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► **To cite this version:**

Yang Hao. Financial Market with Learning from Price under Knightian Uncertainty. 2023. hal-03686748v2

HAL Id: hal-03686748

<https://hal.science/hal-03686748v2>

Preprint submitted on 12 Aug 2023

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LEARNING FROM PRICE WITH KNIGHTIAN UNCERTAINTY

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August 12, 2023

Following the framework proposed by Vives (2014), we reconsidered the problem of learning from equilibrium prices under Knightian uncertainty or ambiguity. Specifically, we examine the situation in which uninformed traders face Knightian uncertainty regarding the number of informed traders, causing them to infer pessimistic market information from prices. With Knightian uncertainty, equilibrium price acts as a signaling device, informing uninformed traders whether the market is crowded with informed traders. It exhibits a two-regime characteristic, in which uninformed traders endogenously believe there are more (less) informed traders trading in the market when observing usual (unusual) price. Consequently, Knightian uncertainty alone can lead to price overreaction or under-reaction, resulting in higher or lower price. In addition, we find that Knightian uncertainty can partially explain demand (in)elasticity, highlighting the role of ambiguity as a micro-foundation for explaining demand (in)elasticity, as discussed previously by Gabaix and Koijen (2021). Finally, we show that reducing ambiguity may not always increase the expected trading profits of uninformed traders, as the effect of ambiguity levels can be nonlinear and nonmonotonic.

JEL: D81; D83; G12;

Yang is with the Swiss Finance Institute (SFI). For helpful discussions and comments, I thank Giovanni Barone Adesi, Julian Cujean, Jérôme Detemple, Semyon Malamud, Antonio Mele, Mohammad Pourmohammadi, Xiaobo Yu. All errors are the author's alone.

"Financial market is the balance of fear and greed."

1. Introduction

Financial technology innovation is blamed for volatility, illiquidity, and inefficiency (see Ben-David, Franzoni, and Moussawi (2018), Zhang (2006)). The advancement of information technology has led to the emergence of new trading platforms, which has caused a substantial rise in market fragmentation within asset markets. This fragmentation refers to the scattering of trading activities across various venues. Decentralized exchanges (DEXs) have emerged as a recent example of trading platforms that operate without a central authority and are built on blockchain smart contracts. Prior to their introduction, nearly all transactions were conducted on centralized exchanges (CEXs) using centralized limit order books (LOBs). The coexistence of the CEX and DEX since 2020 has led to market fragmentation and the possibility of arbitrageurs trading on either or both platforms. This poses a natural concern for uninformed traders on the CEX who are uncertain about the number of informed traders trading on the platform. The presence of difficult-to-identify informed traders who buy or sell large quantities of assets to influence prices and generate short-term profits unrelated to the real economy further exacerbates this concern. As such, the question arises for uninformed traders on CEXs whether they should fear this financial innovation.

In finance literature, most asset pricing and market microstructure models assume that all market participants have full knowledge of the presence and actions of all other traders. However, in reality, it is often difficult to accurately observe the number of traders in real-time, and traders must make educated guesses about the level of participation. The experimental study conducted by Schnitzlein (2002) suggests that when the number of market participants is unknown, the resulting equilibrium outcomes may not align with theoretical predictions. This highlights the importance of considering the impact of uncertainty in the number of market participants in market microstructure models.

The importance of the number of market participants is highlighted by Hayek (1945) , who argued that financial markets are an efficient way to aggregate dispersed information. When both informed and uninformed traders participate in trading for a risky asset, only informed traders possess private information about the future payoff of the asset. The equilibrium price of the risky asset should aggregate all relevant information. Thus, uninformed traders should be able to infer relevant information from the price if they have a thorough understanding of the composition of market participants and the equilibrium forces that generate the observed price. However, most of the current literature assumes that traders are perfectly rational and have complete knowledge about the presence and exact number of all competing traders.

In this paper, we investigate the impact of ambiguity about market participants on a competitive market, utilizing the rational expectations equilibrium model proposed by Vives (2014). By incorporating the possibility of uninformed traders holding incorrect beliefs about the number of market participants, particularly the number of informed traders, we aim to explore the consequences of this uncertainty on the perspective of uninformed traders.

In financial markets, uninformed traders may not have complete knowledge of the number of informed traders, leading to Knightian uncertainty. Consequently, when observing the price, uninformed traders may interpret it pessimistically and incorrectly assume a correlation between the price and the private information of informed traders. As a result, the information that uninformed traders extract from prices is based on a misspecified model of the world, leading to misinference. This, in turn, can result in a deviation of the equilibrium price from the benchmark case where there is no ambiguity regarding the market participants.

Section 2 presents the general model setups and assumptions. Section 3 begins with a benchmark model similar to Vives (2014): Rational expectation equilibrium without ambiguity. I model traders with constant absolute risk aversion (CARA) utility over terminal wealth who solves a portfolio choice problem involving a risky and a riskless asset. M units of traders receive news about the future fundamental value of the asset and are thus informed. N unit of traders is uninformed but can infer information from prices. Since uninformed traders use the correct mapping to infer information from prices, they are able to recover the correct information, which is then aggregated correctly into REE prices (see Grossman (1976), Grossman and Stiglitz (1980)).

In section 4 , we solve for a rational expectation equilibrium with ambiguity. All of the assumptions of Section 4 are identical to those of Section 3, with the addition that we assume each trader is ex ante uncertain about the total number (size) of informed traders M and exhibits preference in the form of the maxmin expected utility (MEU) model of Gilboa and Schmeidler (1989). Due to a lack of private information and a preference for ambiguity aversion , all uninformed traders in this scenario extract information from prices using a misspecified model of the world. Informed traders, on the other hand, resolve their ambiguity immediately after observing the private information.

In our model, the equilibrium price plays a crucial threefold role. First, price has an informational role, which results in strategic complementarities: higher prices reflect better fundamentals, causing uninformed traders to increase their asset demand, which increases prices further, thereby fueling a *information effect*. Second, price serves as a measure of scarcity, which introduces strategic substitutability: higher prices make the asset more expensive, causing traders

to demand less of it, thereby exerting downward pressure on prices. In the existing literature, these two roles are common. In this paper, price plays a role that is novel in the literature. Price acts as a signaling device to tell uninformed traders whether or not the market is crowded with informed traders. When uninformed traders observe an abnormal price (either too high or too low), they conclude endogenously that there are very few informed traders trading in the market. And vice versa, when uninformed traders observe a normal market price (one that is neither too high nor too low), they conclude endogenously that there are a large number of informed traders existing. The interaction of these three forces determines the overall strength of the *information effect*, which in turn influences the properties of equilibrium outcomes: the stronger the *information effect*, the greater the deviations from benchmark REE outcomes with no ambiguity. If the *information effect* is sufficiently strong, it can result in arbitrarily large price fluctuations or deviations.

In Section 5, we examine the properties and implications of equilibrium outcomes under ambiguity, as discussed in Section 4. Specifically, I compare the properties of the price equilibrium under ambiguity with the benchmark result in section 3, and we discuss the relevance of the results to empirical patterns in asset pricing. The most significant result is that ambiguity regarding the number of informed traders alone¹ causes the equilibrium to exhibit a *two-regime* characteristic: it provides a micro-foundation for price over-reactions or under-reactions to news. When prices are abnormally high or low, they have a tendency to overreact to news. This is due to the fact that such extreme prices inform uninformed traders that there are few informed traders. Thus, the uninformed traders interpret the price as if there are few informed traders existing, and this wrong belief and misinterpretation weaken the aforementioned *information effect*, resulting in a price overreaction in equilibrium. Likewise, when prices are normal, neither too high nor too low, they tend to underreact to news. This is due to the fact that such a normal price informs uninformed traders that there are a large number of informed traders trading the market. Thus, uninformed traders interpret the price as if there are a large number of informed traders existing, and this wrong belief and misinterpretation strengthen the *information effect* mentioned previously, resulting in a price under-reaction in equilibrium. In this context, ambiguity regarding market participants also refers to excessive volatility, excessive trading volume, a higher or lower price, demand (in)elasticity, etc. Section 5.7 examines whether the (competitive) REE we derived from proposition 3 is an implementable REE, which is the result of a well-specified game according to Vives (2014).

In section 6, I analyze an alternative circumstance in which traders are uncertain about the size

¹Gertsman (2021) highlights that the behavioral bias "selective learning" can explain both the over-reaction and under-reaction of prices to new information. This behavioral bias can also produce a two-regime outcome similar to that of our paper. In this paper, however, the two-regime result is driven by knightian uncertainty and not "selective learning".

(number) of uninformed traders N . It is intriguing to note that the equilibrium price does not exhibit a two-regime characteristic under these conditions. Uncertainty regarding the number (size) of uninformed traders N always results in a price overreaction and makes the demand function of uninformed traders less elastic. In this context, ambiguity-averse uninformed traders always hold the belief that the market is crowded with many uninformed traders. In section 7, we allow for two-dimensional ambiguity for uninformed traders and demonstrate that, in the context of two-dimensional ambiguity, it is obvious that the equilibrium price combines the characteristics described in sections 4 and 6. All proofs are in the appendix, unless stated otherwise.

1.1. Related Literature

This paper contributes to the literature which study the effect of uncertainty about some market condition parameters on equilibrium outcomes using the rational expectations model. Easley and O'Hara (2009) demonstrates that uninformed trader ambiguity regarding the informed trader's risk-aversion coefficient can lead to non-participation of uninformed traders, and that regulation, particularly regulation of unlikely events, can mitigate the effects of ambiguity, thereby increasing participation and generating welfare gains.

Mele and Sangiorgi (2015) demonstrates that uncertainty about the fundamental value mean leads to strategic complementarities in information acquisition and multiple equilibria. In addition, they demonstrate that swift changes in the demand for information can cause large price swings even after small changes in Knightian uncertainty. Hirshleifer, Huang, and Teoh (2017) demonstrates that in a financial market with ambiguity aversion, investors with ambiguity aversion do not invest in the Value-Weighted Market Portfolio (VWMP), which hinders diversification and risk sharing.

The model that most closely resembles ours is Aliyev and He (2022) (AH henceforth), in which traders are ambiguous regarding the *proportion* of informed traders. AH paper use the alpha-maxmin model to demonstrate that ambiguity generates complementarities in information acquisition and multiple equilibria, and that the ambiguity premium can be positive (respectively negative) if traders are sufficiently (respectively insufficiently) ambiguity averse, leading to both undervaluation and overvaluation in the market. We do not model the information acquisition stage, in contrast to AH's emphasis on information and ambiguity trade-offs. Instead, we consider information to be exogenous and disregard the decision to acquire information.

The focus of this paper is on how ambiguity affects the equilibrium price and related properties.

According to Vives (2014), we employ a unique baseline modeling framework² and different assumptions. In our paper, price is not perfect revealing for uninformed traders. This paper's model demonstrates a two-regime outcome, which is a novel characteristic. We show that ambiguity as a driving factor in trading decisions results in a two-regime outcome. In particular, uninformed traders may believe that too many or too few informed traders are actively participating in the market, resulting in both an price under-reaction and an price over-reaction regime.

In addition, this paper distinguishes between "uncertainty about the number of uninformed traders" and "uncertainty about the number of informed traders." We demonstrate how these two distinct assumptions can significantly influence the equilibrium price and its associated properties in distinct ways. In the context of two-dimensional ambiguity, we also investigate the interaction between these two types of assumptions.

Another closely related paper is Banerjee and Green (2015) (BG henthforce). The BG paper reveals that in cases where some traders are unsure whether informed traders are trading on informative signals or just random noise, this uncertainty results in a nonlinear price response that shows asymmetrical reactions to news.

2. Model

In this section I develop a Rational Expectation Equilibrium (REE) model where traders are ambiguous about market participants .

2.1. Setup of the Model

Our model is static. I consider a similar setup à la Vives (2014), after information acquisition costs have been incurred. I consider a market for a single risky asset which is in fixed supply Z . The economy is populated by two types of price-taking traders, the first one are M units of (identical) informed traders (denoted as I), and the second one are N units of (identical) uninformed traders (denoted as U). Both types of traders' initial wealth are normalized to zero for convenience. The

²The existing literature offers two primary avenues for examining the impact of ambiguity within the Rational Expectation Equilibrium (REE) framework. The first approach involves adopting the noisy REE framework, as demonstrated by Grossman and Stiglitz (1980), which introduces the concept of a noisy asset supply or noise trader, leading to a market that is not entirely revealing. An example of a model built upon this framework is presented in Mele and Sangiorgi (2015). Alternatively, the second approach entails utilizing the standard REE framework, where the equilibrium price is fully revealing. Aliyev and He (2022) exemplify this approach by constructing a model within this framework. In this paper, we propose an innovative approach inspired by Vives (2014), which obviates the need for assuming a noisy asset supply or noise trader. Instead, we assume that informed and uninformed traders have heterogeneous but interdependent valuations. Within this framework, the price still does not perfectly reveal information to uninformed traders. To the best of our knowledge, this represents the first instance of a paper founded upon the framework introduced by Vives (2014). We have obtained several novel results using this framework, distinguishing our findings from those in papers based on alternative frameworks.

fundamental distinction between the informed traders and the uninformed traders is that only the informed traders receive a common noisy signal. Traders are assumed to be risk neutral. The profits of trader $i \in \{\mathbf{I}, \mathbf{U}\}$ when the price is p are

$$(1) \quad \pi_i = (f_i - p) x_i - \frac{k}{2} x_i^2, \quad i \in \{\mathbf{I}, \mathbf{U}\},$$

where x_i is the individual quantity demanded by trader i , f_i is the (fundamental) value idiosyncratic to the trader and kx_i is a marginal transaction, opportunity or limit to arbitrage cost (it could also be interpreted as a proxy for risk aversion³).

I assume f_i has the following structure,

$$(2) \quad f_{\mathbf{I}} = \bar{f} + \theta_{\mathbf{I}} + \varepsilon,$$

$$(3) \quad f_{\mathbf{U}} = \bar{f} + \theta_{\mathbf{U}} + \varepsilon,$$

where \bar{f} is a positive constant, $\theta_{\mathbf{I}}$ and $\theta_{\mathbf{U}}$ are the idiosyncratic components of the traders' valuation. ε is the common shock, $\varepsilon \sim N(0, \tau_\varepsilon^{-1})$, and is independent of $\theta_{\mathbf{I}}$ and $\theta_{\mathbf{U}}$. I further assume $\theta_{\mathbf{I}} \sim N(0, \tau^{-1})$, $\theta_{\mathbf{U}} \sim N(0, \tau^{-1})$ and $\theta_{\mathbf{I}}, \theta_{\mathbf{U}}$ are correlated with correlation coefficient $\rho \in [0, 1]$. We therefore have $\text{cov}(\theta_{\mathbf{I}}, \theta_{\mathbf{U}}) = \rho\tau^{-1}$.

Our information structure encompasses the case of a common value and also that of private value. If $\rho = 1$, the valuation parameters $f_{\mathbf{I}}$ and $f_{\mathbf{U}}$ are perfectly correlated and we are in a common value model. When $0 < \rho < 1$, we are in a private value model. If $\rho = 0$, then the parameters are independent and we are in an independent values model. In this paper, our main focus is on the nontrivial case: $0 < \rho < 1$.

At trading stage, each informed trader receives a common noisy signal $s = \theta_{\mathbf{I}} + u$, where $u \sim N(0, \tau_u^{-1})$. The precision of the signal τ_u is exogenously given in this model. The informativeness of the signal is captured by the signal-to-noise ratio $\lambda \equiv \frac{\text{cov}(s, \theta_{\mathbf{I}})}{\text{var}(s)} = \frac{\tau^{-1}}{\tau^{-1} + \tau_u^{-1}}$. When forming their expectations about the valuation f_i , informed traders use all the information available to them. The information set of informed traders at trading stage is $\mathcal{F}_{\mathbf{I}} = \{s, p\}$, where p is the equilibrium price at trading stage. Uninformed trader receive no signals about the asset payoff. Hence, the uninformed traders' information set at trading stage is $\mathcal{F}_{\mathbf{U}} = \{p\}$.

