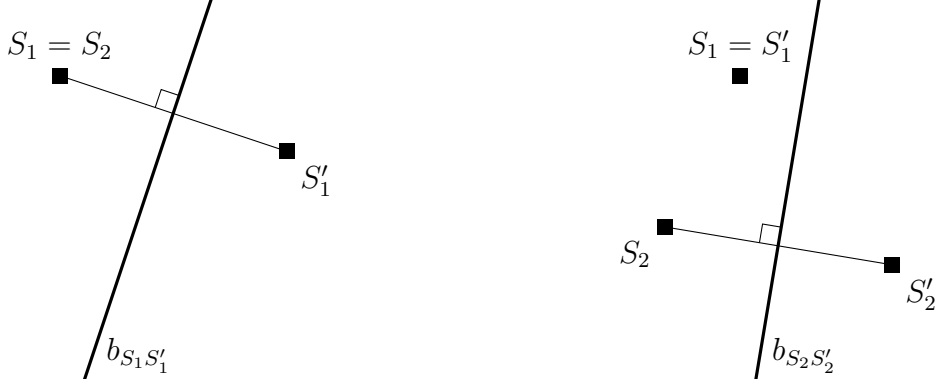


Figure 5: Cases with coinciding sources

So, from now on, we can suppose that $S_1 \neq S_2$ and $S'_1 \neq S'_2$.

Now we consider the case when the two pairs of sources have a point in common (say $S_1 = S'_1$). Note that in this case, we must have $S_2 \neq S'_2$, since otherwise the pairs of sources will not be different (see the left picture in Figure 5). Then, our two pairs of sources provide the same pair of distances to a point D if and only if $\rho(D, S_2) = \rho(D, S'_2)$, which is equivalent to $D \in b_{S_2 S'_2}$. Therefore, all the detectors must belong to the line $b_{S_2 S'_2}$ (and, in particular, they lay on a cross).

So, from now on, we can suppose that the points S_1, S_2, S'_1, S'_2 are all different. In this case, our two pairs of sources provide the same pair of distances to a point D if and only if D belongs at the same time either to the pair of lines $b_{S_1 S'_1}$ and $b_{S_2 S'_2}$, or to the pair of lines $b_{S_1 S'_2}$ and $b_{S_2 S'_1}$ (otherwise speaking, if and only if $D \in (b_{S_1 S'_1} \cap b_{S_2 S'_2}) \cup (b_{S_1 S'_2} \cap b_{S_2 S'_1})$). We need again to distinguish several (sub)cases depending on whether the lines in each of these pairs coincide or not.

First we suppose that $b_{S_1 S'_1} \neq b_{S_2 S'_2}$ and $b_{S_1 S'_2} \neq b_{S_2 S'_1}$ (see the left picture in Figure 6). Then each of these pairs of lines meet in at most one point. So, all the detectors must belong to a set consisting of at most two points (and, therefore, there is at most two detectors and they trivially lay on a cross).

