Hyperexponentially closed fields

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Abstract

We prove that the derivation and composition on the field $\mathbb{L}$ of logarithmic hyperseries of [15] extend to its closure under hyperexponentials. We study the properties of these extensions.

Introduction

Hyperexponentially closed fields

One naturally obtains hyperseries when closing fields of formal power series under derivation, integration, and transfinite sums and products. The first known instance of a thus closed structure is the field $\mathbb{L}$ of logarithmic hyperseries [15]. This is an ordered field containing $\mathbb{R}$, equipped with a derivation

$$\partial\colon \mathbb{L} \longrightarrow \mathbb{L},$$

and an operation

$$\circ\colon \mathbb{L} \times \mathbb{L}^{>\mathbb{R}} \longrightarrow \mathbb{L}; (f, g) \mapsto f \circ g$$

called the composition law. Logarithmic hyperseries are well-based series (i.e. Hahn series as per [23]) built upon formal symbols ℓ_{ω^μ} called hyperlogarithms, where ω^μ is the base omega exponentiation of an arbitrary ordinal $\mu \in \mathbf{On}$. Those hyperlogarithms satisfy the functional equations

$$\forall \mu \in \mathbf{On}, \ell_{\omega^{\mu+1}} \circ \ell_{\omega^\mu} = \ell_{\omega^{\mu+1}} - 1. \tag{1}$$

One of the purposes of fields of formal series with extra structure is to provide a formal framework which retains certain features of analytic or geometric models but which is rid of certain problems related to analytic convergence, the non-existence of canonical solutions to functional or differential equations... A relevant example is the use of so-called logarithmic-exponential transseries by Ecalle [18] as formal counterparts to certain functions involved in Dulac's conjecture, leading to a proof of that conjecture by Ecalle (independently from the previous proof by Ilyashenko [29]).

In logarithmic hyperseries, ℓ_1 acts as a logarithm whereas the terms ℓ_{ω^n} for $n \in \mathbb{N}^{>0}$ correspond to extremely slowly increasing functions on $\mathbb{R}^{\geqslant 0}$. For instance ℓ_ω is related to Kneser's real analytic solution L to Abel's equation in [30]

$$\forall r \geqslant 0, L(\log r) = L(r) - 1,$$

of which (1) is a formal generalization. The inclusion of terms ℓ_{ω^μ} for arbitrary infinite μ (not all of which can correspond to real-valued functions for cardinality reasons) is partly motivated by logic, where one would want to have at one's disposal saturated models of those types of structures.

The rich structure on $\mathbb{L}$ makes it an interesting object to act on other fields. In [7], van der Hoeven, Kaplan and the author introduced the notion of a *hyperserial field* as the action of $\mathbb{L}$ on a field of well-based series by differentiable functions. Fields of well-based series enjoy a notion of transfinite sums which is described in Section 1. An $\mathbb{R}$-linear function between fields of well-based series which commutes with those sums is said strongly linear. A hyperserial field is a field of well-based series $\mathbb{T}$ over $\mathbb{R}$ equipped with an external composition law $\circ_{\mathbb{T}}\colon \mathbb{L}\times\mathbb{T}^{>\mathbb{R}}\longrightarrow\mathbb{L}$ with the following properties (along with a few additional details):

i. For all $s\in\mathbb{T}^{>\mathbb{R}}$, the map $\mathbb{L}\longrightarrow\mathbb{T}$; $f\mapsto f\circ_{\mathbb{T}} s$ is a strongly linear morphism of ordered rings.

ii. For all $f,g\in\mathbb{L}$ with $g>\mathbb{R}$ and all $s\in\mathbb{T}^{>\mathbb{R}}$, we have $f\circ(g\circ_{\mathbb{T}} s)=(f\circ g)\circ_{\mathbb{T}} s$.

iii. For all $s,t\in\mathbb{T}^{>\mathbb{R}}$ and $\mu\in\mathbf{On}$, we have $s<t\Longrightarrow \ell_{\omega^\mu}\circ_{\mathbb{T}} s<\ell_{\omega^\mu}\circ_{\mathbb{T}} t$.

iv. For all $s\in\mathbb{T}^{>\mathbb{R}}$ and $\delta\in\mathbb{T}$ with $\delta\prec s$, we have $f\circ_{\mathbb{T}}(s+\delta)=\sum_{k\in\mathbb{N}}\frac{\partial^k(f)\circ_{\mathbb{T}} s}{k!}\delta^k$.

The class $\mathbb{L}$ itself is a hyperserial field which is closed under integration. However it is not closed under functional inversion. In particular, for any $\mu\in\mathbf{On}$, the hyperlogarithm function $L_{\omega^\mu}\colon\mathbb{L}^{>\mathbb{R}}\longrightarrow\mathbb{L}^{>\mathbb{R}}$; $g\mapsto\ell_{\omega^\mu}\circ g$ is not surjective. In other words, the functional right inverse E_{ω^μ} of L_{ω^μ}, called the hyperexponential function of strength ω^μ, is not totally defined on $\mathbb{L}^{>\mathbb{R}}$. In order to obtain bijective hyperlogarithms, one must extend $\mathbb{L}$, as a hyperserial field, with a formal element $\mathrm{e}_{\omega^\mu}^{\ell_0}$ for each $\mu\in\mathbf{On}$. This symbol $\mathrm{e}_{\omega^\mu}^{\ell_0}$ corresponds as a function to E_{ω^μ}. It was shown [7, Theorem 1.4] that certain hyperserial fields called confluent hyperserial fields admit a closure under those hyperexponential functions. More precisely, we say that a hyperserial field $(\mathbb{T},\circ)$ is hyperexponentially closed if each function $L_{\omega^\mu}\colon\mathbb{T}^{>\mathbb{R}}\longrightarrow\mathbb{T}^{>\mathbb{R}}$ for $\mu\in\mathbf{On}$ is surjective. Then any confluent hyperserial field $\mathbb{T}$ is naturally included in a hyperexponentially closed confluent hyperserial field $\tilde{\mathbb{T}}$ with the following initial property: if $\Phi\colon\mathbb{T}\longrightarrow\mathbb{U}$ is an embedding into a hyperexponentially closed hyperserial field $\mathbb{U}$, then there is a unique embedding $\Psi\colon\tilde{\mathbb{T}}\longrightarrow\mathbb{U}$ which extends Φ.

$$\begin{array}{ccc}\mathbb{T} & \overset{\subseteq}{\hookrightarrow} & \tilde{\mathbb{T}}\\ & {\scriptstyle\Phi}\searrow & \downarrow\exists!\,\Psi\\ & & \mathbb{U}\end{array}$$

In particular, there is a minimal closure $\tilde{\mathbb{L}}$ of logarithmic hyperseries under hyperexponentials, where each hyperexponential function $E_{\omega^\mu}\colon\tilde{\mathbb{L}}^{>\mathbb{R}}\longrightarrow\tilde{\mathbb{L}}^{>\mathbb{R}}$ is defined and bijective. Each series f in $\tilde{\mathbb{L}}$ is obtained by combining hyperlogarithms, hyperexponentials, and well-based sums with real coefficients. This suggests that f should admit a well-defined derivative $\tilde{\partial}(f)$ and should act on $\tilde{\mathbb{L}}^{>\mathbb{R}}$ through a left-composition $g\longmapsto f\mathbin{\tilde{\circ}} g$. The goal of this paper is to prove that this is the case, and to show that the operations $\tilde{\partial}\colon\tilde{\mathbb{L}}\longrightarrow\tilde{\mathbb{L}}$ and $\tilde{\circ}\colon\tilde{\mathbb{L}}\times\tilde{\mathbb{L}}^{>\mathbb{R}}\longrightarrow\tilde{\mathbb{L}}^{>\mathbb{R}}$ retain certain important properties of their respective restrictions to $\mathbb{L}$ and $\mathbb{L}\times\mathbb{L}^{>\mathbb{R}}$.

Our overarching goal, as part of a research program laid out in [5], is to extend these operations to Conway's class **No** of *surreal numbers* [12]. Indeed we showed with van der Hoeven [6] that **No** can be seen, in a natural way, as a hyperexponentially closed confluent hyperserial field that properly contains $\tilde{\mathbb{L}}$. We thus require more general theorems in order to extend derivations and compositions on hyperserial fields to their closure under hyperexponentials. In other words, we must extend Schmeling's work [35] on fields of transseries and their exponential closure to hyperserial fields and their hyperexponential closure. The results of this paper consist in showing that certain properties of a given configuration $\circ\colon \mathbb{U}\times\mathbb{T}^{>\mathbb{R}}\longrightarrow\mathbb{V}$ or $\partial\colon\mathbb{U}\longrightarrow\mathbb{V}$ between hyperserial fields, which are true in the case when $\mathbb{U}=\mathbb{T}=\mathbb{V}=\mathbb{L}$, extend through the closure under hyperexponentials. Their respective statements involve technical conditions on $\mathbb{U}$, $\mathbb{T}$ and $\mathbb{V}$, so it is simpler for now to introduce them in the specific case when $\mathbb{U}=\mathbb{T}=\mathbb{V}=\mathbb{L}$.

The properties of derivations and composition laws which we are looking for are inspired by properties of germs in Hardy fields [10]. A Hardy field is a field $\mathcal{H}\supseteq\mathbb{R}$ of germs at $+\infty$ of real-valued functions, which is closed under derivation. Thus $\mathcal{H}$ comes equipped with a structure of ordered valued differential field $(\mathcal{H},+,\times,<,\prec,\partial)$ as per [1]. As [2, 4] illustrate, many properties of $(\mathcal{H},+,\times,<,\prec,\partial)$ follow from two simple axiomatic properties in the language of ordered valued differential rings:

H1. For all $f\in\mathcal{H}^{>0}$ with $f\succ 1$, we have $\partial(g)>0$.

H2. For all $f\in\mathcal{H}$ with $g\prec 1$, we have $\partial(f)\prec 1$.

Ordered valued differential fields satisfying **H1** and **H2** are called H-fields with small derivation. Given two germs f,g in a Hardy field $\mathcal{H}$ with $g>\mathbb{R}$, i.e. $\lim_{+\infty} g=+\infty$, the germ $f\circ g$ of $r\longmapsto f(g(r))$ is well-defined. If $f\circ g\in\mathcal{H}$ for all such f,g, then we also have a composition law $\circ\colon\mathcal{H}\times\mathcal{H}^{>\mathbb{R}}\longrightarrow\mathcal{H}$, with the following properties:

C1. For all $f\in\mathcal{H}$ and $g,h\in\mathcal{H}^{>\mathbb{R}}$, we have $g\circ h>\mathbb{R}$ and $f\circ(g\circ h)=(f\circ g)\circ h$.

C2. For all $g\in\mathcal{H}^{>\mathbb{R}}$, the function $\mathcal{H}\longrightarrow\mathcal{H}; f\mapsto f\circ g$ is a strictly increasing morphism of rings.

C3. For all $f\in\mathcal{H}^{>\mathbb{R}}$, the function $\mathcal{H}^{>\mathbb{R}}\longrightarrow\mathcal{H}; g\mapsto f\circ g$ is strictly increasing.

In certain cases, such as the field $\mathcal{H}_{\mathrm{an,exp}}$ of germs definable in the o-minimal expansion $\mathbb{R}_{\mathrm{an,exp}}$ of the real ordered field by the exponential and restricted analytic functions [16], we have Taylor approximations for germs:

C4. For all $f\in\mathcal{H}$, $g\in\mathcal{H}^{>\mathbb{R}}$, $\delta\prec g$ with $\delta\prec\frac{f\circ g}{\partial(f)\circ g}$, and for all $n\in\mathbb{N}$, we have

$$f\circ(g+\delta)-\sum_{k=0}^{n}\frac{\partial^k(f)\circ g}{k!}\,\delta^k\prec(\partial^n(f)\circ g)\,\delta^n.$$

One thus expects derivations and composition laws on hyperserial fields to satisfy formal, strongly linear versions of **H1**, **H2**, **C1**, **C2**, **C3** and **C4** such as i, ii, iii and iv above. We will see throughout the paper that straightforward definitions of derivations and composition laws on hyperexponentially closed fields do yield those properties.

Outline of the paper

Let us now describe our results. The main obstacle on our way is the difficult task of dealing with the summability of transfinite families. Which is why we gather a rather large set of tools to that effect in Sections 1 which also defines fields of well-based series over $\mathbb{R}$. Section 2 introduce the notion hyperserial fields. Section 3 focuses on transserial subfields and subgroups, which are substructures of hyperserial fields that are only closd under the logarithm, as opposed to the whole calculus of logartihmic hyperseries.

In Section 5, we show that the derivation $\mathbb{L} \longrightarrow \mathbb{L}$ extends into a derivation $\tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ with similar properties:

Result A (Corollary of Theorem 5.7). [Corollary 5.22] *There is a unique extension of $\partial\colon \mathbb{L} \longrightarrow \mathbb{L}$ into a derivation $\tilde{\partial}\colon \tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ with the following properties:*

i. $\tilde{\partial}$ is strongly linear.

ii. $\tilde{\partial}(f \circ g) = \tilde{\partial}(g) \times \partial(f) \circ g$ for all $f \in \mathbb{L}$ and $g \in \tilde{\mathbb{L}}^{>\mathbb{R}}$.

Moreover $(\tilde{\mathbb{L}}, \tilde{\partial})$ is an H-field with small derivation.

Relying on results from [4, 15], we give a more precise description of the model theory of $\tilde{\mathbb{L}}$ as an ordered differential valued field.

Result B. [Theorem-5.23] *The structure $(\tilde{\mathbb{L}}, +, \times, <, \prec, \tilde{\partial})$ is an elementary extension of Dahn-Göring's [13] and Ecalle's [18] field $(\mathbb{T}_{\mathrm{LE}}, +, \times, <, \prec, \partial)$ of logarithmic-exponential transseries.*

Section 6 regards composition laws. In particular, we prove that the composition law $\circ\colon \mathbb{L} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ extends uniquely into a composition law $\tilde{\mathbb{L}} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$.

Result C (Corollary of Theorem 6.1). [Corollary 6.23] *There is a unique extension of $\circ\colon \mathbb{L} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ into a function $\tilde{\circ}\colon \tilde{\mathbb{L}} \times \tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}$ with the following properties:*

i. For all $g \in \tilde{\mathbb{L}}^{>,\succ}$, the function $f \longmapsto f \circ g\colon \tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ is a strongly linear morphism of ordered rings.

ii. For all $f \in \tilde{\mathbb{L}}$ and $g, h \in \tilde{\mathbb{L}}^{>\mathbb{R}}$, we have $f \mathbin{\tilde{\circ}} (g \mathbin{\tilde{\circ}} h) = (f \mathbin{\tilde{\circ}} g) \mathbin{\tilde{\circ}} h$.

Then we show in Section 7 that the Taylor expansions property iv of $\mathbb{L}$, or **C4** in the context of Hardy fields, extends to $\tilde{\mathbb{L}}$:

Result D (Corollary of Theorem 7.13). [Corollary 8.3] *For all $f \in \tilde{\mathbb{L}}$, $g \in \tilde{\mathbb{L}}^{>\mathbb{R}}$ and $\delta \in \tilde{\mathbb{L}}$ with $\delta \prec g$, we have*

$$\left(\forall \mathfrak{m} \in \operatorname{supp} f, \delta \prec \frac{\mathfrak{m} \mathbin{\tilde{\circ}} g}{\tilde{\partial}(\mathfrak{m}) \mathbin{\tilde{\circ}} g}\right) \Longrightarrow f \mathbin{\tilde{\circ}} (g + \delta) = \sum_{k \in \mathbb{N}} \frac{\tilde{\partial}^k(f) \mathbin{\tilde{\circ}} g}{k!} \delta^k.$$

Section 8 is dedicated to the proof that elements of $\tilde{\mathbb{L}}$ act as monotonous functions:

Result E (Corollary of Theorem 8.16). [Corollary 8.20] *For all $f \in \tilde{\mathbb{L}}^{>\mathbb{R}}$, the function*

$$\tilde{\mathbb{L}}^{>\mathbb{R}} \longrightarrow \tilde{\mathbb{L}}^{>\mathbb{R}};\ g \mapsto f \mathbin{\tilde{\circ}} g$$

is strictly increasing.

It should be noted that the results **A** through **E** are all obtained *via* the same method, which we describe in section 4.3. Finally, we focus on the structure $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \tilde{\circ}, <)$. Adapting arguments from [15, 21, 19], we prove the following:

Result F. [Proposition 8.23 and Theorem 9.16] *The structure $(\tilde{\mathbb{L}}^{>\mathbb{R}}, \tilde{\circ}, <)$ is a bi-ordered group where any two strictly positive elements are conjugate.*

We also apply this in order to solve the inequation $f \circ g \geqslant g \circ f$ for positive elements f, g in the group (Proposition 9.25).

1 Strongly linear algebra

Convention. Before we start, we set a few conventions.

Set theory. We adopt the set-theoretic framework of [7]. In particular, the underlying set theory of this paper is NBG set theory. This is a conservative extension of ZFC which allows us to prove statements about proper classes.

Ordinals. We consider the class $\mathbf{On}$ of ordinals as a "generalized" ordinal. If $\boldsymbol{\nu}$ is a class, then $\boldsymbol{\nu} \leqslant \mathbf{On}$ means that $\boldsymbol{\nu} \in \mathbf{On}$ or $\boldsymbol{\nu} = \mathbf{On}$. For generalized ordinals, we use bold font notations $\boldsymbol{\nu}, \boldsymbol{\mu}, \boldsymbol{\lambda}$ to suggest that $\boldsymbol{\nu}, \boldsymbol{\mu}, \boldsymbol{\lambda}$ may be equal to $\mathbf{On}$, whereas the notations $\alpha, \gamma, \beta, \rho$ and so on are only used for true ordinals $\alpha, \gamma, \beta, \rho \in \mathbf{On}$. We also extend the relations $\leqslant$ and $<$ on $\mathbf{On}$ by making $\mathbf{On}$ maximal, and we set

$$\omega^{\mathbf{On}} := \mathbf{On}.$$

Ordered monoids. If $(\mathcal{M}, <)$ is an ordered monoid such as $\mathbb{N}$, $\mathbb{R}$, one of our hyperserial fields $\mathbb{T}$ or groups of monomials $\mathfrak{M}$, then $\mathcal{M}^>$ denotes its subclass of strictly positive elements in $\mathcal{M}$, whereas $\mathcal{M}^{\neq}$ denotes the class of non-zero elements of $\mathcal{M}$.

1.1 Fields of well-based series

Let $(\mathfrak{M}, \times, 1, \prec)$ be a linearly ordered abelian group, possibly class-sized. We write $\mathbb{R}[[\mathfrak{M}]]$ for the class of functions $s\colon \mathfrak{M} \longrightarrow \mathbb{R}$ whose support

$$\operatorname{supp} s := \{\mathfrak{m} \in \mathfrak{M} : s(\mathfrak{m}) \neq 0\}$$

is a *well-based* set, i.e. a set which is well-ordered in the reverse order $(\mathfrak{M}, \succ)$. The elements of $\mathfrak{M}$ are called monomials, whereas those in $\mathbb{R}^{\neq}\mathfrak{M}$ are called *terms*.

We see elements s of $\mathbb{S}$ as formal *well-based series* $s = \sum_{\mathfrak{m}} s_{\mathfrak{m}}\mathfrak{m}$ where for $\mathfrak{m} \in \mathfrak{M}$, the symbol $s_{\mathfrak{m}}$ denotes the value $s(\mathfrak{m}) \in \mathbb{R}$. If $\operatorname{supp} s \neq \varnothing$, then we write

$$\begin{aligned} \mathfrak{d}_s &:= \max \operatorname{supp} f \in \mathfrak{M} \quad \text{and} \\ \tau_s &:= s_{\mathfrak{d}_s}\mathfrak{d}_s \in \mathbb{R}^{\neq}\mathfrak{M} \end{aligned}$$

respectively for the *dominant monomial* and *dominant term* of s. For $\mathfrak{m} \in \mathfrak{M}$, we set

$$s_{\succ\mathfrak{m}} := \sum_{\mathfrak{n} \succ \mathfrak{m}} s_{\mathfrak{n}}\mathfrak{n},$$

and $s_{\succ} := s_{\succ 1}$. For $s, t \in \mathbb{S}$, we say that t is a *truncation* of s and we write $t \trianglelefteq s$ if $\operatorname{supp}(s-t) \succ \operatorname{supp} s$. The relation $\trianglelefteq$ is a well-founded partial order on $\mathbb{S}$ with minimum 0.

By [23], the class $\mathbb{S}$ is an ordered field under the pointwise sum

$$(s+t) := \sum_{\mathfrak{m}} (s_{\mathfrak{m}} + t_{\mathfrak{m}})\,\mathfrak{m},$$

and the Cauchy product

$$s\,t := \sum_{\mathfrak{m}} \left(\sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}} s_{\mathfrak{u}} t_{\mathfrak{v}} \right) \mathfrak{m},$$

(where each sum $\sum_{\mathfrak{u}\mathfrak{v}=\mathfrak{m}} s_{\mathfrak{u}} t_{\mathfrak{v}}$ has finite support). The positive cone $\mathbb{S}^> = \{s \in \mathbb{S} : s > 0\}$ is

$$\mathbb{S}^> := \{s \in \mathbb{S} : s \neq 0 \wedge s_{\mathfrak{d}_s} > 0\}.$$

We have an embedding of ordered groups

$$(\mathfrak{M}, \times, \prec) \longrightarrow (\mathbb{S}^>, \times, <)\colon \mathfrak{m} \longmapsto \sum_{\mathfrak{n} = \mathfrak{m}} \mathfrak{n}$$

and we identify $\mathfrak{M}$ with its image in $\mathbb{S}^>$.

The order on $\mathfrak{M}$ extends into a strict quasi-order $\prec$ on $\mathbb{S}$ defined by $s \prec t$ if and only if $\mathbb{R}^{>}|s| < |t|$. We write $s \preccurlyeq t$ if $t \prec s$ is false, i.e. if there is $r \in \mathbb{R}^{>}$ with $|s| \leqslant r\,|t|$. We also write $s \asymp t$ if $s \preccurlyeq t$ and $t \preccurlyeq s$, i.e. if there is $r \in \mathbb{R}^{>}$ with $r\,|s| \geqslant |t|$ and $r\,|t| \geqslant s$. Then $\preccurlyeq$ is a dominance relation as per [4, Definition 3.1.1], and it corresponds to the natural valuation on the ordered field $(\mathbb{S}, +, \times, <)$. In particular $(\mathbb{S}, +, \times, <, \preccurlyeq)$ is an ordered valued field with convex valuation ring $\mathbb{S}^{\preccurlyeq} := \{s \in \mathbb{S} : s \preccurlyeq 1\}$.

When s, t are non-zero, we have $s \prec t$ (resp. $s \preccurlyeq t$, resp. $s \asymp t$) if and only if $\mathfrak{d}_s \prec \mathfrak{d}_t$ (resp. $\mathfrak{d}_s \preccurlyeq \mathfrak{d}_t$, resp. $\mathfrak{d}_s = \mathfrak{d}_t$).

We write

$$\begin{aligned} \mathbb{S}_{\succ} &:= \{s \in \mathbb{S} : \operatorname{supp} s \subseteq \mathfrak{M}^{\succ}\} \\ \mathbb{S}^{\prec} &:= \{s \in \mathbb{S} : \operatorname{supp} s \subseteq \mathfrak{M}^{\prec}\} = \{s \in \mathbb{S} : s \prec 1\}, \quad \text{and} \\ \mathbb{S}^{>,\succ} &:= \{s \in \mathbb{S} : s > \mathbb{R}\} = \{s \in \mathbb{S} : s \geqslant 0 \wedge s \succ 1\}. \end{aligned}$$

Series in $\mathbb{S}_{\succ}$, $\mathbb{S}^{\prec}$ and $\mathbb{S}^{>,\succ}$ are respectively said *purely large*, *infinitesimal*, and *positive infinite*. A subclass $\mathfrak{S}$ of $\mathfrak{M}$ is said infinitesimal if all its elements are infinitesimal. We say that $\mathfrak{S}$ is *small* if we have $\mathfrak{s} \preccurlyeq 1$ for all $\mathfrak{s} \in \mathfrak{S}$.

Remark 1.1. On the notation s vs f. The reader will notice that we sometimes write f, g and so on for well-based series, and sometimes rather s, t, and so on. The notation f, g is used to suggest that we consider f and g as functions acting on a field through a composition law, whereas s, t are used to suggest that we are seeing s, t as objects on which certain functions, e.g. right compositions and derivations, act. Sometimes both contexts are relevant, and we have to make a choice.

1.2 Well-based families

Let I be a set. A family $(s_i)_{i \in I}$ in $\mathbb{S}$ is said *well-based* if

i. $\bigcup_{i \in I} \operatorname{supp} s_i$ is well-based, and

ii. $\{i \in I : \mathfrak{m} \in \operatorname{supp} s_i\}$ is finite for all $\mathfrak{m} \in \mathfrak{M}$.

Then we may define the sum $\sum_{i \in I} s_i$ of $(s_i)_{i \in I}$ as the series

$$\sum_{i \in I} s_i := \sum_{\mathfrak{m}} \left(\sum_{i \in I} (s_i)_{\mathfrak{m}} \right) \mathfrak{m}.$$

We have the following consequence of [26, Proposition 3.1(e)]:

Lemma 1.2. *Let I, J be sets and let $(f_{i,j})_{(i,j) \in I \times J}$ be a well-based family. For each $i_0 \in I$ and for each $j_0 \in J$, the families $(f_{i_0,j})_{j \in J}$, $(f_{i,j_0})_{i \in I}$ are well-based. Moreover families $(\sum_{j \in J} f_{i,j})_{i \in I}$ and $(\sum_{i \in I} f_{i,j})_{j \in J}$ are well-based, with*

$$\sum_{i \in I} \left(\sum_{j \in J} f_{i,j} \right) = \sum_{(i,j) \in I \times J} f_{i,j} = \sum_{j \in J} \left(\sum_{i \in I} f_{i,j} \right).$$

One of the main difficulties of our work here is to prove that certain families are well-based. In some cases, the proof can be done using elementary arguments, but they sometimes require more powerful tools. In particular, we will rely on results in [33, 26, 35].

1.3 Neumann's theorems

For $\mathfrak{S} \subseteq \mathfrak{M}$ we write $\mathfrak{S}^n := \underbrace{\mathfrak{S} \cdots \mathfrak{S}}_{n \text{ times}} = \{\mathfrak{s}_1 \cdots \mathfrak{s}_n : \mathfrak{s}_1, \ldots, \mathfrak{s}_n \in \mathfrak{S}\}$ and

$$\mathfrak{S}^{\infty} := \bigcup_{n \in \mathbb{N}} \mathfrak{S}^n = \{\mathfrak{s}_1 \cdots \mathfrak{s}_n : n \in \mathbb{N} \wedge \mathfrak{s}_1, \ldots, \mathfrak{s}_n \in \mathfrak{S}\}.$$

We have the following important results of B. Neumann:

Lemma 1.3. [33, Lemma 3.2 and Corollary 3.21] *Let $\mathfrak{S}, \mathfrak{T} \subseteq \mathfrak{M}$ be well-based. Then the class $\mathfrak{S} \cdot \mathfrak{T}$ is well-based. Moreover, for all $\mathfrak{m} \in \mathfrak{S} \cdot \mathfrak{T}$, the set $\{(\mathfrak{u}, \mathfrak{v}) \in \mathfrak{S} \times \mathfrak{T} : \mathfrak{m} = \mathfrak{u}\,\mathfrak{v}\}$ is finite.*

Lemma 1.4. [33, Theorems 3.4 and 3.5] *Let $\mathfrak{S} \subseteq \mathfrak{M}^{\preccurlyeq}$ be well-based. The class $\mathfrak{S}^\infty$ is well-based. Moreover, for all $\mathfrak{m} \in \mathfrak{S}^\infty$, the set $\{n \in \mathbb{N} : \mathfrak{m} \in \mathfrak{S}^n\}$ is finite.*

A consequence of Lemma 1.4 is that for all $\varepsilon \in \mathbb{S}^{\prec}$, the set $(\operatorname{supp} \varepsilon)^\infty$ is well-based and for all $(r_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ the family $(r_n \varepsilon^k)_{k\in\mathbb{N}}$ is well-based.

Lemma 1.5. *Let $(s_i)_{i\in I}$ be a family in $\mathbb{S}$. Assume that there is a well-based and infinitesimal set $T \subseteq \mathfrak{M}$, a well-based set $S \subseteq \mathfrak{M}$ and a function $\mathrm{N} : I \longrightarrow \mathbb{N}$ such that we have*

$$\operatorname{supp} s_i \subseteq T^{\mathrm{N}(i)} \cdot S \quad \text{for all } i \in I.$$

Assume that $(s_j)_{j\in J}$ is well-based whenever $J \subseteq I$ and $\mathrm{N}(J)$ is finite. Then $(s_i)_{i\in I}$ is well-based.

Proof. Assume for contradiction that $(s_i)_{i\in I}$ is not well-based. So there is an injective sequence $(i_k)_{k\in\mathbb{N}} \in I^{\mathbb{N}}$ and a sequence $(\mathfrak{m}_k)_{k\in\mathbb{N}} \in \mathfrak{M}^{\mathbb{N}}$ with $\mathfrak{m}_0 \preccurlyeq \mathfrak{m}_1 \preccurlyeq \cdots$ and $\mathfrak{m}_k \in \operatorname{supp} s_{i_k}$ for all $k \in \mathbb{N}$. We have $\{\mathfrak{m}_k : k \in \mathbb{N}\} \subseteq T^\infty \cdot S$ where $T^\infty \cdot S$ is well-based by Lemmas 1.3 and 1.4. So $\{\mathfrak{m}_k : k \in \mathbb{N}\}$ is well-based and we may assume that $(\mathfrak{m}_k)_{k\in\mathbb{N}}$ is constant. Fix $\mathfrak{t} \in T^\infty$ and $\mathfrak{s} \in S$ with $\mathfrak{m}_k = \mathfrak{t}\,\mathfrak{s}$ for all $k \in \mathbb{N}$. We have $\mathfrak{t} \in T^{\mathrm{N}(i_k)}$ for all $k \in \mathbb{N}$. By Lemma 1.4, this implies that $\{\mathrm{N}(i_k) : k \in \mathbb{N}\}$ is finite, so $(s_{i_k})_{k\in\mathbb{N}}$ is well-based: a contradiction. □

Corollary 1.6. *Let $(s_{n,m})_{(n,m)\in\mathbb{N}^2}$ be a family in $\mathbb{S}$ such that each $(s_{n,m})_{m\in\mathbb{N}}$ for $n \in \mathbb{N}$ is well-based. Assume that there is a well-based and infinitesimal set $T \subseteq \mathfrak{M}$ and a well-based set $S \subseteq \mathfrak{M}$ with*

$$\forall n, m \in \mathbb{N}, \operatorname{supp} s_{n,m} \subseteq T^n \cdot S.$$

Then $(s_{n,k})_{(n,k)\in\mathbb{N}^2}$ is well-based.

We say that a subclass $\mathfrak{S} \subseteq \mathfrak{U}$ it is *good* if it is well-based, small and if moreover $\mathfrak{S}^\infty \subseteq \mathfrak{S}$. If $\mathfrak{S}$ is small and well-based, then by Lemma 1.4, the class $\mathfrak{S}^\infty \supseteq \mathfrak{S}$ is good.

Proposition 1.7. *Let $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series. Let I be a set and let $f : I \longrightarrow \mathbb{N}$ be an arbitrary function. Let $(s_i)_{i\in I}$ be a well-based family in $\mathbb{S}$ and let $\delta \preccurlyeq 1$. The family $(s_i\, \delta^{f(i)})_{i\in I}$ is well-based.*

Proof. Let $\mathfrak{S} := \bigcup_{i\in I} \operatorname{supp} s_i$. So $\mathfrak{S}$ is well-based. For $(i, k) \in I \times \mathbb{N}$, write $s_{i,k} := s_i \binom{f(i)}{k} r^{f(i)-k} \varepsilon^k$, so

$$\operatorname{supp} s_{i,k} \subseteq \mathfrak{S} \cdot (\operatorname{supp} \varepsilon)^k.$$

If $J \subseteq \mathbb{N}$ is finite, then $(s_{i,k})_{i\in I, k\in J}$ is well-based as a finite union of well-based families. We deduce with Lemma 1.5 that $(s_{i,k})_{i\in I,k\in\mathbb{N}}$ is well-based. In particular $\left(\sum_{k=0}^{f(i)} s_{i,k}\right)_{i\in I} = (s_i\, \delta^{f(i)})_{i\in I}$ is well-based by [26, Proposition 3.1(e)]. □

1.4 Power series

Let $\mathbb{S}, \mathbb{T}, \mathbb{U}$ be fixed fields of well-based series over $\mathbb{R}$. We write $\mathbb{S}[[z]]$ for the ring of power series

$$P = \sum_{k\in\mathbb{N}} P_k\, z^k, (s_k)_{k\in\mathbb{N}} \in \mathbb{S}^{\mathbb{N}}$$

over $\mathbb{S}$. If $P, Q \in \mathbb{S}[[z]]$ with $Q_0 = 0$, we have a composite power series

$$P \circ Q := P_0 + \sum_{k\in\mathbb{N}} \left(\sum_{m_1+\cdots+m_n=k} P_n Q_{m_1} \cdots Q_{m_n} \right) z^k \in \mathbb{S}[[z]].$$

Consider the subdomain $z\,\mathbf{D}[[z]]$ of $\mathbf{D}[[z]]$ of power series $P = \sum_{k\in\mathbb{N}} P_k z^k$ with $P_0 = 0$. We have a composition law $\circ\colon \mathbf{D}[[z]] \times z\,\mathbf{D}[[z]] \longrightarrow \mathbf{D}[[z]]$. Indeed for $P = \sum_{k\in\mathbb{N}} P_k z^k$, $Q = \sum_{k\in\mathbb{N}} Q_k z^k \in \mathbb{S}[[z]]$ with $Q_0 = 0$, we have a *composite power series*

$$P \circ Q := P_0 + \sum_{k\in\mathbb{N}} \left(\sum_{m_1+\cdots+m_n=k} P_n Q_{m_1} \cdots Q_{m_n} \right) z^k \in \mathbb{S}[[z]].$$

For $P \in \mathbf{D}[[z]]$ and $Q, R \in z\,\mathbf{D}[[z]]$, we have $Q \circ R \in z\,\mathbf{D}[[z]]$ and

$$P \circ (Q \circ R) = (P \circ Q) \circ R.$$

1.5 Convergence of power series

Definition 1.8. *Given a power series*

$$P = \sum_{v\in\mathbb{N}^n} P_v z_1^{v_{[1]}} \cdots z_n^{v_{[n]}} \in \mathbb{S}[[z_1, \ldots, z_n]],$$

and $s_1, \ldots, s_n \in \mathbb{S}$, we say that P **converges at** *$(s_1, \ldots, s_n)$ if the family $(P_v s_1^{v_{[1]}} \cdots s_n^{v_{[n]}})_{v\in\mathbb{N}^n}$ is well-based. We then set*

$$\tilde{P}(s_1, \ldots, s_n) := \sum_{v\in\mathbb{N}^n} P_v s_1^{v_{[1]}} \cdots s_n^{v_{[n]}}.$$

We write $\mathrm{Conv}(P)$ *for the class of tuples $(s_1, \ldots, s_n) \in \mathbb{S}^n$ at which P converges.*

Example 1.9. Any real power series $P = \sum_{k\in\mathbb{N}} r_k z^k \in \mathbb{R}[[z]]$ converges on $\mathbb{S}^{\prec}$ by Lemma 1.4. In fact, since the sequence $(s^k)_{k\in\mathbb{N}}$ is $\prec$-increasing whenever $s \succcurlyeq 1$, we have $\mathrm{Conv}(P) = \mathbb{S}^{\prec}$ unless P is a polynomial.

Proposition 1.10. [35, Corollary 1.5.8] *For all $P \in \mathbb{S}[[z]]$, and $\varepsilon, \delta \in \mathbb{S}$ with $\delta \in \mathrm{Conv}(P)$, we have $\varepsilon \preccurlyeq \delta \Longrightarrow \varepsilon \in \mathrm{Conv}(P)$.*

Proof. Write $P = \sum_{k\in\mathbb{N}} P_k z^k$ and $u := \varepsilon/\delta \preccurlyeq 1$. By Proposition 1.7 for $I = \mathbb{N}$ and $f = \mathrm{id}_{\mathbb{N}}$, the family $(P_k \delta^k u^k)_{k\in\mathbb{N}} = (P_k \varepsilon^k)_{k\in\mathbb{N}}$ is well-based. □

Lemma 1.11. *Let $P \in \mathbb{S}[[z]]$ with $\mathrm{Conv}(P) \neq \{0\}$. Then $\mathrm{Conv}(P)$ is open.*

Proof. Given $\varepsilon \in \mathrm{Conv}(P)$, there is a $\delta \in \mathrm{Conv}(P) \setminus \{0\}$, and we have $\varepsilon + \delta \preccurlyeq \delta$ or $\varepsilon + \delta \preccurlyeq \varepsilon$. In any case, we obtain $\varepsilon + \delta \in \mathrm{Conv}(P)$ by Proposition 1.10. Therefore $\mathrm{Conv}(P)$ is open. □

Lemma 1.12. *Let $P = \sum_{k\in\mathbb{N}} P_k z^k \in \mathbb{S}[[z]]$ be a power series. For all $n \in \mathbb{N}$, we have $\mathrm{Conv}(P) = \mathrm{Conv}(P^{(n)})$.*

Proof. It suffices to prove the result for $n = 1$. We have $0 \in \mathrm{Conv}(P) \cap \mathrm{Conv}(P')$. Recall that

$$P' = \sum_{k\in\mathbb{N}} (k+1)\, P_{k+1} z^k.$$

For $\varepsilon \in \mathbb{S}^{\neq}$, we have the following equivalences:

$(P_k\,\varepsilon^k)_{k\in\mathbb{N}}$ is well-based.

$\iff (P_{k+1}\,\varepsilon^{k+1})_{k\in\mathbb{N}}$ is well-based.

$\iff ((k+1)\,P_{k+1}\,\varepsilon^k)_{k\in\mathbb{N}}$ is well-based.

We deduce that $\mathrm{Conv}(P) = \mathrm{Conv}(P')$. □

Proposition 1.13. *Let $P = \sum_{k\in\mathbb{N}} P_k\,z^k \in \mathbb{S}[[z]]$ be a power series and let $\varepsilon, \delta \in \mathrm{Conv}(P)$. Write $P_{+\varepsilon}$ for the power series $P_{+\varepsilon} := \sum_{k\in\mathbb{N}} \frac{\widetilde{P^{(k)}}(\varepsilon)}{k!}\,z^k$. We have $\delta \in \mathrm{Conv}(P_{+\varepsilon})$ and*

$$\widetilde{P_{+\varepsilon}}(\delta) = \tilde{P}(\varepsilon+\delta).$$

Proof. Note that $P_{+0} = P$ and that $P_{+\varepsilon}(0) = P(\varepsilon)$, so we may assume that ε and δ are non-zero. The power series $P_{+\varepsilon}$ is well-defined by Lemma 1.12. We have

$$\bigcup_{i,k\in\mathbb{N}} \mathrm{supp}\,(P_{k+i}\,\varepsilon^{k+i}) = \bigcup_{j\in\mathbb{N}} \mathrm{supp}\,(P_j\,\varepsilon^j),$$

where the right hand set is well-based since $(P_j\varepsilon^j)_{j\in\mathbb{N}}$ is well-based. For each monomial $\mathfrak{m} \in \mathfrak{M}$, the set $I_{\mathfrak{m}} := \{(i,k) \in \mathbb{N}^2 : \mathfrak{m} \in \mathrm{supp}(P_{i+k}\,\delta^{k+i})\}$ is contained in $\{(i,k) \in \mathbb{N}^2 : i+k \in J_{\mathfrak{m}}\}$ where

$$J_{\mathfrak{m}} := \{j \in \mathbb{N} : \mathfrak{m} \in \mathrm{supp}(P_j\,\delta^j)\}.$$

Since $(P_j\varepsilon^j)_{j\in\mathbb{N}}$ is well-based, we deduce that $J_{\mathfrak{m}}$, and hence $I_{\mathfrak{m}}$ are finite. This shows that $(P_{k+i}\,\varepsilon^{k+i})_{i,k\in\mathbb{N}}$ is well-based. Likewise, $(P_{k+i}\,\delta^{k+i})_{i,k\in\mathbb{N}}$ is well-based.

For $k \in \mathbb{N}$, we have

$$\frac{\widetilde{P^{(k)}}(\varepsilon)}{k!}\,\delta^k = \sum_{i\in\mathbb{N}} \binom{k+i}{k} P_{k+i}\,\varepsilon^i\,\delta^k. \tag{1.1}$$

Therefore it suffices to show that the family $(P_{k+i}\,\varepsilon^i\,\delta^k)_{i,k\in\mathbb{N}}$ is well-based in order to prove that $\delta \in \mathrm{Conv}(P_{+\varepsilon})$. For $i,k \in \mathbb{N}$, write

$$\varepsilon^i\,\delta^k = u^{i+k}\,v^k$$

where $(u,v) = (\varepsilon, {}^{\delta}\!/_{\varepsilon})$ if $\delta \preccurlyeq \varepsilon$ and $(u,v) = (\delta, {}^{\varepsilon}\!/_{\delta})$ if $\varepsilon \prec \delta$. In any case, we have $v \preccurlyeq 1$ and the family $(P_{i+k}\,u^{i+k})_{i,k\in\mathbb{N}}$ is well-based. Applying Proposition 1.7 for $I = \mathbb{N}\times\mathbb{N}$ and $f = (a,b) \mapsto a+b$, we see that the family $(P_{i+k}\,u^{i+k}\,v^k)_{i,k\in\mathbb{N}} = (P_{k+i}\,\varepsilon^i\,\delta^k)_{i,k\in\mathbb{N}}$ is well-based.

On the other hand we have $\delta+\varepsilon \preccurlyeq \varepsilon$ or $\delta+\varepsilon \preccurlyeq \delta$, so $\delta+\varepsilon \in \mathrm{Conv}(P)$ and $(P_k\,(\delta+\varepsilon)^k)_{k\in\mathbb{N}}$ is well-based. By Lemma 1.2, we have

$$\begin{aligned}
\sum_{k\in\mathbb{N}} \frac{\widetilde{P^{(k)}}(\varepsilon)}{k!}\,\delta^k &= \sum_{k\in\mathbb{N}}\sum_{i\in\mathbb{N}} \binom{k+i}{k} P_{k+i}\,\varepsilon^i\,\delta^k \\
&= \sum_{i,k\in\mathbb{N}} \binom{k+i}{k} P_{k+i}\,\varepsilon^i\,\delta^k \\
&= \sum_{j\in\mathbb{N}}\sum_{l\leqslant j} \binom{j}{l} P_j\,\varepsilon^{j-l}\,\delta^l \\
&= \sum_{j\in\mathbb{N}} P_j\,(\varepsilon+\delta)^j \\
&= \tilde{P}(\varepsilon+\delta),
\end{aligned}$$

as desired. □

Lemma 1.14. *Let $P=\sum_{k\in\mathbb{N}} P_k\, z^k \in \mathbb{S}[[z]]$ be a power series with $\mathrm{Conv}(P)\neq\{0\}$. Then the function $\tilde{P}$ is infinitely differentiable on $\mathrm{Conv}(P)$ with $\tilde{P}^{(n)}=\widetilde{P^{(n)}}$ on $\mathrm{Conv}(P)$ for all $n\in\mathbb{N}$.*

Proof. Recall by Lemma 1.11 that $\mathrm{Conv}(P)$ is open. We first prove that $\tilde{P}$ is differentiable on $\mathrm{Conv}(P)$ with $\tilde{P}'=\tilde{P}'$. Let $\varepsilon>0$ and let $s\in\mathrm{Conv}(P)$. For all $h\in\mathbb{S}$ with $|h|<|s|$, we have $h\preccurlyeq s$, so Proposition 1.13 yields

$$\begin{aligned}\tilde{P}(s+h)-\tilde{P}(s) &= \sum_{k>0}\frac{\widetilde{P^{(k)}}(s)}{k!}\,h^k\\ &= \tilde{P}'(s)\,h+h^2\,u,\end{aligned}$$

where $u:=\sum_{k\in\mathbb{N}}\frac{\widetilde{P^{(k+2)}}(s)}{(k+2)!}\,h^k$. If $u=0$, then we set $\delta:=|s|$. If $u\neq 0$, then we set $\delta:=\varepsilon/|u|$. In both cases, we obtain $|\tilde{P}(s+h)-\tilde{P}(s)-\tilde{P}'(s)\,h|<\varepsilon\,|h|$ whenever $|h|<\delta$. So $\tilde{P}$ is differentiable at s with $\tilde{P}'(s)=\tilde{P}'(s)$. The result for all n follows by induction. □

1.6 Roots of power series

We next consider roots of power series functions. A *root* of a power series $P\in\mathbb{S}[[z]]$ is an element $s\in\mathrm{Conv}(P)$ with $\tilde{P}(s)=0$.

Lemma 1.15. *Let $P=\sum_{n\in\mathbb{N}} P_n\, z^n\in\mathbb{S}[[z]]$ be a power series and let $\mathcal{R}\subseteq\mathrm{Conv}(P)$ be an uncountable set of roots of P with pairwise distinct dominant terms. We have $P=0$.*

Proof. Assume for contradiction that there is a non-zero term P_n in the sequence and consider $s\in\mathcal{R}$. Since the sum of $(P_n\,s^n)_{n\in\mathbb{N}}$ is zero, for each number m with $P_m\neq 0$, there must exist at least one number $n\neq m$ with $\tau_{P_m}\,\tau_s^m=\tau_{P_n}\,\tau_s^n$. Then $\tau_s=\left(\frac{\tau_{s_m}}{\tau_{s_n}}\right)^{1/(m-n)}$, so we deduce that

$$\mathcal{R}\subseteq\left\{\left(\frac{\tau_{s_m}}{\tau_{s_n}}\right)^q : n,m\in\mathbb{N},\, q\in\mathbb{Q},\, P_m,P_n\neq 0\right\}.$$

Therefore $\mathcal{R}$ is countable: a contradiction. □

Lemma 1.16. *Let $P=\sum_{k\in\mathbb{N}} P_k\, z^k\in\mathbb{S}[[z]]$ be a power series, and let κ be an infinite cardinal. Let $\mathcal{R}\subseteq\mathbb{S}$ be a set of roots of P with cardinal $\geqslant\kappa^+$ such that for each $s\in\mathcal{R}$, the order type of $(\operatorname{supp} s,\succ)$ is $\leqslant\kappa$. Then $P=0$.*

Proof. Assume for contradiction that $P\neq 0$. We will call *large* the subsets X of $\mathcal{R}$ with $|\mathcal{R}\setminus X|\leqslant\kappa$. For $\alpha\leqslant\kappa$ and $s\in\mathbb{S}$, we let $s_{|\alpha}$ denote the $\trianglelefteq$-maximal truncation of s such that the order type of $(\operatorname{supp} s_{|\alpha},\succ)$ is $\leqslant\alpha$, and we write $s_{\alpha|}:=s-s_{|\alpha}$. Let $\mathcal{I}$ denote the set of ordinals $\alpha\leqslant\kappa$ such that there is a large subset $X_\alpha\subseteq\mathcal{R}$ with $t_{|\beta}=u_{|\beta}$ for all $\beta<\alpha$ and $t,u\in X_\alpha$. Notice that $\mathcal{I}$ contains 0 trivially and 1 by Lemma 1.15. We prove that $\kappa\in\mathcal{I}$. Let $\alpha\leqslant\kappa$ with $\beta\in\mathcal{I}$ for all $\beta<\alpha$.

If α is limit, then for each $\beta<\alpha$, pick a large subset $X_\beta\subseteq\mathcal{R}$ satisfying the condition and consider the set $X_\alpha:=\bigcap_{\beta<\alpha}X_\beta$. This set is large since $\alpha<\kappa^+$ and κ^+ is regular. Moreover it satisfies the condition for α by definition. So $\alpha\in\mathcal{I}$.

Assume now that $\alpha = \beta + 1$ where $\beta \in \mathcal{I}$ and $\beta > 0$. We fix a set X_β satisfying the condition for β. For $t \in X_\beta$, since $\beta > 0$, we have $t_{|\beta} \preccurlyeq t$, so $t_{|\beta} \in \mathrm{Conv}(P)$ is defined. We deduce with Lemma 1.14 that $t_{|\beta} \in \mathrm{Conv}(P^{(k)})$ for all $k \in \mathbb{N}$. By Proposition 1.13 we have $\tilde{P}(t) = \tilde{P}(t_{|\beta} + t_{\beta|}) = \widetilde{P_{+t_{|\beta}}}(t_{\beta|})$. Assume for contradiction that $P_{+t_{|\beta}} = 0$. Then $\widetilde{P^{(i)}}(t_{|\beta}) = 0$ for all $i \in \mathbb{N}$, so $\tilde{P}(t_{|\beta} + \varepsilon) = 0$ for all $\varepsilon \in \mathbb{S}$ with $\varepsilon \preccurlyeq t_{|\beta}$. In particular, given $\gamma < \beta$, we have $\tilde{P}(t_{|\gamma} + r\,\mathfrak{d}_{t_{|\gamma}}) = 0$ for all $r \in \mathbb{R}$, which contradicts Lemma 1.15. We deduce that $P_{+t_{|\beta}}$ is non-zero. By Lemma 1.15, there is a co-countable subset of X_β, hence large subset X_α of $\mathcal{R}$ with $(t_{\beta|})_{|1} = (u_{\beta|})_{|1}$, hence $u_{|\alpha} = v_{|\alpha}$ for all $u, v \in X_\alpha$. This proves that $\alpha \in \mathcal{I}$. By induction, we deduce that $\kappa \in \mathcal{I}$. For $u, v \in X_\kappa$, we have $u = u_{|\kappa} = v_{|\kappa} = v$, which contradicts the fact that X_κ is large. □

We note two corollaries to this result.

Corollary 1.17. *Let $P \in \mathbb{S}[[z]]$ and let $\varepsilon \in \mathrm{Conv}(P)$ with $\varepsilon \neq 0$. If $\tilde{P}(\delta) = 0$ for all $\delta \preccurlyeq \varepsilon$ then $P = 0$.*

Proof. Consider the set $\mathcal{S}$ of series $s \in \mathbb{S}$ with $s \preccurlyeq \varepsilon$ and such that the order type of $(\mathrm{supp}\, s, \succ)$ is at most ω. Fix an $\mathfrak{m} \in \mathfrak{M}^{\prec}$ with $\mathfrak{m} \preccurlyeq \varepsilon$. Each binary sequence $u \in 2^{\mathbb{N}}$, we have a yields a single element $\sum_{n \in \mathbb{N}} u(n)\, \mathfrak{m}^n \in \mathcal{S}$, so $\mathcal{S}$ is uncountably infinite. It follows by Lemma 1.16 for $\kappa = \omega$ that $P = 0$. □

Corollary 1.18. *Let $P \in \mathbb{S}[[z]]$ with $\mathrm{Conv}(P) \neq \{0\}$ and let $\delta \in \mathrm{Conv}(P)$. We have $\mathrm{Conv}(P_{+\delta}) = \mathrm{Conv}(P)$ and $P = (P_{+\delta})_{+(-\delta)}$.*

Proof. We may assume that $\delta \neq 0$. Proposition 1.13 shows that $\mathrm{Conv}(P_{+\delta}) \supseteq \mathrm{Conv}(P)$. By $\prec$-initiality of $\mathrm{Conv}(P)$, we have $-\delta \in \mathrm{Conv}(P)$. So $-\delta \in \mathrm{Conv}(P_{+\delta})$, which means that the power series $(P_{+\delta})_{+(-\delta)}$ is well-defined. Since $\mathrm{Conv}(P_{+\delta})$ is $\prec$-initial and contains δ, Proposition 1.13 yields

$$\widetilde{(P_{+\delta})_{+(-\delta)}}(\varepsilon) = \widetilde{P_{+\delta}}(\varepsilon - \delta) = \tilde{P}(\varepsilon)$$

for all $\varepsilon \preccurlyeq \delta$. We deduce by Corollary 1.17 that $P = (P_{+\delta})_{+(-\delta)}$. Applying Proposition 1.13, this time to $(P_{+\delta}, -\delta)$, we get $\mathrm{Conv}(P_{+\delta}) \subseteq \mathrm{Conv}(P)$, hence the equality. □

1.7 Analytic functions

Let $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$, $\mathbb{T}$ and $\mathbb{U}$ be fixed fields of well-based series over $\mathbb{R}$ with $\mathfrak{M} \neq 1$ and $\mathbb{S} \subseteq \mathbb{T} \subseteq \mathbb{U}$. We also fix a non-empty open subclass $\mathbf{O}$ of $\mathbb{S}$.

Definition 1.19. *Let $f : \mathbf{O} \longrightarrow \mathbb{T}$ be a function and let $s \in \mathbf{O}$. We say that f is* **analytic at** *s if there is a power series $f_s \in \mathbb{T}[[z]]$ with $\mathrm{Conv}(f_s) \neq \{0\}$ and a $\delta \in \mathrm{Conv}(f_s) \setminus \{0\}$ such that for all $\varepsilon \preccurlyeq \delta$, we have*

$$(s + \varepsilon \in \mathbf{O}) \Longrightarrow f(s + \varepsilon) = \tilde{f}_s(\varepsilon).$$

We say that f_s is a **Taylor series** *of f at s. We say that f is* **analytic** *if it is analytic at each $s \in \mathbf{O}$.*

Lemma 1.20. *Let $f : \mathbf{O} \longrightarrow \mathbb{S}$ be analytic at $s \in \mathbf{O}$. Then f_s is the unique Taylor series of f at s.*

Proof. Let $P \in \mathbb{S}[[z]]$ and $\delta \in \mathrm{Conv}(P) \setminus \{0\}$ with $s + \varepsilon \in \mathbf{O}$ and $f(s + \varepsilon) = \tilde{P}(\varepsilon)$ for all $\varepsilon \preccurlyeq \delta$. Then the function $\widetilde{f_s - P}$ is zero on the class of series $s \preccurlyeq \delta$, so we have $f_s = P$ by Corollary 1.17. □

If $f\colon \mathbf{O} \longrightarrow \mathbb{S}$ is analytic at $s \in \mathbf{O}$ where $\mathbf{O}$ is open, then we can define

$$\mathrm{Conv}(f)_s := \{t \in \mathbf{O} : t - s \in \mathrm{Conv}(f_s) \wedge f(t) = \tilde{f}_s(t-s)\}.$$

Proposition 1.21. *Let $P \in \mathbb{T}[[z]]$ with $\mathrm{Conv}(P) \neq \{0\}$. Then $\tilde{P}$ is analytic on $\mathrm{Conv}(P)$ with $\tilde{P}_\delta = P_{+\delta}$ and $\mathrm{Conv}(\tilde{P})_\delta = \mathrm{Conv}(P)$ for all $\delta \in \mathrm{Conv}(P)$.*

Proof. Let $\delta \in \mathrm{Conv}(P)$. The class $\mathrm{Conv}(P)$ is open by Lemma 1.11, with $\mathrm{Conv}(P_{+\delta}) = \mathrm{Conv}(P)$. By Proposition 1.13, we have $\tilde{P}(\delta + \varepsilon) = \widetilde{P_{+\delta}(\varepsilon)}$ for all $\varepsilon \in \mathrm{Conv}(P)$, so $\tilde{P}$ is indeed analytic on $\mathrm{Conv}(P)$ with $\mathrm{Conv}(\tilde{P})_\delta \supseteq \mathrm{Conv}(P_{+\delta}) = \mathrm{Conv}(P)$. But we also have $\mathrm{Conv}(\tilde{P})_\delta \subseteq \mathrm{Conv}(P_{+\delta}) = \mathrm{Conv}(P)$ by definition, hence the result. □

Corollary 1.22. *Let $f\colon \mathbf{O} \longrightarrow \mathbb{T}$ be analytic at $s \in \mathbf{O}$. Then there is an open neighborhood $\mathbf{O}_s$ of s such that $f \restriction \mathbf{O}_s$ is analytic.*

Proof. Define $\mathbf{O}_s = \{s + \mathbb{S}^{\prec \delta}\}$ where δ is any element of $\mathrm{Conv}(f_s) \setminus \{0\}$. Then Proposition 1.21 yields the result. □

Proposition 1.23. *Let $f\colon \mathbf{O} \longrightarrow \mathbb{S}$ be analytic at $s \in \mathbf{O}$ and let $\mathbf{U} \subseteq \mathrm{Conv}(f)_s$ be a non-empty open subclass containing 0. Then f is analytic on $s + \mathbf{U}$, with $f_{s+\delta} = (f_s)_{+\delta}$ for all $\delta \in \mathbf{U}$.*

Proof. Let $\delta \in \mathbf{U}$ and set $t := s + \delta$. Since $\mathbf{U} \ni 0$ is open and non-empty, we find a $\rho \neq 0$ with $\delta + \varepsilon \in \mathbf{U}$ for all $\varepsilon \preccurlyeq \rho$. Thus $f(t + \varepsilon) = \tilde{f}_s(\delta + \varepsilon)$ whenever $\varepsilon \preccurlyeq \rho$. But given such ε, we have $\tilde{f}_s(\delta + \varepsilon) = \widetilde{(f_s)_{+\delta}}(\varepsilon)$ by Proposition 1.13, whence

$$f(t + \varepsilon) = \tilde{f}_s(\delta + \varepsilon) = \widetilde{(f_s)_{+\delta}}(\varepsilon).$$

So f is analytic at t with $f_t = (f_s)_{+(t-s)}$. □

Proposition 1.24. *Let $f\colon \mathbf{O} \longrightarrow \mathbb{S}$ be analytic at $s \in \mathbf{O}$. Then f is infinitely differentiable at s, and each $f^{(n)}$ for $n \in \mathbb{N}$ is analytic at s with $\mathrm{Conv}(f^{(n)})_s \supseteq \mathrm{Conv}(f)_s$. Moreover, we have*

$$f_s = \sum_{k \in \mathbb{N}} \frac{f^{(k)}(s)}{k!}\, z^k.$$

Proof. Recall that $\tilde{f}_s$ is infinitely differentiable on $\mathrm{Conv}(f_s)$. By Lemma 1.14, each derivative $\tilde{f}_s^{(n)}$ for $n \in \mathbb{N}$ is a power series function on $\mathrm{Conv}(f_s)$, and is thus analytic on $\mathrm{Conv}(f_s)$ by Proposition 1.21. It follows since $\mathrm{Conv}(f)_s$ is a neighborhood of s that f is infinitely differentiable at s. By Lemma 1.14, given $\delta \in \mathrm{Conv}(f)_s$, we have $f^{(n)}(s + \delta) = \tilde{f}_s^{(n)}(\delta) = \widetilde{(f_s)^{(n)}}(\delta)$. Therefore $f^{(n)}$ is analytic at s with $f_s^{(n)} = (f_s)^{(n)}$ and $\mathrm{Conv}(f^{(n)})_s \supseteq \mathrm{Conv}(f)_s$. Write $f_s = \sum_{k \in \mathbb{N}} s_k\, z^k$. We have $f^{(k)}(s) = \widetilde{(f_s)}^{(k)}(0) = \widetilde{(f_s)^{(k)}}(0) = k!\, s_k$. We deduce that $f_s = \sum_{k \in \mathbb{N}} \frac{f^{(k)}(s)}{k!}\, z^k$. □

Corollary 1.25. *Let $\mathbf{O} \subseteq \mathbb{S}$ be open and non-empty and assume that $\mathbf{O} = \bigsqcup_{i \in \mathbf{I}} \mathbf{O}_i$ where each $\mathbf{O}_i$ is open and non-empty. Let $(s_i)_{i \in \mathbf{I}}$ be a family where $s_i \in \mathbf{O}_i$ for all $i \in \mathbf{I}$. Let $(P_i)_{i \in \mathbf{I}}$ be a family of power series in $\mathbb{S}[[z]]$ with $(s_i + \mathrm{Conv}(P_i)) \supseteq \mathbf{O}_i$. The function $f\colon \mathbf{O} \longrightarrow \mathbb{S}$ such that for all $i \in \mathbf{I}$ and $s \in \mathbf{O}_i$, we have $f(s) = P_i(s - s_i)$ is well-defined and analytic.*

Proof. Let $s \in \mathbf{O}$ and let $i \in \mathbf{I}$ with $s \in \mathbf{O}_i$. We have $s - s_i \in \mathbf{O}_i - s_i \subseteq \mathrm{Conv}(P_i)$ so $\tilde{P}_i(s - s_i)$ is defined. In particular f is well-defined. The class $\mathbf{O}_i - s_i$ is a neighborhood of 0, so there is a $\delta \in \mathrm{Conv}(P_i) \setminus \{0\}$ such that $s_i + \varepsilon \in \mathbf{O}_i$ whenever $\varepsilon \preccurlyeq \delta$. Given $\varepsilon \preccurlyeq \delta$, we have

$$f(s + \varepsilon) = P_i(s + \varepsilon - s_i) = (P_i)_{+(s - s_i)}(\varepsilon)$$

by Proposition 1.13. Therefore f is analytic at s with $f_s = (P_i)_{+(s-s_i)}$. □

We leave it to the reader to check that analyticity, at a point or on an open class, is preserved by sums and products. The following result will be used extensively in the thesis to show that a composition of analytic functions is analytic.