2.2. Ambiguity and Ambiguity Aversion

Our point of departure from the previous literature is the assumption that both informed traders and uninformed traders are ex ante uncertain about the total number (size) of the market

³We may directly assume that investors have CARA utility function. This alternative assumption complicates the analysis without producing any new economic implications. It is reassuring that this more elaborate setting gives similar results as in our current setting.

participants. Throughout the whole paper, we assume that agents display preferences in the form of the maxmin expected utility (MEU) model of Gilboa and Schmeidler (1989). Specifically, we analyze the following case: (case $\mathcal{A} - \mathcal{M}$) both informed traders and uninformed traders are only ambiguous about the total number (size) of the informed traders M . They are unable to assess what M is, but they believe it belongs to some interval, $M \in [M_1, M_2]$, with $M_1 < M < M_2$. We further assume that $M_1 = \mathbf{M} - \Delta M$ and $M_2 = \mathbf{M} + \Delta M$. I use the boldface of \mathbf{M} to denote the true value of M . ΔM is an exogenous parameter that determines the ambiguity. (But here, the total number (size) of the uninformed traders N is known by all of the uninformed traders and this is common knowledge.)

To summarize, the tuple

$$\mathcal{E} = (\bar{f}, \rho, \tau, \tau_\varepsilon, \mathbf{M}, N)$$

defines an economy. We are interested in the implications for asset prices, demand functions, price reactions, volatility, trading volume, etc.

2.3. Solution Concept

Our solution concept is (competitive) Rational Expectation Equilibrium (REE), where each trader optimizes while taking prices as given, as in the usual competitive equilibrium, but infers from prices the relevant information. An REE is price and demand functions that satisfy the optimality (utility maximization) and the market clearing conditions.

DEFINITION 1. *Competitive REE without Ambiguity*

Given the number of informed traders M and the number of uninformed traders N , an REE is a set of functions (p, x_I, x_U) such that:

- (a) (Optimization) The informed demand x_I and the uninformed demand x_U maximize the expected profits of the informed and uninformed traders respectively in the market;
- (b) (Market-Clearing) The price of the risky stock p equates the supply and demand, $M \cdot x_I + N \cdot x_U = Z$.

DEFINITION 2. *Competitive REE with Ambiguity (Case $\mathcal{A} - \mathcal{M}$)*

Given the number of informed traders M , the number of uninformed traders N , traders' ambiguity about M or N , an REE is a set of functions (p, x_I, x_U) such that: an REE is a set of functions (p, x_I, x_U) such that:

- (a) (Optimization) The informed demands of the risky asset x_I maximize the expected profits of informed traders and the uninformed demands x_U maximize the minimum expected profits of the uninformed traders in the market;
- (b) (Market-Clearing) The price of the risky asset p equates the supply and demand of the risky asset, $M \cdot x_I + N \cdot x_U = Z$.

3. Financial Market Equilibrium With Certainty About Market Participants

In this section, I solve for the financial market equilibrium where traders do not suffer from the ambiguity and know the true number of each type of market participants for sure. Specifically, both informed traders and uninformed traders know the total number (size) of the informed traders M and the total number (size) of the uninformed traders N for sure, without ambiguity. The calculation details and proofs are in Appendix A.

PROPOSITION 1. *Equilibrium Price (without Ambiguity)*

When both the number of the informed trader M and the number of the uninformed trader N are known without the ambiguity, there exists a Rational Expectation Equilibrium (REE) in which the price function p is a function of s ,

$$(4) \quad p(s) = \bar{f} + \left(\frac{M + N\rho}{M + N} \right) \lambda \cdot s - \frac{kZ}{M + N}$$

PROOF. All proofs are in Appendix, unless stated otherwise. □

PROPOSITION 2. *Equilibrium Demand Function of Traders (without Ambiguity)*

When both the number of the informed trader M and the number of the uninformed trader N are known without the ambiguity, the equilibrium demand function of informed traders and uninformed traders are characterized as:

$$(5) \quad x_I(s, p) = \frac{\bar{f} + \lambda \cdot s - p}{k}$$

$$x_U(p) = \frac{1}{k} \left[\frac{M(1-\rho)\bar{f} + \rho kZ}{M + N\rho} - \frac{M(1-\rho)}{M + N\rho} \cdot p \right]$$

4. Financial Market Equilibrium with Knightian Uncertainty (or Ambiguity) About Informed traders

In this section, I characterize the financial market equilibrium where traders are only ambiguous about the total number (size) of informed traders M . To find the equilibrium, we need to first characterize the demand function of informed and uninformed traders who exhibit ambiguity aversion with maxmin utility function.

4.1. Demand Function of Informed Traders

By observing the realization of s , informed traders resolve their ambiguity straight away. They choose portfolio holdings x_I to maximize the expected profits π_I

$$(6) \quad \max_{x_I} E \left[(f_I - p) x_I - \frac{k}{2} x_I^2 \mid \mathcal{F}_I = \{s, p\} \right],$$

where p is the observed asset price. Standard arguments yield

$$(7) \quad \begin{aligned} x_I(s, p) &= \frac{E(f_I \mid s, p) - p}{k} \\ &= \frac{\bar{f} + \lambda s - p}{k}, \end{aligned}$$

and the informativeness of the signal is captured by the signal-to-noise ratio:

$$(8) \quad \lambda \equiv \frac{\text{cov}(s, \theta_I)}{\text{var}(s)} = \frac{\tau^{-1}}{\tau^{-1} + \tau_u^{-1}}$$

4.2. Demand Function of Uninformed Traders

At trading stage, for any given M and N , uninformed traders rationally conjecture that the price function is:

$$(9) \quad p = \bar{f} + A(M; N) \cdot s - H(M; N)$$

where the function $A(M; N)$ and $H(M; N)$ will be endogenously determined in equilibrium. Since N is known by the uninformed traders for sure, N is an exogenous parameter here. For simplicity, we write $A(M) \equiv A(M; N)$ and $H(M) \equiv H(M; N)$ and the conjectured price function can be written as:

$$(10) \quad p = \bar{f} + A(M) \cdot s - H(M)$$

Thus, the optimal demand of uninformed traders is determined by

$$(11) \quad \max_{x_U} \min_{M \in [M_1, M_2]} \left(E_M \left[(f_U - p) x_U - \frac{k}{2} x_U^2 \mid \mathcal{F}_U = \{p\} \right] \right),$$

where x_U is the asset demand of uninformed traders, and $E_M(\cdot)$ is the expectation operator taken under the belief that the size/total number of the informed traders is M . The criterion underlying equation (11) is the maxmin expected utility (MEU) axiomatized by Gilboa and Schmeidler (1989). Since uninformed traders are ambiguous about M , they view the stock price p as an ambiguous signal about f_U . This makes the inference problem of the uninformed

traders more interesting and generates the results we will see in the following sections.⁴ Under the belief that the size/total number of the informed traders is M , they map from the observed price p to the extracted signal $G(M)$:

$$(12) \quad G(M) = \frac{p + H(M) - \bar{f}}{A(M)}$$

The conditional moments of f_U taken under a particular belief M , are given by:

$$(13) \quad \begin{aligned} E_M [f_U | p] &= E_M [\bar{f} + \theta_U + \varepsilon | p] \\ &= \bar{f} + \rho\lambda \left[\frac{p + H(M) - \bar{f}}{A(M)} \right] \\ &= \bar{f} + \rho\lambda \cdot G(M; p, \bar{f}, N) \end{aligned}$$

For simplicity, we write

$$G(M) \equiv G(M; p, \bar{f}, N) = \frac{p + H(M) - \bar{f}}{A(M)}$$

The objective function of an uninformed trader can be written as:

$$(14) \quad \begin{aligned} &\min_{M \in [M_1, M_2]} \left((E_M [f_U | p] - p) \cdot x_U - \frac{k}{2} x_U^2 \right) \\ \Rightarrow &\min_{M \in [M_1, M_2]} \left((\bar{f} + \rho\lambda \cdot G(M) - p) \cdot x_U - \frac{k}{2} x_U^2 \right) \\ &= \begin{cases} -\frac{k}{2} x_U^2 + [\bar{f} + \rho\lambda \cdot \{G(M)\}_{\min} - p] \cdot x_U, & \text{if } x_U > 0 \\ 0, & \text{if } x_U = 0 \\ -\frac{k}{2} x_U^2 + [\bar{f} + \rho\lambda \cdot \{G(M)\}_{\max} - p] \cdot x_U, & \text{if } x_U < 0 \end{cases} \end{aligned}$$

where $\{G(M)\}_{\min} = \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\min}$ and $\{G(M)\}_{\max} = \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\max}$ are the minimum and maximum of the function $\{G(M)\}$, given the value of p , respectively.

Thus an uninformed trader's demand function is:

$$(15) \quad x_U(p) = \begin{cases} \frac{\bar{f} + \rho\lambda \cdot \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\min} - p}{k}, & \text{if } \bar{f} + \rho\lambda \cdot \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\min} - p > 0 \\ 0, & \text{if } \bar{f} + \rho\lambda \cdot \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\min} - p \leq 0 \leq \bar{f} + \rho\lambda \cdot \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\max} - p \\ \frac{\bar{f} + \rho\lambda \cdot \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\max} - p}{k}, & \text{if } \bar{f} + \rho\lambda \cdot \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\max} - p < 0 \end{cases}$$

⁴While formulating portfolio decisions, uninformed traders learn from the price and update each of their beliefs. This rule is known as full Bayesian updating. With full Bayesian updating, no learning occurs regarding the original set of priors, which means that the uninformed agents retain all their initial priors.

4.3. Equilibrium

PROPOSITION 3. *Equilibrium Price (with Ambiguity about the Number of the Informed Traders M)*
When the number of the uninformed trader N is known and the number of the informed trader M is unknown with the ambiguity, there exists a competitive Rational Expectation Equilibrium (REE) in which the price function p is piecewise in s and M ,

$$(16) \quad p(s, M) = \begin{cases} \bar{f} + \left[\frac{M(M_1+N\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} \right] \lambda \cdot s - \left[\frac{M_1kZ}{M(M_1+N\rho)+M_1N(1-\rho)} \right], & \text{for } s \in (-\infty, s_1) \cup (s_3, +\infty) \\ \bar{f} + \left[\frac{M(M_2+N\rho)}{M(M_2+N\rho)+M_2N(1-\rho)} \right] \lambda \cdot s - \left[\frac{M_2kZ}{M(M_2+N\rho)+M_2N(1-\rho)} \right], & \text{for } s \in [s_1, s_2) \\ \bar{f} + \lambda s - \frac{kZ}{M}, & \text{for } s \in [s_2, s_3] \end{cases}$$

with $s_1 < s_2 < s_3$,

$$(17) \quad \begin{aligned} s_1 &= \frac{-kZ}{N\lambda(1-\rho)} \\ s_2 &= \left[\frac{1}{M} + \frac{\rho}{M_2(1-\rho)} \right] \frac{kZ}{\lambda} \\ s_3 &= \left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)} \right] \frac{kZ}{\lambda} \end{aligned}$$

PROPOSITION 4. Uninformed Traders' Belief Choice and Extracted Signal

When the number of the uninformed trader N is known and the number of the informed trader M is unknown with the ambiguity, the extracted signal G that uninformed traders map from the observed price characterized as:

$$(18) \quad G = \begin{cases} \frac{(p-\bar{f})(M_1+N)+kZ}{(M_1+N\rho)\lambda}, & \text{Believe the number of the informed traders is } M_1, \text{ for } p \in (-\infty, p_1) \cup (p_3, +\infty) \\ \frac{(p-\bar{f})(M_2+N)+kZ}{(M_2+N\rho)\lambda}, & \text{Believe the number of the informed traders is } M_2, \text{ for } p \in [p_1, p_2) \\ \text{Not Applicable,} & \text{Not Participate, for } p \in [p_2, p_3] \end{cases}$$

with three constants $p_1 < p_2 < p_3$,

$$(19) \quad \begin{aligned} p_1 &= \bar{f} - \left(\frac{1}{1-\rho} \right) \left(\frac{kZ}{N} \right) \\ p_2 &= \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_2} \right) \\ p_3 &= \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_1} \right) \end{aligned}$$

The "too-good-to-be-true effect" and "too-bad-to-be-true effect" can be generated by ambiguity regarding the number of informed traders M . When uninformed traders observe abnormal price, they tend to conclude that there is very less informed traders trading in the market and injecting information in the price.

PROPOSITION 5. *Equilibrium Demand Function of Traders (with Ambiguity about the Number of the Informed Traders M)*

When the number of the uninformed trader N is known and the number of the informed trader M is unknown with the ambiguity, the equilibrium demand function of informed traders and uninformed traders are characterized as:

$$(20) \quad \begin{aligned} x_I(s, p) &= \frac{\bar{f} + \lambda \cdot s - p}{k} \\ x_U(p) &= \begin{cases} \frac{1}{k} \left[\frac{M_1(1-\rho)\bar{f} + \rho kZ}{M_1 + N\rho} - \frac{M_1(1-\rho)}{M_1 + N\rho} \cdot p \right], & \text{Buy, for } p \in (-\infty, p_1) \\ \frac{1}{k} \left[\frac{M_2(1-\rho)\bar{f} + \rho kZ}{M_2 + N\rho} - \frac{M_2(1-\rho)}{M_2 + N\rho} \cdot p \right], & \text{Buy, for } p \in [p_1, p_2) \\ 0, & \text{Not Participate, for } p \in [p_2, p_3] \\ \frac{1}{k} \left[\frac{M_1(1-\rho)\bar{f} + \rho kZ}{M_1 + N\rho} - \frac{M_1(1-\rho)}{M_1 + N\rho} \cdot p \right], & \text{Sell, for } p \in (p_3, +\infty) \end{cases} \end{aligned}$$

with three constants $p_1 < p_2 < p_3$,

$$(21) \quad \begin{aligned} p_1 &= \bar{f} - \left(\frac{1}{1-\rho} \right) \left(\frac{kZ}{N} \right) \\ p_2 &= \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_2} \right) \\ p_3 &= \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_1} \right) \end{aligned}$$

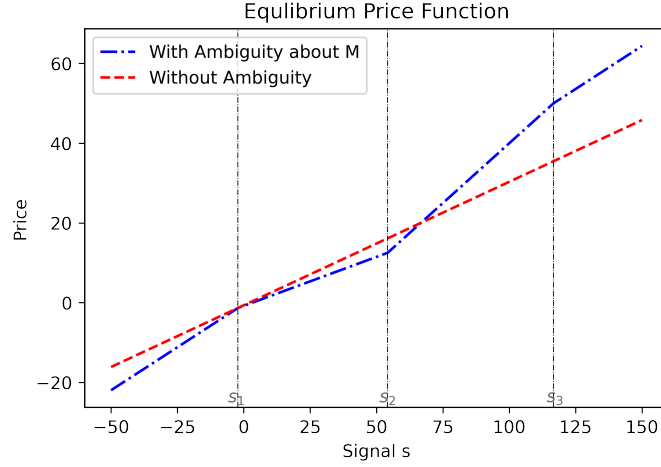


FIGURE 1. This figure depicts the equilibrium asset price given in Proposition 1 (the red line, benchmark case without ambiguity) and Proposition 3 (the blue line, case with ambiguity), as a function of the signal, s . The region delimited by the vertical dashed lines corresponds to the three threshold value s_1 , s_2 and s_3 . The parameter values are $\bar{f} = 0$, $M = 10$, $M_1 = 4$, $M_2 = 16$, $N = 300$, $\rho = 0.5$, $\lambda = 0.6$, $Z = 10$, $k = 20$.

5. Properties of Equilibrium Outcomes

In this section I compare the properties of the price equilibrium with the ambiguity and benchmark result, and we discuss how the outcomes relate to empirical patterns in asset pricing. Specifically, we focus on the role of traders' ambiguity and ambiguity aversion about the market participants on concepts such as: price reaction, volatility, contrarians, trading volume, etc.

5.1. Over-reaction, Under-reaction and Excess Volatility

As shown in equation (15), for uninformed traders, ambiguity about the number of informed traders, and ambiguity aversion preference naturally lead to beliefs that are more extreme than under fully rational inference when learning from the price. Therefore, ambiguity about the market participants provides a micro-foundation to price over-reaction or under-reaction to news, as is clear from comparing equilibrium outcomes from proposition 1 and 3.

