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Some Remarks on Enlargement of Filtration and Finance

Monique Jeanblanc*

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Abstract

In this note, we give a short overview of enlargement (or expansion) of filtration to help the reader who would like to have a survey on known results and some open questions. We try to select some papers (not all!) which contain important results. We present some applications in mathematical finance.

It was a great pleasure to write this paper in honour of my friend Catriona, who so efficiently manages her duties for Springer, providing useful editorial advice, improving the quality of the first versions of submitted books, promoting Springer volumes at many conferences, and contacting authors both old and new. Like many of us, I met Catriona quite often in numerous mathematical workshops. The time we spent together (too short) was always a pleasure. I will miss her during the forthcoming meetings and wish her a pleasant retirement.

1 Introduction

The information about the world is different for all of us. Some of us are specialists in history, others in philosophy and so on. The same is true of financial markets: some of the agents have information, say, in US market, others in European market. We restrict this general framework to the case where two groups of agents on the market may have different information about the dynamics of the same traded asset. If one of them has more information than the remaining ones, and if this new information is useful she can make profit or even arbitrages.

Examples:

- The Monty Hall problem is a well-known situation where you can make profit from extra information:

Suppose you are on a game show, and you are given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" To increase your probability of winning, you have to change your choice.

- William Duer is widely considered the first to have used his privileged knowledge in a scheme that involved speculating on bank stocks in 1789, . Six months later, he resigned from his position after it was discovered that he was taking advantage of his access to confidential information in order to speculate on stocks and bonds (see investopedia).

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- Another example took place on 19th June 1815, the day after the battle of Waterloo. Nathan Rothschild, who knew about Napoleon’s defeat beforehand (thanks to a spy or a carrier pigeon) went to the London Stock Exchange and proceeded to sell his English stocks, causing others to do the same. The resulting Stock Market crash (market impact) enabled Nathan Rothschild’s agents to then buy up these assets.

The American Securities and Exchange Commission (the SEC) defines insider trading as follows: Insider is an officer, director, 10% stockholder or anyone who possesses inside information because of his or her relationship with the Company or with an officer, director or principal stockholder of the Company. Rule 10b-5’s application goes considerably beyond just officers, directors and principal stockholders. This rule also covers any employee who has obtained material non-public corporate information, as well as any person who has received a “tip” from an Insider of the Company concerning information about the Company that is material and nonpublic, and trades (i.e., purchases or sells) the Company’s stock or other securities. This is illegal (see the SEC web page for recent cases and more information).

One of the IFRS (International Financial Reporting Standard) rules is that an entity need not undertake an exhaustive search of all possible markets to identify the principal market or, in the absence of a principal market, the most advantageous market.

In both cases, the notion of different information is advanced, and a goal is to try to model the new information and its impact on the market. The new information mainly concerns the behaviour of prices in the future, for example that the firm will close a part of its activity next month. We assume that a group of agents has access to the information described by a filtration \mathbb{F} and we shall model some kinds of new information. Then the insider can use the knowledge of this information to construct a portfolio and have a better terminal wealth.

Here, we do not distinguish “illegal” insider trading from trading with new information. We are only trying to see the impact of new information on the dynamics of prices.

2 Mathematical Facts

2.1 Problem of enlargement of filtration

At the beginning this problem was purely a mathematical problem. In the 70s, Itô [?] underlined that, in the case of a Brownian motion B with natural filtration \mathbb{F} , in order to give a meaning to $\int_0^t B_1 dB_s$, or more generally to $\int_0^t \theta_s B_1 dB_s$ where θ is an \mathbb{F} -adapted process (note that the process $\Psi = (\Psi_t = B_1, \forall t \geq 0)$ is not \mathbb{F} -adapted) it is natural to enlarge the filtration \mathbb{F} with the random variable B_1 and to obtain the decomposition of B as a semimartingale¹ in this enlarged filtration (this is known as the Brownian bridge). Recall that the set of semimartingales is the larger space of processes which makes it possible to define a “good” stochastic integration (Bitcheler-Dellacherie-Mokobodzki Theorem [?, Section 1.2.1]).

At the same time, independently of each other, P-A. Meyer and D. Williams asked the question: what can be said about \mathbb{F} -martingales when one introduces the smallest filtration containing \mathbb{F} and turns a given random time into a stopping time?

A first fact is that the martingale property is not stable under enlargement of filtration. More precisely, if $(\Omega, \mathcal{G}, \mathbb{P})$ is a probability space endowed with two filtrations \mathbb{F} and \mathbb{K} with $\mathbb{F} \subset \mathbb{K}$ (i.e., $\mathcal{F}_t \subset \mathcal{K}_t, \forall t \geq 0$), and if X is a (\mathbb{P}, \mathbb{F}) -martingale, X can fail to be a (\mathbb{P}, \mathbb{K}) -martingale. The general problem of enlargement of filtration is the following one. Let \mathbb{K} be a filtration larger than \mathbb{F} :

¹A semimartingale is the sum of a local martingale and a process with finite variation. When the finite variation part can be chosen as a predictable process, the semimartingale is said to be special and the decomposition with predictable part is unique.

under which conditions are all \mathbb{F} -martingales \mathbb{K} -semimartingales and obtain the \mathbb{K} -semimartingale decomposition of any \mathbb{F} -martingale. The condition is called the (\mathcal{H}') -hypothesis by Jacod [?] and many other authors.

As usual, for a filtration \mathbb{K} , we denote by $\mathcal{P}(\mathbb{K})$ the predictable σ -algebra and by $\mathcal{O}(\mathbb{K})$ the optional σ -algebra on $\Omega \times \mathbb{R}_+$.

Example 2.1 Let X be an \mathbb{F} -martingale of the form $X_t = \mathbb{E}[X_\infty | \mathcal{F}_t]$ where $X_\infty \in \mathcal{F}_\infty$ is integrable, and $\mathcal{K}_t = \mathcal{F}_\infty, \forall t \geq 0$. Then $\mathbb{E}[X_\infty | \mathcal{K}_t] = X_\infty \neq X_t$, and X is not a \mathbb{K} -martingale.

Example 2.2 Let \mathbb{F} be the filtration generated by a Brownian motion B and $\mathcal{K}_t = \mathcal{F}_{t+\delta}, \forall t \geq 0$, where $\delta > 0$. In that case, B is not a \mathbb{K} -semimartingale (see, e.g., [?, Example 1.19]).

