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# A new definition of the fractal derivative with classical properties

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## Abstract

In this work, we give a new definition of the fractal derivative and some theorems and the integral of the new fractal derivative and some of the classical properties like the fractal chain rule, new fractal integrals, fractal Laplace transform is obtained and the action of fractal derivatives and integrals to each other are discussed. Finally, an example is given to compare the stability of the fractional differential equation to the new definition and other definitions (Hausdorff derivative, YANG derivative, and Caputo derivative). Through comparison, it is indicated that the stretched exponential stability shall serve as an alternative approach for characterizing the decaying patterns of the non-integer order systems.

*Keywords:* Fractal derivative, fractal integral, fractal exponential function, Chain rule.

*MSC(2010):* 26A33.

## 1 Introduction

The fractional calculus [10, 11, 12] attracted many researches in the last and present centuries. The impact of this fractional calculus in both pure and applied branches of science and engineering started to increase substantially during the last two decades apparently. Many researches started to deal with the discrete versions of this fractional calculus benefitting from the theory of time scales (see [13, 14, 15, 16] and the references therein).

The Hausdorff derivative involving the fractal geometry with the Hausdorff measure, proposed by Chinese mathematician Wen Chen [3, 4] has played an important role in the treatment for the mathematical model for the anomalous diffusion process. The Hausdorff integral was suggested in 2018 by Chen and coauthors to develop the three-dimensional diffusion model for fractal porous media [6]. The Hausdorff calculus was used to the mathematical models in the real-world problems for the fractal power law [7]. As is known, the

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Chen Hausdorff calculus is connected with the anomalous transport in porous media [8] and the fractional calculus [9].

The Chen Hausdorff derivative of the function [3, 4]  $x(t)$  is defined as:

$$C^\alpha(x)(t) = \lim_{z \rightarrow t} \frac{x(z) - x(t)}{z^\alpha - t^\alpha}, \quad (1)$$

if  $x$  differentiable, then

$$C^\alpha(x)(t) = \frac{t^{1-\alpha}}{\alpha} x'(t).$$

In [17] we address the general derivatives:

$$\mathbb{F}^\alpha(x)(t) = \frac{1}{\alpha e^{\alpha t}} x'(t).$$

New we present new fractal derivative of function  $x$  is defined as:

$$F^\alpha(x)(t) = \lim_{h \rightarrow t} \frac{x(h) - x(t)}{e^{h^\alpha} - e^{t^\alpha}},$$

if  $x$  differentiable, then

$$F^\alpha(x)(t) = \frac{t^{1-\alpha}}{\alpha e^{t^\alpha}} x'(t).$$

They then proved the product rule, the fractional mean value theorem, and solved some fractal differential equations where the fractal exponential function  $e^{e^{t^\alpha}}$  played an important rule. While in case of well-known fractional calculus Mittag-Leffler functions generalized exponential functions. In this article we continue to settle the basic definitions and concepts of this new theory motivated by the fact that there are certain functions which do not have Taylor power series representation or their Laplace transform cannot be calculated and so forth, but will be possible to do so by the help of the theory of this fractal calculus.

The paper is organized as follows: In Section 2 the new fractal derivative and some of the classical properties like the fractal chain rule, in 3 fractal integrals, and in 4 fractal Laplace transform is obtained and the action of fractal derivatives and integrals to each other are discussed. Finally, the example 4 is necessary for numerical comparison of the stability behavior of the new fractal derivative equation, the Hausdorff derivative equation, and its corresponding classical fractional-order (in the Caputo sense) equation.

## 2 The definition of new fractal derivative

Let us first define a truncated exponential function given by:

$$e_k^t = \sum_{i=1}^k \frac{t^i}{i!},$$

With the help of this definition, we will define another fractional derivative given below.

**Definition 1.** Let a function  $x : I \subset \mathbb{R} \rightarrow \mathbb{R}$ . Then the fractal derivative  $F_k^\alpha$  of  $x$  of order  $\alpha \in (0, 1]$  is defined by:

$$F_k^\alpha(x)(t) = \lim_{h \rightarrow t} \frac{x(h) - x(t)}{e_k^{h^\alpha} - e_k^{t^\alpha}},$$

we say  $x$  a  $\alpha$ -differentiable if  $F_k^\alpha(x)(t)$  exist for all  $t \in I$ .