PROPOSITION 6. *Over-reaction and Under-reaction to News*

The equilibrium price under case A – M and benchmarks case is denoted as $p(s, M)$ and $p(s)$, respectively. Ambiguity about the number of informed traders M makes the equilibrium price display both over-reaction and under-reaction to news,

$$(22) \quad \begin{cases} \frac{\partial p(s, M)}{\partial s} \geq \frac{\partial p(s)}{\partial s}, & (\text{Over-reaction Region}), \text{ for } s \in (-\infty, s_1) \cup [s_2, +\infty) \\ \frac{\partial p(s, M)}{\partial s} \leq \frac{\partial p(s)}{\partial s}, & (\text{Under-reaction Region}), \text{ for } s \in [s_1, s_2) \end{cases}$$

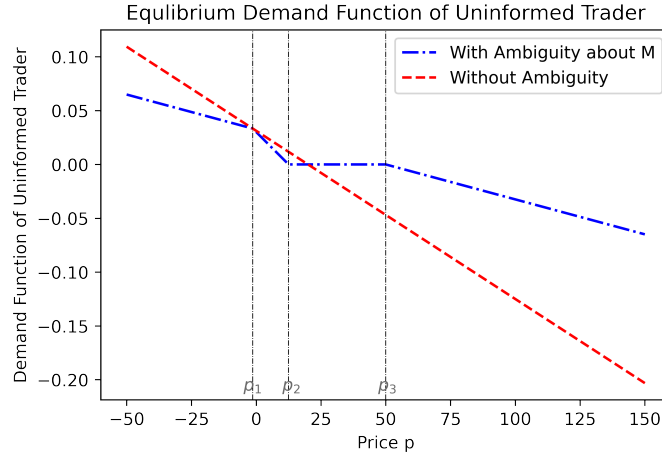


FIGURE 2. This figure depicts the equilibrium demand function of uninformed traders given in Proposition 2 (the red line, benchmark case without ambiguity) and Proposition 4 (the blue line, case with ambiguity), as a function of the price, p . The region delimited by the vertical dashed lines corresponds to the three threshold value p_1 , p_2 and p_3 . The parameter values are $\bar{f} = 0$, $M = 10$, $M_1 = 4$, $M_2 = 16$, $N = 300$, $\rho = 0.5$, $\lambda = 0.6$, $Z = 10$, $k = 20$.

and the equality holds for true only if the true value of M is equal to M_2 .

Ambiguity about the market participants speaks to the long-standing “excess volatility” puzzle (Shiller (1981)): Since beliefs are disproportionately volatile compared to fundamentals, it generates excess volatility.

5.2. Price As An Indicator Of Market Participants

Proposition 11 illustrates the uninformed trader’s belief choice after observing the price and their extracted signal mapped from price. The price serves as a signal to tell uninformed traders whether or not the market is crowded with informed traders. When uninformed traders observe an abnormal price (either too high or too low), they conclude endogenously that there are only a small number of informed traders participating in the market. When uninformed traders observe a normal market price (one that is neither too high nor too low), they conclude endogenously that there are a large number of informed traders.

5.3. The Identity of Contrarians

PROPOSITION 7. *Contrarians*

Uninformed traders are always contrarians (if participating the market) and informed traders always trade in the direction of the signal.

Following good news, informed traders know the asset is good they will trade in the direction of the private signal, while uninformed traders always make decision using worst-case analysis,

and therefore think the asset is over-priced and trade in the opposite direction of the signal. Ambiguity about the market participants speaks to the empirical findings of Luo et al. (2022): A significant portion of retail investors engage in contrarian trading, especially contrarian buying.

5.4. Trading Volume

PROPOSITION 8. *Excessive Trading Volume*

With ambiguity about the market participants, trading volume may be extremely large compared to the benchmark case without ambiguity.

In financial markets, it is a well-established fact that trading volume exceeds what rational models can justify. (Barberis (2018)). Proposition 8 demonstrates that, in the presence of ambiguity regarding market participants, we do not need to rely on extreme signals to achieve excessive trading volume.

5.5. The Impact of Ambiguity On Uninformed Traders' Expected Trading Profits

In this section, I discuss the implications of changes in the extent of ambiguity, ΔM on uninformed traders' expected trading profits, denoted as Π .

$$(23) \quad \Pi = E \left[(f_u - p)x_u^* - \frac{k}{2}x_u^{*2} \right]$$

The explicit expression of uninformed traders' expected trading profits, W is provided in the appendix. Under certain parameter values, the impact of the ambiguity level on uninformed traders' expected trading profits may be nonlinear and non-monotonic. I illustrate this effect with the following numerical example in figure 3. Figure 3 illuminates policy implications by demonstrating that reducing ambiguity is not always beneficial for uninformed traders. From a regulatory standpoint, there exists an optimal value for the ambiguity level that maximizes the expected profits of an uninformed trader.

5.6. Demand Elasticity

Proposition 5 and figure 2 show that ambiguity about the market participants make the uninformed traders' demand function less elastic when $p \in (-\infty, p_1) \cup (p_2, +\infty)$, and more elastic when $p \in [p_1, p_2]$. Ambiguity about the market participants partially speaks to "*The Inelastic Markets Hypothesis*" (Gabaix and Koijen (2021)). This article offers one type of microfoundation to explain demand (in)elasticity.

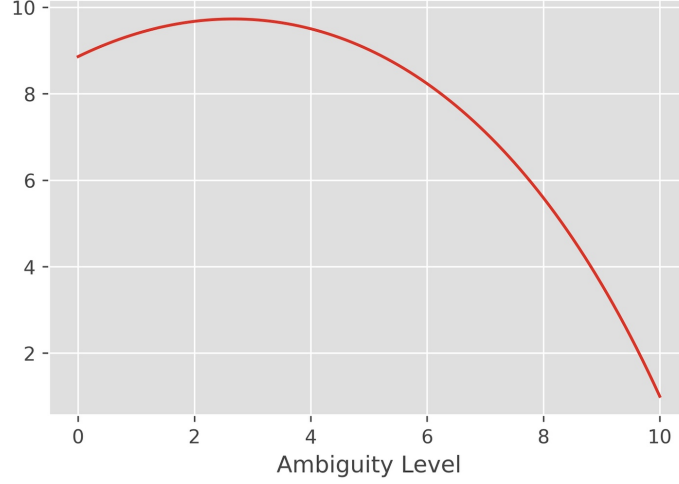


FIGURE 3. This figure depicts the uninformed traders' expected trading profits, as a function of the ambiguity level ΔM . The parameter values are $\bar{f} = 0$, $M = 10$, $N = 300$, $\rho = 0.5$, $\lambda = 0.6$, $Z = 10$, $k = 20$.

5.7. Implementable REE : the Demand Schedule Game with Ambiguity

In this section, I discuss whether the (competitive) REE we obtained in proposition 3 is an implementable REE, which is the outcome of a well-specified game⁵. Following Vives (2014), the natural way to implement (competitive) REE in our context is to consider competition among demand functions in a market where each trader is negligible. Hence, we adopt the definition of Implementable REE with ambiguity as an REE which is associated with a Bayesian Nash equilibrium of the game in demand functions with ambiguity. I focus on the linear demand function equilibrium. We conjecture the strategies of the informed traders and the uninformed traders respectively are as:

$$\begin{aligned}
 x_I(s, p) &= \eta + \beta s - \gamma p \\
 (24) \quad x_U(p) &= \begin{cases} \phi_1 - \omega_1 p, & \text{Buy and believe } M = M_1 \text{ for } p \in (-\infty, p_1) \\ \phi_2 - \omega_2 p, & \text{Buy and believe } M = M_2 \text{ for } p \in [p_1, p_2) \\ 0, & \text{Not Participate, for } p \in [p_2, p_3] \\ \phi_3 - \omega_3 p, & \text{Sell and believe } M = M_1 \text{ for } p \in (p_3, +\infty) \end{cases}
 \end{aligned}$$

DEFINITION 3. *Demand Function Game Equilibrium with Ambiguity*

⁵This is equivalent to find a trading mechanism that delivers the competitive REE.

The demand functions $x_I(s, p)$ and $x_U(p)$ are a Bayesian Nash Equilibrium with ambiguity if:

- (a) (Best Response) Fixing uninformed traders' strategies, for informed traders, (η, β, γ) is a best response to the strategy profile $(\phi_1, \phi_2, \phi_3, \omega_1, \omega_2, \omega_3)$. Fixing informed traders' strategies, for uninformed traders, $(\phi_1, \phi_2, \phi_3, \omega_1, \omega_2, \omega_3)$ is a best response to the strategy profile (η, β, γ) .
- (b) (Optimization) The informed demands of the risky asset x_I **maximize the expected profits** of informed traders and the uninformed demands x_U **maximize the minimum expected profits** of the uninformed traders in the market,

$$(25) \quad x_I^* \in \arg \max_{x_I} \left(E \left[(f_I - p) x_I - \frac{k}{2} x_I^2 \mid \mathcal{F}_I = \{s, p\} \right] \right)$$

$$(26) \quad x_U^* \in \arg \max_{x_U} \min_{M \in [M_1, M_2]} \left(E_M \left[(f_U - p) x_U - \frac{k}{2} x_U^2 \mid \mathcal{F}_U = \{p\} \right] \right)$$

- (c) (Market-Clearing) For each (s, p, M) ,

$$(27) \quad M \cdot x_I + N \cdot x_U = Z$$

PROPOSITION 9. *Implementable REE*

The (competitive) REE of proposition 3 is an implementable REE and it can be implemented through a demand schedule game.

6. Alternative Setup: Financial Market Equilibrium with Knightian Uncertainty (or Ambiguity) About Uninformed traders

In this section, I characterize the financial market equilibrium where traders are only ambiguous about the total number (size) of informed traders N . Specifically, each uninformed traders is ambiguous about the number (size) of uninformed traders N ⁶. They are unable to assess what N is, but they believe it belongs to some interval, $N \in [N_1, N_2]$, with $N_1 < N < N_2$. We further assume that $N_1 = N - \Delta N$ and $N_2 = N + \Delta N$. I use the boldface of N to denote the true value of N . ΔN is an exogenous parameter that determines the ambiguity. (But here, the total number (size) of the informed traders M is known by all of the uninformed traders and this is common knowledge.) The solution method is quite similar with section 4. The calculation details and proofs are in Appendix.

⁶Whether informed traders are ambiguous about the number of uninformed traders N is irrelevant, as they have nothing to learn from the price.

PROPOSITION 10. Equilibrium Price (with Ambiguity about the Number of the Uninformed Traders N)

When the number of the informed trader M is known and the number of the uninformed trader N is unknown with the ambiguity, there exists a Rational Expectation Equilibrium (REE) in which the price function p is a function of s and N ,

$$(28) \quad p(s, N) = \bar{f} + \left[\frac{M + N_2 \rho}{M + N + (N_2 - N) \rho} \right] \lambda \cdot s - \left[\frac{(M + (N_2 - N) \rho) kZ}{M (M + N + (N_2 - N) \rho)} \right]$$

PROPOSITION 11. Uninformed Traders' Belief Choice and Extracted Signal

When the number of the informed trader M is known and the number of the uninformed trader N is unknown with the ambiguity, the extracted signal G that uninformed traders map from the observed price characterized as:

$$(29) \quad G = \frac{(p - \bar{f})(M + N_2) + kZ}{(M + N_2 \rho) \lambda}, \text{ Always believe the number of the uninformed traders is } N_2 \text{ for any observed price } p$$

PROPOSITION 12. Equilibrium Demand Function of Traders (with Ambiguity about the Number of the Uninformed Traders N)

When the number of the informed trader M is known and the number of the uninformed trader N is unknown with the ambiguity, the equilibrium demand function of informed traders and uninformed traders are characterized as:

$$(30) \quad x_I(s, p) = \frac{\bar{f} + \lambda \cdot s - p}{k}$$

$$x_U(p) = \frac{1}{k} \left[\frac{M(1 - \rho)\bar{f} + \rho kZ}{M + N_2 \rho} - \frac{M(1 - \rho)}{M + N_2 \rho} \cdot p \right]$$

Comparing propositions 3, 4, and 5 with propositions 10, 11, and 12, it's intriguing to note that the equilibrium price lacks a two-regime characteristic in this section. In this context, ambiguity-averse uninformed traders consistently assume that the market is saturated with numerous uninformed traders. Uncertainty regarding the count (size) of uninformed traders, denoted as N , consistently leads to price overreactions and reduces the elasticity of the demand function for uninformed traders. In summary, the two assumptions—"ambiguity about the Number of the Uninformed Traders" and "ambiguity about the Number of the Informed Traders"—can notably impact the equilibrium price and its corresponding characteristics in a distinct manner.

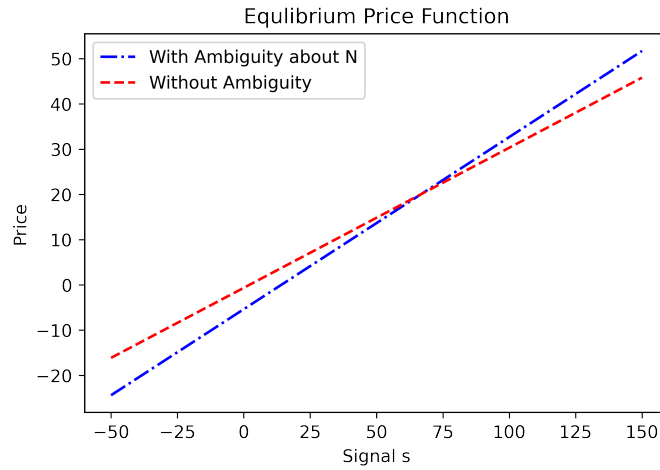


FIGURE 4. This figure depicts the equilibrium asset price given in Proposition 1 (the red line, benchmark case without ambiguity) and Proposition 10 (the blue line, case with ambiguity), as a function of the signal, s . The parameter values are $\bar{f} = 0$, $M = 10$, $N_1 = 100$, $N_2 = 500$, $N = 300$, $\rho = 0.5$, $\lambda = 0.6$, $Z = 10$, $k = 20$.

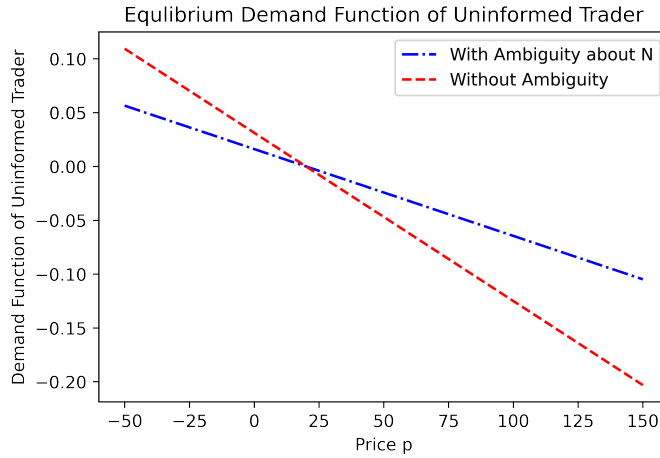


FIGURE 5. This figure depicts the equilibrium demand function of uninformed traders given in Proposition 2 (the red line, benchmark case without ambiguity) and Proposition 11 (the blue line, case with ambiguity), as a function of the price, p . The parameter values are $\bar{f} = 0$, $M = 10$, $N_1 = 100$, $N_2 = 500$, $N = 300$, $\rho = 0.5$, $\lambda = 0.6$, $Z = 10$, $k = 20$.

7. Alternative Setup: Financial Market Equilibrium with Two Dimension Knightian Uncertainty (or Ambiguity)

We now allow for two dimension ambiguity for uninformed traders. In this section, we assume uninformed traders are ambiguous about both the total number (size) of the informed traders M and the total number (size) of the uninformed traders N . The solution method is similar with section 4. The calculation details and proofs are in online Appendix.

PROPOSITION 13. *Equilibrium Price (with Ambiguity about the Number of the Informed Traders M and and Uninformed Traders N)*

When the number of the informed trader M and the number of the uninformed trader N are both unknown with the ambiguity by uninformed traders, there exists a Rational Expectation Equilibrium (REE) in which the price function p is piecewise in s , M and N

$$(31) \quad p(s, M, N) = \begin{cases} \bar{f} + \left[\frac{M(M_1+N_2\rho)}{M(M_1+N_2\rho)+NM_1(1-\rho)} \right] \lambda \cdot s - \left[\frac{[M_1+(N_2-N)\rho] \cdot kZ}{M(M_1+N_2\rho)+NM_1(1-\rho)} \right], & \text{for } s \in (-\infty, s'_1) \cup (s_3, +\infty) \\ \bar{f} + \left[\frac{M(M_2+N_2\rho)}{M(M_2+N_2\rho)+NM_2(1-\rho)} \right] \lambda \cdot s - \left[\frac{[M_2+(N_2-N)\rho] \cdot kZ}{M(M_2+N_2\rho)+NM_2(1-\rho)} \right], & \text{for } s \in [s'_1, s_2] \\ \bar{f} + \lambda s - \frac{kZ}{M}, & \text{for } s \in [s_2, s_3] \end{cases}$$

with $s'_1 < s_2 < s_3$.

$$(32) \quad \begin{aligned} s'_1 &= \frac{-kZ}{N_2\lambda(1-\rho)} \\ s_2 &= \left[\frac{1}{M} + \frac{\rho}{M_2(1-\rho)} \right] \frac{kZ}{\lambda} \\ s_3 &= \left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)} \right] \frac{kZ}{\lambda} \end{aligned}$$

In this two-dimensional ambiguity context, it is evident that the equilibrium price combines the characteristics described in sections 4 and 6. Again, this demonstrates that the two assumptions — "ambiguity about the Number of the Uninformed Traders" and "ambiguity about the Number of the Informed Traders" — can influence the equilibrium price and its corresponding characteristics in significantly different ways.