For financial purposes, semimartingales play an important role: Let S be the \mathbb{F} -adapted price process, (eventually d -dimensional) locally bounded and assume that the riskless asset is constant (equal to 1). The fundamental theorem of asset pricing claims that No Free Lunch with Vanishing Risk (NFLVR) holds in \mathbb{F} if and only if there exists a strictly positive \mathbb{F} -martingale L such that SL is an \mathbb{F} -local martingale, or equivalently if there are no arbitrages (see, e.g., Björk [?, Chapter 10]). Another weaker result states that there is No Unbounded Profits with Bounded Risk (NUPBR) if and only if there exists an \mathbb{F} -strictly positive *local* martingale L such that SL is an \mathbb{F} -local martingale. These two characterisations require that asset prices are semimartingales under historical probability. The (local) martingale L is called a (local) deflator.

The financial definition of NFLVR and NUPBR is too long to give in this note, and would not be particularly useful. Instead we refer the reader to Delbaen & Schachermayer [?] and Björk [?, Chapter 10] for NFLVR and Kabanov, Kardaras & Song [?] for NUPBR.

Despite an extensive literature, very few cases are solved and very few concrete examples are known (see a list of examples in [?] and [?]). Studies are mainly concerned by

- **Initial Enlargement:** A filtration \mathbb{F} being given and ζ being a random variable, one sets $\mathbb{F}^{(\zeta)} = \mathbb{F} \vee \sigma(\zeta)$ (this is the case in $\text{It}\hat{\circ}$, where $\zeta = B_1$). This problem was solved in a quite general setting by Jacod [?]. We shall give a proof under a restrictive condition and recall the general result, without proof.
- **Progressive Enlargement:** A nonnegative random variable τ is given. We denote by A the indicator process $A_t = \mathbb{1}_{\{\tau \leq t\}}$ and by $\mathbb{A} = (\mathcal{A}_t, t \geq 0)$ its natural filtration. A filtration \mathbb{F} being given, one sets $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{A}_t, \forall t \geq 0$ (up to a regularization, so that \mathbb{G} is continuous on right). In other words, \mathbb{G} is the smallest filtration containing \mathbb{F} and turning τ into a stopping time. This corresponds to the question of Meyer and Williams. The first mathematical study was done by Barlow [?] for a specific class of random times, called honest times.
- **Others:** The new information can be the knowledge of a random variable at some random time (see Corcuera & Valkeika [?]) or more generally, two filtrations \mathbb{F} and $\tilde{\mathbb{F}}$ being given, one studies the enlargement of \mathbb{F} with $\tilde{\mathbb{F}}$, i.e., $\mathbb{F} \vee \tilde{\mathbb{F}}$. This general problem was intensively studied by Protter and various coauthors [?, ?].

Up until now, four lecture notes have been dedicated to enlargement of filtration: Aksamit & Jeanblanc [?], Jeulin [?], Jeulin & Yor [?] and Mansuy & Yor [?]. Chapter 20 of Dellacherie et al. [?] contains a very general presentation of enlargement of filtration theory, based on fundamental results of the theory of stochastic processes, developed in the previous chapters and books by the same authors. Chapter X in Jacod [?] presents deep results in a general setting. Chapter 12 in Yor [?] and the book of Mansuy & Yor [?] focus on the case where all martingales in the reference filtration \mathbb{F} are continuous (hypothesis **(C)**). The paper of Nikeghbali [?] also assume hypothesis

(C) and the fact that τ avoids all \mathbb{F} -stopping times (hypothesis (A)). A survey containing many exercises can be found in Mallein & Yor [?, Chapter 10]. Protter [?] and Jeanblanc et al. [?] have devoted a chapter of their books to the subject. The lecture by Song [?] contains a general study of the subject. The book of Hillairet & Jiao [?] contains applications to portfolio optimization.

Quite surprisingly, applications of enlargement of filtration theory to finance started only at the end of the 90s with the thesis of Amendinger [?] and independently in the paper Grorud & Pontier [?].

A basic example is that of point processes with bounded variation, hence the compensated martingale exists in any filtration larger than its natural one. Of course, the compensator depends on the filtration. See [?] and [?].

We shall need the notion of projections, that we recall now. If \mathbb{H} is a filtration satisfying $\mathbb{H} \subset \mathbb{K}$, and Y is a \mathbb{K} -adapted process such that $Y_\vartheta \mathbb{1}_{\{\vartheta < \infty\}}$ is integrable for any \mathbb{H} -stopping time ϑ , the \mathbb{H} -optional projection of Y is the \mathbb{H} -optional process ${}^{\circ, \mathbb{H}}Y$ such that $\mathbb{E}[Y_\vartheta \mathbb{1}_{\{\vartheta < \infty\}} | \mathcal{H}_\vartheta] = {}^{\circ, \mathbb{H}}Y_\vartheta \mathbb{1}_{\{\vartheta < \infty\}}$, for any \mathbb{H} -stopping time ϑ . This optional projection satisfies $\mathbb{E}[Y_t | \mathcal{H}_t] = {}^{\circ, \mathbb{H}}Y_t$, for all $t \geq 0$. If Y is a càdlàg \mathbb{K} -martingale, then ${}^{\circ, \mathbb{H}}Y$ is an \mathbb{H} -martingale. See, e.g., [?, Section 1.3.1].

2.2 Particular cases

2.2.1 Discrete time

In discrete time, any integrable \mathbb{H} -adapted process X is an \mathbb{H} -special semimartingale. Indeed $X_n = M_n + V_n, \forall n \geq 0$ where

$$M_n = M_{n-1} + X_n - \mathbb{E}[X_n | \mathcal{H}_{n-1}], \quad V_n = V_{n-1} + \mathbb{E}[X_n - X_{n-1} | \mathcal{H}_{n-1}]$$

and $M_0 = X_0, V_0 = 0$. The process M is a martingale, V is predictable with finite variation. Therefore, if X is an \mathbb{F} -martingale and \mathbb{K} a larger filtration, X is a \mathbb{K} -special semimartingale and its semimartingale decomposition reduces to computations of the conditional expectations $\mathbb{E}[X_n | \mathcal{K}_{n-1}]$ and $\mathbb{E}[X_{n-1} | \mathcal{K}_{n-1}]$. See Choulli & Deng [?] or Blanchet & Jeanblanc [?] for examples, as well as for initial enlargement and progressive enlargement.

These authors also present the study of arbitrages due to the new information. This is a difficult problem, and the proofs are similar to those in continuous time.