Now, it is easy to see that

$$F_1^\alpha(x)(t) = \lim_{h \rightarrow t} \frac{x(h) - x(t)}{h^\alpha - t^\alpha}, \quad (2)$$

and

$$F_\infty^\alpha(x)(t) = \lim_{h \rightarrow t} \frac{x(h) - x(t)}{e^{h^\alpha} - e^{t^\alpha}}, \quad (3)$$

where the expression on the right-hand-side of (2) is the fractal derivative defined in (1) in [3, 4], while the one on (3) is the  $\alpha$ -fractal derivative defined in this paper. Furthermore, if  $\alpha = 1$  the Equation (2) becomes the classical definition of the first derivative of a function  $x$  at a point  $t$ . While, noting that  $e^{h^\alpha} = 1 + h + O(\epsilon^2)$ , we may also show that the Equation (3) is equivalent to the classical definition of the first derivative of a function  $x$ .

**Definition 2.** Let a function  $x : I \subset \mathbb{R} \rightarrow \mathbb{R}$ . Then the fractal derivative  $F^\alpha$  of  $x$  of order  $\alpha \in (0, 1]$  is defined by:

$$F^\alpha(x)(t) = \lim_{h \rightarrow t} \frac{x(h) - x(t)}{e^{h^\alpha} - e^{t^\alpha}}.$$

We say  $x$  a  $\alpha$ -differentiable if  $F^\alpha(x)(t)$  exist for all  $t \in I$ .

**Example 1.** The fractal derivative of certain functions:

1.  $F^\alpha(1) = 0$ .
2.  $F^\alpha(t^p) = \frac{p}{\alpha e^{t^\alpha}} t^{p-\alpha}$ .
3.  $F^\alpha(e^{e^{t^\alpha}}) = 1$ .
4.  $F^\alpha(e^{\lambda e^{t^\alpha}}) = \lambda e^{\lambda e^{t^\alpha}}$ .

**Remark 1.** Let  $x$   $\alpha$ -differentiable, if  $\alpha = 1$ , then  $x$  is differentiable.

**Theorem 3.** If a function  $x : I \subset \mathbb{R} \rightarrow \mathbb{R}$  is  $\alpha$ -differentiable at  $t_0$ ,  $\alpha \in (0, 1]$ , then  $x$  is continuous at  $t_0$ .

*Proof.* Since

$$x(h) - x(t_0) = \frac{x(h) - x(t_0)}{e^{h^\alpha} - e^{t_0^\alpha}} (e^{h^\alpha} - e^{t_0^\alpha}).$$

Then,

$$\lim_{h \rightarrow t_0} [x(h) - x(t_0)] = \lim_{h \rightarrow t_0} \frac{x(h) - x(t_0)}{e^{h^\alpha} - e^{t_0^\alpha}} \lim_{h \rightarrow t_0} (e^{h^\alpha} - e^{t_0^\alpha}),$$

so

$$\lim_{h \rightarrow t_0} [x(h) - x(t_0)] = F^\alpha(x)(t_0) \cdot 0,$$

which implies that  $\lim_{h \rightarrow t_0} x(h) = x(t_0)$ , hence,  $x$  is continuous at  $t_0$ . □

The  $F^\alpha$  satisfies all the properties in the following theorem.

**Theorem 4.** Let  $f$  and  $g$  are  $\alpha$ -differentiable. We have,

1.  $F^\alpha(af + bg) = aF^\alpha(f) + bF^\alpha(g)$ , for all  $a, b \in \mathbb{R}$ .
2.  $F^\alpha(\lambda) = 0$ , for all constant functions  $f(x) = \lambda$ .

$$3. F^\alpha(fg)(t) = F^\alpha(f)(t)g(t) + f(t)F^\alpha(g)(t).$$

$$4. F^\alpha\left(\frac{f}{g}\right) = \frac{F^\alpha(f)(t)g(t) - F^\alpha(g)(t)f(t)}{g(t)^2}.$$