8. Conclusion

Following Vives (2014), I examine the impact of market participants uncertainty on asset prices. I model uninformed traders as being confronted with Knightian uncertainty regarding the number of informed traders. Thus, uninformed traders may have incorrect views about the number of informed traders chasing the same strategy. Even though equilibrium strategies of uninformed traders depend only on their subjective beliefs, equilibrium prices are determined by the actual market-clearing condition, and thus the perceived number of informed traders may differ from the actual number of informed traders available in the market.

This Knightian uncertainty impedes the ability of uninformed traders to infer information from price fluctuations. With Knightian uncertainty, the equilibrium price shows a two-regime characteristic: When prices are normal, uninformed traders think there are more informed traders around. And when prices are abnormal, they believe there are fewer informed traders. Hence, in our model, price is a signaling mechanism that tells uninformed traders as to whether or not the market is crowded with informed traders. This paper further shows that ambiguity about the market participants can lead to an under- or over-reaction of equilibrium asset prices to news, excessive volatility, excessive trading volume, etc. This paper also sheds light on policy implications: we show that reducing ambiguity is not always beneficial for uninformed traders. However, it depends on the parameter value of the economy.

Moreover, of significant importance, employing the Vives (2014) model as our foundational framework enables us to separately analyze the impacts of "ambiguity regarding the Number of Uninformed Traders" and "ambiguity regarding the Number of Informed Traders" on both the equilibrium price and associated characteristics. Our analysis reveals that these two distinct dimensions of ambiguity do not inevitably lead to identical implications.

To summarize, we emphasize that financial markets represent a delicate equilibrium where ambiguity plays a role in driving pessimism, influencing the trading activity. Our analysis provides a compelling economic rationale for considering deviation from fundamentals as a crucial state variable in understanding market dynamics.

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Appendix A. Proof For Section 3

A.1. Proof of Proposition 1

PROOF. In this section, I solve for the financial market equilibrium where traders do not suffer from the ambiguity and know the true number of each type of market participants for sure. To find the equilibrium, we need to first characterize the demand function of informed and uninformed traders.

A.1.1. Demand Function of Informed Traders

By observing the realization of s , informed traders choose portfolio holdings x_I to maximize the expected profits π_I

$$(A1) \quad \max_{x_I} E \left[(f_I - p) x_I - \frac{k}{2} x_I^2 \mid \mathcal{F}_I = \{s, p\} \right],$$

where p is the observed asset price. Standard arguments yield

$$(A2) \quad \begin{aligned} x_I(s, p) &= \frac{E(f_I \mid s, p) - p}{k} \\ &= \frac{E(\bar{f} + \theta_I + \varepsilon \mid s) - p}{k} \\ &= \frac{\bar{f} + E(\theta_I \mid s) - p}{k} \\ &= \frac{\bar{f} + \frac{\text{cov}(s, \theta_I)}{\text{var}(s)} s - p}{k} \\ &= \frac{\bar{f} + \left(\frac{\tau^{-1}}{\tau^{-1} + \tau_u^{-1}} \right) s - p}{k} \\ &= \frac{\bar{f} + \lambda s - p}{k}, \end{aligned}$$

and the informativeness of the signal is captured by the signal-to-noise ratio:

$$(A3) \quad \lambda \equiv \frac{\text{cov}(s, \theta_I)}{\text{var}(s)} = \frac{\tau^{-1}}{\tau^{-1} + \tau_u^{-1}}$$

A.1.2. Demand Function of Uninformed Traders

At trading stage, uninformed traders rationally conjecture that the price function is:

$$(A4) \quad p = \bar{f} + A \cdot s - H$$

where A and H are two constants, which will be endogenously determined in equilibrium. Thus, the optimal demand of uninformed traders is determined by

$$(A5) \quad \max_{x_U} \left(E \left[(f_U - p) x_U - \frac{k}{2} x_U^2 \mid \mathcal{F}_U = \{p\} \right] \right),$$

where x_U is the asset demand of uninformed traders. Since uninformed traders knows the true value of M and N , they view the stock price p as a noisy signal about f_U . They map from the observed price p to the extracted signal s_p :

$$(A6) \quad s_p = \frac{p + H - \bar{f}}{A}$$

The conditional moments of f_U are given by:

$$\begin{aligned}
E[f_U | p] &= E[\bar{f} + \theta_U + \varepsilon | p] \\
&= \bar{f} + E[\theta_U | p] + 0 \\
&= \bar{f} + E[\theta_U | s = s_p] \\
&= \bar{f} + \frac{\text{cov}(s, \theta_U)}{\text{var}(s)} s_p \\
&= \bar{f} + \left(\frac{\rho \tau^{-1}}{\tau^{-1} + \tau_u^{-1}} \right) s_p \\
&= \bar{f} + \rho \lambda \left[\frac{p + H - \bar{f}}{A} \right]
\end{aligned}
\tag{A7}$$

Thus an uninformed trader's demand function is:

$$\begin{aligned}
x_U(p) &= \frac{E(f_U | p) - p}{k} \\
&= \frac{\bar{f} + \rho \lambda \left[\frac{p + H - \bar{f}}{A} \right] - p}{k},
\end{aligned}
\tag{A8}$$

We insert the equation (A2) and (A8) into the market clearing condition, $M \cdot x_I + N \cdot x_U = Z$ and we get,

$$M \cdot \left(\frac{\bar{f} + \lambda s - p}{k} \right) + N \cdot \left(\frac{\bar{f} + \rho \lambda \cdot \left(\frac{p + H - \bar{f}}{A} \right) - p}{k} \right) = Z$$

Rearrange the terms and we get,

$$\begin{aligned}
\left(M + N - \frac{N\rho\lambda}{A} \right) p &= \left(M + N - \frac{N\rho\lambda}{A} \right) \bar{f} + M\lambda s - \left(kZ - \frac{N\rho\lambda H}{A} \right) \\
p &= \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=A} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda H}{A}}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=H}
\end{aligned}
\tag{A9}$$

Using the undetermined coefficient method and match the coefficient, we get:

$$A = \frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{A} \right) N}
\tag{A10}$$

$$H = \frac{kZ - \frac{N\rho\lambda H}{A}}{M + \left(1 - \frac{\rho\lambda}{A} \right) N}$$

Hence, we can explicitly solve for the fixed point for A and H ,

$$A = \left(\frac{M + N\rho}{M + N} \right) \lambda
\tag{A11}$$

$$H = \frac{kZ}{M + N}$$

To summarize, when both the number of the informed trader M and the number of the uninformed trader N are

known without the ambiguity, there exists a Rational Expectation Equilibrium (REE) in which the price function p is a function of s ,

$$(A12) \quad p(s) = \bar{f} + \left(\frac{M + N\rho}{M + N} \right) \lambda \cdot s - \frac{kZ}{M + N}$$

□

A.2. Proof of Proposition 2

PROOF. According to equation (A2), we find the equilibrium demand function of informed traders,

$$x_I(s, p) = \frac{\bar{f} + \lambda \cdot s - p}{k}$$

Using the equation (A8) and (A11), we find the equilibrium demand function of uninformed traders,

$$x_U(p) = \frac{1}{k} \left[\frac{M(1-\rho)\bar{f} + \rho kZ}{M + N\rho} - \frac{M(1-\rho)}{M + N\rho} \cdot p \right]$$

□

Appendix B. Proofs For Section 4

B.1. Proof of Proposition 3 and 4

PROOF. We define the value of M that causes the function $G(M)$ to reach a minimum as \underline{M} and the value of M that causes the function $G(M)$ to reach a maximum as \bar{M} ,

$$(A13) \quad \begin{aligned} \underline{M} &\equiv \arg \min_{M \in [M_1, M_2]} G(M) = \arg \min_{M \in [M_1, M_2]} \left[\frac{p + H(M) - \bar{f}}{A(M)} \right] \\ \bar{M} &\equiv \arg \max_{M \in [M_1, M_2]} G(M) = \arg \max_{M \in [M_1, M_2]} \left[\frac{p + H(M) - \bar{f}}{A(M)} \right] \end{aligned}$$

We define the values that $H(M)$ and $A(M)$ take on when $M = \underline{M}$ and $M = \bar{M}$ as:

$$(A14) \quad \begin{aligned} \underline{H} &\equiv H(\underline{M}) & \bar{H} &\equiv H(\bar{M}) \\ \underline{A} &\equiv A(\underline{M}) & \bar{A} &\equiv A(\bar{M}) \end{aligned}$$

Note that \underline{H} , \bar{H} , \underline{A} and \bar{A} will be a constant. Hence, we can define the maximum and minimum of $G(M)$ as:

$$(A15) \quad \begin{aligned} \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\min} &\equiv \underline{G} = \frac{p + \underline{H} - \bar{f}}{\underline{A}} \\ \left\{ \frac{p + H(M) - \bar{f}}{A(M)} \right\}_{\max} &\equiv \bar{G} = \frac{p + \bar{H} - \bar{f}}{\bar{A}} \end{aligned}$$

Hence, the optimal demand functions of the uninformed traders, equation (15), can be written as,

$$(A16) \quad x_U(p) = \begin{cases} \frac{\bar{f} + \rho\lambda \cdot \left(\frac{p+H-\bar{f}}{A}\right) - p}{k}, & \text{if } \bar{f} + \rho\lambda \cdot \left(\frac{p+H-\bar{f}}{A}\right) - p > 0 \\ 0, & \text{if } \bar{f} + \rho\lambda \cdot \left(\frac{p+H-\bar{f}}{A}\right) - p \leq 0 \leq \bar{f} + \rho\lambda \cdot \left(\frac{p+\bar{H}-\bar{f}}{A}\right) - p \\ \frac{\bar{f} + \rho\lambda \cdot \left(\frac{p+\bar{H}-\bar{f}}{A}\right) - p}{k}, & \text{if } \bar{f} + \rho\lambda \cdot \left(\frac{p+\bar{H}-\bar{f}}{A}\right) - p < 0 \end{cases}$$

We evaluate the feasibility of the possible demand functions given in equation (??) case by case by inserting it into the market clearing condition.

I. First, suppose $x_U > 0$ and the premise

$$(A17) \quad \bar{f} + \rho\lambda \cdot \frac{p+H-\bar{f}}{A} - p > 0$$

is satisfied, then the demand function of informed traders and uninformed traders are:

$$(A18) \quad \begin{aligned} x_I &= \frac{\bar{f} + \lambda s - p}{k} \\ x_U &= \frac{\bar{f} + \rho\lambda \cdot \left(\frac{p+H-\bar{f}}{A}\right) - p}{k} \end{aligned}$$

We insert the equation (A18) into the market clearing condition, $M \cdot x_I + N \cdot x_U = Z$ and we get,

$$M \cdot \left(\frac{\bar{f} + \lambda s - p}{k}\right) + N \cdot \left(\frac{\bar{f} + \rho\lambda \cdot \left(\frac{p+H-\bar{f}}{A}\right) - p}{k}\right) = Z$$

Rearrange the terms and we get,

$$(A19) \quad \left(M + N - \frac{N\rho\lambda}{A}\right) p = \left(M + N - \frac{N\rho\lambda}{A}\right) \bar{f} + M\lambda s - \left(kZ - \frac{N\rho\lambda H}{A}\right)$$

$$p = \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{A}}\right)}_{=A(M)} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda H}{A}}{M + N - \frac{N\rho\lambda}{A}}\right)}_{=H(M)}$$

Using the undetermined coefficient method and match the coefficient, we get:

$$(A20) \quad \begin{aligned} A(M) &= \frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{A}\right) N} \\ H(M) &= \frac{kZ - \frac{N\rho\lambda H}{A}}{M + \left(1 - \frac{\rho\lambda}{A}\right) N} \end{aligned}$$

Hence, we can express the function $G(M)$ as:

$$\begin{aligned}
G(M) &= \frac{p + H(M) - \bar{f}}{A(M)} \\
&= \frac{p + \left[\frac{kZ - \frac{N\rho\lambda H}{\underline{A}}}{M + \left(1 - \frac{N\rho\lambda}{\underline{A}}\right)N} \right] - \bar{f}}{\left[\frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{\underline{A}}\right)N} \right]} \\
&= \frac{(p - \bar{f}) \left[M + \left(1 - \frac{\rho\lambda}{\underline{A}}\right)N \right] + \left[kZ - \frac{N\rho\lambda H}{\underline{A}} \right]}{M\lambda}
\end{aligned}
\tag{A21}$$

We could find the partial derivative of $G(M)$ w.r.t M ,

$$\begin{aligned}
\frac{\partial G(M)}{\partial M} &= \frac{(p - \bar{f})M - \left\{ (p - \bar{f}) \left[M + \left(1 - \frac{\rho\lambda}{\underline{A}}\right)N \right] + \left[kZ - \frac{N\rho\lambda H}{\underline{A}} \right] \right\}}{M^2\lambda} \\
&= \frac{-N \left[(p - \bar{f}) \left(1 - \frac{\rho\lambda}{\underline{A}}\right) - \frac{\rho\lambda H}{\underline{A}} \right] - kZ}{M^2\lambda} \\
&= \frac{-NL - kZ}{M^2\lambda}
\end{aligned}
\tag{A22}$$

We define $L = (p - \bar{f}) \left(1 - \frac{\rho\lambda}{\underline{A}}\right) - \frac{\rho\lambda H}{\underline{A}}$.

Since at the beginning of this **sub-case I**, our main premise is that

$$\bar{f} + \rho\lambda \cdot \frac{p + H - \bar{f}}{\underline{A}} - p > 0$$

This premise can be written as:

$$- \left[(p - \bar{f}) \left(1 - \frac{\rho\lambda}{\underline{A}}\right) - \frac{\rho\lambda H}{\underline{A}} \right] > 0$$

This implies that $L < 0$, and the sign of $\frac{\partial G(M)}{\partial M}$ will be determined by the sign of $(-NL - kZ)$.

(a) Suppose $(-NL - kZ) > 0$, then $\frac{\partial G(M)}{\partial M} > 0$ and

$$M_1 = \underline{M} = \arg \min_{M \in [M_1, M_2]} G(M)$$

Inserting $\underline{M} = M_1$ into equation (A20), and we get,

$$\underline{A} = A(M_1) = \frac{M_1\lambda}{M_1 + \left(1 - \frac{\rho\lambda}{\underline{A}}\right)N}$$

$$\underline{H} = H(M_1) = \frac{kZ - \frac{N\rho\lambda H}{\underline{A}}}{M_1 + \left(1 - \frac{\rho\lambda}{\underline{A}}\right)N}$$

Solving for \underline{A} and \underline{H} , and we find the explicit expression:

$$\begin{aligned}
\underline{A} &= \left(\frac{M_1 + N\rho}{M_1 + N} \right) \lambda \\
\underline{H} &= \frac{kZ}{M_1 + N}
\end{aligned}
\tag{A23}$$

Under the belief that the size/total number of the informed traders is M_1 , inserting equation (A23) into equation

(A21), we expressed the extracted signal as a function of observed price p ,

$$\begin{aligned} G(M_1) &= \frac{(p - \bar{f}) \left[M_1 + \left(1 - \frac{\rho\lambda}{A}\right) N \right] + \left[kZ - \frac{N\rho\lambda H}{A} \right]}{M_1\lambda} \\ &= \frac{(p - \bar{f}) (M_1 + N) + kZ}{(M_1 + N\rho)\lambda} \end{aligned}$$

Inserting equation (A23) into equation (A19), and we find the price function,

$$\begin{aligned} (A24) \quad p &= \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=A(M)} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda H}{A}}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=H(M)} \\ &= \bar{f} + \left[\frac{M(M_1 + N\rho)}{M(M_1 + N\rho) + M_1N(1 - \rho)} \right] \lambda \cdot s - \left[\frac{M_1kZ}{M(M_1 + N\rho) + M_1N(1 - \rho)} \right] \end{aligned}$$

We need to check that the two premise, equation (A17) and $(-NL - kZ) > 0$, should be satisfied. Equation (A17) and equation (A23) imply that,

$$(A25) \quad p < \bar{f} + \left(\frac{\rho}{1 - \rho} \right) \left(\frac{kZ}{M_1} \right)$$

$(-NL - kZ) > 0$ and equation (A23) imply that,

$$(A26) \quad p < \bar{f} - \left(\frac{1}{1 - \rho} \right) \left(\frac{kZ}{N} \right)$$

When equation (A25) is satisfied, equation (A26) will be automatically satisfied.