Proposition 1.26. *Assume that $\mathfrak{M}$ is densely ordered. Let $\mathbf{U} \subseteq \mathbb{T}$ be open. Let $f : \mathbf{U} \longrightarrow \mathbb{U}$, $g : \mathbf{O} \longrightarrow \mathbf{U}$ and let $s \in \mathbf{O}$ such that g is analytic at s and f is analytic at $g(s)$. Write*

$$f_{g(s)} = \sum_{n\in\mathbb{N}} a_n z^n \qquad \text{and} \qquad g_s = \sum_{n\in\mathbb{N}} b_n z^n.$$

Let $\varepsilon_f \in \mathrm{Conv}(f)_{g(s)}$ and $\varepsilon \in \mathrm{Conv}(g)_s$ with

$$\forall m \in \mathbb{N}^>, b_m \varepsilon^m \prec \varepsilon_f. \tag{1.2}$$

The function $f \circ g$ is analytic at s with $\varepsilon \in \mathrm{Conv}(f \circ g)_s$, and $(f \circ g)_s = f_{g(s)} \circ (g_s - g(s))$.

Proof. For $n \in \mathbb{N}$ and $k \in \mathbb{N}^>$, set $X_{n,k} := \{\mathbb{v} \in (\mathbb{N}^>)^n : |\mathbb{v}| = k\}$.

$$c_{n,k} := \sum_{\mathbb{v}\in X_{n,k}} a_n b_{\mathbb{v}_{[1]}} \cdots b_{\mathbb{v}_{[n]}},$$

so $f_{g(s)} \circ (g_s - g(s)) = f(g(s)) + \sum_{k\in\mathbb{N}^>} \left(\sum_{n\in\mathbb{N}} c_{n,k}\right) z^k$. Note that since $\varepsilon \in \mathrm{Conv}(g)_s \subseteq \mathrm{Conv}(g_s)$, the set

$$\mathfrak{S}_g := \bigcup_{m\in\mathbb{N}} \mathrm{supp}\,(b_m \varepsilon^m)$$

is well-based. We have $\mathfrak{S}_g \prec \varepsilon_f$ by (1.2). Let $\mathfrak{m} \in \mathfrak{M}$ with $\mathfrak{S}_g \prec \mathfrak{m} \preccurlyeq \varepsilon_f$. This exists since $(\mathfrak{M}, \prec)$ is densely ordered. The set $\mathfrak{S}_f := \bigcup_{n\in\mathbb{N}} \mathrm{supp}\,(a_n \mathfrak{m}^n)$ is well-based. For $n \in \mathbb{N}$ and $k \in \mathbb{N}^>$, we have

$$\mathrm{supp}\, c_{n,k} \varepsilon^k \subseteq (\mathfrak{S}_g \cdot \mathfrak{m}^{-1})^n \cdot \mathfrak{S}_f,$$

where $\mathfrak{S}_g \cdot \mathfrak{m}^{-1}$ is well-based and infinitesimal, and $\mathfrak{S}_f$ is well-based. Since each family $(c_{n,k}\varepsilon^k)_{k>0}$ for $n \in \mathbb{N}$ is well-based with sum $(g(s+\varepsilon) - g(s))^n$, we conclude with Corollary 1.6 that $(c_{n,k}\varepsilon^k)_{n\in\mathbb{N},k>0}$ is well-based. We deduce by Lemma 1.2 that

$$\begin{aligned}
f(g(s+\varepsilon)) &= \sum_{n\in\mathbb{N}} a_n (g(s+\varepsilon) - g(s))^n \\
&= \sum_{n\in\mathbb{N}} a_n \left(\sum_{k\in\mathbb{N}^>} b_k \varepsilon^k\right)^n \\
&= f(g(s)) + \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}^>} c_{n,k} \varepsilon^k \\
&= f(g(s)) + \sum_{k\in\mathbb{N}^>} \left(\sum_{n\in\mathbb{N}} c_{n,k}\right) \varepsilon^k \\
&= \widetilde{(f_{g(s)} \circ (g_s - g(s)))}(\varepsilon).
\end{aligned}$$

By Proposition 1.10, we deduce that $f \circ g$ is analytic at s, hence the result. □

Remark 1.27. Note that (1.2) is not optimal since one could expect that the inequality

$$\forall m \in \mathbb{N}^>, b_m \varepsilon^m \preccurlyeq \varepsilon_f,$$

or even

$$g(s+\varepsilon) - g(s) \preccurlyeq \varepsilon_f$$

would suffice. This will not be a problem in the sequel because in the case of hyperseries, there will always be a $\delta_f \succ \varepsilon_f$ with $\delta_f \in \mathrm{Conv}(f)_{g(s)}$.

Remark 1.28. Another well-known type of analytic functions is that of restricted real-analytic functions of [14, 16]. If I is a non-empty interval of $\mathbb{R}$ and $f: I \longrightarrow \mathbb{R}$ is an analytic function, then f extends into a function $\overline{f}: I + \mathbb{S}^{\prec} \longrightarrow \mathbb{R} + \mathbb{S}^{\prec}$ given by

$$\forall r \in I, \forall \varepsilon \prec 1, \overline{f}(r+\varepsilon) := \sum_{k \in \mathbb{N}} \frac{f^{(k)}(r)}{k!} \varepsilon^k.$$

We say that $\overline{f}$ is a restricted real-analytic function on $\mathbb{S}$. The function $\overline{f}$ is in fact analytic.

1.8 Flatness

Let $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series. For $s \in \mathbb{S}^{>}$, we write $s^+ := \max(s, s^{-1})$ and $s^- := (s^+)^{-1}$. So $s^+ = s^{-1}$ if $s < 1$ and $s^+ = s$ otherwise. As in [35, 28], it is useful to consider the following orderings on $\mathbb{S}^{>}$:

Definition 1.29. *Let $s, t \in \mathbb{S}^{>}$. We say that s is* **flatter** *than t and we write*

$$s \prec\!\!\!\prec t \quad \text{if} \quad (s^+)^n < t^+ \quad \text{for all } n \in \mathbb{N}^{>}, \text{ and}$$

$$s \succcurlyeq t \quad \text{if there are } m, n \in \mathbb{N}^{>} \text{ with} \quad t^+ < (s^+)^m < (t^+)^n.$$

We also write $s \preccurlyeq t$ if $s \prec\!\!\!\prec t$ or $s \succcurlyeq t$. We write

$$s \ll t \quad \text{if} \quad s^n < t \quad \text{for all } n \in \mathbb{N}^{>}, \text{ and}$$

$$s \equiv t \quad \text{if there are } m, n \in \mathbb{N}^{>} \text{ with} \quad t < s^m < t^n.$$

We also write $s \lessapprox t$ if $s \ll t$ or $s \equiv t$.

The relations $\prec\!\!\!\prec$ and $\ll$ are partial orderings on $\mathbb{S}^{>}$. We sometimes extend them to $\mathbb{S}^{\neq}$ by writing $s \prec\!\!\!\prec t$ whenever $|s| \prec\!\!\!\prec |t|$, and $s \ll t$ whenever $|s| \ll |t|$. Note that $s \prec\!\!\!\prec t$ if and only if $v\mathfrak{d}_s > v\mathfrak{d}_t$ where v is the natural (or standard, or Archimedean) valuation on the ordered group $\mathfrak{M}$. See [4, p 83–84], for more details.

Lemma 1.30. *Let $L: (\mathbb{S}^{>}, \times) \longrightarrow (\mathbb{S}, +)$ be a strictly increasing morphism. Then for all $s, t \in \mathbb{S}^{\neq}$, we have*

$$\begin{aligned}
s \prec\!\!\!\prec t &\iff L(s) \prec L(t),\\
s \ll t &\iff \mathbb{R}^{>} L(s) < L(t),\\
s \preccurlyeq t &\iff L(s) \preccurlyeq L(t),\\
s \lessapprox t &\iff \exists r \in \mathbb{R}^{>}, r\, L(s) < L(t),\\
s \succcurlyeq t &\iff L(s) \succcurlyeq L(t), \text{ and}\\
s \equiv t &\iff \exists r \in \mathbb{R}^{>}, r\, L(s) \sim L(t).
\end{aligned}$$

Proof. This follows from the relation $L(s^n) = n\, s$ for all $s \in \mathbb{S}^{>}$ and $n \in \mathbb{N}$ and the fact that L is strictly increasing. □

We will frequently use the following consequences of Lemma 1.30, sometimes without mention:

Corollary 1.31. *Assume that there is a strictly increasing morphism $L: (\mathbb{S}^{>}, \times) \longrightarrow (\mathbb{S}, +)$. Then for $s, t, u \in \mathbb{S}^{>}$, we have*

a) $s\, t \preccurlyeq \max(s^+, t^+)$.

b) $s \asymp t \Longrightarrow s \asymp t$.

c) $s \prec\!\!\prec t \Longrightarrow s\, t \asymp t$.

Proof. The assertions *a*), *c*) follow from the classical valuation theoretic properties of $\preccurlyeq$. The assertion *b*) is an immediate consequence of Lemma 1.30. □

1.9 Strong linearity and operator supports

Let $\mathbb{U} = \mathbb{R}[[\mathfrak{U}]]$ and $\mathbb{V} = \mathbb{R}[[\mathfrak{V}]]$ be fields of well-based series. Consider a function $\Phi : \mathbb{S} \longrightarrow \mathbb{U}$ which is $\mathbb{R}$-linear. Then Φ is *strongly linear* if for every well-based family $(s_i)_{i\in I}$ in $\mathbb{S}$, the family $(\Phi(s_i))_{i\in I}$ in $\mathbb{U}$ is well-based, with

$$\Phi\left(\sum_{i\in I} s_i\right) = \sum_{i\in I} \Phi(s_i).$$

By [26, Proposition 3.5], the function Φ is strongly linear if and only if for each $s \in \mathbb{S}$, the family $(\Phi(\mathfrak{m}))_{\mathfrak{m}\in \operatorname{supp} s}$ is well-based with $\Phi(s) = \sum_{\mathfrak{m}\in\operatorname{supp} s} s_{\mathfrak{m}}\, \Phi(\mathfrak{m})$.

A very convenient way to prove that certain family related to certain operators are well-based is to rely on the notion of operator support of [15, p 10] and relative operator support of [7, Definition 2.4]. We recall the definitions here, and then propose a generalization.

Let $\mathbb{S} = \mathbb{R}[[\mathfrak{M}]]$ and $\mathbb{T} = \mathbb{R}[[\mathfrak{N}]]$ be fields of well-based series. A very convenient way to prove that certain family related to certain operators are well-based is to rely on the notion of operator support of [15, p 10] and relative operator support of [7, Definition 2.4]. We recall the definitions here, and then propose a generalization.

Definition 1.32. *Let $\Phi : \mathfrak{M} \longrightarrow \mathbb{T}$ be a function. If $\mathfrak{M} \subseteq \mathfrak{N}$, then the* **support** $\operatorname{supp}\Phi$ *of Φ is the class*

$$\operatorname{supp}\Phi := \bigcup_{\mathfrak{m}\in\mathfrak{M}} \frac{\operatorname{supp}\Phi(\mathfrak{m})}{\mathfrak{m}}.$$

The **relative support** $\operatorname{supp}_{\odot}\Phi$ *of Φ is the class*

$$\operatorname{supp}_{\odot}\Phi := \bigcup_{\mathfrak{m}\in\mathfrak{M}} \frac{\operatorname{supp}\Phi(\mathfrak{m})}{\mathfrak{d}_{\Phi(\mathfrak{m})}}.$$

If $\Psi : \mathbb{S} \longrightarrow \mathbb{T}$ is a linear function, then we define its support and relative support as

$$\begin{aligned} \operatorname{supp}\Psi &:= \operatorname{supp}(\Psi \restriction \mathfrak{M}) \quad \textit{and} \\ \operatorname{supp}_{\odot}\Psi &:= \operatorname{supp}_{\odot}(\Psi \restriction \mathfrak{M}) \quad \textit{respectively.} \end{aligned}$$

We next include two useful results regarding supports and relative supports.

Proposition 1.33. [15, Lemma 2.9] *Let $\Phi : \mathfrak{M} \longrightarrow \mathbb{T}$ have well-based support. Then Φ is well-based.*

Proposition 1.34. [7, Proposition 2.5] *Let $\Phi : \mathfrak{M} \longrightarrow \mathbb{T}$ be relatively well-based. Assume that $0 \notin \Phi(\mathfrak{M})$ and that $\mathfrak{d} \circ \Phi : \mathfrak{M} \longrightarrow \mathfrak{N}$ is strictly increasing. Then Φ is well-based and its strongly linear extension $\hat{\Phi}$ is injective.*

Let $s \in \mathbb{T}^{>}$. If $s \not\asymp 1$, then we write $\mathfrak{M}^{\prec\!\!\prec s}$ for the subclass of $\mathfrak{M}$ of monomials $\mathfrak{m}$ with $\mathfrak{m} \prec\!\!\prec s$. If $s \asymp 1$, then we set $\mathfrak{M}^{\prec\!\!\prec s} := \{1\}$. We write $\mathfrak{M}^{\prec\!\!\prec s, \succ} := \mathfrak{M}^{\prec\!\!\prec s} \cap \mathfrak{M}^{\succ}$. We will only consider this class in contexts where there exists a strictly increasing morphism log: $(\mathbb{T}^{>}, \times) \longrightarrow (\mathbb{T}, +)$. In that case Corollary 1.31(a) applies, so the class $\mathfrak{M}^{\prec\!\!\prec s}$ (resp. $\mathfrak{M}^{\prec\!\!\prec s, \succ}$) is a subgroup (resp. submonoid) of $\mathfrak{M}$.

Definition 1.35. *Let $\Phi\colon \mathfrak{M} \longrightarrow \mathbb{T}$ be a function and let $\mathfrak{W}$ be a subclass of $\mathfrak{N}$. If $\mathfrak{M} \subseteq \mathfrak{N}$, then we say that $\mathfrak{W}$ is a* **near-support** *for Φ if for all $\mathfrak{m} \in \mathfrak{M}$, we have*

$$\operatorname{supp} \Phi(\mathfrak{m}) \subseteq \mathfrak{m} \cdot \mathfrak{N}^{\prec \mathfrak{m}} \cdot \mathfrak{W}.$$

We say that $\mathfrak{W}$ is a **positive near-support** *for Φ if for all $\mathfrak{m} \in \mathfrak{M}$, we have*

$$\operatorname{supp} \Phi(\mathfrak{m}) \subseteq \mathfrak{m} \cdot \mathfrak{N}^{\prec \mathfrak{m}, \succcurlyeq} \cdot \mathfrak{W}.$$

We say that $\mathfrak{W}$ is a **relative near-support** *for Φ if for all $\mathfrak{m} \in \mathfrak{M}$, we have*

$$\operatorname{supp} \Phi(\mathfrak{m}) \subseteq \mathfrak{d}_{\Phi(\mathfrak{m})} \cdot \mathfrak{N}^{\prec \Phi(\mathfrak{m})} \cdot \mathfrak{W}.$$

If $\Psi\colon \mathbb{S} \longrightarrow \mathbb{T}$ is a function, we say that $\mathfrak{S}$ is a near-support (resp. relative near-support) for Ψ if it is a near-support (resp. relative near-support) for $\Psi \restriction \mathfrak{M}$.

Note that the support (resp. relative support) of Φ is a near-support (resp. relative near-support) for Φ. However since $\operatorname{supp} \Phi$ and $\operatorname{supp}_{\odot} \Phi$ may not be well-based, it is sometimes useful to consider other near-supports (resp. relative near-supports) for Φ. As a general rule, we will rely on near-supports when working with derivations, and relative near-supports when working with composition laws.

Lemma 1.36. *Assume that $\mathfrak{M} \subseteq \mathfrak{N}$. Let $\Phi\colon \mathfrak{M} \longrightarrow \mathbb{T}$ be a function, and let $\mathfrak{W}$ be a well-based near support for Φ. If ρ is an ordinal and $(\mathfrak{m}_\gamma)_{\gamma<\rho}$ is a strictly $\ll$-decreasing sequence in $\mathfrak{M}$, then the family $(\Phi(\mathfrak{m}_\gamma))_{\gamma<\rho}$ is well-based.*

Proof. Assume for contradiction that this family is not well-based. So we may assume that there is a nondecreasing sequence of monomials $(\mathfrak{n}_i)_{i\in\mathbb{N}}$ and a strictly increasing sequence of ordinals $(\gamma_i)_{i\in\mathbb{N}}$ with $\mathfrak{n}_i \in \operatorname{supp} \Phi(\mathfrak{m}_{\gamma_i})$ for all $i \in \mathbb{N}$. For $i \in \mathbb{N}$, we fix a $\mathfrak{p}_i \in \mathfrak{N}^{\prec \mathfrak{m}_{\gamma_i}}$ and a $\mathfrak{w}_i \in \mathfrak{W}$ with $\mathfrak{n}_i = \mathfrak{m}_{\gamma_i} \mathfrak{p}_i \mathfrak{w}_i$. Since $\mathfrak{W}$ is well-based, we may assume that $(\mathfrak{w}_i)_{i\in\mathbb{N}}$ is nonincreasing, whence that $(\mathfrak{m}_{\gamma_i}\mathfrak{p}_i)_{i\in\mathbb{N}}$ is nondecreasing. Now for $i \in \mathbb{N}$, we have $\mathfrak{m}_{\gamma_i}\mathfrak{p}_i \equiv \mathfrak{m}_{\gamma_i}$ by Corollary 1.31(c). But $(\mathfrak{m}_{\gamma_i})_{i\in\mathbb{N}}$ is strictly $\ll$-decreasing, so $(\mathfrak{m}_{\gamma_i}\mathfrak{p}_i)_{i\in\mathbb{N}}$ is strictly $\ll$-decreasing, hence strictly $\prec$-decreasing: a contradiction. □

Lemma 1.37. *Let $\Phi\colon \mathfrak{M} \longrightarrow \mathbb{T}^{\neq}$ be a strictly $\ll$-increasing function and let $\mathfrak{W}$ be a well-based near support for Φ. If λ is an ordinal and $(\mathfrak{m}_\gamma)_{\gamma<\rho}$ is a strictly $\ll$-decreasing sequence in $\mathfrak{M}$, then the family $(\Phi(\mathfrak{m}_\gamma))_{\gamma<\rho}$ is well-based.*

Proof. Assume for contradiction that this family is not well-based. So we may assume that there is a nondecreasing sequence of monomials $(\mathfrak{n}_i)_{i\in\mathbb{N}}$ and a strictly increasing sequence of ordinals $(\gamma_i)_{i\in\mathbb{N}}$ with $\mathfrak{n}_i \in \operatorname{supp} \Phi(\mathfrak{m}_{\gamma_i})$ for all $i \in \mathbb{N}$. We write $\mathfrak{u}_i := \mathfrak{d}_{\Phi(\mathfrak{m}_{\gamma_i})}$ for all $i \in \mathbb{N}$. For $i \in \mathbb{N}$, we fix a $\mathfrak{p}_i \in \mathfrak{N}^{\prec \mathfrak{m}_{\gamma_i}}$ and a $\mathfrak{w}_i \in \mathfrak{W}$ with $\mathfrak{n}_i = \mathfrak{u}_i \mathfrak{p}_i \mathfrak{w}_i$. Since $\mathfrak{W}$ is well-based, we may assume that $(\mathfrak{u}_i\mathfrak{p}_i)_{i\in\mathbb{N}}$ is nondecreasing. For $i \in \mathbb{N}$, we have $\mathfrak{u}_i\mathfrak{p}_i \equiv \mathfrak{u}_i$ Corollary 1.31(c), But $(\mathfrak{u}_i)_{i\in\mathbb{N}}$ is strictly $\ll$-decreasing, so $(\mathfrak{u}_i\mathfrak{p}_i)_{i\in\mathbb{N}}$ is strictly $\ll$-decreasing, hence strictly $\prec$-decreasing: a contradiction. □

1.10 Van der Hoeven's theorem and applications

One of the main technical difficulties of the this paper will be to prove that certain families related to hyperseries are well-based. In a number of cases, the arguments in the previous sections will suffice, but we will often require more powerful tools pertaining to the notion of Noetherian ordering, which we next introduce. Most of the results can be found in [25, Appendix A], [35, Chapter 1], [24] and [26]. See [25, Appendix A] for a detailed discussion of their strength and history.

Definition 1.38. *Let $(\mathbf{X},<)$ be a partially ordered class. A* **chain** *in $\mathbf{X}$ is a linearly ordered subclass of $\mathbf{X}$. A* **decreasing chain** *in $\mathbf{X}$ is chain $\mathbf{Y}\subseteq\mathbf{X}$ without minimal element, i.e. with*

$$\forall y\in\mathbf{Y},\exists z\in\mathbf{Y},(z<y).$$

An **antichain** *in $\mathbf{X}$ is a subclass $\mathbf{Y}\subseteq\mathbf{X}$, no two distinct elements of which are comparable, i.e. with*

$$\forall y,z\in\mathbf{Y},y\leqslant z\Longrightarrow y=z.$$

We say that $(\mathbf{X},<)$ is **Noetherian** *if there are no infinite decreasing chains and no infinite antichains in $(\mathbf{X},<)$.*

Noetherianity is a strengthening of well-foundedness, and a weakening of well-orderedness, the latter being equivalent to the conjunction of linearity and Noetherianity. In order to derive results on Noetherian classes, it is convenient to rely on the notion of bad sequence and minimal bad sequence of [32]. If $(\mathbf{X},<_{\mathbf{X}})$ is an ordered class, then a *bad sequence* in $\mathbf{X}$ is a sequence $u\colon\mathbb{N}\longrightarrow\mathbf{X}$ such that there are *no* $i,j\in\mathbb{N}$ with $i<j$ and $u_i\leqslant_{\mathbf{X}}u_j$. Given a function $f\colon\mathbf{X}\longrightarrow\mathbb{N}$, a bad sequence u in $\mathbf{X}$ is said *minimal* for f if for all $i\in\mathbb{N}$, there are no bad sequences v in $\mathbf{X}$ with $(v_0,\dots,v_{i-1})=(u_0,\dots,u_{i-1})$ and $f(v_i)<f(u_i)$.

Lemma 1.39. [35, Theorem 3.5.1] *Let $(X,<)$ be a partially ordered set and let $f\colon X\longrightarrow\mathbb{N}$ be a function. If there is a bad sequence in X, then there is bad sequence in X which is minimal for f.*

Lemma 1.40. [24, Theorem 2.1] *Let $(\mathbf{X},<_{\mathbf{X}})$ be a partially ordered class. The following statements are equivalent*

a) $(\mathbf{X},<_{\mathbf{X}})$ *is Noetherian.*

b) There is no bad sequence in $(\mathbf{X},<_{\mathbf{X}})$.

c) Every sequence in $\mathbf{X}$ *has an increasing subsequence.*

Lemma 1.41. [35, Criterion 1.5.4] *Let I be a* set *and let $S=(s_i)_{i\in I}\in\mathbb{S}^I$ be a family. Consider the set*

$$N_S:=\{(i,\mathfrak{m})\in I\times\mathfrak{M}:\mathfrak{m}\in\operatorname{supp}s_i\},$$

ordered by $(i,\mathfrak{m})\prec_S(j,\mathfrak{n})\Longleftrightarrow\mathfrak{m}\prec\mathfrak{n}$. Then S is well-based if and only if $(N_S,\prec_S)$ is Noetherian.

Proof. Assume that S is well-based. Consider a non-empty chain C for $(N_S,\prec_S)$. Given $(i,\mathfrak{m})\in C$, e have $\mathfrak{m}\in\bigcup_{i\in I}s_i$, and $(i,\mathfrak{m})$ is $\prec_S$ minimal in C if and only if $\mathfrak{m}$ is $\prec$-maximal in $\bigcup_{i\in I}s_i$. Consider an antichain A in $(N_S,\prec_S)$. Since $(\mathfrak{M},\prec)$ is linearly ordered, we must have $A\subseteq I\times\{\mathfrak{m}\}$ for a certain $\mathfrak{m}\in\mathfrak{M}$. But then $A\subseteq(I\times\{\mathfrak{m}\})\cap N_S=I_{\mathfrak{m}}\times\{\mathfrak{m}\}$ where $I_{\mathfrak{m}}=\{i\in I:\mathfrak{m}\in\operatorname{supp}s_i\}$.

In view of the definition of well-based families, we see that $(N_S,\prec_S)$ is Noetherian if and only if S is well-based. □

Let $(X,<)$ be a partially ordered set. We write

$$X^\star:=\bigcup_{n\in\mathbb{N}}X^n=\{(x_1,\dots,x_n):n\in\mathbb{N}\wedge x_1,\dots,x_n\in X\}$$

for the set of so-called *finite words* on X, including the empty word $\varnothing\in X^0$. For any non-empty word $w=(w_0,\dots,w_k)\in X^\star\setminus\{\varnothing\}$, we write $w_\bullet:=w_k\in X$ for the last "letter" of w, and we write $|w|:=k+1$ for its length.

We next state a weaker and simplified version of van der Hoeven's theorem [25, Appendix A.4] on so-called Noetherian choice operators. Let $(\mathbf{X}, <)$ be a partially ordered class. A function ϑ sending each $x \in \mathbf{X}$ to a sub*set* $\vartheta(x)$ of $\mathbf{X}$ is called a *choice operator* on $\mathbf{X}$.

The choice operator ϑ is said *Noetherian* if for all Noetherian sub*sets* $Y \subseteq \mathbf{X}$, the set

$$Y_\vartheta := \{x : \exists y, (y \in Y \wedge x \in \vartheta(y))\} \subseteq \mathbf{X}$$

is Noetherian. It is said *strictly extensive* if for all $x \in \mathbf{X}$, we have

$$x < \vartheta(x).$$

Let $Y \subseteq \mathbf{X}$ be a subset. Let $\vartheta^+(Y)$ denote the set of non-empty finite words $(x_0, \dots, x_k) \in \mathbf{X}^\star \setminus \{\varnothing\}$ where for each $i < k$, we have $x_{i+1} \in \vartheta(x_i)$. We endow $\vartheta^+(Y)$ with the ordering $<_\vartheta$ defined by

$$w <_\vartheta w' \iff w_\bullet < w'_\bullet.$$

Proposition 1.42. (van der Hoeven's theorem) [25, Theorem A.4] *Let $(\mathbf{X}, <)$ be a partially ordered class and let ϑ be a Noetherian and strictly extensive choice operator on $\mathbf{X}$. Then for all Noetherian sub*sets *Y of $\mathbf{X}$, the set $\vartheta^+(Y)$ is Noetherian for $<_\vartheta$.*

Proof. This version follows from an application of [25, Theorem A.4] to a simple case. Nonetheless, let us adapt van der Hoeven's proof to the present simplified setting.

Assume for contradiction that $\vartheta^+(Y)$ is not Noetherian. So there is a minimal bad sequence sequence $(w_i)_{i\in\mathbb{N}} \in \vartheta^+(Y)^{\mathbb{N}}$ for the length function $w \mapsto |w|$. Assume that there is an infinite set $I \subseteq \mathbb{N}$ with $|w_i| \leqslant 2$ for all $i \in I$. Then $\mathcal{Y} := \{x_{i,0} : i \in I\} \subseteq Y$ is Noetherian. Since ϑ is a Noetherian choice operator, the set

$$\mathcal{Y}_\vartheta := \{y : \exists i \in I, y \in \vartheta(x_{i,0})\}$$

is Noetherian. But then $\{w_i : i \in I\}$ is Noetherian for $<_\vartheta$: a contradiction.

So there is a $k \in \mathbb{N}$ with $|w_j| > 2$ for all $j \geqslant k$. For $j \geqslant k$, we write $z_j := (x_{j,0}, \dots, x_{j,|w_j|-1}) \in \vartheta^+(Y)$. We claim that the set $\mathcal{Z} := \{z_j : j \geqslant k\}$, is Noetherian for $<_\vartheta$. Indeed, assume for contradiction that $(z_{j_i})_{i\in\mathbb{N}}$ is a bad sequence in $\mathcal{Z}$ with $j_0 \leqslant j_1 \leqslant \cdots$. We show that

$$z := (w_0, \dots, w_{j_0-1}, z_{j_0}, z_{j_1}, \dots)$$

is a bad sequence, contradicting the minimality of $(w_i)_{i\in\mathbb{N}}$. Indeed assume for contradiction that z isn't bad. Since $(z_{j_i})_{i\in\mathbb{N}}$ is bad, there must exist $i < j_0$ and $p \in \mathbb{N}$ with $w_i \leqslant_\vartheta z_{j_p}$. Since ϑ is strictly extensive, we have

$$(w_{j_p})_\bullet \in \vartheta((z_{j_p})_\bullet) > (z_{j_p})_\bullet,$$

so $w_i <_\vartheta w_k$: a contradiction. Therefore $\mathcal{Z}$ is Noetherian. It follows since ϑ is Noetherian that $\{w_i : i \geqslant k\}$ is Noetherian: a contradiction. □

Corollary 1.43. *Assume that $\mathbf{X}$ is linearly ordered. Let ϑ be a strictly extensive and Noetherian choice operator on $\mathbf{X}$, and let $Y \subseteq \mathbf{X}$ be a well-ordered subset. Define Y_∞ to be the union of sets $Y_n, n \in \mathbb{N}$, where*

$$\begin{aligned} Y_0 &:= Y \quad \textit{and} \\ Y_{n+1} &:= Y_n \cup \bigcup_{y\in Y_n} \vartheta(y) \quad \textit{for all } n \in \mathbb{N}. \end{aligned}$$

Then Y_∞ is well-based.

Proof. By definition, we have $Y_n \subseteq \{w_\bullet : w \in \vartheta^+(Y)\}$ for all $n \in \mathbb{N}$, so $Y_\infty \subseteq \{w_\bullet : w \in \vartheta^+(Y)\}$. Recall that $\mathbf{X}$ is linearly ordered. By van der Hoeven's theorem, the set $\{w_\bullet : w \in \vartheta^+(Y)\}$ is Noetherian, hence well-ordered. So Y_∞ is well-ordered. □

Lemma 1.44. *Let $\mathbb{S}=\mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series and let $\Phi:\mathbb{S}\longrightarrow\mathbb{S}$ be strongly linear with $\Phi(\mathfrak{m})\prec\mathfrak{m}$ for all $\mathfrak{m}\in\mathfrak{M}$. Define a choice operator ϑ_Φ on the reverse order $(\mathfrak{M},\succ)$ by*

$$\forall\mathfrak{m}\in\mathfrak{M},\vartheta_\Phi(\mathfrak{m}):=\operatorname{supp}\Phi(\mathfrak{m}).$$

Then ϑ_Φ is strictly extensive and Noetherian.

Proof. The relation $\forall\mathfrak{m}\in\mathfrak{M},\Phi(\mathfrak{m})\prec\mathfrak{m}$ implies that ϑ_Φ is strictly extensive. Given a Noetherian subset $\mathfrak{Y}\subseteq\mathfrak{M}$, i.e. a well-based subset in $(\mathfrak{M},\prec)$, the set

$$\mathfrak{Y}_{\vartheta_\Phi}=\{\mathfrak{m}:\exists y,(\mathfrak{n}\in\mathfrak{Y}\wedge\mathfrak{m}\in\operatorname{supp}\Phi(\mathfrak{n}))\}\subseteq\bigcup_{\mathfrak{n}\in\mathfrak{Y}}\operatorname{supp}\Phi(\mathfrak{n})$$

is well-based, hence Noetherian in $(\mathfrak{M},\succ)$. So ϑ_Φ is Noetherian. □

Notation 1.45. *Given a function $\Psi:\mathbf{X}\multimap\mathbf{X}$ on a class $\mathbf{X}$ and a $k\in\mathbb{N}$, we will frequently write $\Psi^{[k]}$ for the k-fold iterate of Ψ. So $\Psi^{[k]}$ is the function $\mathbf{X}\longrightarrow\mathbf{X}$ with $\Psi^{[0]}=\Psi$ and $\Psi^{[k+1]}:=\Psi^{[k]}\circ\Psi=\Psi\circ\Psi^{[k]}$ for all $k\in\mathbb{N}$. We will sometimes extend this notation to fractional and real iterates (see Section 9.3) of functions.*

Proposition 1.46. (corollary of [26, Theorem 6.2]) *Let $\mathbb{S}=\mathbb{R}[[\mathfrak{M}]]$ be a field of well-based series and let $\Phi:\mathbb{S}\longrightarrow\mathbb{S}$ be strongly linear with $\Phi(\mathfrak{m})\prec\mathfrak{m}$ for all $\mathfrak{m}\in\mathfrak{M}$. Let $(r_k)_{k\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$. Then for all $s\in\mathbb{S}$, the family $(r_k\,\Phi^{[k]}(s))_{k\in\mathbb{N}}$ is well-based, and the function*

$$\sum_{k\in\mathbb{N}}r_k\,\Phi^{[k]}:\mathbb{S}\longrightarrow\mathbb{S};s\mapsto\sum_{k\in\mathbb{N}}r_k\,\Phi^{[k]}(s)$$

is strongly linear.

Proof. We may assume that $r_k=1$ for all $k\in\mathbb{N}$. Consider the Noetherian, strictly extensive choice operator ϑ_Φ of Lemma 1.44 Let $s\in\mathbb{S}$, write $\mathfrak{S}:=\operatorname{supp}s$, and write F for the family $F=(w_\bullet)_{w\in\vartheta_\Phi^+(\mathfrak{S})}\in\mathfrak{M}^{\vartheta_\Phi^+(\mathfrak{S})}$. By van der Hoeven's theorem, the ordered set $(\vartheta_\Phi^+(\mathfrak{S}),<_{\vartheta_\Phi})$ is Noetherian. But $(\vartheta_\Phi^+(\mathfrak{S}),<_{\vartheta_\Phi})=(N_F,\prec_F)$, so F is well-based by Lemma 1.41. For $\mathfrak{m}\in\mathfrak{S}$ and $k\in\mathbb{N}$, we have

$$\operatorname{supp}\Phi^{[k]}(\mathfrak{m})\subseteq\bigcup_{w\in\vartheta_\Phi^+(\mathfrak{S})}w_\bullet.$$

We deduce that $(\Phi^{[k]}(\mathfrak{m}))_{\mathfrak{m}\in\mathfrak{S}\wedge k\in\mathbb{N}}$ is a subfamily of a well-based family, hence it is well-based as well. □

Corollary 1.47. [26, Corollary 1.4] *Let $\mathbb{S}$ and Φ be as in Proposition 1.46. The function*

$$\mathrm{Id}_{\mathbb{S}}+\Phi:\mathbb{S}\longrightarrow\mathbb{S};s\mapsto s+\Phi(s)$$

is bijective, with functional inverse $\sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}$.

Proof. Let $s\in\mathbb{S}$, and write $t:=\sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}(s)$. We have

$$\begin{aligned}(\mathrm{id}_{\mathbb{S}}+\Phi)(t)&=\sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}(s)+\Phi\left(\sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}(s)\right)\\&=\sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}(s)-\sum_{k\in\mathbb{N}}(-1)^{k+1}\,\Phi^{[k+1]}(s)\quad\text{(by strong linearity of }\Phi)\\&=\sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}(s)-\sum_{k\in\mathbb{N}^>}(-1)^k\,\Phi^{[k]}(s)\\&=s.\end{aligned}$$

Conversely

$$\begin{aligned}\left(\sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}\right)(s+\Phi(s)) &= \sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}(s+\Phi(s)) \qquad \text{(by strong linearity of } \Phi)\\ &= \sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}(s)+(-1)^k\,\Phi^{[k+1]}(s)\\ &= \sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k]}(s)+\sum_{k\in\mathbb{N}}(-1)^k\,\Phi^{[k+1]}(s)\\ &= s\end{aligned}$$

as above. □

This last results, and generalizations thereof can be used to define integrals in differential fields of well-based series (see [3]) strongly linear fixed points operators (see [26, Theorem 6.3]), and solve various functional equations on fields of transseries or hyperseries (see [26, Example 6.7] or Section 9.2).

2 Hyperserial fields

The eponymous *hyperexponentially closed fields* are particular cases of *hyperserial fields*. Those in turn are fields of well-based series $\mathbb{T}$ equipped with an action

$$\circ_{\mathbb{T}}\colon \mathbb{L}\times\mathbb{T}^{\succ,\succ}\longrightarrow\mathbb{T}$$

of the field of $\mathbb{L}$ *logarighmic hyperseries* on $\mathbb{T}$. In this section, we define those notions, starting with logarithmic hyperseries.

2.1 Logarithmic hyperseries

The field $\mathbb{L}$ of *logarithmic hyperseries* of [15] is a field of well-based series $\mathbb{R}[[\mathfrak{L}]]$ whose group of monomials $\mathfrak{L}$ is obtained using formal transfinite products of hyperlogarithms. It equipped with its natural derivation $\partial\colon\mathbb{L}\longrightarrow\mathbb{L}$ and composition law $\circ\colon\mathbb{L}\times\mathbb{L}^{\succ,\succ}\longrightarrow\mathbb{L}$. Here, we recall the definition of $\mathbb{L}$ and some of its properties.

Logarithmic hyperseries Let $\alpha\in\mathbf{On}$. Let $\mathfrak{L}_{<\alpha}$ denote the group of functions $\alpha\longrightarrow\mathbb{R}$ ordered lexicographically. In other words $\mathfrak{L}_{<\alpha}$ is the Hahn product group $\prod_{\gamma<\alpha}(\mathbb{R},+)$. Monomials $\mathfrak{l}\in\mathfrak{L}_{<\alpha}$ are written as formal products $\mathfrak{l}=\Pi_{\gamma<\alpha}\,\ell_\gamma^{\mathfrak{l}_\gamma}$ where for $\gamma<\alpha$, the real term $\mathfrak{l}_\gamma\in\mathbb{R}$ is the value of $\mathfrak{l}\colon\alpha\longrightarrow\mathbb{R}$ at γ. Thus ℓ_γ denotes the monomial such that for $\iota<\alpha$, we have

$$(\ell_\gamma)_\iota=1 \quad\text{if } \iota=\gamma \quad\text{and}\quad (\ell_\gamma)_\iota=0 \quad\text{otherwise.}$$

$\mathbb{L}_{<\alpha}$ is defined as the ordered field of well-based series $\mathbb{L}_{<\alpha}:=\mathbb{R}[[\mathfrak{L}_{<\alpha}]]$. If α,β are ordinals with $\beta<\alpha$, then we let $\mathfrak{L}_{[\beta,\alpha)}$ denote the subgroup of $\mathfrak{L}_{<\alpha}$ of monomials $\mathfrak{l}$ with $\mathfrak{l}_\gamma=0$ whenever $\gamma<\beta$. As in [15], we write

$$\begin{aligned}\mathbb{L}_{[\beta,\alpha)} &:= \mathbb{R}[[\mathfrak{L}_{[\beta,\alpha)}]]\\ \mathfrak{L} &:= \bigcup_{\alpha\in\mathbf{On}}\mathfrak{L}_{<\alpha}\\ \mathbb{L} &:= \mathbb{R}[[\mathfrak{L}]]\end{aligned}$$

We have natural inclusions $\mathfrak{L}_{[\beta,\alpha)}\subseteq\mathfrak{L}_{<\alpha}\subset\mathfrak{L}$, hence natural inclusions $\mathbb{L}_{[\beta,\alpha)}\subseteq\mathbb{L}_{<\alpha}\subset\mathbb{L}$.

Derivation on $\mathbb{L}_{<\alpha}$ The field $\mathbb{L}_{<\alpha}$ is equipped with a derivation $\partial\colon \mathbb{L}_{<\alpha} \longrightarrow \mathbb{L}_{<\alpha}$ which satisfies the Leibniz rule and which is strongly linear. Write $\ell_\gamma^\dagger := \prod_{\iota \leqslant \gamma} \ell_\iota^{-1} \in \mathfrak{L}_{<\alpha}$ for all $\gamma < \alpha$. The derivative of a logarithmic hypermonomial $\mathfrak{l} \in \mathfrak{L}_{<\alpha}$ is defined by

$$\partial(\mathfrak{l}) := \left(\sum_{\gamma<\alpha} \mathfrak{l}_\gamma \ell_\gamma^\dagger \right) \mathfrak{l}. \tag{2.1}$$

So $\partial(\ell_\gamma) = \frac{1}{\prod_{\iota<\gamma} \ell_\iota}$ for all $\gamma < \alpha$. In view of (2.1), the derivation ∂ has well-based support

$$\operatorname{supp} \partial = \{\ell_\gamma^\dagger : \gamma \in \mathbf{On}\}.$$

For $f \in \mathbb{L}_{<\alpha}$ and $k \in \mathbb{N}$, we sometimes write $f^{(k)} := \partial^{\circ k}(f)$.

Composition on $\mathbb{L}_{<\alpha}$ Assume that $\alpha = \omega^\nu$ for a certain ordinal ν. Then the field $\mathbb{L}_{<\alpha}$ is equipped with a function $\circ\colon \mathbb{L}_{<\alpha} \times \mathbb{L}_{<\alpha}^{>,\succ} \longrightarrow \mathbb{L}_{<\alpha}$ where in particular, for all $\mu \in \mathbf{On}$ with $\mu + 1 < \nu$, we have $\ell_{\omega^{\mu+1}} \circ \ell_{\omega^\mu} = \ell_{\omega^{\mu+1}} - 1$ [15, Lemma 5.6]. For $\gamma < \alpha$, the map $\circ_{\ell_\gamma}$: $\mathbb{L}_{<\alpha} \longrightarrow \mathbb{L}_{<\alpha}$ defined for $f \in \mathbb{L}_{<\alpha}$ by $\circ_{\ell_\gamma}(f) := f \circ \ell_\gamma$ is onto $\mathbb{L}_{[\gamma,\alpha)}$ [15, Lemma 5.11]. Given $g \in \mathbb{L}_{[\gamma,\alpha)}$, we write $g^{\uparrow\gamma}$ for the unique series in $\mathbb{L}_{<\alpha}$ with $g^{\uparrow\gamma} \circ \ell_\gamma = g$.

2.2 Hyperserial fields

Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ and set $\boldsymbol{\lambda} := \omega^\nu$. Informally, a hyperserial field of force $\boldsymbol{\nu}$ is the action of $\mathbb{L}_{<\omega^\nu}$ on a field of well-based series by monotonous and analytic functions.

More precisely, let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be an ordered field of well-based series and let $\circ\colon \mathbb{L}_{<\omega^\nu} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}$ be a function. For $r \in \mathbb{R}$ and $\mathfrak{m} \in \mathfrak{M}$, we define $\mathfrak{m}^r$ as follows: set

$$\begin{aligned} 1^r &:= 1, \\ \mathfrak{m}^r &:= \ell_0^r \circ \mathfrak{m} \quad \text{if } \mathfrak{m} \succ 1, \text{ and} \\ \mathfrak{m}^r &:= \ell_0^{-r} \circ \mathfrak{m}^{-1} \quad \text{if } \mathfrak{m} \prec 1. \end{aligned}$$

For $\boldsymbol{\mu} \leqslant \mathbf{On}$, we define $\mathfrak{M}_{\omega^\mu}$ to be the class of series $s \in \mathbb{T}^{>,\succ}$ with $\ell_\gamma \circ s \in \mathfrak{M}^\succ$ for all $\gamma < \omega^\mu$. The elements of $\mathfrak{M}_{\omega^\mu}$ are said $L_{<\omega^\mu}$*-atomic*, and $L_{<\omega}$-atomic series are said *log-atomic*. Finally the elements of $\mathfrak{M}_{\boldsymbol{\lambda}}$ are said *atomic*.

We say that $(\mathbb{T}, \circ)$ is a *hyperserial field of force* $\boldsymbol{\nu}$ if the following axioms are satisfied:

HF1. $\mathbb{L}_{<\omega^\nu} \longrightarrow \mathbb{T}; f \mapsto f \circ s$ is a strongly $\mathbb{R}$-linear ordered field embedding for all $s \in \mathbb{T}^{>,\succ}$.

HF2. $f \circ (g \circ s) = (f \circ g) \circ s$ for all $f \in \mathbb{L}_{<\omega^\nu}$, $g \in \mathbb{L}_{<\omega^\nu}^{>,\succ}$, and $s \in \mathbb{T}^{>,\succ}$.

HF3. $f \circ (t + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ t}{k!} \delta^k$ for all $f \in \mathbb{L}_{<\omega^\nu}$, $t \in \mathbb{T}^{>,\succ}$, and $\delta \in \mathbb{T}$ with $\delta \prec t$.

HF4. $\ell_{\omega^\mu}^{\uparrow\gamma} \circ s < \ell_{\omega^\mu}^{\uparrow\gamma} \circ t$ for all ordinals $\mu < \boldsymbol{\nu}$, all $\gamma < \omega^\mu$, and all $s, t \in \mathbb{T}^{>,\succ}$ with $s < t$.

HF5. The map $\mathbb{R} \times \mathfrak{M} \to \mathfrak{M}; (r, \mathfrak{m}) \mapsto \mathfrak{m}^r$ is a law of ordered $\mathbb{R}$-vector field on $\mathfrak{M}$.

HF6. $\ell_1 \circ (s\,t) = \ell_1 \circ s + \ell_1 \circ t$ for all $s, t \in \mathbb{T}^{>,\succ}$.

HF7. $\operatorname{supp} \ell_1 \circ \mathfrak{m} \succ 1$ for all $\mathfrak{m} \in \mathfrak{M}^\succ$ and $\operatorname{supp} \ell_{\omega^\mu} \circ \mathfrak{a} \succ (\ell_\gamma \circ \mathfrak{a})^{-1}$ for all $1 \leqslant \mu < \boldsymbol{\nu}$, all $\gamma < \omega^\mu$, and all $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$.

The axioms **HF6** and **HF7** are assumed to hold trivially when $\boldsymbol{\nu} = 0$. In most cases we will assume that $\boldsymbol{\nu} > 0$. A consequence of the axioms is that ℓ_0 acts as the identity function:

Lemma 2.1. *Let $\mathbb{T}$ be a hyperserial field of force $\boldsymbol{\nu}$. For all $s \in \mathbb{T}^{>,\succ}$, we have $\ell_0 \circ s = s$.*

Proof. Let $\mathfrak{m} \in \mathfrak{M}^{\succ}$ and $r \in \mathbb{R}^{>}$. We have $\ell_0 \circ \mathfrak{m} = \mathfrak{m}^1$ and $(\mathfrak{m}^1)^1 = \mathfrak{m}^{1\times 1} = \mathfrak{m}^1$ by **HF5**. The function $\mathfrak{M} \longrightarrow \mathfrak{M}; \mathfrak{n} \mapsto \mathfrak{n}^1$ is strictly increasing by **HF5**, hence injective. Thus $\mathfrak{m}^1 = \mathfrak{m}$. We obtain $(r\,\ell_0) \circ \mathfrak{m} = r\,\mathfrak{m}$ by **HF1**. In $\mathbb{L}$, we have $\ell_0 \circ (r\,\ell_0) = r\,\ell_0$, so **HF2** yields $\ell_0 \circ (r\,\mathfrak{m}) = r\,\mathfrak{m}$.

Now let $s \in \mathbb{T}^{>,\succ}$ and write $s = r\,\mathfrak{d}_s + \delta$ where $r \in \mathbb{R}^{>}$ and $\delta \prec \mathfrak{d}_s$. By **HF3**, we have

$$\ell_0 \circ s = \sum_{k\in\mathbb{N}} \frac{\ell_0^{(k)} \circ (r\,\mathfrak{d}_s)}{k!}\,\delta^k = r\,\mathfrak{d}_s + \delta = s. \qquad \square$$

Definition 2.2. *Let $(\mathbb{T}, \circ_{\mathbb{T}})$ and $(\mathbb{U}, \circ_{\mathbb{U}})$ be hyperserial fields of force $\boldsymbol{\nu}$. We say that a strongly linear morphism of ordered rings $\Phi\colon \mathbb{T} \longrightarrow \mathbb{U}$ is a* **hyperserial embedding** *of force $\boldsymbol{\nu}$ if we have*

$$\begin{aligned} \Phi(\mathfrak{M}) &\subseteq \mathfrak{N}, \quad \textit{and} \\ \forall f \in \mathbb{L}, \forall s \in \mathbb{T}^{>,\succ}, \Phi(f \circ_{\mathbb{T}} s) &= f \circ_{\mathbb{U}} \Phi(s). \end{aligned}$$

We say that $(\mathbb{T}, \circ_{\mathbb{T}})$ is a **hyperserial subfield** *of $(\mathbb{U}, \circ_{\mathbb{U}})$ of force $\boldsymbol{\nu}$ and we write $(\mathbb{T}, \circ_{\mathbb{T}}) \subseteq (\mathbb{U}, \circ_{\mathbb{U}})$ if $\mathbb{T} \subseteq \mathbb{U}$ and $\mathrm{id}_{\mathbb{T}}\colon \mathbb{T} \longrightarrow \mathbb{U}$ is a hyperserial embedding of force $\boldsymbol{\nu}$.*

The hyperserial field $(\mathbb{T}, \circ)$ is said *confluent* if $\mathfrak{M} \neq 1$ and if for all $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$ and all $s \in \mathbb{T}^{>,\succ}$, there are an $\mathfrak{a} \in \mathfrak{M}_{\omega^\mu}$ and a $\gamma < \omega^\mu$ with

$$\ell_\gamma \circ s \asymp \ell_\gamma \circ \mathfrak{a}. \tag{2.2}$$

In the sequel, we will mostly work with confluent hyperserial fields.

Example 2.3. Consider the internal composition law $\circ\colon \mathbb{L}_{<\omega^{\boldsymbol{\nu}}} \times \mathbb{L}_{<\omega^{\boldsymbol{\nu}}}^{>,\succ} \longrightarrow \mathbb{L}_{<\omega^{\boldsymbol{\nu}}}$ of Section 2.1. Then [7, Theorems 3.16 and 1.1] the structure $(\mathbb{L}_{<\omega^{\boldsymbol{\nu}}}, \circ)$ is a confluent hyperserial field of force $\boldsymbol{\nu}$.

2.3 Hyperlogarithm functions

Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a confluent hyperserial field of force $\boldsymbol{\nu} > 0$. Given $\gamma < \boldsymbol{\lambda}$, we write L_γ for the function $\mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}; s \mapsto \ell_\gamma \circ s$, called the *hyperlogarithm function* of strength γ.

For each $s \in \mathbb{T}^{>,\succ}$ and $\mu \in \mathbf{On}$ with $\mu \leqslant \boldsymbol{\nu}$, the $L_{<\omega^\mu}$-atomic element $\mathfrak{a}$ in (2.2) is unique, and we write $\mathfrak{d}_{\omega^\mu}(s) := \mathfrak{a}$. In this section, we show how the value of $\ell_{\omega^\mu} \circ s$ is determined by $\ell_{\omega^\mu} \circ \mathfrak{d}_{\omega^\mu}(s)$.

Let $s \in \mathbb{T}^{>}$ and write $s = r_s\,\mathfrak{d}_s\,(1 + \varepsilon_s)$ where $r_s \in \mathbb{R}^{>}$ and $\varepsilon_s := (s - r_s\,\mathfrak{d}_s)\,(r_s\,\mathfrak{d}_s)^{-1}$ is infinitesimal. Let $\iota \in \{-1, 1\}$ with $\mathfrak{d}_s^\iota \succcurlyeq 1$. We set $\ell_1 \circ 1 := 0$. Then by [7, Proposition 4.3], the function $\log\colon \mathbb{T}^{>} \longrightarrow \mathbb{T}$ defined by

$$\log s = \iota\,\ell_1 \circ \mathfrak{d}_s^\iota + \log r_s + \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k+1}\,\varepsilon_s^{k+1},$$

is a strictly increasing morphism $(\mathbb{T}^{>}, \times) \longrightarrow (\mathbb{T}, +)$ which extends L_1. We call it the *logarithm* on $\mathbb{T}$. For any decomposition $s = t\,r\,u$ where $t > 0$, $r \in \mathbb{R}^{>}$ and $u \sim 1$, we have

$$\log s = \log t + \overline{\log}(r\,u) = \log t + \log r + \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k+1}\,(u-1)^{k+1}, \tag{2.3}$$

where $\overline{\log}$ is the restricted analytic logarithm of Remark 1.28.

Proposition 2.4. *The function* $\log\colon \mathbb{T}^{>} \longrightarrow \mathbb{T}$ *is analytic with*

$$\mathrm{Conv}(\log)_s = \mathbb{T}^{\prec s} = \{\delta \in \mathbb{T} : \delta \prec s\}$$

and

$$\log^{(k)}(s) = (-1)^{k-1}\,(k-1)!\,s^{-k}$$

for all $s > 0$.

Proof. Let $s \in \mathbb{T}^{>}$. For $k \in \mathbb{N}$, set $a_{k,s} := (-1)^{k-1}\,(k-1)!\,s^{-k}$. For $\delta \prec s$ and $k > 0$, we have

$$\frac{a_{k,s}\,\delta^k}{k!} = \frac{(-1)^{k-1}}{k}\left(\frac{\delta}{s}\right)^k.$$

Since ${}^{\delta}\!/_{s} \prec 1$, the family $(a_{k,s}\,\delta^k)_{k\in\mathbb{N}^>}$ is well-based with $\sum_{k\in\mathbb{N}^>} \frac{a_{k,s}\,\delta^k}{k!} = \tilde{L}\Big(\frac{\delta}{s}\Big)$. We have

$$\log(s+\delta) = \log\left(s\left(1+\frac{\delta}{s}\right)\right) = \log s + \tilde{L}\left(\frac{\delta}{s}\right).$$

That is, the function log is analytic at s with $\log_s = \log(s) + L \circ (s^{-1}\,z)$ and $\mathrm{Conv}(\log)_s \supseteq \{\delta \in \mathbb{T} : \delta \prec s\}$. Note that $1 \notin \mathrm{Conv}(L)$ so $s \notin \mathrm{Conv}(\log_s)$. It follows since $\mathrm{Conv}(\log)_s$ is $\prec$-initial that $\mathrm{Conv}(\log)_s = \{\delta \in \mathbb{T} : \delta \prec s\}$. By Proposition 1.24, for each $k \in \mathbb{N}$, the series $\frac{\log^{(k)}(s)}{k!}$ is the $(k+1)$-th coefficient $\frac{a_{k,s}}{k!}$ of $\log_s$. So $\log^{(k)}(s) = a_{k,s} = (-1)^{k-1}\,(k-1)!\,s^{-k}$. □

Proposition 2.5. *Let* $(\mathbb{U}, \log)$ *be a hyperserial field of force* 1 *and let* $\Psi\colon \mathbb{T} \longrightarrow \mathbb{U}$ *be a strongly linear and strictly increasing morphism of rings with* $\Psi(\log \mathfrak{m}) = \log \Psi(\mathfrak{m})$ *for all* $\mathfrak{m} \in \mathfrak{M}$. *Then we have*

$$\Psi(\log s) = \log \Psi(s)$$

for all $s \in \mathbb{T}^{>}$.

Proof. Let $s \in \mathbb{T}^{>}$ and write $s = r\,\mathfrak{d}_s\,(1+\varepsilon)$ where $r \in \mathbb{R}^{>}$ and $\varepsilon \prec 1$. Since Ψ is $\mathbb{R}$-linear and strictly increasing, we have $\Psi(r\,(1+\varepsilon)) \preccurlyeq 1$. Since Ψ is strongly linear and preserves products, we have $\Psi(\overline{\log}(r\,(1+\varepsilon))) = \overline{\log}\,\Psi(r\,(1+\varepsilon))$, whence

$$\Psi(\log(r\,(1+\varepsilon))) = \Psi(\overline{\log}(r\,(1+\varepsilon))) = \overline{\log}\,\Psi(r\,(1+\varepsilon)) = \log \Psi(r\,(1+\varepsilon)).$$

It follows that

$$\Psi(\log s) = \Psi(\log \mathfrak{d}_s) + \Psi(\log(r\,(1+\varepsilon))) = \log \Psi(\mathfrak{d}_s) + \log \Psi(r\,(1+\varepsilon)) = \log \Psi(s). \qquad \square$$

Now let $s \in \mathbb{T}^{>,\succ}$ and $\eta < \boldsymbol{\nu}$. There is an ordinal $\gamma < \omega^\eta$ such that $\delta := L_\gamma(s) - L_\gamma(\mathfrak{d}_{\omega^\eta}(s)) \prec L_\gamma(s)$. As a consequence of **HF2** and **HF3**, for any such γ, we have

$$L_{\omega^\eta}(s) = L_{\omega^\eta}(\mathfrak{d}_{\omega^\eta}(s)) + \sum_{k>0} \frac{(\ell_\omega^{\uparrow\gamma}) \circ L_\gamma(\mathfrak{d}_{\omega^\eta}(s))}{k!}\,\delta^k. \tag{2.4}$$

We will often pay close attention the partial functions $L_{\omega^\eta} \restriction \mathfrak{M}_{\omega^\eta}\colon \mathfrak{M}_{\omega^\eta} \longrightarrow \mathbb{T}^{>,\succ}$ for all $\eta < \boldsymbol{\nu}$. The family $(\mathbb{T}, (L_{\omega^\eta} \restriction \mathfrak{M}_{\omega^\eta})_{\eta<\nu})$ is called the *skeleton* of $(\mathbb{T}, \circ)$, and its properties sometimes reflect those of the whole structure $(\mathbb{T}, \circ)$. See for instance [7, Theorem 1.1, Corollary 7.24] and Propositions 5.3 and 6.4.

In view of **HF3**, Propositions 1.23 and 1.24 together imply the following:

Lemma 2.6. *Each $f \in \mathbb{L}_{<\boldsymbol{\lambda}}$ induces an analytic function*

$$\begin{array}{rcll} \mathcal{A}_f\colon \mathbb{T}^{>,\succ} & \longrightarrow & \mathbb{T} & \\ s & \longmapsto & f\circ s, & \text{with} \\ \mathrm{Conv}(\mathcal{A}_f)_s & \supseteq & \mathbb{T}^{\prec s} & \text{and} \\ \mathcal{A}_f^{(k)} & = & \mathcal{A}_{f^{(k)}} & \end{array}$$

for all $s \in \mathbb{T}^{>,\succ}$ and $k \in \mathbb{N}$.

2.4 Hyperexponentiation

For the end of Section 2, we fix a $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. Given $\gamma < \boldsymbol{\lambda}$, the function $L_\gamma\colon \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}$ is strictly increasing, so it has a partially defined left inverse E_γ, defined by $E_\gamma(L_\gamma(s)) = s$ for all $s \in \mathbb{T}^{>,\succ}$. The partial function E_γ is called the *hyperexponential function* of strength γ. The hyperexponential function of force 1, i.e. the partial inverse of log, is denoted exp and called the *exponential function*.

We say that $\mathbb{T}$ is a confluent hyperserial field of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ if each E_γ for $\gamma < \omega^{\boldsymbol{\mu}}$ (or equivalently each E_{ω^η} for $\eta < \boldsymbol{\mu}$) is totally defined on $\mathbb{T}^{>,\succ}$. Note that the relation $\ell_{\omega^{\iota+1}} \circ \ell_{\omega^\iota} = \ell_{\omega^{\iota+1}} - 1$ for all ι with $\iota + 1 < \boldsymbol{\mu}$ yields the functional equations

$$\begin{aligned} \forall s \in \mathbb{T}^{>,\succ}, L_{\omega^{\iota+1}}(L_{\omega^\iota}(s)) &= L_{\omega^{\iota+1}}(s) - 1 \quad \text{and} \\ \forall s \in \mathbb{T}^{>,\succ}, E_{\omega^{\iota+1}}(s+1) &= E_{\omega^\iota}(E_{\omega^{\iota+1}}(s)). \end{aligned} \tag{2.5}$$

In the sequel of this subsection, we assume that $\mathbb{T}$ is a confluent hyperserial field of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$. We will briefly describe how exp and each E_{ω^η} for $\eta < \boldsymbol{\mu}$ act on $\mathbb{T}^{>,\succ}$.

