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► **To cite this version:**

| Laurent Poinot. Multipliers of a semigroup object in a monoidal category. 2022. hal-03685334v1

HAL Id: hal-03685334

<https://hal.science/hal-03685334v1>

Preprint submitted on 2 Jun 2022 (v1), last revised 6 Jul 2022 (v2)

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Multipliers of a semigroup object in a monoidal category

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Abstract

The monoid of multipliers of a semigroup object in a monoidal category is introduced, arising from an abstraction of the definition of the translational hull of an ordinary semigroup or of the multiplier algebra of a Banach algebra and dually, the monoid of comultipliers of a cosemigroup object is obtained. Functoriality of the abstract multiplier monoid construction is studied. Its set-theoretic version, the classical translational hull, is shown to provide a functor from a subcategory of ordinary semigroups to monoids, similar to a left adjoint. The abstract multiplier monoid of a semigroup object is related to the concrete translational hull of its convolution semigroup by a “concretization” homomorphism. For semigroup objects for which this homomorphism is onto, the multiplier construction is functorial and the concretization homomorphisms form a natural epimorphism.

MSC 2020: 18C40, 20M30, 18M05

Keywords: Semigroup object, multiplier, translational hull, convolution semigroup.

1 Introduction

Let $\mathbf{S} = (S, *)$ be a set-theoretic semigroup. By a left (resp. right) translation is meant a map L (resp. R) from S to itself such that $L(x * y) = L(x) * y$ (resp., $R(x * y) = x * R(y)$). Let $LTr(\mathbf{S})$ and $RTr(\mathbf{S})$ be respectively the set of all left and of all right translations of \mathbf{S} . For instance given $x \in S$, $\mathfrak{L}_x: S \rightarrow S$, $y \mapsto x * y$ and $\mathfrak{R}_x: y \mapsto y * x$ are respectively left and right translations. Associativity of $*$ provides for such *inner* translations, the following *linking* relation: $\mathfrak{R}_x(y) * z = y * \mathfrak{L}_x(z)$, $x, y, z \in S$. More generally

a pair (L, R) consisting of a left and a right translations such that for each $y, z \in S$, $R(y) * z = y * L(z)$ is referred to as a *multiplier* of S . The set $\text{TrHull}(S)$ of all multipliers of S is called the *translational hull* of S (see e.g. [3]), and in general it contains over multipliers than only the inner ones, that is, those of the form $\mathfrak{M}_S(x) = (\mathfrak{L}_x, \mathfrak{R}_x)$, $x \in S$. Multipliers may be composed by $(L', R') \star (L, R) := (L' \circ L, R \circ R')$ and (id_S, id_S) acts as an identity for \star , making $\text{TrHull}(S) := (\text{TrHull}(S), \star, (id_S, id_S))$ a monoid, and $\mathfrak{M}_S: S \rightarrow |\text{TrHull}(S)|$, $x \mapsto (\mathfrak{L}_x, \mathfrak{R}_x)$ a homomorphism of semigroups, called the *canonical homomorphism* of S , where $|-|: \mathbf{Mon} \rightarrow \mathbf{Sem}$ is the obvious forgetful functor. In fact, $\text{TrHull}(S)$ is a submonoid of the product monoid $\text{LTr}(S) \times \text{RTr}(S)^{op}$, with $\text{LTr}(S) := (\text{LTr}(S), \circ, id_S)$ and $\text{RTr}(S) := (\text{RTr}(S), \circ, id_S)$, where for a monoid $M = (M, *, 1)$, $M^{op} := (M, *^{op}, 1)$ denotes its *opposite* with $x *^{op} y := y * x$, $x, y \in M$.

The following result is easy and shows that the translational hull is not very interesting for monoids, what we could have suspected by thinking of the translational hull as a kind of unitarization.

1 Lemma *Let $M = (M, *, 1)$ be a monoid. Then, the canonical homomorphism of semigroups $\mathfrak{M}_{|M|}: (M, *) \rightarrow |\text{TrHull}(|M|)|$ lifts to an isomorphism of monoids $\mathfrak{M}_M: M \rightarrow \text{TrHull}(|M|)$, that is, \mathfrak{M}_M is an isomorphism, and $|\mathfrak{M}_M| = \mathfrak{M}_{|M|}$.*

Historically multipliers appeared naturally with the study of ideal extensions of semigroups [6] but did not remain confined to the theory of ordinary semigroups. For instance the Banach algebra of multipliers of a commutative (non-unital) Banach algebra was considered in [11] and the C^* -algebra of multipliers of a (non-unital) C^* -algebra was studied in [2].

S acts on the set S by left (resp. right) translations. Left (right) multipliers then are in fact nothing but endomorphisms of the left (resp. right) S -act S itself. Now given a left (resp. right) translation L (resp. R), $\mathfrak{g}(L): S \times S \rightarrow S$, $(x, y) \mapsto x * L(y)$ (resp. $\mathfrak{d}(R): S \times S \rightarrow S$, $(x, y) \mapsto R(x) * y$) is a homomorphism of left (resp. right) S -acts from $S \times S$ to S , where S acts on $S \times S$ by left (resp. right) multiplication on the first (resp. second) factor. (L, R) then is a multiplier iff $\mathfrak{g}(L) = \mathfrak{d}(R)$, that is, the translational hull of S is nothing but the pullback of \mathfrak{g} along \mathfrak{d} . Such observations are not only true for multipliers of semigroups, but also for those of rings or Banach algebras. These algebraic structures have in common the fact that they are all *semigroup objects* in some category, that of sets for the ordinary semigroups, of abelian groups for the rings, and Banach spaces for Banach algebras. This suggests to use the language of monoidal categories to describe multipliers

more abstractly.

In this contribution, one introduces the (set-theoretic) monoid of multipliers of a semigroup object in *any* monoidal category, which is a generalization of the translational hull in the cartesian monoidal category of sets. In details, in Section 3 are provided the notions of left, right and bi-actions of semigroup objects in a monoidal category, which are used in Section 4 to define an abstract multiplier monoid $\text{Mult}(\mathbf{S})$ for a semigroup object \mathbf{S} in a monoidal category. In Section 4.2 are introduced the *inner multipliers* induced by generalized elements, that is, morphisms from the unit object of the monoidal category to the underlying object of a semigroup. Using one of the main feature of category theory, its dualization principle, that is the process of replacing a category by its opposite, one introduces in Section 4.4 the monoid of *comultipliers* of a cosemigroup object. In Section 5 is given a complete description of (co)multipliers of (co)semigroups in a cocartesian (resp. cartesian) category, that is, a category which is monoidal for its co-product (resp. product). Section 6 focuses on the functorial behavior of our abstract multiplier monoid construction, and in Section 6.2 is introduced the *concretization homomorphism* which relates the multiplier monoid of a semigroup object in some monoidal category to the translational hull of its convolution semigroup. In Section 7.1 is shown that under some conditions, any homomorphism of semigroups $\mathbf{S} \xrightarrow{f} |\text{TrHull}(\mathbf{T})|$, where \mathbf{S}, \mathbf{T} are ordinary semigroups, admits a unique “extension” $\text{TrHull}(\mathbf{S}) \xrightarrow{f^\#} \text{TrHull}(\mathbf{T})$ as a homomorphism of monoids (Theorem 63). This result is used to define a *translational hull functor* from a subcategory of semigroup to monoids, which is very close to be a left adjoint to the forgetful functor from monoids to semigroups (Theorem 66). In Section 7.2, using the notion of *concrete semigroup object*, that is, a semigroup object whose concretization homomorphism is onto, which means that the concrete multipliers of the convolution semigroup of the semigroup object are in fact abstract multipliers, it is proved that under some conditions, any homomorphism of semigroups $\text{Conv}_I(\mathbf{S}) \xrightarrow{f} |\text{Mult}_{\mathbb{D}}(\mathbf{T})|$, where \mathbf{S}, \mathbf{T} are semigroup objects in possibly different monoidal categories \mathbb{C}, \mathbb{D} , and $\text{Conv}_I(\mathbf{S})$ is the convolution semigroup of \mathbf{S} , has a unique “extension” as a homomorphism of monoids $\text{Mult}_{\mathbb{C}}(\mathbf{S}) \xrightarrow{f^M} \text{Mult}_{\mathbb{D}}(\mathbf{T})$ (Theorem 71). This result then is used to provide a *multiplier monoid functor* from a subcategory of semigroup objects in *any* monoidal category to monoids, and turns the concretization homomorphisms into a natural epimorphism between this functor and the composition of the translational hull functor by the convolution semigroup functor.

2 Notations and prerequisites

We next introduce some notions which could be unfamiliar for some readers.

Some notations: Let C be a category and let c, c' be C -objects. $C(c, c')$ denotes the hom-sets of C -morphisms with domain c and codomain c' and given a functor $F: C \rightarrow D$, if needed one denotes by $F_{c, c'}$ the hom-component $C(c, c') \rightarrow D(Fc, Fc')$ of F . **Set** and **Cat** are the categories of sets (and maps) and of categories (and functors).

If C is a category with binary products (resp. coproducts) one denotes $c_1 \xleftarrow{\pi_1} c_1 \times c_2 \xrightarrow{\pi_2} c_2$ (resp. $c_1 \xrightarrow{q_1} c_1 \sqcup c_2 \xleftarrow{q_2} c_2$) the (chosen) canonical product (resp. coproduct) diagram. Given C -morphisms $a \xrightarrow{f_1} c_1$ and $a \xrightarrow{f_2} c_2$ (resp. $c_1 \xrightarrow{f_1} a$ and $c_2 \xrightarrow{f_2} a$) the unique C -morphism $h: a \rightarrow c_1 \times c_2$ (resp. $h: c_1 \sqcup c_2 \rightarrow a$) such that $\pi_i \circ h = f_i$ (resp. $h \circ q_i = f_i$), $i = 1, 2$, is denoted by $\langle f_1, f_2 \rangle$ (resp. $[f_1, f_2]$).

Given functors $F, G: C \rightarrow D$, that α is a natural transformation from F to G is denoted by $\alpha: F \Rightarrow G: C \rightarrow D$.

Monoidal categories and monoidal functors: A monoidal category is denoted $\mathbb{C} = (C, - \otimes -, I, \alpha, \lambda, \rho)$ or sometimes simply $\mathbb{C} = (C, \otimes, I)$, with $- \otimes -: C \times C \rightarrow C$ the tensor product bifunctor, I the unit object and the natural isomorphisms $((a \otimes b) \otimes c \xrightarrow{\alpha_{a,b,c}} a \otimes (b \otimes c))_{a,b,c}$, $(I \otimes c \xrightarrow{\lambda_c} c)_c$ and $(c \otimes I \xrightarrow{\rho_c} c)_c$ which are coherence constraints of associativity, and of left and right unit, respectively. A symmetric monoidal category is denoted by (\mathbb{C}, σ) with symmetry constraint denoted $\sigma = (c \otimes d \xrightarrow{\sigma_{c,d}} d \otimes c)_{c,d}$. \mathbb{C} is called *cartesian* (resp. *cocartesian*) if the tensor product is given categorical product (resp. coproduct) and I is a terminal (resp. initial) object. E.g. the cartesian monoidal category **Set** of sets with underlying category **Set**. In what follows $1 := \{0\}$ stands for a terminal object of **Set**.

Let \mathbb{C} be a monoidal category. Then one may define its *opposite* $\mathbb{C}^{op} := (C^{op}, \otimes^{op}, I, \alpha^{-1}, \lambda^{-1}, \rho^{-1})$ where C^{op} is the usual opposite category of C , and $\otimes^{op}: C^{op} \times C^{op} = (C \times C)^{op} \rightarrow C^{op}$ is the usual opposite of \otimes .

One also defines the *transpose* $\mathbb{C}^t = (C, \otimes^t, I, \alpha^t, \lambda^t, \rho^t)$ of \mathbb{C} with $(c \xrightarrow{f} d) \otimes^t (c' \xrightarrow{f'} d') := c' \otimes c \xrightarrow{f' \otimes f} d' \otimes d$, that is, $\otimes^t: C \times C \simeq C \times C \xrightarrow{\otimes} C$, where the isomorphism is given by $((a, b) \xrightarrow{(f,g)} (c, d)) \mapsto ((b, a) \xrightarrow{(g,f)} (d, c))$, $\alpha_{a,b,c}^t: (a \otimes^t b) \otimes^t c = c \otimes (b \otimes c) \xrightarrow{\alpha_{c,b,a}^{-1}} (c \otimes b) \otimes a = a \otimes^t (b \otimes^t c)$, $\lambda_a^t = I \otimes^t a = a \otimes I \xrightarrow{\rho_a} a$, $\rho_a^t = a \otimes^t I = I \otimes a \xrightarrow{\lambda_a} a$.

Let $\mathbb{C} := (C, \otimes_C, I_C, \alpha, \lambda, \rho)$ to $\mathbb{D} = (D, \otimes_D, I_D, \alpha, \lambda, \rho)$ be monoidal categories. A *lax monoidal functor* from \mathbb{C} to \mathbb{D} is a triple $\mathbb{F} := (F, F^{(2)}, F_0)$ where $F: C \rightarrow D$ is a functor (the *underlying functor* of \mathbb{F}), $(F(c_1) \otimes_D F(c_2) \xrightarrow{F_{c_1, c_2}^{(2)}} F(c_1 \otimes_C c_2))_{c_1, c_2}$ is a natural transformation and $F_0: J \rightarrow FI$ is a D -morphism, subject to coherence conditions (see [9] for details). A lax monoidal functor is called *strong monoidal* (resp. *strict monoidal*) when $F^{(2)}$ and F_0 are isomorphisms (resp. identities).

Let $\mathbb{C} = (C, \otimes_C, I_C) \xrightarrow{\mathbb{F}=(F, F^{(2)}, F_0)} \mathbb{D} = (D, \otimes_D, I_D) \xrightarrow{\mathbb{G}=(G, G^{(2)}, G_0)} \mathbb{E} = (E, \otimes_E, I_E)$ be monoidal functors. Let $\mathbb{G} \circ \mathbb{F} := (G \circ F, (G \circ F)^{(2)}, (G \circ F)_0)$ with $(G \circ F)_{c, d}^{(2)} := G(F_{c, d}^{(2)}) \circ G_{F(c), F(d)}^{(2)}$ and $(G \circ F)_0 := G(F_0) \circ G_0$. With identity $\text{id}_{\mathbb{C}}$ at \mathbb{C} the strict monoidal functor with underlying functor id_C , this provides the category **MonCat** of monoidal categories and lax monoidal functors.

Semigroup objects: A *semigroup object* (or simply *semigroup*) in \mathbb{C} is a pair $(S, S \otimes S \xrightarrow{\mu} S)$ such that the diagram below on the left

$$\begin{array}{ccc} (S \otimes S) \otimes S & \xrightarrow{\mu \otimes \text{id}_S} & S \otimes S & & S \otimes S & \xrightarrow{\mu} & S & (1) \\ \alpha_{S, S, S} \downarrow & & \downarrow \mu & & f \otimes f \downarrow & & \downarrow f & \\ S \otimes (S \otimes S) & \xrightarrow{\text{id}_S \otimes \mu} & S \otimes S & \xrightarrow{\mu} & S & & S' \otimes S' & \xrightarrow{\mu'} & S' \end{array}$$

commutes, while a *semigroup morphism* $(S, \mu) \xrightarrow{f} (S', \mu')$ is a C -morphism $f: S \rightarrow S'$ making the above right diagram commutative. This defines the category **SemC** of semigroups in \mathbb{C} . **Sem** := **Sem**(Set) is the category of ordinary semigroups and their homomorphisms. The category **CosemC** of *cosemigroup objects* (or simply *cosemigroups*) in \mathbb{C} is defined to be $(\mathbf{Sem}(\mathbb{C}^{op}))^{op}$. Let (\mathbb{C}, σ) be a symmetric monoidal category. Given a semigroup $\mathbf{S} = (S, \mu)$ (resp. a cosemigroup $\mathbf{S} = (S, \delta)$) one has its *opposite* (co)semigroup $\mathbf{S}^{op} := (S, \mu \circ \sigma_{S, S})$ (resp. $\mathbf{S}^{op} := (S, \sigma_{S, S} \circ \delta)$). A semigroup (resp. cosemigroup) (S, μ) (resp. (S, δ)) is said to be *commutative* (resp. *co-commutative*) when $\mathbf{S} = \mathbf{S}^{op}$. By ${}_c\mathbf{Sem}(\mathbb{C})$ and ${}_{coc}\mathbf{Cosem}(\mathbb{C})$ we denote the full subcategories of **Sem**(\mathbb{C}) and **Cosem**(\mathbb{C}) respectively spanned by the (co)commutative (co)semigroups. One has ${}_{coc}\mathbf{Cosem}\mathbb{C} = ({}_c\mathbf{Sem}(\mathbb{C}^{op}))^{op}$. The next result is clear.

2 Lemma $\mathbf{Sem}(\mathbb{C}) = \mathbf{Sem}(\mathbb{C}^t)$ and dually $\mathbf{Cosem}(\mathbb{C}) = \mathbf{Cosem}(\mathbb{C}^t)$.

Given a lax monoidal functor $\mathbb{F}: \mathbb{C} \rightarrow \mathbb{C}'$, $\tilde{\mathbb{F}}(S, \mu) := (FS, FS \otimes' FS \xrightarrow{F_{S,S}^{(2)}} F(S \otimes S))$ and $\tilde{\mathbb{F}}(f) := Ff$ define an induced functor $\tilde{\mathbb{F}}: \mathbf{Sem}\mathbb{C} \rightarrow \mathbf{Sem}\mathbb{C}'$, such that the diagram

$$\begin{array}{ccc} \mathbf{Sem}\mathbb{C} & \xrightarrow{\tilde{\mathbb{F}}} & \mathbf{Sem}\mathbb{C}' \\ U_s \downarrow & & \downarrow U'_s \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C}' \end{array} \quad (2)$$

commutes, with U_s, U'_s the obvious forgetful functors. In fact this provides a functor Sem from \mathbf{MonCat} to the category \mathbf{Cat} of categories. For instance, given a symmetric monoidal category (\mathbb{C}, σ) , one has a strong monoidal functor $\Sigma = (id_{\mathbb{C}}, \sigma, id_I): \mathbb{C} \rightarrow \mathbb{C}^t$ and thus an induced functor $\tilde{\Sigma}: \mathbf{Sem}(\mathbb{C}) \rightarrow \mathbf{Sem}(\mathbb{C}^t) = \mathbf{Sem}(\mathbb{C})$. In fact $\tilde{\Sigma}S = S^{op}$.

\mathbf{MonCat} has binary product $\mathbb{C} \times \mathbb{C}'$, with

1. tensor $(C \times C') \times (C \times C') \xrightarrow{\Sigma_{2,3}} (C \times C) \times (C' \times C') \xrightarrow{\otimes \times \otimes'} C \times C'$, where $\Sigma_{2,3}((c, c'), (d, d')) = ((c, d), (c', d'))$.
2. unit object (I, I') ,
3. associativity constraint $\alpha_{(a,a'),(b,b'),(c,c')} = ((a \otimes b) \otimes c, (a' \otimes' b') \otimes' c') \xrightarrow{(\alpha_{a,b,c}, \alpha'_{a',b',c'})} (a \otimes (b \otimes c), a' \otimes (b' \otimes c'))$, left unit constraint $\lambda_{(a,a')} = (I \otimes a, I' \otimes' a') \xrightarrow{(\lambda_a, \lambda'_{a'})} (a, a')$, and right unit constraint $\rho_{(a,a')} = (a \otimes I, a' \otimes' I') \xrightarrow{(\rho_a, \rho'_{a'})} (a, a')$.

\mathbf{MonCat} also has a terminal object namely the monoidal category $\mathbb{1}$ with underlying category $\mathbb{1}$, the category with only one object and its identity arrow. One observes that $\mathbf{Sem}(\mathbb{C} \times \mathbb{C}') \simeq \mathbf{Sem}(\mathbb{C}) \times \mathbf{Sem}(\mathbb{C}')$ canonically.

The hom-functor $C(-, -): C^{op} \times C \rightarrow \mathbf{Set}$, $C(b \xrightarrow{f} a, c \xrightarrow{h} d): C(a, c) \rightarrow C(b, d)$, $g \mapsto h \circ g \circ f$, is a monoidal functor $C_{\text{conv}} = (C(-, -), \Gamma, \gamma): C^{op} \times C \rightarrow \mathbf{Set}$, with $\Gamma_{(a,b),(c,d)}: C(a, b) \times C(c, d) \rightarrow C(a \otimes c, b \otimes d)$, $(f, g) \mapsto f \otimes g$, for $a, b, c, d \in \text{Ob}C$, and $\gamma: 1 \rightarrow C(I, I)$, $0 \mapsto id_I$. Whence C_{conv} induces a functor $\text{Conv} := \widetilde{C_{\text{conv}}}: \mathbf{Cosem}(\mathbb{C})^{op} \times \mathbf{Sem}(\mathbb{C}) \rightarrow \mathbf{Sem}$. Given a semigroup $(S, S \otimes S \xrightarrow{\mu} S)$ and a cosemigroup $(c, c \xrightarrow{\delta} c \otimes c)$ in \mathbb{C} , the the hom-set $C(c, S)$ becomes a semigroup, called the *convolution semigroup* $\text{Conv}((c, \delta), (S, \mu))$ with multiplication $(g, f) \mapsto g \bullet_{\mu, \delta} f = \mu \circ (g \otimes f) \circ \delta$. Since $(I, \lambda_I^{-1} = \rho_I^{-1})$ is always a cosemigroup in any monoidal category, one has a functor $\text{Conv}_I: \mathbf{Sem}(\mathbb{C}) \rightarrow \mathbf{Sem}$ given by $\text{Conv}_I((S, \mu) \xrightarrow{f} (S', \mu')) := (C(I, S), \bullet_{\mu}) \xrightarrow{C(id_I, f)} (C(I, S'), \bullet_{\mu'})$ with $\bullet_{\mu} := \bullet_{\mu, \lambda_I^{-1}}$.