Substituting equation (A24) into equation (A25), we get:

$$(A27) \quad s < \frac{-kZ}{N\lambda(1 - \rho)}$$

To sum up, in this subsection, when $p < \bar{f} - \left(\frac{1}{1 - \rho} \right) \left(\frac{kZ}{N} \right)$, or say, when $s < \frac{-kZ}{N\lambda(1 - \rho)}$, the equilibrium price function is

$$(A28) \quad p = \bar{f} + \left[\frac{M(M_1 + N\rho)}{M(M_1 + N\rho) + M_1N(1 - \rho)} \right] \lambda \cdot s - \left[\frac{M_1kZ}{M(M_1 + N\rho) + M_1N(1 - \rho)} \right]$$

(b) Suppose $(-NL - kZ) < 0$, then $\frac{\partial G(M)}{\partial M} < 0$ and

$$M_2 = \underline{M} = \arg \min_{M \in [M_1, M_2]} G(M)$$

Inserting $\underline{M} = M_2$ into equation (A20), and we get,

$$\begin{aligned} \underline{A} = A(M_2) &= \frac{M_2\lambda}{M_2 + \left(1 - \frac{\rho\lambda}{A}\right) N} \\ \underline{H} = H(M_2) &= \frac{kZ - \frac{N\rho\lambda H}{A}}{M_2 + \left(1 - \frac{\rho\lambda}{A}\right) N} \end{aligned}$$

Solving for \underline{A} and \underline{H} , and we find the explicit expression:

$$(A29) \quad \begin{aligned} \underline{A} &= \left(\frac{M_2 + N\rho}{M_2 + N} \right) \lambda \\ \underline{H} &= \frac{kZ}{M_2 + N} \end{aligned}$$

Under the belief that the size/total number of the informed traders is M_2 , inserting equation (A29) into equation (A21), we expressed the extracted signal as a function of observed price p ,

$$\begin{aligned} G(M_2) &= \frac{(p - \bar{f}) \left[M_2 + \left(1 - \frac{\rho\lambda}{\underline{A}} \right) N \right] + \left[kZ - \frac{N\rho\lambda\underline{H}}{\underline{A}} \right]}{M_2\lambda} \\ &= \frac{(p - \bar{f}) (M_2 + N) + kZ}{(M_2 + N\rho)\lambda} \end{aligned}$$

Inserting equation (A29) into equation (A19), and we find the price function,

$$(A30) \quad \begin{aligned} p &= \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{\underline{A}}} \right)}_{=A(M)} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda\underline{H}}{\underline{A}}}{M + N - \frac{N\rho\lambda}{\underline{A}}} \right)}_{=H(M)} \\ &= \bar{f} + \left[\frac{M(M_2 + N\rho)}{M(M_2 + N\rho) + M_2N(1 - \rho)} \right] \lambda \cdot s - \left[\frac{M_2kZ}{M(M_2 + N\rho) + M_2N(1 - \rho)} \right] \end{aligned}$$

We need to check that the two premise, equation (A17) and $(-NL - kZ) < 0$, should be satisfied. Equation (A17) and equation (A29) imply that,

$$(A31) \quad p < \bar{f} + \left(\frac{\rho}{1 - \rho} \right) \left(\frac{kZ}{M_2} \right)$$

$(-NL - kZ) < 0$ and equation (A29) imply that,

$$(A32) \quad p > \bar{f} - \left(\frac{1}{1 - \rho} \right) \left(\frac{kZ}{N} \right)$$

Substituting equation (A30) into equation (A31), we get

$$(A33) \quad s < \left[\frac{1}{M} + \frac{\rho}{M_2(1 - \rho)} \right] \frac{kZ}{\lambda}$$

Substituting equation (A30) into equation (A33), we get

$$(A34) \quad s > \frac{-kZ}{N\lambda(1 - \rho)}$$

To sum up, in this subsection, when $\bar{f} - \left(\frac{1}{1 - \rho} \right) \left(\frac{kZ}{N} \right) < p < \bar{f} + \left(\frac{\rho}{1 - \rho} \right) \left(\frac{kZ}{M_2} \right)$, or say, when $\frac{-kZ}{N\lambda(1 - \rho)} < s < \left[\frac{1}{M} + \frac{\rho}{M_2(1 - \rho)} \right] \frac{kZ}{\lambda}$, the equilibrium price function is

$$(A35) \quad p = \bar{f} + \left[\frac{M(M_2 + N\rho)}{M(M_2 + N\rho) + M_2N(1 - \rho)} \right] \lambda \cdot s - \left[\frac{M_2kZ}{M(M_2 + N\rho) + M_2N(1 - \rho)} \right]$$

II. Second, suppose $x_U = 0$ and the premise

$$(A36) \quad \bar{f} + \rho\lambda \cdot \left(\frac{p + \underline{H} - \bar{f}}{\underline{A}} \right) - p \leq 0 \leq \bar{f} + \rho\lambda \cdot \left(\frac{p + \bar{H} - \bar{f}}{\bar{A}} \right) - p$$

is satisfied, then the demand function of informed traders and uninformed traders are:

$$(A37) \quad \begin{aligned} x_I &= \frac{\bar{f} + \lambda s - p}{k} \\ x_U &= 0 \end{aligned}$$

We insert the equation (A37) into the market clearing condition, $M \cdot x_I + N \cdot x_U = Z$ and we get,

$$(A38) \quad p = \bar{f} + \lambda s - \frac{kZ}{M}$$

Hence, in this situation,

$$\begin{aligned} A(M) &= \underline{A} = \bar{A} = \lambda \\ H(M) &= \frac{kZ}{M} \end{aligned}$$

Hence, we can express the function $G(M)$, the extracted signal, as:

$$(A39) \quad \begin{aligned} G(M) &= \frac{p + H(M) - \bar{f}}{A(M)} \\ &= \frac{p + \left(\frac{kZ}{M}\right) - \bar{f}}{\lambda} \end{aligned}$$

It is obvious that $\frac{\partial G(M)}{\partial M} < 0$ and

$$\begin{aligned} M_2 = \underline{M} &= \arg \min_{M \in [M_1, M_2]} G(M) \\ M_1 = \bar{M} &= \arg \max_{M \in [M_1, M_2]} G(M) \end{aligned}$$

Hence we can find the explicit expression for \underline{H} and \bar{H} ,

$$\begin{aligned} \underline{H} = H(M_2) &= \frac{kZ}{M_2} \\ \bar{H} = H(M_1) &= \frac{kZ}{M_1} \end{aligned}$$

To satisfy the premise, equation (A36), the following two equations show hold for true.

$$\begin{aligned} \bar{f} + \rho\lambda \cdot \left(\frac{p + \frac{kZ}{M_2} - \bar{f}}{\lambda} \right) - p &\leq 0 \\ \bar{f} + \rho\lambda \cdot \left(\frac{p + \frac{kZ}{M_1} - \bar{f}}{\lambda} \right) - p &\geq 0 \end{aligned}$$

This implies that,

$$(A40) \quad \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_2} \right) \leq p \leq \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_1} \right)$$

Substituting equation (A38) into equation (A40), we get

$$(A41) \quad \left[\frac{1}{M} + \frac{\rho}{M_2(1-\rho)} \right] \frac{kZ}{\lambda} \leq s \leq \left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)} \right] \frac{kZ}{\lambda}$$

To sum up, in this subsection, when $\bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_2} \right) \leq p \leq \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_1} \right)$, or say, when $\left[\frac{1}{M} + \frac{\rho}{M_2(1-\rho)} \right] \frac{kZ}{\lambda} \leq s \leq$

$\left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)} \right] \frac{kZ}{\lambda}$, the equilibrium price function is

$$(A42) \quad p = \bar{f} + \lambda s - \frac{kZ}{M}$$

III. Third, suppose $x_U < 0$ and the premise

$$(A43) \quad \bar{f} + \rho\lambda \cdot \left(\frac{p + \bar{H} - \bar{f}}{A} \right) - p < 0$$

is satisfied, then the demand function of informed traders and uninformed traders are:

$$(A44) \quad \begin{aligned} x_I &= \frac{\bar{f} + \lambda s - p}{k} \\ x_U &= \frac{\bar{f} + \rho\lambda \cdot \left(\frac{p + \bar{H} - \bar{f}}{A} \right) - p}{k} \end{aligned}$$

We insert the equation (A44) into the market clearing condition, $M \cdot x_I + N \cdot x_U = Z$ and we get,

$$M \cdot \left(\frac{\bar{f} + \lambda s - p}{k} \right) + N \cdot \left(\frac{\bar{f} + \rho\lambda \cdot \left(\frac{p + \bar{H} - \bar{f}}{A} \right) - p}{k} \right) = Z$$

Rearrange the terms and we get,

$$\left(M + N - \frac{N\rho\lambda}{A} \right) p = \left(M + N - \frac{N\rho\lambda}{A} \right) \bar{f} + M\lambda s - \left(kZ - \frac{N\rho\lambda\bar{H}}{A} \right)$$

$$(A45) \quad p = \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=A(M)} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda\bar{H}}{A}}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=H(M)}$$

Using the undetermined coefficient method and match the coefficient, we get:

$$(A46) \quad A(M) = \frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{A} \right) N}$$

$$H(M) = \frac{kZ - \frac{N\rho\lambda\bar{H}}{A}}{M + \left(1 - \frac{\rho\lambda}{A} \right) N}$$

Hence, we can express the function $G(M)$ as:

$$(A47) \quad \begin{aligned} G(M) &= \frac{p + H(M) - \bar{f}}{A(M)} \\ &= \frac{p + \left[\frac{kZ - \frac{N\rho\lambda\bar{H}}{A}}{M + \left(1 - \frac{\rho\lambda}{A} \right) N} \right] - \bar{f}}{\left[\frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{A} \right) N} \right]} \\ &= \frac{(p - \bar{f}) \left[M + \left(1 - \frac{\rho\lambda}{A} \right) N \right] + \left[kZ - \frac{N\rho\lambda\bar{H}}{A} \right]}{M\lambda} \end{aligned}$$

We could find the partial derivative of $G(M)$ w.r.t M ,

$$\begin{aligned}
\frac{\partial G(M)}{\partial M} &= \frac{(p - \bar{f})M - \left\{ (p - \bar{f}) \left[M + \left(1 - \frac{\rho\lambda}{A}\right) N \right] + \left[kZ - \frac{N\rho\lambda\bar{H}}{A} \right] \right\}}{M^2\lambda} \\
\text{(A48)} \quad &= \frac{-N \left[(p - \bar{f}) \left(1 - \frac{\rho\lambda}{A}\right) - \frac{\rho\lambda\bar{H}}{A} \right] - kZ}{M^2\lambda} \\
&= \frac{-NL - kZ}{M^2\lambda}
\end{aligned}$$

We define $L = (p - \bar{f}) \left(1 - \frac{\rho\lambda}{A}\right) - \frac{\rho\lambda\bar{H}}{A}$.

Since at the beginning of this **sub-case III**, our main premise is that

$$\bar{f} + \rho\lambda \cdot \frac{p + \bar{H} - \bar{f}}{A} - p < 0$$

This premise can be written as:

$$- \left[(p - \bar{f}) \left(1 - \frac{\rho\lambda}{A}\right) - \frac{\rho\lambda\bar{H}}{A} \right] < 0$$

This implies that $L > 0$, and the sign of $\frac{\partial G(M)}{\partial M}$ will be determined by the sign of $(-NL - kZ)$, which is negative. It is obvious that $\frac{\partial G(M)}{\partial M} < 0$ and

$$M_1 = \bar{M} = \arg \max_{M \in [M_1, M_2]} G(M)$$

Inserting $\bar{M} = M_1$ into equation (A46), and we get,

$$\begin{aligned}
\bar{A} = A(M_1) &= \frac{M_1\lambda}{M_1 + \left(1 - \frac{\rho\lambda}{A}\right) N} \\
\bar{H} = H(M_1) &= \frac{kZ - \frac{N\rho\lambda\bar{H}}{A}}{M_1 + \left(1 - \frac{\rho\lambda}{A}\right) N}
\end{aligned}$$

Solving for \bar{A} and \bar{H} , and we find the explicit expression:

$$\begin{aligned}
\text{(A49)} \quad \bar{A} &= \left(\frac{M_1 + N\rho}{M_1 + N} \right) \lambda \\
\bar{H} &= \frac{kZ}{M_1 + N}
\end{aligned}$$

Under the belief that the size/total number of the informed traders is M_1 , inserting equation (A49) into equation (A47), we expressed the extracted signal as a function of observed price p ,

$$\begin{aligned}
G(M_1) &= \frac{(p - \bar{f}) \left[M_1 + \left(1 - \frac{\rho\lambda}{A}\right) N \right] + \left[kZ - \frac{N\rho\lambda\bar{H}}{A} \right]}{M_1\lambda} \\
&= \frac{(p - \bar{f}) (M_1 + N) + kZ}{(M_1 + N\rho)\lambda}
\end{aligned}$$

Inserting equation (A49) into equation (A45), and we find the price function,

$$\begin{aligned}
\text{(A50)} \quad p &= \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=A(M)} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda\bar{H}}{A}}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=H(M)} \\
&= \bar{f} + \left[\frac{M(M_1 + N\rho)}{M(M_1 + N\rho) + M_1N(1 - \rho)} \right] \lambda \cdot s - \left[\frac{M_1kZ}{M(M_1 + N\rho) + M_1N(1 - \rho)} \right]
\end{aligned}$$

We need to check that the premise, equation (A43), should be satisfied.
Premise (A43) and equation (A49) imply that,

$$(A51) \quad p > \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_1} \right)$$

Substituting equation (A50) into equation (A51), we get

$$(A52) \quad s > \left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)} \right] \frac{kZ}{\lambda}$$

To sum up, in this subsection, when $p > \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_1} \right)$, or say, when $s > \left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)} \right] \frac{kZ}{\lambda}$, the equilibrium price function is

$$(A53) \quad p = \bar{f} + \left[\frac{M(M_1 + N\rho)}{M(M_1 + N\rho) + M_1N(1-\rho)} \right] \lambda \cdot s - \left[\frac{M_1kZ}{M(M_1 + N\rho) + M_1N(1-\rho)} \right]$$

Finally, to summarize, combing the results of **sub-case I, II, III**, when the number of the uninformed trader N is known and the number of the informed trader M is unknown with the ambiguity, there exists a Rational Expectation Equilibrium (REE) in which the price function p is piecewise in s and M ,

$$(A54) \quad p(s, M) = \begin{cases} \bar{f} + \left[\frac{M(M_1 + N\rho)}{M(M_1 + N\rho) + M_1N(1-\rho)} \right] \lambda \cdot s - \left[\frac{M_1kZ}{M(M_1 + N\rho) + M_1N(1-\rho)} \right], & \text{for } s \in (-\infty, s_1) \cup (s_3, +\infty) \\ \bar{f} + \left[\frac{M(M_2 + N\rho)}{M(M_2 + N\rho) + M_2N(1-\rho)} \right] \lambda \cdot s - \left[\frac{M_2kZ}{M(M_2 + N\rho) + M_2N(1-\rho)} \right], & \text{for } s \in [s_1, s_2] \\ \bar{f} + \lambda s - \frac{kZ}{M}, & \text{for } s \in [s_2, s_3] \end{cases}$$

with

$$(A55) \quad \begin{aligned} s_1 &= \frac{-kZ}{N\lambda(1-\rho)} \\ s_2 &= \left[\frac{1}{M} + \frac{\rho}{M_2(1-\rho)} \right] \frac{kZ}{\lambda} \\ s_3 &= \left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)} \right] \frac{kZ}{\lambda} \end{aligned}$$

□

B.2. Proof of Proposition 5

PROOF. (a) When $p < \bar{f} - \left(\frac{1}{1-\rho} \right) \left(\frac{kZ}{N} \right)$, or say, when $s < \frac{-kZ}{N\lambda(1-\rho)}$, using equation (A16) and (A54), we find that the demand function of uninformed trader is:

$$(A56) \quad \begin{aligned} x_U(p) &= \frac{1}{k} \left[\left(1 - \frac{\rho\lambda}{A} \right) (\bar{f} - p) + \frac{\rho\lambda H}{A} \right] \\ &= \frac{1}{k} \left[\frac{M_1(1-\rho)\bar{f} + \rho kZ}{M_1 + N\rho} - \frac{M_1(1-\rho)}{M_1 + N\rho} \cdot p \right] > 0 \end{aligned}$$