Let $\varphi \in \mathbb{T}$. For all $\varepsilon \prec 1$ and $r \in \mathbb{R}$, we have

$$\exp(\varphi + r + \varepsilon) = \exp(r)\exp(\varphi)\left(\sum_{k\in\mathbb{N}} \frac{1}{k!}\varepsilon^k\right), \tag{2.6}$$

where $\exp(r) \in \mathbb{R}^>$ is the standard exponential of $r \in \mathbb{R}$ as a real number, and thus $\exp(r)\left(\sum_{k\in\mathbb{N}} \frac{1}{k!}\varepsilon^k\right)$ is the value at $r+\varepsilon$ of the restricted analytic function $\overline{\exp}$. Moreover, the axioms **HF7** and **HF1** imply that we have

$$\exp(\mathbb{T}_\succ) = \mathfrak{M}. \tag{2.7}$$

Proposition 2.7. *Let $\mathbb{U}$ be a confluent hyperserial field of force $(1,1)$. Then $(\mathbb{U}, +, \times, \exp, <)$ is an elementary extension of $(\mathbb{R}, +, \times, \exp, <)$.*

Proof. By applying (2.6) for $\varphi = 0$, we see that exp extends the real exponential function. We have $\ell_1 \prec \ell_0$ in $\mathbb{L}_{<\boldsymbol{\lambda}}$ which implies by Lemma 2.1, **HF1** and [7, Proposition 4.4] that $\log s \prec s$ for all $s \in \mathbb{U}^{>,\succ}$ and that $\log(s) \leqslant s - 1$ for all $s \in \mathbb{U}^>$. We claim that $\exp(s) > s^n$ for all $n \in \mathbb{N}$ and $s > n^2$. Indeed let $s \in \mathbb{U}$ and $n \in \mathbb{N}$ with $s > n^2$. First assume that $s \preccurlyeq 1$. So $s = r + s_\prec$ for a certain $r \in \mathbb{R}^{\geqslant 0}$ and a $s_\prec \in \mathbb{U}^\prec$. We have $r \geqslant n^2$ so $\exp(r) > r^n$ so $\exp(s) \sim \exp(r) > r^n \sim s^n$ so $\exp(s) > s^n$. Assume now that $s \succ 1$. We have $\exp\left(\frac{1}{n+1}\, s\right) \succ s$ so $\exp(s) > s^n$. This proves that exp satisfies Ressayre's axioms of [34]. By [16, Corollary 4.6], we deduce that $(\mathbb{U}, +, \times, \exp, <)$ can be expanded into a structure $\mathbb{U}_{\mathrm{an,exp}}$ with so-called restricted analytic functions in such a way that $\mathbb{U}_{\mathrm{an,exp}}$ is an elementary extension the structure $\mathbb{R}_{\mathrm{an,exp}}$ of real numbers with restricted analytic functions and the exponential. In particular $(\mathbb{U}, +, \times, \exp, <)$ is a model of $(\mathbb{R}, +, \times, \exp, <)$, hence an elementary extension by Wilkie's theorem [36]. □

Now let $\eta < \boldsymbol{\mu}$ with $\eta > 0$, write $\beta := \omega^\eta$, let $\varphi \in \mathbb{T}^{>,\succ}$ such that $E_\beta(\varphi)$ is defined, and let $\gamma < \beta$. For $k \in \mathbb{N}$, we define series $t_{\gamma,k} \in \mathbb{L}_{<\gamma}$ inductively by

$$t_{\gamma,0} := \ell_\gamma, \quad \text{and} \tag{2.8}$$

$$t_{\gamma,k+1} := \left(\prod_{\rho<\beta} \ell_\rho\right) t'_{\gamma,k}. \tag{2.9}$$

We write E'_β for the function $\mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}^{>,\succ}; s \mapsto t_{0,1} \circ E_\beta(s)$, that is

$$\forall s \in \mathbb{T}^{>,\succ}, E'_\beta(s) = \exp\left(\sum_{\rho<\beta} L_{\rho+1}(s)\right).$$

Then [7, Lemma 7.8] for all $\varepsilon \in \mathbb{T}$ with $\varepsilon \prec \frac{E_\beta(\varphi)}{E'_\beta(\varphi)}$, the family $((t_{\gamma,k} \circ E_\beta(\varphi))\,\varepsilon^k)_{k\in\mathbb{N}}$ is well-based, with

$$L_\gamma(E_\beta(\varphi)) = \sum_{k\in\mathbb{N}} \frac{t_{\gamma,k} \circ E_\beta(\varphi)}{k!}\,\varepsilon^k. \tag{2.10}$$

Lemma 2.8. *For $k \in \mathbb{N}^>$ and $\mathfrak{n} \in \operatorname{supp} t_{\gamma,k}$, there are a $n \in \mathbb{N}$, a $P \in \mathbb{N}[X_1, \dots, X_n]$ and $\gamma_1, \dots, \gamma_n \in (\gamma, \beta)$ with $\mathfrak{n} = \ell_{[\gamma,\beta)} \cdot P(\ell_{[\gamma_1,\beta)}, \dots, \ell_{[\gamma_n,\beta)})$.*

Proof. We prove this by induction on $k \in \mathbb{N}^>$. We obtain the result for $k = 1$ by setting $P = 1$. Assume the result holds for k and let $\mathfrak{n} \in \operatorname{supp} t_{\gamma,k+1}$. So there is $\mathfrak{n}_0 \in \operatorname{supp} t_{\gamma,k}$ and $\mathfrak{m} \in \operatorname{supp} \mathfrak{n}'_0$ with $\mathfrak{n} = \ell_{[0,\beta)}\,\mathfrak{m}$. The inductive assumption yields $\mathfrak{n}_0 = \ell_{[\gamma,\beta)} \cdot P(\ell_{[\gamma_1,\beta)}, \dots, \ell_{[\gamma_n,\beta)})$ for certain $n \in \mathbb{N}$, $P \in \mathbb{N}[X_1, \dots, X_k]$ and $\gamma_1, \dots, \gamma_k \in (\gamma, \beta)$. Recall that for all $\rho < \beta$, we have

$$\ell'_{[\rho,\beta)}\,\ell_{[0,\beta)} = \sum_{\rho\leqslant\sigma<\beta} \ell^\dagger_\sigma\,\ell_{[0,\beta)}\,\ell_{[\gamma,\beta)} = \left(\sum_{\rho\leqslant\sigma<\beta} \ell_{[\sigma+1,\beta)}\right)\ell_{[\rho,\beta)}. \tag{2.11}$$

$$\begin{aligned}
\ell_{[0,\beta)} \cdot \mathfrak{n}'_0 &= \ell_{[0,\beta)}\,\ell'_{[\gamma,\beta)} \cdot P(\ell_{[\gamma_1,\beta)}, \dots, \ell_{[\gamma_n,\beta)}) + \ell_{[\gamma,\beta)}\,\ell_{[0,\beta)} \cdot (P(\ell_{[\gamma_1,\beta)}, \dots, \ell_{[\gamma_n,\beta)}))' \\
&= \ell_{[\gamma,\beta)} \cdot \left(\sum_{\gamma\leqslant\sigma<\beta} \ell_{[\sigma+1,\beta)}\right) \cdot P(\ell_{[\gamma_1,\beta)}, \dots, \ell_{[\gamma_n,\beta)}) + \ell_{[\gamma,\beta)}\,\ell_{[0,\beta)} \cdot (P(\ell_{[\gamma_1,\beta)}, \dots, \\
&\quad \ell_{[\gamma_n,\beta)}))'.
\end{aligned}$$

By (2.11), the support of $\ell_{[0,\beta)} \cdot (P(\ell_{[\gamma_1,\beta)}, \dots, \ell_{[\gamma_n,\beta)}))'$ consists of polynomial combinations of terms $\ell_{[\gamma+1,\beta)}$ for $\gamma \in (\gamma_i, \beta)$ for certain $i \in \{1, \dots, n\}$. The result follows. □

Following [7, Definition 7.10 and Lemma 7.14], we define:

Definition 2.9. *A series $\varphi \in \mathbb{T}^{>,\succ}$ is said* **1-truncated** *if we have $\operatorname{supp} \varphi \succ 1$, i.e. if φ is positive and purely large. For $0 < \eta < \boldsymbol{\mu}$, a series $\varphi \in \mathbb{T}^{>,\succ}$ is said β-***truncated** *if we have*

$$\forall \gamma < \beta, \operatorname{supp} \varphi \succ (L_\gamma(E_\beta(\varphi)))^{-1}.$$

We write $\mathbb{T}_{\succ,\beta}$ for the class of β-truncated series in $\mathbb{T}$. We sometimes write $E_\beta(\varphi) =: E^\varphi_\beta$ when $\varphi \in \mathbb{T}_{\succ,\beta}$. We have $\mathbb{T}_{\succ,\beta} + \mathbb{R}^{\geqslant} \subseteq \mathbb{T}_{\succ,\beta}$ [7, Lemma 7.13]. Thus the axiom **HF7** for η states that $L_\beta(\mathfrak{M}_\beta) \subseteq \mathbb{T}_{\succ,\beta}$. In fact [7, Corollaries 7.21 and 7.24], we have the converse inclusion

$$E_\beta^{\mathbb{T}_{\succ,\beta}} = \mathfrak{M}_\beta, \quad \text{with} \tag{2.12}$$

$$\forall s \in \mathbb{T}^{>,\succ}, L_\beta(\mathfrak{d}_\beta(s)) = \sharp_\beta(L_\beta(s)). \tag{2.13}$$

By [7, Proposition 7.18], for each $s \in \mathbb{T}^{\succ,\succ}$, there is a $\preccurlyeq$-maximal truncation $\sharp_\beta(s)$ of s which is β-truncated, and there is a $\gamma < \beta$ with

$$s - \sharp_\beta(s) \prec \frac{E_\beta(\sharp_\beta(s))}{t_{\gamma,1} \circ E_\beta(\sharp_\beta(s))}. \tag{2.14}$$

So (2.10) applies to $\varphi := \sharp_\beta(s)$ and $\varepsilon := s - \varphi$.

Lemma 2.10. *Let $\eta < \boldsymbol{\nu}$ and $\gamma < \boldsymbol{\lambda}$ with $\gamma\,\omega < \beta$. For all $\mathfrak{a} \in \mathfrak{U}_{\omega^\eta}$, we have $\mathfrak{d}_\beta(L_\gamma(\mathfrak{a})) = \mathfrak{a}$.*

Proof. Set $\rho := \gamma\omega$. We have $\ell_\rho \circ \ell_\gamma \asymp \ell_\rho$ so $L_\rho(L_\gamma(\mathfrak{a})) \asymp L_\rho(\mathfrak{a})$. Since $\rho < \beta$ and $\mathfrak{a}$ is $L_{<\beta}$-atomic, we deduce that $\mathfrak{d}_\beta(L_\gamma(\mathfrak{a})) = \mathfrak{a}$. □

2.5 Hyperexponential closure

Definition 2.11. *Let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu} \leqslant \mathbf{On}$. A confluent hyperserial field* **of force** $(\boldsymbol{\nu}, \boldsymbol{\mu})$ *is a confluent hyperserial field $(\mathbb{T}, \circ)$ of force $\boldsymbol{\nu}$ such that each function $L_{\omega^\mu}\colon \mathbb{T}^{\succ,\succ} \longrightarrow \mathbb{T}^{\succ,\succ}$ is surjective. If $\mathbb{T}$ has force $(\boldsymbol{\nu}, \boldsymbol{\nu})$, then we say that $\mathbb{T}$ is* **hyperexponentially closed**.

As in the case of transseries [25, 35], a confluent hyperserial field can be embedded into a hyperexponentially closed one. More precisely:

Definition 2.12. *Let $\mathbb{T}$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. A* **hyperexponential closure** *of $\mathbb{T}$* **of force** $\boldsymbol{\mu}$ *is a confluent extension $\mathbb{T}_{(<\boldsymbol{\mu})}$ of $\mathbb{T}$ of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ with the following initial property: if $\mathbb{U}$ is another confluent hyperserial skeleton of force $(\boldsymbol{\nu}, \boldsymbol{\mu})$ and if $\Phi\colon \mathbb{T} \longrightarrow \mathbb{U}$ is an embedding of force $\boldsymbol{\nu}$, then there is a unique embedding $\Psi\colon \mathbb{T}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{U}$ of force $\boldsymbol{\nu}$ that extends Φ.*

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\subseteq} & \mathbb{T}_{(<\boldsymbol{\mu})} \\ & {\scriptstyle\Phi}\searrow & \downarrow \exists!\,\Lambda \\ & & \mathbb{U} \end{array}$$

A **hyperexponential closure** *of $\mathbb{T}$ is a hyperexponential closure of $\mathbb{T}$ of force $\boldsymbol{\nu}$.*

Note that a hyperexponential closure of force $\boldsymbol{\mu}$ if it exists is unique up to unique isomorphism. We will write $\mathbb{T}_{(<\boldsymbol{\mu})}$ for the hyperexponential closure of $\mathbb{T}$ of force $\boldsymbol{\mu}$ if it exists, and we set $\tilde{\mathbb{T}} := \mathbb{T}_{(<\boldsymbol{\nu})}$.

Theorem 2.13. [7, Theorem 7.4] *Let $\mathbb{T}$ be a confluent hyperserial skeleton of force $\boldsymbol{\nu} \leqslant \mathbf{On}$ and let $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. Then $\mathbb{T}$ has a hyperexponential closure of force $\boldsymbol{\mu}$.*

In particular, we have the hyperexponential closure $\tilde{\mathbb{L}}$ of $\mathbb{L}$, which we call the class of *finitely nested hyperseries*.

3 Transserial subfields and subgroups

3.1 Transserial subgroups and subfields

It will frequently be convenient to define derivations and compositions on subgroups $\mathbb{R}[[\mathfrak{S}]]$ of a given hyperserial field $\mathbb{U} = \mathbb{R}[[\mathfrak{U}]]$, for $\mathfrak{S} \subseteq \mathfrak{U}$, before we extend them to transserial fields containing $\mathbb{R}[[\mathfrak{S}]]$. This is the purpose of *transserial subgroups* which we next introduce.

Definition 3.1. *Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ with $\boldsymbol{\nu} > 0$. Let $\mathbb{U} = \mathbb{R}[[\mathfrak{U}]]$ be a confluent hyperserial field of force $\boldsymbol{\nu}$. Let $\mathfrak{S} \subseteq \mathfrak{U}$ be a subset with $\log(\mathfrak{S}) \subseteq \mathbb{R}[[\mathfrak{S}]]$. Then we say that $\mathbb{R}[[\mathfrak{S}]]$ is a* **transserial subgroup** *of $\mathbb{T}$. If moreover $\mathfrak{S}$ is a subgroup of $\mathfrak{U}$, then we say that $\mathbb{R}[[\mathfrak{S}]]$ is a* **transserial subfield** *of $\mathbb{T}$.*

Since $(\mathbb{U}, \log)$ is a transserial field, transserial subfields are simply transserial fields contained in $(\mathbb{U}, \log)$. The following notion of exponential extension is also similar to that of [35, Section 2.3.1].

Lemma 3.2. *Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ with $\boldsymbol{\nu} > 0$. Let $\mathbb{U} = \mathbb{R}[[\mathfrak{U}]]$ be a confluent hyperserial field of force $(\boldsymbol{\nu}, 1)$. And let $\mathbb{G} = \mathbb{R}[[\mathfrak{S}]]$ be a transserial subgroup. Then the class*

$$\mathbb{G}^{\exp} := \mathbb{R}\big[\big[E_1^{\mathbb{R}[[\mathfrak{S}^{\succ}]]}\big]\big]$$

is a transserial subfield of $\mathbb{U}$ with $\mathbb{G} \subseteq \mathbb{G}^{\exp}$.

Proof. The class $\mathbb{R}[[\mathfrak{S}^{\succ}]]$ is a subgroup of $\mathbb{U}_{\succ}$ so $E_1^{\mathbb{R}[[\mathfrak{S}^{\succ}]]}$ is a subgroup of $\mathfrak{U}$. Let $\mathfrak{m} \in \mathfrak{S}$. We have $\operatorname{supp} \log \mathfrak{m} \subseteq \mathfrak{S}^{\succ}$ so $\log \mathfrak{m} \in \mathbb{G} \cap \mathbb{U}_{\succ} = \mathbb{R}[[\mathfrak{S}^{\succ}]]$, so $\mathfrak{m} = E_1^{\log \mathfrak{m}} \in E_1^{\mathbb{R}[[\mathfrak{S}^{\succ}]]}$. We deduce that $\mathbb{G} \subseteq \mathbb{G}^{\exp}$, i.e. that $\log E_1^{\mathbb{R}[[\mathfrak{S}^{\succ}]]} \subseteq \mathbb{G}^{\exp}$. Thus $\mathbb{G}^{\exp}$ is a transserial subfield of $\mathbb{U}$. □

As in [35, Section 2.3.4], we may define an increasing tower $(\mathbb{G}_{(\gamma)})_{\gamma \in \mathbf{On}}$ of extensions of $\mathbb{G}$ which, except possibly for $\mathbb{G}_{(0)} = \mathbb{G}$, are transserial subfields of $\mathbb{U}$, and where

$$\mathbb{G}_{(<1)} := \bigcup_{\gamma \in \mathbf{On}} \mathbb{G}_{(\gamma)}$$

is a transserial subfield of $\mathbb{U}$ with a total exponential. Indeed, set $\mathbb{G}_{(\gamma)} := \mathbb{R}[[\mathfrak{S}_{(\gamma)}]]$ where

- $\mathfrak{S}_{(0)} := \mathfrak{S}$,
- $\mathfrak{S}_{(\gamma+1)} := E_1^{\mathbb{R}[[\mathfrak{S}_{(\gamma)}^{\succ}]]}$ for all $\gamma \in \mathbf{On}$, and
- $\mathfrak{S}_{(\gamma)} := \bigcup_{\rho<\gamma} \mathfrak{S}_{(\rho)}$ if γ is a non-zero limit.

We define the *exponential height* $\mathrm{EH}_{\mathbb{G}}(f)$ of $f \in \mathbb{G}_{(<1)}$ over $\mathbb{G}$ as the least ordinal γ with $f \in \mathbb{G}_{(\gamma)}$.

3.2 Extending transserial derivations

Let $\mathbb{U} = \mathbb{R}[[\mathfrak{U}]]$ be a confluent hyperserial field of force $\boldsymbol{\nu}$ and let $\mathbb{G} \subseteq \mathbb{U}$ be a transserial subgroup. A *transserial derivation* $\mathbb{G} \longrightarrow \mathbb{V}$ is a strongly linear function $\partial\colon \mathbb{G} \longrightarrow \mathbb{V}$ with

$$\begin{aligned} \forall s, g \in \mathbb{G}, s\,g \in \mathbb{G} \;&\Longrightarrow\; \partial(s\,g) = \partial(s)\,g + s\,\partial(g), \quad \text{and} \\ \forall \mathfrak{m} \in \mathbb{G} \cap \mathfrak{U}, \partial(\log \mathfrak{m}) \;&=\; \frac{\partial(\mathfrak{m})}{\mathfrak{m}}. \end{aligned}$$

Let $\partial\colon \mathbb{G} \longrightarrow \mathbb{V}$ be a transserial derivation and assume that ∂ has a well-based and positive near-support $\mathfrak{W}_{\partial}$. We extend ∂ into a transserial derivation $\partial\colon \mathbb{G}_{(<1)} \longrightarrow \mathbb{G}$ by induction on the exponential height as follows:

Let $s \in \mathbb{G}_{(<1)}$, set $\gamma := \mathrm{EH}_{\mathbb{G}}(s)$ and assume that $\partial(t)$ is defined for all $t \in \mathbb{G}_{(<1)}$ with $\mathrm{EH}_{\mathbb{G}}(t) < \gamma$. If $\gamma = 0$, i.e. $s \in \mathbb{G}$, then $\partial(s)$ is already defined. Assume that $\gamma > 0$. If ∂ is defined at each $\log \mathfrak{m}$ for $\mathfrak{m} \in \operatorname{supp} s$ and the family $(\partial(\log \mathfrak{m})\,\mathfrak{m})_{\mathfrak{m} \in \operatorname{supp} s}$ is well-based, then ∂ is defined at s, and we set

$$\partial(s) := \sum_{\mathfrak{m} \in \operatorname{supp} s} s_{\mathfrak{m}}\,\partial(\log \mathfrak{m})\,\mathfrak{m}.$$

By induction on $\mathrm{EH}_{\mathbb{G}}(s)$, we see that the definition is warranted.

Lemma 3.3. *If $\mathfrak{m} \in \mathbb{G}_{(<1)} \cap \mathfrak{U}$ and ∂ is defined at $\mathfrak{m}$, then*

$$\operatorname{supp} \partial(\mathfrak{m}) \subseteq \mathfrak{m} \cdot \mathfrak{V}^{\prec \mathfrak{m}, \succcurlyeq} \cdot \mathfrak{W}_{\partial}.$$

Proof. We prove this by induction on $\mathrm{EH}_{\mathbb{G}}(\mathfrak{m})$. This is immediate for $\mathfrak{m} \in \mathbb{G}$. So let $\mathfrak{m} \in \mathbb{G}_{(<1)} \cap \mathfrak{U}$ such that the result holds for all $\mathfrak{n} \in \mathbb{G}_{(<1)} \cap \mathfrak{U}$ with $\mathrm{EH}_{\mathbb{G}}(\mathfrak{n}) < \mathrm{EH}_{\mathbb{G}}(\mathfrak{m})$. Write $\varphi := \log \mathfrak{m}$. So $\mathrm{EH}_{\mathbb{G}}(\mathfrak{n}) < \mathrm{EH}_{\mathbb{G}}(\mathfrak{m})$ for all $\mathfrak{n} \in \operatorname{supp} \varphi$. We have $\partial(\mathfrak{m}) = \partial(\varphi)\, \mathfrak{m}$. Let $\mathfrak{q} \in \operatorname{supp} \partial(\mathfrak{m})$. So $\mathfrak{q} = \mathfrak{m} \cdot \mathfrak{u}$ for a certain $\mathfrak{u} \in \operatorname{supp} \partial(\mathfrak{n})$. By the induction hypothesis, we have $\mathfrak{u} \in \mathfrak{n} \cdot \mathfrak{V}^{\prec \mathfrak{n}, \succcurlyeq} \cdot \mathfrak{W}_{\partial}$ where $\mathfrak{n} \cdot \mathfrak{V}^{\prec \mathfrak{n}, \succcurlyeq} \equiv \mathfrak{n}$, whence $\mathfrak{n} \cdot \mathfrak{V}^{\prec \mathfrak{n}, \succcurlyeq} \subseteq \mathfrak{V}^{\prec \mathfrak{m}, \succcurlyeq}$. It follows that $\mathfrak{q} \in \mathfrak{m} \cdot \mathfrak{V}^{\prec \mathfrak{m}, \succcurlyeq} \cdot \mathfrak{W}_{\partial}$ as desired. □

We will see that ∂ is well-defined. It is easy to see then by induction that it is the unique extension of ∂ into a transserial derivation $\mathbb{G}_{(<1)} \longrightarrow \mathbb{G}$.

Proposition 3.4. *The function ∂ is well-defined on $\mathbb{G}_{(<1)}$.*

Proof. We prove the result by induction on $\mathrm{EH}_{\mathbb{G}}(s)$ for $s \in \mathbb{G}_{(<1)}$. So let $s \in \mathbb{G}_{(<1)}$, write $(\gamma, \boldsymbol{\eta}) := \mathrm{EH}_{\mathbb{G}}(s)$ and assume that we have $g \in \mathcal{D}_{\partial}$ for all $g \in \mathbb{G}_{(<1)}$ with $\mathrm{EH}_{\mathbb{G}}(g) < \gamma$. We may assume that $s \notin \mathbb{G}$. Note that for all $\mathfrak{m} \in \operatorname{supp} s$, we have $\mathrm{EH}_{\mathbb{G}}(\log \mathfrak{m}) < \gamma$ so $\log \mathfrak{m} \in \mathcal{D}_{\partial}$ by the induction hypothesis. So ∂ is defined at $\mathfrak{m}$. Thus it is enough by to prove that the family $(\partial(\mathfrak{m}))_{\mathfrak{m} \in \operatorname{supp} s}$ is well-based.

Assume for contradiction that $(\partial(\mathfrak{m}))_{\mathfrak{m} \in \operatorname{supp} s}$ is not well-based. So there is a strictly decreasing family $(\mathfrak{m}_i)_{i \in \mathbb{N}}$ in $\operatorname{supp} s$ such that $(\partial(\mathfrak{m}_i))_{i \in \mathbb{N}}$ is not well-based. Write $\mathfrak{m}_i = \mathrm{e}^{\varphi_i}$ for each $i \in \mathbb{N}$. There is a family of monomials $(\mathfrak{n}_i)_{i \in \mathbb{N}}$ with $\mathfrak{n}_i \preccurlyeq \mathfrak{n}_{i+1}$ and $\mathfrak{n}_i \in \operatorname{supp} \partial(\mathfrak{m}_i)$ for all $i \in \mathbb{N}$. For all $i \in \mathbb{N}$, there are a $\mathfrak{b}_i \in \operatorname{supp} \varphi_i$ and a $\mathfrak{q}_i \in \operatorname{supp} \partial(\mathfrak{b}_i)$ with $\mathfrak{n}_i = \mathfrak{q}_i\, \mathfrak{m}_i$. By Lemma 3.3, we may write $\mathfrak{q}_i = \mathfrak{b}_i\, \mathfrak{p}_i\, \mathfrak{w}_i$ for a certain $\mathfrak{p}_i \in \mathfrak{U}^{\prec \mathfrak{b}_i, \succcurlyeq}$ and $\mathfrak{w}_i \in \mathfrak{W}_{\partial}$.

Since $\mathfrak{W}_{\partial}$ is well-based, we may assume that $\mathfrak{w}_0 \succcurlyeq \mathfrak{w}_1 \succcurlyeq \cdots$. Let $i, j \in \mathbb{N}$ with $i < j$. Write $\varphi_{i,j}$ for the $\lhd$-maximal common $\lhd$-lower bound or φ_i and φ_j, and write $(\delta_i, \delta_j) = (\varphi_i - \varphi_{i,j}, \varphi_j - \varphi_{i,j})$. Assume for contradiction that $\mathfrak{b}_j \in \operatorname{supp} \delta_j$. The inequalities $\mathfrak{n}_i \preccurlyeq \mathfrak{n}_j$ and $\mathfrak{w}_i \succcurlyeq \mathfrak{w}_j$ imply that $\mathrm{e}^{\delta_i - \delta_j}\, \mathfrak{b}_i\, \mathfrak{p}_i \prec \mathfrak{b}_j\, \mathfrak{p}_j$. Recall that $\mathfrak{p}_i \succcurlyeq 1$ so $\mathrm{e}^{\delta_i - \delta_j}\, \mathfrak{b}_i \prec \mathfrak{b}_j\, \mathfrak{p}_j$, so $\mathrm{e}^{\delta_i - \delta_j}\, \mathfrak{b}_i \preccurlyeq \mathfrak{b}_j$. But since $\varphi_{i,j}$ is $\lhd$-maximal with $\varphi_{i,j} \trianglelefteq \varphi_i, \varphi_j$, we have $\delta_i - \delta_j \succcurlyeq \delta_i$ whence $\mathrm{e}^{\delta_i - \delta_j} \gg \mathfrak{b}_i$ and $\mathrm{e}^{\delta_i - \delta_j} \succ 1$. It follows that $\mathrm{e}^{\delta_i - \delta_j}\, \mathfrak{b}_i \equiv \mathrm{e}^{\delta_i} \not\succcurlyeq \operatorname{supp}(\varphi_{i,j}) \succ \mathfrak{b}_j$: a contradiction. This proves that $\mathfrak{b}_j \in \operatorname{supp} \varphi_{i,j}$. In particular $\mathfrak{b}_j \in \operatorname{supp} \varphi_i$. Therefore, we may assume that $\mathfrak{b}_i \succcurlyeq \mathfrak{b}_{i+1} \succcurlyeq \cdots$.

We have $\mathrm{EH}_{\mathbb{G}}(\varphi_i) < \gamma$ so the family $(\partial(\mathfrak{b}))_{\mathfrak{b} \in \operatorname{supp} \varphi_i}$ is well-based. Since $\mathfrak{q}_i \prec \mathfrak{q}_{i+1} \prec \cdots$, we deduce that there is a $j \geqslant i$ such that $\mathfrak{b}_j = \mathfrak{b}_k$ for all $k > j$. But then $(\mathfrak{q}_k)_{k \geqslant j}$ witnesses that $\operatorname{supp} \partial(\mathfrak{b}_j)$ is not well-based: a contradiction. We deduce that $(\partial(\mathfrak{m}))_{\mathfrak{m} \in \operatorname{supp} s}$ is well-based. □

We next give a strengthening of [35, Proposition 4.1.5]:

Proposition 3.5. *Let $\partial \colon \mathbb{T} \longrightarrow \mathbb{T}$ be a strongly linear function with*

$$\partial(\log \mathfrak{m}) = \frac{\partial(\mathfrak{m})}{\mathfrak{m}} \quad \textit{for all } \mathfrak{m} \in \mathfrak{M}.$$

Then we have

$$(\partial(s\, t) = \partial(s)\, t + s\, \partial(t)) \quad \textit{and}$$

$$\forall u \in \mathbb{T}^{>}, \left(\partial(\log u) = \frac{\partial(u)}{u} \right).$$

for all $s, t \in \mathbb{T}$ *and* $u \in \mathbb{T}^>$.

Proof. We first prove that ∂ satisfies the Leibniz rule. Consider $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$. We have

$$\frac{\partial(\mathfrak{m}\,\mathfrak{n})}{\mathfrak{m}\,\mathfrak{n}} = \partial(\log \mathfrak{m}\,\mathfrak{n}) = \partial(\log \mathfrak{m}) + \partial(\log \mathfrak{n}) = \frac{\partial(\mathfrak{m})}{\mathfrak{m}} + \frac{\partial(\mathfrak{n})}{\mathfrak{n}}.$$

We deduce that $\partial(\mathfrak{m}\,\mathfrak{n}) = \partial(\mathfrak{m})\,\mathfrak{n} + \mathfrak{m}\,\partial(\mathfrak{n})$. Now let $s, t \in \mathbb{T}$. We have

$$\begin{aligned}
\partial(s\,t) &= \partial\left(\sum_{\mathfrak{m},\mathfrak{n}} s_{\mathfrak{m}} t_{\mathfrak{n}}\,\mathfrak{m}\,\mathfrak{n}\right) \\
&= \sum_{\mathfrak{m},\mathfrak{n}} s_{\mathfrak{m}} t_{\mathfrak{n}}\,\partial(\mathfrak{m}\,\mathfrak{n}) \\
&= \sum_{\mathfrak{m},\mathfrak{n}} s_{\mathfrak{m}} t_{\mathfrak{n}}\,\partial(\mathfrak{m})\,\mathfrak{n} + \sum_{\mathfrak{m},\mathfrak{n}} s_{\mathfrak{m}} t_{\mathfrak{n}}\,\mathfrak{m}\,\partial(\mathfrak{n}) \\
&= \left(\sum_{\mathfrak{m}} s_{\mathfrak{m}}\,\partial(\mathfrak{m})\right)\left(\sum_{\mathfrak{n}} t_{\mathfrak{n}}\,\mathfrak{n}\right) + \left(\sum_{\mathfrak{m}} s_{\mathfrak{m}}\,\mathfrak{m}\right)\left(\sum_{\mathfrak{n}} t_{\mathfrak{n}}\,\partial(\mathfrak{n})\right) \\
&= \partial(s)\,t + s\,\partial(t).
\end{aligned}$$

So the Leibniz rule holds for ∂. We deduce that we have $\partial(t^{k+1}) = k\,\partial(t)\,t^k$ for all $t \in \mathbb{T}$ and $k \in \mathbb{N}$. Now let $u \in \mathbb{T}^>$ and write $u = r\,\mathfrak{d}_u\,(1+\varepsilon)$ where $r \in \mathbb{R}^>$ and $\varepsilon \prec 1$. Note that

$$\frac{\partial(u)}{u} = \frac{r\,\partial(\mathfrak{d}_u)\,(1+\varepsilon) + r\,\mathfrak{d}_u\,\partial(\varepsilon)}{r\,\mathfrak{d}_u\,(1+\varepsilon)} = \frac{\partial(\mathfrak{d}_u)}{\mathfrak{d}_u} + \frac{\partial(\varepsilon)}{(1+\varepsilon)}.$$

Recall that $\log u = \log \mathfrak{d}_u + \log r + \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k+1}\varepsilon^{k+1}$, so

$$\begin{aligned}
\partial(\log u) &= \partial(\log \mathfrak{d}_u) + \partial\left(\sum_{k\in\mathbb{N}} \frac{(-1)^k}{k+1}\varepsilon^{k+1}\right) \\
&= \frac{\partial(\mathfrak{d}_u)}{\mathfrak{d}_u} + \sum_{k\in\mathbb{N}} \frac{(-1)^k}{k+1}\,\partial(\varepsilon^{k+1}) \\
&= \frac{\partial(\mathfrak{d}_u)}{\mathfrak{d}_u} + \partial(\varepsilon) \sum_{k\in\mathbb{N}} (-1)^k \varepsilon^k \\
&= \frac{\partial(\mathfrak{d}_u)}{\mathfrak{d}_u} + \frac{\partial(\varepsilon)}{(1+\varepsilon)} \\
&= \frac{\partial(u)}{u}
\end{aligned}$$

as desired. □

3.3 Schmeling's axiom T4

Definition 3.6. *Let* $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ *be a transserial subfield. Then* $\mathbb{T}$ *is a* **transseries subfield** *if it satisfies the following axiom*

T4. *If* $(r_i\,\mathfrak{m}_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\neq}\,\mathfrak{M}^{\neq}$ *is a sequence of terms with* $r_{i+1}\,\mathfrak{m}_{i+1} \in \operatorname{term}\log \mathfrak{m}_i$ *for all* $i \in \mathbb{N}$*, then there is an* $i \in \mathbb{N}$ *with*

$$\begin{aligned}
\mathfrak{m}_{j+1} &= \min \operatorname{supp} \log \mathfrak{m}_j \quad \textit{and} \\
r_{j+1} &\in \{-1, 1\}
\end{aligned}$$

for all $j \geqslant i$.

Note that the axiom **T4** is preserved under taking transserial subgroups. We claim that $\mathbb{L}$ satisfies **T4**. Indeed, consider $(r_i\,\mathfrak{m}_i)_{i\in\mathbb{N}}$ as in Definition 3.6 for $\mathbb{L}$. Writing $\mathfrak{m}_0 = \mathfrak{l} \in \mathfrak{L}$, we have $\mathfrak{m}_1 = \ell_\gamma$ for a certain ordinal γ, whence $r_{1+i}\,\mathfrak{m}_{1+i} = \ell_{\gamma+i}$ for all $i \in \mathbb{N}$. In particular $r_{j+1} = 1 \in \{-1, 1\}$ and $\mathfrak{m}_{j+1} = \min \operatorname{supp} \ell_{\gamma+j}$ for all $j \geqslant i$. This shows that in fact $\mathbb{L}$ satisfies the stronger axiom **TEL4** of [31, Section 5]. It is known [8, Theorem 8.4] that the field **No** of surreal numbers with its natural logarithm satisfies **T4** (see also [22, Theorem 8.1] for a different proof).

3.4 Extending transserial right compositions

Let $\mathbb{U} = \mathbb{R}[[\mathfrak{U}]]$ be a confluent hyperserial field of force $\boldsymbol{\nu}$ and let $\mathbb{G} \subseteq \mathbb{U}$ be a transserial subgroup. A *transserial right composition* $\mathbb{G} \longrightarrow \mathbb{V}$ is a strongly linear function $\triangle\colon \mathbb{G} \longrightarrow \mathbb{V}$ with

$$\begin{aligned} \forall s, g \in \mathbb{G},\, s\,g \in \mathbb{G} \;&\Longrightarrow\; \triangle(s\,g) = \triangle(s)\,\triangle(g), \quad \text{and} \\ \forall \mathfrak{t} \in \mathbb{G} \cap \mathfrak{U},\, \triangle(\mathfrak{t}) > 0 \wedge \triangle(\log \mathfrak{t}) \;&=\; \log \triangle(\mathfrak{t}). \end{aligned}$$

Consider a transserial right composition $\triangle$ on $\mathbb{G}$. We wish to extend $\triangle$ into a transserial right composition $\triangle_1$ on $\mathbb{G}_{(<1)}$. The definition of $\triangle_1$ is already done in [35, Section 5.3] and will coincide with the definition of $\triangle_{\boldsymbol{\mu}}$ on $\mathbb{G}_{(<1)}$ in Proposition 3.7. In order to adapt Schmeling's arguments to this setting our case, we note the following facts:

- Schmeling's proof does not rely on the fact that the transserial field be closed under products, so it works for transserial subgroups,
- any partial right composition of force $\boldsymbol{\nu} \geqslant 1$ is a right composition as per [35, Definition 5.1.1],
- by [35, Theorem 5.3.2], any such right composition extends uniquely into a right composition $\triangle_1\colon \mathbb{G}_{(<1)} \longrightarrow \mathbb{V}_{(<1)}$.

We thus have:

Proposition 3.7. *Assume that* $\mathbb{U}$ *satisfies* **T4**. *Let* $\mathbb{G}$ *be a transserial subgroup of* $\mathbb{U}$ *and let* $\triangle\colon \mathbb{G} \longrightarrow \mathbb{V}$ *be a transserial right composition. Then* $\triangle$ *extends uniquely into a transserial right composition* $\mathbb{G}_{(<1)} \longrightarrow \mathbb{V}_{(<1)}$.

See also [9, Section 9] for a similar extension result in the case of surreal numbers.

4 A proof method

In $\mathbb{L}$, the existence of a well-based support for the derivation and a well-based relative support for the right composition with s allowed van den Dries, Kaplan, van der Hoeven [15] and then those authors and myself [7] to circumvent this problem. Unfortunately, this approach fails in the case of (hyper)exponential extensions, as Schmeling's work [35, Chapters 4 and 5] on the difficult case of exponential extensions already illustrates. Indeed, any monomial

$$\mathfrak{m} = \mathrm{e}^{\varphi}$$

for $\varphi \in \tilde{\mathbb{L}}_{\succ}$ gives rise to a derivative

$$\mathfrak{m}' = \tilde{\partial}(\varphi)\,\mathfrak{m}, \tag{4.1}$$

So the operator support $\operatorname{supp}\tilde{\partial}$ of $\tilde{\partial}$ already contains $\operatorname{supp}\tilde{\partial}(\varphi)$ for all purely large series in $\tilde{\mathbb{L}}$, hence it cannot be well-based. As for the composition law, the fact that the definition of $\exp(\varphi\circ s)$ involves the Taylor expansion of $\mathfrak{m}$ implies that the relative support of $\tilde{\circ_s}$ involves the support of each iterate $\tilde{\partial}^{[k]}$.

However, a more careful study of those phenomena will show that $\tilde{\partial}$ and $\tilde{\circ_s}$ have a well-based near-support and a well-based relative near-support respectively (see Definition 1.35). This fact is not sufficient to justify that those operations are strongly linear, but what it lacks is made for by Schmeling's work. More precisely, it is possible to construe the *hyperexponential* closure $\tilde{\mathbb{L}}$ of $\mathbb{L}$ as the *exponential* closure of a subgroup

$$\mathbb{G}=\mathbb{E}+\mathbb{L}\subseteq\tilde{\mathbb{L}}$$

where $\mathbb{E}=\mathbb{R}[[\mathfrak{S}]]$ is a subclass $\tilde{\mathbb{L}}$ whose monomials $\mathfrak{a}\in\mathfrak{S}$ are all log-atomic. Any two distinct log-atomic series $\mathfrak{a},\mathfrak{b}$ are more than sufficiently far apart from one another that the notion of near-support and relative near-support are conducive to proofs of strong linearity for functions on $\mathbb{E}$. We will combine this with Schmeling's results for extending derivations and compositions through exponential extensions, in order to show strong linearity. Using this trick requires a precise description of (hyper)exponential extensions which is the purpose of this section.

4.1 Internal hyperexponential closure

We fix generalized ordinals $\boldsymbol{\mu}\leqslant\boldsymbol{\nu}\leqslant\mathbf{On}$ with $\boldsymbol{\nu}>0$, a confluent hyperserial field $\mathbb{U}=\mathbb{R}[[\mathfrak{U}]]$ of force $(\boldsymbol{\nu},\boldsymbol{\nu})$. If $\mathbb{T}$ is a confluent subfield of $\mathbb{U}$ of force $\boldsymbol{\nu}$, then by Theorem 2.13, the inclusion $\mathbb{T}\hookrightarrow\mathbb{U}$ extends uniquely into an embedding $\mathbb{T}_{(<\boldsymbol{\mu})}\hookrightarrow\mathbb{U}_{(<\boldsymbol{\mu})}$. But since $\mathbb{U}$ itself has force $(\boldsymbol{\nu},\boldsymbol{\mu})$, we have $\mathbb{U}=\mathbb{U}_{(<\boldsymbol{\mu})}$. So $\mathbb{T}_{(<\boldsymbol{\mu})}$ is naturally included into $\mathbb{U}$. Taking from [7, Section 8], we will show how $\mathbb{T}_{(<\boldsymbol{\mu})}$ can be described as a subclass of $\mathbb{U}$.

Consider the lexicographical order $<_{\mathrm{lex}}$ on the class of ordered pairs $(\gamma,\boldsymbol{\eta})$ where $\gamma\in\mathbf{On}$ and $\boldsymbol{\eta}\leqslant\boldsymbol{\mu}$, that is

$$(\gamma,\boldsymbol{\eta})<_{\mathrm{lex}}(\rho,\boldsymbol{\sigma})\iff((\gamma<\rho)\text{ or }(\gamma=\rho\text{ and }\boldsymbol{\eta}<\boldsymbol{\sigma})).$$

Note that this is a well-ordering.

Let $\mathbb{G}=\mathbb{R}[[\mathfrak{S}]]$ be a transserial subgroup of $\mathbb{U}$. We will define increasing tower $(\mathfrak{S}_{(\gamma,\boldsymbol{\eta})})_{\gamma\in\mathbf{On},\boldsymbol{\eta}\leqslant\boldsymbol{\mu}}$ of extensions of $\mathfrak{S}$. We first define an extension $\mathfrak{S}_{[\eta]}$ of $\mathfrak{S}$ for all $\eta<\boldsymbol{\nu}$. Note that $\mathbb{R}[[\mathfrak{S}^{\succ}]]$ is a transserial subgroup of $\mathbb{T}$. So we may set $\mathfrak{S}_{[0]}=E_1^{\mathbb{R}[[\mathfrak{S}^{\succ}]]}$ as in Section 3. Now let $0<\eta<\boldsymbol{\nu}$ and set $\beta:=\omega^{\eta}$. We write $\mathfrak{S}_{(\eta)}$ for the direct internal product $\mathfrak{S}_{(\eta)}=\mathfrak{S}\cdot\mathfrak{S}_{[\eta]}$ where $\mathfrak{S}_{[\eta]}$ is the class of exponentials

$$\mathfrak{t}=\exp\left(\sum_{\varphi\in\mathbb{T}_{\succ,\beta}\setminus L_\beta(\mathbb{T}^{\succ,\succ})}\log(\mathfrak{t}_\varphi)\circ E_\beta^\varphi\right),\text{ where}$$

- each $\mathfrak{t}_\varphi$ for $\varphi\in(\mathbb{G}\cap\mathbb{U}_{\succ,\beta})\setminus L_\beta(\mathbb{G}\cap\mathbb{U}^{\succ,\succ})$ lies in $\mathfrak{L}_{<\beta}$,
- $\operatorname{hsupp}\mathfrak{t}:=\{\varphi\in(\mathbb{G}\cap\mathbb{U}_{\succ,\beta})\setminus L_\beta(\mathbb{G}\cap\mathbb{U}^{\succ,\succ}):\mathfrak{t}_\varphi\neq1\}$ is a well-based set,
- the set $\{\varphi+\mathbb{Z}:\varphi\in\operatorname{hsupp}\mathfrak{t}\}$ is finite.

In other words, $\mathfrak{S}_{(\eta)}$ is the group generated by $\mathfrak{S}$ and the subgroups

$$\mathfrak{L}_{<\beta}\circ E_\beta^\varphi\subseteq\mathfrak{U},$$

for all $\varphi\in(\mathbb{G}\cap\mathbb{U}_{\succ,\beta})\setminus L_\beta(\mathbb{G}\cap\mathbb{U}^{\succ,\succ})$. Note that for each $\varphi\in(\mathbb{G}\cap\mathbb{U}_{\succ,\beta})\setminus L_\beta(\mathbb{G}\cap\mathbb{U}^{\succ,\succ})$, we have

The definition of $\mathfrak{S}_{(\gamma,\boldsymbol{\eta})}$ is then by induction on $<_{\mathrm{lex}}$, as follows:

Definition 4.1. *Let $\mathbb{G}=\mathbb{R}[[\mathfrak{S}]]\subseteq\mathbb{U}$ be a transserial subgroup. For $\gamma\in\mathbf{On}$ and $\boldsymbol{\eta}\leqslant\boldsymbol{\mu}$, we define*

- $\mathfrak{S}_{(0,0)}:=\mathfrak{S}$.
- $\mathfrak{S}_{(\gamma,\boldsymbol{\eta})}:=(\mathfrak{S}_{(\gamma,\iota)})_{(\iota)}$ *if $\boldsymbol{\eta}=\iota+1$ is a successor.*
- $\mathfrak{S}_{(\gamma,\boldsymbol{\eta})}:=\bigcup_{\sigma<\boldsymbol{\eta}}\mathfrak{S}_{(\gamma,\sigma)}$ *if $\boldsymbol{\eta}$ is a limit.*
- $\mathfrak{S}_{(\gamma,0)}:=\bigcup_{\lambda<\gamma}\mathfrak{S}_{(\lambda,\boldsymbol{\mu})}$ *if $\gamma>0$.*

We set $\mathbb{G}_{(\gamma,\boldsymbol{\eta})}:=\mathbb{R}[[\mathfrak{S}_{(\gamma,\boldsymbol{\eta})}]]$, so $\mathbb{G}_{(0,0)}=\mathbb{G}$. We have an inclusion $\mathbb{G}_{(\lambda,\boldsymbol{\sigma})}\subseteq\mathbb{G}_{(\gamma,\boldsymbol{\eta})}$ whenever $(\gamma,\boldsymbol{\eta})<_{\mathrm{lex}}(\rho,\boldsymbol{\sigma})$. We set

$$\mathfrak{S}_{(<\boldsymbol{\mu})}:=\bigcup_{\gamma\in\mathbf{On}}\mathfrak{S}_{(\gamma,0)},\qquad \mathfrak{S}_{(<\boldsymbol{\mu})}:=\bigcup_{\gamma\in\mathbf{On}}\mathfrak{S}_{(\gamma,0)}.$$

Note that $\mathfrak{S}_{(<0)}=\mathfrak{S}$ and $\mathbb{G}_{(<0)}=\mathbb{G}$. Moreover, we have $\mathbb{G}_{(<\boldsymbol{\mu})}=\mathbb{R}[[\mathfrak{S}_{(<\boldsymbol{\mu})}]]$ by [7, Lemma 2.1]. The following shows that our notation does not conflict with that in [7, Section 8], which it in fact extends.

Proposition 4.2. *Assume that $\mathbb{T}=\mathbb{R}[[\mathfrak{M}]]$ is a confluent subfield of $\mathbb{U}$ of force $\boldsymbol{\nu}$. Then for each $(\gamma,\boldsymbol{\eta})$ where $\boldsymbol{\eta}\leqslant\boldsymbol{\mu}$ and $\gamma\in\mathbf{On}$, the class $\mathbb{T}_{(\gamma,\boldsymbol{\eta})}$ is a confluent subfield of $\mathbb{U}$ of force $\boldsymbol{\nu}$. Moreover $\mathbb{T}_{(<\boldsymbol{\mu})}$ is the smallest confluent subfield of $\mathbb{U}$ of force $(\boldsymbol{\nu},\boldsymbol{\mu})$ which contains $\mathbb{T}$.*

Proof. This follows from the more general construction of $\mathbb{T}_{(<\boldsymbol{\mu})}$ of [7, Section 8]. □

An important feature of the construction is the following:

Proposition 4.3. [7, p 66] *Assume that $\mathbb{T}=\mathbb{R}[[\mathfrak{M}]]$ is a confluent subfield of $\mathbb{U}$ of force $\boldsymbol{\nu}$. For $\gamma\in\mathbf{On}$ and $\eta\leqslant\boldsymbol{\mu}$ with $\eta<\boldsymbol{\nu}$ we have*

$$(\mathfrak{M}_{(\gamma,\boldsymbol{\eta})})_{\omega^\eta}=\mathfrak{M}_{\omega^\eta}.$$

Proposition 4.4. *Let $\boldsymbol{\nu}\leqslant\mathbf{On}$ with $\boldsymbol{\nu}>0$ and let $\mathbb{T}=\mathbb{R}[[\mathfrak{M}]]$ be a hyperserial field of force $\boldsymbol{\nu}$ that satisfies* **T4**. *Then for all $\boldsymbol{\mu}\leqslant\boldsymbol{\nu}$, the field $\mathbb{T}_{(<\boldsymbol{\mu})}$ satisfies* **T4**.

Proof. Since each member of the tower extension $(\mathbb{T}_{(\gamma,\boldsymbol{\eta})})_{\gamma\in\mathbf{On}\wedge\boldsymbol{\eta}\leqslant\boldsymbol{\mu}}$ is a transserial field, it suffices to show the that $\mathbb{T}_{(\eta)}$ satisfies **T4** for all $\eta<\boldsymbol{\mu}$. Let $\eta<\boldsymbol{\mu}$ and let $(r_i\,\mathfrak{m}_i)_{i\in\mathbb{N}}$ be a sequence as in Definition 3.6. If $\eta=0$, then $r_1\,\mathfrak{m}_1\in\mathbb{T}$ so we can conclude using the validity of **T4** in $\mathbb{T}$. Assume that $\eta>0$. If $\mathfrak{m}_1\in\mathbb{T}$, then we conclude as previously. Otherwise, we must have $\mathfrak{m}_1=L_{\gamma+1}\,E_{\omega^\eta}^{\varphi}$ for a certain ω^η-truncated series φ and a certain ordinal γ with $\gamma\omega<\omega^\eta$. Since $\eta>0$, the series $E_{\omega^\eta}^{\varphi}$ is log-atomic, so $r_{1+i}\,\mathfrak{m}_{1+i}=L_{\gamma+1+i}\,E_{\omega^\eta}^{\varphi}$ for all $i\in\mathbb{N}$. This sequence is as specified in **T4**, so **T4** holds in that case. This concludes the proof. □

In particular, the field $\tilde{\mathbb{L}}$ and all its (transserial subfields) satisfies **T4**, and even Kuhlmann-Mantova's axiom **TEL4**.

4.2 Hyperexponential height

Let $\boldsymbol{\nu}\leqslant\mathbf{On}$, let $\boldsymbol{\mu}\leqslant\boldsymbol{\nu}$ and let $\mathbb{U}$ be a hyperserial field of force $\boldsymbol{\nu}$. We define the *hyperexponential height* $\mathrm{HH}_{\mathbb{U}}(f)$ over $\mathbb{U}$ of a series $f\in\mathbb{U}_{(<\boldsymbol{\mu})}$ as the $<_{\mathrm{lex}}$-least ordered pair $(\gamma,\boldsymbol{\eta})$ with $f\in\mathbb{U}_{\gamma,\boldsymbol{\eta}}$. So we have $\mathrm{HH}_{\mathbb{U}}(f)=(0,0)\iff f\in\mathbb{U}$.

Lemma 4.5. *If* $\mathfrak{m} \in \mathfrak{U}_{(<\boldsymbol{\mu})} \setminus \mathbb{U}$, *then* $\boldsymbol{\eta}$ *in* $\mathrm{HH}_{\mathbb{U}}(\mathfrak{m}) = (\gamma, \boldsymbol{\eta})$ *must be a successor ordinal.*

Proof. New monomials are only added at successor stages of the inductive definition of $\mathbb{U}_{(<\boldsymbol{\mu})}$ as per Definition 4.1, hence the result. □

Example 4.6. Consider the extension $\mathbb{L} \hookrightarrow \mathbb{L}_{(<2)}$. For each $n \in \mathbb{N}$, the series $\mathrm{e}^{-\mathrm{e}_n^{\ell_0}}$ appears first in $\mathfrak{L}_{(n,1)}$. So the hyperseries $\ell_0 + 2\,\mathrm{e}^{-\ell_0} + 3\,\mathrm{e}^{-\mathrm{e}^{\ell_0}} + 4\,\mathrm{e}^{-\mathrm{e}^{\mathrm{e}^{\ell_0}}} + \cdots$ first appears in $\mathfrak{L}_{(\omega,0)}$, hence $\mathrm{e}_\omega^{\ell_0 + 2\mathrm{e}^{-\ell_0} + 3\mathrm{e}^{-\mathrm{e}^{\ell_0}} + 4\mathrm{e}^{-\mathrm{e}^{\mathrm{e}^{\ell_0}}} + \cdots}$ first appears in $\mathfrak{L}_{(\omega,2)}$. Therefore we have

$$\mathrm{HH}_{\mathbb{L}}\Big(\mathrm{e}_\omega^{\ell_0 + 2\mathrm{e}^{-\ell_0} + 3\mathrm{e}^{-\mathrm{e}^{\ell_0}} + 4\mathrm{e}^{-\mathrm{e}^{\mathrm{e}^{\ell_0}}} + \cdots}\Big) = (\omega, 2).$$

As a corollary of Proposition 4.3, we obtain:

Corollary 4.7. *Let* $\mathfrak{m} \in \mathfrak{U}_{(<\boldsymbol{\mu})}$ *and* $\boldsymbol{\sigma} \leqslant \boldsymbol{\nu}$ *with* $\mathfrak{m} \in (\mathfrak{U}_{(<\boldsymbol{\mu})})_{\omega^{\boldsymbol{\sigma}}} \setminus \mathbb{U}$ *and write* $\mathrm{HH}_{\mathbb{U}}(\mathfrak{m}) =: (\gamma, \boldsymbol{\eta})$. *Then* $\boldsymbol{\sigma} \leqslant \boldsymbol{\eta}$.

4.3 Decomposition lemmas

We will rely on transserial subgroups $\mathbb{G}$ and their exponential closure $\mathbb{G}_{(<1)}$ in order to extend derivations, compositions and their properties. Here, we explain our recurring method to prove the main results of this paper.

Fix $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. Let $\mathcal{P} \subseteq \mathbb{U}_{(<\boldsymbol{\mu})}$ be a subclass with $\mathcal{P} - 1 \subseteq \mathcal{P}$. We write $\mathbb{E}(\mathcal{P})$ for the class of well-based sums $g \in \mathbb{U}_{(<\boldsymbol{\mu})}$ of the form

$$g = \sum_{\gamma<\rho} r_\gamma L_{\beta_\gamma}(E_{\alpha_\gamma}^{\varphi_\gamma})$$

where ρ is an ordinal, $(L_{\beta_\gamma}(E_{\alpha_\gamma}^{\varphi_\gamma}))_{\gamma<\rho}$ is strictly $\prec$-decreasing, and

- $(\alpha_\gamma)_{\gamma<\rho}$ is a sequence of *infinite* additively indecomposable ordinals,
- $(\beta_\gamma)_{\gamma<\rho}$ is a sequence of ordinals.
- $(r_\gamma)_{\gamma<\rho} \in \mathbb{R}^\rho$,

such that for all $\gamma < \rho$, we have

$$\begin{aligned} \alpha_\gamma &< \boldsymbol{\lambda}, \\ \beta_\gamma \omega &< \alpha_\gamma, \\ \varphi_\gamma &\in (\mathcal{P} \cap (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\alpha_\gamma}) \setminus L_{\alpha_\gamma}(\mathbb{U}^{>,\succ}). \end{aligned}$$

Note that each $L_{\beta_\gamma}(E_{\alpha_\gamma}^{\varphi_\gamma})$ for $\gamma < \rho$ is log-atomic. We write $\mathbb{G}(\mathcal{P}) := \mathbb{E}(\mathcal{P}) + \mathbb{U}_{\succ}$.

Lemma 4.8. *The class* $\mathbb{G}(\mathcal{P})$ *is a transserial subgroup of* $\mathbb{U}_{(<\boldsymbol{\mu})}$.

Proof. We have $\mathbb{E}(\mathcal{P}) \subseteq (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ}$ by definition so $\mathbb{G}(\mathcal{P}) \subseteq (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ}$. Consider a monomial $\mathfrak{m} \in \mathfrak{U}_{(<\boldsymbol{\mu})} \cap \mathbb{E}(\mathcal{P})$ and fix α, β, φ as above with $\mathfrak{m} = L_\beta(E_\alpha^\varphi)$. If $\alpha = \omega$, then $\beta = 0$ and $\log \mathfrak{m} = E_\alpha^{\varphi^{-1}} \in \mathbb{E}(\mathcal{P})$ since $\mathcal{P} - 1 \subseteq \mathcal{P}$. If $\alpha > \omega$, then we have $\log \mathfrak{m} = L_{\beta+1}(E_\alpha^\varphi) \in \mathbb{E}(\mathcal{P})$. So $\mathbb{E}(\mathcal{P})$ is a transserial subgroup of $\mathbb{U}_{(<\boldsymbol{\mu})}$. We deduce that $\mathbb{G}(\mathcal{P})$ is a transserial subgroup of $\mathbb{U}_{(<\boldsymbol{\mu})}$. □

Lemma 4.9. *Let* $\mathfrak{a} \in \mathbb{E}(\mathcal{P}) \cap \mathfrak{U}_{(<\boldsymbol{\mu})}$ *and* $\mathfrak{n} \in \mathfrak{U}_{\succ}$. *We have* $\mathfrak{a} \not\asymp \mathfrak{n}$.

Proof. Write $\mathfrak{a} = L_\beta(E_\alpha^\varphi)$ where $\alpha = \omega^\eta$ for a certain $\eta \in (0, \boldsymbol{\mu})$, a β with $\beta\omega < \alpha$, and a $\varphi \in (\mathcal{P} \cap (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\alpha}) \setminus L_\alpha(\mathbb{U}^{>,\succ})$. We have $E_\alpha^\varphi = \mathfrak{d}_\alpha(\mathfrak{a})$ by Lemma 2.10. Since $\mathbb{U}$ is η-confluent and $E_\alpha^\varphi \notin \mathbb{U}$, we have $\mathfrak{d}_\alpha(\mathfrak{n}) \neq \mathfrak{d}_\alpha(\mathfrak{a})$, whence in particular $\log(\mathfrak{d}_\alpha(\mathfrak{a})) \not\asymp \log(\mathfrak{d}_\alpha(\mathfrak{n}))$. By Lemma 1.30, we deduce that $\mathfrak{a} \not\asymp \mathfrak{n}$. □

Lemma 4.10. *Assume that* $(\mathbb{G}(\mathcal{P}))_{(<1)} \subseteq \mathcal{P}$. *Then* $(\mathbb{G}(\mathcal{P}))_{(<1)} = \mathbb{U}_{(<\boldsymbol{\mu})} = \mathcal{P}$.

Proof. Set $\mathbb{K} := (\mathbb{G}(\mathcal{P}))_{(<1)}$. It is enough to prove that $\mathbb{U}_{(<\boldsymbol{\mu})} \subseteq \mathbb{K}$. The result then follows from the inclusions $\mathbb{K} \subseteq \mathcal{P} \subseteq \mathbb{U}_{(<\boldsymbol{\mu})}$. We will prove by induction on $\mathrm{HH}_{\mathbb{U}}(f)$ that each $f \in \mathbb{U}_{(<\boldsymbol{\mu})}$ lies in $\mathbb{K}$. Let $f \in \mathbb{U}_{(<\boldsymbol{\mu})}$ such that we have $g \in \mathbb{K}$ for all $g \in \mathbb{U}_{(<\boldsymbol{\mu})}$ with $\mathrm{HH}_{\mathbb{U}}(g) <_{\mathrm{lex}} \mathrm{HH}_{\mathbb{U}}(f)$. Note that $\mathbb{K}$ is a field of well-based series so it is enough to prove that $\operatorname{supp} f \subseteq \mathbb{K}$. This is true if $f \in \mathbb{U}$ by definition of $\mathbb{K}$ and Lemma 3.2. Assume that $f \notin \mathbb{U}$. Since $\mathbb{K}$ is closed under exponentiation, it is enough to prove the inclusion

$$\mathfrak{S} := \bigcup_{\mathfrak{m} \in \operatorname{supp} f} \operatorname{supp} \log \mathfrak{m} \subseteq \mathbb{K}.$$

Let $\mathfrak{m} \in \mathfrak{S}$. If $\mathrm{HH}_{\mathbb{U}}(\mathfrak{m}) <_{\mathrm{lex}} \mathrm{HH}_{\mathbb{U}}(f)$, then we have $\mathfrak{m} \in \mathbb{K}$ by the induction hypothesis. Otherwise we can write $\mathfrak{m} = L_{\beta+1}(E_{\omega^\eta}^\varphi)$ where $\eta \in (0, \boldsymbol{\mu})$, $\beta\omega < \omega^\eta$ and $\varphi \in (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\omega^\eta} \setminus L_{\omega^\eta}(\mathbb{U}^{>,\succ})$ satisfies $\mathrm{HH}_{\mathbb{U}}(\varphi) < \mathrm{HH}_{\mathbb{U}}(f)$. If $\eta > 1$, then the induction hypothesis directly yields $\varphi \in \mathbb{K}$, so $\varphi \in \mathcal{P}$, whence $\mathfrak{m} = L_{\beta+1}(E_{\omega^\eta}^\varphi) \in \mathbb{E}(\mathcal{P})$. Assume now that $\eta = 1$, so $\beta = 0$ and $\mathfrak{m} = E_{\omega^\eta}^{\varphi^{-1}}$. The induction hypothesis and the inclusion $\mathcal{P} - 1 \subseteq \mathcal{P}$ yields $\varphi - 1 \in \mathcal{P}$. If $\varphi - 1 \in L_{\omega^\eta}(\mathbb{U}^{>,\succ})$, then we have $\mathfrak{m} \in \mathbb{U}$ so $\mathfrak{m} \in \mathbb{K}$. Otherwise, we have $\mathfrak{m} \in \mathbb{E}(\mathcal{P})$ so $\mathfrak{m} \in \mathbb{K}$. We deduce that $f \in \mathbb{K}$. It follows by induction that $\mathbb{U}_{(<\boldsymbol{\mu})} \subseteq \mathbb{K}$. □

Schmeling showed [35] how to extend derivations, compositions and how to preserve Taylor expansions to exponential closures of transseries fields. We will extend his results to the hyperexponential closure as follows:

- Considering the class $\mathcal{P}$ of series for which a given operation or property is extended, we show that
$$\mathbb{G}(\mathcal{P}) \subseteq \mathcal{P}. \tag{4.2}$$
Monomials in $\mathbb{E}(\mathcal{P})$ are log-atomic, so they satisfy $\mathfrak{a} \succ \mathfrak{b} \Longleftrightarrow \mathfrak{a} \gg \mathfrak{b}$. Combining this with properties of near-supports and relative near-supports, it is often possible to prove (4.2) in an easier fashion than by directly proving that $\mathbb{U}_{(<\boldsymbol{\mu})} \subseteq \mathcal{P}$.
- Applying or extending Schmeling's results, we show that if $\mathbb{G} \subseteq \mathbb{U}_{(<\boldsymbol{\mu})}$ is a transserial subgroup, then we have
$$\mathbb{G} \subseteq \mathcal{P} \Longrightarrow \mathbb{G}_{(<1)} \subseteq \mathcal{P}.$$
- Using Lemma 4.10, we conclude that $\mathcal{P} = \mathbb{U}_{(<\boldsymbol{\mu})}$.

