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CONDITIONING WITH UPPER INTEGRAL. UPPER-SUFFICIENCY.

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-ABSTRACT-

For some problems related with admissibility and definition of posterior measure under partial prior information is necessary to solve the following-- question. Given a nonnegative bounded  $\mathcal{A}$ -measurable function  $f(\theta)$  and the -- probability space  $(\theta, \mathcal{B}, \xi)$ , where  $\mathcal{B}$  is a sub- $\sigma$ -field of  $\mathcal{A}$ , under what conditions does the integral equation  $\int_{\mathcal{B}}^* f(\theta) d\xi(\theta) = \int_{\mathcal{B}} f_{\xi}^{\mathcal{B}}(\theta) d\xi(\theta)$ ,  $\mathcal{B} \in \mathcal{B}$ , where  $\int^*$  denotes the upper integral with respect to the sub- $\sigma$ -field  $\mathcal{B}$ , have a solution  $f_{\xi}^{\mathcal{B}}(\theta)$  which does not depend on  $\xi$ ?. Here  $\xi$  denotes a probability measure belonging to a class  $\mathcal{H}$ . In this paper we give a solution to this problem.

## 1. INTRODUCTION.

Let us consider a statistical problem described by an observable sample space  $(X, \mathcal{S}, P_\theta)$ , where the probability measures  $P_\theta$  belong to the family  $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$ . Let us suppose that  $\mathcal{P} \ll \nu$  and  $\nu$   $\sigma$ -finite. Let  $f(x/\theta)$  be the density at  $x \in X$  of  $P_\theta$ . Furthermore, let  $(\Theta, \mathcal{A})$  be the prior measurable space and let  $(\Theta, \mathcal{B}, \xi)$  be the prior information available, where  $\mathcal{B}$  is a sub- $\sigma$ -field of  $\mathcal{A}$ . Finally let us suppose that  $f(x/\theta)$  is  $\mathcal{A}$ -measurable but not  $\mathcal{B}$ -measurable in  $\theta$  for each  $x \in X$ .

Under the above conditions,  $f(x/\theta)$  is not integrable with respect to the probability measure  $\xi(\cdot)$ . A way to do inference in this situation is to define what must be understood by "posterior measure", (see [6]).

On the other hand, if an  $\mathcal{A}$ -measurable loss function  $L(\theta, d(x))$  is given for each decision  $d \in D$  and  $x \in X$ , associated with the problem, it is not possible to define a prior risk  $r(\xi, d) = E_\xi(\int L(\theta, d(x)) dP_\theta(x))$ , because of the non-integrability of  $\int L(\theta, d(x)) dP_\theta(x)$  with respect to  $\xi(\cdot)$ . Following Kudo (1967), the appropriate risk function (called here mean-max risk) can be defined as

$$r(\mathcal{B}, \xi, d) = \int^* (\int L(\theta, d(x)) dP_\theta(x)) d\xi(\theta).$$

For some questions related with the admissibility of the minimum mean-max risk decisions, and for making precise the definition of posterior measure under

partial prior information as stated above, the following technical problem has to be solved. For sake of brevity, we state it an abstract formulation.

Let  $\theta$  be an unspecified but fixed space,  $A$  a  $\sigma$ -field of subsets of  $\theta$ , and  $\mathcal{B}$  a sub- $\sigma$ -field of  $A$ , on which we have defined a probability measure  $\xi$ . Let  $f(\theta)$  be a bounded  $A$ -measurable and nonnegative function, and consider  $r(\mathcal{B}, \xi, f) = \int_{\mathcal{B}} f(\theta) d\xi(\theta)$ . It can be shown (see Cano, Hernández and Moreno (1984)) that there exists a  $\mathcal{B}$ -measurable function  $f_{\xi}^{\mathcal{B}}(\theta)$ , well defined except for  $\mathcal{B}$ -measurable sets of  $\xi$ -measure zero, solution of the following integral equation

$$(1) \quad \int_{\mathcal{B}} f_{\xi}^{\mathcal{B}}(\theta) d\xi(\theta) = \int_{\mathcal{B}} f(\theta) d\xi(\theta), \quad \mathcal{B} \in \mathcal{B}.$$

Now, if  $\xi$  belong to a class  $H$  of probability measures defined on  $\mathcal{B}$ , can we define  $f_{\xi}^{\mathcal{B}}(\theta)$ , independently of  $\xi \in H$ ? If this can be done we shall say that the transformation  $f(\theta) \rightarrow f_{\xi}^{\mathcal{B}}(\theta)$  defined by equation (1) is an upper-sufficient transformation for the family  $H$ .