2.2.2 Immersion

Let $\mathbb{F} \subset \mathbb{K}$. Immersion holds between \mathbb{F} and \mathbb{K} if any \mathbb{F} -local martingale is a \mathbb{K} -local martingale: this is equivalent to, for any $t \geq 0$, the σ -fields \mathcal{F}_∞ and \mathcal{K}_t being conditionally independent given \mathcal{F}_t , i.e., $\forall t \geq 0, \forall K_t \in \mathcal{K}_t, \forall F_\infty \in \mathcal{F}_\infty$, both being square integrable

$$\mathbb{E}[K_t F_\infty | \mathcal{F}_t] = \mathbb{E}[K_t | \mathcal{F}_t] \mathbb{E}[F_\infty | \mathcal{F}_t].$$

This case (also called the (\mathcal{H}) -hypothesis) was presented in Brémaud & Yor [?] and this hypothesis is assumed in many studies, in particular for progressive enlargement in a credit risk framework. Roughly speaking, it means that the new information contained in \mathcal{K}_t has no influence on the past information \mathcal{F}_t . Note that if $\mathbb{F} \subset \mathbb{K} \subset \mathbb{G}$ and \mathbb{F} is immersed in \mathbb{G} , then \mathbb{F} is immersed in \mathbb{K} (but \mathbb{K} can fail to be immersed in \mathbb{G}). A nice property is that under immersion, NFLVR is preserved. Indeed, if L is an \mathbb{F} -deflator, it is a \mathbb{G} -positive martingale as well and a \mathbb{G} -deflator, the process SL being an \mathbb{F} and a \mathbb{G} -local martingale.

Immersion is not stable under change of probability (see [?]).

Example: An important example is the one introduced by Lando [?]. Given a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and a non negative \mathbb{F} adapted process λ , as well as a random variable Θ , independent of \mathbb{F} with unit exponential law, one defines

$$\tau = \inf\{t : \int_0^t \lambda_s ds \geq \Theta\}.$$

Then, since obviously \mathbb{F} is immersed in $\mathbb{F} \vee \sigma(\Theta)$ (by independence) and $\mathbb{F} \subset \mathbb{G} \subset \mathbb{F} \vee \sigma(\Theta)$, where \mathbb{G} is the progressive enlargement of \mathbb{F} by τ , the filtration \mathbb{F} is immersed in \mathbb{G} .

3 Initial Enlargement

A filtered probability $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and a \mathcal{G} -measurable random variable ζ being given, one sets $\mathbb{F}^{(\zeta)} = \mathbb{F} \vee \sigma(\zeta)$. We assume that \mathcal{F}_0 is trivial, and, if necessary, we take the smallest right-continuous filtration containing $\mathbb{F}^{(\zeta)}$. Note that $\mathcal{F}_0^{(\zeta)} = \sigma(\zeta)$.

This is a generalisation of the problem studied by Itô, for which $\zeta = B_1$.

We now present two important results on measurability:

For any fixed $t > 0$, every $\mathcal{F}_t^{(\zeta)}$ -measurable random variable $Y_t^{(\zeta)}$ is of the form $Y_t^{(\zeta)} = y_t(\omega, \zeta(\omega))$ where $y_t(\cdot, u)$ is, for any u , an \mathcal{F}_t -measurable random variable.

Every $\mathbb{F}^{(\zeta)}$ -predictable process $Y^{(\zeta)}$ is of the form $Y_t^{(\zeta)} = y_t(\omega, \zeta(\omega))$ where $(t, \omega, u) \mapsto y_t(\omega, u)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable function [?, Lemma 3.13].

We shall now simply write $y_t(\zeta)$ for $y_t(\omega, \zeta(\omega))$. The result on predictable processes cannot be extended in full generality to optional processes, but no counterexample is known.

3.1 Jacod's conditions

We start with a particular case, for which the proof is easy, found simultaneously by Grorud & Pontier [?] and Amendinger [?]. Then, we shall state (without proof) the general result of Jacod.

Assumption (\mathcal{E}): The \mathbb{F} -conditional law of ζ is equivalent to η , the law of ζ . More precisely there exists a non-negative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable map $(\omega, t, u) \rightarrow p_t(\omega, u)$ càdlàg in t such that
for every u , the process $(p_t(u))_{t \geq 0}$ is a strictly positive \mathbb{F} -martingale,
for every $t \geq 0$, the measure $p_t(u)\eta(du)$ equals $\mathbb{P}(\tau \in du | \mathcal{F}_t)$, in other words, for any Borel bounded function h , for any $t \geq 0$

$$\mathbb{E}[h(\zeta) | \mathcal{F}_t] = \int_{\mathbb{R}} h(u) p_t(u) \eta(du).$$

Assumption (\mathcal{E}) is also called Jacod's equivalence assumption.

Lemma 3.1 *Under Assumption (\mathcal{E}), the process L defined as $L_t = \frac{1}{p_t(\zeta)}, t \geq 0$ is a $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -martingale. Let \mathbb{P}^* be the probability measure defined on $\mathbb{F}^{(\zeta)}$ as*

$$d\mathbb{P}^*_{|\mathcal{F}_t^{(\zeta)}} = L_t d\mathbb{P}_{|\mathcal{F}_t^{(\zeta)}}. \quad (3.1)$$

Under \mathbb{P}^ , the random variable ζ is independent of \mathcal{F}_t for any $t \geq 0$ and, moreover*

$$\mathbb{P}^*_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t} \text{ for any } t \geq 0, \quad \mathbb{P}^*_{|\sigma(\zeta)} = \mathbb{P}_{|\sigma(\zeta)}.$$

PROOF: Obviously, one has $L_0 := \frac{1}{p_0(\zeta)} = 1$. Setting $L_t(u) := \frac{1}{p_t(u)}$, $\forall t \geq 0$, for any bounded Borel function h and any \mathcal{F}_s -measurable bounded random variable K_s and $s \leq t$, one has

$$\begin{aligned} \mathbb{E}[L_t h(\zeta) K_s] &= \mathbb{E} \left[K_s \int_{\mathbb{R}} L_t(u) h(u) p_t(u) \eta(du) \right] = \mathbb{E} \left[K_s \int_{\mathbb{R}} h(u) \eta(du) \right] \\ &= \mathbb{E}[K_s] \int_{\mathbb{R}} h(u) \eta(du) = \mathbb{E}[K_s] \mathbb{E}[h(\zeta)]. \end{aligned}$$