5 Extending derivations

In this section, we fix $\boldsymbol{\mu}, \boldsymbol{\nu} \leqslant \mathbf{On}$ with $0 < \boldsymbol{\mu} \leqslant \boldsymbol{\nu}$, and we write $\boldsymbol{\lambda} := \omega^{\boldsymbol{\nu}}$. We will see how to extend derivations when taking the hyperexponential closure of a given field.

5.1 Hyperserial derivations

Definition 5.1. *Let* $(\mathbb{U}, \circ)$, $(\mathbb{V}, \circ)$ *be confluent hyperserial fields of force* $\boldsymbol{\nu}$ *with* $(\mathbb{U}, \circ) \subseteq (\mathbb{V}, \circ)$. *A* **hyperserial derivation** *of force* $\boldsymbol{\nu}$ *on* $\mathbb{T}$ *is a function* $\partial\colon \mathbb{U} \longrightarrow \mathbb{V}$ *with*

D1. $\partial\colon \mathbb{U} \longrightarrow \mathbb{V}$ *is strongly linear.*

D2. $\partial(s\,t) = \partial(s)\,t + s\,\partial(t)$ *for all* $s, t \in \mathbb{U}$.

D3. $\partial(\log t) = \frac{\partial(t)}{t}$ *for all* $t \in \mathbb{U}^{>}$.

D4. $\partial(f \circ s) = \partial(s) \times (f' \circ s)$ *for all* $f \in \mathbb{L}_{<\boldsymbol{\lambda}}$ *and* $s \in \mathbb{U}^{>,\succ}$.

Note that hyperserial derivations of force $\boldsymbol{\nu}$ on $\mathbb{U}$ are in particular derivations in the sense of [35, Definition 4.1.1]. If $\mathbb{U}, \partial$ are as above, then for all $s \in \mathbb{U}^{\neq}$, we write

$$s^{\dagger} := \frac{\partial(s)}{s}$$

for the logarithmic derivative of s. Indeed, we have $s^{\dagger} = \partial(\log s)$ whenever $s > 0$. Notice that $(r\,s)^{\dagger} = s^{\dagger}$ and that $(s\,t)^{\dagger} = s^{\dagger} + t^{\dagger}$ for all $s, t \in \mathbb{U}^{\neq}$ and $r \in \mathbb{R}^{\neq}$.

Remark 5.2. On the notation $\partial(f)$ vs f'. We sometimes use the notation f' instead of $\partial(s)$ and and $f^{(k)}$ instead of $\partial^k(s)$. In general, this reflects the distinction of Remark 1.1: the notation f' is favored when we are considering f as a function (through a composition law) acting on a hyperserial field, whereas $\partial(s)$ is favored when we are considering s as an object on which our derivation acts, regardless of how s might be construed as a function.

Consider fixed confluent hyperserial fields $(\mathbb{U}, \circ_{\mathbb{U}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ of force $\boldsymbol{\nu}$. The axioms of hyperserial derivations can be checked on the skeleton of $(\mathbb{U}, \circ_{\mathbb{U}})$, as the following shows.

Proposition 5.3. *Let* $\partial : \mathbb{U} \longrightarrow \mathbb{V}$ *be a strongly linear function with*

$$\partial(\log \mathfrak{m}) = \frac{\partial(\mathfrak{m})}{\mathfrak{m}} \quad \textit{for all } \mathfrak{m} \in \mathfrak{U}, \textit{ and}$$

$$\partial(L_{\omega^\mu}\,\mathfrak{a}) = \partial(\mathfrak{a})\,(\ell'_{\omega^\mu} \circ_{\mathbb{V}} \mathfrak{a}) \quad \textit{for all } 0 < \mu < \boldsymbol{\lambda} \textit{ and } \mathfrak{a} \in \mathfrak{U}_{\omega^\mu}.$$

Then ∂ *is a hyperserial derivation of force* $\boldsymbol{\nu}$.

Proof. By Proposition 3.5, the conditions **D2** and **D3** hold for ∂. We prove by induction on $\mu < \boldsymbol{\lambda}$ that we have $\partial(f \circ s) = \partial(s)\,f' \circ s$ for all $f \in \mathbb{L}_{<\omega^\mu}$ and $s \in \mathbb{U}^{>,\succ}$. Let $\mu < \boldsymbol{\lambda}$ such that the result holds strictly below μ and write $\alpha := \omega^\mu$. We claim, and we will prove in a moment, that for $\beta \in \{0\} \cup \{\omega^\eta : \eta \leqslant \mu\}$ and $s \in \mathbb{U}^{>,\succ}$, we have $\partial(\ell_\gamma \circ s) = \partial(s)\,\ell'_\gamma \circ s$. Notice that this is immediate for $\mu = 0$. For $\mu = 1$, it follows from **D3**. Assume that the claim is true. The chain rule is preserved by composition so we have $\partial(\ell_\gamma \circ s) = \partial(s)\,\ell'_\gamma \circ s$ for all $\gamma < \omega^{\mu+1}$ and $s \in \mathbb{U}^{>,\succ}$. Now let $\mathfrak{l} \in \mathfrak{L}_{<\omega^{\mu+1}}$ and $s \in \mathbb{U}^{>,\succ}$. We have

$$\begin{aligned}
\partial(\mathfrak{l} \circ s) &= \partial\left(\exp\left(\sum_{\gamma<\omega^{\mu+1}} \mathfrak{l}_\gamma\,\ell_{\gamma+1} \circ s\right)\right) \\
&= \partial\left(\sum_{\gamma<\omega^{\mu+1}} \mathfrak{l}_\gamma\,\ell_{\gamma+1} \circ s\right)\exp\left(\sum_{\gamma<\omega^{\mu+1}} \mathfrak{l}_\gamma\,\ell_{\gamma+1} \circ s\right) && \text{(by \textbf{D3})} \\
&= \partial(s)\left(\sum_{\gamma<\omega^{\mu+1}} \mathfrak{l}_\gamma\,\ell'_{\gamma+1} \circ s\right)\mathfrak{l} \circ s \\
&= \partial(s)\left(\left(\sum_{\gamma<\omega^{\mu+1}} \mathfrak{l}_\gamma\,\ell'_{\gamma+1}\right)\mathfrak{l}\right) \circ s \\
&= \partial(s)\,\mathfrak{l}' \circ s.
\end{aligned}$$

We conclude by strong linearity that $\partial(f\circ s)=\partial(s)\,f'\circ s$ for all $f\in\mathbb{L}_{<\omega^{\mu+1}}$ and $s\in\mathbb{U}^{>,\succ}$. So we may assume that $\mu>0$ and it is enough to prove the claim. Let $s\in\mathbb{U}^{>,\succ}$ and write $\mathfrak{a}:=\mathfrak{d}_\alpha(s)$ There is $\beta<\alpha$ such that $\varepsilon:=\ell_\beta\circ s-\ell_\beta\circ\mathfrak{a}$ is infinitesimal. We have $s=E_\beta(L_\beta(\mathfrak{a})+\varepsilon)$ so $\partial(s)=(\partial(\mathfrak{a})\,\ell'_\beta\circ\mathfrak{a}+\partial(\varepsilon))\,\ell_{<\beta}\circ s$ by the induction hypothesis. We deduce that

$$\partial(s)\,\ell'_\alpha\circ s=(\partial(\mathfrak{a})\,\ell'_\beta\circ\mathfrak{a}+\partial(\varepsilon))\,\frac{\ell'_\alpha}{\ell'_\beta}\circ s. \tag{5.1}$$

Note that $\ell'_\alpha\ell'^{-1}_\beta\in\mathfrak{L}_{[\beta,\alpha)}$ so we may consider $f:=(\ell'_\alpha\ell'^{-1}_\beta)^{\uparrow\beta}\in\mathbb{L}_{<\alpha}$. By the chain rule in $\mathbb{L}$, we have $(\ell_\alpha^{\uparrow\beta})'\circ\ell_\beta=\frac{1}{\ell'_\beta}(\ell_\alpha^{\uparrow\beta}\circ\ell_\beta)'=\frac{\ell'_\alpha}{\ell'_\beta}$, so $f=(\ell_\alpha^{\uparrow\beta})$. By **HF3**, we have

$$\frac{\ell'_\alpha}{\ell'_\beta}\circ s=f\circ(\ell_\beta\circ\mathfrak{a}+\varepsilon)=\frac{\ell'_\alpha}{\ell'_\beta}\circ\mathfrak{a}+\sum_{k\in\mathbb{N}}\frac{(\ell_\alpha^{\uparrow\beta})^{(k+1)}\circ\ell_\beta\circ\mathfrak{a}}{k!}\,\varepsilon^k.$$

Recall that

$$\ell_\alpha\circ s=\ell_\alpha\circ\mathfrak{a}+\sum_{k>0}\frac{(\ell_\alpha^{\uparrow\beta})^{(k)}\circ\ell_\beta\circ\mathfrak{a}}{k!}\,\varepsilon^k,$$

by (2.4). For $k\in\mathbb{N}^>$, we have $(\ell_\alpha^{\uparrow\beta})^{(k)}\in\mathbb{L}_{<\alpha}$ so we may apply the induction hypothesis at $\mu,\ell_\beta\circ\mathfrak{a}$, and obtain

$$\partial((((\ell_\alpha^{\uparrow\beta})^{(k)})\circ\ell_\beta\circ\mathfrak{a})=\partial(\ell_\beta\circ\mathfrak{a})\,(\ell_\alpha^{\uparrow\beta})^{(k+1)}\circ\ell_\beta\circ\mathfrak{a}.$$

Therefore

$$\begin{aligned}
\partial(\ell_\alpha\circ s) &= \partial(\ell_\alpha\circ\mathfrak{a})+\sum_{k>0}\frac{\partial((((\ell_\alpha^{\uparrow\beta})^{(k)})\circ\ell_\beta\circ\mathfrak{a})}{k!}\,\varepsilon^k+\partial(\varepsilon)\sum_{k>0}\frac{(\ell_\alpha^{\uparrow\beta})^{(k)}\circ\ell_\beta\circ\mathfrak{a}}{(k-1)!}\,\varepsilon^{k-1}\\
&= \partial(\mathfrak{a})\,\ell'_\alpha\circ\mathfrak{a}+\partial(\ell_\beta\circ\mathfrak{a})\sum_{k>0}\frac{(\ell_\alpha^{\uparrow\beta})^{(k+1)}\circ\ell_\beta\circ\mathfrak{a}}{k!}\,\varepsilon^k+\partial(\varepsilon)\sum_{k=0}\frac{(\ell_\alpha^{\uparrow\beta})^{(k+1)}\circ\ell_\beta\circ\mathfrak{a}}{k!}\,\varepsilon^k\\
&= \partial(\mathfrak{a})\,\ell'_\alpha\circ\mathfrak{a}+\partial(\mathfrak{a})\,\ell'_\beta\circ\mathfrak{a}\left(\frac{\ell'_\alpha}{\ell'_\beta}\circ s-\frac{\ell'_\alpha}{\ell'_\beta}\circ\mathfrak{a}\right)+\partial(\varepsilon)\,\frac{\ell'_\alpha}{\ell'_\beta}\circ s\\
&= (\partial(\mathfrak{a})\,\ell'_\beta\circ\mathfrak{a}+\partial(\varepsilon))\,\frac{\ell'_\alpha}{\ell'_\beta}\circ s\\
&= \partial(s)\,\ell'_\alpha\circ s. && \text{(by (5.1))}
\end{aligned}$$

By induction, this proves **D4** for ∂. □

5.2 H-fields

We are particularly interested in derivations which behave similarly to the derivation of germs in Hardy fields (see [5]). Thus we rely on the notion of H-field with small derivation of [1, 4].

Definition 5.4. *Let $\partial:\mathbb{U}\longrightarrow\mathbb{U}$ be a hyperserial derivation of force $\boldsymbol{\nu}$. We say that $(\mathbb{U},\partial)$ is an* **H-field** *if we have*

H1. *$\partial(s)>0$ for all $s\in\mathbb{U}^{>,\succ}$.*

We say that the derivation ∂ is **small** *if it satisfies*

H2. *$\partial(\varepsilon)\prec 1$ for all $\varepsilon\in\mathbb{U}^{\prec}$.*

If both **H1** *and* **H2** *hold, then we say that $(\mathbb{U},\partial)$ is an H-field with small derivation.*

Lemma 5.5. *Assume that $(\mathbb{U}, \partial)$ is an H-field. Then we have*

i. $\mathrm{Ker}(\partial) = \mathbb{R}$.

ii. For all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{U}^{\neq}$ *we have* $\mathfrak{m} \succ \mathfrak{n} \Longrightarrow \partial(\mathfrak{m}) \succ \partial(\mathfrak{n})$.

iii. For all $\mathfrak{m}, \mathfrak{n} \in \mathfrak{U}^{\neq}$, *we have* $\mathfrak{m} \not\asymp \mathfrak{n} \Longrightarrow \mathfrak{m}^\dagger \not\asymp \mathfrak{n}^\dagger$.

Proof. The assertion *ii* follows as in the proof of [8, Proposition 64(1)]. We next prove *i*. We have $\partial(1) = \partial(1)^2 = 2\,\partial(1)$ by the Leibniz rule, so $\partial(1) = 0$. So for $r \in \mathbb{R}$, we have $\partial(r) = r\,\partial(1) = 0$, whence $\mathbb{R} \subseteq \mathrm{Ker}(\partial)$. Conversely, let $z \in \mathrm{Ker}(\partial)$ and write $z = z_\succ + r + z_\prec$ where $\mathrm{supp}\, z_\succ \succ 1$, $r \in \mathbb{R}$ and $z_\prec \prec 1$. Assume for contradiction that $z_\succ \neq 0$. Then by *ii*, we have $\partial(z) \sim \partial(\tau_z)$. We have $\partial(|\tau_z|) > 0$ by **H1**, so we deduce that $|\partial(z)| = \partial(|z|) \neq 0$: a contradiction. So $z_\succ = 0$, whence $\partial(z) = \partial(z_\prec)$. Assume for contradiction that $z_\prec \neq 0$. By **H1**, we have $\partial((|z_\prec|)^{-1}) > 0$ and $|\partial(z_\prec)| = \partial(|z_\prec|) = \frac{\partial(|z_\prec|)^{-1}}{|z_\prec|^2} > 0$: a contradiction. So $z_\prec = 0$, whence $z = r \in \mathbb{R}$. We now prove *iii*. Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{U}^{\succ}$ with $\mathfrak{m} \not\asymp \mathfrak{n}$. By Lemma 1.30, we have $\log \mathfrak{m} \not\asymp \log \mathfrak{n}$. Since $\mathrm{supp} \log \mathfrak{m} \cup \mathrm{supp} \log \mathfrak{n} \succ 1$, we have $\mathfrak{m}^\dagger = \partial(\log \mathfrak{m}) \not\asymp \partial(\log \mathfrak{n}) = \mathfrak{n}^\dagger$. □

In particular, if $(\mathbb{U}, \partial)$ is an H-field according to Definition 5.4, then it is an H-field according to [1, Definition, p 3].

Example 5.6. The standard derivation $' : \mathbb{L}_{<\boldsymbol{\lambda}} \longrightarrow \mathbb{L}_{<\boldsymbol{\lambda}}$ of [15] is a hyperserial derivation of force $\boldsymbol{\nu}$ and $(\mathbb{L}_{<\boldsymbol{\lambda}}, ')$ is an H-field with small derivation [15, Theorem 1.2].

Our goal in this section is to prove the following theorem:

Theorem 5.7. *Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ and $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ with $\boldsymbol{\mu} > 0$. Let $(\mathbb{U}, \circ)$, $(\mathbb{V}, \circ)$ be hyperserial fields of force $\boldsymbol{\nu}$ with $(\mathbb{U}, \circ) \subseteq (\mathbb{V}, \circ)$. Let $\partial : \mathbb{U} \longrightarrow \mathbb{V}$ be a hyperserial derivation of force $\boldsymbol{\nu}$. Assume that ∂ has a good near-support $\mathfrak{W}_\partial$. Then ∂ extends uniquely into a hyperserial derivation $\partial_{\boldsymbol{\mu}} : \mathbb{U}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{U}_{(<\boldsymbol{\mu})}$ of force $\boldsymbol{\nu}$, and $\mathfrak{W}_\partial$ is a near-support for $\partial_{\boldsymbol{\mu}}$. Moreover, if $(\mathbb{U}, \partial)$ is an H-field with small derivation, then so is $(\mathbb{U}_{(<\boldsymbol{\mu})}, \partial_{\boldsymbol{\mu}})$.*

5.3 Defining the derivation

Fix $(\mathbb{U}, \circ)$, $(\mathbb{V}, \circ)$, ∂ and $\mathfrak{W}_\partial$ as in Theorem 5.7. We inductively define $\partial_{\boldsymbol{\mu}}$ along with the class $\mathcal{D}_\partial$ of series $s \in \mathbb{U}_{(<\boldsymbol{\mu})}$ at which it is defined. The induction is on the exponential height $(\gamma, \boldsymbol{\eta})$ of s over $\mathbb{U}$. We say that $\partial_{\boldsymbol{\mu}}$ is *defined at* $s \in \mathbb{U}_{(<\boldsymbol{\mu})}$ if

i. $s \in \mathbb{U}$. We then set

$$\partial_{\boldsymbol{\mu}}(s) := \partial(s). \tag{5.2}$$

ii. $s \in \mathfrak{U}_{(<\boldsymbol{\mu})}$, $\boldsymbol{\eta} = 1$ and $\log s \in \mathcal{D}_\partial$. We then set

$$\partial_{\boldsymbol{\mu}}(s) := \partial_{\boldsymbol{\mu}}(\log s)\, s. \tag{5.3}$$

iii. $s \in \mathfrak{U}_{(<\boldsymbol{\mu})}$, $\boldsymbol{\eta} = \iota + 1$ for a $\iota > 0$ and $s = \mathfrak{t}\,\mathfrak{m}$ where $\mathfrak{m} \in \mathcal{D}_\partial$ and $\mathrm{hsupp}\, \mathfrak{t} \subseteq \mathcal{D}_\partial$, and the family $(\partial_{\boldsymbol{\mu}}(\varphi)\, E'_{\omega^\iota}(\varphi) \log(\mathfrak{t}_\varphi)' \circ (E_{\omega^\iota}(\varphi)))_{\varphi \in \mathrm{hsupp}\, \mathfrak{t}}$ is well-based. Then we set

$$\partial_{\boldsymbol{\mu}}(s) := \left(\sum_{\varphi \in \mathrm{hsupp}\, \mathfrak{t}} \partial_{\boldsymbol{\mu}}(\varphi)\, E'_{\omega^\iota}(\varphi) \log(\mathfrak{t}_\varphi)' \circ (E_{\omega^\iota}(\varphi)) \right) s\,\mathfrak{m} + s\,\partial_{\boldsymbol{\mu}}(\mathfrak{m}). \tag{5.4}$$

In the case when $\mathfrak{t}\,\mathfrak{m} = L_\gamma(E^\varphi_{\omega^\iota})$ (i.e $\mathfrak{m} = 1$ and $\mathrm{hsupp}\, \mathfrak{t} = \{\varphi\}$), we have

$$\partial_{\boldsymbol{\mu}}(L_\gamma(E^\varphi_{\omega^\iota})) = \partial_{\boldsymbol{\mu}}(\varphi)\, E'_{\omega^\iota}(\varphi)\, \ell'_\gamma \circ E_{\omega^\iota}(\varphi). \tag{5.5}$$

iv. $s \in \mathbb{U}_{(<\boldsymbol{\mu})} \setminus \mathfrak{U}_{(<\boldsymbol{\mu})}$ and $\operatorname{supp} s \subseteq \mathcal{D}_\partial$, and the family $(\partial_{\boldsymbol{\mu}}(\mathfrak{m}))_{\mathfrak{m}\in\operatorname{supp} s}$ is well-based. We then set

$$\partial_{\boldsymbol{\mu}}(s) := \sum_{\mathfrak{m}\in\operatorname{supp} s} s_{\mathfrak{m}}\,\partial_{\boldsymbol{\mu}}(\mathfrak{m}). \tag{5.6}$$

Proposition 5.8. *Let $\eta < \boldsymbol{\nu}$ with, set $\alpha := \omega^\eta$ and let $\mathfrak{m} \in (\mathbb{U}_{<\boldsymbol{\mu}})_\alpha$. We have $\mathfrak{m} \in \mathcal{D}_\partial \iff L_\alpha(\mathfrak{m}) \in \mathcal{D}_\partial$. Moreover, if $\mathfrak{m} \in \mathcal{D}_\partial$ then $\partial_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_\alpha \circ \mathfrak{m}$.*

Proof. We prove this by induction on $\mathrm{EH}_{\mathbb{U}}(\mathfrak{m})$. Let $(\gamma, \boldsymbol{\eta}) := \mathrm{EH}_{\mathbb{U}}(\mathfrak{m})$. If $(\gamma, \boldsymbol{\eta}) = (0,0)$, then $\mathfrak{m} \in \mathbb{U}$ and $L_\alpha(\mathfrak{m}) \in \mathbb{U}$ and $\partial_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = \partial(L_\alpha(\mathfrak{m})) = \partial(\mathfrak{m})\,\ell'_\alpha \circ \mathfrak{m} = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_\alpha \circ \mathfrak{m}$.

Assume that $(\gamma, \boldsymbol{\eta}) >_{\mathrm{lex}} (0,0)$. Then since $\mathfrak{m}$ is a monomial, we know that $\boldsymbol{\eta} = \iota + 1$ is a successor. Write $\beta := \omega^\iota$. We have $\mathfrak{m} = E_\beta^\varphi$ where $\varphi := L_\beta(\mathfrak{m}) \in (\mathbb{U}_{<\boldsymbol{\mu}})_{\succ,\beta}$ and $\mathrm{EH}_{\mathbb{U}}(\varphi) < (\gamma, \boldsymbol{\eta})$. Note that $\alpha \leqslant \beta\omega$ by Corollary 4.7.

Assume first that $\alpha = 1$. If $\beta = 1$, then we have $\mathfrak{m} \in \mathcal{D}_\partial \iff \varphi = L_1(\mathfrak{m}) \in \mathcal{D}_\partial$ by ii. By (5.3) we have $\partial_{\boldsymbol{\mu}}(\mathfrak{m}) = \partial_{\boldsymbol{\mu}}(\varphi)\,\mathfrak{m}$, whence $\partial_{\boldsymbol{\mu}}(L_1(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\varphi)\,\ell'_1 \circ \mathfrak{m}$, if applicable. Recall that $\mathcal{D}_\partial$ is a group, so $\mathcal{D}_\partial - 1 = \mathcal{D}_\partial$. If $\beta = \omega$, then we have $\mathfrak{m} \in \mathcal{D}_\partial \iff \varphi \in \mathcal{D}_\partial \iff \varphi - 1 \in \mathcal{D}_\partial$ by iii. If $L_1(\mathfrak{m}) = E_\omega^{\varphi-1} \in \mathbb{U}$, then we have $\varphi - 1 \in \mathbb{U}$, and $\mathfrak{m} \in \mathcal{D}_\partial$ by ii. Moreover (6.2) yields

$$\partial_{\boldsymbol{\mu}}(L_1(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\varphi - 1)\,E'_\omega(\varphi - 1) = \partial_{\boldsymbol{\mu}}(\varphi)\,\frac{E'_\omega(\varphi)}{E_\omega^\varphi} = \frac{\partial_{\boldsymbol{\mu}}(\mathfrak{m})}{\mathfrak{m}} = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_1 \circ \mathfrak{m}.$$

If $E_\beta^{\varphi-1} \notin \mathbb{U}$, then $E_\beta^{\varphi-1} \in \mathcal{D}_\partial \iff \varphi - 1 \in \mathcal{D}_\partial \iff \mathfrak{m} \in \mathcal{D}_\partial$ by iii. We conclude as above that $\partial_{\boldsymbol{\mu}}(L_1(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_1 \circ \mathfrak{m}$.

If $\beta > \omega$, then we are in case iii and we have $\mathfrak{m} \in \mathcal{D}_\partial \iff \varphi \in \mathcal{D}_\partial \iff L_1(E_\beta^\varphi) \in \mathcal{D}_\partial$. We conclude with (6.4) that $\partial_{\boldsymbol{\mu}}(L_1(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_1 \circ \mathfrak{m}$ if applicable.

This treats the case when $\alpha = 1$. Assume now that $\alpha = \omega$ and $\beta = 1$. We have $\mathfrak{m} \in \mathcal{D}_\partial \iff \varphi \in \mathcal{D}_\partial$ by ii. Since φ is log-atomic, the induction hypothesis yields $\mathfrak{m} \in \mathcal{D}_\partial \iff L_\omega(\varphi) \in \mathcal{D}_\partial$, with $\partial_{\boldsymbol{\mu}}(L_\omega(\varphi)) = \partial_{\boldsymbol{\mu}}(\varphi)\,\ell'_\omega \circ \varphi$ if applicable. We have $L_\omega(\varphi) = L_\omega(\mathfrak{m}) - 1$ so $\mathfrak{m} \in \mathcal{D}_\partial \iff L_\omega(\mathfrak{m}) \in \mathcal{D}_\partial$. Moreover, we have

$$\begin{aligned} \partial_{\boldsymbol{\mu}}(L_\omega(\mathfrak{m})) &= \partial_{\boldsymbol{\mu}}(L_\omega(\varphi)) = \partial_{\boldsymbol{\mu}}(\varphi)\,\ell'_\omega \circ \varphi \\ \partial_{\boldsymbol{\mu}}(\mathfrak{m}) &= \partial_{\boldsymbol{\mu}}(\varphi)\,\mathfrak{m} \quad \text{if } \mathfrak{m}, \varphi \in \mathcal{D}_\partial, \text{ whence} \end{aligned}$$

$$\partial_{\boldsymbol{\mu}}(L_\omega(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\frac{\ell'_\omega \circ \varphi}{\mathfrak{m}} = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\frac{\ell'_\omega \circ \ell_1}{\ell_0} \circ_{\mathbb{V}} \mathfrak{m} = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_\omega \circ \mathfrak{m}$$

if applicable.

In all other cases, the inequality $\alpha\,\omega \leqslant \beta$ implies that $\beta > 1$, so we will only deal with the cases i and iii. Moreover, we have $\mathfrak{m} \in \mathcal{D}_\partial \iff \varphi \in \mathcal{D}_\partial$ by iii.

If $\alpha = \beta\omega$, then φ is $L_{<\alpha}$-atomic. The induction hypothesis yields $\varphi \in \mathcal{D}_\partial \iff L_\alpha(\varphi) \in \mathcal{D}_\partial$, and if applicable. If $\varphi \in \mathcal{D}_\partial$, then we have $\partial_{\boldsymbol{\mu}}(\mathfrak{m}) = \partial_{\boldsymbol{\mu}}(\varphi)\,E'_\beta(\varphi)$ by (5.5). Since $L_\alpha(\varphi) = L_\alpha(\mathfrak{m}) - 1$, we obtain

$$\partial_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(L_\alpha(\varphi)) = \partial_{\boldsymbol{\mu}}(\varphi)\,\ell'_\alpha \circ \varphi = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\frac{\ell'_\alpha \circ \ell_\beta}{\ell_{\beta'}} \circ \mathfrak{m} = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_\alpha \circ \mathfrak{m},$$

if applicable.

If $\alpha = \beta$, then $L_\alpha(\mathfrak{m}) = \varphi$ so by iii, we have $L_\alpha(\mathfrak{m}) \in \mathcal{D}_\partial \iff \mathfrak{m} \in \mathcal{D}_\partial$. If applicable, we have $\partial_{\boldsymbol{\mu}}(\mathfrak{m}) = \partial_{\boldsymbol{\mu}}(\varphi)\,E'_\alpha(\varphi) = \partial_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m}))\,\frac{1}{\ell'_\alpha} \circ \mathfrak{m}$ by (5.5), so $\partial_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_\alpha \circ \mathfrak{m}$.

If $\beta = \alpha\,\omega$, then $L_\alpha(\mathfrak{m}) = E_\beta^{\varphi-1}$. If $E_\beta^{\varphi-1} \notin \mathbb{U}$, then by iii, we have $L_\alpha(\mathfrak{m}) \in \mathcal{D}_\partial \iff \varphi - 1 \in \mathcal{D}_\partial \iff \mathfrak{m} \in \mathcal{D}_\partial$. If applicable, we have $\partial_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\varphi - 1)\,\frac{1}{\ell'_\beta \circ \ell_\alpha} \circ \mathfrak{m} = \partial_{\boldsymbol{\mu}}(\varphi)\,\frac{\ell'_\alpha}{\ell'_\beta} \circ \mathfrak{m}$ and $\partial_{\boldsymbol{\mu}}(\mathfrak{m}) = \partial_{\boldsymbol{\mu}}(\varphi)\,\frac{1}{\ell'_\beta} \circ \mathfrak{m}$ by (5.5). Therefore $\partial_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_\alpha \circ \mathfrak{m}$. If $E_\beta^{\varphi-1} \in \mathbb{U}$, then $\varphi \in \mathbb{U}$, so $\varphi \in \mathcal{D}_\partial$, so $\mathfrak{m} \in \mathcal{D}_\partial$. We obtain $\partial_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\,\ell'_\alpha \circ \mathfrak{m}$ by the same arguments.

If $\beta > \alpha\omega$, then we have $L_\alpha(\mathfrak{m}) = L_\alpha(E_\beta^\varphi)$. Since $\alpha \geqslant \omega$, the series $L_\alpha(\mathfrak{m})$ occurs in iii, and we have $L_\alpha(\mathfrak{m}) \in \mathcal{D}_\partial \iff \varphi \in \mathcal{D}_\partial \iff \mathfrak{m} \in \mathcal{D}_\partial$. If applicable, we have $\partial_{\boldsymbol{\mu}}(\mathfrak{m}) = \partial_{\boldsymbol{\mu}}(\varphi)\, E'_\beta(\varphi) = \partial_{\boldsymbol{\mu}}(\varphi)\, \frac{1}{\ell'_\beta} \circ \mathfrak{m}$ and

$$\partial_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\, E'_\beta(\varphi)\, \ell'_\alpha \circ \mathfrak{m} = \partial_{\boldsymbol{\mu}}(\varphi)\, \frac{\ell'_\alpha}{\ell'_\beta} \circ \mathfrak{m} = \partial_{\boldsymbol{\mu}}(\mathfrak{m})\, \ell'_\alpha \circ \mathfrak{m}.$$

This concludes the proof. □

Lemma 5.9. *For $s, t \in \mathcal{D}_\partial$ with $s\,t \in \mathcal{D}_\partial$, we have $\partial_{\boldsymbol{\mu}}(s\,t) = \partial_{\boldsymbol{\mu}}(s)\,t + s\,\partial_{\boldsymbol{\mu}}(t)$.*

Proof. We have $\operatorname{supp} s \cup \operatorname{supp} t \cup \operatorname{supp} s\,t \subseteq \mathcal{D}_\partial$ by iv. So as in the proof of Proposition 5.3, by (6.5), it is enough to prove the result for $\mathfrak{s}, \mathfrak{t} \in \mathcal{D}_\partial \cap \mathfrak{U}_{(<\boldsymbol{\mu})}$. By Proposition 5.8, we have $\partial_{\boldsymbol{\mu}}(\log \mathfrak{m}) = \frac{\partial_{\boldsymbol{\mu}}(\mathfrak{m})}{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathcal{D}_\partial \cap \mathfrak{U}_{(<\boldsymbol{\mu})}$. Thus

$$\partial_{\boldsymbol{\mu}}(\mathfrak{s}\,\mathfrak{t}) = \partial_{\boldsymbol{\mu}}(\log(\mathfrak{s}\,\mathfrak{t}))\,\mathfrak{s}\,\mathfrak{t} = (\partial_{\boldsymbol{\mu}}(\log \mathfrak{s}) + \partial_{\boldsymbol{\mu}}(\log \mathfrak{t}))\,\mathfrak{s}\,\mathfrak{t} = \left(\frac{\partial_{\boldsymbol{\mu}}(\mathfrak{s})}{\mathfrak{s}} + \frac{\partial_{\boldsymbol{\mu}}(\mathfrak{t})}{\mathfrak{t}}\right)\mathfrak{s}\,\mathfrak{t} = \partial_{\boldsymbol{\mu}}(\mathfrak{s})\,\mathfrak{t} + \mathfrak{s}\,\partial_{\boldsymbol{\mu}}(\mathfrak{t}).$$ □

Proposition 5.10. *Let $\mathbb{G} = \mathbb{R}[[\mathfrak{S}]] \subseteq (\mathbb{U}_{(<\boldsymbol{\mu})})$ be a transserial subgroup with $\mathbb{G} \subseteq \mathcal{D}_\partial$. Then $\mathbb{G}_{(<1)} \subseteq \mathcal{D}_\partial$.*

Proof. This follows from Proposition 3.4. □

5.4 The near-support

We write $\mathcal{P}_\partial$ for the subclass of $\mathcal{D}_\partial$ of series s such that for all $\mathfrak{n} \in \operatorname{supp} s$, we have $\operatorname{supp} \partial_{\boldsymbol{\mu}}(\mathfrak{n}) \subseteq \mathfrak{n} \cdot \mathfrak{V}^{\prec \mathfrak{n}} \cdot \mathfrak{W}_\partial$. For $s \in \mathcal{D}_\partial$, we have $s \in \mathcal{P}_\partial \iff \operatorname{supp} s \subseteq \mathcal{P}_\partial$.

Lemma 5.11. *For $\varphi \in (\mathbb{U}_{(<\boldsymbol{\mu})})_\succ \cap \mathcal{P}_\partial$, we have $E_1^\varphi \in \mathcal{P}_\partial$.*

Proof. We have $\partial_{\boldsymbol{\mu}}(E_1^\varphi) = \partial_{\boldsymbol{\mu}}(\varphi)\, E_1^\varphi$ by Proposition 5.8. Let $\mathfrak{m} \in \operatorname{supp} \partial_{\boldsymbol{\mu}}(E_1^\varphi)$. So there is a $\mathfrak{n} \in \operatorname{supp} \partial_{\boldsymbol{\mu}}(\varphi)$ with $\mathfrak{m} = \mathfrak{n}\, E_1^\varphi$. We have $\mathfrak{n} \in \mathfrak{p} \cdot \mathfrak{V}^{\prec \mathfrak{p}} \cdot \mathfrak{W}_\partial$ for a certain $\mathfrak{p} \in \operatorname{supp} \varphi$. We have $E_1^\varphi \gg \operatorname{supp} \varphi \gg 1$, so $\mathfrak{p} \cdot \mathfrak{V}^{\prec \mathfrak{p}} \subseteq \mathfrak{V}^{\prec E_1^\varphi}$, so $\mathfrak{m} \in E_1^\varphi \cdot \mathfrak{V}^{\prec E_1^\varphi} \cdot \mathfrak{W}_\partial$ as desired. □

Proposition 5.12. *We have $\mathcal{D}_\partial = \mathcal{P}_\partial$.*

Proof. We prove by induction on $\mathrm{EH}_{\mathbb{U}}(s)$ that $\forall s \in \mathbb{U}_{(<\boldsymbol{\mu})}$, $s \in \mathcal{D}_\partial \Longrightarrow s \in \mathcal{P}_\partial$. This is immediate if $\mathrm{EH}_{\mathbb{U}}(s) = (0, 0)$ by our hypothesis that $\mathfrak{W}_\partial$ is a near-support for ∂. Let $s \in \mathbb{U}_{(<\boldsymbol{\mu})}$, set $\mathrm{EH}_{\mathbb{U}}(s) =: (\gamma, \boldsymbol{\eta})$ and assume that we have $g \in \mathcal{P}_\partial$ for all $g \in \mathcal{D}_\partial$ with $\mathrm{EH}_{\mathbb{U}}(g) < (\gamma, \boldsymbol{\eta})$. It is enough to prove that $\operatorname{supp} s \subseteq \mathcal{P}_\partial$, so we may assume that s is a monomial.

If $\boldsymbol{\eta} = 1$, then $s = E_1^\varphi$ where $\varphi \in (\mathbb{U}_{(\gamma,0)})_\succ$, and $\mathrm{EH}_{\mathbb{U}}(\varphi) < (\gamma, \boldsymbol{\eta})$. We conclude with Lemma 5.11 that $s \in \mathcal{P}_\partial$.

If $\boldsymbol{\eta} = \iota + 1$ where $\iota > 0$, then $s = \mathfrak{t}\,\mathfrak{m}$ where in particular $\operatorname{hsupp} \mathfrak{t} \subseteq \mathcal{P}_\partial$. By Lemma 5.11, it is enough to prove that $\operatorname{supp} \log s \subseteq \mathcal{P}_\partial$. So it is enough to show that for $\varphi \in \operatorname{hsupp} \mathfrak{t}$ and $\gamma < \boldsymbol{\lambda}$ with $\gamma\omega < \omega^\iota$, we have $L_\gamma(E_{\omega^\iota}^\varphi) \in \mathcal{P}_\partial$. Given such a monomial $\mathfrak{a} := L_\gamma(E_{\omega^\iota}^\varphi)$, we have $\partial_{\boldsymbol{\mu}}(\mathfrak{a}) = \partial_{\boldsymbol{\mu}}(\varphi)\, E'_{\omega^\iota}(\varphi)\, \ell'_\gamma \circ E_{\omega^\iota}(\varphi)$ by (5.5). Let $\mathfrak{m} \in \operatorname{supp} \partial_{\boldsymbol{\mu}}(\mathfrak{a})$. So there is $\mathfrak{n} \in \operatorname{supp} \partial_{\boldsymbol{\mu}}(\varphi)$ with $\mathfrak{m} = \mathfrak{n}\, E'_{\omega^\iota}(\varphi)\, \ell'_\gamma \circ E_{\omega^\iota}(\varphi)$. Write

$$\mathfrak{q} := E'_{\omega^\iota}(\varphi) \times (\ell_\gamma^\dagger \circ E_{\omega^\iota}(\varphi)),$$

so $\mathfrak{m} = \mathfrak{n}\,\mathfrak{a}\,\mathfrak{q}$. Note that $1 \ll \mathfrak{q} \leqq L_{\gamma+1}(E_{\omega^\iota}^\varphi) \ll \mathfrak{a}$, so $\mathfrak{q} \in \mathfrak{V}^{\prec \mathfrak{a}}$. The induction hypothesis at φ yields $\mathfrak{n} \in \mathfrak{p} \cdot \mathfrak{V}^{\prec \mathfrak{p}} \cdot \mathfrak{W}_\partial$ for a certain $\mathfrak{p} \in \operatorname{supp} \varphi$. We have $\operatorname{supp} \varphi \succ (L_{<\omega^\iota}(E_{\omega^\iota}^\varphi))^{-1}$ since φ is ω^ι-truncated. In particular $\operatorname{supp} \varphi \gg (L_\gamma(E_{\omega^\iota}^\varphi))^{-1} = \mathfrak{a}^{-1}$. Moreover $\varphi \ll \mathfrak{a}$, so $\mathfrak{q}\,\mathfrak{p} \in \mathfrak{V}^{\prec \mathfrak{a}}$. We deduce that $\mathfrak{m} \in \mathfrak{a} \cdot \mathfrak{V}^{\prec \mathfrak{a}} \cdot \mathfrak{W}_\partial$, so $\mathfrak{m} \in \mathcal{P}_\partial$. This concludes the proof. □

Lemma 5.13. *Assume that ∂ has a good near-support $\mathfrak{W}_\partial$ with $\partial(\mathfrak{W}_\partial) \subseteq \mathbb{R}[[\mathfrak{W}_\partial]]$. Then $\mathfrak{W}_\partial$ is a near-support for ∂^k for all $k \in \mathbb{N}$.*

Proof. We prove this by induction on k. For $k \leqslant 1$, this is immediate. Let $k \in \mathbb{N}$ such that the result holds at k, let $s \in \mathbb{U}$ and let $\mathfrak{m} \in \operatorname{supp} s^{(k+1)}$. There is $\mathfrak{n} \in \operatorname{supp} s^{(k)}$ with $\mathfrak{m} \in \operatorname{supp} \partial(\mathfrak{n})$. Now the induction hypothesis yields a $\mathfrak{p}_k \in \mathfrak{U}$ with $\mathfrak{p}_k \prec\!\!\!\prec \mathfrak{d}_s$ with $\mathfrak{n} \in (\mathfrak{d}_s \mathfrak{p}_k) \cdot \mathfrak{W}_\partial$. We deduce by the Leibniz rule that

$$\begin{aligned}
\mathfrak{m} &\in (\operatorname{supp} \partial(\mathfrak{d}_s)) \cdot \mathfrak{p}_k \cdot \mathfrak{W}_\partial \quad \text{or} \\
\mathfrak{m} &\in \mathfrak{d}_s \cdot (\operatorname{supp} \partial(\mathfrak{p}_k)) \cdot \mathfrak{W}_\partial \quad \text{or} \\
\mathfrak{m} &\in \mathfrak{d}_s \cdot \mathfrak{p}_k \cdot \left(\bigcup_{\mathfrak{w} \in \mathfrak{M}_\partial} \operatorname{supp} \partial(\mathfrak{w}) \right).
\end{aligned}$$

Recall that $\mathfrak{W}_\partial$ is a near-support for ∂. Thus, in the first case, we have

$$\mathfrak{m} \in (\mathfrak{d}_s \mathfrak{p}_k) \cdot \mathfrak{V}^{\prec\!\!\!\prec \mathfrak{d}_s} \cdot \mathfrak{W}_\partial^2 \subseteq \mathfrak{d}_s \cdot \mathfrak{V}^{\prec\!\!\!\prec \mathfrak{d}_s} \cdot \mathfrak{W}_\partial.$$

In the second case, we have $\mathfrak{m} \in (\mathfrak{d}_s \mathfrak{p}_k) \cdot \mathfrak{V}^{\prec\!\!\!\prec \mathfrak{p}_k} \cdot \mathfrak{W}_\partial^2$. We have $\mathfrak{V}^{\prec\!\!\!\prec \mathfrak{p}_k} \subseteq \mathfrak{V}^{\prec\!\!\!\prec \mathfrak{d}_s}$ so $\mathfrak{m} \in (\mathfrak{d}_s \mathfrak{p}_{k+1}) \cdot \mathfrak{W}_\partial$ as desired. In the third case, we have $(\bigcup_{\mathfrak{w} \in \mathfrak{M}_\partial} \operatorname{supp} \partial(\mathfrak{w})) \subseteq \mathfrak{W}_\partial$ so we can conclude directly. This proves that $\mathfrak{W}_\partial$ is a near-support for ∂^k. □

Proposition 5.14. *The standard derivation $\partial : \mathbb{L}_{<\boldsymbol{\lambda}} \longrightarrow \mathbb{L}_{<\boldsymbol{\lambda}}$ has a good near-support $\mathfrak{W}_\partial$ with $\partial(\mathfrak{W}_\partial) \subseteq \mathbb{R}[[\mathfrak{W}_\partial]]$.*

Proof. Set $\mathfrak{W}_\partial := (\operatorname{supp} \partial)^\infty = \{\ell_\gamma^\dagger : \gamma < \boldsymbol{\lambda}\}^\infty$. For $\mathfrak{m} \in \mathfrak{W}_\partial$, we have $\operatorname{supp} \partial(\mathfrak{m}) \subseteq \mathfrak{m} \cdot \mathfrak{W}_\partial \subseteq \mathfrak{W}_\partial$, so $\partial(\mathfrak{W}_\partial) \subseteq \mathbb{R}[[\mathfrak{W}_\partial]]$. □

5.5 Extending hyperserial derivations

Proposition 5.15. *We have $\mathbb{E}(\mathcal{P}_\partial) \subseteq \mathcal{P}_\partial$.*

Proof. Since any monomial in $\mathbb{E}(\mathcal{P}_\partial)$ is atomic, this follows from Lemma 1.36. □

Proposition 5.16. *We have $(\mathbb{G}(\mathcal{P}_\partial))_{(<1)} \subseteq \mathcal{P}_\partial$.*

Proof. The class $\mathcal{P}_\partial$ is closed under sums so by Proposition 5.15, we have $\mathbb{G}(\mathcal{P}_\partial) \subseteq \mathcal{P}_\partial$. Lemma 5.9 and Proposition 5.8 imply that $\partial_{\boldsymbol{\mu}} \restriction \mathbb{G}(\mathcal{P}_\partial)$ is a transserial derivation on $\mathbb{G}(\mathcal{P}_\partial)$. We deduce by Proposition 5.10 that $\partial_{\boldsymbol{\mu}} \restriction \mathbb{G}(\mathcal{P}_\partial)$ extends uniquely into a transserial derivation on the field $(\mathbb{G}(\mathcal{P}_\partial))_{(<1)}$. An easy induction using Proposition 5.8 shows that this extension coincides with $\partial_{\boldsymbol{\mu}}$ on $(\mathbb{G}(\mathcal{P}_\partial))_{(<1)}$. Therefore $(\mathbb{G}(\mathcal{P}_\partial))_{(<1)} \subseteq \mathcal{D}_\partial$. We conclude with Proposition 5.12. □

Corollary 5.17. *We have $\mathcal{P}_\partial = \mathbb{U}_{(<\boldsymbol{\mu})}$.*

Proof. Apply Lemma 4.10. □

Corollary 5.18. *The function $\partial_{\boldsymbol{\mu}} : \mathbb{U}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{V}_{(<\boldsymbol{\mu})}$ is a hyperserial derivation of force $\boldsymbol{\nu}$ and $\mathfrak{W}_\partial$ is a good near-support for $\partial_{\boldsymbol{\mu}}$.*

Proof. We already know since $\mathbb{U}_{(<\boldsymbol{\mu})} \subseteq \mathcal{P}_\partial$ that $\partial_{\boldsymbol{\mu}}$ is strongly linear on $\mathbb{U}_{(<\boldsymbol{\mu})}$ with near-support $\mathfrak{W}_\partial$. It satisfies the Leibniz rule by Lemma 5.9. Finally, by Propositions 5.8 and 5.3, it is a hyperserial derivation of force $\boldsymbol{\nu}$. □

Assume that $\mathbb{U}$ is an H-field. Let $\mathcal{P}_H$ denote the subclass of $\mathbb{U}_{(<\boldsymbol{\mu})}$ of series s with

$$\forall \mathfrak{m},\mathfrak{n}\in \operatorname{supp} s\setminus\{1\}, \mathfrak{m}\succ\mathfrak{n}\wedge\mathfrak{m}\succ 1 \implies \partial_{\boldsymbol{\mu}}(\mathfrak{m})\succ\partial_{\boldsymbol{\mu}}(\mathfrak{n}), \quad \text{and} \tag{5.7}$$

$$s>\mathbb{R} \implies \partial_{\boldsymbol{\mu}}(s)>0. \tag{5.8}$$

Note that $\mathcal{P}_H$ contains $\mathbb{U}$ by **H1** and Lemma 5.5.

Proposition 5.19. *Let $\mathbb{G}=\mathbb{R}[[\mathfrak{S}]]$ be a transserial subgroup of $\mathbb{U}_{(<\boldsymbol{\mu})}$ with $\mathfrak{S}=\mathfrak{S}^{-1}$ and $\mathbb{G}\subseteq\mathcal{P}_H$. We have $\mathbb{G}^{\exp}\subseteq\mathcal{P}_H$.*

Proof. Let $s\in\mathbb{G}^{\exp}$ and $\mathfrak{m},\mathfrak{n}\in\operatorname{supp} s\setminus\{1\}$ with $\mathfrak{m}\succ 1,\mathfrak{n}$. Note that $\log\mathfrak{m},\log\mathfrak{n}\in\mathbb{G}\subseteq\mathcal{P}_H$. We have $\operatorname{supp}\log\mathfrak{m}\cup\operatorname{supp}\log\mathfrak{n}\succ 1$ so (5.7) for $\log\mathfrak{m}$ and $\log\mathfrak{n}$ yields

$$\begin{aligned}\partial_{\boldsymbol{\mu}}(\mathfrak{m}) &\sim \partial_{\boldsymbol{\mu}}(\tau_{\log\mathfrak{m}})\,\mathfrak{m} \quad \text{and}\\ \partial_{\boldsymbol{\mu}}(\mathfrak{n}) &\sim \partial_{\boldsymbol{\mu}}(\tau_{\log\mathfrak{n}})\,\mathfrak{n}.\end{aligned}$$

Assume that $\mathfrak{n}\succ 1$. Then $\log\mathfrak{m}\succcurlyeq\log\mathfrak{n}$, so (5.7) for $\mathfrak{d}_{\log\mathfrak{m}}+\mathfrak{d}_{\log\mathfrak{n}}$ yields $\partial_{\boldsymbol{\mu}}(\tau_{\log\mathfrak{m}})\succcurlyeq\partial_{\boldsymbol{\mu}}(\tau_{\log\mathfrak{n}})$. Therefore $\partial_{\boldsymbol{\mu}}(\mathfrak{m})\succ\partial_{\boldsymbol{\mu}}(\mathfrak{n})$. Assume now that $\mathfrak{n}\prec 1$. Set $\mathfrak{v}:=\frac{1}{\mathfrak{d}_{\log\mathfrak{n}}}$. We have $\mathfrak{v}\in\mathfrak{S}$ by our hypothesis on $\mathfrak{S}$. We claim that $\partial_{\boldsymbol{\mu}}(\mathfrak{v})\succ\partial_{\boldsymbol{\mu}}(\mathfrak{n})$. Indeed, by (5.7) for $\log\mathfrak{n}$, we have $\partial_{\boldsymbol{\mu}}(\log\mathfrak{n})\asymp\partial_{\boldsymbol{\mu}}(\mathfrak{d}_{\log\mathfrak{n}})$ so $\partial_{\boldsymbol{\mu}}(\mathfrak{v})\asymp\frac{\partial_{\boldsymbol{\mu}}(\mathfrak{n})}{\mathfrak{n}\,(\log\mathfrak{n})^2}$. We have $\ell_1\prec\!\!\prec\ell_0$ in $\mathbb{L}_{<\boldsymbol{\lambda}}$, so Lemma 2.1 and **HF1** yield $\log\mathfrak{n}\prec\!\!\prec\mathfrak{n}$. We deduce with Corollary 1.31(c) that $\mathfrak{n}\,(\log\mathfrak{n})^2\equiv\mathfrak{n}$. So $\mathfrak{n}\,(\log\mathfrak{n})^2\prec 1$, whence $\partial_{\boldsymbol{\mu}}(\mathfrak{v})\succ\partial_{\boldsymbol{\mu}}(\mathfrak{n})$. Set $\mathfrak{u}:=\mathfrak{d}_{\log\mathfrak{m}}$. We have $\partial_{\boldsymbol{\mu}}(\mathfrak{m})=\partial_{\boldsymbol{\mu}}(\mathfrak{u})\,\mathfrak{m}\succ\partial_{\boldsymbol{\mu}}(\mathfrak{u})$. By (5.7) for $\mathfrak{u}+\mathfrak{v}$ and since $\mathfrak{u}\succ 1\succ\mathfrak{v}$, we have $\partial_{\boldsymbol{\mu}}(\mathfrak{u})\succ\partial_{\boldsymbol{\mu}}(\mathfrak{v})$, whence $\partial_{\boldsymbol{\mu}}(\mathfrak{m})\succ\partial_{\boldsymbol{\mu}}(\mathfrak{n})$. This proves that s satisfies (5.7). We also deduce that $\partial_{\boldsymbol{\mu}}(s)\sim\partial_{\boldsymbol{\mu}}(\tau_s)=\partial_{\boldsymbol{\mu}}(\log\tau_s)\,\tau_s$. Since $\tau_s>0$ and $\mathbb{G}\ni\log\tau_s>\mathbb{R}$, we deduce by (5.8) at $\log\tau_s$ that $\partial_{\boldsymbol{\mu}}(s)>0$. So $s\in\mathcal{P}_H$. □

5.6 Properties of H-field

Theorem 5.20. *Assume that $(\mathbb{U},\partial)$ is an H-field. Then $(\mathbb{U}_{(<\boldsymbol{\mu})},\partial_{\boldsymbol{\mu}})$ is an H-field with small derivation.*

Proof. Since $\mathfrak{W}_\partial$ is small, we have $\partial_{\boldsymbol{\mu}}(\mathfrak{m})\leqslant\mathfrak{m}$ for all $\mathfrak{m}\in\mathfrak{U}_{(<\boldsymbol{\mu})}$. In particular **H2** holds.

We prove by induction on the lexicographic order $<_{\text{lex}}$ that for all $\gamma\in\mathbf{On}$ and $\boldsymbol{\eta}\leqslant\boldsymbol{\mu}$, we have $\mathbb{U}_{(\gamma,\boldsymbol{\eta})}\subseteq\mathcal{P}_H$. Let $(\gamma,\boldsymbol{\eta})$ such that the result holds for all $(\rho,\boldsymbol{\sigma})<_{\text{lex}}(\gamma,\boldsymbol{\eta})$. For $s\in\mathbb{U}_{(<\boldsymbol{\mu})}$, we have $s\in\mathcal{P}_H$ if and only if $\mathfrak{m}+\mathfrak{n}\in\mathcal{P}_H$ for all $\mathfrak{m},\mathfrak{n}\in\operatorname{supp} s$. So $\mathcal{P}_H$ contains $\mathbb{R}[[\bigcup_{\rho<\alpha}\mathfrak{M}_\rho]]$ whenever $\alpha\in\mathbf{On}$ and $(\mathfrak{M}_\rho)_{\rho<\alpha}$ is an increasing sequence of subsets of $\mathfrak{U}_{(<\boldsymbol{\mu})}$ with $\forall\rho<\alpha,\mathfrak{M}_\rho\subseteq\mathcal{P}_H$. So we may assume that $\boldsymbol{\eta}$ is the successor of some ordinal ι. Write $\alpha:=\omega^\iota$ and set

$$\begin{aligned}\mathfrak{S}_+ &:= \bigcup_{\mathfrak{m}\in(\mathfrak{U}_{(\gamma,\iota)})_{[\iota]}} \operatorname{supp}\log\mathfrak{m},\\ \mathfrak{S}_- &:= \bigcup_{\mathfrak{m}\in(\mathfrak{U}_{(\gamma,\iota)})_{[\iota]}} (\operatorname{supp}\log\mathfrak{m})^{-1}, \quad \text{and}\\ \mathfrak{S} &:= \mathfrak{S}_+\cup\mathfrak{S}_-.\end{aligned}$$

So $\mathbb{R}[[\mathfrak{S}]]$ is a transserial subgroup of $\mathbb{U}_{(<\boldsymbol{\mu})}$ with $\mathfrak{S}=\mathfrak{S}^{-1}$. By Proposition 5.19, we may assume that $\iota>0$ and it is enough to prove that $\mathbb{R}[[\mathfrak{S}]]\subseteq\mathcal{P}_H$. So let $s\in\mathbb{R}[[\mathfrak{S}]]$.

Let $\mathfrak{m},\mathfrak{n}\in\operatorname{supp} s$ with $\mathfrak{m},\mathfrak{n}\succ 1$ and $\mathfrak{m}\neq\mathfrak{n}$. By the induction hypothesis, we may assume that $\mathfrak{m}\notin\mathfrak{U}_{(\gamma,\iota)}$. So $\mathfrak{m}=L_\beta(E_\alpha^\varphi)$ for a certain $\varphi\in(\mathbb{U}_{(\eta,\iota)})_{\succ,\alpha}\setminus L_\alpha(\mathbb{U}_{(\eta,\iota)}^{\succ,\succ})$ and a $\beta<\boldsymbol{\lambda}$ with $\beta\,\omega<\alpha$.

We distinguish two cases. First assume that $\mathfrak{n} \notin \mathfrak{U}_{(\gamma,\iota)}$. So $\mathfrak{n} = L_\rho(E^\psi_\alpha)$ for a certain $\psi \in (\mathbb{U}_{(\eta,\iota)})_{\succ,\alpha} \setminus L_\alpha(\mathbb{U}^{>,\succ}_{(\eta,\iota)})$ and a $\rho < \boldsymbol{\lambda}$ with $\rho\,\omega < \alpha$. We may assume that $\mathfrak{m} \succ \mathfrak{n}$. By Lemma 2.10, we have $\varphi > \psi$ or $\varphi = \psi$ and $\rho > \beta$. So we have $\tau_\varphi \geqslant \tau_\psi$ in general. Note that $\tau_\varphi, \tau_\psi \in \mathcal{P}_H$ by the induction hypothesis so $\partial_{\boldsymbol{\mu}}(\tau_\varphi) \succcurlyeq \partial_{\boldsymbol{\mu}}(\tau_\psi)$. Moreover, we have

$$\begin{aligned} \partial_{\boldsymbol{\mu}}(\mathfrak{m}) &\sim \partial_{\boldsymbol{\mu}}(\tau_\varphi) \times \ell_{[\beta,\alpha)} \circ E^\varphi_\alpha \quad \text{and} \\ \partial_{\boldsymbol{\mu}}(\mathfrak{n}) &\sim \partial_{\boldsymbol{\mu}}(\tau_\psi) \times \ell_{[\rho,\alpha)} \circ E^\psi_\alpha. \end{aligned} \tag{5.9}$$

If $\varphi = \psi$, then $\beta < \rho$ so $\partial_{\boldsymbol{\mu}}(\mathfrak{m}) \succcurlyeq \partial_{\boldsymbol{\mu}}(\tau_\varphi) \times \ell_{[\beta,\alpha)} \circ E^\varphi_\alpha \succ \partial_{\boldsymbol{\mu}}(\tau_\varphi) \times \ell_{[\rho,\alpha)} \circ E^\psi_\alpha \succcurlyeq \partial_{\boldsymbol{\mu}}(\mathfrak{n})$. If $\varphi > \psi$ then $\ell_{[\beta,\alpha)} \circ E^\varphi_\alpha \gg \ell_{[\rho,\alpha)} \circ E^\psi_\alpha$ by Lemma 2.10, so likewise $\partial_{\boldsymbol{\mu}}(\mathfrak{m}) \succ \partial_{\boldsymbol{\mu}}(\mathfrak{n})$.

Now assume that $\mathfrak{n} \in \mathfrak{U}_{(\gamma,\iota)}$. Set

$$\begin{aligned} \mathfrak{a} &:= \mathfrak{d}_\alpha(\mathfrak{m}) = E^\varphi_\alpha, && \text{(by Lemma 2.10)} \\ \mathfrak{b} &:= \mathfrak{d}_\alpha(\mathfrak{n}) \in \mathbb{U}_{(\gamma,\iota)} && \text{(since } \mathbb{U}_{(\gamma,\iota)} \text{ is confluent)} \\ \text{and} \quad \psi &:= L_\alpha(\mathfrak{b}). \end{aligned}$$

Note that $\varphi = L_\alpha(\mathfrak{a}) \in \mathbb{U}_{(\gamma,\iota)} \setminus (L_\alpha(\mathbb{U}^{>,\succ}_{(\gamma,\iota)}))$ whereas $\psi \in L_\alpha(\mathbb{U}^{>,\succ}_{(\gamma,\iota)})$, so $\varphi \neq \psi$. For sufficiently large $\rho < \alpha$, we have $L_\rho(\mathfrak{m}) - L_\rho(\mathfrak{a}) \prec 1$ and $L_\rho(\mathfrak{n}) - L_\rho(\mathfrak{b}) \prec 1$. We deduce by **H2** and **D4** that

$$\begin{aligned} \partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{m})) &\sim \partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{a})) = \partial_{\boldsymbol{\mu}}(\varphi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{a} \quad \text{and} \\ \partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{n})) &\sim \partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{b})) = \partial_{\boldsymbol{\mu}}(\psi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{b}. \end{aligned}$$

Assume that $\mathfrak{m} \succ \mathfrak{n}$. Lemma 2.10 implies that $\varphi > \psi$, so as above we have $\partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{m})) \sim \partial_{\boldsymbol{\mu}}(\tau_\varphi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{a} \succ \partial_{\boldsymbol{\mu}}(\tau_\psi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{b} \sim \partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{n}))$. We have $\ell'_\rho \circ \mathfrak{m} \prec \ell'_\rho \circ \mathfrak{n}$ so **D4** yields

$$\partial_{\boldsymbol{\mu}}(\mathfrak{m}) \sim \frac{\partial_{\boldsymbol{\mu}}(\tau_\varphi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{a}}{\ell'_\rho \circ \mathfrak{m}} \succ \frac{\partial_{\boldsymbol{\mu}}(\tau_\psi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{b}}{\ell'_\rho \circ \mathfrak{n}} \sim \partial_{\boldsymbol{\mu}}(\mathfrak{n}).$$

If $\mathfrak{n} \succ \mathfrak{m}$, then symmetric arguments yield $\partial_{\boldsymbol{\mu}}(\mathfrak{n}) \sim \frac{\partial_{\boldsymbol{\mu}}(\tau_\varphi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{b}}{\ell'_\rho \circ \mathfrak{n}} \succ \frac{\partial_{\boldsymbol{\mu}}(\tau_\psi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{a}}{\ell'_\rho \circ \mathfrak{m}} \sim \partial_{\boldsymbol{\mu}}(\mathfrak{n})$.