3 Actions of a semigroup object

Let $\mathbb{C} := (C, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. Let $\mathbf{S} := (S, \mu)$ be a semigroup in \mathbb{C} . Let $c \in \text{Ob}C$, and $\gamma: S \otimes c \rightarrow c$ (resp. $\rho: c \otimes S \rightarrow c$) be a C -morphism. (S, γ) (resp. (S, ρ)) is called a *left \mathbf{S} -act* (resp. *right \mathbf{S} -act*) when the diagram below on the left (resp. right) commutes.

$$\begin{array}{ccc}
 S \otimes (S \otimes c) & \xrightarrow{id_S \otimes \gamma} & S \otimes c \\
 \wr \downarrow & & \downarrow \gamma \\
 (S \otimes S) \otimes c & & c \\
 \mu \otimes id_c \downarrow & \nearrow \gamma & \\
 S \otimes c & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 (c \otimes S) \otimes S & \xrightarrow{\rho \otimes id_S} & c \otimes S \\
 \wr \downarrow & & \downarrow \rho \\
 c \otimes (S \otimes S) & & c \\
 id_c \otimes \mu \downarrow & \nearrow \rho & \\
 c \otimes S & &
 \end{array}
 \quad (3)$$

γ (resp. ρ) then is referred to as the *left* (resp. *right*) \mathbf{S} -action.

3 Example By associativity, (S, μ) is both a left and a right \mathbf{S} -act.

Given left (resp. right) \mathbf{S} -acts $(c, \gamma), (c', \gamma')$ (resp. $(c, \rho), (c', \rho')$) a C -morphism $c \xrightarrow{f} c'$ is said to be a *left* (resp. *right*) \mathbf{S} -morphism when the diagram below on the left (resp. right) commutes.

$$\begin{array}{ccc}
 c & \xrightarrow{f} & c' \\
 \gamma \uparrow & & \uparrow \gamma' \\
 S \otimes c & \xrightarrow{id_S \otimes f} & S \otimes c'
 \end{array}
 \qquad
 \begin{array}{ccc}
 c & \xrightarrow{f} & c' \\
 \rho \uparrow & & \uparrow \rho' \\
 c \otimes S & \xrightarrow{f \otimes id_S} & c' \otimes S
 \end{array}
 \quad (4)$$

This provides the categories ${}_{\mathbf{S}}\mathbf{Act}(\mathbb{C})$ and $\mathbf{Act}_{\mathbf{S}}(\mathbb{C})$ of left and right \mathbf{S} -acts respectively, together with obvious faithful forgetful functors ${}_{\mathbf{S}}|-|: {}_{\mathbf{S}}\mathbf{Act}(\mathbb{C}) \rightarrow C$ and $|-|_{\mathbf{S}}: \mathbf{Act}_{\mathbf{S}}(\mathbb{C}) \rightarrow C$.

Let $\mathbf{S} = (S, \mu)$ and $\mathbf{S}' = (S', \mu')$ be semigroups in \mathbb{C} . Given an object c together with a left \mathbf{S} -action γ and a right \mathbf{S}' -action ρ , (c, γ, ρ) is said to be a \mathbf{S} - \mathbf{S}' -biact when the following diagram commutes.

$$\begin{array}{ccc}
 (S \otimes c) \otimes S' & \xrightarrow{\gamma \otimes id_{S'}} & c \otimes S' \\
 \wr \downarrow & & \downarrow \rho \\
 S \otimes (c \otimes S') & & c \\
 id_S \otimes \rho \downarrow & \nearrow \gamma & \\
 S \otimes c & &
 \end{array}
 \quad (5)$$

When $\mathbf{S} = \mathbf{S}$, a \mathbf{S} - \mathbf{S} -biact is simply called a *two-sided \mathbf{S} -biact*.

4 Example (S, μ, μ) is a \mathbf{S} -biact.

Given S - S' -biacts $(c, \gamma, \rho), (c', \gamma', \rho')$, a C -morphism $c \xrightarrow{f} c'$ which is both a left S -morphism $(c, \gamma) \xrightarrow{f} (c', \gamma')$ and a right S' -morphism $(c, \rho) \xrightarrow{f} (c', \rho')$ is referred to as a S - S' -morphism, and a *two-sided S -morphism* when $S = S'$. This provides the category ${}_S\mathbf{Act}_{S'}(\mathbb{C})$ together with a faithful forgetful functor ${}_S|-|_{S'}: {}_S\mathbf{Act}_{S'}(\mathbb{C}) \rightarrow C$.

One easily obtains the following result.

5 Lemma *Let $S = (S, \mu)$ be a semigroup in \mathbb{C} . Then, $\mathbf{Act}_S(\mathbb{C}) = {}_S\mathbf{Act}(\mathbb{C}^t)$ and ${}_S\mathbf{Act}(\mathbb{C}) = \mathbf{Act}_S(\mathbb{C}^t)$.*

6 Example 1. With S an ordinary semigroup, ${}_S\mathbf{Act}(\mathbf{Set})$ is the usual category of left S -acts.

2. Let R be a commutative ring with a unit. Let ${}_R\mathbf{Mod}$ be the monoidal category of left (unital) R -modules, with the usual tensor product \otimes_R over R . Associative R -algebras and semigroup objects in ${}_R\mathbf{Mod}$ are essentially the same and so are left modules over an algebra and left acts over a semigroup object.

3. Let \mathbf{Ban} be the monoidal category with underlying category \mathbf{Ban} , the category of say complex Banach spaces with bounded linear maps, and with the projective tensor product $\hat{\otimes}$ (see [8]). A semigroup $\mathbf{B} = (B, \mu)$ in \mathbf{Ban} is essentially¹ a non-unital Banach algebra $(B, *)$ with $x * y := \mu(x \otimes y)$, $x, y \in B$. Left \mathbf{B} -acts are left Banach $(B, *)$ -modules.

4. Let C be a category. One has a monoidal category $\mathbf{End}(C)$ of endofunctors of C [4]. A semigroup S in $\mathbf{End}(C)$ then is a *semi-monad*, that is, something similar to a monad [4] but without the requirement of being unital. A left S -act then is similar to an algebra over a monad but without the requirement of being compatible with the unit.

Let (\mathbb{C}, σ) be a symmetric monoidal category. Let S, c, d be C -objects. Let $f: S \otimes c \rightarrow d$ (resp. $g: c \otimes S \rightarrow d$) be a C -morphism. One defines $f' := c \otimes S \xrightarrow{\sigma_{c,S}} S \otimes c \xrightarrow{f} d$ (resp. $g' := S \otimes c \xrightarrow{\sigma_{S,c}} c \otimes S \xrightarrow{g} d$). Of course, $\backslash(f') = f$ and $(\backslash g)' = g$.

7 Proposition *Let $S = (S, \mu)$ be a commutative semigroup in \mathbb{C} . There are functors $A_l: {}_S\mathbf{Act}(\mathbb{C}) \rightarrow \mathbf{Act}_S(\mathbb{C})$, with $|-|_S \circ A_l = {}_S|-|$ and $A_r: \mathbf{Act}_S(\mathbb{C}) \rightarrow {}_S\mathbf{Act}(\mathbb{C})$ with ${}_S|-| \circ A_r = |-|_S$, which are inverse one from the other. There*

¹Up to the change of the original norm by an equivalent sub-multiplicative norm [7].

are full embedding functors $\mathbf{Act}_S(\mathbb{C}) \xrightarrow{B_l} {}_S\mathbf{Act}_S(\mathbb{C}) \xleftarrow{B_r} \mathbf{Act}_S(\mathbb{C})$ such that $s| - |_S \circ B_l = s| - |$ and $s| - |_S \circ B_r = | - |_S$, and the following diagram commutes.

$$\begin{array}{ccc} {}_S\mathbf{Act}(\mathbb{C}) & \xrightarrow{A_l} & \mathbf{Act}_S(\mathbb{C}) \\ & \searrow B_l & \swarrow B_r \\ & {}_S\mathbf{Act}_S(\mathbb{C}) & \end{array} \quad (6)$$

Proof: Let (c, γ) be a left S -act. (c, γ') is a right S -act since each cell of the following diagram commute, so commutes the surrounding diagram.

$$\begin{array}{ccccc} (c \otimes S) \otimes S & \xrightarrow{\sigma_{c,S} \otimes id_S} & (S \otimes c) \otimes S & \xrightarrow{\gamma \otimes id_S} & c \otimes S \\ \downarrow \cong & & \downarrow \cong & \downarrow \cong & \downarrow \sigma_{c,S} \\ c \otimes (S \otimes S) & \xrightarrow{\sigma_{c,S} \otimes S} & S \otimes (S \otimes c) & \xrightarrow{id_S \otimes \gamma} & S \otimes c \\ \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \gamma \\ c \otimes (S \otimes S) & \xrightarrow{\sigma_{c,S} \otimes S} & (S \otimes S) \otimes c & \xrightarrow{\mu \otimes id_c} & (S \otimes S) \otimes c \\ \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \mu \otimes id_c \\ c \otimes S & \xrightarrow{id_c \otimes \mu} & c \otimes S & \xrightarrow{\sigma_{c,S}} & S \otimes c \end{array} \quad (7)$$

(c, γ, γ') is a two-sided S -act. Indeed all internal cells of the following diagram commute, so commutes the surrounding diagram.

$$\begin{array}{ccccc} S \otimes (c \otimes S) & \xrightarrow{id_S \otimes \sigma_{c,S}} & (S \otimes c) \otimes S & \xrightarrow{\gamma \otimes id_S} & c \otimes S \\ \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \sigma_{c,S} \\ S \otimes (S \otimes c) & \xrightarrow{id_S \otimes \sigma_{c,S}} & S \otimes (S \otimes c) & \xrightarrow{id_S \otimes \gamma} & S \otimes c \\ \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \gamma \\ S \otimes (S \otimes c) & \xrightarrow{id_S \otimes \gamma} & (S \otimes S) \otimes c & \xrightarrow{\mu \otimes id_c} & S \otimes c \\ \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \gamma \\ S \otimes c & \xrightarrow{id_S \otimes \gamma} & (S \otimes S) \otimes c & \xrightarrow{\mu \otimes id_c} & S \otimes c \end{array} \quad (8)$$

Let $f \in {}_S\mathbf{Act}(\mathbb{C})((c_1, \gamma_1), (c_2, \gamma_2))$. Then, $f \in \mathbf{Act}_S(\mathbb{C})((c_1, \gamma'_1), (c_2, \gamma'_2))$, and thus $f \in {}_S\mathbf{Act}_S(\mathbb{C})((c_1, \gamma_1, \gamma'_1), (c_2, \gamma_2, \gamma'_2))$. Indeed, the two cells of the fol-

lowing diagram commute so commutes the surrounding diagram.

$$\begin{array}{ccc}
c_1 & \xrightarrow{f} & c_2 \\
\gamma_1 \uparrow & & \uparrow \gamma_2 \\
S \otimes c_1 & \xrightarrow{id_S \otimes f} & S \otimes c_2 \\
\sigma_{c_1, S} \uparrow & & \uparrow \sigma_{c_2, S} \\
c_1 \otimes S & \xrightarrow{f \otimes id_S} & c_2 \otimes S
\end{array} \tag{9}$$

Consequently one may define $A_l((c_1, \gamma_1) \xrightarrow{f} (c_2, \gamma_2)) := (c_1, \gamma'_1) \xrightarrow{f} (c_2, \gamma'_2)$ and $B_l((c_1, \gamma_1) \xrightarrow{f} (c_2, \gamma_2)) := (c_1, \gamma_1, \gamma'_1) \xrightarrow{f} (c_2, \gamma_2, \gamma'_2)$.

That $A_r((c_1, \rho_1) \xrightarrow{f} (c_2, \rho_2)) := (c_1, \rho_1) \xrightarrow{f} (c_2, \rho_2)$ provides a functor from $Act_S(\mathbb{C})$ to ${}_S Act(\mathbb{C})$ and $B_r((c_1, \rho_1) \xrightarrow{f} (c_2, \rho_2)) := (c_1, \rho_1, \rho_1) \xrightarrow{f} (c_2, \rho_2, \rho_2)$ provides a functor from $Act_S(\mathbb{C})$ to ${}_S Act(\mathbb{C})_S$ is obtained by symmetry. \square

4 The multiplier monoid of a semigroup object

In this section we describe the multiplier monoid of a semigroup object together with the inner multipliers, induced by generalized elements. By dualization the comultiplier monoid of a cosemigroup object is also introduced.

Throughout this section $\mathbb{C} = (C, \otimes, I)$ denotes a monoidal category.

4.1 The construction

8 Proposition *Let $S = (S, \mu)$ and $S' = (S', \mu')$ be semigroups in a monoidal category \mathbb{C} . $\otimes : C \times C \rightarrow C$ lifts to a functor ${}_S Act(\mathbb{C}) \times Act_{S'}(\mathbb{C}) \rightarrow {}_S Act_{S'}(\mathbb{C})$, still denoted \otimes , that is, such that the following diagram commutes.*

$$\begin{array}{ccc}
{}_S Act(\mathbb{C}) \times Act_{S'}(\mathbb{C}) & \longrightarrow & {}_S Act_{S'}(\mathbb{C}) \\
s|-|\times|-|s' \downarrow & & \downarrow s|-|s' \\
C \times C & \xrightarrow{\otimes} & C
\end{array} \tag{10}$$

Proof: Let (c, γ) and (c', ρ') be respectively a left S -act and a right S' -act. Let $\gamma_{\rho'} := S \otimes (c \otimes c') \simeq (S \otimes c) \otimes c' \xrightarrow{\gamma \otimes id_{c'}} c \otimes c'$, and let $\gamma_{\rho'} := (c \otimes c') \otimes S' \simeq c \otimes (c' \otimes S') \xrightarrow{id_c \otimes \rho'} c \otimes c'$. Then, one claims that $(c \otimes c', \gamma_{\rho'}, \gamma_{\rho'})$ is a two-sided S - S' -biact. First, $(c \otimes c', \gamma_{\rho'})$ is a left S -act because all the cells of the

following diagram commute.

$$\begin{array}{ccccc}
S \otimes (S \otimes (c \otimes c')) & \xrightarrow{id_S \otimes \alpha_{S,c,c'}^{-1}} & S \otimes ((S \otimes c) \otimes c') & \xrightarrow{id_S \otimes (\gamma \otimes id_{c'})} & S \otimes (c \otimes c') & (11) \\
\alpha_{S,S,c \otimes c'}^{-1} \downarrow & & \alpha_{S,S \otimes c,c'}^{-1} \downarrow & & \alpha_{S,c,c'}^{-1} \downarrow & \\
(S \otimes S) \otimes (c \otimes c') & & (S \otimes (S \otimes c)) \otimes c' & \xrightarrow{(id_S \otimes \gamma) \otimes id_{c'}} & (S \otimes c) \otimes c' & \\
\mu \otimes (id_c \otimes id_{c'}) \downarrow & \searrow \alpha_{S,S,c \otimes c'}^{-1} & \downarrow \alpha_{S,S,c \otimes c'}^{-1} & & \downarrow \gamma \otimes id_{c'} & \\
S \otimes (c \otimes c') & & ((S \otimes S) \otimes c) \otimes c' & & c \otimes c' & \\
\downarrow \alpha_{S,c,c'}^{-1} & \searrow (\mu \otimes id_c) \otimes id_{c'} & \downarrow (\mu \otimes id_c) \otimes id_{c'} & & \nearrow \gamma \otimes id_{c'} & \\
S \otimes (c \otimes c') & & (S \otimes c) \otimes c' & & &
\end{array}$$

Secondly, that $(c \otimes c', \gamma \rho')$ is a right S' -act is obtained similarly. Finally, $(c \otimes c', \gamma \rho', \gamma \rho')$ is a S - S' -biact because all internal cells of the following diagrams commute, so does the surrounding diagram.

$$\begin{array}{ccccc}
(S \otimes (c \otimes c')) \otimes S' & \xrightarrow{\alpha_{S,c,c',S'}^{-1} \otimes id_{S'}} & ((S \otimes c) \otimes c') \otimes S' & \xrightarrow{(\gamma \otimes id_{c'}) \otimes id_{S'}} & (c \otimes c') \otimes S' & (12) \\
\alpha_{S,c \otimes c',S'} \downarrow & & \alpha_{S \otimes c,c',S'} \downarrow & & \downarrow \alpha_{c,c',S'} & \\
S \otimes ((c \otimes c') \otimes S') & \xrightarrow{\alpha_{S,c,c',S'}^{-1}} & (S \otimes c) \otimes (c' \otimes S') & \xrightarrow{\gamma \otimes (id_{c'} \otimes id_{S'})} & c \otimes (c' \otimes S') & \\
id_S \otimes \alpha_{c,c',S'} \downarrow & \nearrow \alpha_{S,c,c',S'}^{-1} & \downarrow \alpha_{S,c,c',S'}^{-1} & & \downarrow id_c \otimes \rho' & \\
S \otimes (c \otimes (c' \otimes S')) & & (id_S \otimes id_c) \otimes \rho' & & c \otimes c' & \\
id_S \otimes (id_c \otimes \rho') \downarrow & & \downarrow (id_S \otimes id_c) \otimes \rho' & & \nearrow \gamma \otimes id_{c'} & \\
S \otimes (c \otimes c') & \xrightarrow{\alpha_{S,c,c'}^{-1}} & (S \otimes c) \otimes c' & & &
\end{array}$$

Let $f \in {}_S \mathbf{Act}((c, \gamma_c), (d, \gamma_d))$ and let $f' \in \mathbf{Act}_{S'}((c', \rho_{c'}), (d', \rho_{d'}))$. Then, $f \otimes f' \in {}_S \mathbf{Act}_{S'}((c \otimes c', (\gamma_c)_{\rho_{c'}}, \gamma_c(\rho_{c'})), (d \otimes d', (\gamma_d)_{\rho_{d'}}, \gamma_d(\rho_{d'})))$ because the internal cells of both following diagrams commute.

$$\begin{array}{ccc}
c \otimes c' \xrightarrow{f \otimes f'} d \otimes d' & & c \otimes c' \xrightarrow{f \otimes f'} d \otimes d' \\
\gamma_c \otimes id_{c'} \uparrow & & id_c \otimes \rho_{c'} \uparrow \\
(S \otimes c) \otimes c' \xrightarrow{(id_S \otimes f) \otimes f'} (S \otimes d) \otimes d' & & c \otimes (c' \otimes S') \xrightarrow{f \otimes (f' \otimes id_{S'})} d \otimes (d' \otimes S') \\
\alpha_{S,c,c'}^{-1} \uparrow & & \alpha_{c,c',S'} \uparrow \\
S \otimes (c \otimes c') \xrightarrow{id_S \otimes (f \otimes f')} S \otimes (d \otimes d') & & (c \otimes c') \otimes S' \xrightarrow{(f \otimes f') \otimes id_{S'}} (d \otimes d') \otimes S'
\end{array} \quad (13)$$

Functoriality of the construction follows from that of \otimes . \square

As a consequence of the associativity condition is the

9 Lemma *Let $\mathbf{S} = (S, \mu)$ be a semigroup in \mathbb{C} . Then, $\mu \in \mathfrak{S}\mathbf{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu) \otimes (S, \mu), (S, \mu, \mu))$.*

Let $\mathbf{S} = (S, \mu)$ be a semigroup in \mathbb{C} . Let us introduce the set-theoretic maps $\mathbf{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu), (S, \mu)) \xrightarrow{\mathfrak{g}_{\mathbb{C}, \mathbf{S}}} \mathfrak{S}\mathbf{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu) \otimes (S, \mu), (S, \mu, \mu))$ using Prop. 8 and Lemma 9 by $\mathfrak{g}_{\mathbb{C}, \mathbf{S}}(L) := \mu \circ (id_S \otimes L)$ and $\mathfrak{S}\mathbf{Act}((S, \mu), (S, \mu)) \xrightarrow{\mathfrak{d}_{\mathbb{C}, \mathbf{S}}} \mathfrak{S}\mathbf{Act}_{\mathbf{S}}((S, \mu) \otimes (S, \mu), (S, \mu, \mu))$ by $\mathfrak{d}_{\mathbb{C}, \mathbf{S}}(R) := \mu \circ (R \otimes id_S)$. Let

$$Mult_{\mathbb{C}}(\mathbf{S}) := (\mathbf{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu), (S, \mu)))_{\mathfrak{g}} \times_{\mathfrak{d}} (\mathfrak{S}\mathbf{Act}(\mathbb{C})((S, \mu), (S, \mu)))$$

(fibre product computed in **Set**). The members of $Mult_{\mathbb{C}}(\mathbf{S})$ are referred to as the *multipliers* of \mathbf{S} .

10 Proposition *Let $\mathbf{S} = (S, \mu)$ be a semigroup in \mathbb{C} . $(Mult_{\mathbb{C}}(\mathbf{S}), \star, (id_S, id_S))$ is a monoid denoted by $Mult_{\mathbb{C}}(\mathbf{S})$, under $(L', R') \star (L, R) := (L' \circ L, R \circ R')$. In fact, $Mult_{\mathbb{C}}(\mathbf{S})$ is a submonoid of the product monoid $\mathbf{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu), (S, \mu)) \times \mathfrak{S}\mathbf{Act}((S, \mu), (S, \mu))^{op}$.*

In what follows $Mult_{\mathbb{C}}(\mathbf{S})$ is referred to as the *multiplier monoid* of the semigroup object \mathbf{S} in \mathbb{C} . The following result is clear.