For $t = s$, we obtain $\mathbb{E}[L_s h(\zeta) K_s] = \mathbb{E}[K_s] \mathbb{E}[h(\zeta)]$, hence $\mathbb{E}[L_t h(\zeta) K_s] = \mathbb{E}[L_s h(\zeta) K_s]$. Since h and K_s are arbitrary and generate $\mathcal{F}_s^{(\zeta)}$, it follows that L is an $\mathbb{F}^{(\zeta)}$ -martingale. Thus, for each $t \geq 0$, we can define the probability measure \mathbb{P}^* on $\mathbb{F}_t^{(\zeta)}$ by $d\mathbb{P}^*|_{\mathcal{F}_t^{(\zeta)}} = L_t d\mathbb{P}|_{\mathcal{F}_t^{(\zeta)}}$. The equivalence of \mathbb{P}^* and \mathbb{P} on $\mathcal{F}_t^{(\zeta)}$ for each $t \in [0, \infty)$ follows from the strict positivity of L_t . For any bounded Borel function h , any \mathcal{F}_t -measurable bounded random variable K_t , and denoting by \mathbb{E}^* the expectation under \mathbb{P}^* , the above computations yield

$$\mathbb{E}^* [h(\zeta) K_t] = \mathbb{E}[h(\zeta)] \mathbb{E}[K_t]. \quad (3.2)$$

For $h = 1$ (resp. $K_t = 1$), one obtains $\mathbb{E}^*[K_t] = \mathbb{E}[K_t]$ (resp. $\mathbb{E}^*[h(\zeta)] = \mathbb{E}[h(\zeta)]$) and the assertions $\mathbb{P}^*|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{P}^*|_{\sigma(\zeta)} = \mathbb{P}|_{\sigma(\zeta)}$ are proven. Thus the identity (3.2) can be rewritten as $\mathbb{E}^* [h(\zeta) K_t] = \mathbb{E}^*[h(\zeta)] \mathbb{E}^*[K_t]$, which shows that the random variable ζ and the σ -field \mathcal{F}_t are independent under \mathbb{P}^* . \square

Corollary 3.2 *Under the probability measure \mathbb{P}^* , \mathbb{F} is immersed in $\mathbb{F}^{(\zeta)}$.*

PROOF: Since under \mathbb{P}^* , the random variable ζ and the σ -field \mathcal{F}_∞ are independent, the assertion follows. \square

Under Assumption (\mathcal{E}) , we study a financial market $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P}, S)$ with null interest rate. If the prices S are (\mathbb{P}, \mathbb{F}) martingales, then \mathbb{P}^* defined above is an equivalent martingale measure for the market $(\Omega, \mathcal{G}, \mathbb{F}^{(\zeta)}, S)$.

Note that if ζ satisfies (\mathcal{E}) under \mathbb{P} , it satisfies (\mathcal{E}) under any probability measure equivalent to \mathbb{P} . If the financial market is such that there exists an \mathbb{F} -equivalent martingale measure \mathbb{Q} , then, denoting $p^{\mathbb{Q}}(\zeta)$ the density of ζ under \mathbb{Q} , it follows that \mathbb{Q}^* is an equivalent probability measure where

$$\mathbb{Q}^*|_{\mathcal{F}_t^{(\zeta)}} = \frac{1}{p_t^{\mathbb{Q}}(\zeta)} \mathbb{Q}|_{\mathcal{F}_t^{(\zeta)}}. \quad (3.3)$$

\square

Proposition 3.3 *Under Assumption (\mathcal{E}) , any (\mathbb{P}, \mathbb{F}) -local martingale X is a $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -special semimartingale with decomposition $X_t = X_t^{(\zeta)} + \int_0^t \frac{d\langle X, p \cdot (u) \rangle_s |_{u=\zeta}}{p_{s-}(\zeta)}$, where $X^{(\zeta)}$ is a $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -local martingale.*

Proof: If X is a (\mathbb{P}, \mathbb{F}) -martingale, it is a $(\mathbb{P}^*, \mathbb{F}^{(\zeta)})$ -martingale. Indeed, since \mathbb{P} and \mathbb{P}^* are equal on \mathbb{F} , X is a $(\mathbb{P}^*, \mathbb{F})$ martingale, hence, using the fact that ζ is \mathbb{P}^* independent of \mathbb{F} , it is a $(\mathbb{P}^*, \mathbb{F}^{(\zeta)})$ martingale. Noting that $d\mathbb{P} = p_t(\zeta) d\mathbb{P}^*$ on $\mathcal{F}_t^{(\zeta)}$, Girsanov's theorem yields that the process $X^{(\zeta)}$, defined by $X_t^{(\zeta)} = X_t - \int_0^t \frac{d\langle X, p \cdot (u) \rangle_s |_{u=\zeta}}{p_{s-}(\zeta)}$, is a $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -martingale.

The general result of Jacod [?] (proved 20 years before the equivalence result) is the following

Theorem 3.4 Assume that there exists a non-negative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable map $(\omega, t, u) \rightarrow p_t(\omega, u)$ càdlàg in t such that

for every u , the process $(p_t(u))_{t \geq 0}$ is a non-negative \mathbb{F} -martingale,
denoting by η the law of ζ , for every $t \geq 0$, the measure $p_t(u)\eta(du)$ equals $\mathbb{P}(\tau \in du | \mathcal{F}_t)$, in other words, for any Borel bounded function h , for any $t \geq 0$

$$\mathbb{E}[h(\zeta) | \mathcal{F}_t] = \int_{\mathbb{R}} h(u) p_t(u) \eta(du).$$

Then, any (\mathbb{P}, \mathbb{F}) -local martingale X is a $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -special semimartingale with canonical decomposition

$$X_t = X_t^{(\zeta)} + \int_0^t \frac{d\langle X, p \cdot(u) \rangle_s |_{u=\zeta}}{p_{s-}(\zeta)}, \quad (3.4)$$

where $X^{(\zeta)}$ is a $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -local martingale.

The proof is more delicate, the process $1/p(\zeta)$ is well defined (since, from Jacod [?, Corollaire 1.11], $p(\zeta) > 0$) but is no longer a martingale. Jacod [?] mentioned (page 25) that it would be possible to use Girsanov-type results in the absolute continuity condition (page 15), but that in any case, the difficulties are due to measurability conditions in both approaches. The assumption of absolute continuity is also called the (\mathcal{J}) -assumption.

Under the (\mathcal{J}) hypothesis

a) every $\mathbb{F}^{(\zeta)}$ -optional process $Y^{(\zeta)}$ is of the form $Y_t^{(\zeta)}(\omega) = y_t(\omega, \zeta(\omega))$ for some $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}^d)$ -optional process $(y_t(\omega, u), t \geq 0)$ (see Fontana [?]).

b) Let $Y_T^{(\zeta)}$ be an $\mathcal{F}_T^{(\zeta)}$ -measurable integrable random variable. Then, for $s < T$:

$$\mathbb{E}(Y_T^{(\zeta)} | \mathcal{F}_s^{(\zeta)}) = \frac{1}{p_s(\zeta)} \mathbb{E}(y_T(u) p_T(u) | \mathcal{F}_s) \Big|_{u=\zeta}.$$

c) Characterization of $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -martingales in terms of (\mathbb{P}, \mathbb{F}) -martingales: The process $Y^{(\zeta)}$ is a $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -martingale if and only if, for any u , the process $y(u)p(u)$ is a (\mathbb{P}, \mathbb{F}) martingale.