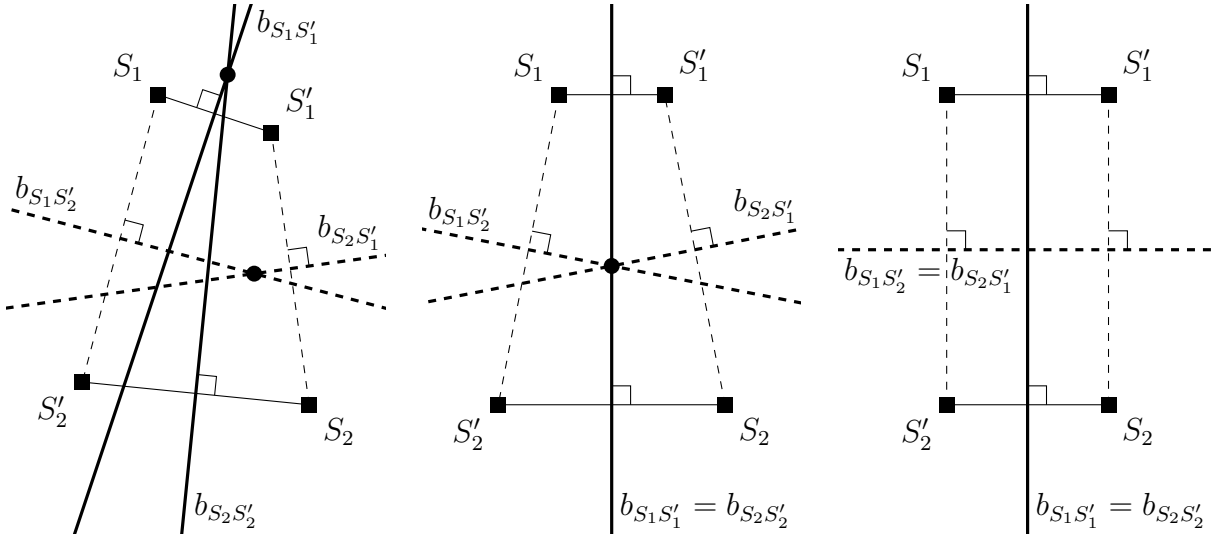


Figure 6: Cases without coinciding sources

Now we suppose that the lines coincide in one of the pairs, and not in the other (say $b_{S_1 S'_1} = b_{S_2 S'_2}$ and $b_{S_1 S'_2} \neq b_{S_2 S'_1}$). Note that in this case, the points S_1 and S'_1 , as well as the points S_2 and S'_2 , are symmetric with respect to the line $b_{S_1 S'_1}$, and hence the lines $b_{S_1 S'_2}$ and $b_{S_2 S'_1}$ either do not meet, or meet in a point belonging to $b_{S_1 S'_1}$ (see the middle picture in Figure 6). So, finally all the detectors must belong to the line $b_{S_1 S'_1}$ (and, in particular, they lay on a cross).

It remains to consider the case when $b_{S_1 S'_1} = b_{S_2 S'_2}$ and $b_{S_1 S'_2} = b_{S_2 S'_1}$ (see the right picture in Figure 6). In this case, as the segments $\overline{S_1 S'_1}$ and $\overline{S_2 S'_2}$ have a common perpendicular bisector, they must, in particular, be parallel (including the case where they lay on a same line). The same reasoning applies to the segments $\overline{S_1 S'_2}$ and $\overline{S_2 S'_1}$. Now, the case when the points S_1, S'_1, S_2, S'_2 lay on a same line is impossible, since these points being all different, we clearly can not have common perpendicular bisectors at the same time for $\overline{S_1 S'_1}$ and $\overline{S_2 S'_2}$ and for $\overline{S_1 S'_2}$ and $\overline{S_2 S'_1}$. Thus, $S_1 S'_1 S_2 S'_2$ is a parallelogram. Moreover, as its opposite sides have common perpendicular bisectors, $S_1 S'_1 S_2 S'_2$ is necessarily a rectangle. But in this case, the perpendicular bisectors $b_{S_1 S'_1}$ and $b_{S_1 S'_2}$ are orthogonal. Hence, as all the detectors belong to $b_{S_1 S'_1} \cup b_{S_1 S'_2}$, they lay again on a cross. \square

4 Discussions

The considered models in the cases of cusp and change point singularities can be easily generalized to the models with signals $S_{i,k}(t - \tau_k(\vartheta))$. The proofs will be more cumbersome but the rates and limit distributions of the studied estimators will be the same.

Remark that the same mathematical models are used in applications related with the detection of the weak optical signals from two sources.

Of course, it will be interesting to see the conditions of identifiability in the situation where the beginning of the emissions of these sources are unknown and have to be estimated together with the positions. In the case of one source such problem was discussed in the work [4].

All useful information about the position of sources according to the statements of this work is contained in the times of arriving of the signals $\tau_{1,k}(\vartheta), \tau_{2,k}(\vartheta), k = 1, \dots, K$. It is possible to study another statement of the problem supposing that these moments are estimated separately by observations $X_k = X_k(t), 0 \leq t \leq T$ for each k , say, $\hat{\tau}_{i,k,n} = \hat{\tau}_{i,k,n}(X_k)$ and then the positions of sources are estimated on the base of the obtained estimators, $\hat{\tau}_{i,k,n}, i = 1, 2; k = 1, \dots, K$. Such approach was considered in the works [4] and [7], see as well [20].

Note that in the works on Poisson source localization sometimes the intensities of Poisson processes are taken in the form

$$\lambda_k(\vartheta, t) = F(\rho_k(\vartheta))S_k(t) + \lambda_0, \quad 0 \leq t \leq T,$$

where $F(\cdot)$ is a known strictly decreasing function of the distance ρ_k between the source and the k -th detector. The developed in the present work approach (smooth case) can be applied for such models too because here all useful information is contained in the distances $\rho_k(\vartheta)$.

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