Some basic properties of that transformation are as follows:

- (i) If  $f(\theta) \leq K$ , then  $f_{\xi}^{\mathcal{B}}(\theta) \leq K$ , a.s. ( $\xi$ ).
- (ii)  $(af(\theta))_{\xi}^{\mathcal{B}} = af_{\xi}^{\mathcal{B}}(\theta)$ , a.s. ( $\xi$ ), where  $a$  is a real constant.
- (iii) Given two functions  $f(\theta)$ ,  $g(\theta)$ , then  $(f(\theta)+g(\theta))_{\xi}^{\mathcal{B}} \leq f_{\xi}^{\mathcal{B}}(\theta)+g_{\xi}^{\mathcal{B}}(\theta)$ , a.s. ( $\xi$ ).
- (iv) If the functions  $f(\theta)$ ,  $g(\theta)$  satisfies  $f(\theta) \geq g(\theta)$ , then  $f_{\xi}^{\mathcal{B}}(\theta) \geq g_{\xi}^{\mathcal{B}}(\theta)$ , a.s. ( $\xi$ ).
- (v)  $E_{\xi} f_{\xi}^{\mathcal{B}}(\theta) = \int_{\mathcal{B}} f(\theta) d\xi(\theta)$ .

(vi) If  $f(\theta)$  is a  $\beta$ -measurable function, then  $f_{\xi}^{\beta}(\theta) = f(\theta)$ , a.s. ( $\xi$ ).

(vii) If  $E_{\xi}^{\beta}f(\theta)$  denotes the conditional expectation of  $f(\theta)$  respect to the sub- $\sigma$ -field  $\beta$  (for which is necessary to suppose  $\xi$  is defined on  $A$ ), then the relation  $f_{\xi}^{\beta}(\theta) \geq E_{\xi}^{\beta}f(\theta)$  a.s. ( $\xi$ ), holds.

The proof of these properties is omitted (see Cano (1983)).

## 2. UPPER-SUFFICIENCY FOR A SPECIAL PARTIAL INFORMATION.

Let us consider the measurable abstract space  $(\theta, A)$ ,  $f(\theta)$  an  $A$ -measurable function and the probability space  $(\theta, \beta, \xi)$ , where  $\beta$  is a sub- $\sigma$ -field of  $A$  generated by a countable partition of  $\theta$ . For this special  $\beta$ , it is obvious that

$$(2) \quad \int^* f(\theta) d\xi(\theta) = \sum_{\theta \in B_n} f(\theta) \xi(B_n)$$

where  $\{B_n\}$  is the partition generating  $\beta$ .

Assume the  $\beta$ -measurable function  $f^{\beta}(\theta)$  defined as

$$(3) \quad f^{\beta}(\theta) = \sum_{\theta \in B_n} f(\theta) I_{B_n}(\theta).$$

This function satisfies the integral equation  $\int_B f^{\beta}(\theta) d\xi(\theta) = \int_B^* f(\theta) d\xi(\theta)$ ,  $\forall B \in \beta$ ,

and consequently,  $f_{\xi}^{\beta}(\theta) = f^{\beta}(\theta)$ , a.s. ( $\xi$ ). Furthermore, for any other probability

space  $(\theta, \beta, \eta)$  it follows that  $f_{\eta}^{\beta}(\theta) = f^{\beta}(\theta)$ , a.s. ( $\eta$ ) holds. Hence we can find a

common version of  $\{f_{\xi}^{\beta}(\theta), \xi \in H\}$ , where  $H$  is the class of all probability measures

on  $\beta$ . Furthermore, that common version is given by the expression (3). Thus, the

transformation defined by (1) is upper-sufficient for the family  $H$ .