*Proof.* Parts (1) through (2) follow directly from the definition. For (3): Now, for fixed  $t$ ,

$$\begin{aligned} F^\alpha(fg)(t) &= \lim_{h \rightarrow t} \frac{f(h)g(h) - f(t)g(t)}{e^{h^\alpha} - e^{t^\alpha}} \\ &= \lim_{h \rightarrow t} \frac{f(h)g(h) - f(t)g(h) + f(t)g(h) - f(t)g(t)}{e^{h^\alpha} - e^{t^\alpha}} \\ &= \lim_{h \rightarrow t} \frac{f(h) - f(t)}{e^{h^\alpha} - e^{t^\alpha}} g(h) + \lim_{h \rightarrow t} f(t) \frac{g(h) - g(t)}{e^{h^\alpha} - e^{t^\alpha}} \\ &= F^\alpha(f)(t)g(t) + F^\alpha(g)(t)f(t). \end{aligned}$$

For (4): Now, for fixed  $t$ ,

$$\begin{aligned} F^\alpha\left(\frac{f}{g}\right)(t) &= \lim_{h \rightarrow t} \frac{\frac{f(h)}{g(h)} - \frac{f(t)}{g(t)}}{e^{h^\alpha} - e^{t^\alpha}} \\ &= \lim_{h \rightarrow t} \frac{\frac{f(h)g(t) - g(h)f(t)}{g(h)g(t)}}{e^{h^\alpha} - e^{t^\alpha}} \\ &= \lim_{h \rightarrow t} \frac{f(h)g(t) - f(t)g(t) + f(t)g(t) - g(h)f(t)}{g(h)g(t)(e^{h^\alpha} - e^{t^\alpha})} \\ &= \lim_{h \rightarrow t} \frac{f(h) - f(t)}{e^{h^\alpha} - e^{t^\alpha}} \frac{g(t)}{g(h)g(t)} - \lim_{h \rightarrow t} \frac{f(t)}{g(h)g(t)} \frac{g(h) - g(t)}{e^{h^\alpha} - e^{t^\alpha}}, \end{aligned}$$

since  $f$  and  $g$  are  $\alpha$ -differentiable so

$$\begin{aligned} F^\alpha\left(\frac{f}{g}\right)(t) &= F^\alpha(f)(t) \frac{g(t)}{g(t)g(t)} - \frac{f(t)}{g(t)g(t)} F^\alpha(g)(t) \\ &= \frac{F^\alpha(f)(t)g(t) - F^\alpha(g)(t)f(t)}{g(t)^2}. \end{aligned}$$

□

**Theorem 5.** (Fractal chain rule) Let  $f$  differentiable and  $g$   $\alpha$ -differentiable. We have,

$$F^\alpha(fog)(t) = f'(g(t))F^\alpha(g)(t).$$

*Proof.*

$$\begin{aligned} F^\alpha(fog)(t) &= \lim_{h \rightarrow t} \frac{f(g(h)) - f(g(t))}{e^{h^\alpha} - e^{t^\alpha}} \\ &= \lim_{h \rightarrow t} \frac{f(g(h)) - f(g(t))}{g(h) - g(t)} \frac{g(h) - g(t)}{e^{h^\alpha} - e^{t^\alpha}} \\ &= f'(g(t))F^\alpha(g)(t). \end{aligned}$$

□

**Lemma 6.** Let a function  $x : \mathbb{R} \rightarrow \mathbb{R}$  differentiable, then

$$F^\alpha(x)(t) = \frac{t^{1-\alpha}}{\alpha e^{t^\alpha}} x'(t).$$

*Proof.* We have

$$\begin{aligned}
F^\alpha(x)(t) &= \lim_{h \rightarrow t} \frac{x(h) - x(t)}{e^{h^\alpha} - e^{t^\alpha}} \\
&= \lim_{h \rightarrow t} \frac{x(h) - x(t)}{h - t} \lim_{h \rightarrow t} \frac{h - t}{e^{h^\alpha} - e^{t^\alpha}} \\
&= \lim_{h \rightarrow t} \frac{x(h) - x(t)}{h - t} \frac{1}{\lim_{h \rightarrow t} \frac{e^{h^\alpha} - e^{t^\alpha}}{h - t}} \\
&= x'(t) \frac{1}{\alpha t^{\alpha-1} e^{t^\alpha}}.
\end{aligned}$$

□

YANG in [17] generalized derivative with respect to another function is defined:

$$\mathbb{F}^\alpha(x)(t) = \frac{1}{h'(t)} x'(t), \quad h'(t) > 0.$$

- The general derivative with respect to power-law function, denoted as [17]  $h(t) = t^\alpha$ , is defined:

$$\mathbb{F}^\alpha(x)(t) = \frac{1}{\alpha t^{\alpha-1}} x'(t).$$

- The general derivative with respect to exponential function (not, YANG derivative), denoted as [17]  $h(t) = e^{\alpha t}$ , is defined:

$$\mathbb{F}^\alpha(x)(t) = \frac{1}{\alpha e^{\alpha t}} x'(t).$$

**Remark 2.** 1. If  $\alpha = 1$ , the YANG derivative  $\mathbb{F}^\alpha$  and new fractal derivative  $F^\alpha$  coincide.

