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# An Immersed Boundary Method by $\phi$ -FEM approach to solve the heat equation

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## Abstract

Thanks to a finite element method, we solve numerically parabolic partial differential equations on complex domains by avoiding the mesh generation, using a background regular mesh, not fitting exactly the domain and the real boundary of the domain like immersed boundary methods. Our technique follows the  $\phi$ -FEM paradigm which suppose that the domain is given by a level-set function. The aim of the present paper is to prove *a priori* error estimates in  $L^2(H^1)$  and  $L^\infty(L^2)$  norms for an implicit Euler discretization in time and to give numerical illustrations to highlight the performances of our technique. The advantage of our approach is that it combines optimal convergence accuracy, easy implementation process and fastness.

**Keywords:** Numerical analysis, finite element method, heat equation, fictitious domain, level-set, Dirichlet conditions

**2000 MSC:** 65M85, 65M60, 35K05, 65M12

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## 1. Introduction

The classical finite element method needs a computational mesh fitting the boundary of the physical domain. In some applications in engineering or bio-mechanics, the construction of such meshes may be very time consuming, or even impossible. Alternative approaches, such as Immersed Boundary Methods (IBM) (see e.g. [1] for a review), can work on unfitted meshes but are usually not very precise. More recent variants, such as CutFEM [2] or SBM [3], demonstrate optimal convergence orders, but are less straightforward to implement than the original IBM. In particular, CutFEM needs special quadrature rules on the cells cut by the boundary.

A new IBM-like method, named  $\phi$ -FEM, combining the optimal convergence and the ease of implementation, was recently proposed in [4, 5]. Initially developed for stationary elliptic PDEs, it has been extended in [6] to a broader class of equations, including the time-dependent parabolic problems, without any theoretical analysis. The goal of the present note is to provide such an analysis in the case of the Heat-Dirichlet problem

$$\partial_t u - \Delta u = f \text{ in } \Omega \times (0, T), \quad u = 0 \text{ on } \Gamma \times (0, T), \quad u(\cdot, 0) = u^0 \text{ in } \Omega, \quad (1)$$

where  $T > 0$ ,  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$  is a bounded domain with a smooth boundary  $\Gamma$  given by a level-set function on  $\mathbb{R}^d$

$$\Omega := \{\phi < 0\} \quad \text{and} \quad \Gamma := \{\phi = 0\}. \quad (2)$$

The Dirichlet boundary conditions are taken into account in our scheme via a product with the level-set function  $\phi$ . An appropriate stabilization is introduced to the finite element discretization in order to obtain well-posed problems. A somewhat unexpected feature of this stabilization is that it works under the constraint on the steps in time and space

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of the type  $\Delta t \geq ch^2$ . This does not affect the practical interest of the scheme since it is normally intended to be used in the regime  $\Delta t \sim h$ . We shall provide *a priori* error estimates for this scheme in  $L^2(H^1)$  and  $L^\infty(L^2)$  norms of similar orders as for the standard FEM, cf. [7]. Moreover, the numerical simulations show that our approach can be considerably faster.

## 2. Definitions, assumptions, description of $\phi$ -FEM and the main result.

We assume that  $\Omega$  lies inside a box  $\mathcal{O} \subset \mathbb{R}^d$  and that  $\Omega$  and  $\Gamma$  are given by (2). The box  $\mathcal{O}$  is covered by a simple quasi-uniform simplicial (typically Cartesian) background mesh denoted by  $\mathcal{T}_h^{\mathcal{O}}$ . We introduce the active computational mesh  $\mathcal{T}_h := \{T \in \mathcal{T}_h^{\mathcal{O}} : T \cap \{\phi_h < 0\} \neq \emptyset\}$  on  $\Omega_h = (\cup_{T \in \mathcal{T}_h} T)^{\circ}$ , the subdomain of  $\mathcal{O}$  composed of mesh cells intersecting  $\Omega$ , cf. Fig. 1. Here,  $\phi_h$  is a piecewise polynomial interpolation of  $\phi$  on  $\mathcal{T}_h^{\mathcal{O}}$ . We shall also need a submesh  $\mathcal{T}_h^\Gamma$ , containing the elements of  $\mathcal{T}_h$  that cut the approximate boundary  $\Gamma_h := \{\phi_h = 0\}$ :  $\mathcal{T}_h^\Gamma = \{T \in \mathcal{T}_h : T \cap \Gamma_h \neq \emptyset\}$ . Finally, we denote by  $\mathcal{F}_h^\Gamma$  the set of the internal facets  $E$  of mesh  $\mathcal{T}_h$  belonging to the set  $\mathcal{T}_h^\Gamma$ ,  $\mathcal{F}_h^\Gamma := \{E \text{ (internal facet of } \mathcal{T}_h) \text{ such that } \exists T \in \mathcal{T}_h : T \cap \Gamma_h \neq \emptyset \text{ and } E \in \partial T\}$ .