## 3. UPPER-SUFFICIENCY FOR A GENERAL PARTIAL INFORMATION.

Now, the problem is stated as follows. Let  $(\theta, A)$  be a measurable space, and  $(\theta, \beta, \xi)$  a probability space where  $\beta$  is an arbitrary sub- $\sigma$ -field of  $A$ . Let  $f(\theta)$  be an  $A$ -measurable function. For getting a similar conclusion to that obtained in section 2 we need some auxiliary lemmas.

From now on we denote by  $P_k$  an arbitrary partition of  $\theta$ , i.e.  $\{B_1, \dots, B_k\}$ , where  $B_i \in \beta$ ,  $i=1, \dots, k$ , and by  $S(P_k, f)$  the summation given by

$$(4) \quad S(P_k, f) = \sum \sup_{\theta \in B_i} f(\theta) \xi(B_i).$$

Let  $f_k(\theta)$  be the simple function  $f_k(\theta) = \sum \sup_{\theta \in B_i} f(\theta) I_{B_i}(\theta)$  associated to  $P_k$ ,

satisfying  $E_{\xi} f_k(\theta) = S(P_k, f)$ .

Lemma 1.

Let  $\{P_n\}$  be a sequence of finite partitions of  $\theta$ , such that  $P_n$  is a refinement of  $P_{n-1}$  for all  $n \geq 2$ , satisfying  $S(P_n, f) + \int f^*(\theta) d\xi(\theta)$ , (it can be shown that such  $\{P_n\}$  always exist). Then the sequence of simple functions  $\{f_n(\theta)\}$  associated to  $\{P_n\}$  as above converge to a version  $\bar{f}_{\xi}^{\beta}(\theta)$  of  $f_{\xi}^{\beta}(\theta)$ , a.s. ( $\xi$ ).

Proof:

By definition  $\{f_n\}$  is a nonincreasing sequence. Let  $f^*(\theta)$  be its limit. Of course  $f^*(\theta)$  is a  $\beta$ -measurable function. Furthermore,  $f^*(\theta) \geq f(\theta)$ . Using the monotone

convergence theorem and the property (iv) stated in section 1, the relations

$$(5) \quad \int_B f_n(\theta) d\xi(\theta) + \int_B f^-(\theta) d\xi(\theta) = \int_B f^*(\theta) d\xi(\theta) \geq \int_B f(\theta) d\xi(\theta)$$

hold for any  $B \in \beta$ . By the other hand, the inequality in the expression (5) above

turn to be in an equality when  $B = \theta$ . It follows that the inequality in expression

(5) must be an equality for all  $B \in \beta$ . This terminates the proof of Lemma 1.

Lemma 2.

Let us consider  $f(\theta)$  and two arbitrary probability measures  $\xi, \eta$  on  $(\theta, \beta)$ .

Then there exists a function  $f_{\xi\eta}^\beta(\theta)$  which coincides with  $f_\xi^\beta(\theta)$  up to a  $\xi$ -equiva-

lence and with  $f_\eta^\beta(\theta)$  up to a  $\eta$ -equivalence.

Proof:

Let  $\{P_n^\xi: n \geq 1\}$  be a sequence of partitions of  $\theta$  with elements in  $\beta$  such that  $P_n^\xi$  is a refinement of  $P_{n-1}^\xi$  for all  $n \geq 2$ , and satisfying  $S(P_n^\xi, f) + \int f^*(\theta) d\xi(\theta)$ .

Let  $\{P_n^\eta: n \geq 1\}$  be a sequence with analogous properties. By Lemma 1, the sequences

$\{f_n^\xi(\theta): n \geq 1\}$  and  $\{f_n^\eta(\theta): n \geq 1\}$  are convergent to some versions  $\bar{f}_\xi^\beta(\theta), \bar{f}_\eta^\beta(\theta)$ , respectively.

Now, let  $P_n$  be the smallest partition of  $\theta$  with elements in  $\beta$  such that the  $\sigma$ -field generated by  $P_n$  contains the  $\sigma$ -fields generated by  $P_n^\xi$  and by  $P_n^\eta$  for each  $n \geq 1$ . Let  $\{P_n: n \geq 1\}$  be the sequence defined as above.