Now let $\mathfrak{n} \in \operatorname{supp} s$ and $\mathfrak{m} \in \operatorname{supp} s$ with $\mathfrak{m} \succ 1 \succ \mathfrak{n}$. We need only prove that $\partial_{\boldsymbol{\mu}}(\mathfrak{m}) \succ \partial_{\boldsymbol{\mu}}(\mathfrak{n})$ to conclude that s satisfies (5.7). Set $\mathfrak{a} := \mathfrak{d}_\alpha(\mathfrak{m})$, $\mathfrak{b} := \mathfrak{d}_\alpha(\mathfrak{n}^{-1})$, and

$$\begin{aligned} \varphi &:= \sharp_\alpha(L_\alpha(\mathfrak{m})) = L_\alpha(\mathfrak{a}) \quad \text{and} \\ \psi &:= \sharp_\alpha(L_\alpha(\mathfrak{n}^{-1})) = L_\alpha(\mathfrak{b}). && \text{(by (2.13))} \end{aligned}$$

If $\mathfrak{m} \in \mathbb{U}_{(\gamma,\iota)}$, then we have $\mathfrak{d}_\alpha(\mathfrak{m}) \in \mathbb{U}_{(\gamma,\iota)}$ by confluence, so $\varphi \in \mathbb{U}_{(\gamma,\iota)}$. If $\mathfrak{m} \notin \mathbb{U}_{(\gamma,\iota)}$, then we have $\mathfrak{m} = L_\beta(E^\varphi_\alpha)$ for an ordinal β with $\beta\,\omega < \alpha$, and a $\varphi_0 \in (\mathbb{U}_{(\gamma,\iota)})_{\succ,\alpha} \setminus L_\alpha(\mathbb{U}^{>,\succ}_{(\gamma,\iota)})$, and then $\varphi = \varphi_0 \in \mathbb{U}_{(\gamma,\iota)}$. So in any case $\varphi \in \mathbb{U}_{(\gamma,\iota)}$. Likewise, we have $\psi \in \mathbb{U}_{(\gamma,\iota)}$. Let $\rho < \alpha$ be large enough, so that $L_\rho(\mathfrak{m}) - L_\rho(\mathfrak{a}) \prec 1$ and $L_\rho(\mathfrak{n}^{-1}) - L_\rho(\mathfrak{b}) \prec 1$. By **H2** and **D4**, we have

$$\begin{aligned} \partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{m})) &\sim \partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{a})) = \partial_{\boldsymbol{\mu}}(\varphi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{a} \quad \text{and} \\ \partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{n}^{-1})) &\sim \partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{b})) = \partial_{\boldsymbol{\mu}}(\psi) \times \ell_{[\rho,\alpha)} \circ \mathfrak{b}. \end{aligned}$$

Applying **D4** and **D2** yields

$$\begin{aligned} \partial_{\boldsymbol{\mu}}(\mathfrak{m}) = \frac{\partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{m}))}{\ell'_\rho \circ \mathfrak{m}} &\asymp \partial_{\boldsymbol{\mu}}(\varphi) \times \frac{\ell_{[\rho,\alpha)} \circ \mathfrak{a}}{\ell'_\rho \circ \mathfrak{m}} \quad \text{and} \\ \partial_{\boldsymbol{\mu}}(\mathfrak{n}) = -\frac{\partial_{\boldsymbol{\mu}}(L_\rho(\mathfrak{n}^{-1}))\,\mathfrak{n}^2}{\ell'_\rho \circ \mathfrak{n}^{-1}} &\asymp \partial_{\boldsymbol{\mu}}(\psi) \times \frac{\mathfrak{n}^2\,(\ell_{(\rho,\alpha)} \circ \mathfrak{b})}{\ell'_\rho \circ \mathfrak{n}^{-1}}, \end{aligned}$$

By (5.7) at $\mathfrak{d}_\varphi + \frac{1}{\mathfrak{d}_\psi} \in \mathbb{U}_{(\gamma,\iota)}$ and $\psi \in \mathbb{U}_{(\gamma,\iota)}$, we have $\partial_{\boldsymbol{\mu}}(\varphi) \succ \partial_{\boldsymbol{\mu}}\Big(\frac{1}{\mathfrak{d}_\psi}\Big) \asymp \frac{\partial_{\boldsymbol{\mu}}(\mathfrak{d}_\psi)}{\mathfrak{d}_\psi^2} \asymp \frac{\partial_{\boldsymbol{\mu}}(\psi)}{\psi^2}$. We have $\ell'_\rho \circ \mathfrak{n}^{-1} \equiv \mathfrak{n}$ and $\ell_{(\rho,\alpha)} \circ \mathfrak{b} \sim L_\rho(\mathfrak{n}^{-1}) \ll \mathfrak{n}^{-1}$, so Corollary 1.31(c) yields

$$\frac{\mathfrak{n}^2\,(\ell_{(\rho,\alpha)} \circ \mathfrak{b})}{\ell'_\rho \circ \mathfrak{n}^{-1}} \equiv \mathfrak{n}\,\ell_{(\rho,\alpha)} \circ \mathfrak{b} \equiv \mathfrak{n} \ll \frac{1}{\psi^2}.$$

It follows that $\partial_{\boldsymbol{\mu}}(\varphi) \succ \partial_{\boldsymbol{\mu}}(\mathfrak{n})$, whence $\partial_{\boldsymbol{\mu}}(\mathfrak{m}) \succ \partial_{\boldsymbol{\mu}}(\mathfrak{n})$. This concludes the proof that s satisfies (5.7).

Assume furthermore that $s > \mathbb{R}$. Note that $\partial_{\boldsymbol{\mu}}(s) \sim \partial_{\boldsymbol{\mu}}(\tau_s)$ by (5.7). If $\mathfrak{d}_s \in \mathbb{U}_{(\gamma,\iota)}$, then $\tau_s \in \mathcal{P}_H$ so $\partial_{\boldsymbol{\mu}}(\tau_s) > 0$. Otherwise we have $\partial_{\boldsymbol{\mu}}(\mathfrak{d}_s) \sim \partial_{\boldsymbol{\mu}}(\tau_\varphi) \times \ell_{[\beta,\alpha)} \circ E_\alpha^\varphi$ as in (5.9) for a certain $\varphi \in \mathcal{P}_H \cap \mathbb{U}_{(<\boldsymbol{\mu})}^{>,\succ}$ and a certain $\beta \in (0,\alpha)$. We deduce that $\partial_{\boldsymbol{\mu}}(\tau_\varphi) > 0$, so $\partial_{\boldsymbol{\mu}}(\mathfrak{d}_s) > 0$, so $\partial_{\boldsymbol{\mu}}(\tau_s) > 0$. Therefore $\partial_{\boldsymbol{\mu}}(s) > 0$. This proves that $s \in \mathcal{P}_H$. So $\mathbb{R}[[\mathfrak{S}]] \subseteq \mathcal{P}_H$. By Proposition 5.19, we have $(\mathbb{R}[[\mathfrak{S}]])^{\exp} \subseteq \mathcal{P}_H$ where $\mathbb{U}_{(\gamma,\boldsymbol{\eta})} \subseteq (\mathbb{R}[[\mathfrak{S}]])^{\exp}$. So this concludes our inductive proof that $\mathbb{U}_{(<\boldsymbol{\mu})} \subseteq \mathcal{P}_H$. In particular **H1** holds for $(\mathbb{U}_{(<\boldsymbol{\mu})}, \partial_{\boldsymbol{\mu}})$ by (5.8). □

In order to complete our proof of Theorem 5.7, we must prove the unicity of $\partial_{\boldsymbol{\mu}}$.

Proposition 5.21. *The function $\partial_{\boldsymbol{\mu}}$ is the only extension of ∂ into a hyperserial derivation $\mathbb{U}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{V}_{(<\boldsymbol{\mu})}$ of force $\boldsymbol{\nu}$.*

Proof. Let $\tilde{\partial}$ be a right composition $\mathbb{U}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{V}_{(<\boldsymbol{\mu})}$ which extends ∂. We claim that $\tilde{\partial} = \partial_{\boldsymbol{\mu}}$, and we prove the result by induction on the exponential height $(\gamma, \boldsymbol{\eta})$ of s over $\mathbb{U}$. We have $\tilde{\partial} = \partial_{\boldsymbol{\mu}}$ on $\mathbb{U}$ by definition. Now let $s \in \mathbb{U}_{(<\boldsymbol{\mu})}$ such that we have $\tilde{\partial}(g) = \partial_{\boldsymbol{\mu}}(g)$ for all $g \in \mathbb{U}_{(<\boldsymbol{\mu})}$ with $\mathrm{HH}_{\mathbb{U}}(g) < \mathrm{HH}_{\mathbb{U}}(s)$. By **D1**, we may assume that s is a monomial. By **D3** and **D1**, it is enough to prove that $\tilde{\partial}(\mathfrak{m}) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{supp} \log s$. Consider such a monomial $\mathfrak{m}$. If $\boldsymbol{\eta} = 1$ or $\boldsymbol{\eta}$ is a limit, then we have $\mathrm{HH}_{\mathbb{U}}(\mathfrak{m}) < \mathrm{HH}_{\mathbb{U}}(s)$, whence $\tilde{\partial}(\mathfrak{m}) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})$. Otherwise write $\boldsymbol{\eta} = \iota + 1$ for a certain $\iota > 0$. We have $\mathfrak{m} = L_\rho(E_{\omega^\iota}^\varphi)$ for certain $\rho < \boldsymbol{\lambda}$ with $\rho\,\omega < \omega^\iota$ and $\varphi \in (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\omega^\iota}$ with $\mathrm{HH}_{\mathbb{U}}(\varphi) < \mathrm{HH}_{\mathbb{U}}(s)$. The induction hypothesis yields $\tilde{\partial}(\varphi) = \partial_{\boldsymbol{\mu}}(\varphi)$. We deduce with **D4** that $\tilde{\partial}(\mathfrak{m}) = \partial_{\boldsymbol{\mu}}(\mathfrak{m})$. By induction, we deduce that $\tilde{\partial}$ and $\partial_{\boldsymbol{\mu}}$ coincide. □

This concludes the proof of Theorem 5.7.

Corollary 5.22. *There is a unique extension of the standard derivation $\mathbb{L} \longrightarrow \mathbb{L}$ into a hyperserial derivation $\tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ of force* **On**. *Moreover, $(\tilde{\mathbb{L}}, +, \times, <, \preccurlyeq, \tilde{\partial})$ is an H-field.*

Proof. This follows from Theorem 5.7 and [4, Theorem 1.2]. □

5.7 Model theory of $(\tilde{\mathbb{L}}, +, \times, <, \prec, ')$

The field $\mathbb{T}_{\mathrm{LE}}$ of logarithmic-exponential transseries of [13, 18] can be realized [9, Theorem 4.11] as a subfield of $(\mathbb{L}_{<\omega})_{(<1)}$, hence $\mathbb{T}_{\mathrm{LE}} \subseteq \tilde{\mathbb{L}}$. The elementary first order theory of $(\mathbb{T}_{\mathrm{LE}}, +, \times, <, \preccurlyeq, \partial)$ is studied in [4]. It is in particular model-complete. Consider the hyperserial derivation $\tilde{\partial}$ of Corollary 5.22 on $\tilde{\mathbb{L}}$. We conclude this section by proving that $\tilde{\mathbb{L}}$ is an elementary extension of $\mathbb{T}_{\mathrm{LE}}$.

Theorem 5.23. *The inclusion $\mathbb{T}_{\mathrm{LE}} \longhookrightarrow \tilde{\mathbb{L}}$ is an elementary embedding for the structures of ordered differential valued fields.*

Proof. By [5, Theorems 15.0.1 and 16.0.1] it suffices to show that $\tilde{\mathbb{L}}$ is the union of an increasing family $(\mathbb{F}_\nu)_{0<\nu<\mathbf{On}}$ of subfields of well-based series $\mathbb{F}_\nu =: \mathbb{R}[[\mathfrak{F}_\nu]]$, where

i. each $\mathbb{F}_\nu$ is closed under $\tilde{\partial}$, and

ii. each $\mathfrak{F}_\nu$ admits an element $\mathfrak{m}_\nu$ with $\mathfrak{F}_\nu \gg \mathfrak{m}_\nu$ and $\mathfrak{m} \in \tilde{\partial}(\tilde{\mathbb{L}})$.

For $\nu \in \mathbf{On}$ with $0 > \nu$, we set $\alpha_\nu := \omega^{\omega\nu}$ and

$$\mathfrak{F}_\nu := (\mathfrak{L}_{<\alpha_\nu})_{(<\omega\nu)} \cdot \mathfrak{L}_{<\alpha_\nu} \circ \mathrm{e}_{\alpha_\nu}^{\ell_0} \subseteq \tilde{\mathfrak{L}}.$$

We claim that $\mathbb{F}_\nu := \mathbb{R}[[\mathfrak{F}_\nu]]$ satisfies the conditions. First note that for $f \in \tilde{\mathbb{L}}$, there is $\nu \in \mathbf{On}$ with $\nu > 0$ and $f \in (\mathbb{L}_{<\alpha_\nu})_{(<\omega\nu)}$, whence $f \in \mathbb{F}_\nu$. So $\tilde{\mathbb{L}} = \bigcup_{0<\nu<\mathbf{On}} \mathbb{F}_\nu$.

Moreover we have $\mathfrak{L}_{<\alpha_\nu} \circ \mathrm{e}_{\alpha_\nu}^{\ell_0} \subseteq (\mathfrak{L}_{<\alpha_\phi})_{(<\omega\phi)}$ whenever $\phi > \nu$ so the family $(\mathbb{F}_\nu)_{0<\nu<\mathbf{On}}$ is increasing for the inclusion. Let $\nu \in \mathbf{On}$ with $\nu > 0$. By **D4**, we have

$$\tilde{\partial}(\mathfrak{L}_{<\alpha_\nu} \circ \mathrm{e}_{\alpha_\nu}^{\ell_0}) \subseteq \tilde{\partial}(\mathrm{e}_{\alpha_\nu}^{\ell_0}) \times \tilde{\partial}(\mathfrak{L}_{<\alpha_\nu}) \circ \mathrm{e}_{\alpha_\nu}^{\ell_0} \subseteq (\mathfrak{L}_{<\alpha_\nu} \circ \mathrm{e}_{\alpha_\nu}^{\ell_0}) \times (\mathbb{L}_{<\alpha_\nu} \circ \mathrm{e}_{\alpha_\nu}^{\ell_0}) \subseteq \mathbb{L}_{<\alpha_\nu} \circ \mathrm{e}_{\alpha_\nu}^{\ell_0}.$$

We deduce that $\partial(\mathfrak{L}_{<\alpha_\nu} \circ \mathrm{e}_{\alpha_\nu}^{\ell_0}) \subseteq \mathbb{F}_\nu$. We also have $\partial((\mathfrak{L}_{<\alpha_\nu})_{(<\omega\nu)}) \subseteq (\mathbb{L}_{<\alpha_\nu})_{(<\omega\nu)}$ by Theorem 5.7. By the Leibniz rule, we deduce that $\mathbb{F}_\nu$ is closed under $\tilde{\partial}$.

For $\mathfrak{m} \in (\mathfrak{L}_{<\alpha_\nu})_{(<\omega\nu)}$ and $\mathfrak{l} \in \mathfrak{L}_{<\alpha_\nu}^{\succ}$, we have $\mathfrak{m} \asymp \mathfrak{l} \circ \mathrm{e}_{\alpha_\nu}^{\ell_0}$. Set $\mathfrak{m}_\nu := \tilde{\partial}(-(\mathrm{e}_{\alpha_\nu}^{\ell_0})^{-1})$. Since $\ell_0^{-1} \preccurlyeq \mathfrak{L}$, the axiom **HF1** yields $\mathfrak{m}_\nu \equiv (\mathrm{e}_{\alpha_\nu}^{\ell_0})^{-1} \preccurlyeq \mathfrak{L}_{<\alpha_\nu} \circ \mathrm{e}_{\alpha_\nu}^{\ell_0}$. So $\mathfrak{m}_\nu \preccurlyeq \mathfrak{F}_\nu$. This concludes the proof. □

6 Extending compositions

We now look into extending composition laws. Instead of always considering binary laws $\circ\colon (f, g) \mapsto f \circ g$, it is sometimes more convenient to work with unary right compositions $\circ_g\colon f \mapsto f \circ g$ for fixed g. Our main goal in this section is to prove the following theorem:

Theorem 6.1. *Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ and $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ with $0 < \boldsymbol{\mu}$. Let $(\mathbb{U}, \circ_{\mathbb{U}})$, $(\mathbb{V}, \circ_{\mathbb{V}})$ be hyperserial fields of force $\boldsymbol{\nu}$ such that $\mathbb{U}$ satisfies Schmeling's axiom* **T4**. *Let $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ be a right composition and assume that $\triangle$ has a good relative near-support $\mathfrak{W}_\triangle$. There is a unique right composition*

$$\triangle_{\boldsymbol{\mu}}\colon \mathbb{U}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{V}_{(<\boldsymbol{\mu})}$$

of force $\boldsymbol{\nu}$ which extends $\triangle$. Moreover $\mathfrak{W}_\triangle$ is a relative near-support for $\triangle_{\boldsymbol{\mu}}$.

6.1 Right compositions

We fix a $\boldsymbol{\nu} \leqslant \mathbf{On}$ and a $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ with $\boldsymbol{\mu} > 0$, and we write $\boldsymbol{\lambda} := \omega^{\boldsymbol{\nu}}$.

Consider two hyperserial fields $(\mathbb{U}, \circ_{\mathbb{U}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ of force $\boldsymbol{\nu}$. A *right composition* $\mathbb{U} \longrightarrow \mathbb{V}$ of force $\boldsymbol{\nu}$ is a function $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ which satisfies the following properties:

RC1. The function $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ is a strongly linear morphism of rings.

RC2. For all $f \in \mathbb{L}_{<\boldsymbol{\lambda}}$ and $s \in \mathbb{U}^{>,\succ}$, we have $\triangle(f \circ_{\mathbb{U}} s) = f \circ_{\mathbb{V}} (\triangle(s))$.

Example 6.2. Consider the field $\mathbb{L}_{<\omega^{\boldsymbol{\nu}}}$, seen as a hyperserial field of force $\boldsymbol{\nu}$. Now let $(\mathbb{T}, \circ)$ be any hyperserial field of force $\boldsymbol{\nu}$. Then for each $s \in \mathbb{T}^{>,\succ}$, **HF1** and **HF2** imply that the function $\circ_s\colon \mathbb{L}_{<\boldsymbol{\nu}} \longrightarrow \mathbb{T}$; $f \mapsto f \circ s$ is a right composition.

We see that hyperserial embeddings $\mathbb{U} \longrightarrow \mathbb{V}$ of force $\boldsymbol{\nu}$ are simply right compositions of force $\boldsymbol{\nu}$ which preserve monomials. Like in the case of hyperserial derivations, the nature of right composition can be checked on the skeleton of $\mathbb{U}$.

Proposition 6.3. *Right compositions $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ of force $\boldsymbol{\nu}$ are strictly increasing.*

Proof. Since $\mathbb{U}$ is a real-closed field and $\mathbb{V}$ is an ordered field, any morphism of rings $\mathbb{U} \longrightarrow \mathbb{V}$ is strictly increasing. □

Proposition 6.4. *Let $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ be a strongly linear function with*

$$\triangle(\ell_{\omega^\mu} \circ_{\mathbb{U}} \mathfrak{g}) = L_{\omega^\mu}(\triangle(\mathfrak{g})) \quad \text{for all } \eta < \boldsymbol{\nu} \text{ and } \mathfrak{g} \in \mathfrak{U}_{\omega^\eta}.$$

for all $\mu < \boldsymbol{\nu}$ and $\mathfrak{g} \in \mathfrak{U}_{\omega^\mu}$. Then $\triangle$ is a right composition of force $\boldsymbol{\nu}$.

Proof. Let $\mathbf{C}$ denote the class of series $f \in \mathbb{L}_{<\boldsymbol{\lambda}}$ with $\triangle(f \circ g) = f \circ \triangle(g)$ for all $g \in \mathbb{U}^{>,\succ}$. We prove that we have $\mathbb{L}_{<\omega^{\boldsymbol{\mu}}} \subseteq \mathbf{C}$ by induction on $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$, starting with $\boldsymbol{\mu} = 1$.

Consider $g \in \mathbb{U}^{>}$ and write $g = r_g\, \mathfrak{d}_g\, (1 + \varepsilon_g)$ where $r_g \in \mathbb{R}^{>}$ and $\varepsilon_g \prec 1$ as in (2.3). We have $\triangle(g) = r_g\, \triangle(\mathfrak{d}_g)\, (1 + \triangle(\varepsilon_g))$ where $\triangle(\varepsilon_g) \prec 1$, so

$$\begin{aligned}
\log g &= \log \mathfrak{d}_g + \log r_g + \sum_{k \in \mathbb{N}} \frac{(-1)^k}{k+1}\, \varepsilon_g^{k+1}, \text{ and} \\
\log \triangle(g) &= \log \triangle(\mathfrak{d}_g) + \log r_g + \sum_{k \in \mathbb{N}} \frac{(-1)^k}{k+1}\, \triangle(\varepsilon_g)^{k+1} \\
&= \triangle(\log \mathfrak{d}_g) + \log r_g + \triangle\left(\sum_{k \in \mathbb{N}} \frac{(-1)^k}{k+1}\, \varepsilon_g^{k+1} \right) \\
&= \triangle(\log g).
\end{aligned}$$

We deduce that $\mathbf{C}$ contains $\mathfrak{l} \in \mathfrak{L}_{<\boldsymbol{\lambda}}$ if and only if it contains $\log \mathfrak{l}$. Note that by strong linearity of $\triangle$, the class $\mathbf{C}$ is closed under sums of well-based families. Moreover, for $f, h \in \mathbf{C}$ with $h > \mathbb{R}$, we have $f \circ h \in \mathbf{C}$. So we need only prove that we have $\ell_{\omega^\eta} \in \mathbf{C}$ for all $\eta < \boldsymbol{\nu}$. Let $\eta > 0$ such that this holds for all $\iota < \eta$. So $\mathbb{L}_{<\omega^\eta} \subseteq \mathbf{C}$ by the previous arguments. Let $g \in \mathbb{T}^{>,\succ}$ and write $\mathfrak{g} := \mathfrak{d}_{\omega^\eta}(g)$. By (2.4), there is $\gamma < \omega^\eta$ such that the number $\varepsilon := \ell_\gamma \circ_{\mathbb{U}} g - \ell_\gamma \circ_{\mathbb{U}} \mathfrak{g}$ is infinitesimal, with

$$\ell_{\omega^\eta} \circ_{\mathbb{U}} g = \ell_{\omega^\eta} \circ_{\mathbb{U}} \mathfrak{g} + \sum_{k>0} \frac{(\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \circ_{\mathbb{U}} \ell_\gamma \circ_{\mathbb{U}} \mathfrak{g}}{k!}\, \varepsilon^k.$$

Note that for $k \in \mathbb{N}^{>}$, we have $(\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \in \mathbb{L}_{<\omega^\eta} \subseteq \mathbf{C}$. Moreover, we have

$$\ell_\gamma \circ_{\mathbb{V}} \triangle(g) - \ell_\gamma \circ_{\mathbb{V}} \triangle(\mathfrak{g}) = \triangle(\ell_\gamma \circ_{\mathbb{U}} g - \ell_\gamma \circ_{\mathbb{U}} \mathfrak{g}) = \triangle(\varepsilon) \prec 1.$$

We deduce that

$$\begin{aligned}
\ell_{\omega^\eta} \circ_{\mathbb{V}} \triangle(g) &= \ell_{\omega^\eta} \circ_{\mathbb{V}} \triangle(\mathfrak{g}) + \sum_{k>0} \frac{(\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \circ_{\mathbb{V}} \ell_\gamma \circ_{\mathbb{V}} \triangle(\mathfrak{g})}{k!}\, \triangle(\varepsilon)^k. \\
&= \triangle(\ell_{\omega^\eta} \circ_{\mathbb{U}} \mathfrak{g}) + \triangle\left(\sum_{k>0} \frac{(\ell_{\omega^\eta}^{\uparrow\gamma})^{(k)} \circ_{\mathbb{U}} \ell_\gamma \circ_{\mathbb{U}} \mathfrak{g}}{k!}\, \varepsilon^k \right) \\
&= \triangle(\ell_{\omega^\eta} \circ_{\mathbb{U}} g).
\end{aligned}$$

We conclude by induction that $\mathbf{C} = \mathbb{L}_{<\boldsymbol{\lambda}}$. □

Corollary 6.5. *Let $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ be a strongly linear function with*

$$\begin{aligned}
\triangle(\log \mathfrak{m}) &= \log \triangle(\mathfrak{m}) \quad \text{for all } \mathfrak{m} \in \mathfrak{U}, \text{ and} \\
\triangle(L_{\omega^\mu}(\mathfrak{a})) &= L_{\omega^\mu}(\triangle(\mathfrak{a})) \quad \text{for all } 0 < \eta < \boldsymbol{\nu} \text{ and } \mathfrak{a} \in \mathfrak{U}_{\omega^\eta}.
\end{aligned}$$

Then $\triangle$ is a right composition of force $\boldsymbol{\nu}$.

Corollary 6.6. *The notions of hyperserial embeddings of* [7, Definition 3.4] *and Definition 2.2 coincide.*

6.2 Extending right compositions

Fix $(\mathbb{U}, \circ_{\mathbb{U}})$, $(\mathbb{V}, \circ_{\mathbb{V}})$, $\triangle$ and $\mathfrak{W}_{\triangle}$ as in Theorem 6.1. As in Section 5.3, we inductively define $\triangle_{\boldsymbol{\mu}}$ along with the class $\mathcal{D}_{\triangle}$ of series $s \in \mathbb{U}_{(<\boldsymbol{\mu})}$ at which $\triangle_{\boldsymbol{\mu}}$ is defined. For $s \in \mathbb{U}_{(<\boldsymbol{\mu})}$ with $\mathrm{HH}_{\mathbb{U}}(s) =: (\gamma, \boldsymbol{\eta})$, we say that $\triangle_{\boldsymbol{\mu}}$ is *defined at* s if

i. $s \in \mathbb{U}$. Then we set
$$\triangle_{\boldsymbol{\mu}}(s) := \triangle(s). \tag{6.1}$$

ii. $s \in \mathfrak{U}_{(<\boldsymbol{\mu})}$, $\boldsymbol{\eta} = 1$ and $\log s \in \mathcal{D}_{\triangle}$. Then we set
$$\triangle_{\boldsymbol{\mu}}(s) = \exp(\triangle_{\boldsymbol{\mu}}(\log s)) \in \mathbb{V}_{(<\boldsymbol{\mu})}. \tag{6.2}$$

iii. $s \in \mathfrak{U}_{(<\boldsymbol{\mu})}$, $\boldsymbol{\eta} = \iota + 1$ for a certain $\iota > 0$ and $s = \mathfrak{t}\,\mathfrak{m}$ where $\mathfrak{m} \in \mathcal{D}_{\triangle}$, $\operatorname{hsupp} \mathfrak{t} \subseteq \mathcal{D}_{\triangle}$ and the family $(\log(\mathfrak{t}_{\varphi}) \circ_{\mathbb{V}} (\triangle_{\boldsymbol{\mu}}(E^{\varphi}_{\omega^{\iota}})))_{\varphi \in \operatorname{hsupp} \mathfrak{t}}$ is well-based. Then we set
$$\triangle_{\boldsymbol{\mu}}(s) := \exp\left(\sum_{\varphi \in \operatorname{hsupp} \mathfrak{t}} \log(\mathfrak{t}_{\varphi}) \circ_{\mathbb{V}} (E_{\omega^{\iota}}(\triangle_{\boldsymbol{\mu}}(\varphi))) \right) \triangle_{\boldsymbol{\mu}}(\mathfrak{m}) \in \mathbb{V}_{(<\boldsymbol{\mu})}. \tag{6.3}$$
A simple computation in the case when $\mathfrak{t}\,\mathfrak{m} = L_{\gamma}(E^{\varphi}_{\omega^{\iota}})$ yields
$$\triangle_{\boldsymbol{\mu}}(L_{\gamma}(E^{\varphi}_{\omega^{\iota}})) = L_{\gamma}(E_{\omega^{\iota}}(\triangle_{\boldsymbol{\mu}}(\varphi))). \tag{6.4}$$

iv. $s \notin \mathfrak{U}_{(<\boldsymbol{\mu})}$, $\operatorname{supp} s \subseteq \mathcal{D}_{\triangle}$ and the family $(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))_{\mathfrak{m} \in \operatorname{supp} s}$ is well-based. Then we set
$$\triangle_{\boldsymbol{\mu}}(s) := \sum_{\mathfrak{m} \in \operatorname{supp} s} s_{\mathfrak{m}} \triangle_{\boldsymbol{\mu}}(\mathfrak{m}). \tag{6.5}$$

This definition is warranted by induction on $\mathrm{HH}_{\mathbb{U}}(s)$. The existence in Theorem 6.1 reduces to the identity $\mathcal{D}_{\triangle} = \mathbb{U}_{(<\boldsymbol{\mu})}$. From iii. above, it follows that $\mathcal{D}_{\triangle}$ is a subgroup of $\mathbb{U}_{(<\boldsymbol{\mu})}$ which contains $\mathbb{U}$. We first justify that $\mathcal{D}_{\triangle}$ is closed under various operations.

Proposition 6.7. *Let $\eta < \boldsymbol{\nu}$, set $\alpha := \omega^{\eta}$ and let $\mathfrak{m} \in (\mathfrak{U}_{<\boldsymbol{\mu}})_{\alpha}$ or $\mathfrak{m} \in \mathfrak{U}$ if $\alpha = 1$. We have $\mathfrak{m} \in \mathcal{D}_{\triangle} \iff L_{\alpha}(\mathfrak{m}) \in \mathcal{D}_{\triangle}$. Moreover, if $\mathfrak{m} \in \mathcal{D}_{\triangle}$ then $\triangle_{\boldsymbol{\mu}}(L_{\alpha}(\mathfrak{m})) = L_{\alpha}(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$.*

Proof. We proceed by induction on $\mathrm{HH}_{\mathbb{U}}(\mathfrak{m})$. Let $(\gamma, \boldsymbol{\eta}) := \mathrm{HH}_{\mathbb{U}}(\mathfrak{m})$. If $(\gamma, \boldsymbol{\eta}) = (0, 0)$, then $\mathfrak{m} \in \mathbb{U}$ and $L_{\alpha}(\mathfrak{m}) \in \mathbb{U}$ and $\triangle_{\boldsymbol{\mu}}(L_{\alpha}(\mathfrak{m})) = \triangle(L_{\alpha}(\mathfrak{m})) = L_{\alpha}(\triangle(\mathfrak{m})) = L_{\alpha}(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$.

Assume that $(0, 0) <_{\mathrm{lex}} (\gamma, \boldsymbol{\eta})$ and that the result holds for monomials with exponential height $<_{\mathrm{lex}}(\gamma, \boldsymbol{\eta})$ over $\mathbb{U}$. Since $\mathfrak{m}$ is a monomial, we know that $\boldsymbol{\eta} = \iota + 1$ is a successor. Write $\beta := \omega^{\iota}$. We have $\mathfrak{m} = E^{\varphi}_{\beta}$ where $\varphi := L_{\beta}(\mathfrak{m}) \in (\mathbb{U}_{<\boldsymbol{\mu}})_{\succ, \beta}$ and $\mathrm{HH}_{\mathbb{U}}(\varphi) <_{\mathrm{lex}} (\gamma, \boldsymbol{\eta})$. Note that $\alpha \leqslant \beta\omega$ by Corollary 4.7. There will be many cases to consider.

Assume first that $\alpha = 1$. If $\beta = 1$, then we have $\mathfrak{m} \in \mathcal{D}_{\triangle} \iff \varphi = L_1(\mathfrak{m}) \in \mathcal{D}_{\triangle}$ by ii. By (6.2) we have $\triangle_{\boldsymbol{\mu}}(\mathfrak{m}) = E_1(\triangle_{\boldsymbol{\mu}}(\varphi))$, whence $\triangle_{\boldsymbol{\mu}}(L_1(\mathfrak{m})) = L_1(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$, if applicable. Recall that $\mathcal{D}_{\triangle}$ is a group, so $\mathcal{D}_{\triangle} - 1 = \mathcal{D}_{\triangle}$. If $\beta = \omega$, then we have $\mathfrak{m} \in \mathcal{D}_{\triangle} \iff \varphi \in \mathcal{D}_{\triangle} \iff \varphi - 1 \in \mathcal{D}_{\triangle}$ by iii. If $L_1(\mathfrak{m}) = E^{\varphi - 1}_{\beta} \in \mathbb{U}$, then we have $\varphi - 1 \in \mathbb{U}$, and $\mathfrak{m} \in \mathcal{D}_{\triangle}$ by ii. Moreover (6.2) yields $\triangle_{\boldsymbol{\mu}}(\mathfrak{m}) = E_1(\triangle(L_1(\mathfrak{m})))$, so $\triangle(L_1(\mathfrak{m})) = L_1(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$. We obtain $\triangle_{\boldsymbol{\mu}}(L_1(\mathfrak{m})) = \triangle(L_1(\mathfrak{m})) = L_1(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$ by (6.1). If $E^{\varphi - 1}_{\beta} \notin \mathbb{U}$, then $E^{\varphi - 1}_{\beta} \in \mathcal{D}_{\triangle} \iff \varphi - 1 \in \mathcal{D}_{\triangle} \iff \mathfrak{m} \in \mathcal{D}_{\triangle}$ by iii. Moreover, we have $\triangle_{\boldsymbol{\mu}}(L_1(\mathfrak{m})) = L_1(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$ by (6.4) if applicable.

If $\beta > \omega$, then we are in case iii and we have $\mathfrak{m} \in \mathcal{D}_\triangle \iff \varphi \in \mathcal{D}_\triangle \iff L_1 E_\beta^\varphi \in \mathcal{D}_\triangle$. We conclude with (6.4) that $\triangle_{\boldsymbol{\mu}}(L_1(\mathfrak{m})) = L_1(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$ if applicable.

This treats the case when $\alpha = 1$. Assume now that $\alpha = \omega$ and $\beta = 1$. We have $\mathfrak{m} \in \mathcal{D}_\triangle \iff \varphi \in \mathcal{D}_\triangle$ by ii. Since φ is log-atomic, the induction hypothesis yields $\mathfrak{m} \in \mathcal{D}_\triangle \iff L_\omega(\varphi) \in \mathcal{D}_\triangle$, with $\triangle_{\boldsymbol{\mu}}(L_\omega(\varphi)) = L_\omega(\triangle_{\boldsymbol{\mu}}(\varphi))$ if applicable. We have $L_\omega(\varphi) = L_\omega(\mathfrak{m}) - 1$ so $\mathfrak{m} \in \mathcal{D}_\triangle \iff L_\omega(\mathfrak{m}) \in \mathfrak{D}_\triangle$, and $\triangle_{\boldsymbol{\mu}}(L_\omega(\mathfrak{m})) = L_\omega(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$ if applicable.

In all other cases, the inequality $\alpha\,\omega \leqslant \beta$ implies that $\beta > 1$, so we will only deal with the cases i and iii. Moreover, we have $\mathfrak{m} \in \mathcal{D}_\triangle \iff \varphi \in \mathcal{D}_\triangle$ by iii.

If $\alpha = \beta\omega$, then φ is $L_{<\alpha}$-atomic. The induction hypothesis yields $\varphi \in \mathcal{D}_\triangle \iff L_\alpha(\varphi) \in \mathcal{D}_\triangle$, and if applicable. Since $L_\alpha(\varphi) = L_\alpha(\mathfrak{m}) - 1$, we obtain $(\triangle_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m}))) = L_\alpha(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$ if applicable.

If $\alpha = \beta$, then $L_\alpha(\mathfrak{m}) = \varphi$ so by iii, we have $L_\alpha(\mathfrak{m}) \in \mathcal{D}_\triangle \iff \mathfrak{m} \in \mathcal{D}_\triangle$. If applicable, we have $\triangle_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = L_\alpha(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$ by (6.4).

If $\beta = \alpha\,\omega$, then $L_\alpha(\mathfrak{m}) = E_\beta^{\varphi-1}$. If $E_\beta^{\varphi-1} \notin \mathbb{U}$, then by iii, we have $L_\alpha(\mathfrak{m}) \in \mathcal{D}_\triangle \iff \varphi - 1 \in \mathcal{D}_\triangle \iff \mathfrak{m} \in \mathcal{D}_\triangle$. If applicable, we have $\triangle_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = L_\alpha(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$ by (6.4). If $E_\beta^{\varphi-1} \in \mathbb{U}$, then $\varphi \in \mathbb{U}$, so $\varphi \in \mathcal{D}_\triangle$, so $\mathfrak{m} \in \mathcal{D}_\triangle$. We have $\triangle_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = L_\alpha(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$ by (6.1) and (6.4).

If $\beta > \alpha\,\omega$, then we have $L_\alpha(\mathfrak{m}) = L_\alpha\, E_\beta^\varphi$. Since $\alpha \geqslant \omega$, the series $L_\alpha(\mathfrak{m})$ occurs in iii, and we have $L_\alpha(\mathfrak{m}) \in \mathcal{D}_\triangle \iff \varphi \in \mathcal{D}_\triangle \iff \mathfrak{m} \in \mathcal{D}_\triangle$. If applicable, we have $\triangle_{\boldsymbol{\mu}}(L_\alpha(\mathfrak{m})) = L_\alpha(\triangle_{\boldsymbol{\mu}}(\mathfrak{m}))$ by (6.4). This concludes the inductive proof. □

Lemma 6.8. *For $s, t \in \mathcal{D}_\triangle$ with $s\,t \in \mathcal{D}_\triangle$, we have $\triangle_{\boldsymbol{\mu}}(s\,t) = \triangle_{\boldsymbol{\mu}}(s)\,\triangle_{\boldsymbol{\mu}}(t)$.*

Proof. We have $\operatorname{supp} s \cup \operatorname{supp} t \cup \operatorname{supp} s\,t \subseteq \mathcal{D}_\triangle$ by iv, so by (6.5), it is enough to prove the result for $\mathfrak{s}, \mathfrak{t} \in \mathcal{D}_\triangle \cap \mathfrak{U}_{(<\boldsymbol{\mu})}$. In that case we have $\log(\mathfrak{s}), \log(\mathfrak{t}), \log(\mathfrak{s}\,\mathfrak{t}) \in \mathcal{D}_\triangle$ and

$$\begin{aligned} \triangle_{\boldsymbol{\mu}}(\log \mathfrak{s}) &= \log \triangle_{\boldsymbol{\mu}}(\mathfrak{s}), \\ \triangle_{\boldsymbol{\mu}}(\log \mathfrak{t}) &= \log \triangle_{\boldsymbol{\mu}}(\mathfrak{s}), \text{ and} \\ \triangle_{\boldsymbol{\mu}}(\log(\mathfrak{s}\,\mathfrak{t})) &= \log \triangle_{\boldsymbol{\mu}}(\mathfrak{s}\,\mathfrak{t}), \end{aligned}$$

by Proposition 6.7. Therefore

$$\begin{aligned} \triangle_{\boldsymbol{\mu}}(\mathfrak{s}\,\mathfrak{t}) &= \exp(\triangle_{\boldsymbol{\mu}}(\log(\mathfrak{s}\,\mathfrak{t}))) \\ &= \exp(\triangle_{\boldsymbol{\mu}}(\log \mathfrak{s}) + \triangle_{\boldsymbol{\mu}}(\log \mathfrak{t})) \\ &= \exp(\triangle_{\boldsymbol{\mu}}(\log \mathfrak{s}))\exp(\triangle_{\boldsymbol{\mu}}(\log \mathfrak{t})) \\ &= \triangle_{\boldsymbol{\mu}}(\mathfrak{s})\,\triangle_{\boldsymbol{\mu}}(\mathfrak{t}). \end{aligned}$$

This concludes the proof. □

By Proposition 3.7, we have:

Proposition 6.9. *Let $\mathbb{G}$ be a transserial subgroup of $\mathbb{U}_{(<\boldsymbol{\mu})}$ with $\mathbb{G} \subseteq \mathcal{D}_\triangle$. Then $\mathbb{G}_{(<1)} \subseteq \mathcal{D}_\triangle$.*

6.3 The relative near-support

Let $\mathcal{P}_\triangle$ denote the subclass of series $s \in \mathcal{D}_\triangle$ such that for all $\mathfrak{m} \in \operatorname{supp} s$, we have

$$\operatorname{supp} \triangle_{\boldsymbol{\mu}}(\mathfrak{m}) \subseteq \mathfrak{d}_{\triangle_{\boldsymbol{\mu}}(\mathfrak{m})} \cdot \mathfrak{V}^{\prec \triangle_{\boldsymbol{\mu}}(\mathfrak{m})} \cdot \mathfrak{W}_\triangle.$$

So our hypothesis that $\mathfrak{W}_\triangle$ is a relative near-support for $\triangle$ translates as the inclusion $\mathbb{U} \subseteq \mathcal{P}_\triangle$. We will prove that $\mathcal{D}_\triangle = \mathcal{P}_\triangle = \mathbb{U}_{(<\boldsymbol{\mu})}$.

Lemma 6.10. *Let $\eta \in (0, \boldsymbol{\mu})$, set $\alpha := \omega^\eta$ and let $\varphi \in \mathcal{P}_\triangle \cap (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\alpha}$. There are a $\sigma_0 < \eta$ and an $n_0 < \omega$ such that $L_{\omega^\sigma n}(E_\alpha^\varphi) \in \mathcal{P}_\triangle$ whenever $\sigma \in [\sigma_0, \eta)$ and $n \in [n_0, \omega)$.*

Proof. By (2.14), there are a $\sigma_0 < \eta$ and an $n_0 \in \mathbb{N}^>$ such that the series $\varepsilon := \triangle_{\boldsymbol{\mu}}(\varphi) - \sharp_\alpha(\triangle_{\boldsymbol{\mu}}(\varphi))$ satisfies $\varepsilon \prec (L_{\omega^{\sigma_0}(n_0-1)}(E_\alpha(\triangle_{\boldsymbol{\mu}}(\varphi))))^{-1}$. Let $\sigma < \eta$ and $n \in \mathbb{N}$ with $\omega^\sigma n \geqslant \omega^{\sigma_0} n_0$. We write $\beta := \omega^\sigma n$, $\beta^- := \omega^\sigma (n-1)$ and $\sigma := \sharp_\alpha(\triangle_{\boldsymbol{\mu}}(\varphi))$. Write $\mathfrak{a} := L_\beta(E_\alpha^\varphi)$. We have $\varepsilon \prec (L_\beta(E_\alpha(\triangle_{\boldsymbol{\mu}}(\varphi))))^{-1}$. By Proposition 6.7, we have a family $(t_{\beta,k})_{k\in\mathbb{N}}$ as in (2.8, 2.9) with

$$\triangle_{\boldsymbol{\mu}}(\mathfrak{a}) = \sum_{k\in\mathbb{N}} \frac{t_{\beta,k} \circ_{\mathbb{V}} E_\alpha^\sigma}{k!} \varepsilon^k.$$

In particular $\mathfrak{d}_{\triangle_{\boldsymbol{\mu}}(\mathfrak{a})} = L_\beta(E_\alpha^\sigma)$. Let $\mathfrak{m} \in \operatorname{supp} \triangle_{\boldsymbol{\mu}}(\mathfrak{a})$. We have $\mathfrak{m} = \mathfrak{q}_k \circ_{\mathbb{V}} E_\alpha^\sigma \cdot \mathfrak{n}$ for certain $k \in \mathbb{N}$, $\mathfrak{q}_k \in \operatorname{supp} t_{\beta,k}$ and for a certain $\mathfrak{n} \in \operatorname{supp} \varepsilon^k$. Set $\mathfrak{v}_k = \frac{\mathfrak{q}_k \circ_{\mathbb{V}} E_\alpha^\sigma}{L_\beta(E_\alpha^\sigma)}$, so that $\mathfrak{m} = \mathfrak{d}_{\triangle_{\boldsymbol{\mu}}(\mathfrak{a})} \mathfrak{v}_k \mathfrak{n}$. In order to conclude, we must prove that $\mathfrak{v}_k \mathfrak{n} \in \mathfrak{V}^{\prec\triangle_{\boldsymbol{\mu}}(\mathfrak{a})} \cdot \mathfrak{W}_\triangle$.

By Lemma 2.8, we have $1 \ll \mathfrak{v}_k \leqslant L_{\beta+1}(E_\alpha^\sigma)$, so $\mathfrak{v}_k \in \mathfrak{V}^{\prec\triangle_{\boldsymbol{\mu}}(\mathfrak{a})}$. We now turn to $\mathfrak{n}$. We have $\operatorname{supp} \varepsilon \subseteq \operatorname{supp} \triangle_{\boldsymbol{\mu}}(\varphi)$, so since $\varphi \in \mathcal{P}_\triangle$, there are $\mathfrak{n}_1, \dots, \mathfrak{n}_k \in \mathfrak{d}_{\operatorname{supp}\triangle_{\boldsymbol{\mu}}(\varphi)}$ with

$$\mathfrak{n} \in (\mathfrak{n}_1 \cdots \mathfrak{n}_k) \cdot \mathfrak{V}^{\prec\mathfrak{n}_1} \cdots \mathfrak{V}^{\prec\mathfrak{n}_1} \cdot \mathfrak{W}_\triangle^k = (\mathfrak{n}_1 \cdots \mathfrak{n}_k) \cdot \mathfrak{V}^{\prec\mathfrak{n}_1} \cdots \mathfrak{V}^{\prec\mathfrak{n}_1} \cdot \mathfrak{W}_\triangle.$$

Consider $m \in \{1, \dots, k\}$. We have $\mathfrak{n}_m \in \mathfrak{d}_{\operatorname{supp}\triangle_{\boldsymbol{\mu}}(\varphi)}$ where φ is α-truncated. We deduce that $\mathfrak{n}_m \succ (L_{<\alpha}(E_\alpha(\triangle_{\boldsymbol{\mu}}(\varphi))))^{-1}$, so $\mathfrak{n}_m \succ (L_{<\alpha} E_\alpha^\sigma)^{-1}$. We also have $\mathfrak{n}_m \preccurlyeq \mathfrak{d}_{\operatorname{supp}\triangle_{\boldsymbol{\mu}}(\varphi)} \ll L_{<\alpha}(E_\alpha^\sigma)$. Thus in total

$$L_\beta (E_\alpha^\sigma)^{-1} \ll \mathfrak{n}_m \ll L_\beta(E_\alpha^\sigma).$$

In other words $\mathfrak{V}^{\prec\mathfrak{n}_m} \subseteq \mathfrak{V}^{\prec\triangle_{\boldsymbol{\mu}}(\mathfrak{a})}$. We deduce that $\mathfrak{v}_k \cdot \mathfrak{V}^{\prec\mathfrak{n}_1} \cdots \mathfrak{V}^{\prec\mathfrak{n}_k} \subseteq \mathfrak{V}^{\prec\triangle_{\boldsymbol{\mu}}(\mathfrak{a})}$, so

$$\mathfrak{v}_k \mathfrak{n} \in \mathfrak{V}^{\prec\triangle_{\boldsymbol{\mu}}(\mathfrak{a})} \cdot \mathfrak{W}_\triangle$$

This concludes the proof. □

Lemma 6.11. *Let $\varphi \in \mathcal{P}_\triangle \cap (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,1}$. We have $E_1^\varphi \in \mathcal{P}_\triangle$.*

Proof. Write $\triangle_{\boldsymbol{\mu}}(\varphi) = \triangle_{\boldsymbol{\mu}}(\varphi)_\succ + r + \triangle_{\boldsymbol{\mu}}(\varphi)_\prec$ where $\operatorname{supp} \triangle_{\boldsymbol{\mu}}(\varphi)_\succ \succ 1$, $r \in \mathbb{R}$ and $\triangle_{\boldsymbol{\mu}}(\varphi)_\prec \prec 1$. By Proposition 6.7, we have

$$\triangle_{\boldsymbol{\mu}}(E_1^\varphi) = \exp(\triangle_{\boldsymbol{\mu}}(\varphi)) = \exp(r) \exp(\triangle_{\boldsymbol{\mu}}(\varphi)_\succ) \left(\sum_{k\in\mathbb{N}} \frac{1}{k!} (\triangle_{\boldsymbol{\mu}}(\varphi)_\prec)^k \right). \tag{6.6}$$

Note that $\exp(\triangle_1(\varphi)_\succ) = \mathfrak{d}_{\triangle_{\boldsymbol{\mu}}(E_1^\varphi)}$. Let $\mathfrak{v} \in \operatorname{supp} \triangle_{\boldsymbol{\mu}}(E_1^\varphi)$. So there is $k \in \mathbb{N}$ such that $\mathfrak{v} = \mathfrak{d}_{\triangle_{\boldsymbol{\mu}}(E_1^\varphi)} \mathfrak{m}$ for a certain $\mathfrak{m} \in \operatorname{supp} (\triangle_{\boldsymbol{\mu}}(\varphi)_\prec)^k$. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_k \in \operatorname{supp} \triangle_{\boldsymbol{\mu}}(\varphi)$ with $\mathfrak{m}_1, \dots, \mathfrak{m}_k \prec 1$ and $\mathfrak{m} = \mathfrak{m}_1 \cdots \mathfrak{m}_k$. By the induction hypothesis, for each $i \in \{1, \dots, k\}$, there is $\mathfrak{n}_i \in \operatorname{supp} \varphi$ with $\mathfrak{m}_i \in \mathfrak{d}_{\triangle_{\boldsymbol{\mu}}(\mathfrak{n}_i)} \cdot \mathfrak{V}^{\prec\triangle_{\boldsymbol{\mu}}(\mathfrak{n}_i)} \cdot \mathfrak{W}_\triangle$. Since $\operatorname{supp} \varphi \succ 1$ we have $\mathfrak{n}_i \prec \mathrm{e}^\varphi$ for all $i \in \{1, \dots, k\}$. We deduce that we have $\mathfrak{d}_{\triangle_{\boldsymbol{\mu}}(\mathfrak{n}_i)} \cdot \mathfrak{V}^{\prec\triangle_{\boldsymbol{\mu}}(\mathfrak{n}_i)} \subseteq \mathfrak{V}^{\prec\triangle_{\boldsymbol{\mu}}(E_1^\varphi)}$ for all $i \in \{1, \dots, k\}$. Thus $\mathfrak{m} \in \mathfrak{d}_{\triangle_{\boldsymbol{\mu}}(E_1^\varphi)} \cdot \mathfrak{V}^{\prec\triangle_{\boldsymbol{\mu}}(E_1^\varphi)} \cdot \mathfrak{W}_\triangle$. This concludes the proof. □

Corollary 6.12. *Let $\eta < \boldsymbol{\mu}$ and let $\varphi \in \mathcal{P}_\triangle \cap (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\omega^\eta}$. We have $E_{\omega^\eta}^\varphi \in \mathcal{P}_\triangle$.*

Proof. We prove the result by induction on η. The case $\eta = 0$ is done in Lemma 6.11. Let $\eta < \boldsymbol{\mu}$ with $\eta > 0$ such that the result holds for all $\sigma < \eta$, set $\alpha := \omega^\eta$ and let $\varphi \in \mathcal{P}_\triangle \cap (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\alpha}$. By Lemma 6.10, there are $\sigma < \mu$ and $n \in \mathbb{N}$ with $L_{\omega^\sigma n}(E_\alpha^\varphi) \in \mathcal{P}_\triangle$. Note that for $k \in \{0, \dots, n\}$, the series $L_{\omega^\sigma k}(E_\alpha^\varphi)$ is an infinite monomial, so it is ω^σ-truncated. By the induction hypothesis, we deduce that $E_{\omega^\sigma n}(L_{\omega^\sigma k}(E_\alpha^\varphi)) = E_\alpha^\varphi \in \mathcal{P}_\triangle$, hence the result by induction. □

Proposition 6.13. *Let $\eta<\boldsymbol{\mu}$ and let $\varphi\in\mathcal{P}_\triangle\cap(\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\omega^\eta}$. For $\gamma<\omega^\eta$, we have $L_\gamma(E^\varphi_{\omega^\eta})\in\mathcal{P}_\triangle$.*

Proof. We prove this by induction on η. The case $\eta=0$ is vacuously true by Lemma 6.11. Let $\eta<\boldsymbol{\mu}$ with $\eta>0$ such that the result holds for all $\sigma<\eta$, set $\alpha:=\omega^\eta$ and let $\varphi\in\mathcal{P}_\triangle\cap(\mathbb{U}_{(<\boldsymbol{\lambda})})_{\succ,\alpha}$ and $\gamma<\alpha$. Write $\beta:=\alpha$ if η is a limit, and $\beta:=\omega^\iota$ if $\eta=\iota+1$. Considering $\varphi-m\in\mathcal{P}_\triangle$ for a certain m with $\gamma=\beta m+\gamma'$ with $\gamma'<\beta$, we may assume that $\gamma\omega<\alpha$. Consider by Lemma 6.10 a $\sigma<\eta$ and a $n\in\mathbb{N}^>$ with $\psi:=L_{\omega^\sigma n}(E^\varphi_\alpha)\in\mathcal{P}_\triangle$. Choosing σ large enough, we have $\gamma<\omega^\sigma$ by the previous argument. We have $\psi\in\mathcal{P}_\triangle\cap(\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\omega^\sigma}$, so $E^\psi_{\omega^\sigma(n-1)}\in\mathcal{P}_\triangle\cap(\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\omega^\sigma}$ by Corollary 6.12. So by the induction hypothesis, we have

$$L_\gamma\Big(E^{E^\psi_{\omega^\sigma(n-1)}}_{\omega^\sigma}\Big)=L_\gamma(E^\varphi_\alpha)\in\mathcal{P}_\triangle.$$

This concludes the proof. □

6.4 The extension theorem for right compositions

Proposition 6.14. *We have $\mathbb{E}(\mathcal{P}_\triangle)\subseteq\mathcal{P}_\triangle$.*

Proof. Since each monomial in $\mathbb{E}(\mathcal{P}_\triangle)$ is atomic, this follows from Lemma 1.37. □

Proposition 6.15. *We have $(\mathbb{G}(\mathcal{P}_\triangle))_{(<1)}\subseteq\mathcal{P}_\triangle$.*

Proof. By Proposition 6.14, we have $\mathbb{E}(\mathcal{P}_\triangle)\subseteq\mathcal{P}_\triangle$, hence $\mathbb{G}(\mathcal{P}_\triangle)\subseteq\mathcal{P}_\triangle$. We obtain $(\mathbb{G}(\mathcal{P}_\triangle))_{(<1)}\subseteq\mathcal{D}_\triangle$ by Proposition 6.9. An easy induction using Proposition 6.13 shows that $(\mathbb{G}(\mathcal{P}_\triangle))_{(<1)}\subseteq\mathcal{P}_\triangle$. □

Corollary 6.16. *We have $\mathcal{P}_\triangle=\mathbb{U}_{(<\boldsymbol{\mu})}$.*

Proof. Apply Lemma 4.10. □

Corollary 6.17. *The function $\triangle_{\boldsymbol{\mu}}\colon\mathbb{U}_{(<\boldsymbol{\mu})}\longrightarrow\mathbb{V}_{(<\boldsymbol{\mu})}$ is a right composition of force $\boldsymbol{\nu}$ and $\mathfrak{W}_\triangle$ is a good relative near-support for $\triangle_{\boldsymbol{\mu}}$.*

Proof. We already know since $\mathbb{U}_{(<\boldsymbol{\mu})}\subseteq\mathcal{P}_\triangle$ that $\triangle_{\boldsymbol{\mu}}$ is strongly linear on $\mathbb{U}_{(<\boldsymbol{\mu})}$ with relative near-support $\mathfrak{W}_\triangle$. It is a ring morphism by Lemma 6.8. Finally, by Propositions 6.7 and 6.4, it is a hyperserial composition of force $\boldsymbol{\nu}$. □

In order to complete our proof of Theorem 6.1, we must prove the unicity of $\triangle_{\boldsymbol{\mu}}$.

Proposition 6.18. *The extension of $\triangle$ to $\mathbb{U}_{(<\boldsymbol{\mu})}$ is unique.*

Proof. Let $\triangledown$ be a right composition $\mathbb{U}_{(<\boldsymbol{\mu})}\longrightarrow\mathbb{V}_{(<\boldsymbol{\mu})}$ which extends $\triangle$. We claim that $\triangledown=\triangle_{\boldsymbol{\mu}}$, and we prove the result by induction on the hyperexponential height $(\gamma,\boldsymbol{\eta})$ of s over $\mathbb{U}$. We have $\triangledown=\triangle_{\boldsymbol{\mu}}$ on $\mathbb{U}$ by definition. Now let $s\in\mathbb{U}_{(<\boldsymbol{\mu})}$ such that we have $\triangledown(g)=\triangle_{\boldsymbol{\mu}}(g)$ for all $g\in\mathbb{U}_{(<\boldsymbol{\mu})}$ with $\mathrm{HH}_{\mathbb{U}}(g)<_{\mathrm{lex}}(\gamma,\boldsymbol{\eta})$. By **RC1**, we may assume that s is a monomial. By **RC2**, it is enough to prove that $\triangledown(\mathfrak{m})=\triangle_{\boldsymbol{\mu}}(\mathfrak{m})$ for all $\mathfrak{m}\in\operatorname{supp}\log s$. Consider such a monomial $\mathfrak{m}$. If $\boldsymbol{\eta}=1$ or $\boldsymbol{\eta}$ is a limit, then we have $\mathrm{HH}_{\mathbb{U}}(\mathfrak{m})<_{\mathrm{lex}}(\gamma,\boldsymbol{\eta})$, whence $\triangledown(\mathfrak{m})=\triangle_{\boldsymbol{\mu}}(\mathfrak{m})$. Otherwise write $\boldsymbol{\eta}=\iota+1$ for a certain $\iota>0$. We have $\mathfrak{m}=L_\rho(E^\varphi_{\omega^\iota})$ for certain $\rho<\boldsymbol{\lambda}$ with $\rho\omega<\omega^\iota$ and $\varphi\in(\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\omega^\iota}$ with $\mathrm{HH}_{\mathbb{U}}(\varphi)<_{\mathrm{lex}}(\gamma,\boldsymbol{\eta})$. The induction hypothesis yields $\triangledown(\varphi)=\triangle_{\boldsymbol{\mu}}(\varphi)$. We deduce with **RC2** that $\triangledown(\mathfrak{m})=\triangle_{\boldsymbol{\mu}}(\mathfrak{m})$. By induction, we deduce that $\triangledown$ and $\triangle_{\boldsymbol{\mu}}$ coincide. □

6.5 Hyperserial composition laws

Let $(\mathbb{U}, \circ_{\mathbb{U}})$, $(\mathbb{T}, \circ_{\mathbb{T}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ be confluent hyperserial fields of force $\boldsymbol{\nu}$. A *hyperserial composition law* $\circ\colon \mathbb{U} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{V}$ of force $\boldsymbol{\nu}$ is a function such that for each $s \in \mathbb{T}^{>,\succ}$, the function

$$\circ_s\colon \mathbb{U} \longrightarrow \mathbb{V};\ f \mapsto f \circ s$$

is a right composition of force $\boldsymbol{\nu}$. The function $\circ_s$ is called the *right composition with* s.

Example 6.19. By **HF1** and **HF2**, the law $\circ_{\mathbb{U}}\colon \mathbb{L}_{<\boldsymbol{\lambda}} \times \mathbb{U}^{>,\succ} \longrightarrow \mathbb{U}$ is a hyperserial composition of force $\boldsymbol{\nu}$. If $(\mathbb{U}, \circ_{\mathbb{U}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$, then we have a *trivial composition law*

$$\mathbb{U} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{V};\ (f, s) \mapsto f,$$

all of whose right compositions are the inclusion $\mathbb{U} \hookrightarrow \mathbb{V}$.

Remark 6.20. We do not ask that hyperserial composition laws be associative with respect to an eventual internal composition law $\mathbb{U} \times \mathbb{U}^{>,\succ} \longrightarrow \mathbb{U}$. Indeed, studying the right context for that would take us beyond the scope of this paper. However we will see that such an extended associativity follows immediately in the case when $\mathbb{U} = \tilde{\mathbb{L}}$. Some additional properties of hyperserial compositions laws pertaining to the structure of hyperserial field on $\mathbb{T}$ are considered in Section 7 and 8.

As immediate corollaries of Theorem 6.1, we have:

Corollary 6.21. *Let $\boldsymbol{\mu} \leqslant \mathbf{On}$ with $0 < \boldsymbol{\mu} \leqslant \boldsymbol{\nu}$. If each $\circ_s$ for $s \in \mathbb{T}^{>,\succ}$ has a good relative near-support, then $\circ$ extends uniquely into a hyperserial composition $\mathbb{U}_{(<\boldsymbol{\mu})} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{V}_{(<\boldsymbol{\mu})}$ of force $\boldsymbol{\nu}$.*

Corollary 6.22. *Let $\boldsymbol{\mu} \leqslant \mathbf{On}$ with $0 < \boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ and let $(\mathbb{T}, \circ_{\mathbb{T}})$ be a confluent hyperserial field of force $\boldsymbol{\nu}$. There is a unique extension of $\circ_{\mathbb{T}}$ into a hyperserial composition $(\mathbb{L}_{<\boldsymbol{\lambda}})_{(<\boldsymbol{\mu})} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{T}_{(<\boldsymbol{\mu})}$ of force $\boldsymbol{\nu}$.*

Corollary 6.23. *There is a unique extension of the standard composition $\mathbb{L} \times \tilde{\mathbb{L}}^{>,\succ} \longrightarrow \tilde{\mathbb{L}}$ into a hyperserial composition law $\circ\colon \tilde{\mathbb{L}} \times \tilde{\mathbb{L}}^{>,\succ} \longrightarrow \tilde{\mathbb{L}}$ of force $\mathbf{On}$. Moreover, for all $f \in \tilde{\mathbb{L}}$ and $g, h \in \tilde{\mathbb{L}}^{>,\succ}$, we have $f \circ (g \circ h) = (f \circ g) \circ h$.*

Proof. The existence and unicity of $\circ$ follows from Corollary 6.22. Given $\xi \in \tilde{\mathbb{L}}^{>,\succ}$, let $\triangle_\xi$ denote the right composition $\tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ with ξ. Let $g, h \in \tilde{\mathbb{L}}^{>,\succ}$ be fixed. By **RC2** for $\triangle_h$, we have $\triangle_{g\circ h} \restriction \mathbb{L} = (\triangle_h \circ \triangle_g) \restriction \mathbb{L}$. So $\triangle_{g\circ h}$ and $\triangle_h \circ \triangle_g$ are both right compositions of force $\mathbf{On}$ which extend $\triangle_{g\circ h} \restriction \mathbb{L}$. From the unicity in Theorem 6.1, we deduce that $\triangle_{g\circ h} = \triangle_h \circ \triangle_g$, i.e. $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f \in \tilde{\mathbb{L}}$. □

We next see that right compositions on subfields of $\tilde{\mathbb{L}}$ really are right compositions $\circ_\xi$ with certain positive infinite series ξ, with respect to a suited hyperserial composition law $\circ$.