Introduce a uniform partition of  $[0, T]$  into time steps  $0 = t_0 < t_1 < \dots < t_N = T$  with  $t_n = n\Delta t$ . The basic idea of  $\phi$ -FEM is to introduce the new unknown  $w = w(x, t)$  and to set  $u = \phi w$  so that the Dirichlet condition  $u = 0$  is automatically satisfied on  $\Gamma$  since  $\phi$  vanishes there. Using an implicit Euler scheme to discretize (1) in time and denoting  $f^n(\cdot) = f(\cdot, t_n)$ , we get the following discretization in time: given  $u^n = \phi w^n$  find  $u^{n+1} = \phi w^{n+1}$  such that

$$\Delta t^{-1}(\phi w^{n+1} - \phi w^n) - \Delta(\phi w^{n+1}) = f^{n+1}. \quad (3)$$

To discretize in space, we introduce the finite element space of degree  $k$  on  $\Omega_h$ :  $V_h^{(k)} = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_k(T), \forall T \in \mathcal{T}_h\}$  for some  $k \geq 1$ . Supposing that  $f$  and  $u^0$  are actually well defined on  $\Omega_h$  (rather than on  $\Omega$  only), we can finally introduce the  $\phi$ -FEM scheme for (1) as follows: find  $w_h^{n+1} \in V_h^{(k)}$ ,  $n = 0, 1, \dots, N-1$  such that for all  $v_h \in V_h^{(k)}$

$$\begin{aligned} & \int_{\Omega_h} \frac{\phi_h w_h^{n+1}}{\Delta t} \phi_h v_h + \int_{\Omega_h} \nabla(\phi_h w_h^{n+1}) \cdot \nabla(\phi_h v_h) - \int_{\partial\Omega_h} \frac{\partial}{\partial n}(\phi_h w_h^{n+1}) \phi_h v_h + \sigma h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E \left[ \frac{\partial(\phi_h w_h^{n+1})}{\partial n} \right] \left[ \frac{\partial(\phi_h v_h)}{\partial n} \right] \\ & - \sigma h^2 \sum_{K \in \mathcal{T}_h^\Gamma} \int_K \left( \frac{\phi_h w_h^{n+1}}{\Delta t} - \Delta(\phi_h w_h^{n+1}) \right) \Delta(\phi_h v_h) = \int_{\Omega_h} \left( \frac{u_h^n}{\Delta t} + f^{n+1} \right) \phi_h v_h - \sigma h^2 \sum_{K \in \mathcal{T}_h^\Gamma} \int_K \left( \frac{u_h^n}{\Delta t} + f^{n+1} \right) \Delta(\phi_h v_h) \end{aligned} \quad (4)$$

with  $u_h^n = \phi_h w_h^n$  for  $n \geq 1$  and  $u_h^0 \in V_h^{(k)}$  an interpolant of  $u^0$ .  $\phi_h$  is the piecewise polynomial interpolation of  $\phi$  in  $V_h^{(k)}$ . This scheme contains two stabilization terms: the ghost stabilization (the sum of the facets in  $\mathcal{F}_h^\Gamma$ ) as in [8], and a least-square stabilization that reinforces (3) on the mesh cells in  $\mathcal{T}_h^\Gamma$ .

**Hypothesis 1.** We assume the regularity hypothesis on the mesh (quasi-uniform is sufficient) and on  $\Gamma$  of [4, Ass. 1-2] (the regularity of  $\Gamma$  is related to the one of  $\phi$  which needs to be  $C^k$ ). We assume moreover  $\Omega \subset \Omega_h$ .