We now present the propagation of the predictable representation property.

We assume that there exists a (\mathbb{P}, \mathbb{F}) -local martingale X such that any (\mathbb{P}, \mathbb{F}) -local martingale Y can be represented as

$$Y_t = Y_0 + \int_0^t \varphi_s dX_s \quad (3.5)$$

for some $\varphi \in \mathcal{P}(\mathbb{F})$.

Then (see Fontana [?]), under the (\mathcal{J}) -hypothesis, every $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -martingale $Y^{(\zeta)}$ admits a representation

$$Y_t^{(\zeta)} = Y_0^{(\zeta)} + \int_0^t \Phi_s dX_s^{(\zeta)} \quad (3.6)$$

where $\Phi \in \mathcal{P}(\mathbb{F}^{(\zeta)})$ and $Y_0^{(\zeta)} \in \mathcal{F}_0^{(\zeta)}$. Here $X^{(\zeta)}$ is the $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -martingale part (given in (3.4)) of the $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -semimartingale X introduced in (3.5).

3.2 Brownian bridge

The Brownian Bridge is obtained when studying the initial enlargement of a Brownian filtration \mathbb{F} (generated by the Brownian motion B) with the random variable B_1 . Note that Jacod's absolute

condition is not satisfied at time 1. Nevertheless, it is not difficult to prove (see Jeulin [?, Th. 3.23] or [?, Proposition 4.1]) that

$$B_t^{(B_1)} := B_t - \int_0^{t \wedge 1} \frac{B_1 - B_s}{1-s} ds, \quad 0 \leq t \leq 1$$

is an $\mathbb{F}^{(B_1)}$ -martingale, a first step being to prove the existence of the integral. Then, using elementary computations, it is easy to prove that \widehat{B} is a martingale, and by Lévy's Theorem, it is a Brownian motion. This main example presents another point of interest: even if B is a semimartingale, not all \mathbb{F} -martingales are \mathbb{G} -semimartingales (see [?]).

Theorem 3.5 *Let X be an \mathbb{F} -local martingale with representation $X_t = X_0 + \int_0^t \varphi_s dB_s$ for an \mathbb{F} -predictable process φ satisfying $\int_0^1 \varphi_s^2 ds < \infty$ a.s. Then, the following conditions are equivalent:*

(a) *the process X is an $\mathbb{F}^{(B_1)}$ -semimartingale;*

(b) $\int_0^1 |\varphi_s| \frac{|B_1 - B_s|}{1-s} ds < \infty$ \mathbb{P} -a.s.;

(c) $\int_0^1 \frac{|\varphi_s|}{\sqrt{1-s}} ds < \infty$ \mathbb{P} -a.s.

If these conditions are satisfied, the $\mathbb{F}^{(B_1)}$ -semimartingale decomposition of X is, for $t \leq 1$,

$$X_t = X_0 + \int_0^{t \wedge 1} \varphi_s dB_s^{(B_1)} + \int_0^{t \wedge 1} \varphi_s \frac{B_1 - B_s}{1-s} ds. \quad (3.7)$$

This is an example where some \mathbb{F} -martingales are $\mathbb{F}^{(B_1)}$ -semimartingales, but not all of them.

Note that, in a Brownian filtration, Yor's criterion [?, Section 4.3] is more general than Jacod's condition.

Application: Consider a financial market with null interest rate and risky asset $dS_t = S_t(bdt + \sigma dB_t)$, $S_0 = x$, driven by a Brownian motion B and $\zeta = S_T$. The arbitrage is obvious (no need of mathematics) and the conditional density does not exist on $[0, T]$. If one takes $\zeta = S_T + \epsilon$ where ϵ is a discrete random variable, independent of \mathbb{F} , Jacod's absolute continuity assumption is satisfied (see Amendinger et al. [?]).

4 Progressive enlargement

Here \mathbb{F} is a given filtration, τ a *finite* random time and \mathbb{G} the progressive enlargement: roughly speaking, $\mathcal{G}_t = \mathcal{F}_t$ on $t < \tau$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau)$ after τ .

There are typically two cases: before τ and after τ . Before τ , there is no new information, except τ has not yet occurred. After τ , the time when τ has occurred is known. It is easy to illustrate with a "financial" example. Let S be the price of a risky asset (e.g., a Black and Scholes dynamic, and assume zero interest rate. Let $\tau = \inf\{t : S_t = \sup_{u \leq T} S_u\}$. If an agent has access to the progressive enlargement: before τ she will buy the stock, say at time 0 at price S_0 , and wait till τ , when she will sell the stock making arbitrage. She can also realize an arbitrage after τ : at time τ , she short sells the stock at price S_τ and delivers it at price $S_t < S_\tau$ at any time t after τ . This kind of random time is called an honest time (see below).

The \mathbb{F} -dual optional projection A° of A is the optional process such that for any non-negative bounded \mathbb{F} -optional process Y such that Y_τ is integrable

$$\mathbb{E}[Y_\tau] = \mathbb{E} \left[\int_{[0, \infty)} Y_s dA_s^\circ \right].$$

The \mathbb{F} -dual predictable projection A^p of A is the \mathbb{F} -predictable process such that for any non-negative bounded \mathbb{F} -predictable process Y , such that Y_τ is integrable,

$$\mathbb{E}[Y_\tau] = \mathbb{E} \left[\int_{[0, \infty)} Y_s dA_s^p \right]. \quad (4.1)$$

Two processes are important: the optional projection of $1 - A$, denoted Z , and the optional projection of $1 - A_-$, denoted \tilde{Z} , i.e., $Z = {}^o(1 - A)$, and $\tilde{Z} = {}^o(1 - A_-)$.