Lemma 6.24. *The atomic elements of $\widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$ are ℓ_0 if $\boldsymbol{\nu}$ is a limit and the series $\ell_{\omega^{\eta_n}}$ and $\mathrm{e}^{\ell_0}_{\omega^{\eta_n}}$ for all $n \in \mathbb{N}$ if $\boldsymbol{\nu} = \eta + 1$ is a successor.*

Proof. By [7, Theorem 3.16 and 4.1], the atomic elements of $\mathbb{L}_{<\boldsymbol{\lambda}}$ are ℓ_0 if $\boldsymbol{\nu}$ is a limit, and and the series $\ell_{\omega^{\eta_n}}$ for all $n \in \mathbb{N}$ if $\boldsymbol{\nu} = \eta + 1$ is a successor. If $\boldsymbol{\nu}$ is a limit, then by Proposition 4.3, there is no atomic element in $\widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}} \setminus \mathbb{L}_{<\boldsymbol{\lambda}}$.

Assume that $\boldsymbol{\nu} = \eta + 1$ is a successor. If $(\mathbb{T}, \circ)$ is a confluent hyperserial field of force $\boldsymbol{\nu}$, then by [7, Remark 8.23], the atomic elements of $\mathbb{T}_{(\eta)}$ are those of the form $\mathfrak{a} = E^{\mathfrak{b}}_{\omega^\eta}$ where $\mathfrak{b} \in \mathbb{T}_{(\gamma,\iota)}$ is atomic. For $\iota < \eta$, the atomic elements of $\mathbb{T}_{(\iota)}$ are those of $\mathbb{T}$. It follows by induction according to Definition 4.1 that the atomic elements of $\widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$ are the series $E^{\mathfrak{b}}_{\omega^{\eta_n}}$ where $\mathfrak{b} \in \mathbb{L}_{<\boldsymbol{\lambda}}$ is atomic and $n \in \mathbb{N}$. □

Proposition 6.25. *Assume that* $(\mathbb{U}, \circ_{\mathbb{U}})$ *be a confluent hyperserial subfield of* $\widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$ *of force* $\boldsymbol{\nu}$ *and that* $(\mathbb{V}, \circ_{\mathbb{V}})$ *is a confluent hyperserial field of force* $(\boldsymbol{\nu}, \boldsymbol{\nu})$*. Then there is a hyperserial composition law* $\circ\colon \mathbb{U} \times \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}$ *of force* $\boldsymbol{\nu}$ *such that each right composition* $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ *of force* $\boldsymbol{\nu}$ *is a right composition* $\circ_\xi$ *with a unique* $\xi \in \mathbb{V}^{>,\succ}$*.*

Proof. By Corollary 6.22, the hyperserial composition $\circ_{\mathbb{V}}\colon \mathbb{L}_{<\boldsymbol{\lambda}} \times \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}$ extends uniquely into a hyperserial composition law $(\circ_{\mathbb{V}})_{\boldsymbol{\nu}}\colon \widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}} \times \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}$ of force $\boldsymbol{\nu}$ which restricts to a hyperserial composition law $\circ\colon \mathbb{U} \times \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}$ of force $\boldsymbol{\nu}$. Consider a right composition $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ of force $\boldsymbol{\nu}$. Likewise $\triangle$ extends uniquely into a right composition $\triangle_{\boldsymbol{\nu}}\colon \mathbb{U}_{(<\boldsymbol{\nu})} \longrightarrow \mathbb{V}_{(<\boldsymbol{\nu})}$. We claim that $\mathbb{U}_{(<\boldsymbol{\nu})} = \widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$. Indeed, we have $\mathbb{U}_{(<\boldsymbol{\nu})} \subseteq \widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$ since $\mathbb{U} \subseteq \widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$. Now let $\mathfrak{a} \in \mathbb{U}$ be atomic. As a consequence of Lemma 6.24, there is a $\gamma < \boldsymbol{\lambda}$ with $\mathfrak{a} = \ell_\gamma$ or $\mathfrak{a} = \mathrm{e}_\gamma^{\ell_0}$. We deduce that $\ell_0 \in \mathbb{U}_{(<\boldsymbol{\nu})}$, so $\mathbb{L}_{<\boldsymbol{\lambda}} = \mathbb{L}_{<\boldsymbol{\lambda}} \circ \ell_0 \subseteq \mathbb{U}_{(<\boldsymbol{\nu})}$, so $\widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}} \subseteq \mathbb{U}_{(<\boldsymbol{\nu})}$ as claimed. For $f \in \mathbb{L}_{<\boldsymbol{\lambda}} \subseteq \mathbb{U}_{(<\boldsymbol{\nu})}$, we have

$$\triangle_{\boldsymbol{\nu}}(f) = \triangle(f \circ_{\mathbb{U}} \ell_0) = f \circ_{\mathbb{V}} \triangle(\ell_0)$$

by **RC2**. Thus $\triangle_{\boldsymbol{\nu}} \restriction \mathbb{L}_{<\boldsymbol{\lambda}}$ is the right composition $\mathbb{L}_{<\boldsymbol{\lambda}} \longrightarrow \mathbb{V}$ by $\triangle_{\boldsymbol{\nu}}(\ell_0)$. We deduce by the unicity in Theorem 6.1 that $\triangle$ is the right composition $(\triangle \restriction \mathbb{L}_{<\boldsymbol{\lambda}})_{\boldsymbol{\nu}}\colon \widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}} \longrightarrow \mathbb{V}_{(<\boldsymbol{\nu})}$ by $\triangle_{\boldsymbol{\nu}}(\ell_0)$ restricted to $\mathbb{U}$. □

In this setting, hyperserial embeddings $\mathbb{U} \longrightarrow \mathbb{V}$ of force $\boldsymbol{\nu}$ are right compositions with $L_{<\omega^{\boldsymbol{\nu}}}$-atomic series. The converse is not true. Indeed, consider $\mathbb{L}_{<\omega^2}$ as a hyperserial field of force 1. So ℓ_1 is atomic in $\mathbb{L}_{<\omega^2}$. But the right composition $\mathbb{L}_{<\omega^2} \longrightarrow \mathbb{L}_{<\omega^2}$ with ℓ_1 is not a hyperserial embedding of force 2 since $\ell_\omega \circ \ell_1 = \ell_\omega - 1$ is not a monomial.

Remark 6.26. Consider in particular the case when $\mathbb{U} = \mathbb{V} \subseteq \widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$. Proposition 6.25 shows that there is a natural bijective correspondence between right compositions $\mathbb{U} \longrightarrow \mathbb{U}$ and series in $\mathbb{U}^{>,\succ}$. There are two obstructions to generalizing this to any confluent hyperserial field $\mathbb{U}$.

The first one is that there might not be a way to define a hyperserial composition law $\mathbb{U} \times \mathbb{U}^{>,\succ} \longrightarrow \mathbb{U}$ if one wants this right composition to have additional properties such as Taylor expansions with respect to a given hyperserial derivation (which the trivial composition doesn't have). Yet there always exists a right composition $\mathbb{U} \longrightarrow \mathbb{U}$ which is the identity.

The second one is that there may be non-isomorphic (in the expected sense) hyperserial composition laws $\circ^i\colon \mathbb{U} \times \mathbb{U}^{>,\succ} \longrightarrow \mathbb{U}$ for $i \in \{0, 1\}$, each one yielding its own "sheave" of right compositions $\circ_s^i$, $s \in \mathbb{U}^{>,\succ}$. We expect that this will be the case for surreal numbers.

6.6 The chain rule

Let $(\mathbb{U}, \circ_{\mathbb{U}})$, $(\mathbb{T}, \circ_{\mathbb{T}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ be confluent hyperserial fields of force $\boldsymbol{\nu}$ with $(\mathbb{T}, \circ_{\mathbb{T}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$. Let $\circ\colon \mathbb{U} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{V}$ be a composition law of force $\boldsymbol{\nu}$ with $\mathbb{U} \circ \mathbb{T}^{>,\succ} \subseteq \mathbb{V}$. Let $\partial_{\mathbb{V}}\colon \mathbb{V} \longrightarrow \mathbb{V}$ and $'\colon \mathbb{U} \longrightarrow \mathbb{U}$ be hyperserial derivations of force $\boldsymbol{\nu}$ on. We say that a series $f \in \mathbb{U}$ satisfies the *chain rule* if for all $s \in \mathbb{T}^{>,\succ}$, we have

CR. $\partial_{\mathbb{V}}(f \circ s) = \partial_{\mathbb{V}}(s) \times f' \circ s$.

Assume that each $f^{(k)}, k \in \mathbb{N}$ satisfies the chain rule and let $s \in \mathbb{T}^{>,\succ}$. For $(n, k) \in \mathbb{N}$, write

$$X_{n,k} := \{v \in (\mathbb{N}^>)^n : |v| = v_{[1]} + \cdots + v_{[n]} = k\}.$$

Then for $k \in \mathbb{N}$, we have Faà di Bruno's formula

$$\frac{(f \circ s)^{(k)}}{k!} = \sum_{n \in \mathbb{N}} \sum_{v \in X_{n,k}} \frac{f^{(n)} \circ s}{n!} \frac{\partial_{\mathbb{V}}^{v_{[1]}}(s)}{v_{[1]}!} \cdots \frac{\partial_{\mathbb{V}}^{v_{[n]}}(s)}{v_{[n]}!} \tag{6.7}$$

Theorem 6.27. *Let $(\mathbb{U}, \circ_{\mathbb{U}})$, $(\mathbb{T}, \circ_{\mathbb{T}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ be confluent hyperserial fields of force $\boldsymbol{\nu}$ with $(\mathbb{T}, \circ_{\mathbb{T}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$. Let $\circ\colon \mathbb{U}_{(<\boldsymbol{\mu})} \times \mathbb{T}^{\succ,\succ} \longrightarrow \mathbb{V}_{(<\boldsymbol{\mu})}$ be a composition law of force $\boldsymbol{\nu}$ with $\mathbb{U} \circ \mathbb{T}^{\succ,\succ} \subseteq \mathbb{V}$. Assume that there are hyperserial derivations $\partial_{\mathbb{V}}$ and $'$ of force $\boldsymbol{\nu}$ on $\mathbb{V}_{(<\boldsymbol{\mu})}$ and $\mathbb{U}_{(<\boldsymbol{\mu})}$ respectively with $\partial_{\mathbb{V}}(\mathbb{V}) \subseteq \mathbb{V}$ and $\mathbb{U}' \subseteq \mathbb{U}$. If each $f \in \mathbb{U}$ satisfies the chain rule, then each $f \in \mathbb{U}_{(<\boldsymbol{\mu})}$ satisfies the chain rule.*

Proof. We prove the result by induction on $\mathrm{HH}_{\mathbb{U}}(f)$. Write $(\gamma, \boldsymbol{\eta}) := \mathrm{HH}_{\mathbb{U}}(f)$ and assume that the result holds for all $g \in \mathbb{U}_{(<\boldsymbol{\mu})}$ with $\mathrm{HH}_{\mathbb{U}}(g) <_{\mathrm{lex}} (\gamma, \boldsymbol{\eta})$. We may assume that $(\gamma, \boldsymbol{\eta}) >_{\mathrm{lex}} (0,0)$. Since $'$ is a hyperserial derivation of force $\boldsymbol{\nu} > 0$, it satisfies $(\exp(g))' = g' \exp(g)$ for all $g \in \log(\mathbb{U}^{\succ})$, so it is enough to prove the result for $\log f$. Since $'$ is strongly linear, we may assume that $\log f$ is a monomial. So $\boldsymbol{\eta} = \iota + 1$ for a certain ordinal ι and $\log f = L_{\beta+1}(E^{\varphi}_{\omega^\iota})$ for certain $\varphi \in (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\omega^\iota}$ with $\mathrm{HH}_{\mathbb{U}}(\varphi) < (\gamma, \boldsymbol{\eta})$ and a β with $\beta\omega < \omega^\iota$. It follows that

$$\begin{aligned}
\partial_{\mathbb{V}}((\log f) \circ s) &= \partial_{\mathbb{V}}((\ell_{\beta+1} \circ \mathrm{e}_{\omega^\iota}) \circ_{\mathbb{V}} (\varphi \circ s)) && \text{(by \textbf{RC2} for } \circ_s) \\
&= \partial_{\mathbb{V}}(\varphi \circ s) \times (\ell_{\beta+1} \circ \mathrm{e}_{\omega^\iota})' \circ_{\mathbb{V}} (\varphi \circ s) && \text{(by \textbf{D4} for } \partial_{\mathbb{V}}) \\
&= \partial_{\mathbb{V}}(s) \times (\ell_{\beta+1} \circ \mathrm{e}_{\omega^\iota})' \circ_{\mathbb{V}} (\varphi \circ s) \times \varphi' \circ s && \text{(by \textbf{CR} for } \varphi) \\
&= \partial_{\mathbb{V}}(s) \times ((\ell_{\beta+1} \circ \mathrm{e}_{\omega^\iota})' \circ \varphi) \circ s \times \varphi' \circ s && \text{(by \textbf{RC2} for } \circ_s) \\
&= \partial_{\mathbb{V}}(s) \times (\log f)' \circ s. && \text{(by \textbf{D4} for } ')
\end{aligned}$$

We conclude by induction that every $f \in \mathbb{U}_{(<\boldsymbol{\mu})}$ satisfies the chain rule. □

Corollary 6.28. *Let $(\mathbb{T}, \circ)$ be a confluent hyperserial field of force $\boldsymbol{\nu}$, let $\partial\colon \mathbb{T} \longrightarrow \mathbb{T}$ be a hyperserial derivation of force $\boldsymbol{\nu}$ and assume that each $f \in \mathbb{L}_{<\boldsymbol{\lambda}}$ satisfies the chain rule. Write $\circ\colon (\mathbb{L}_{<\boldsymbol{\lambda}})_{(<\boldsymbol{\nu})} \times \mathbb{T}^{\succ,\succ} \longrightarrow \mathbb{T}$ for the unique extension of $\circ$ into a hyperserial composition of force $\boldsymbol{\nu}$. Then we have*

$$\partial(f \circ g) = \partial(g) \times f' \circ g$$

for all $f \in (\mathbb{L}_{<\boldsymbol{\lambda}})_{(<\boldsymbol{\nu})}$ and $g \in (\mathbb{L}_{<\boldsymbol{\lambda}})^{\succ,\succ}_{(<\boldsymbol{\nu})}$.

6.7 Large supports and monomial values

Let $\mathbb{U} = \mathbb{R}[[\mathfrak{U}]]$, $\mathbb{V} = \mathbb{R}[[\mathfrak{V}]]$ be confluent hyperserial fields of force $\boldsymbol{\nu}$ and let $\xi \in \mathbb{V}$. Let $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ be a right composition of force $\boldsymbol{\nu}$ with a good relative near-support $\mathfrak{W}_{\triangle}$ and let $\partial\colon \mathbb{U} \longrightarrow \mathbb{U}$ be a hyperserial derivation of force $\boldsymbol{\nu}$ with a $\{1\}$ as a good near-support.

Let $\mathfrak{m} \in \mathfrak{U}$. In Section 7, we will see that in certain cases, Taylor expansions determine how $\triangle(\mathfrak{m})$ may decompose as a series. In the opposite direction, we now give a criterion for $\triangle(\mathfrak{m})$ to be a monomial.

Given $\xi \in \mathbb{V}$, we consider the following condition on a subclass $\mathfrak{W} \subseteq \mathfrak{V}$:

$$\forall \mathfrak{w} \in \mathfrak{W}, \exists \mathfrak{m} \in (\operatorname{supp} \xi)^{\prec}, \mathfrak{w} \preceq \mathfrak{m}. \tag{6.8}$$

So (6.8) states that $(\operatorname{supp} \xi)^{\prec}$ has no strict $\prec\!\!\prec$-upper bound in $\mathfrak{W}$. Note that (6.8) is preserved under products as well as under the operation $\mathfrak{W} \mapsto \mathfrak{W}^{\infty}$.

Theorem 6.29. *Assume that $\mathfrak{W}_{\triangle}$ satisfies (6.8) with respect to ξ. Then for all $\mathfrak{m} \in \mathfrak{U}$, if $\operatorname{supp} \xi \gg \frac{\triangle(\mathfrak{n})}{\triangle(\mathfrak{n}')}$ for all $\mathfrak{n} \in \operatorname{supp} \log \mathfrak{m}$, then we have $\triangle(\mathfrak{m}) \in \mathfrak{V}$.*

Proof. Let $\mathfrak{m} \in \mathfrak{U}$ and write $\varphi := \log \mathfrak{m}$. Assume that $\operatorname{supp} \xi \gg \frac{\triangle(\mathfrak{n})}{\triangle(\partial(\mathfrak{n}))}$ for all $\mathfrak{n} \in \operatorname{supp} \varphi$. Let $\mathfrak{q} \in \operatorname{supp} \triangle(\varphi)$. So

$$\mathfrak{q} \in \mathfrak{d}_{\triangle(\mathfrak{n})} \cdot \mathfrak{V}^{\prec\!\!\prec \triangle(\mathfrak{n})} \cdot \mathfrak{w}$$

for a certain $\mathfrak{n} \in \operatorname{supp} \varphi$ and a certain $\mathfrak{w} \in \mathfrak{W}_{\triangle}$. Since $\{1\}$ is a near-support for ∂, we have $\frac{\operatorname{supp} \partial(\mathfrak{n})}{\mathfrak{d}_{\partial(\mathfrak{n})}} \prec\!\!\prec \mathfrak{d}_{\partial(\mathfrak{n})}$. We obtain $\frac{\triangle(\mathfrak{n})}{\triangle(\partial(\mathfrak{n}))} \prec\!\!\prec \triangle(\mathfrak{n})$ by Proposition 6.3 and **RC1**, whence $(\operatorname{supp} \xi)^{\prec} \prec\!\!\prec \triangle(\mathfrak{n})$. By (6.8), we have $\mathfrak{w} \prec\!\!\prec \triangle(\mathfrak{n})$. Therefore $\mathfrak{w} \prec\!\!\prec \triangle(\mathfrak{n})$, so $\mathfrak{q} \equiv \triangle(\mathfrak{n})$ by Corollary-1.31(c). It follows that $\operatorname{supp} \triangle(\varphi) \succ 1$. We deduce that $\triangle(\mathfrak{m}) = \triangle(\mathrm{e}^{\varphi}) = \exp(\triangle(\varphi)) \in \mathfrak{V}$. □

Proposition 6.30. *Let $\mathbb{T} = \mathbb{R}[[\mathfrak{M}]]$ be a confluent hyperserial field of force $\boldsymbol{\nu}$ and let $s \in \mathbb{T}^{>,\succ}$. The right composition $\circ_s \colon \mathbb{L}_{<\boldsymbol{\lambda}} \longrightarrow \mathbb{T}$ with s admits a good relative support $\mathfrak{W}_{\boldsymbol{\nu},s}$ which satisfies*

a) (6.8) with respect to s.

b) $\left(\operatorname{supp} s_{\prec} \cup \frac{\operatorname{supp} s}{\mathfrak{d}_s}\right) \subseteq \mathfrak{W}_{\boldsymbol{\nu},s}$.

c) $\mathfrak{W}_{\boldsymbol{\nu},\mathfrak{d}_s} \subseteq \mathfrak{W}_{\boldsymbol{\nu},s} \subseteq \mathfrak{W}_{\boldsymbol{\nu},\mathfrak{d}_s} \cdot (\operatorname{supp} s_{\prec})^{\infty} \cdot \left(\frac{\operatorname{supp} s}{\mathfrak{d}_s}\right)^{\infty}$.

Proof. We will construct such a relative support $\mathfrak{W}_{\boldsymbol{\eta},s}$ for each $\boldsymbol{\eta} \leqslant \boldsymbol{\nu}$ by induction. First assume that $\boldsymbol{\nu} = 0$ and let $s \in \mathbb{T}^{>,\succ}$. By [7, Lemma 5.1], we can set $\mathfrak{W}_{0,s} := (\operatorname{supp} \varepsilon_s \cup \operatorname{supp} s_{\prec})^{\infty}$ where $\varepsilon_s := (s - \tau_s)\, \tau_s^{-1} \prec 1$. For $\mathfrak{m} \in \operatorname{supp} \varepsilon_s$, there is $\mathfrak{n} \in \operatorname{supp} s$ with $\mathfrak{m} = \frac{\mathfrak{n}}{\mathfrak{d}_s}$, so $\mathfrak{m} \prec\!\!\prec \mathfrak{n}$ or $\mathfrak{m} \prec\!\!\prec \mathfrak{d}_s$ by Corollary 1.31(a). We deduce that $\mathfrak{W}_{0,s}$ satisfies (6.8), whereas c holds by definition.

Assume that the result holds for all for all $\eta < \boldsymbol{\nu}$, for all confluent hyperserial fields $\mathbb{U}$ of force η and $s \in \mathbb{U}^{>,\succ}$. For all $\eta < \boldsymbol{\nu}$ and $n < \omega$, we will prove the following additional statement by induction on $\omega^{\eta} n$:

$$\forall t \in \mathbb{T}^{>,\succ}, \forall \mathfrak{n} \in \operatorname{supp} L_{\omega^{\eta} n}(t), \exists \mathfrak{m} \in \operatorname{supp} t, \mathfrak{n} \prec\!\!\prec \mathfrak{m}, \tag{6.9}$$

as well

That is, (6.9) states that $\operatorname{supp} t$ has no strict $\prec\!\!\prec$-upper bound in $\operatorname{supp} L_{\omega^{\eta} n}(t)$. Note that if η is an ordinal such that (6.9) holds for ω^{η}, then it holds for $\omega^{\eta} n$ for all $n \in \mathbb{N}^{>}$ by transitivity of $\prec\!\!\prec$. So considering $\eta < \boldsymbol{\nu}$ such that the result holds for all $\omega^{\iota} n$ for $\iota < \eta$ and $n \in \mathbb{N}^{>}$, it is enough to prove that (6.9) holds for ω^{η}. We first assume that $\eta = 0$. Let $t \in \mathbb{T}^{>,\succ}$. In view of (2.3), we have

$$\operatorname{supp} L_1(t) \subseteq (\operatorname{supp} \ell_1 \circ \mathfrak{d}_t) \cup (\operatorname{supp} \varepsilon_t)^{\infty},$$

where $\varepsilon_t = \frac{t - \tau_t}{\tau_t}$. As above $\operatorname{supp} t$ has no strict $\prec\!\!\prec$-upper bound in $(\operatorname{supp} \varepsilon_t)^{\infty}$. By **HF1**, **HF7** and Lemma 2.1, we have $1 \prec \operatorname{supp} \ell_1 \circ \mathfrak{d}_t \prec\!\!\prec \mathfrak{d}_t$ so likewise $\operatorname{supp} t$ has no strict $\prec\!\!\prec$-upper bound in $\operatorname{supp} \ell_1 \circ \mathfrak{d}_t$. This proves (6.9) in this case. Assume now that $\eta > 0$ and let $t \in \mathbb{T}^{>,\succ}$. Recall that there are a $\iota < \eta$ and an $n \in \mathbb{N}^{>}$ such that setting $\beta = \omega^{\mu} n$ and $\delta := L_{\beta}(\mathfrak{d}_{\omega^{\eta}}(t)) - L_{\beta}(t)$, we have $\delta \prec 1$ and

$$L_{\omega^{\eta}}(t) = L_{\omega^{\eta}}(\mathfrak{d}_{\omega^{\eta}}(t)) + \sum_{k>0} \frac{(\ell_{\omega^{\eta}}^{\uparrow\beta})^{(k)} \circ (L_{\beta}(\mathfrak{d}_{\omega^{\eta}}(t)))}{k!}\, \delta^k.$$

We have $\operatorname{supp} L_{\omega^{\eta}}(\mathfrak{d}_{\omega^{\eta}}(t)) \succ (L_{<\omega^{\eta}}(t))^{-1}$ by **HF7**. In particular $\operatorname{supp} L_{\omega^{\eta}}(\mathfrak{d}_{\omega^{\eta}}(t))$ has no strict $\prec\!\!\prec$-upper bound in $\operatorname{supp} t$. For $k \in \mathbb{N}^{>}$, we have

$$\operatorname{supp} ((\ell_{\omega^{\eta}}^{\uparrow\beta})^{(k)} \circ (L_{\beta}(\mathfrak{d}_{\omega^{\eta}}(t)))) \subseteq \mathfrak{W} \circ \mathfrak{d}_{\omega^{\eta}}(t)$$

where $\mathfrak{W} := (\ell_{\omega^\eta}^{\uparrow\beta})^{(k)} \circ \ell_\beta$ is a well-based subset of $\mathfrak{L}_{[\beta,\omega^\eta)}$ [7, Lemma 5.9]. Since $L_\gamma(\mathfrak{d}_{\omega^\eta}(t)) \asymp L_\gamma(t) \prec\!\!\prec t$ for all $\gamma \in [\beta, \omega^\eta)$ and $\mathfrak{d}_{\omega^\eta}(t)$ is $L_{<\omega^\eta}$-atomic, the set $\operatorname{supp} t$ has no strict $\prec\!\!\prec$-upper bound in $\mathfrak{L}_{[\beta,\omega^\eta)} \circ \mathfrak{d}_{\omega^\eta}(t)$. Finally, we have $\operatorname{supp}\delta \subseteq \operatorname{supp} L_\beta(t)$ so the induction hypothesis on η implies that $\operatorname{supp} t$ has no strict $\prec\!\!\prec$-upper bound in $\operatorname{supp}\delta$. This concludes the proof of (6.9).

Now let $s \in \mathbb{T}^{\succ,\succ}$. As in [7, Proposition 5.11], there are $\eta < \boldsymbol{\nu}$, $n < \omega$ with $\varepsilon_s := L_{\omega^\eta n}(s) - L_{\omega^\eta n}(\mathfrak{d}_{\boldsymbol{\lambda}}(s)) \prec 1$, such that

$$\mathfrak{W}_{\boldsymbol{\nu},s} := \left(\prod_{k<n} \mathfrak{W}_{\eta, L_{\omega^\eta k}(s)}\right) \cdot (\operatorname{supp}\varepsilon)^\infty$$

contains the relative support for $\circ_s$ on $\mathbb{L}_{<\boldsymbol{\lambda}}$. By our induction hypothesis on $\boldsymbol{\nu}$ each $\operatorname{supp} L_{\omega^\eta k}(s)$ for $k < n$ has no strict $\prec\!\!\prec$-upper bound in $\mathfrak{W}_{\eta, L_{\omega^\eta k}(s)}$. Hence by (6.9), neither has $\operatorname{supp} s$. By (6.9), the set $\operatorname{supp} s$ has no strict $\prec\!\!\prec$-upper bound in $(\operatorname{supp}\varepsilon)^\infty$. We deduce that $\mathfrak{W}_{\boldsymbol{\nu},s}$ satisfies (6.8). Note that $\mathfrak{W}_{\eta,s} \subseteq \mathfrak{W}_{\boldsymbol{\nu},s}$ so b holds for $\mathfrak{W}_{\boldsymbol{\nu},s}$. It remains to show that c holds for $\mathfrak{W}_{\boldsymbol{\nu},s}$. Note that the same η, n and ε can be chosen for $\mathfrak{d}_s$, yielding

$$\mathfrak{W}_{\boldsymbol{\nu},\mathfrak{d}_s} = \left(\prod_{k<n} \mathfrak{W}_{\eta, L_{\omega^\eta k}(\mathfrak{d}_s)}\right) \cdot (\operatorname{supp}\varepsilon)^\infty.$$

So

$$\begin{aligned}
\mathfrak{W}_{\boldsymbol{\nu},s} &\subseteq \left(\prod_{k<n} \mathfrak{W}_{\eta, \mathfrak{d}_{L_{\omega^\eta k}(s)}}\right) \cdot \left((\operatorname{supp} s_\prec)^\infty \cdot \left(\frac{\operatorname{supp} s}{\mathfrak{d}_s}\right)^\infty\right) \cdot (\operatorname{supp}\varepsilon)^\infty && \text{(by c at } \eta) \\
&\subseteq \left(\prod_{k<n} \mathfrak{W}_{\eta, \mathfrak{d}_{L_{\omega^\eta k}(s)}}\right) \cdot \mathfrak{W}_{\boldsymbol{\nu},\mathfrak{d}_s} \cdot (\operatorname{supp} s_\prec)^\infty \cdot \left(\frac{\operatorname{supp} s}{\mathfrak{d}_s}\right)^\infty \\
&\subseteq \left(\prod_{k<n} \mathfrak{W}_{\eta, L_{\omega^\eta k}(\mathfrak{d}_s)}\right) \cdot \mathfrak{W}_{\boldsymbol{\nu},\mathfrak{d}_s} \cdot (\operatorname{supp} s_\prec)^\infty \cdot \left(\frac{\operatorname{supp} s}{\mathfrak{d}_s}\right)^\infty && \text{(by c at } \eta) \\
&\subseteq \mathfrak{W}_{\boldsymbol{\nu},\mathfrak{d}_s} \cdot (\operatorname{supp} s_\prec)^\infty \cdot \left(\frac{\operatorname{supp} s}{\mathfrak{d}_s}\right)^\infty.
\end{aligned}$$

Thus c holds at $\boldsymbol{\nu}$. This concludes the proof. □

We will call the class $\mathfrak{W}_s := \mathfrak{W}_{\boldsymbol{\nu},s}$ obtained in the proof the *canonical relative support* for $\circ_s$. We finally note the following property which is required in extending our results to the case of surreal numbers.

Proposition 6.31. *For $s, \sigma \in \mathbb{T}^{\succ,\succ}$ with $\sigma \trianglelefteq s$, we have $\mathfrak{W}_\sigma \subseteq \mathfrak{W}_s$.*

Proof. We have $\mathfrak{d}_\sigma = \mathfrak{d}_s$ and $\operatorname{supp}\sigma \subseteq \operatorname{supp} s$, so Proposition 6.30(c) for $\mathfrak{W}_\sigma$ gives

$$\begin{aligned}
\mathfrak{W}_{\boldsymbol{\nu},\sigma} &\subseteq \mathfrak{W}_{\boldsymbol{\nu},\mathfrak{d}_\sigma} \cdot (\operatorname{supp}\sigma_\prec)^\infty \cdot \left(\frac{\operatorname{supp}\sigma}{\mathfrak{d}_\sigma}\right)^\infty \\
&\subseteq \mathfrak{W}_{\boldsymbol{\nu},\mathfrak{d}_s} \cdot (\operatorname{supp} s_\prec)^\infty \cdot \left(\frac{\operatorname{supp} s}{\mathfrak{d}_s}\right)^\infty \\
&\subseteq \mathfrak{W}_{\boldsymbol{\nu},s} \cdot (\operatorname{supp} s_\prec)^\infty \cdot \left(\frac{\operatorname{supp} s}{\mathfrak{d}_s}\right)^\infty \\
&\subseteq \mathfrak{W}_{\boldsymbol{\nu},s},
\end{aligned}$$

as desired. □

7 Taylor expansions

Our goal in this section is to study the existence of Taylor expansions of hyperseries around a series. We fix $\boldsymbol{\nu} \leqslant \mathbf{On}$ and $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ with $0 < \boldsymbol{\mu}$. Let $(\mathbb{U}, \circ_{\mathbb{U}})$, $(\mathbb{T}, \circ_{\mathbb{T}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ be confluent hyperserial fields of force $\boldsymbol{\nu}$ such that $(\mathbb{T}, \circ_{\mathbb{T}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$ and $(\mathbb{L}_{<\boldsymbol{\lambda}}, \circ) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$. Let $\circ\colon \mathbb{U} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{V}$ be a composition law of force $\boldsymbol{\nu}$. We also assume that there is a hyperserial derivation

$$\partial\colon \mathbb{U} \longrightarrow \mathbb{U};\ f \mapsto f'$$

of force $\boldsymbol{\nu}$ which extends the standard derivation $'\colon \mathbb{L}_{<\boldsymbol{\lambda}} \longrightarrow \mathbb{L}_{<\boldsymbol{\lambda}}$, and assume that ∂ has a near-support $\mathfrak{W}_\partial \preccurlyeq \ell_0$. Finally, assume that $(\partial, \circ)$ satisfies the chain rule. Given $f \in \mathbb{U}$, $s \in \mathbb{T}^{>,\succ}$ and $\delta \in \mathbb{T}$, we study conditions under which $f \circ (s + \delta)$ is given by the Taylor series $\sum_{k\in\mathbb{N}} \frac{f^{(k)} \circ s}{k!}\, z^k$ at δ. That is, we want to find conditions under which the family $((f^{(k)} \circ s)\, \varepsilon^k)_{k\in\mathbb{N}}$ is well-based, with

$$f \circ (g + \varepsilon) = \sum_{k\in\mathbb{N}} \frac{f^{(k)} \circ s}{k!}\, \delta^k.$$

The existence of such expansions is a crucial feature of the local behavior of transseries and hyperseries. Our main result regarding these expansions, which for technical reasons will be proved in the next section, is the following:

Theorem (Corollary 8.3). *Consider the field $\tilde{\mathbb{L}}$ equipped with the derivation of Corollary 5.22 and the composition law of Corollary 6.23. For all $f, \delta \in \tilde{\mathbb{L}}$ and $g \in \tilde{\mathbb{L}}^{>,\succ}$ with $\delta \prec g$ and $(\mathfrak{m}^\dagger \circ g)\, \delta \prec 1$ for all $\mathfrak{m} \in \operatorname{supp} f$, the family $((f^{(k)} \circ g)\, \delta^k)_{k\in\mathbb{N}}$ is well-based, with*

$$f \circ (g + \delta) = \sum_{k\in\mathbb{N}} \frac{f^{(k)} \circ g}{k!}\, \delta^k.$$

This means that the function $\mathcal{A}_f\colon h \mapsto f \circ h$ is analytic on $\tilde{\mathbb{L}}^{>,\succ}$, with $\mathcal{A}_f^{(n)} = \mathcal{A}_{f^{(n)}}$ for all $n \in \mathbb{N}$ and

$$\operatorname{Conv}(\mathcal{A}_f)_g \supseteq \{\varepsilon \in \tilde{\mathbb{L}} : \varepsilon \prec g \wedge (\forall \mathfrak{m} \in \operatorname{supp} f, ((\mathfrak{m}^\dagger \circ g)\, \delta \prec 1))\} \tag{7.1}$$

for all $g \in \tilde{\mathbb{L}}^{>,\succ}$. The convergence domain (7.1) is optimal in that for $\delta \succcurlyeq \operatorname{Conv}(\mathcal{A}_f)_g$, there is a monomial $\mathfrak{m} \in \operatorname{supp} f$ for which the family $((\mathfrak{m}^{(k)} \circ g)\, \delta^k)_{k\in\mathbb{N}}$ is not well-based. There are various known results about Taylor expansions in fields of transseries. The history of these result is less linear than one might think, so we feel it is appropriate to briefly discuss those results in chronological order:

- Ecalle [18, 4.1.26bis] considered Taylor expansions of grid-based transseries or logarithmic-exponential transseries. His propositions for the domains of convergence are sometimes too small to be used appropriately (see [17, (6.32)]).
- Van den Dries, Macintyre and Marker [17, (6.8)-(4)] showed that logarithmic-exponential transseries in $\mathbb{T}_{\mathrm{LE}}$ have Taylor expansions of non-optimal radius.
- Schmeling [35, Section 6] showed that transseries in $\widetilde{\mathbb{L}_{<\omega}}$ act on hyperserial fields of force $(1, 1)$ and have Taylor expansions with optimal radius. Unfortunately, his proof is incomplete.
- Van der Hoeven [27, Proposition 5.11(c)] showed that the theorem above is valid in the field of grid-based transseries.

- Berarducci and Mantova defined a composition law $\circ\colon \widetilde{\mathbb{L}_{<\omega}} \times \mathbf{No}^{>\mathbb{R}} \longrightarrow \mathbf{No}$ on the class $\mathbf{No}$ of surreal numbers [9, Theorem 6.3] and showed [9, Theorem 7.5] that a transseries $f \in \widetilde{\mathbb{L}_{<\omega}}$ has a Taylor expansion

$$f \circ (\xi + \delta) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ \xi}{k!} \delta^k$$

at every $\xi \in \mathbf{No}^{>,\succ}$ for small enough (but undetermined) $\delta \in \mathbf{No}$ depending on f and a.

- Van den Dries, van der Hoeven and Kaplan [15, Proposition 8.1] showed that the theorem above is valid in logarithmic hyperseries, for which the condition $(\mathfrak{m}^\dagger \circ g)\,\delta \prec 1$ for all $\mathfrak{m} \in \operatorname{supp} f$ is redundant with $\delta \prec g$ whenever $f \notin \mathbb{R}$.
- With van der Hoeven, we defined [6, Theorem 1.1] a composition law $\circ\colon \mathbb{L} \times \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}$ for which $(\mathbf{No}, \circ)$ is a hyperexponentially closed confluent hyperserial field. In particular the composition law $\circ\colon \mathbb{L} \times \mathbf{No}^{>,\succ} \longrightarrow \mathbf{No}$ satisfies the same Taylor expansion property as that of $\circ\colon \mathbb{L} \times \mathbb{L}^{>,\succ} \longrightarrow \mathbb{L}$.

7.1 Taylor series

We require a few results on Taylor series in the context of fields of well-based series. Generally speaking, we will call Taylor series power series of the form $\sum_{k \in \mathbb{N}} \frac{\triangle(f^{(k)})}{k!} z^k$ where $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ is a strongly linear morphism of ordered rings.

Lemma 7.1. *Let $\mathfrak{m} \in \mathfrak{U}$. We either have $\mathfrak{m} \preccurlyeq \ell_0$ and then $\operatorname{supp} \mathfrak{m}' \preccurlyeq \ell_0$, or $\mathfrak{m} \succcurlyeq \ell_0$ and then $\operatorname{supp} \mathfrak{m}' \succcurlyeq \mathfrak{m}$.*

Proof. We have $\operatorname{supp} \mathfrak{m}' \subseteq \mathfrak{m} \cdot \mathfrak{V}^{\prec \mathfrak{m}} \cdot \mathfrak{W}_\partial$. Assume that $\mathfrak{m} \preccurlyeq \ell_0$. Then $\mathfrak{V}^{\prec \mathfrak{m}} \preccurlyeq \ell_0$ so then $\operatorname{supp} \mathfrak{m}' \preccurlyeq \ell_0$. Assume that $\mathfrak{m} \succcurlyeq \ell_0$. Then $\mathfrak{W}_\partial \prec \mathfrak{m}$ so $\frac{\operatorname{supp} \mathfrak{m}'}{\mathfrak{m}} \prec \mathfrak{m}$, whence in particular $\operatorname{supp} \mathfrak{m}' \succcurlyeq \mathfrak{m}$. □

We next adapt Schmeling's arguments in [35, Section 6.1.3] in order to prove that Taylor series converge in certain cases.

Proposition 7.2. *Let $\triangle\colon \mathbb{U} \longrightarrow \mathbb{V}$ be a strongly linear morphism of ordered rings. Let $f \in \mathbb{U}$ and $\varepsilon \in \mathbb{V}$ with $\varepsilon \prec \triangle(\ell_0)$ and $\triangle(\mathfrak{m}')\,\varepsilon \prec \triangle(\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{supp} f$. Set*

$$\mathbf{X} := \{(\mathfrak{m}, m) : \mathfrak{m} \in \mathfrak{U} \wedge m \in \mathbb{N} \wedge \triangle(\mathfrak{m}')\,\varepsilon \prec \triangle(\mathfrak{m})\}.$$

Consider the ordering on $\mathbf{X}$ given by

$$(\mathfrak{m}, m) <_{\mathbf{X}} (\mathfrak{n}, n) \Longleftrightarrow \triangle(\mathfrak{m})\,\varepsilon^m \succ \triangle(\mathfrak{n})\,\varepsilon^n.$$

Then the function $\vartheta\colon \mathbf{X} \longrightarrow \mathcal{P}(\mathfrak{U} \times \mathbb{N})$ given by

$$\vartheta(\mathfrak{m}, m) := \{(\mathfrak{n}, m+1) : \mathfrak{n} \in \operatorname{supp} \partial(\mathfrak{m})\}$$

is a strictly extensive choice operator.

Proof. We first prove that ϑ is a choice operator. Let $(\mathfrak{m}, m) \in \mathbf{X}$ and let $\mathfrak{n} \in \operatorname{supp} \mathfrak{m}'$. If $\mathfrak{m} \preccurlyeq \ell_0$, then we have $\mathfrak{n} \preccurlyeq \ell_0$ by Lemma 7.1. It follows that $\frac{\partial(\mathfrak{n})}{\mathfrak{n}} \preccurlyeq \ell_0$. We deduce since $\varepsilon \prec \triangle(\ell_0)$ that $\triangle(\partial(\mathfrak{n}))\,\varepsilon \prec \triangle(\mathfrak{n})$, so $(\mathfrak{n}, m+1) \in \mathbf{X}$. If $\mathfrak{m} \succcurlyeq \ell_0$, then Lemma 7.1 yields $\mathfrak{n} \succcurlyeq \mathfrak{m}$, whence $\frac{\mathfrak{n}'}{\mathfrak{n}} \asymp \frac{\mathfrak{m}'}{\mathfrak{m}}$. We deduce since $\triangle(\mathfrak{m}')\,\varepsilon \prec \triangle(\mathfrak{m})$ that $\triangle(\mathfrak{n}')\,\delta \prec \triangle(\mathfrak{n})$, so $(\mathfrak{n}, m+1) \in \mathbf{X}$.

We next prove that ϑ is strictly extensive. Let $(\mathfrak{m}, m+1) \in \mathbf{X}$ and let $\mathfrak{n} \in \operatorname{supp} \mathfrak{m}'$. We have $\triangle(\mathfrak{m}')\,\varepsilon \prec \triangle(\mathfrak{m})$ and $\mathfrak{n} \preccurlyeq \mathfrak{m}'$ so $\triangle(\mathfrak{n})\,\varepsilon^{m+1} \prec \triangle(\mathfrak{m})\,\varepsilon^m$, i.e. $(\mathfrak{n}, m+1) <_{\mathbf{X}} (\mathfrak{m}, m)$. □

Corollary 7.3. *If $\triangle$, f and ε are as in Proposition 7.2 above, then we have*

$$\triangle(f) \succ \triangle(f')\,\varepsilon \succ \triangle(f'')\,\varepsilon^2 \succ \cdots.$$

Corollary 7.4. *If $\triangle$, f and ε are as in Proposition 7.2 above and if moreover the family $(\triangle(f^{(k)})\,\varepsilon^k)_{k\in\mathbb{N}}$ is well-based, then*

$$\sum_{k\geqslant m} \frac{\triangle(f^{(k)})}{k!}\,\varepsilon^k \sim \frac{\triangle(f^{(m)})}{m!}\,\varepsilon^m$$

for all $m\in\mathbb{N}$.

Proof. This follows from Corollary 7.3. □

Theorem 7.5. *Let $\triangle\colon \mathbb{U}\longrightarrow\mathbb{V}$ be a strongly linear morphism of ordered rings. Let $f\in\mathbb{V}$ and $\varepsilon\in\mathbb{V}$ with $\varepsilon\prec\triangle(\ell_0)$ and $\triangle(\mathfrak{m}')\,\varepsilon\prec\triangle(\mathfrak{m})$ for all $\mathfrak{m}\in\operatorname{supp} f$. Assume that $\triangle$ preserves monomials and that*

$$\triangle(\mathfrak{m})' = \triangle(\ell_0)\,\triangle(\mathfrak{m}') \tag{7.2}$$

for all $\mathfrak{m}\in\operatorname{supp} f$. Then the family $(\triangle(\mathfrak{m}^{(k)}))_{\mathfrak{m}\in\operatorname{supp} f,k\in\mathbb{N}}$ is well-based.

Proof. By Proposition 1.7, we may assume that $\varepsilon=\mathfrak{d}_\varepsilon$ is a monomial. Consider the strictly extensive choice operator ϑ on $(\mathbf{X},<_{\mathbf{X}})$ as in Proposition 7.2. We will prove that ϑ is Noetherian. So given a Noetherian subset $Y\subseteq\mathbf{X}$, we want to prove that the set

$$Z := \{x : \exists y, (y\in Y\wedge x\in\vartheta(y))\}\subseteq\mathbf{X}$$

is Noetherian. We have $Z=\{(\mathfrak{n},m+1):\exists\mathfrak{m}\in\mathfrak{U},(\mathfrak{m},n)\in Y\wedge\mathfrak{n}\in\operatorname{supp}\mathfrak{m}'\}$. Let $(\mathfrak{n}_i,m_i+1)_{i\in\mathbb{N}}$ be a sequence in Z, and let $(\mathfrak{m}_i,m_i)_{i\in\mathbb{N}}$ be a sequence in Y with $\mathfrak{n}_i\in\operatorname{supp}\mathfrak{m}_i'$ for all $i\in\mathbb{N}$. For each $i\in\mathbb{N}$, we have $\triangle(\mathfrak{m}_i)\in\mathfrak{V}$, whence by Noetherianity of Y, we may assume that the family of monomials $(\triangle(\mathfrak{m}_i)\,\mathfrak{d}_\varepsilon^{m_i})_{i\in\mathbb{N}}$ is well-based. By strong linearity of ∂, so is $((\triangle(\mathfrak{m}_i)\,\mathfrak{d}_\varepsilon^{m_i})')_{i\in\mathbb{N}}$. By (7.2), the family $(\triangle(\ell_0)\,(\triangle(\mathfrak{m}_i'\circ s)\,\mathfrak{d}_\varepsilon^{m_i})_{i\in\mathbb{N}})$ is the sum of the families $((\triangle(\mathfrak{m}_i)\,\mathfrak{d}_\varepsilon^{m_i})')_{i\in\mathbb{N}}$ and $(-\mathfrak{d}_\varepsilon^{\dagger}\,m_i\,\triangle(\mathfrak{m}_i)\,\mathfrak{d}_\varepsilon^{m_i})_{i\in\mathbb{N}}$, so it is well-based. Therefore the family $(\triangle(\mathfrak{m}_i')\,\mathfrak{d}_\varepsilon^{m_i})_{i\in\mathbb{N}}$ is also well-based. In particular, there are $i,j\in\mathbb{N}$ with $i<j$ and $\triangle(\mathfrak{n}_i)\,\mathfrak{d}_\varepsilon^{m_i}\succ\triangle(\mathfrak{n}_j)\,\mathfrak{d}_\varepsilon^{m_j}$, whence $\triangle(\mathfrak{n}_i)\,\varepsilon^{m_i+1}\succ\triangle(\mathfrak{n}_j)\,\varepsilon^{m_j+1}$, that is, $(\mathfrak{n}_i,\,m_i+1)>_{\mathbf{X}}(\mathfrak{n}_j,\,m_j+1)$. This proves that Z is Noetherian.

By Theorem 1.42, we deduce that the family $(x_k)_{(x_0,\ldots,x_k)\in\vartheta^+(\operatorname{supp} f\times\{0\})}$ is Noetherian. Writing $x_k=(x_{i,1},k)\in\mathfrak{V}\times\mathbb{N}$ for all $(x_0,\ldots,x_k)\in\vartheta^+(\operatorname{supp} f\times\{0\})$, this means that the family $(\triangle(x_{k,1})\,\mathfrak{d}_\varepsilon^k)_{(x_0,\ldots,x_k)\in\vartheta^+(\operatorname{supp} f\times\{0\})}$ is well-based. For $\mathfrak{m}\in\operatorname{supp} f$ and $k\in\mathbb{N}$, we have

$$\triangle(\mathfrak{m}^{(k)})\,\mathfrak{d}_\varepsilon^k = \sum_{i\leqslant k}\ \sum_{(x_0,\ldots,x_k)\in\vartheta^+(\{(\mathfrak{m},0)\})} \triangle(x_{k,1})\,\mathfrak{d}_\varepsilon^k.$$

We deduce that $(\triangle(\mathfrak{m}^{(k)})\,\varepsilon^k)_{\mathfrak{m}\in\operatorname{supp} f,k\in\mathbb{N}}$ is well-based. □

Theorem 7.6. *Let $\triangle\colon\mathbb{V}\longrightarrow\mathbb{V}$ be a strongly linear morphism of ordered rings. Let $f\in\mathbb{V}$ and $\varepsilon\in\mathbb{V}$ with $\varepsilon\prec\triangle(\ell_0)$ and $\triangle(\mathfrak{m}')\,\varepsilon\prec\triangle(\mathfrak{m})$ for all $\mathfrak{m}\in\operatorname{supp} f$. Assume that $\triangle$ is bijective and that $\triangle(\mathfrak{m})'=\triangle(\ell_0)\,\triangle(\mathfrak{m}')$ for all $\mathfrak{m}\in\operatorname{supp} f$. Then the family $(\triangle(\mathfrak{m}^{(k)})\,\varepsilon^k)_{\mathfrak{m}\in\operatorname{supp} f,k\in\mathbb{N}}$ is well-based.*

Proof. Note that the functional inverse $\triangle^{\mathrm{inv}}$ of $\triangle$ is an $\mathbb{R}$-linear morphism or ordered rings. So setting $\delta:=\triangle^{\mathrm{inv}}(\varepsilon)$, we have $\mathfrak{m}'\,\delta\prec\mathfrak{m}$ for all $\mathfrak{m}\in\operatorname{supp} f$. Applying Theorem 7.5 to $(\mathrm{id}_{\mathbb{V}},f,\delta)$, we obtain that $(\mathfrak{m}^{(k)}\,\delta^k)_{\mathfrak{m}\in\operatorname{supp} f,k\in\mathbb{N}}$ is well-based. By strong linearity of $\triangle$, so is $(\triangle(\mathfrak{m}^{(k)})\,\varepsilon^k)_{\mathfrak{m}\in\operatorname{supp} f,k\in\mathbb{N}}$. □

7.2 Properties of Taylor series

We will use Taylor series in two ways: in order to define composition laws and to study the local behavior of hyperseries as functions.

Proposition 7.7. *Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ with $\boldsymbol{\nu} > 0$. Let $\mathbb{U}$ be a confluent hyperserial field of force $\boldsymbol{\nu}$, and let $\partial\colon \mathbb{U} \longrightarrow \mathbb{U}$ be a hyperserial derivation of force $\boldsymbol{\nu}$. Let $\mathbb{W}$ be a transserial subfield of $\mathbb{U}$ with a total exponential and with $L_{\omega^\mu}(\mathfrak{a}) \in \mathbb{W}$ for all $\mu < \boldsymbol{\nu}$ and $\mathfrak{a} \in \mathbb{W} \cap \mathfrak{U}_{\omega^\mu}$. Assume that $\mathbb{W}$ is closed under ∂. Let $\triangle\colon \mathbb{W} \longrightarrow \mathbb{V}$ be a transserial right composition with*

$$\triangle(L_{\omega^\mu}(\mathfrak{a})) = L_{\omega^\mu}(\triangle(\mathfrak{a})) \tag{7.3}$$

for all $\mu < \boldsymbol{\nu}$ and $\mathfrak{a} \in \mathbb{W} \cap \mathfrak{U}_{\omega^\mu}$.

Let $\mu < \boldsymbol{\nu}$, $\mathfrak{a} \in \mathbb{W} \cap \mathfrak{U}_{\omega^\mu}$ and $\varepsilon \in \mathbb{V}$ such that the family $(\triangle(\mathfrak{a}^{(k)})\,\varepsilon^k)_{k\in\mathbb{N}}$ is well-based with

$$\forall k > 0, \triangle(\mathfrak{a}) \succ \triangle(\mathfrak{a}^{(k)})\,\varepsilon^k. \tag{7.4}$$

Then the family $(\triangle((L_{\omega^\mu}\mathfrak{a})^{(k)})\,\varepsilon^k)_{k\in\mathbb{N}}$ is well-based, with

$$L_{\omega^\mu}\left(\sum_{k\in\mathbb{N}} \frac{\triangle(\mathfrak{a}^{(k)})}{k!}\,\varepsilon^k\right) = \sum_{k\in\mathbb{N}} \frac{\triangle((L_{\omega^\mu}(\mathfrak{a}))^{(k)})}{k!}\,\varepsilon^k.$$

Proof. We may assume that $\varepsilon \neq 0$. By Proposition 1.23, the function $\mathcal{A}\colon \mathbb{V}^{\preccurlyeq\varepsilon} \longrightarrow \mathbb{V}$ given for $\delta \preccurlyeq \varepsilon$ by

$$\mathcal{A}(\delta) \;:=\; \sum_{k\in\mathbb{N}} \frac{\triangle(\mathfrak{a}^{(k)})}{k!}\,\delta^k$$

is analytic on $\mathbb{V}^{\preccurlyeq\varepsilon}$. Our goal is to show that $L_{\omega^\mu}(\mathcal{A}(\varepsilon)) = \tilde{P}(\varepsilon)$ where

$$P := \sum_{k\in\mathbb{N}} \frac{\triangle((L_{\omega^\mu}(\mathfrak{a}))^{(k)})}{k!}\,z^k \in \mathbb{V}[[z]].$$

The function L_{ω^μ} is analytic at $\triangle(\mathfrak{a})$ with $\mathrm{Conv}(L_{\omega^\mu})_{\triangle(\mathfrak{a})} = \mathbb{V}^{\prec\triangle(\mathfrak{a})}$. For $n \in \mathbb{N}$ and $k > 0$, we set

$$\begin{aligned} X_{n,k} \;&:=\; \{v \in (\mathbb{N}^>)^n : |v| := v_{[1]} + \cdots + v_{[n]} = k\} \quad \text{and} \\ c_{k,n} \;&:=\; \sum_{v\in X_{n,k}} \frac{\ell_{\omega^\mu}^{(n)} \circ \triangle(\mathfrak{a})}{n!} \frac{\triangle\big(\mathfrak{a}^{(v_{[1]})}\big)}{v_{[1]}!} \cdots \frac{\triangle\big(\mathfrak{a}^{(v_{[n]})}\big)}{v_{[n]}!}. \end{aligned}$$

The monomial group of $\mathbb{V}$ is densely ordered. Furthermore we have $\mathcal{A}(\delta) - \triangle(\mathfrak{a}) \prec \triangle(\mathfrak{a})$ by (7.4), so may apply Proposition 1.26 and see that $L_{\omega^\mu} \circ \mathcal{A}$ is analytic on $\mathbb{V}^{\preccurlyeq\varepsilon}$. Moreover, the family $(c_{k,n}\,\varepsilon^k)_{n\in\mathbb{N},k>0}$ is well-based, with

$$L_{\omega^\mu} \circ \mathcal{A}(\varepsilon) = L_{\omega^\mu}(\triangle(\mathfrak{a})) + \sum_{n\in\mathbb{N},k>0} c_{n,k}\,\varepsilon^k. \tag{7.5}$$

So by Lemma 1.2, the family $(\sum_{n\in\mathbb{N}} c_{k,n}\,\varepsilon^k)_{k>0}$ is well-based, and

$$\sum_{n\in\mathbb{N},k>0} c_{n,k}\,\varepsilon^k = \sum_{k>0} \left(\sum_{n\in\mathbb{N}} c_{k,n}\right)\varepsilon^k.$$

Since $L_{\omega^\mu}(\triangle(\mathfrak{a})) = \triangle(L_{\omega^\mu}(\mathfrak{a}))$ and in view of (7.5), it suffices to show that $\sum_{n\in\mathbb{N}} c_{k,n} = \frac{(L_{\omega^\mu}(\mathfrak{a}))^{(k)}}{k!}$ for all $k > 0$.

By iterating (7.3), we have $\triangle(L_\gamma(\mathfrak{a})) = L_\gamma(\triangle(\mathfrak{a}))$ for all $\gamma < \omega^\mu$. Note that each $\ell_{\omega^\mu}^{(n)}$ for $n > 0$ lies in $\mathbb{L}_{<\omega^\mu}$. Since $\triangle$ is a transserial right composition, we deduce that $\triangle\big(\ell_{\omega^\mu}^{(n)} \circ \mathfrak{a}\big) = \ell_{\omega^\mu}^{(n)} \circ \triangle(\mathfrak{a})$ for all $n \in \mathbb{N}$. Note that each $\ell_{\omega^\mu}^{(k)}, k \in \mathbb{N}$ satisfies the chain rule at $\mathfrak{a}$, so (6.7) yields

$$\frac{(L_{\omega^\mu}(\mathfrak{a}))^{(k)}}{k!} = \sum_{n\in\mathbb{N}} \sum_{v\in X_{n,k}} \frac{\ell_{\omega^\mu}^{(n)} \circ \mathfrak{a}}{n!} \frac{\mathfrak{a}^{(v_{[1]})}}{v_{[1]}!} \cdots \frac{\mathfrak{a}^{(v_{[n]})}}{v_{[n]}!}.$$

Therefore

$$\frac{\triangle((L_{\omega^\mu}(\mathfrak{a}))^{(k)})}{k!} = \sum_{n\in\mathbb{N}} \sum_{v\in X_{n,k}} \frac{\ell_{\omega^\mu}^{(n)} \circ \triangle(\mathfrak{a})}{n!} \frac{\triangle\big(\mathfrak{a}^{(v_{[1]})}\big)}{v_{[1]}!} \cdots \frac{\triangle\big(\mathfrak{a}^{(v_{[n]})}\big)}{v_{[n]}!} = \sum_{n\in\mathbb{N}} c_{n,k}.$$

This concludes the proof. □

7.3 Taylor expansions

Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ with $\boldsymbol{\nu} > 0$ and write $\boldsymbol{\lambda} := \omega^{\boldsymbol{\nu}}$. Let $(\mathbb{U}, \circ_{\mathbb{U}})$, $(\mathbb{T}, \circ_{\mathbb{T}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ be confluent hyperserial fields of force $\boldsymbol{\nu}$ with $(\mathbb{T}, \circ_{\mathbb{T}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$ and $(\mathbb{U}, \circ_{\mathbb{U}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$. Let $\circ\colon \mathbb{U} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{V}$ be a composition law of force $\boldsymbol{\nu}$. Let $\partial\colon \mathbb{V} \longrightarrow \mathbb{V}; f \mapsto f'$ be a derivation of force $\boldsymbol{\nu}$ with $\partial(\mathbb{U}) \subseteq \mathbb{U}$, such that $(\mathbb{U}, \partial)$ is an H-field, and which satisfies the chain rule

$$\forall f \in \mathbb{U}, \forall t \in \mathbb{T}^{>,\succ}, (f \circ t)' = t'\, f' \circ t.$$

Definition 7.8. *Let $s \in \mathbb{T}^{>,\succ}$ and $f \in \mathbb{U}$. We say that f* **has a Taylor expansion** *at s (with respect to $(\circ, \partial)$) if the following holds:*

TE. *If $\delta \in \mathbb{T}$ satisfies $\delta \prec s$ and $(\mathfrak{m}' \circ s)\,\delta \prec \mathfrak{m} \circ s$ for all $\mathfrak{m} \in \operatorname{supp} f$, then the family $((f^{(k)} \circ s)\,\delta^k)_{k\in\mathbb{N}}$ is well-based, with*

$$f \circ (s + \delta) = \sum_{k\in\mathbb{N}} \frac{f^{(k)} \circ s}{k!}\, \delta^k.$$

We say that f has Taylor expansions with respect to $(\partial, \circ)$ if it has a Taylor expansion at each $s \in \mathbb{T}^{>,\succ}$ with respect to $(\partial, \circ)$. We say that $(\circ, \partial)$ has Taylor expansions if each $f \in \mathbb{U}$ has Taylor expansions with respect to $(\circ, \partial)$.

Remark 7.9. Assume that f satisfies **TE** at $s \in \mathbb{T}^{>,\succ}$ and that the set

$$\left\{ \frac{\mathfrak{m} \circ s}{\mathfrak{m}' \circ s} : \mathfrak{m} \in \operatorname{supp} f \setminus \{1\} \right\}$$

is not $\prec$-coinitial in $\mathbb{V}^{\neq}$. This is for instance the case in $\tilde{\mathbb{L}}$ or **No**. Then we have

$$f' \circ s = \lim_{\varepsilon \to 0} \frac{f \circ (s + \varepsilon) - f \circ s}{\varepsilon}.$$

So f' is determined by the composition law.

Lemma 7.10. *Let $\eta < \boldsymbol{\mu}$ and $\gamma < \omega^\eta =: \beta$. Defining $t_{\gamma,k}$ as in (2.8, 2.9), we have*

$$t_{\gamma,k} \circ E_\beta(s) = (\ell_\gamma \circ \mathrm{e}_\beta^{\ell_0})^{(k)} \circ s$$

for all $k \in \mathbb{N}$ and $s \in \mathbb{T}^{>,\succ}$.