**Remark 1.** Our approach can be easily generalized to non-homogeneous Dirichlet boundary conditions  $u = u_D$  on  $\Gamma \times (0, T)$ . We can pose then  $u_h^n = \phi_h w_h^n + I_h u_g(\cdot, t_n)$  where  $u_g$  is some lifting of  $u_D$  from  $\Gamma$  to  $\Omega_h$  and  $I_h$  stands for an interpolation by finite elements. Scheme (4) should be then modified accordingly, adding some terms depending on  $u_g$  in the right-hand side.

**Theorem 1.** Assume Hypothesis 1,  $f \in H^1(0, T; H^{k-1}(\Omega_h))$  and  $u \in H^2(0, T; H^{k-1}(\Omega))$  being the exact solution to (1),  $u^n(\cdot) = u(\cdot, t_n)$  and  $w_h^n$  be the solution to (4) for  $n = 1, \dots, N$ . For  $\sigma$  large enough, there exist the space-mesh independent constants  $c, C > 0$  (depending on  $T$ ) such that if  $\Delta t \geq ch^2$  then

$$\left( \sum_{n=0}^N \Delta t \|u^n - \phi_h w_h^n\|_{H^1(\Omega)}^2 \right)^{\frac{1}{2}} \leq C \|u^0 - u_h^0\|_{L^2(\Omega_h)} + C(h^k + \Delta t) \left( \|u\|_{H^2(0, T; H^{k-1}(\Omega))} + \|f\|_{H^1(0, T; H^{k-1}(\Omega_h))} \right)$$

and

$$\max_{1 \leq n \leq N} \|u^n - \phi_h w_h^n\|_{L^2(\Omega)} \leq C \|u^0 - u_h^0\|_{L^2(\Omega_h)} + C(h^{k+\frac{1}{2}} + \Delta t) \left( \|u\|_{H^2(0, T; H^{k-1}(\Omega))} + \|f\|_{H^1(0, T; H^{k-1}(\Omega_h))} \right).$$

Note that the hypotheses on  $f$  and  $u$  in the Theorem above imply  $u \in H^1(0, T; H^{k+1}(\Omega))$  thanks to the elliptic regularity at each time  $t$ . However, it would not be sufficient to demand the regularity  $H^2(0, T; H^{k-1}(\Omega)) \cap H^1(0, T; H^{k+1}(\Omega))$  from  $u$  alone. Indeed, this would imply  $f \in H^1(0, T; H^{k-1}(\Omega))$  but, in  $\phi$ -FEM,  $f$  should be defined and sufficiently regular on  $\Omega_h$  rather than on  $\Omega$ . The other way round, the Theorem's hypothesis on  $f$ , combined with a compatibility condition between  $f$  and  $u^0$ , would imply the required regularity of  $u$  if  $k = 1$ , cf. [9, Theorems 5 and 6, Chapter 7.1]. However, one would need to control higher derivatives in time of  $f$  in order to assure the regularity of  $u$  for  $k > 1$ . In summary, we need indeed to assume enough regularity of both  $u$  and  $f$ .

Before the proof of Theorem 1, we recall the results from [4] about  $\phi$ -FEM for the Poisson equation with Dirichlet boundary conditions.

**Lemma 2** (cf. [4, Lemma 3.7]). *Consider the bilinear form*

$$a_h(u, v) = \int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\partial\Omega_h} \frac{\partial u}{\partial n} v + \sigma h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E \left[ \frac{\partial u}{\partial n} \right] \left[ \frac{\partial v}{\partial n} \right] + \sum_{K \in \mathcal{T}_h^\Gamma} \sigma h^2 \int_K \Delta u \Delta v.$$

*Provided  $\sigma$  is chosen big enough, there exists an  $h$ -independent constant  $\alpha > 0$  such that*

$$a_h(\phi_h v_h, \phi_h v_h) \geq \alpha |\phi_h v_h|_{H^1(\Omega_h)}^2, \quad \forall v_h \in V_h^{(k)}.$$