2. The Definition 1 equivalence the general derivative in [17] ( $h(t) = e_k^{t^\alpha}$ ).

### 3 Fractal integral

When it comes to integration, the most important class of functions to define the integral is the space of continuous functions.

**Definition 7.** Let  $\alpha \in (0, 1]$ ,

$$I_\alpha(f)(t) = \int_0^t \alpha s^{\alpha-1} e^{s^\alpha} f(s) ds.$$

One of the nice results is the following.

**Theorem 8.**

$$F^\alpha I_\alpha(f)(t) = f(t),$$

for  $t$ , where  $f$  is any continuous function in the domain of  $I_\alpha$ .

*Proof.* Since  $f$  is continuous, then  $I_\alpha(f)(t)$  is clearly differentiable. Hence,

$$\begin{aligned} F^\alpha(I_\alpha(f))(t) &= \frac{1}{\alpha t^{\alpha-1} e^{t^\alpha}} \frac{d}{dt} I_\alpha(f)(t) \\ &= \frac{1}{\alpha t^{\alpha-1} e^{t^\alpha}} \frac{d}{dt} \int_0^t \alpha s^{\alpha-1} e^{s^\alpha} f(s) ds \\ &= \frac{1}{\alpha t^{\alpha-1} e^{t^\alpha}} \alpha t^{\alpha-1} e^{t^\alpha} f(t) \\ &= f(t). \end{aligned}$$

□

**Theorem 9.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable and  $0 < \alpha \leq 1$ . Then,

$$I_\alpha(F^\alpha(f)(t)) = f(t) - f(0).$$

*Proof.* we have

$$\begin{aligned} I_\alpha(F^\alpha(f)(t)) &= \int_0^t \alpha s^{\alpha-1} e^{s^\alpha} F^\alpha(f)(s) ds. \\ &= \int_0^t \alpha s^{\alpha-1} e^{s^\alpha} \frac{1}{\alpha s^{\alpha-1} e^{s^\alpha}} f'(s) ds \\ &= \int_0^t f'(s) ds \\ &= f(t) - f(0). \end{aligned}$$

□

## 4 The fractal Laplace transform

**Definition 10.** Let  $0 < \alpha \leq 1$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be real valued function. Then the fractal Laplace transform of order  $\alpha$  of  $f$  is defined by:

$$L_\alpha\{f(t)\}(s) = F_\alpha(s) = \alpha \int_0^\infty e^{-se^{t^\alpha}} f(t) t^{\alpha-1} e^{t^\alpha} dt.$$

**Theorem 11.** Let  $0 < \alpha \leq 1$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable real valued function. Then

$$L_\alpha\{F^\alpha(f)(t)\}(s) = sF_\alpha(s) - e^{-s} f(0).$$

*Proof.* We have

$$\begin{aligned} L_\alpha\{F^\alpha(f)(t)\}(s) &= \alpha \int_0^\infty e^{-se^{t^\alpha}} F^\alpha(f)(t) t^{\alpha-1} e^{t^\alpha} dt \\ &= \alpha \int_0^\infty e^{-se^{t^\alpha}} \frac{1}{\alpha t^{\alpha-1} e^{t^\alpha}} f'(t) t^{\alpha-1} e^{t^\alpha} dt \\ &= \int_0^\infty e^{-se^{t^\alpha}} f'(t) dt \\ &= [e^{-se^{t^\alpha}} f(t)]_0^\infty - \alpha s \int_0^\infty -e^{-se^{t^\alpha}} f(t) t^{\alpha-1} e^{t^\alpha} dt \\ &= -e^{-s} f(0) + sF_\alpha(s). \end{aligned}$$

□

The following lemma relates the fractal Laplace transform to the usual Laplace transform.