11 Lemma *Let \mathbb{C} be a monoidal category and let $\mathbf{S} = (S, \mu)$ be a semigroup object in \mathbb{C} . Then, $Mult_{\mathbb{C}^t}(\mathbf{S}) = \{(R, L) : (L, R) \in Mult_{\mathbb{C}}(\mathbf{S})\}$ and $(L, R) \mapsto (R, L)$ provides an isomorphism of monoids from $Mult_{\mathbb{C}}(\mathbf{S})$ to $Mult_{\mathbb{C}^t}(\mathbf{S})$.*

4.2 Inner multipliers induced by generalized elements

Since they might be no elements for an object S in an arbitrary monoidal category \mathbb{C} , one cannot define directly the inner multipliers as in the set-theoretic situation. One uses instead the *generalized elements*, that is, the members of the hom-set $C(I, S)$.

12 Lemma *Let $\mathbf{S} = (S, \mu)$ be a semigroup in \mathbb{C} . Let $f \in C(I, S)$. Let $L_f := S \xrightarrow{\lambda_S^{-1}} I \otimes S \xrightarrow{f \otimes id_S} S \otimes S \xrightarrow{\mu} S$. Then, $L_f \in \mathbf{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu), (S, \mu))$.*

Proof: All the cells commute in the diagram below, so commutes the surrounding diagram. (The left down triangle commutes by coherence.)

$$\begin{array}{ccccc}
S & \xrightarrow{\lambda_S^{-1}} & I \otimes S & \xrightarrow{f \otimes id_S} & S \otimes S & \xrightarrow{\mu} & S & (14) \\
\uparrow \mu & & \uparrow id_I \otimes \mu & & \uparrow id_S \otimes \mu & & \uparrow \mu & \\
S \otimes S & \xrightarrow{\lambda_{S \otimes S}^{-1}} & I \otimes (S \otimes S) & \xrightarrow{f \otimes (id_S \otimes id_S)} & S \otimes (S \otimes S) & & & \\
\downarrow \lambda_{S \otimes S}^{-1} \otimes id_S & & \downarrow \wr & & \downarrow \wr & & & \\
S \otimes S & \xrightarrow{\lambda_S^{-1} \otimes id_S} & (I \otimes S) \otimes S & \xrightarrow{(f \otimes id_S) \otimes id_S} & (S \otimes S) \otimes S & \xrightarrow{\mu} & S \otimes S &
\end{array}$$

□

By symmetry,

13 Lemma Let $\mathbf{S} = (S, \mu)$ be a semigroup in \mathbb{C} . Let $f \in C(I, S)$. Let $R_f := S \xrightarrow{\rho_S^{-1}} S \otimes I \xrightarrow{id_S \otimes f} S \otimes S \xrightarrow{\mu} S$. Then, $R_f \in {}_S \mathbf{Act}(\mathbb{C})((S, \mu), (S, \mu))$.

14 Lemma Let $\mathbf{S} = (S, \mu)$ be a semigroup in \mathbb{C} . Let $f \in C(I, S)$. Then, $(L_f, R_f) \in \mathbf{Mult}_{\mathbb{C}}(\mathbf{S})$.

Proof:

$$\begin{aligned}
\mu \circ (R_f \otimes id_S) &= \mu \circ (\mu \otimes id_S) \circ ((id_S \otimes f) \otimes id_S) \circ (\rho_S^{-1} \otimes id_S) \\
&= \mu \circ (id_S \otimes \mu) \circ \alpha_{S,S,S} \circ ((id_S \otimes f) \otimes id_S) \circ (\rho_S^{-1} \otimes id_S) \\
&\quad \text{(by associativity of } \mu \text{)} \\
&= \mu \circ (id_S \otimes \mu) \circ (id_S \otimes (f \otimes id_S)) \circ \alpha_{S,I,S} \circ (\rho_S^{-1} \otimes id_S) \\
&\quad \text{(by naturality of } \alpha \text{)} \\
&= \mu \circ (id_S \otimes \mu) \circ (id_S \otimes (f \otimes id_S)) \circ (id_S \otimes \lambda_S^{-1}) \\
&\quad \text{(by coherence)} \\
&= \mu \circ (id_S \otimes L_f).
\end{aligned} \tag{15}$$

□

According to Lemma 14, for each semigroup \mathbf{S} in \mathbb{C} , one has a map $C(I, S) \xrightarrow{\mathcal{M}_{\mathbb{C}, \mathbf{S}}} \mathbf{Mult}_{\mathbb{C}}(\mathbf{S})$ given by $f \mapsto (L_f, R_f)$. The image of this map is the set of all *inner multipliers* of \mathbf{S} in \mathbb{C} and is denoted $\mathbf{InnMult}_{\mathbb{C}}(\mathbf{S}) := \mathcal{M}_{\mathbf{S}}(C(I, S))$. Moreover L_f and R_f are respectively called the *inner left translation* and the *inner right translation* induced by f . $\mathcal{M}_{\mathbb{C}, \mathbf{S}}$ plays the role of the canonical homomorphism $\mathfrak{M}_{\mathbf{S}}$ for an ordinary semigroup \mathbf{S} .

15 Proposition Let $\mathbf{S} = (S, \mu)$ be a semigroup in \mathbb{C} . Then, $\mathcal{M}_{\mathbf{S}}: \mathbf{Conv}_I(\mathbf{S}) \rightarrow |\mathbf{Mult}_{\mathbb{C}}(\mathbf{S})| = (\mathbf{Mult}_{\mathbb{C}}(\mathbf{S}), \star)$ is a homomorphism of semigroups and in particular, $\mathbf{InnMult}_{\mathbb{C}}(\mathbf{S})$ is a sub-semigroup of $(\mathbf{Mult}_{\mathbb{C}}(\mathbf{S}), \star)$.

Proof: Let $f, g \in C(I, S)$. One has

$$\begin{aligned}
L_{g \bullet_\mu f} &= \mu \circ (\mu \otimes id_S) \circ ((g \otimes f) \otimes id_S) \circ (\lambda_I^{-1} \otimes id_S) \circ \lambda_S^{-1} \\
&= \mu \circ (id_S \otimes \mu) \circ \alpha_{S,S,S} \circ ((g \otimes f) \otimes id_S) \circ (\lambda_I^{-1} \otimes id_S) \circ \lambda_S^{-1} \\
&\quad \text{(by associativity)} \\
&= \mu \circ (id_S \otimes \mu) \circ \alpha_{S,S,S} \circ ((g \otimes id_S) \otimes id_S) \circ ((id_I \otimes f) \otimes id_S) \circ (\lambda_I^{-1} \otimes id_S) \circ \lambda_S^{-1} \\
&\quad \text{(by functoriality)} \\
&= \mu \circ (id_S \otimes \mu) \circ (g \otimes (id_S \otimes id_S)) \circ \alpha_{I,S,S} \circ ((id_I \otimes f) \otimes id_S) \circ (\lambda_I^{-1} \otimes id_S) \circ \lambda_S^{-1} \\
&\quad \text{(by naturality of } \alpha) \\
&= \mu \circ (g \otimes id_S) \circ (id_I \otimes \mu) \circ \alpha_{I,S,S} \circ ((id_I \otimes f) \otimes id_S) \circ (\lambda_I^{-1} \otimes id_S) \circ \lambda_S^{-1} \\
&\quad \text{(by functoriality of } \otimes) \\
&= \mu \circ (g \otimes id_S) \circ (id_I \otimes \mu) \circ (id_{\otimes}(f \otimes id_S)) \circ \alpha_{I,I,S} \circ (\lambda_I^{-1} \otimes id_S) \circ \lambda_S^{-1} \\
&\quad \text{(by naturality of } \alpha) \\
&= \mu \circ (g \otimes id_S) \circ (id_I \otimes \mu) \circ (id_{\otimes}(f \otimes id_S)) \circ (id_I \otimes \lambda_S^{-1}) \circ \lambda_S^{-1} \\
&\quad \text{(by coherence)} \\
&= \mu \circ (g \otimes id_S) \circ (id_I \otimes \mu) \circ \lambda_{S \otimes S}^{-1} \circ (f \otimes id_S) \circ \lambda_S^{-1} \\
&\quad \text{(by naturality of } \lambda) \\
&= \mu \circ (g \otimes id_S) \circ \lambda_S^{-1} \circ \mu \circ (f \otimes id_S) \circ \lambda_S^{-1} \\
&\quad \text{(by naturality of } \lambda) \\
&= L_g \circ L_f.
\end{aligned} \tag{16}$$

By symmetry one also has $R_{g \bullet_\mu f} = R_f \circ R_g$. \square

16 Remark The proof of Prop. 15 also shows that the maps $C(I, S) \xrightarrow{\mathcal{L}_{\mathbb{C}, S}} \mathbf{Act}_S(\mathbb{C})((S, \mu), (S, \mu))$, $f \mapsto L_f$, and $C(I, S) \xrightarrow{\mathcal{R}_{\mathbb{C}, S}} {}_S\mathbf{Act}(\mathbb{C})((S, \mu), (S, \mu))$, $f \mapsto R_f$, are homomorphisms $\mathbf{Conv}_I(S) \rightarrow (\mathbf{Act}_S(\mathbb{C})((S, \mu), (S, \mu)), \circ)$ and $\mathbf{Conv}_I(S)^{op} \rightarrow ({}_S\mathbf{Act}(\mathbb{C})((S, \mu), (S, \mu)), \circ)$ respectively.

17 Example Let $S := (S, *)$ be a semigroup. Then, $\mathbf{Mult}_{\mathbf{Set}}(S)$ is the ordinary translational hull $\mathbf{TrHull}(S)$ of S . Moreover, $S \simeq \mathbf{Conv}_1(S)$ under $x \mapsto (gen_S(x): 1 \rightarrow S)$ with $gen_S(x)(0) := x$. (When $S = \emptyset$, then gen_S is the empty map.) Moreover $\mathfrak{M}_S = \mathcal{M}_{\mathbf{Set}, S} \circ gen_S$.

18 Example Let S be a semigroup object in \mathbf{Ban} , considered as a Banach algebra. Then, $\mathbf{Mult}_{\mathbf{Ban}}(S)$ is the underlying semigroup of the multiplier Banach algebra of S (see e.g. [5, Theorem 1.2.4, p. 29]).

Let $\mathbf{S} = (S, \mu)$ be a semigroup object in \mathbb{C} . Let $f, g \in C(I, S)$. Then,

$$\begin{aligned} L_f \circ g &= \mu \circ (f \otimes id_S) \circ \lambda_S^{-1} \circ g \\ &= \mu \circ (f \otimes id_S) \circ (id_I \otimes g) \circ \lambda_I^{-1} \\ &= f \bullet_{\mu} g \end{aligned} \quad (17)$$

and almost identically,

$$R_f \circ g = g \bullet_{\mu} f. \quad (18)$$

From these computations one deduces the following

19 Lemma *Let $\mathbf{S} = (S, \mu)$ be a semigroup object in \mathbb{C} . Assume that for each $f, g \in C(I, S)$, if for each $h \in C(I, S)$, $f \bullet_{\mu} h = g \bullet_{\mu} h$, then $f = g$, or for each $f, g \in C(I, S)$, if for each $h \in C(I, S)$, $h \bullet_{\mu} f = h \bullet_{\mu} g$, then $f = g$. Then, $\mathcal{M}_{\mathbb{C}, \mathbf{S}}: \text{Conv}_I(\mathbf{S}) \rightarrow (\text{Mult}_{\mathbb{C}}(\mathbf{S}), \star)$ is one-to-one.*

20 Proposition *Let us assume that (\mathbb{C}, σ) is a symmetric monoidal category. Let \mathbf{S} be a commutative semigroup in \mathbf{S} . Then, one has $\{(L, L): L \in {}_{\mathbf{S}}\text{Act}(\mathbb{C})((S, \mu), (S, \mu))\} = \{(R, R): R \in \text{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu), (S, \mu))\} \subseteq \text{Mult}_{\mathbb{C}}(\mathbf{S})$. When furthermore μ is an epimorphism, then $\text{Mult}_{\mathbb{C}}(\mathbf{S}) = \{(L, L): L \in {}_{\mathbf{S}}\text{Act}(\mathbb{C})((S, \mu), (S, \mu))\}$. In this case, the multiplier monoid $\text{Mult}_{\mathbb{C}}(\mathbf{S})$ and the endomorphism monoid $({}_{\mathbf{S}}\text{Act}(\mathbb{C})((S, \mu), (S, \mu)), \circ, id_S)$ are isomorphic under $(L, L) \mapsto L$.*

Proof: The first equality is a consequence of Prop. 7 since one may notice that the isomorphism $A_l: {}_{\mathbf{S}}\text{Act}(\mathbb{C}) \rightarrow \text{Act}_{\mathbf{S}}(\mathbb{C})$ restricts to the identity from ${}_{\mathbf{S}}\text{Act}(\mathbb{C})((S, \mu), (S, \mu))$ to $\text{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu), (S, \mu))$ (because $A_l(S, \mu) = (S, \mu \circ \sigma_{S, S}) = (S, \mu)$). Let $L \in {}_{\mathbf{S}}\text{Act}(\mathbb{C})((S, \mu), (S, \mu))$. Then, $\mu \circ (id_S \otimes L) = \mu \circ \sigma_{S, S} \circ (L \otimes id_S) = \mu \circ (L \otimes id_S)$. Since $L \in \text{Act}_{\mathbf{S}}((S, \mu), (S, \mu))$ it follows that $(L, L) \in \text{Mult}_{\mathbb{C}}(\mathbf{S})$. Let us assume that μ is an epimorphism. Let $(L, R) \in \text{Mult}_{\mathbb{C}}(\mathbf{S})$. Then, $R \circ \mu = \mu \circ (id_S \otimes R) = \mu \circ \sigma_{S, S} \circ (R \otimes id_S) = \mu \circ (R \otimes id_S) = \mu \circ (id_S \otimes L) = \mu \circ \sigma_{S, S} \circ (L \otimes id_S) = \mu \circ (L \otimes id_S) = L \circ \mu$, and since μ is an epimorphism, $R = L$. \square

4.3 The case of monoids

A *monoid object*, or simply a *monoid* in a monoidal category \mathbb{C} is a triple $\mathbf{M} = (M, \mu, \eta)$ where (M, μ) is a semigroup in \mathbb{C} and $\eta: I \rightarrow M$ is a C -morphism such that the diagrams

$$\begin{array}{ccccc} I \otimes M & \xrightarrow{\eta \otimes id_M} & M \otimes M & \xleftarrow{id_M \otimes \eta} & M \otimes I \\ & \searrow \lambda_M & \downarrow \mu & \swarrow \rho_M & \\ & & M & & \end{array} \quad (19)$$

commute. When \mathbf{M} is a monoid, the convolution semigroup $\text{Conv}_I(M, \mu)$ becomes an ordinary monoid $\text{Conv}_I(\mathbf{M}) := (C(I, M), \bullet_{\mu}, \eta)$.

21 Remark An ordinary monoid $\mathbf{M} = (M, *, 1_M)$ is a monoid object in Set with $\eta := \text{gen}_{|\mathbf{M}|}(1_M)$ (see Example 17). Besides $\text{gen}_{|\mathbf{M}|}: |\mathbf{M}| \simeq \text{Conv}_1(M, *)$ lifts to an isomorphism $\text{gen}_{\mathbf{M}}: \mathbf{M} \simeq \text{Conv}_1(\mathbf{M})$ of monoids.

Similarly to the set-theoretic setting (Lemma 1) one has the

22 Proposition *Let $\mathbf{M} = (M, \mu, \eta)$ be a monoid in \mathbb{C} . Then, $\mathcal{M}_{\mathbb{C}, (M, \mu)}$ is the underlying homomorphism of semigroups of an isomorphism of monoids $\mathcal{M}_{\mathbb{C}, \mathbf{M}}: \text{Conv}_I(\mathbf{M}) \simeq \text{Mult}_{\mathbb{C}}(|\mathbf{M}|)$. In particular a monoid object has only inner multipliers.*

Proof: Let $L \in \mathbf{Act}_{(M, \mu)}(\mathbb{C})((M, \mu), (M, \mu))$. Then, $L = L_{L \circ \eta}$. Indeed,

$$\begin{aligned}
M &\xrightarrow{\lambda_M^{-1}} I \otimes M \xrightarrow{\eta \otimes id_M} M \otimes M \xrightarrow{L \otimes id_M} M \otimes M \xrightarrow{\mu} M \\
&= M \xrightarrow{\lambda_M^{-1}} I \otimes M \xrightarrow{\eta \otimes id_M} M \otimes M \xrightarrow{\mu} M \xrightarrow{L} M \\
&\text{(since } L \text{ is a right } (M, \mu)\text{-morphism)} \\
&= M \xrightarrow{id_M} M \xrightarrow{L} M. \\
&\text{(since } \eta \text{ is the unit of } \mathbf{M})
\end{aligned} \tag{20}$$

By symmetry for $R \in {}_{(M, \mu)}\mathbf{Act}(\mathbb{C})((M, \mu), (M, \mu))$, $R = R_{R \circ \eta}$. Now assume that $(L, R) \in \text{Mult}_{\mathbb{C}}(M, \mu)$. Let us check that $R \circ \eta = L \circ \eta$. For each $f \in C(M, M)$, one has

$$\begin{aligned}
\mu \circ ((f \otimes \eta) \otimes \eta) &= \mu \circ (id_M \otimes \eta) \circ ((f \circ \eta) \otimes id_I) \\
&= \rho_M \circ ((f \circ \eta) \otimes id_I) \\
&= (f \circ \eta) \circ \rho_I \\
&\text{(by naturality of } \rho)
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\mu \circ (\eta \otimes (f \circ \eta)) &= \mu \circ (\eta \otimes id_M) \circ (id_I \otimes (f \circ \eta)) \\
&= \lambda_M \circ (id_I \otimes (f \circ \eta)) \\
&= (f \circ \eta) \circ \lambda_I. \\
&\text{(by naturality of } \lambda)
\end{aligned} \tag{22}$$

Furthermore,

$$\begin{aligned}
\mu \circ ((R \circ \eta) \otimes \eta) &= \mu \circ (R \otimes id_M) \circ (\eta \otimes \eta) \\
&= \mu \circ (id_M \otimes L) \circ (\eta \otimes \eta) \\
&= \mu \circ (\eta \otimes id_M) \circ (id_I \otimes (L \circ \eta)) \\
&= \lambda_M \circ (id_I \otimes (L \circ \eta)) \\
&= L \circ \eta \circ \lambda_I
\end{aligned} \tag{23}$$

Therefore by the above $(R \circ \eta) \circ \rho_I = L \circ \eta \circ \lambda_I$ so that $R \circ \eta = L \circ \eta \circ \lambda_I \circ \rho_I^{-1} = L \circ \eta$. So with $f := L \circ \eta = R \circ \eta$, one thus has $\mathcal{M}_{\mathbb{C},(M,\mu)}(f) := (L, R)$ which shows that $\mathcal{M}_{\mathbb{C},(M,\mu)}: C(I, M) \rightarrow \text{Mult}_{\mathbb{C}}(M, \mu)$ is onto.

Let $f, g \in C(I, M)$ so that $L_f = L_g$. Then, because η is the unit of $\text{Conv}_I(\mathbb{M})$, $f = \mu \circ (f \otimes \text{id}_M) \circ \lambda_M^{-1} \circ \eta = L_f \circ \eta = L_g \circ \eta = \mu \circ (g \otimes \text{id}_M) \circ \lambda_M^{-1} \circ \eta = g$. Assume that $R_f = R_g$. Then, $f = \mu \circ (\text{id}_M \otimes f) \circ \rho_M^{-1} \circ \eta = R_f \circ \eta = R_g \circ \eta = \mu \circ (\text{id}_M \otimes g) \circ \rho_M^{-1} \circ \eta = g$.

As a consequence $\mathcal{M}_{\mathbb{C},(M,\mu)}$ is also one-to-one, and hence provides an isomorphism of semigroups $\text{Conv}_I(M, \mu) \rightarrow (\text{Mult}(M, \mu), \star)$. As, $\mathcal{M}_{\mathbb{C},(M,\mu)}(\eta) = (L_\eta, R_\eta) = (\text{id}_M, \text{id}_M)$ since $L_\eta = \mu \circ (\eta \otimes \text{id}_M) \circ \lambda_M^{-1} = \lambda_M \circ \lambda_M^{-1} = \text{id}_M$ and $R_\eta = \mu \circ (\text{id}_M \otimes \eta) \circ \rho_M^{-1} = \rho_M \circ \rho_M^{-1} = \text{id}_M$, $\mathcal{M}_{\mathbb{C},(M,\mu)}$ is an isomorphism of monoids from $\text{Conv}_I(\mathbb{M})$ to $\text{Mult}_{\mathbb{C}}(M, \mu)$. \square

4.4 Comultipliers by dualization

By a simple dualization process, that is, essentially by replacing a category by its opposite, one now summarizes without proof some results about coactions of cosemigroups, deduced from that of actions of semigroups and one introduces the notion of comultipliers.

Let $S := (S, \delta)$ be a cosemigroup in \mathbb{C} , that is, a semigroup in \mathbb{C}^{op} . A *left* (resp. *right*) S -*coact* (c, β) (resp. (c, τ)) is an object of ${}_S\text{Act}(\mathbb{C}^{op})$ (resp. $\text{Act}_S(\mathbb{C}^{op})$). In what follows β (resp. τ) is referred to as a *left* (resp. *right*) S -*coaction*.

23 Example By coassociativity, (C, δ) is both a left and a right S -coact.

Now one defines the category ${}_S\mathbf{Coact}(\mathbb{C}) := {}_S\mathbf{Act}(\mathbb{C}^{op})^{op}$ (respectively, $\mathbf{Coact}_S(\mathbb{C}) := \mathbf{Act}_S(\mathbb{C}^{op})^{op}$) of left (resp. right) S -coacts.