**Lemma 3** (cf. [4, Theorem 2.3]). *For any  $f \in H^{k-1}(\Omega_h)$ , let  $w_h \in V_h^{(k)}$  be the solution to*

$$a_h(\phi_h w_h, \phi_h v_h) = \int_{\Omega_h} f \phi_h v_h - \sigma h^2 \sum_{K \in \mathcal{T}_h^\Gamma} \int_K f \Delta(\phi_h v_h)$$

*and  $u \in H^{k+1}(\Omega)$  be the solution to*

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma$$

*extended to  $\tilde{u} \in H^{k+1}(\Omega_h)$  so that  $u = \tilde{u}$  on  $\Omega$ . Provided  $\sigma$  is chosen big enough, there exists an  $h$ -independent constant  $C > 0$  such that*

$$|\tilde{u} - \phi_h w_h|_{H^1(\Omega_h)} \leq Ch^k \|f\|_{H^{k-1}(\Omega_h)} \quad \text{and} \quad \|\tilde{u} - \phi_h w_h\|_{L^2(\Omega_h)} \leq Ch^{k+\frac{1}{2}} \|f\|_{H^{k-1}(\Omega_h)}.$$

**Remark 2.** This result is proven in [4] under the more stringent assumption  $f \in H^k(\Omega_h)$  which was used to assure  $\tilde{u} \in H^{k+2}(\Omega_h)$  and to provide an interpolation error of  $\tilde{u}$  by a product  $\phi_h w_h$ . However, in [10, Lemma 6] we have proven a better interpolation estimate  $\|\tilde{u} - \phi_h I_h w\|_{H^s(\Omega_h)} \leq Ch^{k+1-s} \|f\|_{H^{k-1}(\Omega_h)}$  ( $s = 0, 1$ ) for  $\tilde{u} = \phi w$  and the Scott Zhang interpolant  $I_h$ . Thus,  $f \in H^{k-1}(\Omega_h)$  is actually sufficient.

**Lemma 4** (see e.g. [4, Lemma 3.4] together of the Poincare inequality). *There exists an  $h$ -independent constant  $C > 0$  such that for all  $v_h \in V_h^{(k)}$*

$$\|\phi_h v_h\|_{L^2(\Omega_h)} \leq C_P |\phi_h v_h|_{H^1(\Omega_h)}.$$

**PROOF OF THEOREM 1.** Assuming Hypothesis 1, there exists  $\tilde{u} \in H^2(0, T; H^{k-1}(\Omega_h))$ , an extension of  $u$  to  $\Omega_h$ , such that

$$\|\tilde{u}\|_{H^2(0, T; H^{k-1}(\Omega_h))} \leq C \|u\|_{H^2(0, T; H^{k-1}(\Omega))} \quad (5)$$

Let  $w_h^n$  be the solution to our scheme, which we rewrite as

$$\begin{aligned} \int_{\Omega_h} \phi_h \frac{w_h^{n+1} - w_h^n}{\Delta t} \phi_h v_h + a_h(\phi_h w_h^{n+1}, \phi_h v_h) - \sum_{T \in \mathcal{T}_h^\Gamma} \sigma h^2 \int_T \phi_h \frac{w_h^{n+1} - w_h^n}{\Delta t} \Delta(\phi_h v_h) \\ = \int_{\Omega_h} f^{n+1} \phi_h v_h - \sum_{T \in \mathcal{T}_h^\Gamma} \sigma h^2 \int_T f^{n+1} \Delta(\phi_h v_h) \end{aligned} \quad (6)$$

for  $n \geq 1$  while  $\phi_h w_h^0$  should be replaced with  $u_h^0$  for  $n = 0$ .