**Lemma 12.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that  $L_\alpha\{f(t)\}(s) = F_\alpha(s)$  exists. Then*

$$F_\alpha(s) = \mathfrak{L}\{f((\log(t))^{\frac{1}{\alpha}})\}(s),$$

where  $\mathfrak{L}\{g(t)\}(s) = \int_1^\infty e^{-st}g(t)dt$ .

*Proof.* The proof follows easily by setting  $u = e^{t^\alpha}$ . □

**Theorem 13.** *Let  $0 < \alpha \leq 1$ . So, we have*

1.  $L_\alpha\{1\}(s) = \frac{e^{-s}}{s}, s > 0$ .
2.  $L_\alpha\{e^{e^{t^\alpha}}\}(s) = \frac{e^{-s+1}}{s-1}, s > 1$ .
3.  $L_\alpha\{e^{\lambda e^{t^\alpha}}\} = \frac{e^{\lambda-s}}{s-\lambda}, s > \lambda$ .

*Proof.* From definition directly. □

**Proposition 14.** *1. If the functions  $f$  and  $g$  are transformable, then there is the transform of the sum and is equal to the sum of the transforms, that is:*

$$L_\alpha\{f + g\} = L_\alpha\{f\} + L_\alpha\{g\}.$$

*2. If the function  $f$  is transformable and  $\lambda$  is a real number, then there is the transform of product of  $\lambda$  by  $f$  and is equal to product of  $\lambda$  by the transform of  $f$ , that is:*

$$L_\alpha\{\lambda f\} = \lambda L_\alpha\{f\}.$$

*Proof.* Taking into account the two previous propositions, we say that  $L_\alpha$  is a linear operator. □

**Example 2.** *Consider the fractal initial value problem:*

$$F^\alpha(x)(t) = \lambda x(t), \quad x(0) = x_0, t \geq 0, \quad (4)$$

the exact solution of this is  $x(t) = e^{\lambda e^{t^\alpha}} e^{-\lambda} x_0$ .

*Proof.* Applying the fractal Laplace Transform to both sides of equation (4), we get

$$L_\alpha\{F^\alpha(x)(t)\}(s) = L_\alpha\{\lambda x(t)\}(s),$$

from Theorem 11 and Proposition 14, we have

$$sF_\alpha(s) - e^{-s}x_0 = \lambda F_\alpha(s).$$

Simplifying this we get

$$F_\alpha(s) = \frac{e^{-s}}{s-\lambda}x_0. \quad (5)$$

Taking the inverse fractal Laplace transform to (5), we get  $x(t) = e^{\lambda e^{t^\alpha}} e^{-\lambda} x_0$ . The solution of (4), obtained from fractal Laplace transformation method, are shown in Figure 1 for different values of  $\alpha$ .

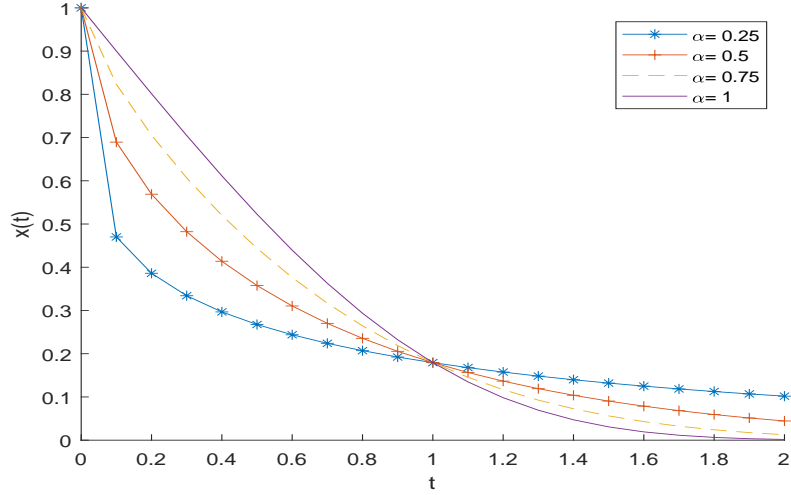


Figure 1: Fractal Laplace solution of (5) for different values of  $\alpha$  with  $\lambda = -1$  and  $x_0 = 1$ .