Proof. The result holds trivially for $k=0$ and by the chain rule for ∂ for $k=1$. Note that ∂ is a hyperserial derivation of force $\boldsymbol{\nu}$, that $t_{\gamma,k}\in\mathbb{L}_{<\boldsymbol{\lambda}}$, and $(\mathrm{e}_\beta^{\ell_0})'=t_{\gamma,1}\circ\mathrm{e}_\beta^{\ell_0}=t_{\gamma,1}\circ\mathrm{e}_\beta^{\ell_0}$. We deduce by induction that for all $k>1$, we have

$$\begin{aligned}(\ell_\gamma\circ\mathrm{e}_\beta^{\ell_0})^{(k+1)}\circ s &= (t_{\gamma,1}\circ\mathrm{e}_\beta^{\ell_0}\times t'_{\gamma,k}\circ\mathrm{e}_\beta^{\ell_0})\circ s\\ &= (t_{\gamma,k+1}\circ\mathrm{e}_\beta^{\ell_0})\circ s\\ &= t_{\gamma,k+1}\circ E_\beta(s),\end{aligned}$$

hence the result. □

Proposition 7.11. *Let $\mu<\nu$, write $\alpha:=\omega^\mu$ and let $\beta<\alpha$. Let $\varphi\in\mathbb{U}_{\succ,\alpha}$ and $s\in\mathbb{T}^{>,\succ}$ such that φ has Taylor expansions at s. Then $L_\beta E_\alpha^\varphi$ has Taylor expansions at s.*

Proof. We first assume that $\alpha=1$, so $\beta=0$. Write $\mathfrak{n}:=E_1^\varphi$. Let $\delta\in\mathbb{T}$ with $\delta\prec s$ and

$$\delta\prec\frac{\mathfrak{n}\circ s}{\mathfrak{n}'\circ s}=\frac{1}{\varphi'\circ s}.$$

Let $\mathfrak{m}\in\operatorname{supp}\varphi$. We have $\mathfrak{m}\preccurlyeq\varphi$ so $\mathfrak{m}'\preccurlyeq\varphi'$ by Lemma 5.5(ii). Since $\operatorname{supp}\varphi\succcurlyeq 1$, we deduce that $\frac{1}{\varphi'\circ s}\preccurlyeq\frac{1}{\mathfrak{m}'\circ s}\preccurlyeq\frac{\mathfrak{m}\circ s}{\mathfrak{m}'\circ s}$. By **TE**, the function $\mathbb{T}^{>,\succ}\longrightarrow\mathbb{V},t\mapsto\varphi\circ t$ is analytic at s and $\varphi\circ(s+\delta)-\varphi\circ s\preccurlyeq\delta\,(\varphi'\circ s)\prec 1$. Now the exponential is analytic at $\varphi\circ s$ with $\operatorname{Conv}(\exp)_{\varphi\circ s}=\mathbb{V}^\prec$. It follows with Proposition 1.26 and (2.6) that $\mathcal{A}:t\mapsto\mathfrak{n}\circ t$ is analytic at s with

$$\mathfrak{n}\circ(s+\delta)=\exp(\varphi\circ s)+\sum_{k>0}c_k\,\delta^k,$$

where

$$c_k=\sum_{n\in\mathbb{N}}\sum_{v\in X_{n,k}}\frac{(E_1^\varphi\circ s)}{n!}\frac{\varphi^{(v_{[1]})}\circ s}{v_{[1]}!}\cdots\frac{\varphi^{(v_{[n]})}\circ s}{v_{[n]}!}.$$

Now by (6.7), we have $c_k=\frac{(E_1^\varphi)^{(k)}}{k!}$ for all $k\in\mathbb{N}$. So $((\mathfrak{n}^{(k)}\circ s)\,\delta^k)_{k\in\mathbb{N}}$ is well-based with

$$\mathfrak{n}\circ(s+\delta)=\sum_{k\in\mathbb{N}}\frac{\mathfrak{n}^{(k)}\circ s}{k!}.$$

Assume now that $\alpha>0$. Since ∂ is a hyperserial derivation of force $\boldsymbol{\nu}$, we have

$$(L_\beta(E_\alpha^\varphi))'=\varphi'\,(\mathrm{e}'_\alpha\circ\varphi)\,((\ell'_\beta\circ\mathrm{e}_\alpha)\circ\varphi)=\varphi'\times\ell_{[\beta,\alpha)}\circ E_\alpha^\varphi.$$

Let $\varepsilon\prec s$ with $\varepsilon\prec\frac{L_\beta(E_\alpha^\varphi)\circ s}{(L_\beta(E_\alpha^\varphi))'\circ s}$. Let $\mathfrak{m}\in\operatorname{supp}\varphi$. We have $(\ell_{(\beta,\alpha)}\circ E_\alpha^\varphi)^{-1}\prec\mathfrak{m}$ because φ is truncated. So

$$\frac{\mathfrak{m}}{\mathfrak{m}'}\succcurlyeq\frac{\mathfrak{m}}{\varphi'}\succcurlyeq\frac{1}{(\ell_{(\beta,\alpha)}\circ E_\alpha^\varphi)\,\varphi'}=\frac{L_\beta(E_\alpha^\varphi)}{(L_\beta(E_\alpha^\varphi))'}.$$

We deduce that $\varepsilon\prec\frac{\mathfrak{m}\circ s}{\mathfrak{m}'\circ s}$ for each $\mathfrak{m}\in\operatorname{supp}\varphi$. So we have $\varphi\circ(s+\varepsilon)=\varphi\circ s+\delta$ where

$$\delta:=\sum_{k>0}\frac{\varphi^{(k)}\circ s}{k!}\varepsilon^k.$$

Corollary 7.4 yields $\delta\preccurlyeq\varepsilon\,(\varphi'\circ s)\prec(\ell_{(\beta,\alpha)}\circ E_\alpha^\varphi)\circ s$. We deduce with (2.10) that the family $((\ell_\beta\circ\mathrm{e}_\alpha)^{(k)}\circ_{\mathbb{V}}(\varphi\circ s)\,\delta^k)_{k\in\mathbb{N}}$ is well-based. We conclude again with Proposition 1.26, (6.7) and Lemma 7.10. □

Lemma 7.12. *Assume that $(\circ, \partial)$ has Taylor expansions. For each $f \in \mathbb{U}$, the relation*

$$s \sim_f t \Longleftrightarrow (\forall \mathfrak{m} \in \operatorname{supp} f, (\mathfrak{m}' \circ t)\,(t-s) \prec \mathfrak{m} \circ t)$$

is an equivalence relation on $\mathbb{T}^{>,\succ}$.

Proof. If $f = 0$ then the relation is trivial. We assume that $f \neq 0$. The relation $\sim_f$ is clearly reflexive. We next prove that it is symmetric. Let $s, t \in \mathbb{T}^{>,\succ}$, assume that $s \sim_f t$ and let $\mathfrak{m} \in \operatorname{supp} f$. We have

$$\begin{aligned}
\mathfrak{m} \circ t - \mathfrak{m} \circ s &= \mathfrak{m} \circ (t + (s-t)) - \mathfrak{m} \circ s \\
&= \sum_{k>0} \frac{\mathfrak{m}^{(k)} \circ t}{k!} (s-t)^k \\
&\sim (\mathfrak{m}' \circ t)\,(s-t) \qquad \text{(by Corollary 7.4)} \\
&\prec \mathfrak{m} \circ t.
\end{aligned}$$

So $\mathfrak{m} \circ t \sim \mathfrak{m} \circ s$. We have $\mathfrak{m}' \preccurlyeq \mathfrak{m}$ so the same applies to $\mathfrak{m}'$, yielding $\mathfrak{m}' \circ t \sim \mathfrak{m}' \circ s$. We deduce that $t \sim_f s$, so the relation is symmetric. Now consider $s_0, s_1, s_2 \in \mathbb{T}^{>,\succ}$ with $s_0 \sim_f s_1$ and $s_1 \sim_f s_2$. In particular, we have

$$\frac{\mathfrak{m} \circ s_0}{\mathfrak{m}' \circ s_0} \sim \frac{\mathfrak{m} \circ s_1}{\mathfrak{m}' \circ s_1} \sim \frac{\mathfrak{m} \circ s_2}{\mathfrak{m}' \circ s_2}$$

for all $\mathfrak{m} \in \operatorname{supp} f \setminus \{1\}$.

For $\mathfrak{m} \in \operatorname{supp} f \setminus \{1\}$, we have $s_2 - s_0 = (s_2 - s_1) + (s_1 - s_0)$ where $s_2 - s_1, s_1 - s_0 \prec \frac{\mathfrak{m} \circ s_2}{\mathfrak{m}' \circ s_2}$, so $s_2 - s_0 \prec \frac{\mathfrak{m} \circ s_2}{\mathfrak{m}' \circ s_2}$. We deduce that $s_0 \sim_f s_2$. Therefore $\sim_f$ is an equivalence relation. □

7.4 The extension theorem for Taylor expansions

We now prove the following result:

Theorem 7.13. *Let $\boldsymbol{\nu} \leqslant \mathbf{On}$ and $\boldsymbol{\mu} \leqslant \boldsymbol{\nu}$ with $0 < \boldsymbol{\mu}$. Let $(\mathbb{U}, \circ_{\mathbb{U}})$, $(\mathbb{T}, \circ_{\mathbb{T}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ be confluent hyperserial fields of force $\boldsymbol{\nu}$ with $(\mathbb{T}, \circ_{\mathbb{T}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$ and $(\mathbb{L}_{<\boldsymbol{\lambda}}, \circ) \subseteq (\mathbb{U}, \circ_{\mathbb{U}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$. Let $\circ : \mathbb{U}_{(<\boldsymbol{\mu})} \times \mathbb{T}^{>,\succ} \longrightarrow \mathbb{V}_{(<\boldsymbol{\mu})}$ be a composition law of force $\boldsymbol{\nu}$. Let $\partial : \mathbb{V}_{(<\boldsymbol{\mu})} \longrightarrow \mathbb{V}_{(<\boldsymbol{\mu})}$; $f \mapsto f'$ be a derivation of force $\boldsymbol{\nu}$ with $\partial(\mathbb{U}_{(<\boldsymbol{\mu})}) \subseteq \mathbb{U}_{(<\boldsymbol{\mu})}$, which extends the standard derivation on $\mathbb{L}_{<\boldsymbol{\lambda}}$, and satisfies the chain rule*

$$\forall f \in \mathbb{U}_{(<\boldsymbol{\mu})}, \forall t \in \mathbb{T}^{>,\succ}, \partial(f \circ t) = \partial(t)\,\partial(f) \circ t.$$

Assume that ∂ has a near-support $\mathfrak{W}_\partial$ with $\mathfrak{W}_\partial \preccurlyeq \ell_0$. Let $s \in \mathbb{T}^{>,\succ}$ such that $\circ_s$ satisfies one of the two following conditions:

a) $\circ_s$ preserves monomials, or

b) $\circ_s$ extends into a strongly linear and bijective morphism or ordered rings $\mathbb{V} \longrightarrow \mathbb{V}$.

If $(\circ \restriction (\mathbb{U} \times \mathbb{T}^{>,\succ}), \partial \restriction \mathbb{U})$ has Taylor expansions at s, then so does $(\circ, \partial)$.

We fix $\boldsymbol{\nu}$, $\boldsymbol{\mu}$, $(\mathbb{U}, \circ_{\mathbb{U}})$, $(\mathbb{T}, \circ_{\mathbb{T}})$, $(\mathbb{V}, \circ_{\mathbb{V}})$, ∂, $\circ$ and s as in the statement of Theorem 7.13. For all $f \in \mathbb{U}_{(<\boldsymbol{\mu})}$ and $k \in \mathbb{N}$, we write $f^{(k)} := \partial^k(f)$. We consider the class $\mathcal{P}_{\partial,\circ,s}$ of series $f \in \mathbb{U}_{(<\boldsymbol{\mu})}$ that have Taylor expansions at s with respect to $(\partial, \circ)$. We will prove that $\mathcal{P}_{\partial,\circ,s} = \mathbb{U}_{(<\boldsymbol{\mu})}$.

Proposition 7.14. *For $f \in \mathbb{U}_{(<\boldsymbol{\mu})}$, we have* $\operatorname{supp} f \subseteq \mathcal{P}_{\partial,\circ,s} \Longrightarrow f \in \mathcal{P}_{\partial,\circ,s}$.

Proof. Let $\varepsilon \prec s$ with $\varepsilon \prec \frac{\mathfrak{m} \circ s}{\mathfrak{m}' \circ s}$ for all $\mathfrak{m} \in \operatorname{supp} f \setminus \{1\}$. By Theorem 7.5 if the condition Theorem 7.13(a) holds, or Theorem 7.6 if the condition Theorem 7.13(b) holds, the family $((\mathfrak{m}^{(k)} \circ s)\, \varepsilon^k)_{k \in \mathbb{N} \wedge \mathfrak{m} \in \operatorname{supp} f}$ is well-based. For $\mathfrak{m} \in \operatorname{supp} f$, our hypothesis that $\mathfrak{m} \subseteq \mathcal{P}_{\partial,\circ}$ implies that $\mathfrak{m} \circ (s+\varepsilon) = \sum_{k \in \mathbb{N}} \frac{\mathfrak{m}^{(k)} \circ s}{k!} \varepsilon^k$. Moreover, we have

$$f \circ (s+\delta) = \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\, \mathfrak{m} \circ (s+\varepsilon) \qquad \text{and} \qquad f^{(k)} \circ s = \sum_{\mathfrak{m} \in \operatorname{supp} f} f_{\mathfrak{m}}\, \mathfrak{m}^{(k)} \circ s$$

for all $k \in \mathbb{N}$. We deduce with Lemma 1.2 that $f \circ (s+\varepsilon) = \sum_{k \in \mathbb{N}} \frac{f^{(k)} \circ s}{k!} \varepsilon^k$. Thus $f \in \mathcal{P}_{\partial,\circ,s}$. □

Proposition 7.15. *Let $\mathbb{G} \subseteq \mathbb{U}_{(<\boldsymbol{\mu})}$ be a transserial subgroup with $\mathbb{G} \subseteq \mathcal{P}_{\partial,\circ}$. We have $\mathbb{G}_{(<1)} \subseteq \mathcal{P}_{\partial,\circ,s}$.*

Proof. By Proposition 7.14 and by induction on γ in $\mathbb{G}_{(<1)} = \bigcup_{\gamma \in \mathbf{On}} \mathbb{G}_{(\gamma)}$ it is enough to prove that each monomial in $\mathbb{G}^{\exp}$ lies in $\mathcal{P}_{\partial,\circ,s}$. This follows from Proposition 7.11. □

Proposition 7.16. *Let $\eta < \boldsymbol{\nu}$ with $\mu > 0$ and set $\alpha := \omega^\eta$. Let $\beta < \alpha$ and let $\varphi \in \mathcal{P}_{\partial,\circ,s} \cap (\mathbb{U}_{(<\boldsymbol{\mu})})_{\succ,\alpha}$. We have $L_\beta(E_\alpha^\varphi) \in \mathcal{P}_{\partial,\circ,s}$.*

Proof. This follows from Proposition 7.11. □

Proposition 7.17. *We have $\mathbb{E}(\mathcal{P}_{\partial,\circ,s}) \subseteq \mathcal{P}_{\partial,\circ,s}$.*

Proof. This is a direct consequence of Propositions 7.14 and 7.16. □

Proposition 7.18. *We have $(\mathbb{G}(\mathcal{P}_{\partial,\circ,s}))_{(<1)} \subseteq \mathcal{P}_{\partial,\circ,s}$.*

Proof. By Proposition 7.17, we have $\mathbb{G}(\mathcal{P}_{\partial,\circ,s}) \subseteq \mathcal{P}_{\partial,\circ,s}$, whence $(\mathbb{G}_{\partial,\circ,s})_{(<1)} \subseteq \mathcal{P}_{\partial,\circ,s}$ by Proposition 7.15. □

Corollary 7.19. *We have $\mathcal{P}_{\partial,\circ,s} = \mathbb{U}_{(<\boldsymbol{\mu})}$.*

Proof. Apply Lemma 4.10. □

This proves Theorem 7.13.

8 Finitely nested hyperseries as an ordered group

We now focus on the class $\tilde{\mathbb{L}}^{>,\succ}$ of positive infinite finitely nested hyperseries. We will show that it is a group under composition, and then that it is a linearly ordered group for the ordering $<$ on $\tilde{\mathbb{L}}$. This last result amounts to proving that each function $\tilde{\mathbb{L}}^{>,\succ} \longrightarrow \tilde{\mathbb{L}}$; $g \mapsto f \circ g$ for fixed $f \in \tilde{\mathbb{L}}$ is either constant or strictly monotonous. We will proceed by proving that this property is preserved under the hyperexponential closure in certain cases. Throughout this section, we fix a $\boldsymbol{\nu} \leqslant \mathbf{On}$ with $\boldsymbol{\nu} > 0$.

Definition 8.1. *Let $\mathbb{U}$ and $\mathbb{V}$ be hyperserial fields of force $\boldsymbol{\nu}$ with $\mathbb{U} \subseteq \mathbb{V}$, let $\circ\colon \mathbb{U} \times \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}$ be a hyperserial composition of force $\boldsymbol{\nu}$ and let $f \in \mathbb{U}$. We say that f* **acts as a strictly increasing function on** *$\mathbb{V}^{>,\succ}$ if we have*

$$\forall s, t \in \mathbb{V}^{>,\succ}, s < t \Longrightarrow f \circ s < f \circ t.$$

8.1 Functional inverses

Theorem 8.2. *Let ν be a non-zero limit ordinal. Then $(\mathbf{G}(\nu), \circ, \ell_0)$ is a group. As a consequence $(\tilde{\mathbb{L}}^{>,\succ}, \circ, \ell_0)$ is a group.*

Proof. Fix an ordinal $\mu \in (0,\nu)$, an $n \in \mathbb{N}^>$ and set $\beta := \omega^\mu$. Consider the subgroup

$$\mathfrak{N}(\mu) := \{\mathfrak{m} \in (\mathfrak{L}_{<\beta\omega})_{(<\mu+1)} : \mathfrak{m} \preccurlyeq \ell_0\}$$

of $(\mathfrak{L}_{<\beta\omega})_{(<\mu+1)}$. We also write $\mathbb{S}(\mu) := \mathbb{R}[[\mathfrak{N}(\mu)]]$. Recall that $\mathfrak{W}_\partial := \{\ell_\gamma^\dagger : \gamma \in \mathbf{On}\}^\infty$ is a good near-support for the derivation $\tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$, with $\mathfrak{W}_\partial \preccurlyeq \ell_0$. So for $\mathfrak{m} \in \mathfrak{N}(\mu)$, we have

$$\operatorname{supp} \partial(\mathfrak{m}) \subseteq \mathfrak{m} \cdot \tilde{\mathfrak{L}}^{\prec\mathfrak{m}} \cdot \mathfrak{W}_\partial \preccurlyeq \ell_0.$$

It follows since $\partial((\mathbb{L}_{<\beta\omega})_{(<\mu+1)}) \subseteq (\mathbb{L}_{<\beta\omega})_{(<\mu+1)}$ that $\partial(\mathbb{S}(\mu)) \subseteq \mathbb{S}(\mu)$. We also have $\mathbb{S}(\mu) \circ \ell_{\beta n} \subseteq \mathbb{S}(\mu)$ since $\ell_{\beta n}$ is $L_{<\beta\omega}$-atomic.

Let $\varepsilon \in \mathbb{S}(\mu)$ with $\varepsilon \prec 1$ and write $\triangle_{n,\varepsilon}$ for the restriction to $\mathbb{S}(\mu)$ of the right composition $\tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ with $\ell_{\beta n} + \varepsilon$. Let $f \in \mathbb{S}(\mu)$ and $\mathfrak{m} \in \operatorname{supp} f \setminus \{1\}$. We have $\mathfrak{m} \preccurlyeq \ell_0$, so $\mathfrak{m}^\dagger \preccurlyeq \ell_0^{-1}$, whence $\varepsilon \prec 1 \prec \frac{\mathfrak{m} \circ \ell_{\beta n}}{\mathfrak{m}' \circ \ell_{\beta n}}$. By Theorem 7.13(a), we have $\triangle_{n,\varepsilon} = \Psi_{n,\alpha} + H_{n,\varepsilon}$ where

$$H_{n,\varepsilon} \colon \mathbb{S}(\mu) \longrightarrow \tilde{\mathbb{L}};\ f \mapsto \sum_{k>0} \frac{f^{(k)} \circ \ell_{\beta n}}{k!}\, \varepsilon^k,$$

and $\Psi_{n,\alpha}$ is the right-composition with $\ell_{\beta n}$ restricted to $\mathbb{S}(\mu)$. Write $\Phi_{n,\alpha}$ for the right composition with $\mathrm{e}_{\alpha n}^{\ell_0}$ on $\tilde{\mathbb{L}}$. The function $H_{n,\varepsilon}$ is strongly linear with $H_{n,\varepsilon}(f) \prec \Psi_{n,\alpha}(f)$ for all $f \in \mathbb{S}(\mu)$, so $\Phi_{n,\alpha} \circ H_{n,\varepsilon}$ is strongly linear with $(\Phi_{n,\alpha} \circ H_{n,\varepsilon})(f) \prec f$ for all $f \in \mathbb{S}(\mu)$. Furthermore, the inclusion $\partial(\mathbb{S}(\nu)) \subseteq \mathbb{S}(\mu)$ yields $(\Phi_{n,\alpha} \circ H_{n,\varepsilon})(\mathbb{S}(\mu)) \subseteq \mathbb{S}(\mu)$.

By Corollary 1.47, the function

$$I_{n,\varepsilon} \colon \mathbb{S}(\mu) \longrightarrow \mathbb{S}(\mu),\ g \mapsto \sum_{k \in \mathbb{N}} (-1)^k\, (\Phi_{n,\alpha} \circ H_{n,\varepsilon})^{\circ k}(g)$$

is well-defined, with $(\mathrm{id}_{\mathbb{S}(\nu)} + \Phi_{n,\alpha} \circ H_{n,\varepsilon}) \circ I_{n,\varepsilon}(g) = g$ for all $g \in \mathbb{S}(\mu)$. We deduce that

$$((\Phi_{n,\alpha} \circ I_{n,\varepsilon})(g)) \circ (\ell_{\beta n} + \varepsilon) = g$$

for all $g \in \mathbb{S}(\mu)$. In particular, the number $(\Phi_{n,\alpha} \circ I_{n,\varepsilon})(\ell_0) \in \mathbb{S}(\mu)$ is a left-inverse of $\ell_{\beta n} + \varepsilon$ in $(\tilde{\mathbb{L}}^{>,\succ}, \circ)$.

Now let $f \in \mathbf{G}(\nu)$. Considering a sufficiently large non-zero ordinal $\mu < \nu$, we claim that $f \in \mathbb{S}(\mu) \circ \mathrm{e}_{\omega^\mu}^{\ell_0}$. Indeed take $\mu = \eta + 1$ where $f \in (\mathbb{L}_{<\omega^\eta})_{(<\eta)}$. Then $\operatorname{supp} f \prec\!\!\prec \mathrm{e}_{\omega^\mu}^{\ell_0}$, so since ℓ_{ω^μ} is atomic in $(\mathbb{L}_{<\omega^\eta})_{(<\eta)}$, we get $\operatorname{supp} f \circ \ell_{\omega^\mu} \preccurlyeq \ell_0$. This proves the claim. The series $\mathrm{e}_{\omega^\mu}^{\ell_0}$ has an inverse ℓ_{ω^μ} in $\mathbf{G}(\nu)$, so it suffices to show that $g := f \circ \ell_{\omega^\mu} \in \mathbb{S}(\mu)$ has an inverse in $\mathbf{G}(\nu)$. Write again $\alpha = \omega^\mu$. We have $g \in (\mathbb{L}_{<\omega^{\mu+1}})_{(<\mu+1)}$ so by Lemma 6.24, there are $m, p \in \mathbb{N}^>$ with $\ell_{\alpha m} \circ g = \ell_{\alpha p} + \varepsilon$ for a certain $\varepsilon \prec 1$. We have

$$\operatorname{supp} \varepsilon \subseteq \operatorname{supp} \ell_{\alpha m} \circ g \subseteq (L_{\alpha p}\, \omega) \cdot \tilde{\mathbb{L}}^{\prec \ell_{\alpha n}} \cdot \mathfrak{W}_g$$

where $\mathfrak{W}_g$ is the canonical relative near-support for the right composition with g. For all $\mathfrak{w} \in \mathfrak{W}_g$, there is a $\mathfrak{m} \in \operatorname{supp} g_\prec$ with $\mathfrak{w} \preccurlyeq \mathfrak{m}$. In particular $\mathfrak{W}_g \subseteq \mathfrak{N}(\mu)$, so $\varepsilon \in \mathbb{S}(\mu)$. We deduce that there is an $h \in \mathbb{S}(\mu)$ with $h \circ (\ell_{\alpha m} \circ g) = \omega$. We have $(g \circ (h \circ \ell_{\alpha m})) \circ g = g$ so $g \circ (h \circ \ell_{\alpha m}) = \ell_0$ by injectivity of $\circ_g$. Thus $h \circ \ell_{\alpha m} \in \mathbb{S}(\mu) \subseteq \mathbf{G}(\nu)$ is the inverse of g. We deduce since $\tilde{\mathbb{L}}^{>,\succ} = \bigcup_{\nu \in \mathbf{On}^>_{\mathrm{Lim}}} \mathbf{G}(\nu)$ that $(\tilde{\mathbb{L}}^{>,\succ}, \circ, \ell_0)$ is also a group. □

We will write f^{inv} for the inverse of a series f in $(\tilde{\mathbb{L}}^{>,\succ},\circ)$. We can now apply Theorem 7.13(b) to each right composition $\tilde{\mathbb{L}}\longrightarrow\tilde{\mathbb{L}}$ with a series $s\in\tilde{\mathbb{L}}^{>,\succ}$. We thus have

Corollary 8.3. *The hyperserial derivation $\tilde{\mathbb{L}}\longrightarrow\tilde{\mathbb{L}}$ of Corollary 5.22 and the hyperserial composition law $\tilde{\mathbb{L}}\times\tilde{\mathbb{L}}^{>,\succ}\longrightarrow\tilde{\mathbb{L}}$ of Corollary 6.23 have Taylor expansions.*

Given $\nu\in\mathbf{On}^{>}_{\mathrm{Lim}}$, write $\mathbf{G}(\nu):=\bigcup_{\mu<\nu}(\mathbb{L}_{<\omega^\mu})^{>,\succ}_{(<\mu)}$. Note that $\mathbf{G}(\nu)$ is closed under $\circ$. We will show that $(\mathbf{G}(\nu),\circ,\ell_0)$ is a group.

8.2 The approximate mean value inequality

We rely on the following weakened mean value inequality for differentiable real-valued functions. The link between monotonicity and this sort of inequality was suggested to us by Vincenzo Mantova who also gave us the proof of Lemma 8.8 below.

Definition 8.4. *Let $\mathbb{U}$ and $\mathbb{V}$ be hyperserial fields of force $\boldsymbol{\nu}$ with $\mathbb{U}\subseteq\mathbb{V}$ and let $\circ\colon\mathbb{U}\times\mathbb{V}^{>,\succ}\longrightarrow\mathbb{V}$ be a hyperserial composition of force $\boldsymbol{\nu}$. Let $\partial\colon\mathbb{V}\longrightarrow\mathbb{V}$ be a hyperserial derivation of force $\boldsymbol{\nu}$. Given $s,t\in\mathbb{V}^{>,\succ}$ with $s<t$, consider the following statement for a series $f\in\mathbb{U}^{>,\succ}$:*

amvi. $(f'\circ s)\,(t-s)\preccurlyeq f\circ t-f\circ s\preccurlyeq(f'\circ t)\,(t-s)$.

We say that f satisfies the **asymptotic mean value inequality** *(or amvi) at (s,t) if* **amvi** *holds for (f,s,t). We say that f* **satisfies the amvi** *if it satisfies the amvi at all $(s,t)\in(\mathbb{V}^{>,\succ})^2$ with $s<t$.*

Lemma 8.5. *Assume that each $f\in\mathbb{U}$ satisfies the chain rule. Let $s,t\in\mathbb{V}^{>,\succ}$ with $s<t$. Let $f,g\in\mathbb{U}^{>,\succ}$ with $f\circ g\in\mathbb{U}^{>,\succ}$. If g satisfies the* **amvi** *at (s,t), if $g\circ s<g\circ t$ and if f satisfies the* **amvi** *at $(g\circ s,g\circ t)$, then $f\circ g$ satisfies the* **amvi** *at (s,t).*

Proof. We have $\frac{(f\circ g)\circ t-(f\circ g)\circ s}{t-s}=\frac{f\circ(g\circ t)-f\circ(g\circ s)}{g\circ t-g\circ s}\times\frac{g\circ t-g\circ s}{t-s}$. Our hypotheses yield

$$f'\circ(g\circ t)\times g'\circ t\preccurlyeq\frac{(f\circ g)\circ t-(f\circ g)\circ s}{t-s}\preccurlyeq f'\circ(g\circ t)\times g'\circ t.$$

The chain rule $(f\circ g)'=g'\times f'\circ g$ yields the result. □

Lemma 8.6. *We consider the hyperserial composition law $\circ\colon\widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}\times\mathbb{V}^{>,\succ}\longrightarrow\mathbb{V}_{(<\boldsymbol{\nu})}$ of force $\boldsymbol{\nu}$ given by Corollary 6.22. Let $\mu<\boldsymbol{\nu}$ and let $\gamma<\boldsymbol{\lambda}$ with $\gamma<\omega^\mu$. The series $\mathfrak{a}:=\ell_\gamma\circ\mathrm{e}^{\ell_0}_{\omega^\mu}\in\widetilde{\mathbb{L}_{<\boldsymbol{\lambda}}}$ satisfies the* **amvi**.

Proof. We prove the result by induction on μ. For $\mu=0$, we have $\gamma=0$, so $\mathfrak{a}=\mathrm{e}^{\ell_0}$. Thus in view of Proposition 2.7, the **amvi** for $\mathfrak{a}$ is a consequence of the mean value theorem for exp and the fact that exp is strictly increasing. Assume that $\mu>0$ and that the result holds for all $\eta<\mu$. In particular, by Lemma 8.5, each $\ell_\rho\circ\mathrm{e}^{\ell_0}_{\omega^{\eta}n}$ for $\eta<\mu$, $n<\omega$ and $\rho<\omega^\eta$ satisfies the **amvi**.

Let $s,t\in\mathbb{V}^{>,\succ}$ with $s<t$. We distinguish two cases. First assume that $\sharp_{\omega^\mu}(t)=\sharp_{\omega^\mu}(s)$. Thus there is a $\beta<\omega^\mu$ with $t-s\prec\frac{1}{L_\beta(E_\alpha(t))}$. We may choose $\beta\geqslant\gamma$ with $\beta=\omega^\eta n$ for certain $\eta<\mu$ and $n<\omega$. Write $\mathfrak{b}:=\ell_\beta\circ\mathrm{e}^{\ell_0}_{\alpha}$ and $\mathfrak{c}:=\ell_\gamma\circ\mathrm{e}^{\ell_0}_{\beta}$, so $\mathfrak{a}=\mathfrak{c}\circ\mathfrak{b}$. By our previous arguments, the series $\mathfrak{c}$ satisfies the **amvi** at $\mathfrak{b}\circ s,\mathfrak{b}\circ t$. So Lemma 8.5 yields

$$(\mathfrak{a}'\circ s)\,(t-s)\preccurlyeq\mathfrak{a}\circ t-\mathfrak{a}\circ s\preccurlyeq(\mathfrak{a}'\circ t)\,(t-s).$$

Now assume that $\sharp_{\omega^\mu}(s) \neq \sharp_{\omega^\mu}(t)$. So $\mathfrak{d}_{\omega^\mu}(E_{\omega^\mu}(s)) < \mathfrak{d}_{\omega^\mu}(E_{\omega^\mu}(t))$ whence in particular $\mathfrak{a} \circ s \ll \mathfrak{a} \circ t$. Thus $\mathfrak{a} \circ t - \mathfrak{a} \circ s \asymp \mathfrak{a} \circ t \preccurlyeq (\mathfrak{a}' \circ t)\,(t-s)$. We have $\mathfrak{a}' \circ t \equiv \mathfrak{a} \circ t \gg \mathfrak{a} \circ s \equiv \mathfrak{a}' \circ s$ and $\mathfrak{a}' \circ t \gg t \gtrsim t - s$, so $\mathfrak{a}' \circ t \gg (\mathfrak{a}' \circ s)\,(t-s)$, so $\mathfrak{a} \circ t - \mathfrak{a} \circ s \succcurlyeq (\mathfrak{a}' \circ s)\,(t-s)$.

Thus $\mathfrak{a}$ satisfies the **amvi**. The result follows by induction. □

8.3 Monotonicity and exponential extensions

Let $(\mathbb{U}, \circ_{\mathbb{U}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ be confluent hyperserial fields of force $\boldsymbol{\nu}$ with $(\mathbb{U}, \circ_{\mathbb{U}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}})$. Let $\circ\colon \mathbb{U} \times \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}$ be a composition law of force $\boldsymbol{\nu}$. Let $'\colon \mathbb{U} \longrightarrow \mathbb{U}$ be a derivation of force $\boldsymbol{\nu}$ such that

- $(\mathbb{U}, ')$ is an H-field with small derivation.
- $\{1\}$ is a positive near-support for $'$.
- $(\circ, ')$ has Taylor expansions.
- $(\circ, ')$ satisfies the chain rule.

Given $s, t \in \mathbb{V}^{>,\succ}$ with $s < t$, write $\mathcal{P}^{>,\succ}_{\nearrow,s,t}$ for the class of positive infinite series $f \in \mathbb{U}^{>,\succ}$ which satisfy the **amvi** at (s,t), as well as:

$$\begin{aligned} f \circ t &> f \circ s \quad \text{and} \\ f \circ t - f \circ s &\sim \tau_f \circ t - \tau_f \circ s. \end{aligned}$$

Write

$$\mathcal{P}_{\nearrow,s,t} := \mathcal{P}^{>,\succ}_{\nearrow,s,t} \cup (\mathbb{U} \setminus \mathbb{U}^{>,\succ})$$

and

$$\mathcal{P}_{\nearrow} := \bigcap_{s<t} \mathcal{P}_{\nearrow,s,t}.$$

Proposition 8.7. *Let $s, t \in \mathbb{V}^{>,\succ}$ with $s < t$. Let $\mathfrak{S}$ be a subclass of $\mathfrak{U}^{\succ} \cap \mathcal{P}_{\nearrow,s,t}$. We have $\mathbb{R}[[\mathfrak{S}]] \subseteq \mathcal{P}_{\nearrow,s,t}$.*

Proof. Let $f \in \mathbb{R}[[\mathfrak{S}]]$ with $f > \mathbb{R}$. We claim that

$$f \circ t - f \circ s \sim \tau_f \circ t - \tau_f \circ s. \tag{8.1}$$

Assume for a moment that (8.1) holds. Since $\tau_f > \mathbb{R}$ and $\tau_f \in \mathcal{P}_{\nearrow,s,t}$, we have $f \circ t - f \circ s > 0$. Moreover since $(\mathbb{U}, \partial)$ is an H-field, we have $f' \sim \tau_f'$, so the **amvi** for τ_f yields

$$f' \circ s \asymp \tau_f' \circ s \preccurlyeq \frac{f \circ t - f \circ s}{t - s} \asymp \frac{\tau_f \circ t - \tau_f \circ s}{t - s} \preccurlyeq \tau_f' \circ t \asymp f' \circ t.$$

Thus it suffices to prove our claim.

Assume first that there is a term τ in f with $\tau \prec \tau_f$ and $\tau' \circ t \succcurlyeq \tau_f' \circ s$. Then $\tau_f' \circ t \succ \tau_f' \circ s$, which implies since τ_f' has Taylor expansions that $t - s \succcurlyeq \frac{\tau_f' \circ s}{\tau_f'' \circ s}$. Since $\{1\}$ is a positive near support for ∂ and $(\mathbb{U}, \partial)$ is an H-field, we have $\mathfrak{m}^\dagger \asymp (\mathfrak{m}')^\dagger$ for all $\mathfrak{m} \in \mathfrak{U}^{\succ}$. So we also have $t - s \succcurlyeq \frac{\tau_f \circ s}{\tau_f' \circ s}$. For $\mathfrak{m} \in (\operatorname{supp} f \setminus \{\mathfrak{d}_f\})$, we have $\tau_f \circ t - \tau_f \circ s \succcurlyeq (t-s)\,\tau_f' \circ s \succcurlyeq \tau_f \circ s$. Recall that $\mathfrak{m}$ acts as a strictly increasing function on $\mathbb{V}^{>,\succ}$, so $\mathfrak{m} \circ t - \mathfrak{m} \circ s \preccurlyeq \mathfrak{m} \circ t$. We deduce since $\sim$ is an equivalence relation that

$$\tau_f \circ t - \tau_f \circ s \succcurlyeq \tau_f \circ t \succ \tau \circ t \succcurlyeq \tau \circ t - \tau \circ s.$$

So $f \circ t - f \circ s \sim \tau_f \circ t - \tau_f \circ s$.

Assume now that $\tau' \circ t \prec \tau'_f \circ s$ for all terms $\tau \in \operatorname{term} f$ with $\tau \prec \tau_f$. Since

$$\frac{\mathfrak{m} \circ t - \mathfrak{m} \circ s}{t-s} \preccurlyeq \mathfrak{m}' \circ t \prec \tau'_f \circ s \preccurlyeq \frac{\tau_f \circ t - \tau_f \circ s}{t-s}$$

for all $\mathfrak{m} \in (\operatorname{supp} f \setminus \{\mathfrak{d}_f\})$, we deduce that $\frac{f \circ t - f \circ s}{t-s} \sim \frac{\tau_f \circ t - \tau_f \circ s}{t-s}$, whence $f \circ t - f \circ s \sim \tau_f \circ t - \tau_f \circ s$. □

The following proof is from Vincenzo Mantova.

Lemma 8.8. *Let $s, t \in \mathbb{V}^{>,\succ}$ with $s < t$ and let $\mathbb{G} \subseteq \mathbb{U}$ be a transserial subgroup such that each $f \in \mathbb{G}^{>}_{\succ}$ satisfies $f \circ s < f \circ t$. Then for all $f \in (\mathbb{G}_{(<1)})^{>}_{\succ}$ and $s, t \in \mathbb{V}^{>,\succ}$ with $s < t$, we have*

$$f \circ t - f \circ s \sim \tau_f \circ t - \tau_f \circ s.$$

In particular, each $f \in (\mathbb{G}_{(<1)})^{>}_{\succ}$ acts as a strictly increasing function on $\mathbb{V}^{>,\succ}$.

Proof. We prove by induction on $\mathrm{EH}_{\mathbb{G}}(f)$. Consider $s, t \in \mathbb{T}^{>,\succ}$ with $s < t$ and let $f \in (\mathbb{G}_{(<1)})^{>}_{\succ}$ such that for all $g \in (\mathbb{G}_{(<1)})^{>}_{\succ}$ with $\mathrm{EH}_{\mathbb{G}}(g) < \mathrm{EH}_{\mathbb{G}}(f)$, we have $g \circ t - g \circ s \sim \tau_g \circ t - \tau_g \circ s$ and $g \circ t > g \circ s$. If $f \in \mathbb{G}$, then also $\operatorname{supp} f \subseteq \mathbb{G}^{>}_{\succ}$. So for $\mathfrak{m} \in \operatorname{supp} f \setminus \{\mathfrak{d}_f\}$ and $r \in \mathbb{R}^{>}$, we have $(\tau_f - r\,\mathfrak{m}) \circ t > (\tau_f - r\,\mathfrak{m}) \circ s$. We deduce that $(\tau_f \circ t - \tau_f \circ s) \succ (\mathfrak{m} \circ t - \mathfrak{m} \circ s)$ so $f \circ t - f \circ s \sim \tau_f \circ t - \tau_f \circ s$. Thus we may assume that $f \notin \mathbb{G}$.

Consider $\mathfrak{m} \in \operatorname{supp} f$ with $\mathfrak{m} \prec \mathfrak{d}_f$. Write $\varphi := \log \mathfrak{m}$ and $\varphi_0 := \log \mathfrak{d}_f$, so $\varphi, \varphi_0 - \varphi \in (\mathbb{G}_{(<1)})^{>}_{\succ}$. Our assumption that $f \notin \mathbb{G}$ gives $\mathrm{EH}_{\mathbb{G}}(\varphi), \mathrm{EH}_{\mathbb{G}}(\varphi_0 - \varphi) < \mathrm{EH}_{\mathbb{G}}(f)$, so $(\varphi_0 - \varphi) \circ t > (\varphi_0 - \varphi) \circ s$ and $\varphi \circ t - \varphi \circ s > 0$ so

$$\varphi_0 \circ t - \varphi_0 \circ s > \varphi \circ t - \varphi \circ s > 0.$$

This yields $\exp(\varphi_0 \circ t - \varphi_0 \circ s) - 1 > \exp(\varphi \circ t - \varphi \circ s) - 1 > 0$, whence

$$\exp(\varphi_0 \circ t - \varphi_0 \circ s) - 1 \succcurlyeq \exp(\varphi \circ t - \varphi \circ s) - 1. \tag{8.2}$$

We also have

$$\mathrm{e}^{\varphi_0} \circ s \succ \mathrm{e}^{\varphi} \circ s. \tag{8.3}$$

Multiplying (8.2) and (8.3), we obtain

$$\mathrm{e}^{\varphi_0} \circ t - \mathrm{e}^{\varphi_0} \circ s \succ \mathrm{e}^{\varphi} \circ t - \mathrm{e}^{\varphi} \circ s.$$

We deduce that $f \circ t - f \circ s \sim \tau_f \circ t - \tau_f \circ s$. Moreover since $\varphi_0 \circ t > \varphi_0 \circ s$ acts as a strictly increasing function, we have $\tau_f \circ t - \tau_f \circ s > 0$, whence $f \circ t - f \circ s > 0$. □

Proposition 8.9. *Let $s, t \in \mathbb{V}^{>,\succ}$ with $s < t$. If $\mathfrak{S} \subseteq \mathfrak{U}$ is a subclass with $\mathfrak{S} \subseteq \mathcal{P}_{\nearrow,s,t}$ and $\log \mathfrak{S} \subseteq \mathbb{R}[[\mathfrak{S}]]$ (that is, if $\mathbb{R}[[\mathfrak{S}]]$ is a transserial subgroup), then $(\mathbb{R}[[\mathfrak{S}]])_{(<1)} \subseteq \mathcal{P}_{\nearrow,\nu}$.*

Proof. Write $\mathbb{G} := \mathbb{R}[[\mathfrak{S} \cap \mathfrak{U}^{\succ}]]$, so $\mathbb{G} \subseteq \mathcal{P}_{\nearrow,s,t}$ by Proposition 8.7. Note that $(\mathbb{R}[[\mathfrak{S}]])_{(<1)} = \mathbb{G}_{(<1)}$. We first prove that $(\mathbb{G}_{(<1)})_{\succ} \subseteq \mathcal{P}_{\nearrow,\nu}$ by induction on the hyperexponential height. Let $f \in (\mathbb{G}_{(<1)})_{\succ} \setminus \mathbb{G}$ such that any $g \in (\mathbb{G}_{(<1)})_{\succ}$ with $\mathrm{EH}_{\mathbb{G}}(g) < \mathrm{EH}_{\mathbb{G}}(f)$ lies in $\mathcal{P}_{\nearrow,\nu}$ and let $s, t \in \mathbb{V}^{>,\succ}$ with $s < t$. We may assume that $f > 0$. By Lemma 8.8, we have $f \circ t - f \circ s \sim \tau_f \circ t - \tau_f \circ s$. Now $\mathrm{EH}_{\mathbb{G}}(\log \mathfrak{d}_f) <_{\mathrm{lex}} \mathrm{EH}_{\mathbb{G}}(f)$, so by the induction hypothesis $\log \mathfrak{d}_f$ satisfies the **amvi**. We deduce with Lemmas 8.5 and 8.6 that $\mathfrak{d}_f$ satisfies the **amvi**, so $(\mathfrak{d}'_f \circ s)\,(t-s) \preccurlyeq \mathfrak{d}_f \circ t - \mathfrak{d}_f \circ s \preccurlyeq (\mathfrak{d}'_f \circ t)\,(t-s)$. Therefore $(f' \circ s)\,(t-s) \preccurlyeq f \circ t - f \circ s \preccurlyeq (f' \circ t)\,(t-s)$, i.e. f satisfies the **amvi**. The result for all $f \in (\mathbb{G}_{(<1)})_{\succ}$ follows by induction.

Now let $f \in \mathbb{G}_{(<1)}$ and write $f = f_\succ + r + f_\prec$ where $f_\succ \in (\mathbb{G}_{(<1)})_\succ$, $r \in \mathbb{R}$ and $f_\prec$ is infinitesimal. We may assume that $r = 0$ and that $f_\succ > 0$. Let $\mathfrak{m} \in \operatorname{supp} f_\prec$. We claim that $\mathfrak{m} \circ s - \mathfrak{m} \circ t \preccurlyeq (\mathfrak{m}' \circ s)\,(t-s)$.

By our previous arguments, we have $\mathfrak{m}^{-1} \in \mathcal{P}_\nearrow$, so $\mathfrak{m} \circ s > \mathfrak{m} \circ t$. Assume first that $t - s \prec \frac{\mathfrak{m} \circ s}{\mathfrak{m}' \circ s}$. So by **TE** and Corollary 7.4, we have $\mathfrak{m} \circ t - \mathfrak{m} \circ s \sim (\mathfrak{m}' \circ s)\,(t-s)$, hence the result. Assume now that $t - s \succcurlyeq \frac{\mathfrak{m} \circ s}{\mathfrak{m}' \circ s}$. Assume for contradiction that $\mathfrak{m} \circ t \sim \mathfrak{m} \circ s$. So $\mathfrak{m}^{-1} \circ t \sim \mathfrak{m}^{-1} \circ s$, that is, $\mathfrak{m}^{-1} \circ t - \mathfrak{m}^{-1} \circ s \prec \mathfrak{m}^{-1} \circ s$. Now the **amvi** for $\mathfrak{m}^{-1}$ yields $\mathfrak{m}^{-1} \circ t - \mathfrak{m}^{-1} \circ s \succcurlyeq ((\mathfrak{m}^{-1})' \circ s)\,(t-s)$. We deduce that $((\mathfrak{m}^{-1})' \circ s)\,(t-s) \prec \mathfrak{m}^{-1} \circ s$, i.e.

$$t - s \prec \frac{\mathfrak{m}^{-1} \circ s}{(\mathfrak{m}^{-1})' \circ s} \asymp \frac{\mathfrak{m} \circ s}{\mathfrak{m}' \circ s}: \quad \text{a contradiction.}$$

Therefore $\mathfrak{m} \circ t \asymp \mathfrak{m} \circ s$. We have $\mathfrak{m} \circ s > \mathfrak{m} \circ t > 0$ so $\mathfrak{m} \circ t - \mathfrak{m} \circ s \asymp \mathfrak{m} \circ s \preccurlyeq (\mathfrak{m}' \circ s)\,(t-s)$ by our assumption on $t - s$.

Since $\mathfrak{m} \prec 1$ and $(\mathbb{U}, ')$ is an H-field with small derivation, we have $\mathfrak{m}' \prec 1$, so $\mathfrak{m} \circ t - \mathfrak{m} \circ s \prec t - s$. On the other hand, we have $f_\succ \circ t - f_\succ \circ s \succcurlyeq (f' \circ s)\,(t-s)$ where $f' \equiv f \succ 1$ as a consequence of Lemma 8.15. In particular $f_\prec \circ t - f_\prec \circ s \prec t - s \prec f_\succ \circ t - f_\succ \circ s$, so $f \circ t - f \circ s \sim f_\succ \circ t - f_\succ \circ s$. Since $f_\succ \in \mathcal{P}_{\nearrow,\nu}$, we deduce that $f \in \mathcal{P}_{\nearrow,\nu}$. This concludes the proof. □

8.4 The extension theorem for monotonicity

We now to prove the following theorem:

Theorem 8.10. *Let $(\mathbb{U}, \circ_\mathbb{U})$ and $(\mathbb{V}, \circ_\mathbb{V})$ be confluent hyperserial fields of forces $\boldsymbol{\nu}$ and $(\boldsymbol{\nu}, \boldsymbol{\nu})$ respectively, with $(\mathbb{U}, \circ_\mathbb{U}) \subseteq (\mathbb{V}, \circ_\mathbb{V})$. Let $\circ : \tilde{\mathbb{U}} \times \mathbb{V}^{\succ,\succ} \longrightarrow \mathbb{V}$ be a composition law of force $\boldsymbol{\nu}$. Let $' : \tilde{\mathbb{U}} \longrightarrow \tilde{\mathbb{U}}$ be a derivation of force $\boldsymbol{\nu}$ such that*

- *$(\tilde{\mathbb{U}}, ')$ is an H-field with small derivation.*
- *$\{1\}$ is a positive near-support for $'$.*
- *$(\circ, ')$ has Taylor expansions.*
- *$(\circ, ')$ satisfies the chain rule.*

Let $s, t \in \mathbb{V}^{\succ,\succ}$ with $s < t$ and $\mathbb{U} \subseteq \mathcal{P}_{\nearrow,s,t}$ where $\mathcal{P}_{\nearrow,s,t}$ is as in Section 8.3. Then $\tilde{\mathbb{U}} \subseteq \mathcal{P}_{\nearrow,s,t}$.

We fix $(\mathbb{U}, \circ_\mathbb{U})$, $(\mathbb{V}, \circ_\mathbb{V})$, $'$, $\circ$ and s, t as in the statement of Theorem 8.10.

Lemma 8.11. *We have $\mathbb{E}(\mathcal{P}_{\nearrow,s,t}) \subseteq \mathcal{P}_{\nearrow,s,t}$.*

Proof. By Proposition 8.7 and since $\mathbb{E}(\mathcal{P}_{\nearrow,s,t}) \subseteq \tilde{\mathbb{U}}_\succ$, it is enough to show that $\mathbb{E}(\mathcal{P}_{\nearrow,s,t}) \cap \mathfrak{U} \subseteq \mathcal{P}_{\nearrow,s,t}$. Let $\mathfrak{a} := L_\beta(E_\alpha^\varphi)$ be such a monomial, with $\varphi \in \tilde{\mathbb{U}}_{\succ,\alpha} \cap \mathcal{P}_{\nearrow,s,t}$ and $\beta\,\omega < \alpha$. Lemmas 8.6 and 8.5 imply that $\mathfrak{a}$ satisfies the **amvi** at (s, t). We have $\mathfrak{a} \circ t > \mathfrak{a} \circ s$ since $L_\beta \circ E_\alpha$ is strictly increasing. So $\mathfrak{a} \in \mathcal{P}_{\nearrow,s,t}$. □

Proposition 8.12. *We have $(\mathbb{G}(\mathcal{P}_{\nearrow,s,t}))_{(<1)} \subseteq \mathcal{P}_{\nearrow,s,t}$.*

Proof. By Proposition 8.9, it is enough to prove that $\mathbb{G}(\mathcal{P}_{\nearrow,s,t}) \subseteq \mathcal{P}_{\nearrow,s,t}$. But this follows from Proposition 8.9 and Lemma 8.11. □

Proposition 8.13. *We have $\tilde{\mathbb{U}} = \mathcal{P}_{\nearrow,s,t}$.*

Proof. Apply Lemma 4.10. □

This proves Theorem 8.10.

8.5 Monotonicity and right compositions with atomic elements

Let $(\mathbb{U}, \circ_{\mathbb{U}})$ and $(\mathbb{V}, \circ_{\mathbb{V}})$ be confluent hyperserial fields of force **On** and $(\mathbf{On}, \mathbf{On})$ respectively with

$$(\mathbb{L}, \circ) \subseteq (\mathbb{U}, \circ_{\mathbb{U}}) \subseteq (\mathbb{V}, \circ_{\mathbb{V}}).$$

Let $\circ\colon \tilde{\mathbb{U}} \times \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}$ be a hyperserial composition of force **On**. Let $'\colon \tilde{\mathbb{U}} \longrightarrow \tilde{\mathbb{U}}$ be a hyperserial derivation of force **On** extending the standard derivation on $\mathbb{L}$, and such that $(\mathbb{U}, ')$ is an H-field with small derivation. Assume that $(\circ, ')$ has Taylor expansions and satisfies the chain rule.

We fix an increasing union $(\mathbb{U}_\nu)_{0<\nu<\mathbf{On}}$ where each $\mathbb{U}_\nu$ is a hyperserial subfield of $\mathbb{U}$ of force ν, and a sequence $(\mathfrak{a}_\nu)_{0<\nu<\mathbf{On}} \in \mathbb{V}$ such that

- each $\mathfrak{a}_\nu$ for $0<\nu<\mathbf{On}$ is $L_{<\omega^\nu}$-atomic and the right composition $\mathbb{U} \longrightarrow \mathbb{V}; f \mapsto f \circ \mathfrak{a}_\nu$ preserves monomials,
- for each $\nu < \mathbf{On}$ with $\nu > 0$, the function $s \mapsto \mathfrak{a}_\nu \circ s\colon \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}^{>,\succ}$ is a strictly increasing bijection,
- for each $0<\nu<\mathbf{On}$, each $f \in \mathbb{U}_\nu^{>,\succ} \circ \mathfrak{a}_\nu$ satisfies the **amvi**
- the set $\{1\}$ is a positive near-support for $'$ on $\mathbb{U}_\nu \circ \mathfrak{a}_\nu$.

Moreover, we assume that

$$\tilde{\mathbb{U}} = \bigcup_{0<\nu<\mathbf{On}} (\mathbb{U}_\nu)_{(<\nu)}. \tag{8.4}$$

We write $\mathbb{S}_\nu := \mathbb{U}_\nu \circ \mathfrak{a}_\nu$ for all $\nu \in \mathbf{On}$. So $\mathbb{S}_\nu$ is a confluent hyperserial subfield of $\mathbb{U}_\nu$ of force ν.

Lemma 8.14. *For $\nu \in \mathbf{On}$ with $\nu > 0$, we have $(\mathbb{U}_\nu)_{(<\nu)} \circ \mathfrak{a}_\nu = (\mathbb{S}_\nu)_{(<\nu)}$.*

Proof. On the one hand, the right composition $\circ_{\mathfrak{a}_\nu}\colon \mathbb{U}_\nu \longrightarrow \mathbb{S}_\nu$ preserves monomials, so it is a hyperserial embedding of force ν. Furthermore this function is bijective by definition. Thus by the initial property of the closure under hyperexponentials, it extends uniquely into a bijective hyperserial embedding $\Phi\colon (\mathbb{U}_\nu)_{(<\nu)} \longrightarrow (\mathbb{S}_\nu)_{(<\nu)}$ of force ν.

On the other hand, by Theorem 6.1, there is a unique right composition $\circ\widetilde{_{\mathfrak{a}_\nu}}\colon (\mathbb{U}_\nu)_{(<\nu)} \longrightarrow (\mathbb{S}_\nu)_{(<\nu)}$ of force ν which extends $\circ_{\mathfrak{a}_\nu}$. Since $\circ_{\mathfrak{a}_\nu}$ is a hyperserial embedding, we have $\operatorname{supp} \circ_{\mathfrak{a}_\nu} = \{1\}$. Theorem 6.1 also yields $\operatorname{supp} \circ\widetilde{_{\mathfrak{a}_\nu}} = \{1\}$, which trivially satisfies (6.8) with respect to any series. By Theorem 6.29, the function $\circ\widetilde{_{\mathfrak{a}_\nu}}$ preserves monomials. In other words $\circ\widetilde{_{\mathfrak{a}_\nu}}$ is a hyperserial embedding $(\mathbb{U}_\nu)_{(<\nu)} \longrightarrow (\mathbb{S}_\nu)_{(<\nu)}$ of force ν which extends $\circ_{\mathfrak{a}_\nu}$. We deduce that $(\mathbb{U}_\nu)_{(<\nu)} \circ \mathfrak{a}_\nu = \Phi((\mathbb{U}_\nu)_{(<\nu)}) = (\mathbb{S}_\nu)_{(<\nu)}$. □

Lemma 8.15. *The set $\{1\}$ is a near-support for $'$ on $(\mathbb{S}_\nu)_{(<\nu)}$.*

Proof. Note that $\{1\}$ is good. By Theorem 5.7, it follows that $\partial \restriction \mathbb{S}_\nu$ extends uniquely into a hyperserial derivation of force ν on $(\mathbb{S}_\nu)_{(<\nu)}$ with near support $\{1\}$. Thus ∂ itself has near-support $\{1\}$. □

Theorem 8.16. *For all $f \in \tilde{\mathbb{U}}^{>,\succ}$ and $s, t \in \mathbb{V}^{>,\succ}$ with $s < t$ we have*

$$\begin{aligned} f \circ s &< f \circ t \quad \text{and} \\ f \circ t - f \circ s &\sim \tau_f \circ t - \tau_f \circ s. \end{aligned}$$

Proof. There is a $\nu>0$ with $f\in(\mathbb{U}_\nu)_{(<\nu)}$ by (8.4), so $g:=f\circ\mathfrak{a}_\nu\in(\mathbb{S}_\nu)_{(<\nu)}$ by Lemma 8.14. We have $g\in\mathcal{P}_{\nearrow,\nu}$ by Proposition 8.13. In particular, the series g acts as a strictly increasing function on $\mathbb{V}^{>,\succ}$. Since $\mathbb{V}^{>,\succ}\longrightarrow\mathbb{V}^{>,\succ};s\mapsto\mathfrak{a}_\nu\circ s$ is bijective and strictly increasing, there are $u,v\in\mathbb{V}^{>,\succ}$ with $u<v$ and $(\mathfrak{a}_\nu\circ u,\mathfrak{a}_\nu\circ v)=(s,t)$. So

$$f\circ s=g\circ u<g\circ v=f\circ t$$

and

$$f\circ t-f\circ s=g\circ v-g\circ u\sim\tau_g\circ v-\tau_g\circ u=\tau_f\circ t-\tau_f\circ s.$$

This concludes the proof. □

Corollary 8.17. *For $f\in\tilde{\mathbb{U}}$, the function $F:\mathbb{V}^{>,\succ}\longrightarrow\mathbb{V};s\mapsto f\circ s$ is strictly increasing if and only if $f'>0$, strictly decreasing if and only if $f'<0$, constant if and only if $f'=0$.*

Proof. We first treat the case when $f\succ 1$. If $f>\mathbb{R}$ then $f'>0$ by **H1**, and F is strictly increasing by Theorem 8.16. It follows that if $f<\mathbb{R}$, then $f'<0$ and F is strictly decreasing. Assume now that $f\asymp 1$, so $f=r+\varepsilon$ for a $r\in\mathbb{R}$ and a $\varepsilon\in\tilde{\mathbb{U}}^{\prec}$. We have $f'=\varepsilon'$ and the monotony of F is that of the function $s\mapsto\varepsilon\circ s$. Assume that $\varepsilon\neq 0$. The function $s\mapsto\varepsilon\circ s$ is strictly increasing (resp. decreasing) if and only if $s\mapsto\varepsilon^{-1}\circ s$ is strictly decreasing (resp. increasing), we obtain the result by applying the previous arguments to $\varepsilon^{-1}\succ 1$. We have $f'=0$ if and only if $\varepsilon=0$, in which case F is constant with constant value r. □

8.6 The case of finitely nested hyperseries

Lemma 8.18. *Write $\lambda:=\omega^\nu$. The set $\{1\}$ is a near-support for $'$ on $\mathbb{L}_{<\lambda}\circ\mathrm{e}_\lambda^{\ell_0}$.*

Proof. Consider a monomial $\mathfrak{h}$ in $\mathbb{L}_{<\lambda}\circ\mathrm{e}_\lambda^{\ell_0}$. There is an $\mathfrak{l}\in\mathfrak{L}_{<\lambda}$ with $\mathfrak{h}=\mathfrak{l}\circ\mathrm{e}_\lambda^{\ell_0}$, and we have $\partial(\mathfrak{h})=\partial(\mathrm{e}_\lambda^{\ell_0})\,\partial(\mathfrak{l})\circ\mathrm{e}_\lambda^{\ell_0}$. We have $\operatorname{supp}\partial(\mathfrak{l})\subseteq\mathfrak{l}\cdot\{\ell_\gamma^\dagger:\gamma_0\leqslant\gamma<\lambda\}$ where $\gamma_0<\lambda$ is minimal with $\mathfrak{l}_{\gamma_0}\neq 0$. Note that $\mathfrak{l}\equiv\ell_{\gamma_0}$. Therefore

$$\operatorname{supp}\partial(\mathfrak{h})\subseteq\partial(\mathrm{e}_\lambda^{\ell_0})\,(\mathfrak{l}\circ\mathrm{e}_\lambda^{\ell_0})\cdot\{\ell_\gamma^\dagger\circ\mathrm{e}_\lambda^{\ell_0}:\gamma_0\leqslant\gamma<\lambda\}.$$

For $\gamma<\lambda$, we have

$$1\preccurlyeq\ell_\gamma^\dagger\circ\mathrm{e}_\lambda^{\ell_0}\,\partial(\mathrm{e}_\lambda^{\ell_0})\equiv\ell_{\gamma+1}\circ\mathrm{e}_\lambda^{\ell_0}\ll\ell_{\gamma_0}\circ\mathrm{e}_\lambda^{\ell_0}\equiv\mathfrak{l}\circ\mathrm{e}_\lambda^{\ell_0}$$

so $\partial(\mathrm{e}_\lambda^{\ell_0})\cdot\{\ell_\gamma^\dagger\circ\mathrm{e}_\lambda^{\ell_0}:\gamma_0\leqslant\gamma<\lambda\}\prec\!\!\prec\mathfrak{h}$. This proves that $\{1\}$ is a near-support for ∂ on $\mathbb{L}_{<\lambda}\circ\mathrm{e}_\lambda^{\ell_0}$. □

Corollary 8.19. *Let $\mathbb{V}$ be a confluent hyperserial field of force $(\mathbf{On},\mathbf{On})$, and assume that each $f\in\tilde{\mathbb{L}}$ has Taylor expansions with respect to $':\tilde{\mathbb{L}}\longrightarrow\tilde{\mathbb{L}}$ and the law $\circ:\tilde{\mathbb{L}}\times\mathbb{V}^{>,\succ}\longrightarrow\mathbb{V}$ given by Corollary 6.21. For $f\in\tilde{\mathbb{L}}^{>,\succ}$ and $s,t\in\mathbb{V}^{>,\succ}$ with $s<t$ we have*

$$\begin{aligned} f\circ s &< f\circ t \quad \textit{and}\\ f\circ t-f\circ s &\sim \tau_f\circ t-\tau_f\circ s.\end{aligned}$$

Proof. We need only justify that $\mathbb{L}$ satisfies the conditions imposed on $\mathbb{U}$ in Section 8.5. We know by Corollary 6.28 and [15, Proposition 7.8] that each $f\in\tilde{\mathbb{L}}$ satisfies the chain rule. We know by Corollary 5.22 that $(\tilde{\mathbb{L}},')$ is an H-field with small derivations. For each $\nu\in\mathbf{On}$ with $\nu>0$, we set $\mathbb{L}_\nu:=\mathbb{L}_{<\omega^\nu}$, and $\mathfrak{a}_\nu:=\mathrm{e}_{\omega^\nu}^{\ell_0}$. Since $\mathfrak{a}_\nu$ is $L_{<\omega^\nu}$-atomic, the right composition with $\mathfrak{a}_\nu$ preserves monomials, so it is a hyperserial embedding of force $\boldsymbol{\nu}$. Note that $\mathbb{V}^{>,\succ}\longrightarrow\mathbb{V}^{>,\succ};s\mapsto\mathfrak{a}_\nu\circ s$ is strictly increasing. Fix $\nu\in\mathbf{On}$ with $\nu>0$. The same arguments as in Lemma 8.11 using Lemma 8.6 yield $(\log\mathfrak{L}_{<\omega^\nu})\circ\mathrm{e}_{\omega^\nu}^{\ell_0}\subseteq\mathcal{P}_{\nearrow,\nu}$. We deduce with Proposition 8.9 that $\mathbb{L}_{<\omega^\nu}\circ\mathrm{e}_{\omega^\nu}^{\ell_0}\subseteq\mathcal{P}_{\nearrow,\nu}$. Finally, we have $\tilde{\mathbb{L}}=\bigcup_{0<\nu<\mathbf{On}}(\mathbb{L}_{<\omega^\nu})_{(<\nu)}$, so Theorem 8.16 applies. □

Corollary 8.20. *Each $f \in \tilde{\mathbb{L}}^{>,\succ}$ acts as a strictly increasing function on $\tilde{\mathbb{L}}^{>,\succ}$.*

Since $(\tilde{\mathbb{L}}, \partial)$ is an H-field, we have the following equivalences for $f \in \tilde{\mathbb{L}}$:

$$\begin{aligned} f' > 0 &\iff f \in \tilde{\mathbb{L}}^{>,\succ} \quad \text{or} \quad f \in \mathbb{R} - (\tilde{\mathbb{L}}^{>,\succ})^{-1}, \\ f' < 0 &\iff f \in -\tilde{\mathbb{L}}^{>,\succ} \quad \text{or} \quad f \in \mathbb{R} + (\tilde{\mathbb{L}}^{>,\succ})^{-1}, \quad \text{and} \\ f' \in 0 &\iff f \in \mathbb{R}. \end{aligned}$$

It is then easy to deduce the following result.