For any time  $t \in [0, T]$ , introduce  $\tilde{w}_h(t, \cdot) = \tilde{w}_h \in V_h^{(k)}$ , as in Lemma 3, with  $f$  replaced by  $f - \partial_t \tilde{u}$  evaluated at time  $t$ :

$$a_h(\phi_h \tilde{w}_h, \phi_h v_h) = \int_{\Omega_h} (f - \partial_t \tilde{u}) \phi_h v_h - \sigma h^2 \sum_{K \in \mathcal{T}_h^1} \int_K (f - \partial_t \tilde{u}) \Delta(\phi_h v_h). \quad (7)$$

Let  $\tilde{w}_h^n = \tilde{w}_h(t_n)$  and  $e_h^n := \phi_h(w_h^n - \tilde{w}_h^n)$  for  $n \geq 1$  and  $e_h^0 := u_h^0 - \phi_h \tilde{w}_h^0$ . Taking the difference between (6) and (7) at time  $t_{n+1}$ , we get

$$\begin{aligned} & \int_{\Omega_h} \frac{e_h^{n+1} - e_h^n}{\Delta t} \phi_h v_h + a_h(e_h^{n+1}, \phi_h v_h) - \sum_{T \in \mathcal{T}_h^1} \sigma h^2 \int_T \frac{e_h^{n+1} - e_h^n}{\Delta t} \Delta(\phi_h v_h) \\ &= \int_{\Omega_h} \left( \partial_t \tilde{u}^{n+1} - \phi_h \frac{\tilde{w}_h^{n+1} - \tilde{w}_h^n}{\Delta t} \right) \phi_h v_h - \sum_{T \in \mathcal{T}_h^1} \sigma h^2 \int_T \left( \partial_t \tilde{u}^{n+1} - \phi_h \frac{\tilde{w}_h^{n+1} - \tilde{w}_h^n}{\Delta t} \right) \Delta(\phi_h v_h). \end{aligned}$$

Taking  $v_h = w_h^n - \tilde{w}_h^n$ , i.e.  $\phi_h v_h = e_h^{n+1}$ , applying the equality

$$\|e_h^{n+1}\|_{L^2(\Omega_h)}^2 - (e_h^n, e_h^{n+1})_{L^2(\Omega_h)} = \frac{\|e_h^{n+1}\|_{L^2(\Omega_h)}^2 - \|e_h^n\|_{L^2(\Omega_h)}^2 + \|e_h^{n+1} - e_h^n\|_{L^2(\Omega_h)}^2}{2},$$

and estimating the terms in the RHS by Cauchy-Schwarz and inverse inequalities ( $\|\Delta e_h^{n+1}\|_{L^2(T)} \leq Ch^{-2} \|e_h^{n+1}\|_{L^2(T)}$ ) we deduce that

$$\begin{aligned} & \frac{\|e_h^{n+1}\|_{L^2(\Omega_h)}^2 - \|e_h^n\|_{L^2(\Omega_h)}^2 + \|e_h^{n+1} - e_h^n\|_{L^2(\Omega_h)}^2}{2\Delta t} + \overbrace{a_h(e_h^{n+1}, e_h^{n+1})}^{(I)} - \overbrace{\sigma h^2 \int_{\Omega_h^1} \frac{e_h^{n+1} - e_h^n}{\Delta t} \Delta e_h^{n+1}}^{(II)} \\ & \leq C \underbrace{\left\| \partial_t \tilde{u}^{n+1} - \phi_h \frac{\tilde{w}_h^{n+1} - \tilde{w}_h^n}{\Delta t} \right\|_{L^2(\Omega_h)}}_{(III)} \|e_h^{n+1}\|_{L^2(\Omega_h)}. \quad (8) \end{aligned}$$

Thanks to the coercivity lemma 2, the term (I) can be bounded from below by  $\alpha |e_h^{n+1}|_{H^1(\Omega_h)}^2$ . We now use the Young inequality (with some  $\varepsilon > 0$ ) and the inverse inequality  $\|\Delta e_h^{n+1}\|_{L^2(T)} \leq C_I h^{-1} |e_h^{n+1}|_{H^1(T)}$  to bound the term (II):

$$(I) - (II) \geq \alpha |e_h^{n+1}|_{H^1(\Omega_h)}^2 - \frac{\sigma h^2}{2\varepsilon(\Delta t)^2} \|e_h^{n+1} - e_h^n\|_{L^2(\Omega_h^1)}^2 - \frac{\varepsilon \sigma C_I^2}{2} |e_h^{n+1}|_{H^1(\Omega_h^1)}^2 \geq \frac{3}{4} \alpha |e_h^{n+1}|_{H^1(\Omega_h)}^2 - \frac{1}{2\Delta t} \|e_h^{n+1} - e_h^n\|_{L^2(\Omega_h^1)}^2,$$

where we have chosen  $\varepsilon$  so that  $\varepsilon \sigma C_I^2 / 2 = \alpha / 4$  and then assumed  $\sigma h^2 / (\varepsilon \Delta t) \leq 1$ . This will allow us to control the negative term above by the similar positive term in (8), and leads to the restriction  $\Delta t \geq ch^2$  with  $c = \sigma / \varepsilon$ .