Let $S = (S, \delta)$ and $S' = (S', \delta')$ be cosemigroups in \mathbb{C} . A S - S' -*bi-coact* is a S - S' -biact in \mathbb{C}^{op} , and one defines ${}_S\mathbf{Coact}_{S'}(\mathbb{C}) := {}_S\mathbf{Act}_{S'}(\mathbb{C}^{op})^{op}$. When $S = S'$, a S - S' -bi-coact is simply called a (*two-sided*) S -*bi-coact*.

24 Example (S, δ, δ) is a S -bi-coact.

25 Lemma Let $S = (S, \delta)$ be a cosemigroup in \mathbb{C} . Then, $\mathbf{Coact}_S(\mathbb{C}) = {}_S\mathbf{Coact}(\mathbb{C}^t)$.

26 Proposition Let $S = (S, \delta)$ and $S' = (S', \delta')$ be cosemigroups in a monoidal category \mathbb{C} . $\otimes: C \times C \rightarrow C$ lifts to a functor ${}_S\mathbf{Coact}(\mathbb{C}) \times$

$\mathbf{Coact}_{S'}(\mathbb{C}) \rightarrow {}_S\mathbf{Coact}_{S'}(\mathbb{C})$, that is, such that the following diagram commutes.

$$\begin{array}{ccc} {}_S\mathbf{Coact}(\mathbb{C}) \times \mathbf{Coact}_{S'}(\mathbb{C}) & \longrightarrow & {}_S\mathbf{Coact}_{S'}(\mathbb{C}) \\ \downarrow \scriptstyle |-\!|^{op} \times |-\!|_{S'}^{op} & & \downarrow \scriptstyle |-\!|_{S'}^{op} \\ C \times C & \xrightarrow{\otimes} & C \end{array} \quad (24)$$

27 Lemma $\delta \in {}_S\mathbf{Coact}_S(\mathbb{C})((S, \delta), (S, \delta) \otimes (S, \delta))$ for a cosemigroup $S = (S, \delta)$ in \mathbb{C} .

Let $S = (S, \delta)$ be a cosemigroup in \mathbb{C} . Define $Comult_{\mathbb{C}}(S) := Mult_{\mathbb{C}^{op}}(S)$. In other words,

$$Comult_{\mathbb{C}}(S) := \mathbf{Coact}_S(\mathbb{C})((S, \delta), (S, \delta))_{\mathbf{g}_{\mathbb{C}^{op}, S}} \times_{\mathbf{d}_{\mathbb{C}^{op}, S}} \mathbf{Coact}(\mathbb{C})((S, \delta), (S, \delta))$$

with $\mathbf{g}_{\mathbb{C}^{op}, S}(L) = \mathbf{g}(L) := (id_S \otimes L) \circ \delta$ and $\mathbf{d}_{\mathbb{C}^{op}, S}(R) = (R \otimes id_S) \circ \delta$. The members of $Comult_{\mathbb{C}}(S)$ are referred to as the *comultipliers* of S .

28 Proposition Let $S = (S, \delta)$ be a cosemigroup in \mathbb{C} . Then, $Comult_{\mathbb{C}}(S) := (Comult_{\mathbb{C}}(S), \star, (id_S, id_S))$ is a monoid under $(L', R') \star (L, R) := (L \circ L', R' \circ R)$. $Comult_{\mathbb{C}}(S)$ is in fact a submonoid of $\mathbf{Coact}_S(\mathbb{C})((S, \delta), (S, \delta))^{op} \times {}_S\mathbf{Coact}(\mathbb{C})((S, \delta), (S, \delta))$ (product monoid).

In what follows $Comult_{\mathbb{C}}(S)$ is referred to as the *comultiplier monoid* of S in \mathbb{C} .

29 Lemma Let $C = (S, \delta)$ be a cosemigroup in \mathbb{C} . Let $f \in C(S, I)$.

1. $L_f := S \xrightarrow{\delta} S \otimes S \xrightarrow{f \otimes id_S} I \otimes S \xrightarrow{\lambda_S} S$. Then, $L_f \in \mathbf{Coact}_S(\mathbb{C})((S, \delta), (S, \delta))$.
2. $R_f := S \xrightarrow{\delta} S \otimes S \xrightarrow{id_S \otimes f} S \otimes I \xrightarrow{\rho_S} S$. Then, $R_f \in {}_S\mathbf{Coact}(\mathbb{C})((S, \delta), (S, \delta))$.
3. Moreover $(L_f, R_f) \in Comult_{\mathbb{C}}(S)$.

Let $S = (S, \delta)$ be a cosemigroup in \mathbb{C} . Let $\mathcal{C}_{\mathbb{C}, S} := \mathcal{M}_{\mathbb{C}^{op}, S}: C(S, I) \rightarrow Comult_{\mathbb{C}}(S)$ be given by $\mathcal{C}_{\mathbb{C}, S}(f) := (L_f, R_f)$. The image of $\mathcal{C}_{\mathbb{C}, S}$ into $Comult_{\mathbb{C}}(S)$ is the set $InnComult_{\mathbb{C}}(S)$ of all *inner comultipliers* of S .

Recall that $C(S, I) = C^{op}(I, S)$ becomes a semigroup under convolution which here means $g \bullet_{\delta} f := S \xrightarrow{\delta} S \otimes S \xrightarrow{g \otimes f} I \otimes I \xrightarrow{\lambda_I = \rho_I} I$.

30 Proposition Let $S = (S, \delta)$ be a cosemigroup in \mathbb{C} . $\mathcal{C}_{\mathbb{C}, S}: (C(S, I), \bullet_{\delta}) \rightarrow (Comult_{\mathbb{C}}(S), \star)$ is a homomorphism of semigroups and $InnComult_{\mathbb{C}}(S)$ is a sub-semigroup of $(Comult_{\mathbb{C}}(S), \star)$.

Let $\mathbf{S} = (S, \delta)$ be a cosemigroup object in \mathbb{C} . Let $f, g \in C(S, I)$. Then,

$$g \circ L_f = f \bullet_{\delta} g \text{ and } g \circ R_f = g \bullet_{\delta} f. \quad (25)$$

As a consequence one has the

31 Lemma *Let $\mathbf{S} = (S, \delta)$ be a cosemigroup object in \mathbb{C} . Assume that for each $f, g \in C(S, I)$, if for each $h \in C(S, I)$, $f \bullet_{\delta} h = g \bullet_{\delta} h$, then $f = g$, or for each $f, g \in C(S, I)$, if for each $h \in C(S, I)$, $h \bullet_{\delta} f = h \bullet_{\delta} g$, then $f = g$. Then, $\mathcal{C}_{\mathbb{C}, \mathbf{S}}: (C(S, I), \bullet_{\delta}) \rightarrow (\text{Comult}_{\mathbb{C}}(\mathbf{S}), \star)$ is one-to-one.*

5 The cocartesian (cartesian) monoidal setting

In this section we describe entirely the comultiplier monoid of a cosemigroup object in a cartesian monoidal category, and dually the multiplier monoid of a semigroup object in a cocartesian monoidal category.

By rewriting the axioms of (co)associativity and of (co)commutativity of (co)semigroups using the (co)projections of the (co)products one obtains rather easily the

32 Proposition 1. *Let $\mathbb{C} = (C, \times, 1)$ be a cartesian monoidal category. Let $S \in \text{Ob}C$. The bijection $C(S, S \times S) \simeq C(S, S) \times C(S, S)$, $\gamma \mapsto (\pi_1 \circ \gamma, \pi_2 \circ \gamma)$, with inverse $(s, t) \mapsto \langle s, t \rangle$, restricts and co-restricts to a bijection between the C -morphisms $\delta: S \rightarrow S \times S$ that make (C, δ) a cosemigroup and the pairs $(S \xrightarrow{f} S, S \xrightarrow{g} S)$ of C -morphisms such that $f \circ f = f$, $g \circ g = g$ and $f \circ g = g \circ f$. Under this bijection, (S, δ) is cocommutative iff $f = g$. Moreover if $\mathbf{S} = (S, \delta)$ is a cosemigroup, then $\text{InnComult}_{\mathbb{C}}(\mathbf{S}) = \{(\pi_2 \circ \delta, \pi_1 \circ \delta)\}$.*

2. Let $\mathbb{C} = (C, \sqcup, 0)$ be a cocartesian monoidal category. Let $S \in \text{Ob}C$. The bijection $C(S \sqcup S, S) \simeq C(S, S) \times C(S, S)$, $\gamma \mapsto (\gamma \circ q_1, \gamma \circ q_2)$, with inverse $(s, t) \mapsto [s, t]$, restricts and co-restricts to a bijection between the C -morphisms $\mu: S \sqcup S \rightarrow S$ that make (S, μ) a semigroup and the pairs $(S \xrightarrow{f} S, S \xrightarrow{g} S)$ of C -morphisms such that $f \circ f = f$, $g \circ g = g$ and $f \circ g = g \circ f$. Under this bijection, (S, μ) is commutative iff $f = g$. Moreover if $\mathbf{S} = (S, \mu)$ is a semigroup, then $\text{InnMult}_{\mathbb{C}}(\mathbf{S}) = \{(\mu \circ q_2, \mu \circ q_1)\}$.

Proof: Let us only prove the last assertion of the first item. Let $!_S: S \rightarrow 1$ be the terminal morphism out of S . Then, of course $\text{InnComult}_{\mathbb{C}}(\mathbf{S}) = \{(L_{!_S}, R_{!_S})\}$. Now $L_{!_S} = S \xrightarrow{\delta} S \times S \xrightarrow{!_S \times \text{id}_S} 1 \times S \xrightarrow{\lambda_S = \pi_2} S = S \xrightarrow{\delta} S \times S \xrightarrow{\pi_2} S \xrightarrow{\text{id}_S} S = S \xrightarrow{\delta} S \times S \xrightarrow{\pi_2} S$ and $R_{!_S} = S \xrightarrow{\delta} S \times S \xrightarrow{\text{id}_S \times !_S} S \times 1 \xrightarrow{\pi_1} S = S \xrightarrow{\delta} S \times S \xrightarrow{\pi_1} S$. \square

In what follows for a C -morphism $a \xrightarrow{f} b \times c$ (resp. $a \sqcup b \xrightarrow{f} c$) one denotes $f_i := \pi_i \circ f$ (resp. $f_i := f \circ q_i$), $i = 1, 2$.

33 Lemma *Let $\mathbb{C} = (C, \times, 1)$ be a cartesian monoidal category. Let $\mathbf{S} = (S, \delta)$ and $\mathbf{S}' = (S', \delta')$ be cosemigroups in \mathbb{C} . A C -morphism $\phi: S \rightarrow S'$ is a morphism of cosemigroups from \mathbf{S} to \mathbf{S}' iff $\phi \circ \delta_1 = \delta'_1 \circ \phi$ and $\phi \circ \delta_2 = \delta'_2 \circ \phi$.*

Proof: Because $(\phi \times \phi) \circ \delta = (\phi \times \phi) \circ \langle \delta_1, \delta_2 \rangle = \langle \phi \circ \delta_1, \phi \circ \delta_2 \rangle$ and $\delta' \circ \phi = \langle \delta'_1, \delta'_2 \rangle \circ \phi = \langle \delta'_1 \circ \phi, \delta'_2 \circ \phi \rangle$. \square

Dually,

34 Lemma *Let $\mathbb{C} = (C, \sqcup, 0)$ be a cocartesian monoidal category. Let $\mathbf{S} = (S, \mu)$ and $\mathbf{S}' = (S', \mu')$ be semigroups in \mathbb{C} . A C -morphism $\phi: S \rightarrow S'$ is a morphism of semigroups from \mathbf{S} to \mathbf{S}' iff $\phi \circ \mu_1 = \mu'_1 \circ \phi$ and $\phi \circ \mu_2 = \mu'_2 \circ \phi$.*

Let C be a category. Define the category $\mathbf{Idem}(C)$ of *idempotent endomorphisms* with objects the pairs (c, f) where $c \in \mathit{Ob}C$, $f \in C(c, c)$ and $f^2 = f$, hom-sets $\mathbf{Idem}(C)((c, f), (c', f')) := \{ \phi \in C(c, c') : f' \circ \phi = \phi \circ f \}$, composition inherited from C and $id_{(c, f)} := id_c$, $c \in \mathit{Ob}C$.

Let $\mathbf{Idem}_2(C)$ be the category of *commuting idempotent endomorphisms* with objects the triples (c, f, g) , with $c \in \mathit{Ob}C$, $f, g \in C(c, c)$ such that $f \circ f = f$, $g \circ g = g$ and $f \circ g = g \circ f$, hom-sets $\mathbf{Idem}_2(C)((c, f, g), (c', f', g')) := \{ \phi \in C(c, c') : f' \circ \phi = \phi \circ f, g' \circ \phi = \phi \circ g \}$, composition inherited from C and $id_{(c, f, g)} = id_c$, $c \in \mathit{Ob}C$.

There is an embedding functor $D: \mathbf{Idem}(C) \rightarrow \mathbf{Idem}_2(C)$ given by $((c, f) \xrightarrow{\phi} (c', f')) \mapsto ((c, f, f) \xrightarrow{\phi} (c', f', f'))$.

The above results are used to get the

35 Proposition *Let $\mathbb{C} = (C, \times, 1)$ (resp. $\mathbb{C} = (C, \sqcup, 0)$) be a cartesian (resp. cocartesian) monoidal category. Then, $\mathbf{Idem}_2(C)$ is isomorphic to $\mathbf{Cosem}(\mathbb{C})$ (resp. $\mathbf{Sem}(\mathbb{C})$) and $\mathbf{Idem}(C)$ is isomorphic to ${}_{\mathit{coc}}\mathbf{Cosem}(\mathbb{C})$ (resp. ${}_c\mathbf{Sem}(\mathbb{C})$).*

Using the projections out of a product, one easily obtains the following result.

36 Lemma *Let $\mathbb{C} = (C, \times, 1)$ be a cartesian monoidal category. Let $\mathbf{S} = (S, \delta)$ be a cosemigroup in \mathbb{C} . Let c be a C -object.*

1. *Let $\beta: c \rightarrow S \times c$. (c, β) is a left \mathbf{S} -coact iff $\delta_1 \circ \beta_1 = \beta_1$, $\beta_1 \circ \beta_2 = \delta_2 \circ \beta_1$ and $\beta_2 \circ \beta_2 = \beta_2$.*

2. Let $\tau: c \rightarrow c \times S$. (c, τ) is a right \mathbf{S} -coact iff $\delta_2 \circ \tau_2 = \tau_2$, $\tau_2 \circ \tau_1 = \delta_1 \circ \tau_2$ and $\tau_1 \circ \tau_1 = \tau_1$.

And dually, using the coprojections into a coproduct,

37 Lemma Let $\mathbb{C} = (C, \sqcup, 0)$ be a cartesian monoidal category. Let $\mathbf{S} = (S, \mu)$ be a cosemigroup in \mathbb{C} . Let c be a C -object.

1. Let $\gamma: S \times c \rightarrow c$. (c, γ) is a left \mathbf{S} -act iff $\gamma_1 \circ \mu_1 = \gamma_1$, $\gamma_2 \circ \gamma_1 = \gamma_2 \circ \mu_2$ and $\gamma_2 \circ \gamma_2 = \gamma_2$.

2. Let $\rho: c \times S \rightarrow c$. (c, ρ) is a right \mathbf{S} -act iff $\rho_2 \circ \mu_2 = \rho_2$, $\rho_1 \circ \rho_2 = \rho_2 \circ \mu_1$ and $\rho_1 \circ \rho_1 = \rho_1$.

38 Proposition Let $\mathbb{C} = (C, \times, 1)$ be a cartesian monoidal category. Let $\mathbf{S} = (S, \delta)$ be a cosemigroup in \mathbb{C} . Then, $\text{Comult}_{\mathbb{C}}(\mathbf{S}) = \{(L, R) \in C(S, S)^2: L \circ \delta_1 = \delta_1 \circ L, L \circ \delta_2 = \delta_2 = \delta_2 \circ L, R \circ \delta_1 = \delta_1 = \delta_1 \circ R, R \circ \delta_2 = \delta_2 \circ R\}$. If \mathbf{S} is cocommutative, that is, $\delta_1 = \delta_2$, and δ_1 is either a monomorphism or an epimorphism, then $\text{Comult}_{\mathbb{C}}(\mathbf{S}) = \{(L, L): L \in C(S, S), L \circ \delta_1 = \delta_1 \circ L = \delta_1\}$.

Proof: The commutativity of the diagram for endomorphisms of the right \mathbf{S} -act (S, δ)

$$\begin{array}{ccc} S & \xrightarrow{L} & S \\ \delta \downarrow & & \downarrow \delta \\ S \times S & \xrightarrow{L \times id_S} & S \times S \end{array} \quad (26)$$

is equivalent to the equations $L \circ \delta_1 = L \circ \pi_1 \circ \delta = \pi_1 \circ (L \times id_S) \circ \delta = \pi_1 \circ \delta \circ L = \delta_1 \circ L$ and $\delta_2 = \pi_2 \circ \delta = \pi_2 \circ (L \times id_S) \circ \delta = \pi_2 \circ \delta \circ L = \delta_2 \circ L$, while the commutativity of the diagram for endomorphisms of the left \mathbf{S} -act (S, δ)

$$\begin{array}{ccc} S & \xrightarrow{R} & S \\ \delta \downarrow & & \downarrow \delta \\ S \times S & \xrightarrow{id_S \times R} & S \times S \end{array} \quad (27)$$

is equivalent to the equations $\delta_1 = \pi_1 \circ \delta = \pi_1 \circ (id_S \times R) \circ \delta = \pi_1 \circ \delta \circ R = \delta_1 \circ R$ and $R \circ \delta_2 = R \circ \pi_2 \circ \delta = \pi_2 \circ (id_S \times R) \circ \delta = \pi_2 \circ \delta \circ R = \delta_2 \circ R$.

Finally (L, R) is a comultiplier iff $(R \times id_S) \circ \delta = (id_S \times L) \circ \delta$ iff $R \circ \delta_1 = R \circ \pi_1 \circ \delta = \pi_1 \circ (R \times id_S) \circ \delta = \pi_1 \circ (id_S \times L) \circ \delta = \pi_1 \circ \delta = \delta_1$ and $\delta_2 = \pi_2 \circ \delta = \pi_2 \circ (R \times id_S) \circ \delta = \pi_2 \circ (id_S \times L) \circ \delta = L \circ \pi_2 \circ \delta = L \circ \delta_2$.

Assume that (S, δ) is cocommutative, that is, $\delta_1 = \delta_2$. Let $(L, R) \in \text{Comult}_{\mathbb{C}}(S, \delta)$. Then, $L \circ \delta_2 = \delta_2 \circ L = \delta_2 = \delta_1 = R \circ \delta_1 = \delta_1 \circ R$. Therefore when δ_1 is either a monomorphism or an epimorphism, $L = R$. Since the converse is obvious, the last assertion is proved. \square

Finally by dualization:

39 Proposition *Let $\mathbb{C} = (C, \sqcup, 0)$ be a cocartesian monoidal category. Let $S = (S, \mu)$ be a semigroup in \mathbb{C} . Then, $\text{Mult}_{\mathbb{C}}(S) = \{(L, R) \in C(S, S)^2 : L \circ \mu_1 = \mu_1 \circ L, L \circ \mu_2 = \mu_2 = \mu_2 \circ L, R \circ \mu_1 = \mu_1 = \mu_1 \circ R, R \circ \mu_2 = \mu_2 \circ R\}$. If S is commutative, that is, $\mu_1 = \mu_2$, and μ_1 is either a monomorphism or an epimorphism, then $\text{Mult}_{\mathbb{C}}(S) = \{(L, L) : L \in C(S, S), L \circ \mu_1 = \mu_1 \circ L = \mu_1\}$.*

6 Functoriality

In this section one describes two domain categories which make Mult a functor. This will be further explored in Section 7.2.

6.1 Functorial behavior of the Mult construction

Let $\mathbb{C} = (C, \otimes_C, I_C)$ and $\mathbb{D} = (D, \otimes_D, I_D)$ be monoidal categories, and $\mathbb{F} = (F, \Phi, \phi) : \mathbb{C} \rightarrow \mathbb{D}$ be a lax monoidal functor.