(b) When $\bar{f} - \left(\frac{1}{1-\rho} \right) \left(\frac{kZ}{N} \right) < p < \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_2} \right)$, or say, when $\frac{-kZ}{N\lambda(1-\rho)} < s < \left[\frac{1}{M} + \frac{\rho}{M_2(1-\rho)} \right] \frac{kZ}{\lambda}$, using equation (A16)

and (A54), we find that the demand function of uninformed trader is:

$$(A57) \quad \begin{aligned} x_U(p) &= \frac{1}{k} \left[\left(1 - \frac{\rho\lambda}{A}\right) (\bar{f} - p) + \frac{\rho\lambda H}{A} \right] \\ &= \frac{1}{k} \left[\frac{M_2(1-\rho)\bar{f} + \rho kZ}{M_2 + N\rho} - \frac{M_2(1-\rho)}{M_2 + N\rho} \cdot p \right] > 0 \end{aligned}$$

(c) When $\bar{f} + \left(\frac{\rho}{1-\rho}\right) \left(\frac{kZ}{M_2}\right) \leq p \leq \bar{f} + \left(\frac{\rho}{1-\rho}\right) \left(\frac{kZ}{M_1}\right)$, or say, when $\left[\frac{1}{M} + \frac{\rho}{M_2(1-\rho)}\right] \frac{kZ}{\lambda} \leq s \leq \left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)}\right] \frac{kZ}{\lambda}$, using equation (A16) and (A54), we find that the demand function of uninformed trader is:

$$(A58) \quad x_U(p) = 0$$

(d) When $p > \bar{f} + \left(\frac{\rho}{1-\rho}\right) \left(\frac{kZ}{M_1}\right)$, or say, when $s > \left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)}\right] \frac{kZ}{\lambda}$, using equation (A16) and (A54), we find that the demand function of uninformed trader is:

$$(A59) \quad \begin{aligned} x_U(p) &= \frac{1}{k} \left[\left(1 - \frac{\rho\lambda}{A}\right) (\bar{f} - p) + \frac{\rho\lambda H}{A} \right] \\ &= \frac{1}{k} \left[\frac{M_1(1-\rho)\bar{f} + \rho kZ}{M_1 + N\rho} - \frac{M_1(1-\rho)}{M_1 + N\rho} \cdot p \right] < 0 \end{aligned}$$

Finally, to summarize, combing the results of **1,2,3,4**, when the number of the uninformed trader N is known and the number of the informed trader M is unknown with the ambiguity, in equilibrium, the demand function of uninformed traders is characterized as,

$$(A60) \quad x_U(p) = \begin{cases} \frac{1}{k} \left[\frac{M_1(1-\rho)\bar{f} + \rho kZ}{M_1 + N\rho} - \frac{M_1(1-\rho)}{M_1 + N\rho} \cdot p \right], & \text{Buy, for } p \in (-\infty, p_1) \\ \frac{1}{k} \left[\frac{M_2(1-\rho)\bar{f} + \rho kZ}{M_2 + N\rho} - \frac{M_2(1-\rho)}{M_2 + N\rho} \cdot p \right], & \text{Buy, for } p \in [p_1, p_2) \\ 0, & \text{Not Participate, for } p \in [p_2, p_3] \\ \frac{1}{k} \left[\frac{M_1(1-\rho)\bar{f} + \rho kZ}{M_1 + N\rho} - \frac{M_1(1-\rho)}{M_1 + N\rho} \cdot p \right], & \text{Sell, for } p \in (p_3, +\infty) \end{cases}$$

with p_1, p_2 and p_3 are three constants,

$$(A61) \quad \begin{aligned} p_1 &= \bar{f} - \left(\frac{1}{1-\rho}\right) \left(\frac{kZ}{N}\right) \\ p_2 &= \bar{f} + \left(\frac{\rho}{1-\rho}\right) \left(\frac{kZ}{M_2}\right) \\ p_3 &= \bar{f} + \left(\frac{\rho}{1-\rho}\right) \left(\frac{kZ}{M_1}\right) \end{aligned}$$

□

Appendix C. Proof For Section 5.1 - 5.6

C.1. Proof of Proposition 6

PROOF. From proposition 1 and 3, we find that,

$$(A62) \quad \frac{\partial p(s)}{\partial s} = \left(\frac{M + N\rho}{M + N}\right) \lambda$$

$$(A63) \quad \begin{cases} \frac{\partial p(s,M)}{\partial s} = \left[\frac{M(M_1+N\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} \right] \lambda, & \text{for } s \in (-\infty, s_1) \cup [s_3, +\infty) \\ \frac{\partial p(s,M)}{\partial s} = \left[\frac{M(M_2+N\rho)}{M(M_2+N\rho)+M_2N(1-\rho)} \right] \lambda, & \text{for } s \in [s_1, s_2) \\ \frac{\partial p(s,M)}{\partial s} = \lambda, & \text{for } s \in [s_2, s_3) \end{cases}$$

It is easy to show that,

$$(A64) \quad \left[\frac{M(M_1+N\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} \right] \lambda - \left(\frac{M+N\rho}{M+N} \right) \lambda = \frac{N^2\rho(1-\rho)(M-M_1)}{[M(M_1+N\rho)+M_1N(1-\rho)][M+N]} \lambda \geq 0$$

$$(A65) \quad \left[\frac{M(M_2+N\rho)}{M(M_2+N\rho)+M_2N(1-\rho)} \right] \lambda - \left(\frac{M+N\rho}{M+N} \right) \lambda = \frac{N^2\rho(1-\rho)(M-M_2)}{[M(M_2+N\rho)+M_2N(1-\rho)][M+N]} \lambda \leq 0$$

$$(A66) \quad \lambda > \frac{M+N\rho}{M+N} \lambda$$

□

C.2. Proof of Proposition 7

PROOF. **I. Benchmark** Using proposition 1 and 2, we find that,

$$(A67) \quad \frac{\partial x_I}{\partial s} = \frac{N(1-\rho)\lambda}{k(M+N)} > 0$$

$$(A68) \quad \frac{\partial x_U}{\partial s} = -\frac{M(1-\rho)\lambda}{k(M+N)} < 0$$

Therefore, in the benchmark case, informed traders' equilibrium asset demand is increasing in the signal s , while uninformed traders' equilibrium asset demand is decreasing in the signal s .

II. Case $\mathcal{A} - \mathcal{M}$ Using proposition 3 and 5, we find that,

$$(A69) \quad \frac{\partial x_I}{\partial s} = \begin{cases} \frac{1}{k} \left[\frac{M_1N(1-\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} \right] \lambda > 0, & \text{for } s \in (-\infty, s_1) \\ \frac{1}{k} \left[\frac{M_2N(1-\rho)}{M(M_2+N\rho)+M_2N(1-\rho)} \right] \lambda > 0, & \text{for } s \in [s_1, s_2) \\ 0, & \text{for } s \in [s_2, s_3] \\ \frac{1}{k} \left[\frac{M_1N(1-\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} \right] \lambda > 0, & \text{for } s \in (s_3, +\infty) \end{cases}$$

$$(A70) \quad \frac{\partial x_U}{\partial s} = \begin{cases} -\frac{1}{k} \left[\frac{M_1(1-\rho)}{M_1+N\rho} \right] \left[\frac{M(M_1+N\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} \right] \lambda < 0, & \text{for } s \in (-\infty, s_1) \\ -\frac{1}{k} \left[\frac{M_2(1-\rho)}{M_2+N\rho} \right] \left[\frac{M(M_2+N\rho)}{M(M_2+N\rho)+M_2N(1-\rho)} \right] \lambda < 0, & \text{for } s \in [s_1, s_2) \\ 0, & \text{for } s \in [s_2, s_3] \\ -\frac{1}{k} \left[\frac{M_1(1-\rho)}{M_1+N\rho} \right] \left[\frac{M(M_1+N\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} \right] \lambda < 0, & \text{for } s \in (s_3, +\infty) \end{cases}$$

with $s_1 = \frac{-kZ}{N\lambda(1-\rho)}$, $s_2 = \left[\frac{1}{M} + \frac{\rho}{M_2(1-\rho)} \right] \frac{kZ}{\lambda}$, $s_3 = \left[\frac{1}{M} + \frac{\rho}{M_1(1-\rho)} \right] \frac{kZ}{\lambda}$.

Therefore, in the case $\mathcal{A} - \mathcal{M}$, informed traders' equilibrium asset demand is increasing in the signal s , while uninformed traders' equilibrium asset demand is decreasing in the signal s . \square

C.3. Proof of Proposition 8

PROOF. Trading volume is formally defined as: $\mathbf{V} = M |x_I| + N |x_U|$.

I. Benchmark Using proposition 1 and 2, we find that,

$$(A71) \quad \begin{aligned} \mathbf{V}^{\text{Benchmark}} &= M |x_I| + N |x_U| \\ &= M \left| \frac{1}{k(M+N)} [kZ + N(1-\rho)\lambda s] \right| + N \left| \frac{1}{k(M+N)} [kZ - M(1-\rho)\lambda s] \right| \\ &= \frac{M}{k(M+N)} \cdot |kZ + N(1-\rho)\lambda s| + \frac{N}{k(M+N)} \cdot |kZ - M(1-\rho)\lambda s| \end{aligned}$$

II. Case $\mathcal{A} - \mathcal{M}$ Using proposition 3 and 5, we find that,

$$(A72) \quad |x_I| = \begin{cases} \frac{M_1}{k[M(M_1+N\rho)+M_1N(1-\rho)]} \cdot |N(1-\rho)\lambda s + kZ|, & \text{for } s \in (-\infty, s_1) \\ \frac{M_2}{k[M(M_2+N\rho)+M_2N(1-\rho)]} \cdot |N(1-\rho)\lambda s + kZ|, & \text{for } s \in [s_1, s_2) \\ \frac{Z}{M}, & \text{for } s \in [s_2, s_3] \\ \frac{M_1}{k[M(M_1+N\rho)+M_1N(1-\rho)]} \cdot |N(1-\rho)\lambda s + kZ|, & \text{for } s \in (s_3, +\infty) \end{cases}$$

$$(A73) \quad |x_U| = \begin{cases} \frac{1}{k(M_1+N\rho)} \left| \left(\frac{M_1^2(1-\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} + \rho \right) \cdot kZ - \left(\frac{M_1M(M_1+N\rho)(1-\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} \right) \cdot \lambda s \right|, & \text{for } s \in (-\infty, s_1) \\ \frac{1}{k(M_2+N\rho)} \left| \left(\frac{M_2^2(1-\rho)}{M(M_2+N\rho)+M_2N(1-\rho)} + \rho \right) \cdot kZ - \left(\frac{M_2M(M_2+N\rho)(1-\rho)}{M(M_2+N\rho)+M_2N(1-\rho)} \right) \cdot \lambda s \right|, & \text{for } s \in [s_1, s_2) \\ 0, & \text{for } s \in [s_2, s_3] \\ \frac{1}{k(M_1+N\rho)} \left| \left(\frac{M_1^2(1-\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} + \rho \right) \cdot kZ - \left(\frac{M_1M(M_1+N\rho)(1-\rho)}{M(M_1+N\rho)+M_1N(1-\rho)} \right) \cdot \lambda s \right|, & \text{for } s \in (s_3, +\infty) \end{cases}$$

Hence we could find the expression of the trading volume in case $\mathcal{A} - \mathcal{M}$ as

$$\mathbf{v}^{\mathcal{A} - \mathcal{M}} = M |x_I| + N |x_U|$$

□

C.4. Derivation of Equation (23)

$$\begin{aligned}
(A74) \quad \Pi &= E \left[(f_u - p)x_u^* - \frac{k}{2}x_u^{*2} \right] \\
&= E \left[(f_u - p) \left(\frac{f_u - p}{k} \right) - \frac{k}{2} \left(\frac{f_u - p}{k} \right)^2 \right] \\
&= E \left[\frac{(f_u - p)^2}{2k} \right] \\
&= E \left[\frac{(\bar{f} + \theta_U + \varepsilon - p)^2}{2k} \right] \\
&= E \left[\frac{(\theta_U + \varepsilon + H - A \cdot s)^2}{2k} \right]
\end{aligned}$$

Denote $\theta_U + \varepsilon$ as θ . Hence, θ and s are said to have a bivariate normal distribution,

$$\begin{pmatrix} \theta \\ s \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tau^{-1} + \tau_\varepsilon^{-1} & \rho \sqrt{\tau^{-1} + \tau_\varepsilon^{-1}} \sqrt{\tau^{-1} + \tau_u^{-1}} \\ \rho \sqrt{\tau^{-1} + \tau_\varepsilon^{-1}} \sqrt{\tau^{-1} + \tau_u^{-1}} & \tau^{-1} + \tau_u^{-1} \end{pmatrix} \right)$$

and their joint PDF is given by

$$\begin{aligned}
f_{\theta s}(\theta, s) &= \frac{1}{2\pi \sqrt{\tau^{-1} + \tau_\varepsilon^{-1}} \sqrt{\tau^{-1} + \tau_u^{-1}} \sqrt{1 - \rho^2}} \\
&\exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[\left(\frac{\theta}{\sqrt{\tau^{-1} + \tau_\varepsilon^{-1}}} \right)^2 + \left(\frac{s}{\sqrt{\tau^{-1} + \tau_u^{-1}}} \right)^2 - \frac{2\rho\theta s}{\sqrt{\tau^{-1} + \tau_\varepsilon^{-1}} \sqrt{\tau^{-1} + \tau_u^{-1}}} \right] \right\}
\end{aligned}$$

Hence, the equation (A74) can be rewritten as,

$$\begin{aligned}
(A75) \quad \Pi &= E \left[\frac{(\theta_U + \varepsilon + H - A \cdot s)^2}{2k} \right] = E \left[\frac{(\theta + H - A \cdot s)^2}{2k} \right] \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{s_1} \frac{(\theta + H_1 - A_1 \cdot s)^2}{2k} f_{\theta s}(\theta, s) ds d\theta + \int_{-\infty}^{+\infty} \int_{s_1}^{s_2} \frac{(\theta + H_2 - A_2 \cdot s)^2}{2k} f_{\theta s}(\theta, s) ds d\theta \\
&\quad + \int_{-\infty}^{+\infty} \int_{s_2}^{s_3} \frac{(\theta + H_3 - A_3 \cdot s)^2}{2k} f_{\theta s}(\theta, s) ds d\theta + \int_{-\infty}^{+\infty} \int_{s_3}^{+\infty} \frac{(\theta + H_1 - A_1 \cdot s)^2}{2k} f_{\theta s}(\theta, s) ds d\theta
\end{aligned}$$

where $A_1 = \left[\frac{M(M_1 + N\rho)}{M(M_1 + N\rho) + M_1 N(1 - \rho)} \right] \lambda$, $A_2 = \left[\frac{M(M_2 + N\rho)}{M(M_2 + N\rho) + M_2 N(1 - \rho)} \right] \lambda$, $A_3 = \lambda$ and $H_1 = \left[\frac{M_1 kZ}{M(M_1 + N\rho) + M_1 N(1 - \rho)} \right]$, $H_2 = \left[\frac{M_2 kZ}{M(M_2 + N\rho) + M_2 N(1 - \rho)} \right]$, $H_3 = \frac{kZ}{M}$ and $s_1 = \frac{-kZ}{N\lambda(1 - \rho)}$, $s_2 = \left[\frac{1}{M} + \frac{\rho}{M_2(1 - \rho)} \right] \frac{kZ}{\lambda}$, $s_3 = \left[\frac{1}{M} + \frac{\rho}{M_1(1 - \rho)} \right] \frac{kZ}{\lambda}$.

Appendix D. Proof For Section 5.7

To prove proposition 9, we need to first find the demand function game equilibrium with ambiguity following the definition 3 and then show that this equilibrium result coincides with the equilibrium result of proposition 3.