Comment 4.1 One can prove (see, e.g., [?, Proposition 1.4]) that $Z = m - A^o = M - A^p$, where m and M are \mathbb{F} -martingales and that $\tilde{Z} = m - A_-^o$. The decomposition $Z = M - A^p$ is the Doob-Meyer decomposition of Z ,

Note that

$$Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t), \quad \tilde{Z}_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t). \quad (4.2)$$

Defining Z as in (4.2) can create some difficulties. Indeed the equality is valid a.s. for any t , and (except if Z is continuous) prevents us for defining the process Z (the union of negligible sets can fail to be negligible)

4.1 Before τ

Lemma 4.2 For any \mathcal{G}_t -measurable random variable $Y_t^{\mathbb{G}}$, there exists an \mathcal{F}_t -measurable random variable Y such that $Y_t^{\mathbb{G}} \mathbb{1}_{\{t < \tau\}} = Y_t \mathbb{1}_{\{t < \tau\}}$. If X_T is integrable and \mathcal{F}_T -measurable, one has

$$\mathbb{E}[X_T \mathbb{1}_{\{T < \tau\}} | \mathcal{G}_t] = \mathbb{1}_{\{t < \tau\}} \frac{\mathbb{E}[X_T Z_T | \mathcal{F}_t]}{Z_t}.$$

PROOF: By definition of \mathbb{G} , the existence of Y is obvious. The uniqueness is not granted. The second assertion follows from the first, taking conditional expectation with respect to \mathcal{F}_t the equality $\mathbb{E}[X_T \mathbb{1}_{\{T < \tau\}} | \mathcal{G}_t] = \mathbb{1}_{\{t < \tau\}} Y_t$. Note that $Z > 0$ on $\{t < \tau\}$. See Elliott et al. [?, Section 3.1]. \square

The \mathbb{G} -predictable processes can be described in terms of a family of \mathbb{F} -predictable processes:

Lemma 4.3 For any \mathbb{G} -predictable bounded process $Y^{\mathbb{G}}$, there exists a bounded \mathbb{F} -predictable process Y and a map $y : \mathbb{R}^+ \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$, which is $\mathcal{B}(\mathbb{R}^+) \times \mathcal{P}(\mathbb{F})$ -measurable and bounded such that $Y_t^{\mathbb{G}} = Y_t \mathbb{1}_{\{t \leq \tau\}} + y(t, \tau) \mathbb{1}_{\{\tau < t\}}$. (See Jeulin [?, Lemma 4.4].)

The \mathbb{G} -compensator of A is the \mathbb{G} -predictable increasing process $\Lambda^{\mathbb{G}}$ such that

$$\tilde{M} := A - \Lambda^{\mathbb{G}} \quad (4.3)$$

is a \mathbb{G} -martingale; this process is flat after τ (i.e., $\Lambda_{t \wedge \tau}^{\mathbb{G}} = \Lambda_t^{\mathbb{G}}$). From Lemma 4.3, there exists an \mathbb{F} -predictable increasing process Λ such that $\Lambda_t^{\mathbb{G}} = \Lambda_{t \wedge \tau}, \forall t \geq 0$. Furthermore, $\Lambda_t^{\mathbb{G}} \mathbb{1}_{\{t \leq \tau\}} = \mathbb{1}_{\{t \leq \tau\}} \int_0^t \frac{dA_s^p}{Z_{s-}}$ (see, e.g., Proposition 2.15, page 37 in [?]). The process Λ is not uniquely defined after τ (except if $Z_- > 0$) and, hereafter, we choose

$$d\Lambda_t = \frac{dA_t^p}{Z_{t-}} \mathbb{1}_{\{Z_{t-} > 0\}}, \quad \forall t \geq 0, \quad \Lambda_0 = 0. \quad (4.4)$$

As an application of the above and by definition of dual projections, we obtain the following result (see, e.g., Jeanblanc & Li [?]) which is useful for pricing defaultable claims:

For any bounded \mathbb{F} -predictable process K ,

$$\mathbb{E}[K_\tau \mathbb{1}_{\{\tau \leq T\}} | \mathcal{G}_t] = K_\tau \mathbb{1}_{\{\tau < t\}} + \mathbb{1}_{\{\tau \geq t\}} \frac{\mathbb{E}[\int_t^T K_u dA_u^p | \mathcal{F}_t]}{Z_t}.$$

For any bounded \mathbb{F} -optional process K ,

$$\mathbb{E}[K_\tau \mathbb{1}_{\{\tau \leq T\}} | \mathcal{G}_t] = K_\tau \mathbb{1}_{\{\tau \leq t\}} + \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}[\int_t^T K_u dA_u^o | \mathcal{F}_t]}{Z_t}.$$

Lemma 4.4 *Under the two assumptions (A) and (C), any (càdlàg) (\mathbb{P}, \mathbb{F}) -local martingale X stopped at time τ is a (\mathbb{P}, \mathbb{G}) -semimartingale with decomposition*

$$X_{t \wedge \tau} = X_t^{\mathbb{G}} + \int_0^{t \wedge \tau} \frac{d\langle X, M \rangle_s}{Z_{s-}}$$

where $X^{\mathbb{G}}$ is a (\mathbb{P}, \mathbb{G}) -local martingale. Here M is the martingale part in the Doob-Meyer decomposition of Z .

PROOF: Let $Y_s^{\mathbb{G}}$ be a \mathcal{G}_s -measurable random variable. There exists an \mathcal{F}_s -measurable random variable y_s such that $Y_s^{\mathbb{G}} \mathbb{1}_{\{s < \tau\}} = y_s \mathbb{1}_{\{s < \tau\}}$, hence, if X is an \mathbb{F} -martingale, for $s < t$,

$$\begin{aligned} \mathbb{E}(Y_s^{\mathbb{G}}(X_{t \wedge \tau} - X_{s \wedge \tau})) &= \mathbb{E}(Y_s^{\mathbb{G}} \mathbb{1}_{\{s < \tau\}}(X_{t \wedge \tau} - X_{s \wedge \tau})) \\ &= \mathbb{E}(y_s \mathbb{1}_{\{s < \tau\}}(X_{t \wedge \tau} - X_{s \wedge \tau})) \\ &= \mathbb{E}(y_s (\mathbb{1}_{\{s < \tau \leq t\}}(X_\tau - X_s) + \mathbb{1}_{\{t < \tau\}}(X_t - X_s))) . \end{aligned}$$

From the definition of Z and (4.1),

$$\mathbb{E}(y_s \mathbb{1}_{\{s < \tau \leq t\}} X_\tau) = -\mathbb{E}\left(y_s \int_s^t X_u dZ_u\right).$$

From the integration by parts formula (taking into account the continuity of Z and X)

$$\int_s^t X_u dZ_u = -X_s Z_s + Z_t X_t - \int_s^t Z_u dX_u - \langle X, Z \rangle_t + \langle X, Z \rangle_s.$$