40 Proposition *For all semigroups $S = (S, \mu), S' = (S', \mu')$ in \mathbb{C} , \mathbb{F} induces functors $\mathbf{Act}_S(\mathbb{C}) \rightarrow \mathbf{Act}_{\tilde{\mathbb{F}}(S)}(\mathbb{D}), {}_S\mathbf{Act}(\mathbb{C}) \rightarrow \tilde{\mathbb{F}}(S)\mathbf{Act}(\mathbb{D})$ and ${}_S\mathbf{Act}_{S'}(\mathbb{C}) \rightarrow \tilde{\mathbb{F}}(S)\mathbf{Act}_{\tilde{\mathbb{F}}(S')}\mathbb{D}$, all denoted by \tilde{F} , and the following diagrams commute.*

$$\begin{array}{ccc}
\mathbf{Act}_S(\mathbb{C}) & \xrightarrow{\tilde{F}} & \mathbf{Act}_{\tilde{\mathbb{F}}(S)}(\mathbb{D}) \\
|-|_S \downarrow & & \downarrow |-|_{\tilde{\mathbb{F}}(S)} \\
C & \xrightarrow{F} & D \\
s|-|_{S'} \uparrow & & \uparrow \tilde{\mathbb{F}}(S) |-|_{\tilde{\mathbb{F}}(S')} \\
{}_S\mathbf{Act}_{S'}(\mathbb{C}) & \xrightarrow{\tilde{F}} & \tilde{\mathbb{F}}(S)\mathbf{Act}_{\tilde{\mathbb{F}}(S')}\mathbb{D}
\end{array}
\qquad
\begin{array}{ccc}
{}_S\mathbf{Act}(\mathbb{C}) & \xrightarrow{\tilde{F}} & \tilde{\mathbb{F}}(S)\mathbf{Act}(\mathbb{D}) \\
s|-| \downarrow & & \downarrow \tilde{\mathbb{F}}(S) |-| \\
C & \xrightarrow{F} & D
\end{array}
\quad (28)$$

Proof: Let (c, γ) be a left S -act. One claims that $\tilde{F}(c, \gamma) := (F(c), F(\gamma) \circ \Phi_{S,c})$ is a left $\tilde{\mathbb{F}}(S)$ -act. This follows from the commutativity of the following diagram. (The top left cell commutes by coherence and the bottom right

cell commutes since γ is a left action.)

$$\begin{array}{ccccc}
F(S) \otimes_D (F(S) \otimes_D F(c)) & \xrightarrow{id_{F(S)} \otimes \Phi_{S,c}} & F(S) \otimes_D F(S \otimes_C c) & & \\
\downarrow \alpha_{F(S), F(S), F(c)}^{-1} & & \downarrow \Phi_{S, S \otimes_C c} & \searrow id_{F(S)} \otimes_D F(\gamma) & \\
(F(S) \otimes_D F(S)) \otimes_D F(c) & & F(S \otimes_C (S \otimes_C c)) & & F(S) \otimes_D F(c) \\
\downarrow \Phi_{S, S \otimes_D id_{F(c)}} & & \downarrow F(\alpha_{S, S, c}^{-1}) & \searrow F(id_S \otimes_C \gamma) & \downarrow \Phi_{S, c} \\
F(S \otimes_C S) \otimes_D F(c) & \xrightarrow{\Phi_{S \otimes_C S, c}} & F((S \otimes_C S) \otimes_C c) & & F(S \otimes_C c) \\
\downarrow F(\mu) \otimes id_{F(c)} & & \downarrow F(\mu \otimes_C id_c) & & \downarrow F(\gamma) \\
F(S) \otimes_D F(c) & \xrightarrow{\Phi_{S, c}} & F(S \otimes_C c) & \xrightarrow{F(\gamma)} & F(c)
\end{array} \tag{29}$$

Now let $f \in {}_S \mathbf{Act}(\mathbb{C})((c, \gamma), (c', \gamma'))$. Then, $\tilde{F}(f) := F(f)$ belongs to $\tilde{\mathbb{F}}(S) \mathbf{Act}(\mathbb{D})((F(c), F(\gamma) \circ \Phi_{S, c}), (F(c'), F(\gamma') \circ \Phi_{S, c'}))$ as it is shown by the commutativity of the following diagram.

$$\begin{array}{ccc}
F(c) & \xrightarrow{F(f)} & F(c') \\
F(\gamma) \uparrow & & \uparrow F(\gamma') \\
F(S \otimes_C c) & \xrightarrow{F(id_S \otimes_C f)} & F(S \otimes_C c') \\
\Phi_{S, c} \uparrow & & \uparrow \Phi_{S, c'} \\
F(S) \otimes_D F(c) & \xrightarrow{id_{F(S)} \otimes_D F(f)} & F(S) \otimes_D F(c')
\end{array} \tag{30}$$

One thus obtains the desired functor $\tilde{F}: {}_S \mathbf{Act}(\mathbb{C}) \rightarrow \tilde{\mathbb{F}}(S) \mathbf{Act}(\mathbb{D})$.

Let (c, ρ) be a right S -act. By symmetry, $\tilde{F}(c, \rho) := (F(c), F(\rho) \circ \Phi_{c, S})$ is a right $\tilde{\mathbb{F}}(S)$ -act and given $f \in \mathbf{Act}_S(\mathbb{C})((c, \rho), (c', \rho'))$, $\tilde{F}(f) := F(f) \in \mathbf{Act}_{\tilde{\mathbb{F}}(S)}(\mathbb{D})((F(c), F(\rho) \circ \Phi_{c, S}), (F(c'), F(\rho') \circ \Phi_{c', S}))$. One thus obtains the desired functor $\tilde{F}: \mathbf{Act}_S(\mathbb{C}) \rightarrow \mathbf{Act}_{\tilde{\mathbb{F}}(S)}(\mathbb{D})$.

Let (c, γ, ρ) be a S - S' -biact. That $(F(c), F(\gamma) \circ \Phi_{S, c}, F(\rho) \circ \Phi_{S', c})$ is a $\tilde{\mathbb{F}}(S)$ - $\tilde{\mathbb{F}}(S')$ -biact follows from the commutativity of the following diagram. (The top right cell commutes by coherence. The bottom left diagram com-

mates because (c, γ, ρ) is a S - S' -biact.)

$$\begin{array}{ccccc}
(F(S) \otimes_D F(c)) \otimes_D \tilde{F}(S') & \xrightarrow{\Phi_{S,f} \otimes_D id_{F(S')}} & \tilde{F}(S \otimes_C c) \otimes_D F(S') & \xrightarrow{\tilde{F}(\gamma) \otimes_D id_{F(S')}} & F(c) \otimes_D F(S') \\
\alpha_{F(S), F(c), F(S')} \downarrow & & \Phi_{S \otimes_C c, S'} \downarrow & & \downarrow \Phi_{c, S'} \\
F(S) \otimes_D (F(c) \otimes_D F(S')) & & F((S \otimes_C c) \otimes_C S') & & \\
id_{F(S)} \otimes_D \Phi_{c, S'} \downarrow & & F(\alpha_{S, c, S'}) \downarrow & & \\
F(S) \otimes_D F(c \otimes_C S') & \xrightarrow{\Phi_{S, c \otimes_C S'}} & \tilde{F}(S \otimes_C (c \otimes_C S')) & \xrightarrow{\tilde{F}(\gamma \otimes_C id_{S'})} & F(c \otimes_C S') \\
id_{F(S)} \otimes_D F(\rho) \downarrow & & F(id_S \otimes \rho) \downarrow & & \downarrow F(\rho) \\
F(S) \otimes_D F(c) & \xrightarrow{\Phi_{S, c}} & F(S \otimes_C c) & \xrightarrow{F(\gamma)} & F(c)
\end{array} \tag{31}$$

One may thus define a functor $\tilde{F}: {}_S Act_{S'}(\mathbb{C}) \rightarrow \tilde{\mathbb{F}}(S) Act_{\tilde{\mathbb{F}}(S')}(\mathbb{D})$ acting as $\tilde{F}((c, \gamma, \rho) \xrightarrow{f} (c', \gamma', \rho')) = (F(c), F(\gamma) \circ \Phi_{S, c}, F(\rho) \circ \Phi_{c, S'}) \xrightarrow{F(f)} (F(c'), F(\gamma') \circ \Phi_{S, c'}, F(\rho') \circ \Phi_{c', S'})$. \square

41 Lemma *Let $S = (S, \mu), S' = (S', \mu')$ be semigroups in \mathbb{C} . Let (c, γ) be a left S -act and let (c', ρ') be a right S' -act. Then, $\Phi_{c, c'} \in \tilde{\mathbb{F}}(S) Act_{\tilde{\mathbb{F}}(S')}(\mathbb{D})(\tilde{F}(c, \gamma) \otimes_D \tilde{F}(c', \rho'), \tilde{F}((c, \gamma) \otimes_C (c', \rho')))$.*

Proof: According to Proposition 8, $(c, \gamma) \otimes_C (c', \rho') = (c \otimes_C c', (\gamma \otimes id_{c'}) \circ \alpha_{S, c, c'}^{-1}, (id_c \otimes \rho') \circ \alpha_{c, c', S'})$ whence $\tilde{F}((c, \gamma) \otimes_C (c', \rho')) = (F(c \otimes_C c'), F(\gamma \otimes id_{c'}) \circ F(\alpha_{S, c, c'}^{-1}) \circ \Phi_{S, c \otimes_C c'}, F(id_c \otimes \rho') \circ F(\alpha_{c, c', S'}) \circ \Phi_{c \otimes_C c', S'})$.

Now $\tilde{F}(c, \gamma) \otimes_D \tilde{F}(c', \rho') = (F(c), F(\gamma) \circ \Phi_{S, c}) \otimes_D (F(c'), F(\rho') \circ \Phi_{c', S'}) = (F(c) \otimes_D F(c'), (F(\gamma) \otimes_D id_{F(c')}) \circ (\Phi_{S, c} \otimes id_{F(c')}) \circ \alpha_{F(S), F(c), F(c')}^{-1}, (id_{F(c)} \otimes_D F(\rho')) \circ (id_{F(c)} \otimes_D \Phi_{c', S'}) \circ \alpha_{F(c), F(c'), F(S')})$.

The following diagram commutes. (The top cell commutes by naturality while the bottom cell commutes by coherence.)

$$\begin{array}{ccc}
F(c) \otimes_D F(c') & \xrightarrow{\Phi_{c, c'}} & F(c \otimes_C c') \\
F(\gamma) \otimes_D id_{F(c')} \uparrow & & \uparrow F(\gamma \otimes_C id_{c'}) \\
F(S \otimes_C c) \otimes_D F(c') & \xrightarrow{\Phi_{S \otimes_C c, c'}} & F((S \otimes_C c) \otimes_C c') \\
\Phi_{S, c} \otimes_D id_{F(c')} \uparrow & & \uparrow F(\alpha_{S, c, c'}^{-1}) \\
(F(S) \otimes_D F(c)) \otimes_D F(c') & & F(S \otimes_C (c \otimes_C c')) \\
\alpha_{F(S), F(c), F(c')}^{-1} \uparrow & & \uparrow \Phi_{S, c \otimes_C c'} \\
F(S) \otimes_D (F(c) \otimes_D F(c')) & \xrightarrow{id_{F(S)} \otimes_D \Phi_{c, c'}} & F(S) \otimes_D F(c \otimes_C c')
\end{array} \tag{32}$$

Whence $\Phi_{c, c'}$ is a left $\tilde{\mathbb{F}}(S)$ -morphism. Symmetrically, $\Phi_{c, c'}$ is a right $\tilde{\mathbb{F}}(S')$ -morphism. Consequently, $\Phi_{c, c'} \in \tilde{\mathbb{F}}(S) \mathbf{Act}_{\tilde{\mathbb{F}}(S')}(\mathbb{D})(\tilde{F}(c, \gamma) \otimes_D \tilde{F}(c', \rho'), \tilde{F}((c, \gamma) \otimes_C (c', \rho')))$. \square

Let $\mathbb{S} = (S, \mu)$ be a semigroup in \mathbb{C} . Let $L \in \mathbf{Act}_{\mathbb{S}}(\mathbb{C})((S, \mu), (S, \mu))$. By Prop. 40, $\tilde{F}(L) = F(L) \in \mathbf{Act}_{\tilde{\mathbb{F}}(\mathbb{S})}(\mathbb{D})(\tilde{F}(S, \mu), \tilde{F}(S, \mu))$. Now

$$\begin{aligned} \tilde{F}(\mathfrak{g}_{\mathbb{C}, \mathbb{S}}(L)) \circ \Phi_{S, S} &= F(\mu) \circ F(id_S \otimes_C L) \circ \Phi_{S, S} \\ &= F(\mu) \circ \Phi_{S, S} \circ (id_{F(S)} \otimes_D F(L)) \\ &= \mathfrak{g}_{\mathbb{D}, \tilde{\mathbb{F}}(\mathbb{S})}(\tilde{F}(L)). \end{aligned} \quad (33)$$

In a similar way one sees that for each $R \in {}_{\mathbb{S}}\mathbf{Act}(\mathbb{C})((S, \mu), (S, \mu))$, $\tilde{F}(d_{\mathbb{C}, \mathbb{S}}(R)) = d_{\mathbb{D}, \tilde{\mathbb{F}}(\mathbb{S})}(\tilde{F}(R))$ as members of ${}_{\tilde{\mathbb{F}}(\mathbb{S})}\mathbf{Act}_{\tilde{\mathbb{F}}(\mathbb{S})}(\tilde{F}(S, \mu) \otimes_D \tilde{F}(S, \mu), \tilde{F}(S, \mu))$.

Now let $(L, R) \in \mathbf{Mult}_{\mathbb{C}}(\mathbb{S})$. Then $(\tilde{F}(L), \tilde{F}(R)) = (F(L), F(R)) \in \mathbf{Mult}_{\mathbb{D}}(\tilde{\mathbb{F}}(\mathbb{S}))$ since

$$\begin{aligned} \mathfrak{g}_{\mathbb{D}, \tilde{\mathbb{F}}(\mathbb{S})}(\tilde{F}(L)) &= \tilde{F}(\mathfrak{g}_{\mathbb{C}, \mathbb{S}}(L)) \circ \Phi_{S, S} \\ &= \tilde{F}(d_{\mathbb{C}, \mathbb{S}}(R)) \circ \Phi_{S, S} \\ &= d_{\mathbb{D}, \tilde{\mathbb{F}}(\mathbb{S})}(\tilde{F}(R)). \end{aligned} \quad (34)$$

Now one thus easily obtains the following

42 Proposition $(L, R) \mapsto (F(L), F(R))$ is a homomorphism $\mathbf{Mult}_{disc}(\mathbb{S}; \mathbb{F})$ of monoids from $\mathbf{Mult}_{\mathbb{C}}(\mathbb{S})$ to $\mathbf{Mult}_{\mathbb{D}}(\tilde{\mathbb{F}}(\mathbb{S}))$. If F is faithful, then this homomorphism is one-to-one.

43 Corollary Let (\mathbb{C}, σ) be a symmetric monoidal category. Then for each semigroup \mathbb{S} in \mathbb{C} , $\mathbf{Mult}_{\mathbb{C}}(\mathbb{S}) \simeq \mathbf{Mult}_{\mathbb{C}}(\mathbb{S}^{op})$, $(L, R) \mapsto (R, L)$. In particular if \mathbb{S} is commutative, then $(L, R) \mapsto (R, L)$ provides an involutive automorphism of $\mathbf{Mult}_{\mathbb{C}}(\mathbb{S})$.

Proof: It suffices to apply Prop. 42 with the monoidal functor $\Sigma := (Id_{\mathbb{C}}, \sigma, id_I): \mathbb{C} \rightarrow \mathbb{C}^t$ to obtain the homomorphism $\mathbf{Mult}_{disc}(\mathbb{S}; \Sigma): \mathbf{Mult}_{\mathbb{C}}(\mathbb{S}) \rightarrow \mathbf{Mult}_{\mathbb{C}^t}(\mathbb{S}^{op})$, and then to note that $(\mathbb{S}^{op})^{op} = \mathbb{S}$ and $(\mathbb{C}^t)^t = \mathbb{C}$ so that $\Sigma: \mathbb{C}^t \rightarrow \mathbb{C}$, which then provides a homomorphism of monoids from $\mathbf{Mult}_{\mathbb{C}^t}(\mathbb{S}^{op})$ to $\mathbf{Mult}_{\mathbb{C}}(\mathbb{S})$, inverse from the first one. One concludes using Lemma 11. \square

44 Example Assume that \mathbb{C}' is a monoidal subcategory of \mathbb{C} , that is, the canonical inclusion functor is strictly monoidal. Then for any semigroup \mathbb{S} in \mathbb{C}' , $\mathbf{Mult}_{\mathbb{C}'}(\mathbb{S}) \subseteq \mathbf{Mult}_{\mathbb{C}}(\mathbb{S})$. If the inclusion is full, then one has even $\mathbf{Mult}_{\mathbb{C}'}(\mathbb{S}) = \mathbf{Mult}_{\mathbb{C}}(\mathbb{S})$.

45 Lemma Let \mathbb{S} be a semigroup in \mathbb{C} .

1. $\mathbf{Mult}_{disc}(\mathbb{S}; \mathbb{1}_{\mathbb{C}}) = id_{\mathbf{Mult}_{\mathbb{C}}(\mathbb{S})}$.

2. Let $\mathbb{F}: \mathbb{A} \rightarrow \mathbb{B}$ and $\mathbb{G}: \mathbb{B} \rightarrow \mathbb{C}$ be lax monoidal functors. Then one has $Mult_{disc}(\mathbb{S}; \mathbb{G} \circ \mathbb{F}) = Mult_{disc}(\widetilde{\mathbb{F}}(\mathbb{S}); \mathbb{G}) \circ Mult_{disc}(\mathbb{S}; \mathbb{F}): \mathbf{Mult}_{\mathbb{A}}(\mathbb{S}) \rightarrow \mathbf{Mult}_{\mathbb{C}}(\widetilde{\mathbb{G} \circ \mathbb{F}}(\mathbb{S}))$.

What kind of functoriality is stated above? By Grothendieck's construction (see e.g. [1]) the functor $Sem: \mathbf{MonCat} \rightarrow \mathbf{Cat}$ from Section 2 provides a split opfibration over \mathbf{MonCat} , that is, an opfibred category $\mathbf{IntSem} \xrightarrow{\Pi} \mathbf{MonCat}$ with a choice of cartesian morphisms, where \mathbf{IntSem} is the category of *all* internal semigroups. In details, \mathbf{IntSem} has objects the pairs (\mathbb{C}, \mathbb{S}) consisting of a monoidal category \mathbb{C} together with a semigroup \mathbb{S} in \mathbb{C} , morphisms $(\mathbb{C}, \mathbb{S}) \xrightarrow{(\mathbb{F}, f)} (\mathbb{D}, \mathbb{T})$ the pairs (\mathbb{F}, f) consisting of a monoidal functor $\mathbb{F}: \mathbb{C} \rightarrow \mathbb{D}$ and a morphism $f \in \mathbf{Sem}(\mathbb{D})(\widetilde{\mathbb{F}}(\mathbb{S}), \mathbb{T})$, composition: $(\mathbb{C}, \mathbb{S}) \xrightarrow{(\mathbb{F}, f)} (\mathbb{D}, \mathbb{T}) \xrightarrow{(\mathbb{G}, g)} (\mathbb{E}, \mathbb{U}) = (\mathbb{C}, \mathbb{S}) \xrightarrow{(\mathbb{G} \circ \mathbb{F}, g \circ \widetilde{\mathbb{G}}(f))} (\mathbb{E}, \mathbb{U})$, and identities $id_{(\mathbb{C}, \mathbb{S})} := (\mathbb{C}, \mathbb{S}) \xrightarrow{(id_{\mathbb{C}}, id_{\mathbb{S}})} (\mathbb{C}, \mathbb{S})$. Moreover the projection functor $\Pi: \mathbf{IntSem} \rightarrow \mathbf{MonCat}$ is a split opfibration given by $\Pi((\mathbb{C}, \mathbb{S}) \xrightarrow{(\mathbb{F}, f)} (\mathbb{D}, \mathbb{T})) := \mathbb{C} \xrightarrow{\mathbb{F}} \mathbb{D}$ and with cleavage described as follows: given a monoidal functor $\mathbb{F}: \mathbb{C} \rightarrow \mathbb{D}$ and a semigroup \mathbb{S} in \mathbb{C} , $(\mathbb{C}, \mathbb{S}) \xrightarrow{(\mathbb{F}, id_{\widetilde{\mathbb{F}}(\mathbb{S})})} (\mathbb{D}, \widetilde{\mathbb{F}}(\mathbb{S}))$ is an opcartesian morphism over \mathbb{F} with domain (\mathbb{C}, \mathbb{S}) . The fibre at a monoidal category \mathbb{C} , that is, the subcategory of \mathbf{IntSem} consisting of morphisms of the form $(\mathbb{C}, \mathbb{S}) \xrightarrow{(id_{\mathbb{C}}, f)} (\mathbb{C}, \mathbb{T})$, is isomorphic to $\mathbf{Sem}(\mathbb{C})$.

\mathbf{IntSem} has two subcategories of particular interest for us. Let \mathbf{IntSem}^{disc} be the subcategory of \mathbf{IntSem} consisting only of its opcartesian morphisms. \mathbf{IntSem}^{disc} is in fact (isomorphic to) the category of elements (see [4]) of the functor $\mathbf{MonCat} \xrightarrow{Sem} \mathbf{Cat} \xrightarrow{Ob} \mathbf{Set}$ and thus corresponds to a discrete opfibration. Lemma 45 tells us that there is a functor $Mult_{disc}: \mathbf{IntSem}^{disc} \rightarrow \mathbf{Mon}$, acting as $Mult_{disc}((\mathbb{C}, \mathbb{S}) \xrightarrow{(\mathbb{F}, id_{\widetilde{\mathbb{F}}(\mathbb{S})})} (\mathbb{D}, \widetilde{\mathbb{F}}(\mathbb{S}))) := \mathbf{Mult}_{\mathbb{C}}(\mathbb{S}) \xrightarrow{Mult_{disc}(\mathbb{S}; \mathbb{F})} \mathbf{Mult}_{\mathbb{D}}(\widetilde{\mathbb{F}}(\mathbb{S}))$.

The second interesting subcategory \mathbf{IntSem}^{iso} of \mathbf{IntSem} will be described hereafter. With the notation A_{iso} for a category A is meant the subcategory of A with the same objects and with only A -isomorphisms as morphisms. The following easy result tells us that the multiplier construction enables transport of structures.