We turn now to the RHS of (8), i.e. term (III). By triangle inequality

$$\left\| \partial_t \tilde{u}^{n+1} - \phi_h \frac{\tilde{w}_h^{n+1} - \tilde{w}_h^n}{\Delta t} \right\|_{L^2(\Omega_h)} \leq \left\| \partial_t \tilde{u}^{n+1} - \frac{\tilde{u}^{n+1} - \tilde{u}^n}{\Delta t} \right\|_{L^2(\Omega_h)} + \left\| \frac{\tilde{u}^{n+1} - \tilde{u}^n}{\Delta t} - \phi_h \frac{\tilde{w}_h^{n+1} - \tilde{w}_h^n}{\Delta t} \right\|_{L^2(\Omega_h)}. \quad (9)$$

By Taylor's theorem with integral remainder

$$\tilde{u}^n(\cdot) = \tilde{u}^{n+1}(\cdot) - \Delta t \partial_t \tilde{u}^{n+1}(\cdot) - \int_{t_n}^{t_{n+1}} \partial_{tt} \tilde{u}(t, \cdot) (t_n - t) dt$$

so that

$$\left\| \partial_t \tilde{u}^{n+1} - \frac{\tilde{u}^{n+1} - \tilde{u}^n}{\Delta t} \right\|_{L^2(\Omega_h)} = \frac{1}{\Delta t} \left\| \int_{t_n}^{t_{n+1}} \partial_{tt} \tilde{u}(t, \cdot) (t_n - t) dt \right\|_{L^2(\Omega_h)} \leq \sqrt{\Delta t} \|\partial_{tt} \tilde{u}\|_{L^2(t_n, t_{n+1}; L^2(\Omega_h))}.$$

Differentiating  $-\Delta u = f - \partial_t u$  and (7) in time, thanks to Lemma 3,

$$\|\partial_t(\tilde{u}(t) - \phi_h \tilde{w}_h)(t)\|_{L^2(\Omega_h)} \leq Ch^{k+\frac{1}{2}} \|(\partial_t f - \partial_{tt} \tilde{u})(t)\|_{H^{k-1}(\Omega_h)}.$$

Thus, for the second term in (9), we get by the last interpolation estimate:

$$\left\| \frac{\tilde{u}^{n+1} - \tilde{u}^n}{\Delta t} - \phi_h \frac{\tilde{w}_h^{n+1} - \tilde{w}_h^n}{\Delta t} \right\|_{L^2(\Omega_h)} = \frac{1}{\Delta t} \left\| \int_{t_n}^{t_{n+1}} \partial_t(\tilde{u}(t, \cdot) - \phi_h \tilde{w}_h(t, \cdot)) dt \right\|_{L^2(\Omega_h)} \leq \frac{Ch^{k+\frac{1}{2}}}{\sqrt{\Delta t}} \|\partial_t f - \partial_{tt} \tilde{u}\|_{L^2(t_n, t_{n+1}; H^{k-1}(\Omega_h))}.$$

Collecting these estimates and applying the Young inequality with some  $\delta > 0$  and Poincaré inequality from Lemma 4, we get

$$(III) \leq \frac{C}{\delta} \left( \Delta t \|\partial_{tt} \tilde{u}\|_{L^2(t_n, t_{n+1}; L^2(\Omega_h))}^2 + \frac{h^{2k+1}}{\Delta t} \|\partial_t f - \partial_{tt} \tilde{u}\|_{L^2(t_n, t_{n+1}; H^{k-1}(\Omega_h))}^2 \right) + \frac{\delta C_P^2}{2} |e_h^{n+1}|_{H^1(\Omega_h)}^2.$$