In that situation, the sequence of simple functions  $\{f_n^{\xi\eta}(\theta)\}$  associated to  $\{P_n\}$

converges to a common version of  $f_{\xi}^{\beta}(\theta)$  and  $f_{\eta}^{\beta}(\theta)$ , hence result.

To extend the conclusion of the Lemma 2 to more than two measures, we use an idea due to Speed (1976). We develop it in the next Lemma.

Lemma 3.

Let  $\mathcal{P}_0$  be a countable family of probability measures on  $(\theta, \mathcal{B})$ , and let  $f(\theta)$  be an  $\mathcal{A}$ -measurable function. Then, there exists a  $\mathcal{B}$ -measurable function  $f_{*}^{\beta}(\theta)$  satisfying the relations

$$(6) \quad f_{*}^{\beta}(\theta) = f_{\xi}^{\beta}(\theta), \text{ a.s. } (\xi),$$

for any  $\xi \in \mathcal{P}_0$ .

Proof:

Let us consider two arbitrary probability measures  $\xi, \eta$  belonging to  $\mathcal{P}_0$ . By

Lemma 2, we can take a common version  $f_{\xi\eta}^{\beta}(\theta)$  satisfying

$$(7) \quad f_{\xi\eta}^{\beta}(\theta) = f_{\xi}^{\beta}(\theta), \text{ a.s. } (\xi)$$

$$(8) \quad f_{\xi\eta}^{\beta}(\theta) = f_{\eta}^{\beta}(\theta), \text{ a.s. } (\eta).$$

Let us consider the functions  $f_{*}^{\beta}(\theta)$  and  $f_{**}^{\beta}(\theta)$  defined as

$$f_{*}^{\beta}(\theta) = \sup_{\xi} \inf_{\eta} f_{\xi\eta}^{\beta}(\theta); \quad f_{**}^{\beta}(\theta) = \inf_{\eta} \sup_{\xi} f_{\xi\eta}^{\beta}(\theta),$$

where the sup and inf is taken in the class  $\mathcal{P}_0$ . That functions are  $\mathcal{B}$ -measurable,

because of the countability of the class  $\mathcal{P}_0$ . Furthermore, by construction of  $f_{\xi\eta}^{\beta}(\theta)$

it follows that  $f_{\xi\eta}^{\beta}(\theta) = f_{\eta\xi}^{\beta}(\theta)$ . Thus we have the relations

The proof of Theorem 1 is complete.

$$(9) \quad \inf_{\eta} f_{\xi\eta}^{\beta}(\theta) \leq \sup_{\xi} \inf_{\eta} f_{\xi\eta}^{\beta}(\theta) \leq \inf_{\eta} \sup_{\xi} f_{\xi\eta}^{\beta}(\theta) = \inf_{\xi} \sup_{\eta} f_{\xi\eta}^{\beta}(\theta) \leq \sup_{\eta} \inf_{\xi} f_{\xi\eta}^{\beta}(\theta) \leq \sup_{\eta} f_{\xi\eta}^{\beta}(\theta)$$

On the other hand, from the expression (7) it follows that,

$$(10) \quad \inf_{\eta} f_{\xi\eta}^{\beta}(\theta) = f_{\xi}^{\beta}(\theta), \text{ a.s. } (\xi)$$

$$(11) \quad \sup_{\eta} f_{\xi\eta}^{\beta}(\theta) = f_{\xi}^{\beta}(\theta), \text{ a.s. } (\xi).$$

Using (10) and (11) in (9) we conclude that

$$f_{*}^{\beta}(\theta) = f_{*}^{\beta}(\theta) = f_{\xi}^{\beta}(\theta), \text{ a.s. } (\xi).$$

Since  $\xi$  is an arbitrary probability measure in the class  $\mathcal{P}_0$ ; the version we seek

can be taken as  $f_{*}^{\beta}(\theta) = f_{*}^{\beta}(\theta) = f_{*}^{\beta}(\theta)$  and Lemma 3 is proved.

Thus, the mapping  $f(\theta) \rightarrow f_{\xi}^{\beta}(\theta)$  given by expression (1) is upper-sufficient for  $\mathcal{P}_0$ . With the help of this Lemma 3 and of the next Lemma due to Halmos and Savage, we can show that this transformation is upper-sufficient for the family of all probability measures on  $\hat{\mathcal{B}}$  which are  $\mu$ -continuous, where  $\mu$  is some  $\sigma$ -finite measure on  $\mathcal{B}$ .