D.1. Proof of Proposition 9

PROOF. I first solve for the demand function game equilibrium with ambiguity. Following Kyle (1989), I focus on linear market equilibrium, where informed trader's strategy and uninformed trader's strategy are,

$$(A76) \quad x_I(s, p) = \eta + \beta s - \gamma p$$

$$(A77) \quad x_U(p) = \begin{cases} \phi_1 - \omega_1 p, & \text{Buy and believe } M = M_1 \text{ for } p \in (-\infty, p_1) \\ \phi_2 - \omega_2 p, & \text{Buy and believe } M = M_2 \text{ for } p \in [p_1, p_2) \\ 0, & \text{Not Participate, for } p \in [p_2, p_3] \\ \phi_3 - \omega_3 p, & \text{Sell and believe } M = M_1 \text{ for } p \in (p_3, +\infty) \end{cases}$$

Using the method of undetermined coefficients and matching the coefficient of equation (7) and equation (A76), we solve for the expression for η , β and γ .

$$(A78) \quad \eta = \frac{\bar{f}}{k}, \quad \beta = \frac{\lambda}{k}, \quad \gamma = \frac{1}{k}$$

In equilibrium, price will clear the market:

$$(A79) \quad \begin{cases} M(\eta + \beta s - \gamma p) + N(\phi_1 - \omega_1 p) = Z, & \text{for } p \in (-\infty, p_1) \\ M(\eta + \beta s - \gamma p) + N(\phi_2 - \omega_2 p) = Z, & \text{for } p \in [p_1, p_2) \\ M(\eta + \beta s - \gamma p) = Z, & \text{for } p \in [p_2, p_3] \\ M(\eta + \beta s - \gamma p) + N(\phi_3 - \omega_3 p) = Z, & \text{for } p \in (p_3, +\infty) \end{cases}$$

$$\Rightarrow \begin{cases} p = \frac{M\beta}{M\gamma + N\omega_1} \cdot s + \frac{M\eta + N\phi_1 - Z}{M\gamma + N\omega_1}, & \text{for } p \in (-\infty, p_1) \\ p = \frac{M\beta}{M\gamma + N\omega_2} \cdot s + \frac{M\eta + N\phi_2 - Z}{M\gamma + N\omega_2}, & \text{for } p \in [p_1, p_2) \\ p = \frac{\beta}{\gamma} \cdot s + \frac{M\eta - Z}{M\gamma}, & \text{for } p \in [p_2, p_3] \\ p = \frac{M\beta}{M\gamma + N\omega_3} \cdot s + \frac{M\eta + N\phi_3 - Z}{M\gamma + N\omega_3}, & \text{for } p \in (p_3, +\infty) \end{cases}$$

For the uninformed trader, price p is informationally equivalent as s_M ,

$$(A80) \quad s_M = \begin{cases} \frac{\gamma p - \eta}{\beta} + \frac{N\omega_1 p - N\phi_1 + Z}{M\beta}, & \text{for } p \in (-\infty, p_1) \\ \frac{\gamma p - \eta}{\beta} + \frac{N\omega_2 p - N\phi_2 + Z}{M\beta}, & \text{for } p \in [p_1, p_2) \\ \text{N/A}, & \text{for } p \in [p_2, p_3] \\ \frac{\gamma p - \eta}{\beta} + \frac{N\omega_3 p - N\phi_3 + Z}{M\beta}, & \text{for } p \in (p_3, +\infty) \end{cases}$$

Hence, for uninformed traders, the conditional moments of f_U taken under a particular belief M , are given by:

$$(A81) \quad \begin{aligned} E_M [f_U | p] &= E_M [\bar{f} + \theta_U + \varepsilon | p] \\ &= \bar{f} + E_M [\theta_U | p] + 0 \\ &= \bar{f} + E_M [\theta_U | s = s_M] \\ &= \bar{f} + \frac{\text{cov}(s, \theta_U)}{\text{var}(s)} s_M \\ &= \bar{f} + \left(\frac{\rho\tau^{-1}}{\tau^{-1} + \tau_u^{-1}} \right) s_M \end{aligned}$$

The objective function of an uninformed trader can be written as:

$$(A82) \quad \begin{aligned} &\min_{M \in [M_1, M_2]} \left((E_M [f_U | p] - p) \cdot x_U - \frac{k}{2} x_U^2 \right) \\ &\Rightarrow \min_{M \in [M_1, M_2]} \left((\bar{f} + \rho\lambda \cdot s_M - p) \cdot x_U - \frac{k}{2} x_U^2 \right) \\ &= \begin{cases} -\frac{k}{2} x_U^2 + [\bar{f} + \rho\lambda \cdot \{s_M\}_{\min} - p] \cdot x_U, & \text{if } x_U > 0 \\ 0, & \text{if } x_U = 0 \\ -\frac{k}{2} x_U^2 + [\bar{f} + \rho\lambda \cdot \{s_M\}_{\max} - p] \cdot x_U, & \text{if } x_U < 0 \end{cases} \end{aligned}$$

where $\{s_M\}_{\min}$ and $\{s_M\}_{\max}$ are the minimum and maximum of the function $\{s_M\}$, given the value of p , respectively.

Thus an uninformed trader's demand function is:

$$(A83) \quad x_U(p) = \begin{cases} \frac{\bar{f} + \rho\lambda \cdot \{s_M\}_{\min} - p}{k}, & \text{if } \bar{f} + \rho\lambda \cdot \{s_M\}_{\min} - p > 0 \\ 0, & \text{if } \bar{f} + \rho\lambda \cdot \{s_M\}_{\min} - p \leq 0 \leq \bar{f} + \rho\lambda \cdot \{s_M\}_{\max} - p \\ \frac{\bar{f} + \rho\lambda \cdot \{s_M\}_{\max} - p}{k}, & \text{if } \bar{f} + \rho\lambda \cdot \{s_M\}_{\max} - p < 0 \end{cases}$$

Inserting equation C.4 into C.4, and then using the method of undetermined coefficients, we find the demand function game equilibrium with ambiguity.

(a) **Case I Uninformed traders buy and believe $M = M_1$ for $p \in (-\infty, p_1)$**

Using equation (A78), (A81) and (A83), solving for the x_U and match the coefficients gives the solution for ϕ_1 and

ω_1 .

$$(A84) \quad \frac{\bar{f} + \rho\lambda \cdot \left(\frac{\gamma p - \eta}{\beta} + \frac{N\omega_1 p - N\phi_1 + Z}{M_1\beta} \right) - p}{k} = \phi_1 - \omega_1 p$$

$$(A85) \quad \Rightarrow \begin{cases} \frac{\frac{\rho\lambda\gamma}{\beta} + \frac{\rho\lambda N\omega_1}{M_1\beta} - 1}{k} = -\omega_1 \\ \frac{\bar{f} + \rho\lambda \left(\frac{-\eta}{\beta} + \frac{-N\phi_1 + Z}{M_1\beta} \right)}{k} = \phi_1 \end{cases}$$

$$(A86) \quad \Rightarrow \begin{cases} \omega_1 = \frac{1}{k} \left[\frac{M_1(1-\rho)}{M_1+N\rho} \right] \\ \phi_1 = \frac{1}{k} \left[\frac{M_1(1-\rho)\bar{f} + \rho k Z}{M_1+N\rho} \right] \end{cases}$$

We need to check that two premise should be satisfied.

$$(A87) \quad \begin{cases} N\omega_1 p - N\phi_1 + Z < 0 \\ \bar{f} + \rho\lambda \cdot \left(\frac{\gamma p - \eta}{\beta} + \frac{N\omega_1 p - N\phi_1 + Z}{M_1\beta} \right) - p > 0 \end{cases}$$

Inserting equation (A78) and (A86) into (A87), we get

$$(A88) \quad p < \bar{f} - \left(\frac{1}{1-\rho} \right) \left(\frac{kZ}{N} \right) \equiv p_1$$

(b) **Case II Uninformed traders buy and believe $M = M_2$ for $p \in [p_1, p_2]$**

Using equation (A78), (A81) and (A83), solving for the x_U and match the coefficients gives the solution for ϕ_2 and ω_2 .

$$(A89) \quad \frac{\bar{f} + \rho\lambda \cdot \left(\frac{\gamma p - \eta}{\beta} + \frac{N\omega_2 p - N\phi_2 + Z}{M_2\beta} \right) - p}{k} = \phi_2 - \omega_2 p$$

$$(A90) \quad \Rightarrow \begin{cases} \frac{\frac{\rho\lambda\gamma}{\beta} + \frac{\rho\lambda N\omega_2}{M_2\beta} - 1}{k} = -\omega_2 \\ \frac{\bar{f} + \rho\lambda \left(\frac{-\eta}{\beta} + \frac{-N\phi_2 + Z}{M_2\beta} \right)}{k} = \phi_2 \end{cases}$$

$$(A91) \quad \Rightarrow \begin{cases} \omega_2 = \frac{1}{k} \left[\frac{M_2(1-\rho)}{M_2+N\rho} \right] \\ \phi_2 = \frac{1}{k} \left[\frac{M_2(1-\rho)\bar{f} + \rho k Z}{M_2+N\rho} \right] \end{cases}$$

We need to check that two premise should be satisfied.

$$(A92) \quad \begin{cases} N\omega_2 p - N\phi_2 + Z > 0 \\ \bar{f} + \rho\lambda \cdot \left(\frac{\gamma p - \eta}{\beta} + \frac{N\omega_2 p - N\phi_2 + Z}{M_2 \beta} \right) - p > 0 \end{cases}$$

Inserting equation (A78) and (A91) into (A92), we get

$$(A93) \quad p_1 \equiv \bar{f} - \left(\frac{1}{1-\rho} \right) \left(\frac{kZ}{N} \right) \leq p < \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_2} \right) \equiv p_2$$

(c) **Case III Uninformed traders sell and believe $M = M_1$ for $p \in (p_3, +\infty)$**

Using equation (A78), (A81) and (A83), solving for the x_U and match the coefficients gives the solution for ϕ_3 and ω_3 .

$$(A94) \quad \frac{\bar{f} + \rho\lambda \cdot \left(\frac{\gamma p - \eta}{\beta} + \frac{N\omega_3 p - N\phi_3 + Z}{M_1 \beta} \right) - p}{k} = \phi_3 - \omega_3 p$$

$$(A95) \quad \Rightarrow \begin{cases} \frac{\frac{\rho\lambda\gamma}{\beta} + \frac{\rho\lambda N\omega_3 - 1}{M_1 \beta}}{k} = -\omega_3 \\ \frac{\bar{f} + \rho\lambda \left(\frac{-\eta}{\beta} + \frac{-N\phi_3 + Z}{M_1 \beta} \right)}{k} = \phi_3 \end{cases}$$

$$(A96) \quad \Rightarrow \begin{cases} \omega_3 = \frac{1}{k} \left[\frac{M_1(1-\rho)}{M_1 + N\rho} \right] = \omega_1 \\ \phi_3 = \frac{1}{k} \left[\frac{M_1(1-\rho)\bar{f} + \rho kZ}{M_1 + N\rho} \right] = \phi_1 \end{cases}$$

We need to check that two premise should be satisfied.

$$(A97) \quad \begin{cases} N\omega_1 p - N\phi_1 + Z > 0 \\ \bar{f} + \rho\lambda \cdot \left(\frac{\gamma p - \eta}{\beta} + \frac{N\omega_1 p - N\phi_1 + Z}{M_1 \beta} \right) - p < 0 \end{cases}$$

Inserting equation (A78) and (A96) into (A97), we get

$$(A98) \quad p > \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_1} \right) \equiv p_3$$

(d) **Case IV** This case is trivial, for uninformed traders do not participate in stock markets. Hence, $x_U = 0$ in this case. We need to check that two premise should be satisfied.

$$(A99) \quad \begin{cases} \bar{f} + \rho\lambda \cdot \left(\frac{\gamma p - \eta}{\beta} + \frac{N\omega_2 p - N\phi_2 + Z}{M_2\beta} \right) - p < 0 \\ \bar{f} + \rho\lambda \cdot \left(\frac{\gamma p - \eta}{\beta} + \frac{N\omega_3 p - N\phi_3 + Z}{M_1\beta} \right) - p > 0 \end{cases}$$

Inserting equation (A78), (A91) and (A96) into (A99), we get

$$(A100) \quad p_2 \equiv \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_2} \right) < p < \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M_1} \right) \equiv p_3$$

To summarize, combing the results of **case I, II, III, IV**, we find the demand function game equilibrium with ambiguity. This demand function game equilibrium coincides with the proposition 3 and 5. Hence, we conclude that the (competitive) REE of proposition 3 is an implementable REE and it can be implemented through a demand schedule game. □

Appendix E. Proof For Section 6

To find the equilibrium, we need to first characterize the demand function of uninformed traders who exhibit ambiguity aversion with maxmin utility function.

E.1. Demand Function of Informed Traders

By observing the realization of s , informed traders resolve their ambiguity straight away. They choose portfolio holdings x_I to maximize the expected profits π_I

$$(A101) \quad \max_{x_I} E \left[(f_I - p) x_I - \frac{k}{2} x_I^2 \mid \mathcal{F}_I = \{s, p\} \right],$$

where p is the observed asset price. Standard arguments yield

$$(A102) \quad \begin{aligned} x_I(s, p) &= \frac{E(f_I \mid s, p) - p}{k} \\ &= \frac{E(\bar{f} + \theta_I + \varepsilon \mid s) - p}{k} \\ &= \frac{\bar{f} + E(\theta_I \mid s) - p}{k} \\ &= \frac{\bar{f} + \frac{\text{cov}(s, \theta_I)}{\text{var}(s)} s - p}{k} \\ &= \frac{\bar{f} + \left(\frac{\tau^{-1}}{\tau^{-1} + \tau_u^{-1}} \right) s - p}{k} \\ &= \frac{\bar{f} + \lambda s - p}{k}, \end{aligned}$$

and the informativeness of the signal is captured by the signal-to-noise ratio:

$$(A103) \quad \lambda \equiv \frac{\text{cov}(s, \theta_I)}{\text{var}(s)} = \frac{\tau^{-1}}{\tau^{-1} + \tau_u^{-1}}$$

E.2. Demand Function of Uninformed Traders

At trading stage, for any given M and N , uninformed traders rationally conjecture that the price function is:

$$(A104) \quad p = \bar{f} + A(N; M) \cdot s - H(N; M)$$

where the function $A(N; M)$ and $H(N; M)$ will be endogenously determined in equilibrium. Since M is known by the uninformed traders for sure, M is an exogenous parameter here. For simplicity, we write $A(N) \equiv A(N; M)$ and $H(N) \equiv H(N; M)$ and the conjectured price function can be written as:

$$(A105) \quad p = \bar{f} + A(N) \cdot s - H(N)$$

Thus, the optimal demand of uninformed traders is determined by

$$(A106) \quad \max_{x_U} \min_{N \in [N_1, N_2]} \left(E_N \left[(f_U - p) x_U - \frac{k}{2} x_U^2 \mid \mathcal{F}_U = \{p\} \right] \right),$$

where x_U is the asset demand of uninformed traders, and $E_N(\cdot)$ is the expectation operator taken under the belief that the size/total number of the uninformed traders is N . The criterion underlying equation (A106) is the maxmin expected utility axiomatized by Gilboa and Schmeidler (1989). Since uninformed traders are ambiguous about N ,

they view the stock price p as an ambiguous signal about f_U . Under the belief that the size/total number of the uninformed traders is N , they map from the observed price p to the extracted signal s_N :

$$(A107) \quad s_N = \frac{p + H(N) - \bar{f}}{A(N)}$$

The conditional moments of f_U taken under a particular belief N , are given by:

$$(A108) \quad \begin{aligned} E_N [f_U | p] &= E_N [\bar{f} + \theta_U + \varepsilon | p] \\ &= \bar{f} + E_N [\theta_U | p] + 0 \\ &= \bar{f} + E_N [\theta_U | s = s_N] \\ &= \bar{f} + \frac{\text{cov}(s, \theta_U)}{\text{var}(s)} s_N \\ &= \bar{f} + \left(\frac{\rho \tau^{-1}}{\tau^{-1} + \tau_u^{-1}} \right) s_N \\ &= \bar{f} + \rho \lambda \left[\frac{p + H(N) - \bar{f}}{A(N)} \right] \\ &= \bar{f} + \rho \lambda \cdot G(N; p, \bar{f}, M) \end{aligned}$$

For simplicity, we write $G(N; p, \bar{f}, M) \equiv G(N)$ and the objective function of an uninformed trader can be written as:

$$(A109) \quad \begin{aligned} &\min_{N \in [N_1, N_2]} \left((E_N [f_U | p] - p) \cdot x_U - \frac{k}{2} x_U^2 \right) \\ &\Rightarrow \min_{N \in [N_1, N_2]} \left((\bar{f} + \rho \lambda \cdot G(N) - p) \cdot x_U - \frac{k}{2} x_U^2 \right) \\ &= \begin{cases} -\frac{k}{2} x_U^2 + [\bar{f} + \rho \lambda \cdot \{G(N)\}_{\min} - p] \cdot x_U, & \text{if } x_U > 0 \\ 0, & \text{if } x_U = 0 \\ -\frac{k}{2} x_U^2 + [\bar{f} + \rho \lambda \cdot \{G(N)\}_{\max} - p] \cdot x_U, & \text{if } x_U < 0 \end{cases} \end{aligned}$$

where $\{G(N)\}_{\min} = \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\min}$ and $\{G(N)\}_{\max} = \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\max}$ are the minimum and maximum of the function $\{G(N)\}$, respectively.