Using these estimates in (8) and taking  $\delta$  so that  $\delta C_P^2 = \alpha/2$  yields

$$\frac{\|e_h^{n+1}\|_{L^2(\Omega_h)}^2 - \|e_h^n\|_{L^2(\Omega_h)}^2}{2\Delta t} + \frac{\alpha}{2} |e_h^{n+1}|_{H^1(\Omega_h)}^2 \leq C \left( \Delta t \|\partial_{tt} \tilde{u}\|_{L^2(t_n, t_{n+1}; L^2(\Omega_h))}^2 + \frac{h^{2k+1}}{\Delta t} \|\partial_t f - \partial_{tt} \tilde{u}\|_{L^2(t_n, t_{n+1}; H^{k-1}(\Omega_h))}^2 \right).$$

Multiplying this by  $2\Delta t$  and summing on  $n = 0, \dots, N-1$ , we get

$$\|e_h^N\|_{L^2(\Omega_h)}^2 + \alpha \Delta t \sum_{n=1}^N |e_h^n|_{H^1(\Omega_h)}^2 \leq \|e_h^0\|_{L^2(\Omega_h)}^2 + C(\Delta t^2 \|\partial_{tt} \tilde{u}\|_{L^2(0, T; L^2(\Omega_h))}^2 + h^{2k+1} \|\partial_t f - \partial_{tt} \tilde{u}\|_{L^2(0, T; H^{k-1}(\Omega_h))}^2).$$

Thus, observing that the sum above can be stopped at any number  $n \leq N$ , we get

$$\max_{n=1, \dots, N} \|e_h^n\|_{L^2(\Omega_h)} + \left( \Delta t \sum_{n=1}^N |e_h^n|_{H^1(\Omega_h)}^2 \right)^{\frac{1}{2}} \leq C \|e_h^0\|_{L^2(\Omega_h)} + C \left( \Delta t \|\partial_{tt} \tilde{u}\|_{L^2(0, T; L^2(\Omega_h))}^2 + h^{k+\frac{1}{2}} \|\partial_t f - \partial_{tt} \tilde{u}\|_{L^2(0, T; H^{k-1}(\Omega_h))}^2 \right).$$

Combining this with Lemma 3 applied to  $-\Delta u = f - \partial_t u$  in  $\Omega$ , which can be rewritten as

$$\begin{aligned} \max_{n=0, \dots, N} \|\tilde{u}^n - \phi_h \tilde{w}_h^n\|_{L^2(\Omega_h)} &\leq Ch^{k+1/2} \|f - \partial_t \tilde{u}\|_{C([0, T], H^{k-1}(\Omega_h))}, \\ \left( \Delta t \sum_{n=1}^N \|\tilde{u}^n - \phi_h \tilde{w}_h^n\|_{H^1(\Omega_h)}^2 \right)^{\frac{1}{2}} &\leq Ch^k \|f - \partial_t \tilde{u}\|_{C([0, T], H^{k-1}(\Omega_h))} \end{aligned}$$

the regularity of  $\tilde{u}$ , cf. (5) and  $f$ , together with the bound  $\|\cdot\|_{C([0, T], \cdot)} \leq C \|\cdot\|_{H^1(0, T; \cdot)}$  gives the announced result where  $C$  depends on  $T$ .  $\square$

### 3. Simulation

We will now illustrate the convergence of our method on a test case in a domain of Fig. 1 (Left). The level-set function  $\phi$  is the signed distance to the boundary of the domain. Its approximation  $\phi_h$  is numerically evaluated thanks to the distance of the Gauss points to the boundary of the domain. We take the exact manufactured solution  $u(x, y, t) = \exp(x) \sin(2\pi y) \sin(t)$  which gives the right-hand side  $f$  and the initial condition  $u^0$ . The non-homogeneous boundary conditions are taken into account in  $\phi$ -FEM as in Remark 1 via the lifting  $u_g = u(1 + \phi)$ .