We have also

$$\begin{aligned} \mathbb{E}(y_s \mathbb{1}_{\{s < \tau \leq t\}} X_s) &= \mathbb{E}(y_s X_s (Z_s - Z_t)) \\ \mathbb{E}(y_s \mathbb{1}_{\{t < \tau\}} (X_t - X_s)) &= \mathbb{E}(y_s Z_t (X_t - X_s)) \end{aligned}$$

hence, from the martingale property of X ,

$$\begin{aligned} \mathbb{E}(Y_s^{\mathbb{G}}(X_{t \wedge \tau} - X_{s \wedge \tau})) &= \mathbb{E}(y_s (\langle X, M \rangle_t - \langle X, M \rangle_s)) \\ &= \mathbb{E}\left(y_s \int_s^t \frac{d\langle X, M \rangle_u}{Z_u} Z_u\right) = \mathbb{E}\left(y_s \int_s^t \frac{d\langle X, M \rangle_u}{Z_u} \mathbb{E}(\mathbb{1}_{\{u < \tau\}} | \mathcal{F}_u)\right) \\ &= \mathbb{E}\left(y_s \int_s^t \frac{d\langle X, M \rangle_u}{Z_u^\tau} \mathbb{1}_{\{u < \tau\}}\right) = \mathbb{E}\left(y_s \int_{s \wedge \tau}^{t \wedge \tau} \frac{d\langle X, M \rangle_u}{Z_u}\right). \end{aligned}$$

The result follows. \square

The general case is more delicate. See, e.g., [?, Section 76] or [?, Theorem 5.1].

Theorem 4.5 *Every càdlàg \mathbb{F} -local martingale X stopped at time τ is a special \mathbb{G} -semimartingale with the canonical decomposition*

$$X_t^\tau = X_t^{\mathbb{G}} + \int_0^{t \wedge \tau} \frac{d\langle X, m \rangle_s}{Z_{s-}}, \quad (4.5)$$

where $X^{\mathbb{G}}$ is a \mathbb{G} -local martingale and m is as defined in Comments 4.3.

Arbitrages before τ : Introduce $\tilde{R} := R\mathbb{1}_{\{\tilde{Z}_R=0 < Z_{R-}\}} + \infty\mathbb{1}_{\{\tilde{Z}_R=0 < Z_{R-}\}^c}$, where $R := \inf\{t : Z_t = 0\}$. The following conditions are equivalent.

- (1) The \mathbb{F} -stopping time \tilde{R} is infinite.
- (2) For any \mathbb{F} -local martingale X , there exists a non-negative \mathbb{G} -local martingale ζ such that $X^\tau\zeta$ is a \mathbb{G} -local martingale, where X^τ is the stopped process (non arbitrage of the first kind). See [?, Theorem 5.46] for a proof.

Arbitrages under the (\mathcal{E}) hypothesis: Under the (\mathcal{E}) hypothesis, if discounted prices are (\mathbb{P}, \mathbb{F}) -martingales, \mathbb{P}^* (defined in (3.1)) is an equivalent martingale measure on \mathbb{G} . Otherwise, if there exists an equivalent martingale measure \mathbb{Q} for \mathbb{F} -adapted discounted prices, \mathbb{Q}^* defined in (3.3) is an equivalent martingale measure in \mathbb{G} .

4.2 Some facts on the predictable representation property

We assume that the predictable representation property holds in the filtration \mathbb{F} , i.e., there exists an \mathbb{F} -local martingale X such that every \mathbb{F} -local martingale Y can be represented as $Y_t = Y_0 + \int_0^t \varphi_s dX_s$ for some $\varphi \in \mathcal{P}(\mathbb{F})$

Under some conditions, the predictable representation property propagates to \mathbb{G} . For example, (see Fontana [?]) under the (\mathcal{J}) -hypothesis $(X^\mathbb{G}, \tilde{M})$ has the predictable representation property in \mathbb{G} where $X^\mathbb{G}$ is the \mathbb{G} martingale part of the \mathbb{G} -semimartingale X and \tilde{M} is the \mathbb{G} -martingale defined in (4.3).

We refer to [?, Section 5.6] for more information.

We now study the relationship between \mathbb{G} -martingales and \mathbb{F} -martingales obtained in [?, proposition 2.2]

Proposition 4.6 *Under the (\mathcal{J}) -hypothesis, a \mathbb{G} -optional process of the form $Y^\mathbb{G} := \tilde{y}\mathbb{1}_{[0, \tau[} + \hat{y}(\tau)\mathbb{1}_{[\tau, \infty[}$ where \tilde{y} and $\hat{y}(u)$ are \mathbb{F} -optional processes, is a \mathbb{G} -martingale if and only if the following two conditions are satisfied*

- a) for η -a.e u , $(\hat{y}_t(u)p_t(u), t \geq u)$ is an \mathbb{F} -martingale;
- b) the process y is an \mathbb{F} -martingale, where

$$y_t := \mathbb{E}(Y_t | \mathcal{F}_t) = \tilde{y}_t Z_t + \int_0^t \hat{y}_t(u) p_t(u) \eta(du). \quad (4.6)$$

Under the (\mathcal{J}) -hypothesis, if the function $(\omega, u) \rightarrow X(\omega, u)$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable and bounded, then

$$\mathbb{E}[X(\tau) | \mathcal{G}_t] = \mathbb{1}_{\{t < \tau\}} \frac{1}{Z_t} \mathbb{E} \left[\int_{]t, \infty]} X(u) p_T(u) \eta(du) | \mathcal{F}_t \right] + \mathbb{1}_{\{\tau \leq t\}} \frac{1}{p_t(\tau)} \mathbb{E}[X(u) p_T(u) | \mathcal{F}_t], \text{ for } t \leq T$$

and

$$\mathbb{E}[X(\tau) | \mathcal{G}_t] = \mathbb{1}_{\{t < \tau\}} \frac{1}{Z_t} \int_{]t, \infty]} X(u) p_t(u) \eta(du) + \mathbb{1}_{\{\tau \leq t\}} X(\tau), \text{ for } T < t$$

(see e.g., [?, lemma 5.24]).

4.3 Immersion

Immersion is easily characterized in a progressive enlargement setting: \mathbb{F} is immersed in \mathbb{G} if and only if

$$\mathbb{P}(\tau > t | \mathcal{F}_t) = \mathbb{P}(\tau > t | \mathcal{F}_\infty)$$

(see, e.g., [?, lemma 3.8]). This implies that Z is decreasing and $Z = 1 - A^\circ$ (see [?, Proposition 3.9]).

Many models of “default risk” are constructed as follows (see Gueye & Jeanblanc [?]). Let \mathbb{F} be a given filtration and K a càdlàg increasing \mathbb{F} -adapted process, and define

$$\tau = \inf\{t : K_t \geq \Theta\}$$

where Θ is a random variable independent of \mathbb{F} with unit exponential law. Then $\mathbb{P}(\tau > t | \mathcal{F}_t) = e^{-K_t}$, and immersion holds between \mathbb{F} and \mathbb{G} . If K is continuous, τ avoids \mathbb{F} stopping times. If not, the jump times of K are the \mathbb{F} -stopping times not avoided by τ .