46 Lemma *For each monoidal category \mathbb{C} , there is a conjugation functor $Conj_{\mathbb{C}}: \mathbf{Sem}(\mathbb{C})_{iso} \rightarrow \mathbf{Mon}_{iso}$ given by $Conj_{\mathbb{C}}\left((S, \mu) \xrightarrow{\phi} (S', \mu')\right) := \mathbf{Mult}_{\mathbb{C}}(S, \mu) \xrightarrow{Conj_{\mathbb{C}}(\phi)} \mathbf{Mult}_{\mathbb{C}}(S', \mu')$, $(L, R) \mapsto (\phi \circ L \circ \phi^{-1}, \phi \circ R \circ \phi^{-1})$.*

Let $(-)_{iso}: \mathbf{Cat} \rightarrow \mathbf{Cat}$ be the functor acting as $F \mapsto F_{iso}$, with F_{iso} the restriction and co-restriction of a functor F . Let us consider the composition $\mathbf{MonCat} \xrightarrow{Sem} \mathbf{Cat} \xrightarrow{(-)_{iso}} \mathbf{Cat}$. This gives rise to a split opfibration $\mathbf{IntSem}^{iso} \rightarrow \mathbf{MonCat}$ in groupoids, that is, its fibres are groupoids, as follows: the objects of \mathbf{IntSem}^{iso} are the same as those of \mathbf{IntSem} , the morphisms $(\mathbb{C}, \mathbb{S}) \xrightarrow{(F, \phi)} (\mathbb{D}, \mathbb{T})$ are those of \mathbf{IntSem} such that $\phi \in \mathbf{Sem}(\mathbb{C})_{iso}(\tilde{\mathbb{F}}(\mathbb{S}), \mathbb{T})$. Composition and identities are inherited from \mathbf{IntSem} . Consequently, one has inclusions of categories $\mathbf{IntSem}^{disc} \hookrightarrow \mathbf{IntSem}^{iso} \hookrightarrow \mathbf{IntSem}$.

47 Proposition *One has a functor $Mult_{iso}$ from \mathbf{IntSem}^{iso} to \mathbf{Mon} , given by $Mult_{iso}((\mathbb{C}, \mathbb{S}) \xrightarrow{(F, \phi)} (\mathbb{D}, \mathbb{T})) := Mult_{\mathbb{C}}(\mathbb{S}) \xrightarrow{Mult_{disc}(\mathbb{S}; F)} Mult_{\mathbb{D}}(\tilde{\mathbb{F}}(\mathbb{S})) \xrightarrow{Conj_{\mathbb{D}}(\phi)} Mult_{\mathbb{D}}(\mathbb{T})$.*

Proof: This follows from the fact that any \mathbf{IntSem}^{iso} -morphism $(\mathbb{C}, \mathbb{S}) \xrightarrow{(F, \phi)} (\mathbb{D}, \mathbb{T})$ factors as $(\mathbb{C}, \mathbb{S}) \xrightarrow{(F, id_{\tilde{\mathbb{F}}(\mathbb{S})})} (\mathbb{D}, \tilde{\mathbb{F}}(\mathbb{S})) \xrightarrow{(id_{\mathbb{D}}, \phi)} (\mathbb{D}, \mathbb{T})$ where the factor on the left is an opcartesian morphism of \mathbf{IntSem} so a \mathbf{IntSem}^{disc} -morphism and the factor on the right is essentially a $\mathbf{Sem}(\mathbb{D})_{iso}$ -morphism. \square

6.2 The concretization homomorphism

Let $\mathbb{F} := (F, \Phi, \phi): \mathbb{C} = (C, \otimes_C, I) \rightarrow \mathbb{D} = (D, \otimes_D, J)$ be a monoidal functor. Let S be a C -object and let T be a D -object. Let $f \in D(F(S), T)$. Then there is a map $D(\phi, f) \circ F_{I, S}$ from $C(I, S)$ to $D(J, T)$ given by $g \mapsto f \circ F(g) \circ \phi$.

Now let $\mathbb{S} = (S, \mu_S)$ and $\mathbb{T} = (T, \mu_T)$ be respectively semigroups in \mathbb{C} and in \mathbb{D} . Let $f \in \mathbf{Sem}(\mathbb{D})(\tilde{\mathbb{F}}(\mathbb{S}), \mathbb{T})$. Then, $D(\phi, f) \circ F_{I, S}$ is the underlying map of homomorphism of semigroups from $\mathbf{Conv}_I(\mathbb{S})$ to $\mathbf{Conv}_J(\mathbb{T})$. Indeed,

let $g, h \in C(I, S)$. Then,

$$\begin{aligned}
D(\phi, f)(F_{I,S}(h \bullet_{\mu_S} g)) &= D(\phi, f)(F_{I,S}(\mu_S \circ (h \otimes_C g) \circ \lambda_I^{-1})) \\
&= f \circ F(\mu_S) \circ F(h \otimes_C g) \circ F(\lambda_I)^{-1} \circ \phi \\
&= f \circ F(\mu_S) \circ F(h \otimes_C g) \circ \Phi_{I,I} \circ (\phi \otimes_D id_{F(I)}) \circ \lambda_{F(I)}^{-1} \circ \phi \\
&\text{(by coherence)} \\
&= f \circ F(\mu_S) \circ \Phi_{S,S} \circ (F(h) \otimes_D F(g)) \circ (\phi \otimes_D id_{F(I)}) \circ \lambda_{F(I)}^{-1} \circ \phi \\
&\text{(by naturality of } \Phi) \\
&= \mu_T \circ (f \otimes_D f) \circ (F(h) \otimes_D F(g)) \circ (\phi \otimes_D id_{F(I)}) \circ \lambda_{F(I)}^{-1} \circ \phi \\
&\text{(since } f \in \mathbf{Sem}(\mathbb{D})(\widetilde{\mathbb{F}}(S), \mathbb{T})) \\
&= \mu_T \circ ((f \circ F(h)) \otimes_D (f \circ F(g))) \circ (\phi \otimes_D id_{F(I)}) \circ (id_J \otimes_D \phi) \circ \lambda_J^{-1} \\
&\text{(by naturality of } \lambda^{-1}) \\
&= \mu_T \circ ((f \circ F(h)) \otimes_D (f \circ F(g))) \circ (\phi \otimes_D \phi) \circ \lambda_J^{-1} \\
&= \mu_T \circ ((f \circ F(h) \circ \phi) \otimes_D (f \circ F(g) \circ \phi)) \circ \lambda_J^{-1} \\
&= \mu_T \circ (D(\phi, f)(F_{I,S}(h)) \otimes_D D(\phi, f)(F_{I,S}(g))) \circ \lambda_J^{-1} \\
&= D(\phi, f)(F_{I,S}(h)) \bullet_{\mu_T} D(\phi, f)(F_{I,S}(g)).
\end{aligned} \tag{35}$$

One so obtains easily

48 Proposition *There is a functor $\mathbf{IntSem} \xrightarrow{\text{Conv}} \mathbf{Sem}$ such that $((\mathbb{C}, S) \xrightarrow{(\mathbb{F}, f)} (\mathbb{D}, \mathbb{T})) \mapsto (\text{Conv}_I(S) \xrightarrow{D(\phi, f) \circ F_{I,S}} \text{Conv}_J(\mathbb{T}))$.*

Now let us assume furthermore that in fact $f \in \mathbf{Sem}(\mathbb{D})_{iso}(\widetilde{\mathbb{F}}(S), \mathbb{T})$, that is, $f: \widetilde{\mathbb{F}}(S) \rightarrow \mathbb{T}$ is an isomorphism of semigroups in \mathbb{D} . Then the following diagram commutes in \mathbf{Sem} , where one recalls that $|-|$ stands for the forgetful functor $\mathbf{Mon} \rightarrow \mathbf{Sem}$. In other words, one has a natural transformation $(\mathcal{M}_{\mathbb{C}, S})_{\mathbb{C}, S}: \text{Conv}_{iso} \Rightarrow |-| \circ \text{Mult}_{iso}: \mathbf{IntSem}_{iso} \rightarrow \mathbf{Sem}$, where $\text{Conv}_{iso} := \mathbf{IntSem}_{iso} \hookrightarrow \mathbf{IntSem} \xrightarrow{\text{Conv}} \mathbf{Sem}$.

$$\begin{array}{ccc}
\text{Conv}_I(S) & \xrightarrow{\text{Conv}(\mathbb{F}, f)} & \text{Conv}_J(\mathbb{T}) \\
\mathcal{M}_{\mathbb{C}, S} \downarrow & & \downarrow \mathcal{M}_{\mathbb{D}, \mathbb{T}} \\
(\text{Mult}_{\mathbb{C}}(S), \star) & \xrightarrow{[\text{Mult}_{iso}(\mathbb{F}, f)]} & (\text{Mult}_{\mathbb{D}}(\mathbb{T}), \star)
\end{array} \tag{36}$$

Let us now define the following monoidal functor $\mathbb{G}_I = (G_I, \Gamma, \gamma): \mathbb{C} \rightarrow \mathbb{C}^{op} \times \mathbb{C}$ as follows: its underlying functor is $G_I: C \rightarrow C^{op} \times C$, $G_I(c \xrightarrow{f} d) := (I \xrightarrow{id_I} I, c \xrightarrow{f} d)$, $\Gamma_{c,d}: G_I(c) \otimes G_I(d) = (I \otimes I, c \otimes d) \xrightarrow{(\lambda_I^{-1}, id_{c \otimes d})} G_I(c \otimes d) = (I, c \otimes d)$, and $\gamma = (I, I) \xrightarrow{(id_I, id_I)} G_I(I) = (I, I)$.

Given a semigroup $S = (S, \mu)$ in \mathbb{C} , one thus has a semigroup $\widetilde{\mathbb{G}}_I(S) = ((I, S), (\lambda_I^{-1}, \mu))$ in $\mathbb{C}^{op} \times \mathbb{C}$ and then with $\widetilde{\text{Conv}}(\widetilde{\mathbb{G}}_I(S))$ is recovered $\text{Conv}_I(S)$. According to Prop. 42 one has a homomorphism of monoids, called the *concretization homomorphism*,

$$\text{Conc}_{\mathbb{C}}(S) := \text{Mult}_{\mathbb{C}}(S) \xrightarrow{\text{Mult}_{disc}(S; \text{Conv} \circ \mathbb{G}_I)} \text{Mult}_{\text{Set}}(\text{Conv}_I(S)) = \text{TrHull}(\text{Conv}_I(S))$$

which acts as $(L, R) \mapsto (C(id_I, L), C(id_I, R))$. Of course, $C(id_I, L): C(I, S) \rightarrow C(I, S)$, $f \mapsto L \circ f$ and $C(id_I, R): C(I, S) \rightarrow C(I, S)$, $f \mapsto R \circ f$.

49 Remark When $C(I, -)$ is faithful, then $\text{Mult}_{\mathbb{C}}(S)$ is (isomorphic to) a submonoid of $\text{TrHull}(\text{Conv}_I(S))$.

50 Definition A semigroup S in \mathbb{C} . S is said to be *concrete* when the concretization homomorphism $\text{Conc}_{\mathbb{C}}(S): \text{Mult}_{\mathbb{C}}(S) \rightarrow \text{TrHull}(\text{Conv}_I(S))$ is onto.

51 Remark When S is a concrete semigroup in \mathbb{C} , then $\text{TrHull}(\text{Conv}_I(S)) \simeq \text{Mult}_{\mathbb{C}}(S) / \ker \text{Conc}_{\mathbb{C}}(S)$, where $\ker f$ is the *kernel congruence* $\{ (x, y): f(x) = f(y) \}$ for a homomorphism of monoids $f: M \rightarrow N$.

52 Remark *How to check concreteness?* Let C be a category with a forgetful functor $C \xrightarrow{|\cdot|} \mathbf{Set}$ represented by some C -object I . So there is a natural isomorphism $\alpha: |\cdot| \Rightarrow C(I, -): C \rightarrow \mathbf{Set}$. In this case for each C -object c , $\mathbf{Set}(|c|, |c|) \simeq \mathbf{Set}(C(I, c), C(I, c))$ under $f \mapsto \alpha_c \circ f \circ \alpha_c^{-1}$. So in particular, the composition $C(c, c) \xrightarrow{|\cdot|} \mathbf{Set}(|c|, |c|) \simeq \mathbf{Set}(C(I, c), C(I, c))$ is $f \mapsto \alpha_c \circ |f| \circ \alpha_c^{-1} = C(c, f)$ (by naturality), that is, it is equal to $C(I, -)_{c,c}$.

Now let us assume that $\mathbb{C} = (C, \otimes, I)$ is a monoidal category with $|\cdot|: C \rightarrow \mathbf{Set}$ as above. In this case, for each semigroup $S = (S, \mu)$ in \mathbb{C} , $(C(I, S), \bullet_{\mu}) \simeq (|S|, *_{\mu})$ as semigroups, by transporting \bullet_{μ} along α_S^{-1} , that is, $x *_{\mu} y = \alpha_S^{-1}(\alpha_S(x) \bullet_{\mu} \alpha_S(y))$, $x, y \in |S|$. Consequently, $\text{TrHull}(\text{Conv}_I(S)) = \text{Mult}_{\text{Set}}(\text{Conv}_I(S)) \simeq \text{Mult}_{\text{Set}}(|S|, *_{\mu}) = \text{TrHull}(|S|, *_{\mu})$ as monoids, under $\text{Conj}_{\text{Set}}(\alpha_S^{-1}): (L, R) \mapsto (\alpha_S^{-1} \circ L \circ \alpha_S, \alpha_S^{-1} \circ R \circ \alpha_S)$. So one has the composite homomorphism $\text{Mult}_{\mathbb{C}}(S) \xrightarrow{\text{Conc}_{\mathbb{C}}(S)} \text{TrHull}(\text{Conv}_I(S)) \xrightarrow{\text{Conj}_{\text{Set}}(\alpha_S^{-1})} \text{TrHull}(|S|, *_{\mu})$, $(L, R) \mapsto (|L|, |R|)$. Note that $\text{Conv} \circ \mathbb{G}_I$ is a monoidal functor with underlying functor $C(I, -)$ so that $|\cdot|$ may be lifted to a monoidal functor say \mathbb{U} using α , and in fact $\widetilde{\mathbb{U}}(S) = (|S|, *_{\mu})$. In this situation, S is concrete when each multiplier $(L, R) \in \text{TrHull}(|S|, *_{\mu})$ is of the form $(|\ell|, |r|)$ for some $(\ell, r) \in \text{Mult}_{\mathbb{C}}(S)$.

- 53 Example** 1. Every ordinary semigroup is concrete. In fact, $Conc_{\mathbf{Set}}(\mathbf{S}) = Conj_{\mathbf{Set}}(gen_{\mathbf{S}}): \mathbf{TrHull}(\mathbf{S}) \simeq \mathbf{TrHull}(\mathbf{Conv}_1(\mathbf{S}))$ (see Example 17).
2. The underlying semigroup object of any monoid object in any monoidal category is concrete. Indeed according to Prop. 22, for each monoid \mathbf{M} , $\mathcal{M}_{\mathbb{C}, \mathbf{M}}: \mathbf{Conv}_I(|\mathbf{M}|) \simeq \mathbf{Mult}_{\mathbb{C}}(\mathbf{M})$. Moreover $\mathfrak{M}_{\mathbf{Conv}_I(\mathbf{M})}: \mathbf{Conv}_I(\mathbf{M}) \simeq \mathbf{Mult}(|\mathbf{Conv}_I(\mathbf{M})|)$ is also an isomorphism of monoids (see the Introduction). Consequently, $Conc_{\mathbb{C}}(|\mathbf{M}|) = \mathfrak{M}_{\mathbf{Conv}_I(|\mathbf{M}|)} \circ \mathcal{M}_{\mathbb{C}, |\mathbf{M}|}^{-1} = |\mathfrak{M}_{\mathbf{Conv}_I(\mathbf{M})} \circ \mathcal{M}_{\mathbb{C}, \mathbf{M}}^{-1}|$ is an isomorphism of semigroups.
3. Let R be a unital and commutative ring. Then, any R -algebra $\mathbf{A} = (A, \mu)$, that is, a semigroup in ${}_R\mathbf{Mod} = ({}_R\mathbf{Mod}, \otimes_R, R)$, faithful as a module over itself, is concrete. Indeed, the usual forgetful functor ${}_R\mathbf{Mod} \rightarrow \mathbf{Set}$ is represented by the free module of rank one, that is, R itself (as a module). So one may use Remark 52 to check concreteness. Let $(L, R) \in \mathbf{TrHull}(|A|, *)$, with $x * y := \mu(x \otimes y)$. Let $x, y, z \in |A|$ and let $\alpha \in R$. Then, $x * L(\alpha y + z) = R(x) * (\alpha y + z) = R(x) * (\alpha y) + R(x) * z = \alpha(R(x) * y) + R(x) * z = \alpha(x * L(y)) + x * L(z) = x * (\alpha L(y) + L(z))$ so that $L(\alpha y + z) = \alpha L(y) + L(z)$ since A is faithful. Similarly R is also linear. Consequently, $(L, R) \in \mathbf{Mult}_{\Delta_{\mathbb{B}}}(\mathbf{A})$.
4. According to [11] any commutative Banach algebra $\mathbf{B} = (B, *)$ without order (that is, $x * y = 0$ for all x implies $x = 0$) and such that $B^2 = B$ (that is, $*$ is onto), is concrete in $\mathbf{Ban} = (\mathbf{Ban}, \hat{\otimes}, \mathbb{C})$. In fact it is proven in [11] that under the condition that \mathbf{B} is without order and with $|-|$ the forgetful functor into \mathbf{Set} , $L \in \mathbf{Act}_{(|B|, *)}((|B|, *), (|B|, *))$ iff $(L, L) \in \mathbf{TrHull}(|B|, *)$ iff $(L, L) \in \mathbf{Mult}_{\mathbf{Ban}}(\mathbf{B})$. So if one assumes furthermore that $B^2 = B$, that is, $*$ is epi, and thus so is the multiplication as a morphism $B \hat{\otimes} B \rightarrow B$, since faithful functors reflect epimorphisms, then $(L, R) \in \mathbf{Mult}_{\mathbf{Ban}}(\mathbf{B})$ implies $L = R$ by Prop. 20.

Let us finally consider the composite $\mathbf{Conv}_I(\mathbf{S}) \xrightarrow{\mathcal{M}_{\mathbb{C}, \mathbf{S}}} |\mathbf{Mult}_{\mathbb{C}}(\mathbf{S})| \xrightarrow{Conc_{\mathbb{C}}(\mathbf{S})} |\mathbf{TrHull}(\mathbf{Conv}_I(\mathbf{S}))|$. It is the map $f \mapsto (C(I, L_f), C(I, R_f))$. Using Eqs. (17) and (18), $C(I, L_f) = \mathfrak{L}_f$, and $C(I, R_f) = \mathfrak{R}_f$. Therefore, the above composition is nothing but $\mathfrak{M}_{\mathbf{Conv}_I(\mathbf{S})}: \mathbf{Conv}_I(\mathbf{S}) \rightarrow |\mathbf{TrHull}(\mathbf{Conv}_I(\mathbf{S}))|$.

7 Around ordinary semigroups

In this section one first turns the ordinary translational hull construction into a functor from a subcategory of semigroups to monoids, which behaves

almost like a left adjoint to the forgetful functor from monoids to semigroups. Secondly, one extends the abstract multiplier monoid construction into a functor from a subcategory of **IntSem** to monoids, and the concretization homomorphisms then become a natural transformation.

7.1 The translational hull functor

54 Definition Let $S = (S, *)$ be an ordinary semigroup. It is said to be

1. *globally idempotent* when $S * S = S$, that is, $S = \{x * y : x, y \in S\}$ or in other terms $*$: $S \times S \rightarrow S$ is onto,
2. *right non-degenerate* (resp. *left non-degenerate*) when $x * y = x * z$ (resp. $y * x = z * x$) for all $x \in S$ implies $y = z$. It is *non-degenerate* when it is both left and right non-degenerate.

The first easy results below are stated without proofs.

55 Lemma $S = (S, *)$ is right (resp. left) non-degenerate iff $\mathfrak{L}_S: S \rightarrow \text{LTr}(S)$, $x \mapsto \mathfrak{L}_x$ (resp. $\mathfrak{R}_S: S \rightarrow \text{RTr}(S)$, $x \mapsto \mathfrak{R}_x$) is one-to-one. If S is non-degenerate, then \mathfrak{M}_S is one-to-one.

56 Lemma Let $M = (M, *, 1)$ be a monoid. Then, $|M|$ is globally idempotent and non-degenerate.

57 Definition Let S be a set and let $(T, *)$ be a semigroup.

1. Let $f: S \rightarrow T$ be a map. It is said to be *non-degenerate* if $T = \{f(s) * t : s \in S, t \in T\} = \{t * f(s) : s \in S, t \in T\}$.
2. Let $f: S \rightarrow \text{TrHull}(T, *)$ be a map. Let $f_L := p_1 \circ f$ and $f_R := p_2 \circ f$ where $\text{LTr}(T, *) \xleftarrow{p_1} \text{TrHull}(T, *) \xrightarrow{p_2} \text{RTr}(T, *)$ is the canonical pullback diagram. In other words for each $s \in S$, $f(s) = (f_L(s), f_R(s))$.
3. Let $f: S \rightarrow \text{TrHull}(T, *)$ be a map. It is said to be *translation non-degenerate* when $T = \{f_L(s)(t) : s \in S, t \in T\} = \{f_R(s)(t) : s \in S, t \in T\}$.

58 Lemma Let $S = (S, *)$ be a semigroup. S is globally idempotent iff id_S is non-degenerate.

Let $f: S \rightarrow T$ be a map, and let $\mathbb{T} = (T, \cdot)$ be a semigroup. Noticing that $(\mathfrak{M}_{\mathbb{T}} \circ f)_L = \mathfrak{L}_{\mathbb{T}} \circ f$ and $(\mathfrak{M}_{\mathbb{T}} \circ f)_R = \mathfrak{R}_{\mathbb{T}} \circ f$, one obtains the following result and its corollary (also using Lemma 58).

59 Lemma Let S be a set and let $T = (T, \cdot)$ be a semigroup. Let $f: S \rightarrow T$ be a map. f is a non-degenerate map iff $\mathfrak{M}_T \circ f: S \rightarrow |\text{TrHull}(T)|$ is translation non-degenerate.