The results obtained by  $\phi$ -FEM are represented at Fig. 2. We compare them with a standard FEM on quasi-uniform fitted meshes of similar sizes. In both cases, we use  $\mathbb{P}^1$  finite elements ( $k = 1$ ) and set  $\Delta t = h$ .  $\phi_h$  is a  $\mathbb{P}_3$  function. We compare the  $L^2(H^1)$  and  $L^\infty(L^2)$  relative errors of the two methods, with respect to the mesh size  $h$  and also the computation time (this is just the sum of time needed to solve the finite element systems at each time step, without the time used to construct the meshes). The numerical results fit well the theoretically convergence orders announced in Theorem 1, namely order 1 for both  $L^\infty(L^2)$  and  $L^2(H^1)$  errors. In fact, the  $L^\infty(L^2)$  error behaves even better than in theory since the slope is close to 1.5. Moreover,  $\phi$ -FEM turn out to be significantly faster than conforming FEM on similar meshes. We remark that even if the domain does not satisfy the regularity hypothesis (its boundary is only piecewise  $C^1$ ), we observe the optimal convergence of the method.

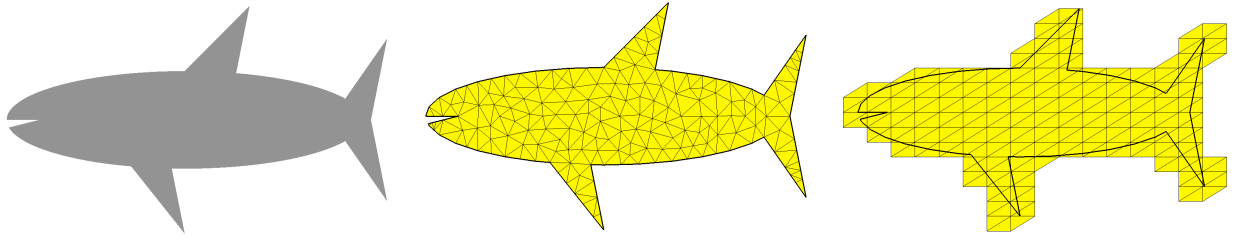


Figure 1: Left: considered domain. Center: a conforming mesh for the standard FEM. Right: a uniform Cartesian mesh for  $\phi$ -FEM.

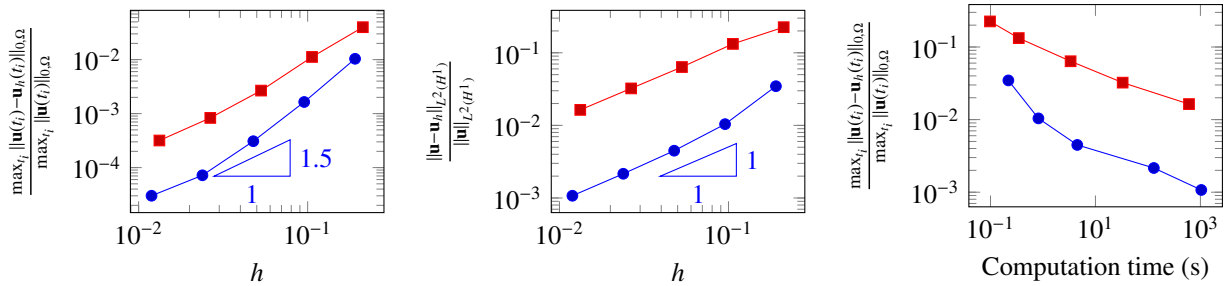


Figure 2: Heat equation for Standard FEM (red squares) and  $\phi$ -FEM (blue dots) with  $\Delta t = h$ ,  $\mathbb{P}_1$  elements,  $\sigma = 20$ . Left:  $L^\infty(0, T; L^2(\Omega))$  relative errors against  $h$ . Center:  $L^2(0, T; H^1(\Omega))$  relative errors against  $h$ . Right:  $L^\infty(0, T; L^2(\Omega))$  relative errors against the computation time.

**Remark 3.** We have here only considered the case of a first order in time discretization, using an implicit Euler scheme but our method also seems to converge optimally in the case of a second order time discretization even if it is not included in the present paper. Indeed, the scheme introduced in (4) can be easily adapted to Crank-Nicolson or BDF2 time discretizations, instead of implicit Euler. Furthermore interesting numerical results on such schemes are available in the github repository<sup>1</sup> as the code of the simulations.

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<sup>1</sup><https://github.com/KVuillemot/An-Immersed-Boundary-Method-by-phi-FEM-approach-to-solve-the-heat-equation>