Thus an uninformed trader's demand function is:

$$(A110) \quad x_U(p) = \begin{cases} \frac{\bar{f} + \rho \lambda \cdot \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\min} - p}{k}, & \text{if } \bar{f} + \rho \lambda \cdot \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\min} - p > 0 \\ 0, & \text{if } \bar{f} + \rho \lambda \cdot \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\min} - p \leq 0 \leq \bar{f} + \rho \lambda \cdot \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\max} - p \\ \frac{\bar{f} + \rho \lambda \cdot \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\max} - p}{k}, & \text{if } \bar{f} + \rho \lambda \cdot \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\max} - p < 0 \end{cases}$$

E.3. Proof of Proposition 10, 11 and 12

PROOF. We define the value of N that causes the function $G(N)$ to reach a minimum as \underline{N} and the value of N that causes the function $G(N)$ to reach a maximum as \bar{N} ,

$$(A111) \quad \begin{aligned} \underline{N} &\equiv \arg \min_{N \in [N_1, N_2]} G(N) = \arg \min_{N \in [N_1, N_2]} \left[\frac{p + H(N) - \bar{f}}{A(N)} \right] \\ \bar{N} &\equiv \arg \max_{N \in [N_1, N_2]} G(N) = \arg \max_{N \in [N_1, N_2]} \left[\frac{p + H(N) - \bar{f}}{A(N)} \right] \end{aligned}$$

We define the values that $H(N)$ and $A(N)$ take on when $N = \underline{N}$ and $N = \bar{N}$ as:

$$(A112) \quad \begin{aligned} \underline{H} &\equiv H(\underline{N}) & \bar{H} &\equiv H(\bar{N}) \\ \underline{A} &\equiv A(\underline{N}) & \bar{A} &\equiv A(\bar{N}) \end{aligned}$$

Note that \underline{H} , \bar{H} , \underline{A} and \bar{A} will be a constant. Hence, we can define the maximum and minimum of $G(N)$ as:

$$(A113) \quad \begin{aligned} \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\min} &\equiv \underline{G} = \frac{p + \underline{H} - \bar{f}}{\underline{A}} \\ \left\{ \frac{p + H(N) - \bar{f}}{A(N)} \right\}_{\max} &\equiv \bar{G} = \frac{p + \bar{H} - \bar{f}}{\bar{A}} \end{aligned}$$

Hence, the optimal demand functions of the uninformed traders, equation (??), can be written as,

$$(A114) \quad x_U(p) = \begin{cases} \frac{\bar{f} + \rho\lambda \cdot \left(\frac{p + \underline{H} - \bar{f}}{\underline{A}} \right) - p}{k}, & \text{if } \bar{f} + \rho\lambda \cdot \left(\frac{p + \underline{H} - \bar{f}}{\underline{A}} \right) - p > 0 \\ 0, & \text{if } \bar{f} + \rho\lambda \cdot \left(\frac{p + \underline{H} - \bar{f}}{\underline{A}} \right) - p \leq 0 \leq \bar{f} + \rho\lambda \cdot \left(\frac{p + \bar{H} - \bar{f}}{\bar{A}} \right) - p \\ \frac{\bar{f} + \rho\lambda \cdot \left(\frac{p + \bar{H} - \bar{f}}{\bar{A}} \right) - p}{k}, & \text{if } \bar{f} + \rho\lambda \cdot \left(\frac{p + \bar{H} - \bar{f}}{\bar{A}} \right) - p < 0 \end{cases}$$

We evaluate the feasibility of the possible demand functions given in equation (A114) case by case by inserting it into the market clearing condition.

I. First, suppose $x_U > 0$ and the premise

$$(A115) \quad \bar{f} + \rho\lambda \cdot \frac{p + \underline{H} - \bar{f}}{\underline{A}} - p > 0$$

is satisfied, then the demand function of informed traders and uninformed traders are:

$$(A116) \quad \begin{aligned} x_I &= \frac{\bar{f} + \lambda s - p}{k} \\ x_U &= \frac{\bar{f} + \rho\lambda \cdot \left(\frac{p + \underline{H} - \bar{f}}{\underline{A}} \right) - p}{k} \end{aligned}$$

We insert the equation (A116) into the market clearing condition, $M \cdot x_I + N \cdot x_U = Z$ and we get,

$$M \cdot \left(\frac{\bar{f} + \lambda s - p}{k} \right) + N \cdot \left(\frac{\bar{f} + \rho\lambda \cdot \left(\frac{p + \underline{H} - \bar{f}}{\underline{A}} \right) - p}{k} \right) = Z$$

Rearrange the terms and we get,

$$(A117) \quad \left(M + N - \frac{N\rho\lambda}{A}\right) p = \left(M + N - \frac{N\rho\lambda}{A}\right) \bar{f} + M\lambda s - \left(kZ - \frac{N\rho\lambda H}{A}\right)$$

$$p = \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{A}}\right)}_{=A(N)} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda H}{A}}{M + N - \frac{N\rho\lambda}{A}}\right)}_{=H(N)}$$

Using the undetermined coefficient method and match the coefficient, we get:

$$(A118) \quad A(N) = \frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{A}\right) N}$$

$$H(N) = \frac{kZ - \frac{N\rho\lambda H}{A}}{M + \left(1 - \frac{\rho\lambda}{A}\right) N}$$

Hence, we can express the function $G(N)$ as:

$$(A119) \quad G(N) = \frac{p + H(N) - \bar{f}}{A(N)}$$

$$= \frac{p + \left[\frac{kZ - \frac{N\rho\lambda H}{A}}{M + \left(1 - \frac{\rho\lambda}{A}\right) N}\right] - \bar{f}}{\left[\frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{A}\right) N}\right]}$$

$$= \frac{(p - \bar{f}) \left[M + \left(1 - \frac{\rho\lambda}{A}\right) N\right] + \left[kZ - \frac{N\rho\lambda H}{A}\right]}{M\lambda}$$

We could find the partial derivative of $G(N)$ w.r.t N ,

$$\frac{\partial G(N)}{\partial N} = \frac{1}{M\lambda} \underbrace{\left[(p - \bar{f}) \left(1 - \frac{\rho\lambda}{A}\right) - \frac{\rho\lambda H}{A}\right]}_{\equiv L}$$

We define $L = (p - \bar{f}) \left(1 - \frac{\rho\lambda}{A}\right) - \frac{\rho\lambda H}{A}$.

Since at the beginning of this **sub-case I**, our main premise is that

$$\bar{f} + \rho\lambda \cdot \frac{p + H - \bar{f}}{A} - p > 0$$

This premise can be written as:

$$- \left[(p - \bar{f}) \left(1 - \frac{\rho\lambda}{A}\right) - \frac{\rho\lambda H}{A}\right] > 0$$

This implies that $L < 0$, $\frac{\partial G(N)}{\partial N} < 0$ and

$$N_2 = \underline{N} = \arg \min_{N \in [N_1, N_2]} G(N)$$

Inserting $\underline{N} = N_2$ into equation (A118), and we get,

$$\underline{A} = A(N_2) = \frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{\underline{A}}\right) N_2}$$

$$\underline{H} = H(N_2) = \frac{kZ - \frac{N_2\rho\lambda\underline{H}}{\underline{A}}}{M + \left(1 - \frac{\rho\lambda}{\underline{A}}\right) N_2}$$

Solving for \underline{A} and \underline{H} , and we find the explicit expression:

$$(A120) \quad \underline{A} = \left(\frac{M + N_2\rho}{M + N_2}\right) \lambda$$

$$\underline{H} = \frac{kZ}{M + N_2}$$

Inserting equation (A120) into equation (A117), and we find the price function,

$$(A121) \quad p = \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{\underline{A}}}\right)}_{=A(N)} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda\underline{H}}{\underline{A}}}{M + N - \frac{N\rho\lambda}{\underline{A}}}\right)}_{=H(N)}$$

$$= \bar{f} + \left[\frac{M + N_2\rho}{M + N + (N_2 - N)\rho}\right] \lambda \cdot s - \left[\frac{(M + (N_2 - N)\rho) kZ}{M(M + N + (N_2 - N)\rho)}\right]$$

We need to check that the premise, equation (A115), should be satisfied. This premise and equation (A120) imply that,

$$(A122) \quad p < \bar{f} + \left(\frac{\rho}{1 - \rho}\right) \left(\frac{kZ}{M}\right)$$

Inserting equation (A121) into (A122), this is equivalent with

$$(A123) \quad s < \left[\frac{M + N_2\rho}{M + N + (N_2 - N)\rho}\right]^{-1} \left[\frac{\rho}{M(1 - \rho)} + \frac{(M + (N_2 - N)\rho)}{M(M + N + (N_2 - N)\rho)}\right] \frac{kZ}{\lambda} \equiv \frac{kz}{M\lambda(1 - \rho)}$$

II. Second, suppose $x_U = 0$ and the premise

$$(A124) \quad \bar{f} + \rho\lambda \cdot \left(\frac{p + \underline{H} - \bar{f}}{\underline{A}}\right) - p \leq 0 \leq \bar{f} + \rho\lambda \cdot \left(\frac{p + \bar{H} - \bar{f}}{\bar{A}}\right) - p$$

is satisfied, then the demand function of informed traders and uninformed traders are:

$$(A125) \quad x_I = \frac{\bar{f} + \lambda s - p}{k}$$

$$x_U = 0$$

We insert the equation (A125) into the market clearing condition, $M \cdot x_I + N \cdot x_U = Z$ and we get,

$$(A126) \quad p = \bar{f} + \lambda s - \frac{kZ}{M}$$

Hence, in this situation,

$$A(N) = \underline{A} = \bar{A} = \lambda$$

$$H(N) = \underline{H} = \bar{H} = \frac{kZ}{M}$$

To satisfy the premise, equation (A124), the following equation should hold for true.

$$\bar{f} + \rho\lambda \cdot \left(\frac{p + \frac{kZ}{M} - \bar{f}}{\lambda} \right) - p = 0$$

This implies that,

$$(A127) \quad p = \bar{f} + \left(\frac{\rho}{1 - \rho} \right) \left(\frac{kZ}{M} \right)$$

This is equivalent with

$$(A128) \quad s = \frac{kz}{M\lambda(1 - \rho)}$$

III. Third, suppose $x_U < 0$ and the premise

$$(A129) \quad \bar{f} + \rho\lambda \cdot \left(\frac{p + \frac{\bar{H} - \bar{f}}{A}}{\lambda} \right) - p < 0$$

is satisfied, then the demand function of informed traders and uninformed traders are:

$$(A130) \quad \begin{aligned} x_I &= \frac{\bar{f} + \lambda s - p}{k} \\ x_U &= \frac{\bar{f} + \rho\lambda \cdot \left(\frac{p + \frac{\bar{H} - \bar{f}}{A}}{\lambda} \right) - p}{k} \end{aligned}$$

We insert the equation (A130) into the market clearing condition, $M \cdot x_I + N \cdot x_U = Z$ and we get,

$$M \cdot \left(\frac{\bar{f} + \lambda s - p}{k} \right) + N \cdot \left(\frac{\bar{f} + \rho\lambda \cdot \left(\frac{p + \frac{\bar{H} - \bar{f}}{A}}{\lambda} \right) - p}{k} \right) = Z$$

Rearrange the terms and we get,

$$\begin{aligned} \left(M + N - \frac{N\rho\lambda}{A} \right) p &= \left(M + N - \frac{N\rho\lambda}{A} \right) \bar{f} + M\lambda s - \left(kZ - \frac{N\rho\lambda\bar{H}}{A} \right) \\ p &= \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=A(N)} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda\bar{H}}{A}}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=H(N)} \end{aligned}$$

Using the undetermined coefficient method and match the coefficient, we get:

$$(A131) \quad A(N) = \frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{A} \right) N}$$

$$H(N) = \frac{kZ - \frac{N\rho\lambda\bar{H}}{A}}{M + \left(1 - \frac{\rho\lambda}{A} \right) N}$$

Hence, we can express the function $G(N)$ as:

$$\begin{aligned}
G(N) &= \frac{p + H(N) - \bar{f}}{A(N)} \\
&= \frac{p + \left[\frac{kZ - \frac{N\rho\lambda\bar{H}}{A}}{M + \left(1 - \frac{N\rho\lambda}{A}\right)N} \right] - \bar{f}}{\left[\frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{A}\right)N} \right]} \\
&= \frac{(p - \bar{f}) \left[M + \left(1 - \frac{\rho\lambda}{A}\right)N \right] + \left[kZ - \frac{N\rho\lambda\bar{H}}{A} \right]}{M\lambda}
\end{aligned}
\tag{A132}$$

We could find the partial derivative of $G(N)$ w.r.t N ,

$$\frac{\partial G(N)}{\partial N} = \frac{1}{M\lambda} \underbrace{\left[(p - \bar{f}) \left(1 - \frac{\rho\lambda}{A}\right) - \frac{\rho\lambda\bar{H}}{A} \right]}_{\equiv L}$$

We define $L = (p - \bar{f}) \left(1 - \frac{\rho\lambda}{A}\right) - \frac{\rho\lambda\bar{H}}{A}$.

Since at the beginning of this **sub-case II**, our main premise is that

$$\bar{f} + \rho\lambda \cdot \frac{p + \bar{H} - \bar{f}}{A} - p < 0$$

This premise can be written as:

$$- \left[(p - \bar{f}) \left(1 - \frac{\rho\lambda}{A}\right) - \frac{\rho\lambda\bar{H}}{A} \right] < 0$$

This implies that $L > 0$, $\frac{\partial G(N)}{\partial N} > 0$ and

$$N_2 = \bar{N} = \arg \max_{N \in [N_1, N_2]} G(N)$$

Inserting $\bar{N} = N_2$ into equation (A131), and we get,

$$\bar{A} = A(N_2) = \frac{M\lambda}{M + \left(1 - \frac{\rho\lambda}{A}\right)N_2}$$

$$\bar{H} = H(N_2) = \frac{kZ - \frac{N_2\rho\lambda\bar{H}}{A}}{M + \left(1 - \frac{\rho\lambda}{A}\right)N_2}$$

Solving for \bar{A} and \bar{H} , and we find the explicit expression:

$$\begin{aligned}
\bar{A} &= \left(\frac{M + N_2\rho}{M + N_2} \right) \lambda \\
\bar{H} &= \frac{kZ}{M + N_2}
\end{aligned}
\tag{A133}$$

Inserting equation xxx into equation xxx, and we find the price function,

$$\begin{aligned}
p &= \bar{f} + \underbrace{\left(\frac{M\lambda}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=A(N)} s - \underbrace{\left(\frac{kZ - \frac{N\rho\lambda\bar{H}}{A}}{M + N - \frac{N\rho\lambda}{A}} \right)}_{=H(N)} \\
&= \bar{f} + \left[\frac{M + N_2\rho}{M + N + (N_2 - N)\rho} \right] \lambda \cdot s - \left[\frac{(M + (N_2 - N)\rho)kZ}{M(M + N + (N_2 - N)\rho)} \right]
\end{aligned}
\tag{A134}$$

We need to check that the premise, equation (A129), should be satisfied. This implies that,

$$(A135) \quad p > \bar{f} + \left(\frac{\rho}{1-\rho} \right) \left(\frac{kZ}{M} \right)$$

This is equivalent with

$$(A136) \quad s > \left[\frac{M + N_2 \rho}{M + N + (N_2 - N) \rho} \right]^{-1} \left[\frac{\rho}{M(1-\rho)} + \frac{(M + (N_2 - N) \rho)}{M(M + N + (N_2 - N) \rho)} \right] \frac{kZ}{\lambda} \equiv \frac{kz}{M\lambda(1-\rho)}$$

To summarize, combing the results of **sub-case I, II, III**, when the number of the informed trader M is known and the number of the uninformed trader N is unknown with the ambiguity, there exists a Rational Expectation Equilibrium (REE) in which the price function p is a function of s and N ,

$$(A137) \quad p(s, N) = \bar{f} + \underbrace{\left[\frac{M + N_2 \rho}{M + N + (N_2 - N) \rho} \right]}_{=A(N)} \lambda \cdot s - \underbrace{\left[\frac{(M + (N_2 - N) \rho) kZ}{M(M + N + (N_2 - N) \rho)} \right]}_{=H(N)}$$

□