60 Corollary Let $S = (S, *)$ be a globally idempotent semigroup. Then, $\mathfrak{M}_S: S \rightarrow |\text{TrHull}(S)|$ is translation non-degenerate.

Let us state for a later use the following two easy lemmas.

61 Lemma Let S, T, U be semigroups. Let $S \xrightarrow{f} T$ and $T \xrightarrow{g} U$ be non-degenerate homomorphism of semigroups. Then, so is $S \xrightarrow{g \circ f} U$.

62 Lemma Let $M = (M, *, 1)$ and $N = (N, *, 1)$ be monoids and let $f: M \rightarrow N$ be a homomorphism of monoids. Then, $|f|: (M, *) \rightarrow (N, *)$ is a non-degenerate homomorphism of semigroups.

63 Theorem Let $S = (S, *)$ and $T = (T, \cdot)$ be semigroups, with T which is non-degenerate. Let $f: S \rightarrow |\text{TrHull}(T)|$ be a homomorphism of semigroups which is translation non-degenerate. Then, there is a unique homomorphism of monoids $f^\sharp: \text{TrHull}(S) \rightarrow \text{TrHull}(T)$ such that $|f^\sharp| \circ \mathfrak{M}_S = f$. Moreover $|f^\sharp|: |\text{TrHull}(S)| \rightarrow |\text{TrHull}(T)|$ is also translation non-degenerate.

Proof: Let $s \in S$. Then, $\mathfrak{M}_S(L(s)) = (\mathfrak{L}_S(L(s)), \mathfrak{R}_S(L(s)))$. Let $s' \in S$. Then, $\mathfrak{L}_S(L(s))(s') = L(s) * s' = L(s * s') = L(\mathfrak{L}_S(s)(s'))$ so that $\mathfrak{L}_S(L(s)) = L \circ \mathfrak{L}_S(s)$. Also $\mathfrak{R}_S(L(s))(s') = s' * L(s) = R(s') * s = \mathfrak{R}_S(s)(R(s'))$ so that $\mathfrak{R}_S(L(s)) = \mathfrak{R}_S(s) \circ R$. Therefore, $\mathfrak{M}_S(L(s)) = (L \circ \mathfrak{L}_S(s), \mathfrak{R}_S(s) \circ R) = (L, R) \star \mathfrak{M}_S(s)$. Similarly, $\mathfrak{M}_S(R(s)) = (L, R) \star \mathfrak{M}_S(s)$.

Assume that f^\sharp exists. Then, for each $(L, R) \in \text{TrHull}(S)$, $s \in S$ and $t \in T$, $f_L(L(s))(t) = f_L^\sharp(\mathfrak{M}_S(L(s)))(t) = f_L^\sharp((L, R) \circ \mathfrak{M}_S(s))(t) = f_L^\sharp(L, R)(f_L^\sharp(\mathfrak{M}_S(s)))(t) = f_L^\sharp(L, R)(f_L(s)(t))$. In a way similar, $f_R(R(s))(t) = f_R^\sharp(L, R)(f_R(s)(t))$. Since f is non-degenerate, it follows that f^\sharp is unique.

Now let $(L, R) \in \text{TrHull}(S)$. Let $u \in T$. Then there are $s \in S$ and $t \in T$ such that $u = f_L(s)(t)$. Define $f_L^\sharp(L, R)(u) := f_L(L(s))(t)$. There are also $r \in S$ and $v \in T$ such that $u = f_R(r)(v)$, and one also defines $f_R^\sharp(L, R)(u) := f_R(R(r))(v)$.

To be well-defined one has to check that for each $(L, R) \in \text{TrHull}(S)$ if $f_L(s)(t) = f_L(s')(t')$, then $f_L(L(s))(t) = f_L(L(s'))(t')$ and for each $(L, R) \in \text{TrHull}(S)$ if $f_R(s)(t) = f_R(s')(t')$, then $f_R(R(s))(t) = f_R(R(s'))(t')$.

Now let $(L, R) \in \text{TrHull}(\mathbf{S})$. Assume that $f_R(s)(t) = f_R(s')(t')$. Let $u \in S$, $v \in T$. Then,

$$\begin{aligned}
f_R(R(s))(t) \cdot f_L(u)(v) &= f_R(u)(f_R(R(s)))(t) \cdot v \\
&= f_R(R(s) * u)(t) \cdot v \\
&= f_R(s * L(u))(t) \cdot v \\
&= f_R(L(u))(f_R(s)(t)) \cdot v \\
&= f_R(L(u))(f_R(s')(t')) \cdot v \quad (37) \\
&= f_R(s' * L(u))(t') \cdot v \\
&= f_R(R(s') * u)(t') \cdot v \\
&= f_R(u)(f_R(R(s')))(t') \cdot v \\
&= f_R(R(s'))(t') \cdot f_L(u)(v).
\end{aligned}$$

Since f is non-degenerate, this is equivalent to $f_R(R(s))(t) \cdot w = f_R(R(s'))(t') \cdot w$ for each $w \in T$, and since \mathbf{T} is both left and right non-degenerate, $f_R(R(s))(t) = f_R(R(s'))(t')$. Consequently, f^\sharp is well-defined.

One observes that $f_L^\sharp(\mathfrak{M}_S(a))(u) = f_L^\sharp(\mathfrak{M}_S(a))(f_L(s)(t)) = f_L(\mathfrak{L}_S(a)(s))(t) = f_L(a * s)(t) = f_L(a)(f_L(s)(t)) = f_L(a)(u)$ when $u = f_L(s)(t)$, that is, $f_L^\sharp \circ \mathfrak{M}_S = f_L$ and also $f_R^\sharp(\mathfrak{M}_S(a))(u) = f_R(a)(u)$ when $u = f_R(s)(t)$, that is, $f_R^\sharp \circ \mathfrak{M}_S = f_R$. Consequently, $f^\sharp \circ \mathfrak{M}_S = f$.

Now let us check that for each $(L, R) \in \text{TrHull}(\mathbf{S})$, $(f_L^\sharp(L, R), f_R^\sharp(L, R)) \in \text{TrHull}(\mathbf{T})$. One has $f_L^\sharp(L, R)(u \cdot v) = f_L^\sharp(L, R)(f_L(s)(t) \cdot v) = f^\sharp(L, R)(f_L(s)(t \cdot v)) = f_L(L(s))(t \cdot v) = f_L(L(s))(t) \cdot v = f_L^\sharp(L, R)(u) \cdot v$ when $u = f_L(s)(t)$, and similarly $f_R^\sharp(L, R)(u \cdot v) = u \cdot f_R^\sharp(L, R)(v)$ when $v = f_R(s)(t)$. Let $u = f_R(s)(t)$ and $u' = f_L(s')(t')$. Then,

$$\begin{aligned}
f_R^\sharp(L, R)(u) \cdot u' &= f_R(R(s))(t) \cdot u' \\
&= t \cdot f_L(R(s))(u') \\
&= t \cdot f_L(R(s))(f_L(s')(t')) \\
&= t \cdot f_L(R(s) * s')(t') \\
&= t \cdot f_L(s * L(s'))(t') \quad (38) \\
&= t \cdot f_L(s)(f_L(L(s'))(t')) \\
&= t \cdot f_L(s)(f_L^\sharp(L, R)(u')) \\
&= f_R(s)(t) \cdot f_L^\sharp(L, R)(u') \\
&= u \cdot f_L^\sharp(L, R)(u').
\end{aligned}$$

Finally let us check that f^\sharp is a homomorphism: let $(L, R), (L', R') \in \text{TrHull}(\mathbf{S})$. Let $s \in S$ and $t \in T$. Then,

$$\begin{aligned}
f_L^\sharp(L' \circ L, R \circ R')(f_L(s)(t)) &= f_L(L'(L(s)))(t) \\
&= f_L^\sharp(L', R')(f_L(L(s))(t)) \\
&= f_L^\sharp(L', R')(f_L^\sharp(L, R)(f_L(s)(t))).
\end{aligned} \tag{39}$$

Likewise one also has $f_R^\sharp(L' \circ L, R \circ R')(f_R(s)(t)) = f_R^\sharp(L, R)(f_R^\sharp(L', R')(f_R(s)(t)))$. Consequently, f^\sharp is a homomorphism of semigroups. Since it evidently preserves the units, it is in fact a homomorphism of monoids.

Let $u \in T$. Since f is translation non-degenerate, there are $s \in S$ and $t \in T$ such that $f_L(s)(t) = u$. Then, $f_L^\sharp(\mathfrak{M}_S(s))(t) = f_L(s)(t) = u$. The last assertion now follows easily. \square

By defining $TrHull(f) := (\mathfrak{M}_T \circ f)^\sharp$ one obtains the

64 Corollary *Let S and T be semigroups, with T non-degenerate. Let $f: S \rightarrow T$ be a homomorphism of semigroups which is non-degenerate. Then, there is a unique homomorphism of monoids $TrHull(f): TrHull(S) \rightarrow TrHull(T)$ such that the following diagram commutes in **Sem**. Moreover $|TrHull(f)|$ is translation non-degenerate.*

$$\begin{array}{ccc}
|TrHull(S)| & \xrightarrow{|TrHull(f)|} & |TrHull(T)| \\
\mathfrak{M}_S \uparrow & & \uparrow \mathfrak{M}_T \\
S & \xrightarrow{f} & T
\end{array} \tag{40}$$

Let \mathbf{Sem}_{nd} be the subcategory of semigroups whose objects are the globally idempotent and non-degenerate semigroups, and $\mathbf{Sem}_{nd}(S, T)$ consists of all homomorphisms of semigroups which are non-degenerate. (From Lemma 58 it follows that $id_S \in \mathbf{Sem}_{nd}(S, S)$ and from Lemma 61 that composition of non-degenerate homomorphisms is a non-degenerate homomorphism.)

65 Corollary *$TrHull$ provides a functor from \mathbf{Sem}_{nd} to \mathbf{Mon} , and $\mathfrak{M} = (\mathfrak{M}_S)_S: J \Rightarrow |-| \circ TrHull: \mathbf{Sem}_{nd} \rightarrow \mathbf{Sem}$, where $J: \mathbf{Sem}_{nd} \rightarrow \mathbf{Sem}$ is the canonical (non-full) embedding functor.*

According to Lemmas 56 and 62, the forgetful functor $|-|: \mathbf{Mon} \rightarrow \mathbf{Sem}$ co-restricts to a functor still denoted $|-|$ from \mathbf{Mon} to \mathbf{Sem}_{nd} . Note that $|-|: \mathbf{Mon} \rightarrow \mathbf{Sem}_{nd}$ is still faithful because so is the embedding functor $\mathbf{Sem}_{nd} \hookrightarrow \mathbf{Sem}$.

Nevertheless \mathfrak{M} is not a natural transformation $id_{\mathbf{Sem}_{nd}} \Rightarrow |-| \circ TrHull$ from \mathbf{Sem}_{nd} to \mathbf{Sem}_{nd} because $\mathfrak{M}_{\mathfrak{S}}$ while translation non-degenerate, may fail to be non-degenerate. Indeed, let $s \in S$ and let $(L, R) \in TrHull(S)$. Then, $\mathfrak{M}_{\mathfrak{S}}(s) \star (L, R) = \mathfrak{M}_{\mathfrak{S}}(R(s))$. So if it happens that $TrHull(S)$ contains other multipliers than the inner ones, $\mathfrak{M}_{\mathfrak{S}}$ cannot be non-degenerate.

Nevertheless $TrHull: \mathbf{Sem}_{nd} \rightarrow \mathbf{Mon}$ is very close to be a left adjoint to $|-|: \mathbf{Mon} \rightarrow \mathbf{Sem}_{nd}$ as is shown in the next theorem (the only missing part is that \mathfrak{M} cannot be a unit for this “adjunction”), and in any case, $TrHull$ may be thought to as a *unitarization* functor.

66 Theorem *Let $S = (S, *)$ be a globally idempotent and non-degenerate semigroup and let $M = (M, *, 1)$ be a monoid. Let $f: S \rightarrow |M|$ be a non-degenerate homomorphism of semigroups. Then, there is a unique homomorphism of monoids $Mult(S) \xrightarrow{f^b} M$ such that $|f^b| \circ \mathfrak{M}_{\mathfrak{S}} = f$.*

Proof: The homomorphism of semigroups $S = (S, *) \xrightarrow{f} |M| = (M, *) \xrightarrow{\mathfrak{M}_{|M|}} |TrHull(|M|)|$ is translation non-degenerate by Lemma 59 since f is non-degenerate, and since $(M, *) = |M|$ is non-degenerate by Lemma 56, Theorem 63 tells us that there is a unique homomorphism of monoids $g_f := (\mathfrak{M}_{|M|} \circ f)^{\sharp}: TrHull(S) \rightarrow TrHull(|M|)$ such that $|g_f| \circ \mathfrak{M}_{\mathfrak{S}} = \mathfrak{M}_{|M|} \circ f$. According to Lemma 1, $\mathfrak{M}_M: M \rightarrow TrHull(|M|)$ is an isomorphism, and thus one may consider the homomorphism of monoids $f^b := \mathfrak{M}_M^{-1} \circ g_f: TrHull(S) \rightarrow M$. It satisfies $|f^b| \circ \mathfrak{M}_{\mathfrak{S}} = \mathfrak{M}_{|M|}^{-1} \circ \mathfrak{M}_{|M|} \circ f = f$. Uniqueness is easily checked. \square

One may also use $TrHull$ to define another category in order to turn the translational hull construction into an adjoint. Let S, T, U be semigroups, where T, U are non-degenerate. Let $f: S \rightarrow |TrHull(T)|$ and $g: T \rightarrow |TrHull(U)|$ be translation non-degenerate homomorphisms of semigroups. Then, $|g^{\sharp}| \circ f: S \rightarrow |TrHull(U)|$ is of course a homomorphism of semigroups. Let us check that it is translation non-degenerate too. One first notices that $(g^{\sharp} \circ f)(s) = g^{\sharp}(f_L(s), f_R(s)) = (g_L^{\sharp}(f_L(s), f_R(s)), g_R^{\sharp}(f_L(s), f_R(s)))$, $s \in S$. Consequently, $(g^{\sharp} \circ f)_L = g_L^{\sharp} \circ f$ and $(g^{\sharp} \circ f)_R = g_R^{\sharp} \circ f$. Now let $u \in U$. One has to show that there are $s \in S$ and $v \in U$ such that $g_L^{\sharp}(f_L(s), f_R(s))(v) = u$. For this $u \in U$ there are $t \in T$ and $v \in U$ such that $g_L(t)(v) = u$. Moreover there are $s \in S$ and $r \in T$. Then, $f_L(s)(r) = t$. Whence $g_L(f_L(s)(r))(v) = u$. By definition of g^{\sharp} , $g^{\sharp}(f(s))(g_L(r)(v)) = g_L(f_L(s)(r))(v) = g_L(t)(v) = u$. In a same way given $u \in U$, there are $s \in S$ and $v \in U$ such that $g_R^{\sharp}(f_L(s), f_R(s))(v) = u$.

Now let S be a globally idempotent semigroup. Then, according to Corollary 60, $\mathfrak{M}_{\mathfrak{S}}: S \rightarrow |TrHull(S)|$ is translation non-degenerate.

By the above one can define the category $\mathbf{MultSem}_{nd}$ whose objects are globally idempotent, non-degenerate semigroups, $\mathbf{MultSem}_{nd}(\mathbf{S}, \mathbf{T})$ consists of the homomorphisms of semigroups $\mathbf{S} \rightarrow |\mathbf{TrHull}(\mathbf{T})|$ which are translation non-degenerate, $g \bullet f := |g^\sharp| \circ f$, and $id_{\mathbf{S}} := \mathfrak{M}_{\mathbf{S}}$. (Note that $\mathbf{MultSem}_{nd}$ is *not* a subcategory of \mathbf{Sem} .) The morphisms of $\mathbf{MultSem}_{nd}$ are similar to that used in [10] to define multiplier Hopf algebras.

According to Lemma 59 one has a functor $E: \mathbf{Sem}_{nd} \rightarrow \mathbf{MultSem}_{nd}$, $(\mathbf{S} \xrightarrow{f} \mathbf{T}) \mapsto (\mathbf{S} \xrightarrow{\mathfrak{M}_{\mathbf{T}} \circ f} \mathbf{T})$. Since for each non-degenerate semigroup \mathbf{S} , $\mathfrak{M}_{\mathbf{S}}$ is one-to-one it follows that the functor E is faithful and thus provides an embedding of \mathbf{Sem}_{nd} into $\mathbf{MultSem}_{nd}$.

Now one may also define a functor $\widetilde{TrHull}: \mathbf{MultSem}_{nd} \rightarrow \mathbf{Mon}$ as follows: $\widetilde{TrHull}(\mathbf{S} \xrightarrow{f} |\mathbf{TrHull}(\mathbf{T})|) := \mathbf{TrHull}(\mathbf{S}) \xrightarrow{f^\sharp} \mathbf{TrHull}(\mathbf{T})$. And one recovers $TrHull: \mathbf{Sem}_{nd} \rightarrow \mathbf{Mon}$ as $\mathbf{Sem}_{nd} \xrightarrow{E} \mathbf{MultSem}_{nd} \xrightarrow{\widetilde{TrHull}} \mathbf{Mon}$.

Let $\mathbf{M} = (M, *, 1)$, and $\mathbf{N} = (N, *, 1)$ be monoids and let $\mathbf{M} \xrightarrow{f} \mathbf{N}$ be a homomorphism of monoids. Then, $|f|: |\mathbf{M}| = (M, *) \rightarrow |\mathbf{N}| = (N, *)$ is non-degenerate by Lemma 62, and thus $\mathfrak{M}_{|\mathbf{N}|} \circ |f|: |\mathbf{M}| \rightarrow |Mult(|\mathbf{M}|)|$ is translation non-degenerate (Lemma 59). Since any monoid is automatically globally idempotent and non-degenerate, it is clear that $(\mathfrak{M}_{|\mathbf{N}|} \circ |f|)$ may be considered as a $\mathbf{MultSem}_{nd}$ -morphisms from $|\mathbf{M}|$ to $|\mathbf{N}|$. So one may consider the unique homomorphism of monoids $(\mathfrak{M}_{|\mathbf{N}|} \circ |f|)^\sharp: \mathbf{TrHull}(|\mathbf{M}|) \rightarrow \mathbf{TrHull}(|\mathbf{N}|)$ such that $|(\mathfrak{M}_{|\mathbf{N}|} \circ |f|)^\sharp| \circ \mathfrak{M}_{|\mathbf{M}|} = \mathfrak{M}_{|\mathbf{N}|} \circ |f|$, that is, $(\mathfrak{M}_{|\mathbf{N}|} \circ |f|)^\sharp = \widetilde{TrHull}(\mathfrak{M}_{|\mathbf{N}|} \circ |f|) = TrHull(|f|)$. Now let \mathbf{P} be also a monoid and let $g: \mathbf{N} \rightarrow \mathbf{P}$ be a homomorphism of monoids. Then, $(\mathfrak{M}_{|\mathbf{P}|} \circ |g|) \bullet (\mathfrak{M}_{|\mathbf{N}|} \circ |f|) = |(\mathfrak{M}_{|\mathbf{P}|} \circ |g|)^\sharp| \circ (\mathfrak{M}_{|\mathbf{N}|} \circ |f|) = \mathfrak{M}_{|\mathbf{P}|} \circ |g| \circ |f| = \mathfrak{M}_{|\mathbf{P}|} \circ |g \circ f|$.

All of this provides a functor $F: \mathbf{Mon} \rightarrow \mathbf{MultSem}_{nd}$ given by $F(\mathbf{M} \xrightarrow{f} \mathbf{N}) := |\mathbf{M}| \xrightarrow{|f|} |\mathbf{N}| \xrightarrow{\mathfrak{M}_{|\mathbf{N}|}} |\mathbf{TrHull}(|\mathbf{N}|)|$. One has the factorisation $F = \mathbf{Mon} \xrightarrow{|\cdot|} \mathbf{Sem}_{nd} \xrightarrow{E} \mathbf{MultSem}_{nd}$.

67 Proposition F is a left adjoint of \widetilde{TrHull} .

Proof: Let $\mathbf{M} = (M, *, 1)$ be a monoid. Let \mathbf{S} be a globally idempotent and non-degenerate semigroup and let $\mathbf{M} \xrightarrow{f} \mathbf{TrHull}(\mathbf{S})$ be a homomorphism of monoids. Let us consider the homomorphism of monoids $\tilde{f} := \mathbf{TrHull}(|\mathbf{M}|) \xrightarrow{\mathfrak{M}_{\mathbf{M}}^{-1}} \mathbf{M} \xrightarrow{f} \mathbf{TrHull}(\mathbf{S})$. Then, $|\tilde{f}| \in \mathbf{Sem}(|\mathbf{TrHull}(|\mathbf{M}|)|, |\mathbf{TrHull}(\mathbf{S})|) = \mathbf{MultSem}_{nd}(|\mathbf{TrHull}(|\mathbf{M}|)|, \mathbf{S})$ is such that $|\tilde{f}| \circ \mathfrak{M}_{|\mathbf{M}|} = |f|$. Therefore, as $|f|$ is non-degenerate, $\tilde{f} = |f|^\sharp = \widetilde{TrHull}(|f|)$. So by construction $|\widetilde{TrHull}(|f|)| \circ$

$|\mathfrak{M}_{\mathbf{M}}| = |\widetilde{\text{TrHull}}(|f|)| \circ \mathfrak{M}_{|\mathbf{M}|} = |f|$ as homomorphisms of semigroups and thus $\widetilde{\text{TrHull}}(|f|) \circ \mathfrak{M}_{\mathbf{M}} = f$ as homomorphisms of monoids since the forgetful functor is faithful. Now let $\phi \in \mathbf{MultSem}_{nd}(|\text{TrHull}(|\mathbf{M}|)|, \mathbf{S})$ such that $\widetilde{\text{TrHull}}(\phi) \circ \mathfrak{M}_{\mathbf{M}} = f$. Then $\widetilde{\text{TrHull}}(\phi) = f \circ \mathfrak{M}_{\mathbf{M}}^{-1} = |f|^{\sharp}$ and thus $\phi = |f|$. Consequently, $|f| \in \mathbf{MultSem}_{nd}(|\mathbf{M}|, \mathbf{S})$ is the unique morphism such that $\widetilde{\text{TrHull}}(|f|) \circ \mathfrak{M}_{\mathbf{M}} = f$.