Lemma 4 (Halmos and Savage, 1949).

Let  $\mathcal{P}$  be a family of probability measures defined on  $(\theta, \mathcal{B})$  such that  $\mathcal{P} \ll \mu$ , where  $\mu$  is  $\sigma$ -finite. Then, there exists a countable family  $\mathcal{P}_0 \subset \mathcal{P}$  which is dense in  $\mathcal{P}$ .

## Theorem 1.

Let  $\mathcal{P}$  be a family of probability measures on  $\mathcal{B}$  which are  $\mu$ -continuous, where  $\mu$  is some  $\sigma$ -finite measure on  $\mathcal{B}$ . Let  $f(\theta)$  be a bounded nonnegative and  $\mathcal{A}$ -measurable function. Then there exists a version  $f_{*}^{\mathcal{B}}(\theta)$  such that the relations

$$f_{*}^{\mathcal{B}}(\theta) = f_{\xi}^{\mathcal{B}}(\theta), \text{ a.s. } (\xi),$$

hold for all  $\xi \in \mathcal{P}$ .

## Proof:

Since  $\mathcal{P} \ll \mu$ , with  $\mu$   $\sigma$ -finite we can take a countable subfamily  $\mathcal{P}_0$  of  $\mathcal{P}$  which is dense in  $\mathcal{P}$ . Let  $f_{*}^{\mathcal{B}}(\theta)$  be the  $\mathcal{B}$ -measurable version associated to  $\mathcal{P}_0$  defined as in Lemma 3. Let  $\eta$  be an arbitrary probability measure belonging to  $\mathcal{P} - \mathcal{P}_0$ , and

let  $\mathcal{P}_1 = \mathcal{P}_0 \cup \{\eta\}$ . By Lemma 3, there exists a version  $f_{**}^{\mathcal{B}}(\theta)$  associated to  $\mathcal{P}_1$

which satisfies

$$f_{**}^{\mathcal{B}}(\theta) = f_{\xi}^{\mathcal{B}}(\theta), \text{ a.s. } (\xi),$$

for all  $\xi \in \mathcal{P}_1$ . It implies that

$$f_{**}^{\mathcal{B}}(\theta) = f_{*}^{\mathcal{B}}(\theta), \text{ a.s. } (\xi),$$

for all  $\xi \in \mathcal{P}_0$ . Hence, it follows that

$$f_{**}^{\mathcal{B}}(\theta) = f_{*}^{\mathcal{B}}(\theta), \text{ a.s. } (\eta).$$

Thus the version  $f_{*}^{\mathcal{B}}(\theta)$  associated with  $\mathcal{P}_0$  (dense in  $\mathcal{P}$ ) is such that

$$\int_B f_{*}^{\mathcal{B}}(\theta) d\xi(\theta) = \int_B f_{\xi}^{\mathcal{B}}(\theta) d\xi(\theta), \quad B \in \mathcal{B},$$

for all  $\xi \in \mathcal{P}$ . The proof of Theorem 1 is complete.

## REFERENCES

- [1] Cano, J.A.(1983)."El método media-max de inferencia con informacion a priori parcial: Estudio comparativo y convergencias".Tesis doctoral. Univ.Granada.
- [2] Cano, J.A.;Hernández, A. and Moreno,E.(1984)."On minimum risk parameter estimation under partial prior information". (To appear).
- [3] Kudō,H.(1967)."On partial prior information and the property of parametric sufficiency". Proc.Fith Berkeley Symp.Stat.Prob.1; 251-265.
- [4] Speed, T.P.(1976)."A note on pairwise sufficiency and completions".Sankya, 38, A;194-197.
- [5] Halmos, P.R. and Savage, L.J.(1949)."Application of the Radon-Nikodym theorem to the theory of sufficient statistics". Ann.Math.Stat.20; 225-241.
- [6] Cano, J.A., Hernández,A. and Moreno, E.(1985)."Posterior measure under partial prior information". (to appear).

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