Corollary 8.21. *Let $\mathbb{V}$ be a confluent hyperserial field of force $(\mathbf{On}, \mathbf{On})$, and assume that each $f \in \tilde{\mathbb{L}}$ has Taylor expansions with respect to $': \tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ and the law $\circ: \tilde{\mathbb{L}} \times \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}$ given by Corollary 6.21. For $f \in \tilde{\mathbb{L}}^{>,\succ}$ and $s, t \in \mathbb{V}^{>,\succ}$ with $s < t$ we have*

$$\begin{aligned} f \circ t > f \circ s &\iff f' > 0, \\ f \circ t < f \circ s &\iff f' < 0, \ \textit{and} \\ f \circ s = f \circ t &\iff f' = 0. \end{aligned}$$

Corollary 8.22. *Let $\mathbb{V} \supseteq \tilde{\mathbb{L}}$ be a confluent hyperserial field of force $(\mathbf{On}, \mathbf{On})$, equipped with the composition law $\circ: \tilde{\mathbb{L}} \times \mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}$ of Corollary 6.21. Assume that $(', \circ)$ has Taylor expansions where $': \tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$, and that $(\partial, \circ)$ satisfies the chain rule. Then each function*

$$\mathbb{V}^{>,\succ} \longrightarrow \mathbb{V}^{>,\succ}; s \mapsto f \circ s$$

for $f \in \tilde{\mathbb{L}}^{>,\succ}$ is bijective.

Proof. This function is strictly increasing by Theorem 8.16, hence injective. We have a $f^{\mathrm{inv}} \in \tilde{\mathbb{L}}^{>,\succ}$ with $f^{\mathrm{inv}} \circ f = \ell_0$. So by Lemma 2.1 and Corollary 6.23, the function $s \mapsto f^{\mathrm{inv}} \circ s$ is the functional inverse of $s \mapsto f \circ s$. □

Proposition 8.23. *The structure $(\tilde{\mathbb{L}}^{>,\succ}, \circ, <)$ is a linearly ordered group.*

Proof. We know that $(\tilde{\mathbb{L}}^{>,\succ}, <)$ is a linearly ordered class and that $(\tilde{\mathbb{L}}^{>,\succ}, \circ, \ell_0)$ is a group, so it is enough to prove the following statement

$$\forall f, g, h \in \tilde{\mathbb{L}}^{>,\succ}, g > \ell_0 \Longrightarrow (f \circ g > f \wedge g \circ h > h).$$

Let $f, g, h \in \tilde{\mathbb{L}}^{>,\succ}$ with $g > \ell_0$. Since f acts as a strictly increasing function on $\tilde{\mathbb{L}}^{>,\succ}$, we have $f \circ g > f \circ \ell_0 = f$. The right composition by h is strictly increasing, so $g \circ h > \ell_0 \circ h = h$. □

9 Conjugacy

In [19], Ecalle studies what he calls the natural growth scale. This is a (somewhat informally defined) group $\mathcal{G}$, under composition, of germs at $+\infty$ of real quasi-analytic functions. The elements in $\mathcal{G}$ involve hyperexponential and hyperlogarithmic functions $\exp_{\omega^k}, \log_{\omega^k}$, $k \in \mathbb{N}$ which satisfy the same conjugation equations as ℓ_{ω^k} and $\mathrm{e}_{\omega^k}^{\ell_0}$, i.e.

$$\begin{aligned} \exp_{\omega^{k+1}}(r+1) &= \exp_{\omega^k}(\exp_{\omega^{k+1}}(r)) \quad \text{and} \\ \log_{\omega^{k+1}}(\log_{\omega^k}(r)) &= \log_{\omega^{k+1}}(r) \end{aligned}$$

for large enough $r \in \mathbb{R}$ (see also [35, Appendix A] for a construction of those functions).

Thus the group $(\tilde{\mathbb{L}}^{>,\succ},\circ)$ and $(\mathcal{G},\circ)$ can be regarded respectively as formal and geometric substantiations of the same idea. Ecalle gives formulas for conjugacy relations within $(\mathcal{G},\circ)$. In order to make sense of those formulas in our formal setting, we rely on G. A. Edgar's work [21]. In [21, Section 4], Edgar shows that each transseries $f\in\mathbb{T}_{\mathrm{LE}}^{>,\succ}\subseteq\tilde{\mathbb{L}}$ of exponentiality 0 (i.e. with $\mathfrak{d}_\omega(f)=\ell_0$) with $f>\ell_0$ is a conjugate of ℓ_0+1 [21, Theorem 4.4]. Edgar's proofs apply in our case with a few adjustments that will be made below.

We will prove that any two $f,g\in\tilde{\mathbb{L}}^{>,\succ}$ with $f,g>\ell_0$ are conjugate in the group $(\tilde{\mathbb{L}}^{>,\succ},\circ)$, i.e. that there is a $V\in\tilde{\mathbb{L}}^{>,\succ}$ with $V\circ f=g\circ V$. It is enough to prove that each $f\in\tilde{\mathbb{L}}^{>,\succ}$ with $f>\ell_0$ is a conjugate of a fixed series $>\ell_0$. We choose ℓ_0+1, which we see as the simplest one. So for $a\in\tilde{\mathbb{L}}$ with $a>\ell_0$, we are looking for a $V\in\tilde{\mathbb{L}}^{>,\succ}$ with

$$V\circ f=V+1.$$

9.1 Edgar's method

Besides the properties of $\mathbb{T}_{\mathrm{LE}}$ as a differential field (in particular, that it is an H-field), Edgar relies on properties of an integral operator $\int$ on $\mathbb{T}_{\mathrm{LE}}$. So we must introduce it in our setting. Recall that $\partial(\tilde{\mathbb{L}})=\tilde{\mathbb{L}}$ and $\operatorname{Ker}(\partial)=\mathbb{R}$, so for each $f\in\tilde{\mathbb{L}}$, there is a unique $F\in\tilde{\mathbb{L}}$ with $1\notin\operatorname{supp}F$ and $F'=f$. We write $\int f:=F$. For $s,t\in\tilde{\mathbb{L}}^{>,\succ}$, we also write

$$\int_s^t f:=\left(\int f\right)\circ t-\left(\int f\right)\circ s.$$

Lemma 9.1. *For $s,t\in\tilde{\mathbb{L}}^{>,\succ}$, the functions $\int:\tilde{\mathbb{L}}\longrightarrow\tilde{\mathbb{L}}$ and $\int_s^t:\tilde{\mathbb{L}}\longrightarrow\tilde{\mathbb{L}}$ are strongly linear.*

Proof. Recall that $(\tilde{\mathbb{L}},\partial)$ is an H-field, so given $\mathfrak{m}\in\mathfrak{K}$, there is a unique $\mathfrak{n}\in\mathfrak{K}$ and a unique $r\in\mathbb{R}^{\neq}$ with $\mathfrak{m}\sim r\,\mathfrak{n}'$. We then write $\mathcal{I}(\mathfrak{m}):=r\,\mathfrak{n}$. Note that $\mathcal{I}:\mathfrak{K}\longrightarrow\mathfrak{K}$ is strictly $\prec$-increasing so it extends uniquely into a strongly linear function $\mathcal{I}:\tilde{\mathbb{L}}\longrightarrow\tilde{\mathbb{L}}$. Now the strongly linear function

$$\begin{aligned}\Psi:\tilde{\mathbb{L}} &\longrightarrow \tilde{\mathbb{L}}\\ f &\longmapsto f-(\partial\circ\mathcal{I})(f)\end{aligned}$$

is contracting, i.e. satisfies $\Psi(\mathfrak{m})\prec\mathfrak{m}$ for all $\mathfrak{m}\in\mathfrak{K}$. So by Corollary 1.47, the function $\partial\circ\mathcal{I}=\mathrm{id}_{\tilde{\mathbb{L}}}-\Psi$ has a strongly linear functional inverse

$$(\partial\circ\mathcal{I})^{\mathrm{inv}}=\sum_{k\in\mathbb{N}}\Psi^{[k]}.$$

We have $(\mathcal{I}\circ(\partial\circ\mathcal{I})^{\mathrm{inv}})(1)=\mathcal{I}(0)=0$. We deduce that $\int=\mathcal{I}\circ(\partial\circ\mathcal{I})^{\mathrm{inv}}$ is strongly linear, whence also $\int_s^t$ is strongly linear. □

We next prove elementary properties of the integral operator. The reader can see [20, Corollary 3.17 and Proposition 3.18] for similar results in the case of transseries.

Lemma 9.2. *Let $f,g\in\tilde{\mathbb{L}}$ and $s,t\in\tilde{\mathbb{L}}^{>,\succ}$ with $s\leqslant t$. If $0\leqslant f\leqslant g$ then $0\leqslant\int_s^t f\leqslant\int_s^t g$.*

Proof. By linearity, it is enough to show that $\int_s^t f\geqslant 0$. Since $(\int f)'\geqslant 0$, the function $\tilde{\mathbb{L}}^{>,\succ}\to\tilde{\mathbb{L}};u\mapsto(\int f)\circ u$ is non-decreasing by Corollary 8.20. Therefore $\int_s^t f=(\int f)\circ t-(\int f)\circ s\geqslant 0$. □

Corollary 9.3. *For $f,g\in\tilde{\mathbb{L}}$ and $s,t\in\tilde{\mathbb{L}}^{>,\succ}$ with $s\neq t$, we have $f\prec g\Longrightarrow\int_s^t f\prec\int_s^t g$.*

Proof. Write $s_0 := \min(s,t)$ and $t_0 := \max(s,t)$. For all $r \in \mathbb{R}^>$, we have $|f| < r\,|f|$. Therefore

$$\left|\int_s^t f\right| = \int_{s_0}^{t_0} |f| \leqslant \int_{s_0}^{t_0} r\,|g| = r\int_{s_0}^{t_0} |g| = \left|\int_s^t g\right|$$

by Lemmas 9.2 and 9.1. Note that $g \neq 0$, so $\int g \notin \mathbb{R}$. Since $s \neq t$, we have $(\int g)\circ s \neq (\int g)\circ t$ by Corollary 8.20, whence $|\int_s^t g| > 0$. We deduce that $|\int_s^t f| \prec |\int_s^t g|$, whence $\int_s^t f \prec \int_s^t g$. □

Corollary 9.4. *For $f, g \in \tilde{\mathbb{L}}^{\neq}$ and $s,t \in \tilde{\mathbb{L}}^{>,\succ}$ with $s \neq t$, we have $f \sim g \Longrightarrow \int_s^t f \sim \int_s^t g$.*

An important step in Edgar's proof of [21, Theorem 4.4] is to conjugate an $f \in \mathbb{T}_{\mathrm{LE}}^{>,\succ}$ of exponentiality 0 to a series $x + \varepsilon$ where $\varepsilon \prec 1$ is a so-called log-free transseries. We thus need to define a field $\mathbb{T}$ which plays the same role in $\tilde{\mathbb{L}}$ as the field of log-free transseries does in $\mathbb{T}_{\mathrm{LE}}^{>,\succ}$. We write $\mathfrak{M}$ for the group generated by $\ell_0^{\mathbb{R}}$ and all groups $\widetilde{\mathfrak{L}_{<\omega^\nu}} \circ \mathrm{e}_{\omega^\nu}^{\ell_0}$ for $\nu \in \mathbf{On}^>$. We also write $\mathbb{T} := \mathbb{R}[[\mathfrak{M}]]$. We have $\partial(\ell_0^{\mathbb{R}}) \subseteq \mathbb{L}_{<1} \circ \omega$. For $\nu \in \mathbf{On}^>$, we have

$$\begin{aligned}
\partial(\widetilde{\mathfrak{L}_{<\omega^\nu}} \circ \mathrm{e}_{\omega^\nu}^{\ell_0}) &\subseteq E_{\omega^\nu}^{\ell_0\prime} \times \partial(\widetilde{\mathfrak{L}_{<\omega^\nu}}) \circ \mathrm{e}_{\omega^\nu}^{\ell_0} \\
&\subseteq (\widetilde{\mathfrak{L}_{<\omega^\nu}} \circ \mathrm{e}_{\omega^\nu}^{\ell_0}) \cdot (\widetilde{\mathfrak{L}_{<\omega^\nu}} \circ \mathrm{e}_{\omega^\nu}^{\ell_0}) \\
&\subseteq (\widetilde{\mathfrak{L}_{<\omega^\nu}}) \circ \mathrm{e}_{\omega^\nu}^{\ell_0}.
\end{aligned}$$

We deduce with the Leibniz rule that $\partial(\mathbb{T}) \subseteq \mathbb{T}$.

Lemma 9.5. *For all $\mu, \nu \in \mathbf{On}$ with $0 < \mu < \nu$ and all $(\mathfrak{m}_0, \dots, \mathfrak{m}_n) \in \ell_0^{\mathbb{R}} \times \widetilde{\mathfrak{L}_{<\omega^\nu}} \circ \mathrm{e}_{\omega^{\nu_1}}^{\ell_0} \times \cdots \times \widetilde{\mathfrak{L}_{<\omega^\nu}} \circ \mathrm{e}_{\omega^{\nu_p}}^{\ell_0}$, we have $\mathfrak{m}_0 \cdots \mathfrak{m}_n \equiv \mathfrak{m}_i$ where $i = \max\{j \in \{0,\dots,p\} : \mathfrak{m}_j \neq 1\}$.*

Proof. By Corollary 1.31(c), it is enough to note that $\widetilde{\mathfrak{L}_{<\omega^\mu}}^{\neq} \circ \mathrm{e}_{\omega^\mu}^{\ell_0} \prec\!\!\prec (\widetilde{\mathfrak{L}_{<\omega^\nu}})^{\neq} \circ \mathrm{e}_{\omega^\nu}^{\ell_0}$ whenever $0 < \mu < \nu$ and that $\ell_0^{\mathbb{R}^{\neq}} \prec\!\!\prec \widetilde{\mathfrak{L}_{<\omega^\mu}}^{\neq} \circ \mathrm{e}_{\omega^\mu}^{\ell_0}$. □

Lemma 9.6. *For all $\mathfrak{m} \in \mathfrak{M}^{\prec}$, we have $\operatorname{supp} \mathfrak{m}' \equiv \mathfrak{m}$.*

Proof. Write $\mathfrak{m} = \mathfrak{m}_0 \cdots \mathfrak{m}_n$ for $(\mathfrak{m}_0, \dots, \mathfrak{m}_n) \in \ell_0^{\mathbb{R}} \times \widetilde{\mathfrak{L}_{<\omega^\nu}}^{\neq} \circ \mathrm{e}_{\omega^{\nu_1}}^{\ell_0} \times \cdots \times \widetilde{\mathfrak{L}_{<\omega^\nu}}^{\neq} \circ \mathrm{e}_{\omega^{\nu_p}}^{\ell_0}$ where $p \in \mathbb{N}$ and $\nu_1 < \cdots < \nu_p$. We have $\mathfrak{m}' = \mathfrak{m} \sum_{i=0}^p \mathfrak{m}_i^\dagger$ where $\mathfrak{m} \equiv \mathfrak{m}_p$ by Lemma 9.5. By Lemmas 8.18 and 8.15, the set $\{1\}$ is a near-support for ∂ on each $\mathbb{R}[[\mathfrak{M}(\nu)]] \circ E_{\omega^\nu}^{\ell_0}$, so $\operatorname{supp} \mathfrak{m}_i^\dagger \prec\!\!\prec \mathfrak{m}_i$ for all $i \in \{1,\dots,p\}$. If $p \neq 0$, then we also have $\operatorname{supp} \mathfrak{m}_0^\dagger \subseteq \{1, \ell_0^{-1}\} \prec\!\!\prec \mathfrak{m}$. So we have $\operatorname{supp} \mathfrak{m}' \equiv \mathfrak{m}$ in that case. If $p = 0$ then $\mathfrak{m} = \mathfrak{m}_0$, and since $\mathfrak{m}$ is infinitesimal, we have $\mathfrak{m}_0' \equiv \mathfrak{m}_0$. □

We require some technical lemmas whose proof in [21] relies on the specific inductive definition of $\mathbb{T}_{\mathrm{LE}}$, and must therefore be proved in a different way. The results which are relevant to our case are [21, Lemmas 3.11(g), 3.14(a,b), 3.20, 3.21 3.23, Theorems 3.8 and 4.1, and Propositions 4.2 and 4.3]. Using the adapted versions of those results, we will prove an adapted version of [21, Theorem 4.4].

Lemma 9.7. (adapted from [21, Lemma 3.11(g)]) *If $\mathfrak{b} \in \mathfrak{M}^{\neq}$ and $\mathfrak{n} \in \operatorname{supp}(\ell_0\,\mathfrak{b})'$, then $\mathfrak{n}^\dagger \sim \mathfrak{b}^\dagger$.*

Proof. We have $(\ell_0\,\mathfrak{b})' = \mathfrak{b} + \ell_0\,\mathfrak{b}'$ so we may assume that $\mathfrak{n} \in \ell_0 \cdot \operatorname{supp} \mathfrak{b}'$. If $\mathfrak{b} \in \mathfrak{L}_{<1}$, then since $\mathfrak{b} \neq 1$, we have $\mathfrak{b}' \in \mathbb{R}^{\neq}\,\mathfrak{b}\,\ell_0^{-1}$ so $\mathfrak{n} \asymp \mathfrak{b}$, whence $\mathfrak{n}^\dagger \sim \mathfrak{b}^\dagger$. Otherwise $\mathfrak{b} \succ\!\!\!\succ \ell_0$. We have $\log((\operatorname{supp} \mathfrak{b}') \cdot \ell_0) \subseteq (\log(\operatorname{supp} \mathfrak{b}')) + \log \ell_0$ where $\log(\operatorname{supp} \mathfrak{b}') \asymp \log \mathfrak{b} \succ \log \ell_0$ by Lemma 9.6, whence $(\log(\operatorname{supp} \mathfrak{b}')) \succ \log \ell_0$. In particular $\log \mathfrak{n} \sim \log \mathfrak{b}$ so $\mathfrak{n}^\dagger \sim \mathfrak{b}^\dagger$. □

Lemma 9.8. (adapted from [21, Lemma 3.14(a,b)]) *Let* $\mathfrak{m} \in \mathfrak{K}$ *and set*

$$\begin{aligned} \mathfrak{B} &:= \{\mathfrak{n} \in \mathfrak{K} : \mathfrak{n}^\dagger \prec \mathfrak{m}\} \quad \textit{and} \\ \overline{\mathfrak{B}} &:= \{\mathfrak{n} \in \mathfrak{K} : \mathfrak{n}^\dagger \preccurlyeq \mathfrak{m}\}. \end{aligned}$$

Then $\mathfrak{B}$ *and* $\overline{\mathfrak{B}}$ *are subgroups of* $\mathfrak{K}$*. Moreover, if* $\mathfrak{g} \in \mathfrak{M}^\prec$*, then we have*

$$\begin{aligned} \mathfrak{g} \in \mathfrak{B} &\implies \operatorname{supp}(\ell_0\,\mathfrak{g})' \subseteq \mathfrak{B} \quad \textit{and} \\ \mathfrak{g} \in \overline{\mathfrak{B}} &\implies \operatorname{supp}(\ell_0\,\mathfrak{g})' \subseteq \overline{\mathfrak{B}}. \end{aligned}$$

Proof. The fact that $\mathfrak{B}$ and $\overline{\mathfrak{B}}$ are subgroups follows from the inequality

$$(\mathfrak{n}_0\,\mathfrak{n}_1)^\dagger = \mathfrak{n}_0^\dagger + \mathfrak{n}_1^\dagger \preccurlyeq \max_{\preccurlyeq}(\mathfrak{n}_0^\dagger, \mathfrak{n}_1^\dagger) \quad \text{for all } \mathfrak{n}_0, \mathfrak{n}_1 \in \mathfrak{K}.$$

The last two statements follow from Lemma 9.7. □

Lemma 9.9. (adapted from [21, Lemma 3.20]) *Let* $\mathfrak{B} \subseteq \mathfrak{K}$ *be well-based. Let* $\mathfrak{g} \in \mathfrak{B}$*. There are finitely many pairs* $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathfrak{B}$ *with* $\mathfrak{g} \in \operatorname{supp}(\ell_0\,\mathfrak{g}_1)'\,\mathfrak{g}_2$*.*

Proof. The family $(\ell_0\,\mathfrak{m})_{\mathfrak{m}\in\mathfrak{B}}$ is well-based. Since ∂ is strongly linear, it follows that $((\ell_0\,\mathfrak{m})')_{\mathfrak{m}\in\mathfrak{B}}$ is well-based. So $((\ell_0\,\mathfrak{m})'\,\mathfrak{n})_{\mathfrak{m},\mathfrak{n}\in\mathfrak{B}}$ is well-based by Lemma 1.3. □

Lemma 9.10. (adapted from [21, Lemma 3.21]) *Let* $\mathfrak{e} \in \mathfrak{K}^\prec$*, and set*

$$\begin{aligned} \mathfrak{A} &:= \{\mathfrak{g} \in \mathfrak{K} : \mathfrak{g} \preccurlyeq \mathfrak{e} \wedge \mathfrak{g}^\dagger \prec (\ell_0\,\mathfrak{e})^{-1}\} \quad \textit{and} \\ \overline{\mathfrak{A}} &:= \{\mathfrak{g} \in \mathfrak{K} : \mathfrak{g} \preccurlyeq \mathfrak{e} \wedge \mathfrak{g}^\dagger \preccurlyeq (\ell_0\,\mathfrak{e})^{-1}\}. \end{aligned}$$

Then $\mathbb{R}[[\mathfrak{A}]]$ *and* $\mathbb{R}[[\overline{\mathfrak{A}}]]$ *are closed under the operations*

$$\begin{aligned} \mathfrak{g}_0, \mathfrak{g}_1 &\mapsto \mathfrak{g}_0\,\mathfrak{g}_1, \\ \mathfrak{g}_0, \mathfrak{g}_1 &\mapsto (\ell_0\,\mathfrak{g}_0)'\,\mathfrak{g}_1, \quad \textit{and} \\ \mathfrak{g} &\mapsto \ell_0\,\mathfrak{e}\,\mathfrak{g}'. \end{aligned}$$

Proof. Apply Edgar's proof of [21, Lemma 3.21], using Lemma 9.8 instead of [21, Lemma 3.14(a,b)]. □

Lemma 9.11. (adapted from [21, Lemma 3.23]) *Let* $\mathfrak{B} \subseteq \mathfrak{M}$ *be non-empty, well-based and infinitesimal. Write* $\mathfrak{e} = \max \mathfrak{B}$ *and assume that* $\mathfrak{g}^\dagger \preccurlyeq (\ell_0\,\mathfrak{e})^{-1}$ *for all* $\mathfrak{g} \in \mathfrak{B}$*. Let* $\overline{\mathfrak{B}}$ *denote the smallest set of monomials such that*

i. $\overline{\mathfrak{B}} \supseteq \mathfrak{B}$*,*

ii. if $\mathfrak{g}_1, \mathfrak{g}_2 \in \overline{\mathfrak{B}}$*, then* $\mathfrak{g}_1\,\mathfrak{g}_2 \in \overline{\mathfrak{B}}$*,*

iii. if $\mathfrak{g}_1, \mathfrak{g}_2 \in \overline{\mathfrak{B}}$*, then* $\operatorname{supp}(\ell_0\,\mathfrak{g}_1)'\,\mathfrak{g}_2 \subseteq \overline{\mathfrak{B}}$*.*

Then $\overline{\mathfrak{B}}$ *is well-based.*

Proof. We need to prove that the least set $\mathfrak{B}_1$ of monomials with $\mathfrak{B}_1 \supseteq \mathfrak{B} \cup \{\mathfrak{e}^2\}$ with $\forall \mathfrak{g} \in \mathfrak{B}_1, \operatorname{supp}(\ell_0\,\mathfrak{e}\,\mathfrak{g}') \subseteq \mathfrak{B}_1$ is well-based. To that end write

$$\begin{aligned} \mathfrak{S} &:= \{\mathfrak{m} \in \mathfrak{M} : \mathfrak{m}^\dagger \prec (\ell_0\,\mathfrak{e})^{-1}\} \quad \text{and} \\ \overline{\mathfrak{S}} &:= \{\mathfrak{m} \in \mathfrak{M} : \mathfrak{m}^\dagger \preccurlyeq (\ell_0\,\mathfrak{e})^{-1}\}. \end{aligned}$$

Consider the derivation $\partial_\mathfrak{e} := \ell_0\,\mathfrak{e}\,\partial$ on $\tilde{\mathbb{L}}$. We claim that

$$\begin{aligned} \partial_\mathfrak{e}(\mathbb{R}[[\mathfrak{S}]]) &\subseteq \mathbb{R}[[\mathfrak{S}]] \quad \text{and} \\ \partial_\mathfrak{e}(\mathbb{R}[[\overline{\mathfrak{S}}]]) &\subseteq \mathbb{R}[[\overline{\mathfrak{S}}]]. \end{aligned}$$

Given $\mathfrak{m} \in \mathfrak{S}$, we have $\operatorname{supp} \mathfrak{m}' \equiv \mathfrak{m}$ by Lemma 9.6. We deduce that $(\operatorname{supp} \mathfrak{m}')^\dagger \asymp \mathfrak{m}^\dagger \preccurlyeq (\ell_0\, \mathfrak{e})^{-1}$ with a strict inequality if $\mathfrak{m} \in \mathfrak{S}$. Thus $(\operatorname{supp} \partial_{\mathfrak{e}}(\mathfrak{m}))^\dagger \subseteq \{\ell_0^{-1}, \mathfrak{e}^\dagger\} \cup (\operatorname{supp} \mathfrak{m}')^\dagger \preccurlyeq (\ell_0\, \mathfrak{e})^{-1}$, with strict inequality if $\mathfrak{m} \in \mathfrak{S}$. So $\partial_{\mathfrak{e}}$ restricts to strongly linear maps $\mathbb{R}[[\mathfrak{S}]] \longrightarrow \mathbb{R}[[\mathfrak{S}]]$ and $\mathbb{R}[[\widetilde{\mathfrak{S}}]] \longrightarrow \mathbb{R}[[\widetilde{\mathfrak{S}}]]$. Moreover we have $\partial_{\mathfrak{e}}(\mathfrak{m}) \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{S}$. Thus we have a strictly extensive and Noetherian choice operator

$$\forall \mathfrak{m} \in \mathfrak{G}, \vartheta(\mathfrak{m}) := \operatorname{supp} \partial_{\mathfrak{e}}(\mathfrak{m})$$

on $\mathbb{R}[[\mathfrak{G}]]$. Given $\mathfrak{X} \subseteq \mathfrak{M}$, we write $\mathrm{Cl}(\mathfrak{X})$ for the union of classes $\mathrm{Cl}_n(\mathfrak{X}), n \in \mathbb{N}$ where

$$\begin{aligned} \mathrm{Cl}_0(\mathfrak{X}) &:= \mathfrak{X} \quad \text{and} \\ \mathrm{Cl}_{n+1}(\mathfrak{X}) &:= \mathrm{Cl}_n(\mathfrak{X}) \cup \bigcup_{\mathfrak{m} \in \mathrm{Cl}_n(\mathfrak{X})} \operatorname{supp} \partial_{\mathfrak{e}}(\mathfrak{m}) \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

By Corollary 1.43, for each well-based subset $\mathfrak{W}$ of $\mathfrak{S}$, the set $\mathrm{Cl}(\mathfrak{W})$ is well-based.

Let $\mathfrak{C} := (\mathfrak{B} \cup \{\mathfrak{e}^2\}) \cap \mathfrak{S}$ and $\mathfrak{D} := (\mathfrak{B} \cup \{\mathfrak{e}^2\}) \setminus \mathfrak{S} = \{\mathfrak{m} \in \mathfrak{B} \cup \{\mathfrak{e}^2\} : \mathfrak{m}^\dagger \asymp (\ell_0\, \mathfrak{e})^{-1}\}$. Since $\partial_{\mathfrak{e}}(\mathfrak{S}) \subseteq \mathbb{R}[[\mathfrak{S}]]$, writing $\mathfrak{E} = \bigcup_{\mathfrak{m} \in \mathfrak{D}} \operatorname{supp} \partial_{\mathfrak{e}}(\mathfrak{m}) \setminus \{\mathfrak{m}\}$, we have $\mathfrak{E} \subseteq \mathfrak{S}$ and $\mathrm{Cl}(\mathfrak{D}) = \mathfrak{D} \cup \mathrm{Cl}(\mathfrak{E})$. So

$$\mathfrak{B}_1 = \mathrm{Cl}(\mathfrak{B}) \subseteq \mathrm{Cl}(\mathfrak{C}) \cup \mathrm{Cl}(\mathfrak{D}) \subseteq \mathrm{Cl}(\mathfrak{C}) \cup \mathfrak{D} \cup \mathrm{Cl}(\mathfrak{E})$$

is well-based. Now apply Edgar's proof of [21, Lemma 3.23], using Lemma 9.10 instead of [21, Lemma 3.21(c)]. □

9.2 Solving conjugacy equations

Proposition 9.12. (adapted from [21, Theorems 3.8 and 4.1]) *Let $f \in \mathbb{T}^{>,\succ}$ of the form $f = \ell_0\,(1 + r\,\mathfrak{e} + \delta)$ where $r \in \mathbb{R}^>$, $\delta \prec \mathfrak{e} \prec 1$, $(\operatorname{supp} \delta)^\dagger \preccurlyeq (\ell_0\, \mathfrak{e})^{-1}$ and $\mathfrak{g}^\dagger \asymp (\ell_0\, \mathfrak{e})^{-1}$ for a $\mathfrak{g} \in \operatorname{supp} \delta$. Then there is a $V \in \tilde{\mathbb{L}}^{>,\succ}$ with $V' \sim (r\, \ell_0\, \mathfrak{e})^{-1}$ and*

$$V \circ f = V + 1.$$

Proof. Note that $\mathfrak{e}, \delta \in \mathbb{T}$. By the arguments in the proof of [21, Theorems 3.8], using Lemma 9.11 instead of [21, Lemma 3.23] and Lemma 9.9 instead of [21, Lemma 3.20], we obtain a series $\Phi_1(0, \ell_0) \sim r\, \ell_0\, \mathfrak{e}$ with

$$\frac{f'}{\Phi_1(0, \ell_0) \circ f} = \frac{1}{\Phi_1(0, \ell_0)}.$$

So setting $V := \int \frac{1}{\Phi_1(0, \ell_0)}$, we have $f' \times V' \circ f = V'$, whence $(V \circ f)' = V'$ by the chain rule. So $V \circ f = V + r_0$ for a certain $r_0 \in \mathbb{R}$. It is enough in order to conclude to prove that $r_0 = 1$. Set $A = \int \frac{1}{r\, \ell_0\, \mathfrak{e}}$ and write τ for the dominant term of A. So

$$\begin{aligned} V \circ f - V &= \int_{\ell_0}^{f} \frac{1}{\Phi_1(0, \ell_0)} \sim \int \frac{1}{r\, \ell_0\, \mathfrak{e}} \quad \text{and} && \text{(by Corollary 9.4)} \\ \int \frac{1}{r\, \ell_0\, \mathfrak{e}} &= A \circ f - A \sim \tau \circ f - \tau. && \text{(by Corollary 8.20)} \end{aligned}$$

We have $\frac{1}{r\, \ell_0\, \mathfrak{e}} \succ \ell_0^{-1} = (\log \ell_0)'$ so $A \succ 1$. It follows that $f - \ell_0 \asymp \mathfrak{e}\, \ell_0 \prec \frac{\tau}{\tau'}$. Thus by **TE** and Corollary 7.4, we have $\tau \circ f - \tau \sim \tau'\,(f - \ell_0) \sim A'\,(f - \ell_0) \sim \frac{r\, \ell_0\, \mathfrak{e}}{r\, \ell_0\, \mathfrak{e}} \sim 1$. So $r_0 \sim 1$. But r_0 is a real number, so $r_0 = 1$. □

Proposition 9.13. (adapted from [21, Theorem 4.2]) *Let $f \in \mathbb{T}^{>,\succ}$ of the form $f = \ell_0 + \varepsilon$ where $\varepsilon \prec 1$ and $\varepsilon > 0$. There are a $V \in \tilde{\mathbb{L}}^{>,\succ}$ and a $\delta \in \tilde{\mathbb{L}}^{\prec}$ with $\delta^\dagger \succ 1$ and*

$$V \circ f \circ V^{\mathrm{inv}} = \ell_0 + 1 + \delta.$$

Proof. The proof is the same as in [21, Proposition 4.2], using Proposition 9.12 instead of [21, Theorem 4.1]. □

Proposition 9.14. (adapted from [21, Proposition 4.3]) *Let $f \in \tilde{\mathbb{L}}^{>,\succ}$ of the form $f = \ell_0 + 1 + \delta$ where $\delta \prec 1$ and $\delta^\dagger \succ 1$. There is a $V \in \tilde{\mathbb{L}}^{>,\succ}$ with*

$$V \circ f = V + 1. \tag{9.1}$$

Proof. Write Ψ for the operator $\tilde{\mathbb{L}} \longrightarrow \tilde{\mathbb{L}}$ defined by

$$\Psi(g) := (\ell_0 + g) \circ f - \ell_0 - 1 = g \circ f + \delta$$

It suffices to show that Ψ has a fixed point g_0; then $V := \ell_0 + g_0$ satisfies (9.1).

To that end, we will show that there is a subclass $\mathfrak{S}$ of $\mathfrak{K}$ such that $\Psi(\mathbb{R}[[\mathfrak{S}]]) \subseteq \mathbb{R}[[\mathfrak{S}]]$ and that $\mathfrak{m} \circ f \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{S}$. Consider the class $\mathfrak{S}$ of monomials $\mathfrak{m}$ with $\mathfrak{m} \prec 1$ and $\mathfrak{m}^\dagger \succ 1$. We have $\delta \in \mathbb{R}[[\mathfrak{S}]]$ since $\mathfrak{d}_\delta^\dagger \succ 1$ and the logarithmic derivative is strictly $\prec$-decreasing on $\mathfrak{K}^{\prec 1}$. Fix an $\mathfrak{m} \in \mathfrak{S}$. We have

$$\begin{aligned} \log \frac{\mathfrak{m} \circ f}{\mathfrak{m}} &= \log(\mathfrak{m} \circ f) - \log \mathfrak{m} \\ &= \int_{\ell_0}^{f} \mathfrak{m}^\dagger \\ &\succ \int_{\ell_0}^{f} 1. \qquad \text{(by Corollary 9.3)} \end{aligned}$$

So $\log \frac{\mathfrak{m} \circ f}{\mathfrak{m}} \succ f - \ell_0 \sim 1$. Since $f > \ell_0$, and $\mathfrak{m}$ is positive and infinitesimal, we have $\mathfrak{m} \circ f < \mathfrak{m}$. We deduce that $\log \frac{\mathfrak{m} \circ f}{\mathfrak{m}} < 0$ whence $\log \frac{\mathfrak{m} \circ f}{\mathfrak{m}} < \mathbb{R}$, so $\mathfrak{m} \circ f \prec \mathfrak{m}$. In particular, we have $\mathfrak{m} \circ f \in \mathbb{R}[[\mathfrak{S}]]$. By Proposition 1.46, the family $\delta, \delta \circ f, \delta \circ f \circ f, \ldots$ is well-based. The series

$$g_0 := \delta + \delta \circ f + \delta \circ f \circ f + \cdots$$

satisfies $g_0 \circ f = g_0 - \delta$, whence $\Psi(g_0) = g_0$ as desired. □

Lemma 9.15. *Let $f \in \tilde{\mathbb{L}}^{>,\succ}$. For sufficiently large $\nu \in \mathbf{On}$ there is an infinitesimal $\varepsilon \in \widetilde{\mathfrak{L}_{<\omega^\nu}} \circ \mathrm{e}_{\omega^\nu}^{\ell_0}$ with*

$$\ell_{\omega^\nu} \circ f \circ \mathrm{e}_{\omega^\nu}^{\ell_0} = \ell_0 + \varepsilon.$$

Proof. Fix a sufficiently large limit ordinal $\nu \in \mathbf{On}$ with $f \in \widetilde{\mathbb{L}_{<\omega^\nu}}$. We deduce with Lemma 6.24 that $\mathfrak{d}_{\omega^\nu}(f) = \ell_0$. So there are a $\mu < \nu$ and a $\delta \in \widetilde{\mathbb{L}_{<\omega^\nu}}$ with $\delta \prec \ell_{\omega^\mu}$ and $\ell_{\omega^\mu} \circ f = \ell_{\omega^\mu} + \delta$. By **TE**, we have

$$\ell_{\omega^\nu} \circ f - \ell_{\omega^\mu} = \ell_{\omega^\nu}^{\uparrow \omega^\mu} \circ (\ell_{\omega^\mu} + \delta) - \ell_{\omega^\nu} = \sum_{k>0} \frac{(\ell_{\omega^\nu}^{\uparrow \omega^\mu})^{(k)} \circ \ell_{\omega^\mu}}{k!} \delta^k.$$

Now each $(\ell_{\omega^\nu}^{\uparrow \omega^\mu})^{(k)} \in \mathbb{L}_{<\omega^\nu}$ for all $k > 0$ so $\sum_{k>0} \frac{\left(\ell_{\omega^\nu}^{\uparrow \omega^\mu}\right)^{(k)} \circ \ell_{\omega^\mu}}{k!} \delta^k \in \widetilde{\mathbb{L}_{<\omega^\nu}}$. The series

$$\varepsilon := \left(\sum_{k>0} \frac{(\ell_{\omega^\nu}^{\uparrow \omega^\mu})^{(k)} \circ \ell_{\omega^\mu}}{k!} \delta^k \right) \circ \mathrm{e}_{\omega^\nu}^{\ell_0}$$

satisfies the conditions. □

Theorem 9.16. *Any two $f, g \in \tilde{\mathbb{L}}^{>,\succ}$ with $f, g > \ell_0$ are conjugate.*

Proof. It suffices to prove that each $f > \ell_0$ is a conjugate of $\ell_0 + 1$. By Lemma 9.15, the series f is a conjugate of $\ell_0 + \varepsilon$ where $\varepsilon \in \mathbb{T}^{\prec}$ and $\varepsilon > 0$. By Proposition 9.13, the series f is a conjugate of $\ell_0 + 1 + \delta$ where $\delta \prec 1$ and $(\operatorname{supp}\delta)^{\dagger} \succ 1$. By Proposition 9.14, the series f is a conjugate of $\ell_0 + 1$. □

We see by taking inverses that any two series $f, g \in \tilde{\mathbb{L}}^{>,\succ}$ with $f, g < \ell_0$ are conjugate. Since $(\tilde{\mathbb{L}}^{>,\succ}, \circ, <)$ is an ordered group, no two series $f, g \in \tilde{\mathbb{L}}^{>,\succ}$ with $f < \ell_0 < g$ can be conjugate. This shows that there are exactly three conjugacy classes in $(\tilde{\mathbb{L}}^{>,\succ}, \circ)$ including $\{\ell_0\}$. In particular, the group $(\tilde{\mathbb{L}}^{>,\succ}, \circ)$ is simple. Note furthermore that the positive cone $\tilde{\mathbb{L}}^{>\ell_0}$ (and thus the ordering on $\tilde{\mathbb{L}}^{>,\succ}$) is first-order definable with parameters in $(\tilde{\mathbb{L}}^{>,\succ}, \circ)$ as the class of series that are conjugates of $\ell_0 + 1$.

Remark 9.17. Conversely, let $(\mathcal{G}, \cdot, 1)$ be a group with exactly three conjugacy classes $\{1\}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\mathcal{C}_1 = \mathcal{C}_2^{-1}$ and $\mathcal{C}_1 \cdot \mathcal{C}_1 \subseteq \mathcal{C}_1$. Then the class $\mathcal{C}_1$ is a positive cone for $\mathcal{G}$ and for the resulting order, the group $\mathcal{G}$ is linearly bi-ordered. As far as we know, the existence of such a linearly bi-ordered group is an open problem (see [11, Problem 3.31]). In order to obtain a set-sized solution $\mathcal{G}$ to this problem, it is enough to consider the closure under solutions V of $V \circ f = V + 1$ for $f > \ell_0$, composition and inverses, of $\{\ell_0 + 1\}$. So set $\mathcal{G}_0 := \{\ell_0 + \mathbb{R}\}$, and for $n \in \mathbb{N}$, define $\mathcal{G}_{n+1}$ as the subgroup of $\tilde{\mathbb{L}}^{>,\succ}$ generated by

$$\mathcal{G}_n \cup \{V \in \tilde{\mathbb{L}}^{>,\succ} : \exists f \in \mathcal{G}_n, (f > \ell_0 \wedge V \circ f = V + 1)\}.$$

For any fixed $f > \ell_0$, the class of series V with $V \circ f = V + 1$ is a set (see (9.3) below), so each $\mathcal{G}_n$ is a set. Thus $\mathcal{G} := \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$ is a set-sized solution.

9.3 Real iterates

Let us consider and solve the simple inequation

$$f \circ g \geqslant f \circ g, \tag{9.2}$$

for $f, g \in \tilde{\mathbb{L}}^{>,\succ}$ with $f, g > \ell_0$.

Consider for a moment the idea that hyperseries are akin to very regularly monotonous differentiable functions, like germs in Hardy fields with composition. Then one expects that when f is large, a slight increment of its argument, from ℓ_0 to g, should result in a relatively sharp increment in its value, which ought to exceed the increment from f to $g \circ f$. In other words, the series g being fixed, the inequality (9.2) ought to hold for large enough f. In order to understand how large f should be with respect to g, we are first led to consider the case when f and g commute, i.e. when $f \circ g = g \circ f$. Indeed this case should be elucidated first. As in [21], it turns out that the class of series which commute with g can be described using fractional and real iterates of g. Starting with the simpler case when $g = \ell_0 + 1$, we will show how those real iterates allow us to solve (9.2).

Lemma 9.18. *Let $f \in \tilde{\mathbb{L}}^{>,\succ}$. We have*

$$\begin{aligned} f \circ (\ell_0 + 1) &> f + 1 \quad \text{if } f > \ell_0 + \mathbb{R}. \\ f \circ (\ell_0 + 1) &< f + 1 \quad \text{if } f < \ell_0 + \mathbb{R}. \end{aligned}$$

Proof. Assume that $f > \ell_0 + \mathbb{R}$, and fix an $f \in \mathbb{R}^{>}$. The inequality $f - \ell_0 > \mathbb{R}$ implies by **H1** that $f' > 1$. If $f' \sim 1$, then $f = \ell_0 + \delta$ for $\delta := f - \ell_0 > \mathbb{R}$. We have

$$f \circ (\ell_0 + 1) - f = \ell_0 + 1 - \ell_0 + \delta \circ (\ell_0 + 1) - \delta = 1 + \delta \circ (\ell_0 + 1) - \delta$$

By Corollary 8.20, we have $\delta\circ(\ell_0+1)-\delta>0$, whence $f\circ(\ell_0+1)>f+1$. If $f'\asymp 1$ and $f'\nsim 1$, then $f=r\,\ell_0+\varepsilon$ for a certain $r\in\mathbb{R}$ with $r>1$ and a certain $\varepsilon\prec\ell_0$. By Corollary 8.19, we have

$$f\circ(\ell_0+1)-f\sim\tau_f\circ(\ell_0+1)-\tau_f\sim r.$$

We deduce that $f\circ(\ell_0+1)-f>1$, hence the result.

Assume now that $f'\succ 1$. By Corollary 8.19, we have

$$f\circ\left(\ell_0+\frac{2}{\tau_f'}\right)-f\sim\tau_f\circ\left(\ell_0+\frac{2}{\tau_f'}\right)-\tau_f.$$

We have ${}^{2}\!/_{\tau_f'}\prec{}^{\mathfrak{d}_f}\!/_{\mathfrak{d}_f'}$ so **TE** and Corollary 7.4 yield

$$\tau_f\circ\left(\ell_0+\frac{2}{\tau_f'}\right)-\tau_f\sim\tau_f'\frac{2}{\tau_f'}=2.$$

In particular, $f\circ\left(\ell_0+\frac{2}{\tau_f'}\right)-f>1$. Since $\tau_f'>\mathbb{R}$, we have $f\circ(\ell_0+1)-f>f\circ\left(\ell_0+{}^{2}\!/_{\tau_f'}\right)-f$ by Corollary 8.20, hence $f\circ(\ell_0+1)-f>1$. The statement when $f<\ell_0+\mathbb{R}$ follows from symmetric arguments. □

Lemma 9.19. *Let $f\in\tilde{\mathbb{L}}^{>,\succ}$ with $f\not<\ell_0+\mathbb{R}$ and $f\not>\ell_0+\mathbb{R}$. Then there is a unique $r_f\in\mathbb{R}$ such that $\delta:=f-\ell_0+r_f$ is infinitesimal, and for all $r\in\mathbb{R}^{>}$, we have*

$$\begin{aligned} f\circ(f+r) &< f+r \quad \text{if } \delta>0\\ f\circ(f+r) &> f+r \quad \text{if } \delta<0.\end{aligned}$$

Proof. We must have $f-\ell_0\asymp 1$ so $f-\ell_0=r_f+\delta$ for unique $r_f\in\mathbb{R}$ and $\delta\prec 1$. Let $r\in\mathbb{R}^{>}$. So

$$f\circ(\ell_0+r)-f+g=r\circ(\ell_0+r)-\delta.$$

If $\delta>0$, then $\delta^{-1}>0$ so δ^{-1} acts as a strictly increasing function on $\tilde{\mathbb{L}}^{>,\succ}$ by Corollary 8.20. We deduce that $\delta\circ(\ell_0+r)-\delta<0$, so $f\circ(\ell_0+r)<f+r$. The case when $\delta<0$ is symmetric. □

For $g\in\tilde{\mathbb{L}}^{>,\succ}$, we write $\mathcal{C}(g):=\{h\in\tilde{\mathbb{L}}^{>,\succ}:h\circ g=g\circ h\}$. The class $\mathcal{C}(g)$ is a subgroup of $(\tilde{\mathbb{L}}^{>,\succ},\circ)$ which contains g.

Corollary 9.20. *We have $\mathcal{C}(\ell_0+1)=\ell_0+\mathbb{R}$.*

Proof. We have $\mathcal{C}(\ell_0+1)\subseteq\ell_0+\mathbb{R}$ by Lemmas 9.18 and 9.19, whereas the converse inclusion $\mathcal{C}(\ell_0+1)\supseteq\ell_0+\mathbb{R}$ is immediate. □

Let $h\in\tilde{\mathbb{L}}^{>,\succ}$ with $h>\ell_0$ and let $U,V\in\tilde{\mathbb{L}}^{>,\succ}$ with

$$\begin{aligned} V\circ h &= V+1 \quad \text{and}\\ U\circ h &= U+1.\end{aligned}$$

The existence of V and U follows from Theorem 9.16. We have $V\circ h\circ V^{\mathrm{inv}}=\ell_0+1=U\circ h\circ U^{\mathrm{inv}}$, so $(U\circ V^{\mathrm{inv}})\circ(\ell_0+1)=(\ell_0+1)\circ(U\circ V^{\mathrm{inv}})$. Thus by Corollary 9.20, there exists an $s\in\mathbb{R}$ with

$$V=U+s. \tag{9.3}$$

It follows that for all $r \in \mathbb{R}$, the series $V^{\mathrm{inv}} \circ (V + r)$ does not depend on the choice of V. We write

$$h^{[r]} := V^{\mathrm{inv}} \circ (V + r) \quad \text{for any } V \in \tilde{\mathbb{L}}^{>,\succ} \text{ with } V \circ h = V + 1.$$

We also write $(h^{\mathrm{inv}})^{[r]} := h^{[-r]}$ and $\ell_0^{[r]} = \ell_0$ for all $r \in \mathbb{R}$, so the operation

$$^{[]} \colon \mathbb{R} \times \tilde{\mathbb{L}}^{>,\succ} \longrightarrow \tilde{\mathbb{L}}^{>,\succ};\ (r, h) \mapsto h^{[r]}$$

is well-defined. This can be interpreted as a law of ordered $\mathbb{R}$-vector space on the non-commutative linearly ordered group $(\tilde{\mathbb{L}}^{>,\succ}, \circ, <)$.

Remark 9.21. Another consequence of (9.3) is that $V - U \in \mathbb{R}$. So $V' = U'$, and there is a unique finitely nested hyperseries denoted $h^{[-\omega]}$ with constant term 0 and $h^{[-\omega]} \circ h = h^{[-\omega]} + 1$. For instance $(\ell_0^2)^{[-\omega]} = \frac{\ell_2}{\log 2}$ and $(\mathrm{e}_{\omega^\mu}^{\ell_0})^{[-\omega]} = \ell_{\omega^{\mu+1}}$ for all $\mu \in \mathbf{On}$.

Proposition 9.22. *For all $h \in \tilde{\mathbb{L}}^{>,\succ} \setminus \{\ell_0\}$, the function $r \mapsto h^{[r]}$ is an isomorphism of ordered groups $(\mathbb{R}, +, <) \longrightarrow (\mathcal{C}(h), \circ, <)$ with $h^{[1]} = h$.*

Proof. Fix a V with $V \circ h = V + 1$. So $h = V^{\mathrm{inv}} \circ (V + 1) = h^{[1]}$ by definition. For all $\varphi \in \tilde{\mathbb{L}}^{>,\succ}$ we have

$$\begin{aligned}
\varphi \circ (\ell_0 + 1) = (\ell_0 + 1) \circ \varphi &\iff V^{\mathrm{inv}} \circ \varphi \circ (V + 1) = V^{\mathrm{inv}} \circ (\ell_0 + 1) \circ \varphi \circ V \\
&\iff (V^{\mathrm{inv}} \circ \varphi \circ V) \circ V^{\mathrm{inv}} \circ (V + 1) = V^{\mathrm{inv}} \circ (V + 1) \circ (V^{\mathrm{inv}} \circ \varphi \circ V) \\
&\iff (V^{\mathrm{inv}} \circ \varphi \circ V) \circ h = h \circ (V^{\mathrm{inv}} \circ \varphi \circ V).
\end{aligned}$$

Therefore $\mathcal{C}(h) = V^{\mathrm{inv}} \circ \mathcal{C}(\ell_0 + 1) \circ V = V^{\mathrm{inv}} \circ (\ell_0 + \mathbb{R}) \circ V = \{h^{[r]} : r \in \mathbb{R}\}$. For all $r, s \in \mathbb{R}$, we have $(\ell_0 + r) \circ (\ell_0 + s) = \ell_0 + r + s$ so $h^{[r+s]} = h^{[r]} \circ h^{[s]}$. Furthermore, we have $r < s \Longrightarrow V + r < V + s \Longrightarrow h^{[r]} < h^{[s]}$ by Corollary 8.20. So $h \mapsto h^{[r]}$ is an isomorphism of ordered groups. □

For each $n \in \mathbb{N}$, the series $h^{[n]}$ is the n-fold compositional iterate of h, whereas $h^{[-n]}$ is the n-fold compositional iterate of h^{inv}. Thus for $q \in \mathbb{Q}$, the series $h^{[q]}$ is a fractional iterate of h. For instance, we have a solution $h^{[1/2]}$ to the formal Schröder equation in y:

$$y \circ y = h. \tag{9.4}$$

Proposition 9.23 below shows that $h^{[1/2]}$ is in fact the unique solution of (9.4) in $\tilde{\mathbb{L}}^{>,\succ}$. As for the values of $h^{[r]}$ for $r \in \mathbb{R} \setminus \mathbb{Q}$, one can consider them as "real iterates" of h. The existence and properties of real iterates of so-called grid-based transseries of exponentiality 0 were studied in detail by Edgar [21]. Our results in the case of hyperseries are similar to his.

Proposition 9.23. *For $h \in \tilde{\mathbb{L}}^{>,\succ}$ and $r, s \in \mathbb{R}$. We have $(h^{[r]})^{[s]} = h^{[rs]}$.*

Proof. We treat the case when $h > \ell_0$ and $r, s > 0$. The other cases follow from the identities $\varphi^{[-t]} = (\varphi^{\mathrm{inv}})^{[t]} = (\varphi^{[t]})^{\mathrm{inv}}$ for all $\varphi \in \tilde{\mathbb{L}}^{>,\succ}$ and $t \in \mathbb{R}$. Let $V \in \tilde{\mathbb{L}}^{>,\succ}$ with $V \circ h = V + 1$ and set $U := (r^{-1}\ell_0) \circ V$. We have $(r^{-1}\ell_0) \circ (\ell_0 + r) = (\ell_0 + 1) \circ (r^{-1}\ell_0)$, so

$$\begin{aligned}
U \circ h^{[r]} &= (r^{-1}\ell_0) \circ V \circ V^{\mathrm{inv}} \circ (\ell_0 + r) \circ V \\
&= (r^{-1}\ell_0) \circ (\ell_0 + r) \circ V \\
&= U + 1.
\end{aligned}$$

So $(h^{[r]})^{[s]} = U^{\mathrm{inv}} \circ (U + s) = V^{\mathrm{inv}} \circ (r\ell_0) \circ (\ell_0 + s) \circ (r^{-1}\ell_0) \circ V = V^{\mathrm{inv}} \circ (\ell_0 + rs) \circ V = h^{[rs]}$. □

Proposition 9.24. *For all $h \in \tilde{\mathbb{L}}^{>,\succ}$, the function $r \mapsto h^{[r]}$ is the unique non-decreasing group morphism $(\mathbb{R}, +) \longrightarrow (\mathcal{C}(h), \circ)$ with $h^{[1]} = h$.*

Proof. By Proposition 9.22, it is enough to prove the unicity. Let $\Psi : \mathbb{R} \longrightarrow \mathcal{C}(h)$ be a non-decreasing morphism of ordered groups with $\Psi(1) = h$. Let $q = \frac{k}{n} \in \mathbb{Q}$ where $n \in \mathbb{N}^{>}$ and $k \in \mathbb{Z}$. We have $\Psi(k/n)^{[n]} = h^{[k]}$, whence $\Psi(k/n) = h^{[k/n]}$ by Proposition 9.23. We deduce that Ψ and $r \mapsto h^{[r]}$ coincide on $\mathbb{Q}$. Since Ψ is non-decreasing and $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, it follows that $\Psi(r) = h^{[r]}$ for all $r \in \mathbb{R}$. □

We now solve (9.2).

Proposition 9.25. *For $f, g > \ell_0$, the solution to the inequation $f \circ g \geqslant g \circ f$ is as follows:*

a) If $f > \mathcal{C}(g)$, then $f \circ g > g \circ f$.

b) If $f < \mathcal{C}(g)$, then $f \circ g < g \circ f$.

c) If $f \not\succ \mathcal{C}(g)$ and $f \not\prec \mathcal{C}(g)$, then for $g_f := \sup\{h \in \mathcal{C}(g) : h \leqslant f\}$, we have

$$\begin{aligned} f \circ g &< g \circ f \quad \text{if } f > g_f \\ f \circ g &> g \circ f \quad \text{if } f < g_f, \text{ and} \\ f \circ g &= g \circ f \quad \text{if } f = g_f. \end{aligned}$$

Proof. Let $V \in \tilde{\mathbb{L}}^{>,\succ}$ with $V \circ g = V + 1$ and write $h := V \circ f \circ V^{\mathrm{inv}}$. Note that

$$f \circ g \geqslant g \circ f \Longleftrightarrow h \circ (\ell_0 + 1) \geqslant h + 1.$$

So we may assume that $g = \ell_0 + 1$ and $f = h$. Then the statements *a)* and *b)* follow from Lemma 9.18 and Proposition 9.22. As for *c)*, note that the number r_f in Lemma 9.19 is the supremum of $\{r \in \mathbb{R} : \ell_0 + r \leqslant f\}$. So *c)* follows from Lemma 9.19 and Proposition 9.22. □

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Glossary

$\mathbf{On}$	class of ordinals	5
$\boldsymbol{\nu} \leqslant \mathbf{On}$	$\boldsymbol{\nu} \in \mathbf{On}$ or $\boldsymbol{\nu} = \mathbf{On}$	5
$\mathcal{M}^{>}$	class of strictly positive elements in $(\mathcal{M}, <)$	5
$M^{\neq}$	class of non-zero elements in M	5
$\mathbb{R}[[\mathfrak{M}]]$	field of well-based series with real coefficients over $\mathfrak{M}$	5
$\operatorname{supp} f$	support of a series	5
$\mathfrak{d}_s$	dominant monomial $\max \operatorname{supp} s$ of s	5
τ_s	dominant term $\mathfrak{d}_s\, s_{\mathfrak{d}_s}$ of s	5
$f_{\succ \mathfrak{m}}$	truncation $\sum_{\mathfrak{n} \succ \mathfrak{m}} f_{\mathfrak{n}}\, \mathfrak{n}$ of f	5
$f_{\succ}$	$f_{\succ 1}$	5
$f \trianglelefteq g$	$\operatorname{supp} f \prec g - f$	5
$f \prec g$	$\mathbb{R}^{>}\, \lvert f \rvert < \lvert g \rvert$	6
$s \preccurlyeq t$	$\exists r \in \mathbb{R}^{>}, \lvert s \rvert < r \lvert t \rvert$	6
$s \asymp t$	$s \preccurlyeq t$ and $t \preccurlyeq s$	6
$\mathbb{S}^{\preccurlyeq}$	class of series s with $s \preccurlyeq 1$	6
$\mathbb{S}_{\succ}$	class of series $s \in \mathbb{S}$ with $\operatorname{supp} s \succ 1$	6
$\mathbb{S}^{\prec}$	class of series $s \in \mathbb{S}$ with $s \prec 1$	6
$\mathbb{S}^{\geqslant, \succ}$	class of series $s \in \mathbb{S}$ with $s \geqslant 0$ and $s \succ 1$	6
$\mathfrak{S}^{\infty}$	the class $\bigcup_{n \in \mathbb{N}} \mathfrak{S}^n$	6
$P \circ Q$	composite power series	8
$\operatorname{Conv}(P)$	class of series ε where $P(\varepsilon)$ is defined	8
$\operatorname{Conv}(f)_s$	class of series t with $f(t) = \tilde{f}_s(t - s)$	12
s^+	$\max(s, s^{-1})$	14
s^-	$\min(s, s^{-1})$	14
$s \precdot\!\!\prec t$	$(s^+)^n < t^+$ for all $n \in \mathbb{N}^{>}$	14
$s \asymp\!\!\prec t$	$t^+ < (s^+)^m < (t^+)^n$ for some $m, n \in \mathbb{N}^{>}$	14