By direct inspection one checks that $\mathfrak{M} := (\mathfrak{M}_{\mathbf{M}})_{\mathbf{M}}: id_{\mathbf{Mon}} \Rightarrow \widetilde{\text{TrHull}} \circ F: \mathbf{Mon} \rightarrow \mathbf{Mon}$. \square

68 Remark 1. By composition of adjunctions, one has an adjunction

$F \circ (-)^1 \dashv | - | \circ \widetilde{\text{TrHull}}: \mathbf{Sem} \rightarrow \mathbf{MultSem}_{nd}$, where $\mathbf{Sem} \xrightarrow{(-)^1} \mathbf{Mon}$ is the usual unitarization functor.

2. Since the unit of $F \dashv \widetilde{\text{Mult}}$ is the natural isomorphism $\mathfrak{M} := (\mathfrak{M}_{\mathbf{M}})_{\mathbf{M}}$, it follows that $F: \mathbf{Mon} \rightarrow \mathbf{MultSem}_{nd}$ is fully faithful.

7.2 The multiplier functor

Let $\mathbb{C} = (C, \otimes_C, I)$ be a monoidal category. Let $\mathbf{S} = (S, \mu)$ be a semigroup in \mathbb{C} and let X be a set. For a map $X \xrightarrow{f} \text{Mult}_{\mathbb{C}}(\mathbf{S})$, one defines $f_L: X \rightarrow \mathbf{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu), (S, \mu))$ and $f_R: X \rightarrow {}_{\mathbf{S}}\mathbf{Act}(\mathbb{C})((S, \mu), (S, \mu))$ by $f_L := p_1 \circ f$ and $f_R := p_2 \circ f$, where $\mathbf{Act}_{\mathbf{S}}(\mathbb{C})((S, \mu), (S, \mu)) \xleftarrow{p_1} \text{Mult}_{\mathbb{C}}(\mathbf{S}) \xrightarrow{p_2} {}_{\mathbf{S}}\mathbf{Act}(\mathbb{C})((S, \mu), (S, \mu))$ is the canonical pullback diagram.

A map $X \xrightarrow{f} \text{Mult}_{\mathbb{C}}(\mathbf{S})$ is said to be *multiplier non-degenerate* when $C(I, S) = \{f_L(x) \circ s : x \in X, s \in C(I, S)\} = \{f_R(x) \circ s : x \in X, s \in C(I, S)\}$.

69 Lemma Let $f: X \rightarrow C(I, S)$ be a map. f is non-degenerate iff $X \xrightarrow{f} C(I, S) \xrightarrow{\mathcal{M}_{\mathbb{C}, \mathbf{S}}} \text{Mult}_{\mathbb{C}}(\mathbf{S})$ is multiplier non-degenerate.

Proof: One first notes that $(\mathcal{M}_{\mathbb{C}, \mathbf{S}} \circ f)_L = \mathcal{L}_{\mathbb{C}, \mathbf{S}} \circ f$ and $(\mathcal{M}_{\mathbb{C}, \mathbf{S}} \circ f)_R = \mathcal{R}_{\mathbb{C}, \mathbf{S}} \circ f$. Since for each $x \in X$ and $s \in C(I, S)$, $f(x) \bullet_{\mu} s = L_{f(x)} \circ s$ and $s \bullet_{\mu} R_{f(x)} = f(x) \circ s$, the expected result follows easily. \square

70 Remark Let $f: X \rightarrow C(I, S)$ be a map. f is non-degenerate iff $X \xrightarrow{f} C(I, S) \xrightarrow{\mathcal{M}_{\mathbb{C}, \mathbf{S}}} \text{Mult}_{\mathbb{C}}(\mathbf{S})$ is multiplier non-degenerate iff $X \xrightarrow{f} C(I, S) \xrightarrow{\mathfrak{M}_{\text{Conv}_I(\mathbf{S})}} \text{TrHull}(C(I, S))$ is translation non-degenerate (since $\mathfrak{M}_{\text{Conv}_I(\mathbf{S})} = \text{Conc}_{\mathbb{C}}(\mathbf{S}) \circ \mathcal{M}_{\mathbb{C}, \mathbf{S}}$, see Section 6.2).

71 Theorem Let $\mathbf{S} = (S, \mu_S)$ be a semigroup in $\mathbb{C} = (C, \otimes_C, I)$ and let $\mathbf{T} = (T, \mu_T)$ be a semigroup in $\mathbb{D} = (D, \otimes_D, J)$. Let us assume that the convolution semigroup $\text{Conv}_J(\mathbf{T})$ is non-degenerate, and \mathbf{T} is concrete.

Let $f: \text{Conv}_I(\mathbf{S}) \rightarrow (\text{Mult}_{\mathbb{D}}(\mathbf{T}), \star)$ be a homomorphism of semigroups which is translation non-degenerate. Then, there is a unique homomorphism of monoids $f^M: \text{Mult}_{\mathbb{C}}(\mathbf{S}) \rightarrow \text{Mult}_{\mathbb{D}}(\mathbf{T})$ such that $|f^M| \circ \mathcal{M}_{\mathbb{C}, \mathbf{S}} = f$. Moreover, $|\text{Conc}_{\mathbb{D}}| \circ f: \text{Conv}_I(\mathbf{S}) \rightarrow |\text{TrHull}(\text{Conv}_J(\mathbf{D}))|$ is multiplier non-degenerate and $(|\text{Conc}_{\mathbb{D}}(\mathbf{T})| \circ f)^\sharp \circ \text{Conc}_{\mathbb{C}}(\mathbf{S}) = \text{Conc}_{\mathbb{D}}(\mathbf{T}) \circ f^M$.

Proof: Preamble: let $(L, R) \in \text{Mult}_{\mathbb{C}}(\mathbf{S})$ and let $s \in C(I, S)$. Then, $\mathcal{M}_{\mathbb{C}, \mathbf{S}}(L \circ s) = (L \circ L_s, R_s \circ R)$ because $L_{L \circ s} = L \circ L_s$ and $R_{L \circ s} = R_s \circ R$. Therefore $\mathcal{M}_{\mathbb{C}, \mathbf{S}}(L \circ s) = (L, R) \star \mathcal{M}_{\mathbb{C}, \mathbf{S}}(s)$. Also, $\mathcal{M}_{\mathbb{C}, \mathbf{S}}(R \circ s) = (L_s \circ L, R \circ R_s) = \mathcal{M}_{\mathbb{C}, \mathbf{S}}(s) \star (L, R)$.

Now let us assume that the homomorphism $f^M: \text{Mult}_{\mathbb{C}}(\mathbf{S}) \rightarrow \text{Mult}_{\mathbb{D}}(\mathbf{T})$ such that $|f^M| \circ \mathcal{M}_{\mathbb{C}, \mathbf{S}} = f$ exists. Let $(L, R) \in \text{Mult}_{\mathbb{C}}(\mathbf{S})$ and $s \in C(I, S)$. Then,

$$\begin{aligned} f_L(L \circ s) &= f_L^M(\mathcal{M}_{\mathbb{C}, \mathbf{S}}(L \circ s)) \\ &= f_L^M((L, R) \star \mathcal{M}_{\mathbb{C}, \mathbf{S}}(s)) \\ &= f_L^M(L, R) \circ f_L^M(\mathcal{M}_{\mathbb{C}, \mathbf{S}}(s)) \\ &= f_L^M(L, R) \circ f_L(s). \end{aligned} \tag{41}$$

Now let $t \in D(J, T)$. Then, $f_L(L \circ s) \circ t = (f_L^M(L, R) \circ f_L(s)) \circ t = f_L^M(L, R) \circ (f_L(s) \circ t)$. So if $g: \text{Mult}_{\mathbb{C}}(\mathbf{S}) \rightarrow \text{Mult}_{\mathbb{D}}(\mathbf{T})$ is a homomorphism such that $|f^M| \circ \mathcal{M}_{\mathbb{C}, \mathbf{S}} = f$, then for each $t \in D(J, T)$, each $s \in C(I, S)$, $g_L(L, R) \circ (f_L(s) \circ t) = f_L^M(L, R) \circ (f_L(s) \circ t)$. Since f is assumed non-degenerate, this is equivalent to the fact that for each $u \in D(J, T)$, $g_L(L, R) \circ u = f_L^M(L, R) \circ u$. Since $\text{Conv}_J(\mathbf{T})$ is assumed non-degenerate, it follows that $g_L(L, R) = f_L^M(L, R)$ which guarantees uniqueness of f_L^M . Similarly, one obtains uniqueness of f_R^M , and so that of f^M .

Let $(L, R) \in \text{Mult}_{\mathbb{C}}(\mathbf{S})$ and let $u \in D(J, T)$. Define $G_f(L, R)(u) := f_L(L \circ s) \circ t$ when $u = f_L(s) \circ t$, and define $D_f(L, R)(u) := f_R(R \circ s) \circ t$ when $u = f_R(s) \circ t$.

Let us check that both $G_f(L, R)(u)$ and $D_f(L, R)(u)$ are well-defined. So let $s, s' \in C(I, S)$ and $t, t' \in D(J, T)$ such that $f_L(s) \circ t = f_L(s') \circ t'$. Let

$u \in C(I, S)$ and $v \in D(J, T)$. Then,

$$\begin{aligned}
(f_R(u) \circ v) \bullet_{\mu_T} (f_L(L \circ s) \circ t) &= D(J, f_R(u))(v) \bullet_{\mu_T} (f_L(L \circ s) \circ t) \\
&= v \bullet_{\mu_T} D(J, f_L(u))(f_L(L \circ s) \circ t) \\
&= v \bullet_{\mu_T} (f_L(u) \circ f_L(L \circ s) \circ t) \\
&= v \bullet_{\mu_T} (f_L(u \bullet_{\mu_S} (L \circ s)) \circ t) \\
&= v \bullet_{\mu_T} (f_L(u \bullet_{\mu_S} C(I, L)(s)) \circ t) \\
&= v \bullet_{\mu_T} (f_L(C(I, R)(u) \bullet_{\mu_S} s) \circ t) \\
&= v \bullet_{\mu_T} (f_L((R \circ u) \bullet_{\mu_S} s) \circ t) \\
&= v \bullet_{\mu_T} (f_L(R \circ u) \circ f_L(s) \circ t) \\
&= v \bullet_{\mu_T} (f_L(R \circ u) \circ f_L(s') \circ t') \\
&= v \bullet_{\mu_T} (f_L((R \circ u) \bullet_{\mu_S} s') \circ t') \\
&= v \bullet_{\mu_T} (f_L(C(I, R)(u) \bullet_{\mu_S} s') \circ t') \\
&= v \bullet_{\mu_T} (f_L(u \bullet_{\mu_S} C(I, L)(s')) \circ t') \\
&= v \bullet_{\mu_T} (f_L(u \bullet_{\mu_S} (L \circ s')) \circ t') \\
&= v \bullet_{\mu_T} (f_L(u) \circ f_L(L \circ s') \circ t') \\
&= v \bullet_{\mu_T} (D(J, f_L(u))(f_L(L \circ s') \circ t')) \\
&= D(J, f_R(u))(v) \bullet_{\mu_T} (f_L(L \circ s') \circ t') \\
&= (f_R(u) \circ v) \bullet_{\mu_T} (f_L(L \circ s') \circ t').
\end{aligned} \tag{42}$$

Since f is non-degenerate, this means that for each $w \in D(J, T)$, $w \bullet_{\mu_T} (f_L(L \circ s) \circ t) = w \bullet_{\mu_T} (f_L(L \circ s') \circ t')$ and since $\text{Conv}_J(D)$ is non-degenerate, $f_L(L \circ s) \circ t = f_L(L \circ s') \circ t'$, and $G_f(L, R)(u)$ is well-defined. Likewise $D_f(L, R)(u)$ is well-defined.

Now let us show that $(G_f(L, R), D_f(L, R)) \in \text{TrHull}(\text{Conv}_J(\mathbb{T}))$. Let $u, v \in D(J, T)$. Let $s \in C(I, S)$ and $t \in D(J, T)$ such that $u = f_L(s) \circ t$. Then,

$$\begin{aligned}
G_f(L, R)(u \bullet_{\mu_T} v) &= G(L, R)((f_L(s) \circ t) \bullet_{\mathbb{T}} v) \\
&= G_f(L, R)(D(J, f_L(s))(t) \bullet_{\mu_T} v) \\
&= G_f(L, R)(D(J, f_L(s))(t \bullet_{\mu_T} v)) \\
&= G_f(L, R)(f_L(s) \circ (t \bullet_{\mu_T} v)) \\
&= f_L(L \circ s) \circ (t \bullet_{\mu_T} v) \\
&= D(J, f_L(L \circ s))(t \bullet_{\mu_T} v) \\
&= D(J, f_L(L \circ s))(t) \bullet_{\mu_T} v \\
&= (f_L(L \circ s) \circ t) \bullet_{\mu_T} v \\
&= G_f(L, R)(u) \bullet_{\mu_T} v.
\end{aligned} \tag{43}$$

Likewise $D_f(L, R)(u \bullet_{\mu_T} v) = u \bullet_{\mu_T} D_f(L, R)(v)$, $u, v \in D(J, T)$.

Let $u, u' \in D(J, T)$ and let $s, s' \in C(I, S)$, $t, t' \in D(J, T)$ such that

$u = f_R(s) \circ t$ and $u' = f_L(s') \circ t'$. Then,

$$\begin{aligned}
D(L, R)(u) \bullet_{\mu_T} u' &= (f_R(R \circ s) \circ t) \bullet_{\mu_T} u' \\
&= D(J, f_R(R \circ s))(t) \bullet_{\mu_T} u' \\
&= t \bullet_{\mu_T} D(J, f_L(R \circ s))(u') \\
&= t \bullet_{\mu_T} (f_L(R \circ s) \circ u') \\
&= t \bullet_{\mu_T} (f_L(R \circ s) \circ f_L(s') \circ t') \\
&= t \bullet_{\mu_T} (f_L((R \circ s) \bullet_{\mu_S} s') \circ t') \\
&= t \bullet_{\mu_T} (f_L(C(I, R)(s) \bullet_{\mu_S} s') \circ t') \\
&= t \bullet_{\mu_T} (f_L(s \bullet_{\mu_S} C(I, L)(s')) \circ t') \\
&= t \bullet_{\mu_T} (f_L(s \bullet_{\mu_S} (L \circ s')) \circ t') \\
&= t \bullet_{\mu_T} (f_L(s) \circ f_L(L \circ s') \circ t') \\
&= t \bullet_{\mu_T} D(J, f_L(s))(f_L(L \circ s') \circ t') \\
&= D(J, f_R(s))(t) \bullet_{\mu_T} (f_L(L \circ s') \circ t') \\
&= (f_R(s) \circ t) \bullet_{\mu_T} G(L, R)(u') \\
&= u \bullet_{\mu_T} G(L, R)(u').
\end{aligned} \tag{44}$$

Consequently, $(G(L, R), D(L, R)) \in \text{TrHull}(\text{Conv}_J(\mathbb{D}))$. By the concreteness assumption then so exists $f^M(L, R) \in \text{Mult}_{\mathbb{D}}(\mathbb{T})$ such that $D(J, f^M(L, R)) = (G_f(L, R), D_f(L, R))$, that is, for each $u \in D(J, T)$, $f_L^M(L, R) \circ u = G_f(L, R)(u)$ and $f_R^M(L, R) \circ u = D_f(L, R)(u)$.

Let us assume that for each $(L, R) \in \text{Mult}_{\mathbb{C}}(\mathbb{S})$ is chosen $f^M(L, R) \in \text{Mult}_{\mathbb{D}}(\mathbb{T})$ as above. One then defines a map $f^M: (L, R) \mapsto f^M(L, R)$. Let $a \in C(I, S)$ and let $u = f_L(s) \circ t \in D(J, T)$ for $s \in C(I, S)$ and $t \in D(J, T)$. Then,

$$\begin{aligned}
f_L^M(\mathcal{M}_{\mathbb{C}, \mathbb{S}}(a)) \circ u &= G(L_a, R_a)(f_L(s) \circ t) \\
&= f_L(L_a \circ s) \circ t \\
&= f_L(a \bullet_{\mu_S} s) \circ t \\
&= f_L(a) \circ f_L(s) \circ t \\
&= f_L(a) \circ u.
\end{aligned} \tag{45}$$

By non-degeneracy of $\text{Conv}_J(\mathbb{T})$, $f_L^M(\mathcal{M}_{\mathbb{C}, \mathbb{S}}(a)) = f_L(a)$. Similarly, $f_R^M(\mathcal{M}_{\mathbb{C}, \mathbb{S}}(a)) \circ u = f_R(a) \circ u$. One so obtains that $f^M \circ \mathcal{M}_{\mathbb{C}, \mathbb{S}} = f$.

Let $(L, R), (L', R') \in \text{Mult}_{\mathbb{C}}(\mathbb{S})$ and let $u, t \in D(J, T)$, $s \in C(I, S)$ such that $f_L(s) \circ t = u$. Then,

$$\begin{aligned}
f_L^M((L', R') \star (L, R)) \circ u &= f_L^M(L' \circ L, R \circ R') \circ f_L(s) \circ t \\
&= f_L(L' \circ L \circ s) \circ t \\
&= f_L^M(L', R') \circ f_L(L \circ s) \circ t \\
&= f_L^M(L', R') \circ f_L^M(L, R) \circ f_L(s) \circ t \\
&= f_L^M(L', R') \circ f_L^M(L, R) \circ u.
\end{aligned} \tag{46}$$

By non-degeneracy of $\mathbf{Conv}_J(\mathbf{T})$, it then follows that $f_L^M((L', R') \star (L, R)) = f_L^M(L', R') \circ f_L^M(L, R)$. Similarly, $f_R^M((L', R') \star (L, R)) = f_R^M(L, R) \circ f_R^M(L', R')$. Consequently, $f^M((L', R') \star (L, R)) = f^M(L', R') \star f^M(L, R)$. Finally, since it is quite clear that $f^M(id_S, id_S) = (id_T, id_T)$, $f^M \in \mathbf{Mon}(\mathbf{Mult}_{\mathbb{C}}(\mathbf{S}), \mathbf{Mult}_{\mathbb{D}}(\mathbf{T}))$.

The remaining assertion is checked by direct inspection. \square

72 Definition Let \mathbf{IntSem}_{nd} be the subcategory of \mathbf{IntSem} consisting of those objects (\mathbb{C}, \mathbf{S}) with $\mathbf{Conv}_{\mathbb{C}}(\mathbf{S})$ globally idempotent and non-degenerate, and morphisms $(F, f): (\mathbb{C}, \mathbf{S}) \rightarrow (\mathbb{D}, \mathbf{T})$ such that $\mathbf{Conv}(F, f): \mathbf{Conv}_{\mathbb{C}}(\mathbf{S}) \rightarrow \mathbf{Conv}_{\mathbb{D}}(\mathbf{T})$ is non-degenerate.

Let $_{conc}\mathbf{IntSem}_{nd}$ be the full subcategory of \mathbf{IntSem}_{nd} spanned by the concrete semigroup objects.

73 Remark By definition, $\mathbf{Conv}: \mathbf{IntSem} \rightarrow \mathbf{Sem}$ restricts and co-restricts as a functor $\mathbf{Conv}_{nd}: \mathbf{IntSem}_{nd} \rightarrow \mathbf{Sem}_{nd}$.

74 Proposition *There is a functor $Mult: _{conc}\mathbf{IntSem}_{nd} \rightarrow \mathbf{Mon}$ such that $Mult(\mathbb{C}, \mathbf{S}) = \mathbf{Mult}_{\mathbb{C}}(\mathbf{S})$.*

Proof: Let $(F, f) \in _{conc}\mathbf{IntSem}_{nd}((\mathbb{C}, \mathbf{S}), (\mathbb{D}, \mathbf{S}))$. It suffices to define $Mult(F, f) := (\mathcal{M}_{\mathbb{D}, \mathbf{T}} \circ \mathbf{Conv}(F, f))^M: \mathbf{Mult}_{\mathbb{C}}(\mathbf{S}) \rightarrow \mathbf{Mult}_{\mathbb{D}}(\mathbf{T})$ using Theorem 71. \square

Let $Conc := (Conc_{\mathbb{C}}(\mathbf{S}))_{(\mathbb{C}, \mathbf{S}) \in Ob_{_{conc}\mathbf{IntSem}_{nd}}}$. Using Theorem 71, one easily shows that $Conc$ is a natural transformation $Mult \Rightarrow \mathbf{TrHull} \circ \mathbf{Conv}_{nd}$ from $_{conc}\mathbf{IntSem}_{nd}$ to \mathbf{Mon} . As a consequence of the above lemma, the following diagram commutes up to the natural epimorphism $Conc$.

$$\begin{array}{ccc}
 \mathbf{IntSem}_{nd} & \xrightarrow{\mathbf{TrHull} \circ \mathbf{Conv}_{nd}} & \mathbf{Mon} \\
 \uparrow & \nearrow^{Mult} & \\
 {conc}\mathbf{IntSem}{nd} & &
 \end{array} \tag{47}$$

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