Comment 4.7 Let us point out a “technical” difficulty. Assume that K has no jumps at constant time, which implies $\mathbb{P}(\tau = t) = 0, \forall t > 0$. This does not imply that $Z = \tilde{Z}$ (the equality meaning that the two processes are indistinguishable). Indeed, $Z = 1 - A^o$ whereas $\tilde{Z} = 1 - A^o_-$.

We have underlined that immersion is not stable under change of probability. However, let us point out that if a price process S is given on $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and the interest rate is null, if the corresponding financial market satisfies NFLVR, and if the $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P}, S)$ financial market satisfies NFLVR, then one can choose a \mathbb{G} -equivalent martingale measure such that immersion holds. If the market $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ is complete and discounted prices are (\mathbb{P}, \mathbb{F}) -martingales, under any \mathbb{G} -equivalent martingale measure immersion holds.

4.4 Honest times

Honest times were introduced by Barlow [?].

Definition 4.8 A random time τ is an \mathbb{F} -honest time if, for every $t > 0$, there exists an \mathcal{F}_t -measurable random variable τ_t such that $\tau = \tau_t$ on $\{\tau < t\}$.

Fontana et al. [?] assume that τ is honest, and the following conditions (a), (b) and (c)

- (a) The restricted financial market $(\Omega, \mathbb{F}, \mathbb{P}, S)$ satisfies NFLVR.
- (b) The random time τ avoids all \mathbb{F} -stopping times (Condition **(A)**).
- (c) The martingale part of the semimartingale S is a continuous \mathbb{F} -local martingale $M^S = (M_t^S)_{t \geq 0}$ which has the \mathbb{F} -predictable representation property in the filtration \mathbb{F} .

Then, they prove that the $(\mathbb{F}, \mathbb{P}, S)$ market is complete and

- (1) NUPBR holds in the enlarged market on the time horizon $[0, \tau]$,
- (2) there exists an explicit arbitrage opportunity in the enlarged market on the time horizon $[0, \tau]$ and on the interval $[\tau, \vartheta]$ for an explicit \mathbb{G} stopping time ϑ (see [?, Theorem 3],
- (3) NFLVR fails to hold in the enlarged market on the time horizon $[0, \tau]$,
- (4) NUPBR fails to hold in the enlarged financial market on the global time horizon $[0, \infty]$.

In the case of honest time, for a (\mathbb{P}, \mathbb{F}) -local martingale X

$$X_t = X_t^{\mathbb{G}} + \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\langle X, m \rangle_s - \int_{\tau}^{\tau \vee t} \frac{1}{1 - Z_{s-}} d\langle X, m \rangle_s,$$

where $X^{\mathbb{G}}$ is a (\mathbb{P}, \mathbb{G}) -local martingale.

Any \mathbb{G} -optional process can be written

$$Y = L\mathbb{1}_{[0, \tau]} + J\mathbb{1}_{[\tau, \infty]} + K\mathbb{1}_{[\tau, \infty[},$$

where L and K are \mathbb{F} -optional processes and J is an \mathbb{F} -progressively measurable process.

Example 4.9 We recall Barlow’s counterexample given in [?, p. 319] to show that a \mathbb{G} -optional process cannot always be decomposed as $L\mathbb{1}_{[0,\tau]} + K\mathbb{1}_{[\tau,\infty]}$, where L and K are \mathbb{F} -optional processes. Let B be a Brownian motion, \mathbb{F} its natural filtration $\vartheta = \inf\{t : |B_t| = 1\}$, $\tau = \sup\{t \leq \vartheta : B_t = 0\}$ and \mathbb{G} the progressive enlargement of \mathbb{F} with τ . The process X defined as $X_t = \mathbb{1}_{\{t \geq \tau\}} \text{sgn}(B_{\vartheta})$ is right-continuous and \mathbb{G} -adapted, hence \mathbb{G} -optional. Moreover X is a \mathbb{G} -martingale. Obviously, if the pair (L, K) exists, then $L = 0$ and one can choose K to be \mathbb{F} -predictable, since $\mathcal{O}(\mathbb{F}) = \mathcal{P}(\mathbb{F})$. Then $\Delta X_{\tau} = K_{\tau}$ would be $\mathcal{G}_{\tau-}$ -measurable, which contradicts the \mathbb{G} -martingale property of X .

Lemma 4.10 *Assume that $(\Omega, \mathbb{F}, \mathbb{P}, S)$ is a complete market satisfying NFLVR on the time horizon $[0, T]$. If τ is a finite honest time which satisfies **(A)**, there are classical arbitrages before τ for $(\Omega, \mathbb{G}, \mathbb{P}, S)$ and classical arbitrages after τ for $(\Omega, \mathbb{G}, \mathbb{P}, S)$.*

PROOF: See [?, Section 5.8.1] and the examples in [?]. □

5 Information drift

Assume that B is an \mathbb{F} -Brownian motion and $B_t^{\mathbb{K}} = B_t + \int_0^t k_s ds$ a \mathbb{K} -Brownian motion where $\mathbb{F} \subset \mathbb{K}$. When S is the \mathbb{F} -adapted price of an asset, one has (in the Brownian case)

$$\begin{aligned} dS_t &= S_t(b_t dt + \sigma_t dB_t) \\ &= S_t((b_t \sigma_t + k_t) dt + \sigma_t dB_t^{\mathbb{K}}). \end{aligned}$$

The quantity k is called the information drift. See [?] or [?] for more information.

In the case of portfolio optimisation, when the interest rate is null, denoting by X the wealth associated to a self financing portfolio, i.e., $dX_t = \pi_t dS_t$, $X_0 = x$, one computes easily

$$\sup_{\pi \in \mathbb{F}} \mathbb{E}[\ln(X_T)] = \ln x, \quad \sup_{\pi \in \mathbb{F}^{\mathbb{K}}} \mathbb{E}[\ln(X_T)] = \ln x + \mathbb{E}\left[\int_0^T k_t^2 dt\right].$$

6 Conclusion and open problems

We hope to have given a presentation of enlargement problems. As we mentioned at the beginning, many problems remain to be solved. For example, solve an optimal stopping problem in an enlarged filtration, compare the solution of a BSDE in two filtrations, give the \mathbb{G} -decomposition of any martingale when τ is a random time in a Poisson filtration (see [?] for some examples). It would be interesting to provide some tests to detect insider trading (as in [?]). The reverse problem of shrinkage is to give the \mathbb{F} -decomposition of the optional projection of a \mathbb{K} -semimartingale and has no general solution.

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