□

**Example 3.** Consider the system

$$F^\alpha(x)(t) = Ax(t) + f(t), \quad x(0) = x_0, t \geq 0, \quad (6)$$

where  $x, f : [0, b) \rightarrow \mathbb{R}^n$  are vector functions and  $A$  is an  $n \times n$  matrix. We use the fractal fundamental exponential matrix to express the solution of fractal linear systems. The general solution of the fractal nonhomogeneous system (6) is expressed by:

$$x(t) = e^{Ae^{t^\alpha}} e^{-A} x_0 + \alpha \int_0^t e^{A(e^{t^\alpha} - e^{s^\alpha})} s^{\alpha-1} e^{s^\alpha} ds.$$

**Example 4.** In this example is necessary numerical comparisons of the stability behavior of the new fractal derivative system, Hausdorff derivative system [1] and its corresponding classical fractional-order (in the Caputo sense [2]) system. For the following one dimensional dynamical system [1]:

$$F^\alpha(x)(t) = -x(t), \quad (7)$$

where  $\alpha \in (0, 1]$  and  $x(t) : [0, \infty) \rightarrow \mathbb{R}$ . The equilibrium point of system (7) is  $x = 0$ . A comparison of the numerical solutions of system (7) and its corresponding system when replacing the new fractal derivative  $C^\alpha$  by Caputo derivative [2] and Chen Hausdorff derivative [3, 4] is presented in Figure 2. From Figure 2, it is clear that the new fractal derivative system converges to zero much faster than its corresponding fractional-order system.

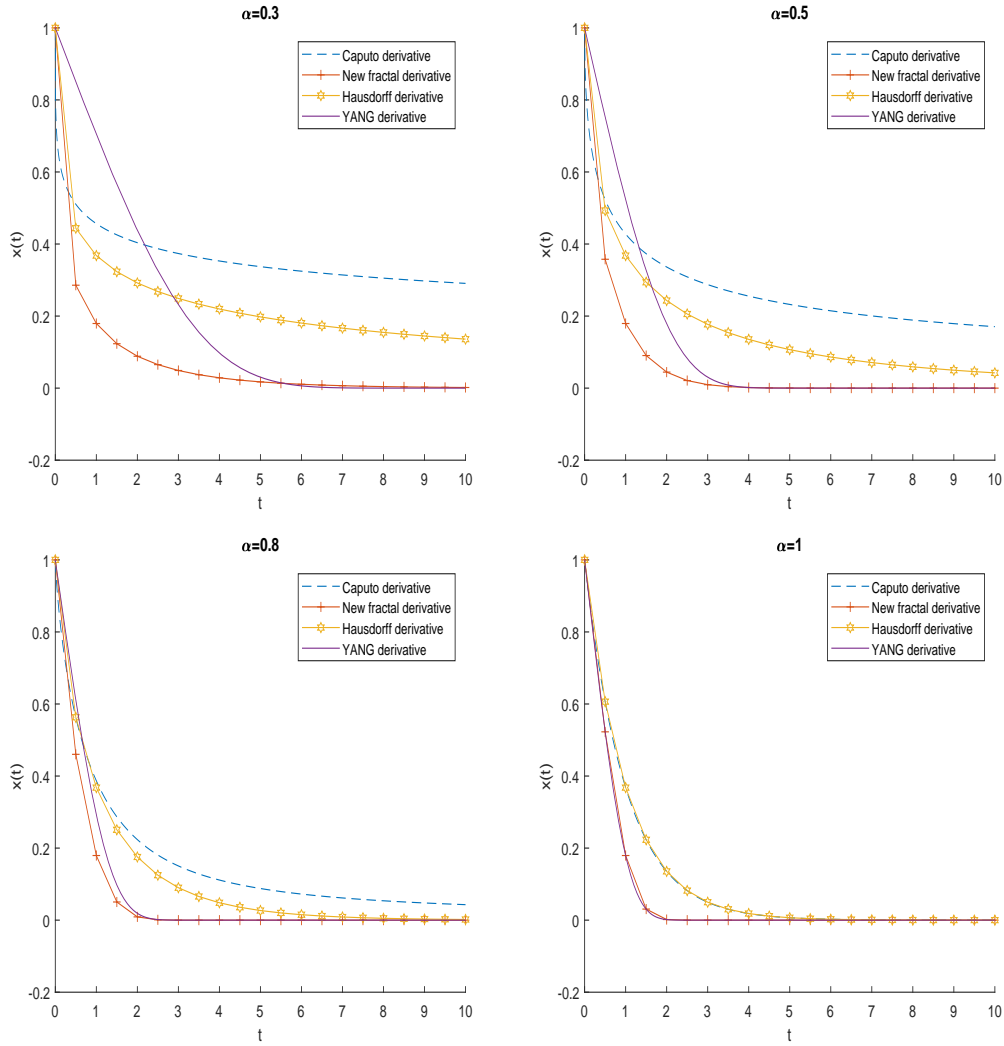


Figure 2: Comparisons of numerical solutions of new fractal derivative system and fractional-order system when they have the same initial condition  $x(0) = 1$  and parameter  $\alpha = \{0.3, 0.5, 0.8, 1\}$ .

## 5 Conclusion

In this work, we presented a new definition of the fractal derivative and some theorems and the integral of the new fractal derivative. we obtained important results with respect to the properties of the integer-order derivative. Through comparison, it is indicated that the stretched exponential stability shall serve as an alternative approach for characterizing the decaying patterns of the non-integer order systems. More specifically, our future works will consider applications of the proposed new definition. We conclude the paper with the following question, which has yet to be answered. What is the physical meaning or geometric interpretation of the  $\alpha$ -derivative?

## Data availability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

## Declarations

### Conflict of interest

The authors declare that they have no conflict of interest.

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## References

- [1] Hu, D. L., Chen, W., & Sun, H. G. (2020). Power-law stability of Hausdorff derivative nonlinear dynamical systems. *International Journal of Systems Science*, 51(4), 601-607.
- [2] Oldham, K., & Spanier, J. (1974). *The fractional calculus theory and applications of differentiation and integration to arbitrary order*. Elsevier.
- [3] Chen, W. (2006). Timespace fabric underlying anomalous diffusion. *Chaos, Solitons & Fractals*, 28(4), 923-929.
- [4] Chen, W., Sun, H., Zhang, X., & Koroak, D. (2010). Anomalous diffusion modeling by fractal and fractional derivatives. *Computers & Mathematics with Applications*, 59(5), 1754-1758.
- [5] Chen, W., Hei, X., Sun, H., & Hu, D. (2018). Stretched exponential stability of nonlinear Hausdorff dynamical systems. *Chaos, Solitons & Fractals*, 109, 259-264.
- [6] Cai, W., Chen, W., & Wang, F. (2018). Three-dimensional Hausdorff derivative diffusion model for isotropic/anisotropic fractal porous media. *Thermal Science*, 22(Suppl. 1), 1-6.
- [7] Liang, Yingjie, Wen Chen, and Wei Cai. *Hausdorff calculus: applications to fractal systems*. Vol. 6. Walter de Gruyter GmbH & Co KG, 2019.
- [8] Liang, Yingjie, Ninghu Su, and Wen Chen. "A time-space Hausdorff derivative model for anomalous transport in porous media." *Fractional Calculus and Applied Analysis* 22.6 (2019): 1517-1536.
- [9] Atangana, Abdon. "Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system." *Chaos, solitons & fractals* 102 (2017): 396-406.

- [10] Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). Fractional integrals and derivatives (Vol. 1). Yverdon-les-Bains, Switzerland: Gordon and Breach Science Publishers, Yverdon.
- [11] Podlubny, I. (1998). Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier.
- [12] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and applications of fractional differential equations (Vol. 204). Elsevier.
- [13] Gray, H. L., & Zhang, N. F. (1988). On a new definition of the fractional difference. *Mathematics of Computation*, 50(182), 513-529.
- [14] Atici, F., & Eloe, P. (2009). Initial value problems in discrete fractional calculus. *Proceedings of the American Mathematical Society*, 137(3), 981-989.
- [15] Abdeljawad, T., & Atici, F. M. (2012, January). On the definitions of nabla fractional operators. In *Abstract and Applied Analysis* (Vol. 2012). Hindawi.
- [16] Abdeljawad, T. (2013). Dual identities in fractional difference calculus within Riemann. *Advances in Difference Equations*, 2013(1), 1-16.
- [17] Yang, X. J. (2019). New general calculi with respect to another functions applied to describe the Newton-like dashpot models in anomalous viscoelasticity. *Thermal Science*, 23(6 Part B), 3751-3757.