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Abstract: This paper investigates the problems of stability analysis and control design for continuous-time Lur'e systems with slope bounded nonlinearities. Starting from a stability analysis condition from the literature, based on the real positivity and a bound to the $\mathscr{L}_1$ norm of a certain transfer function, new sufficient conditions are proposed in an augmented parameter space for the simultaneous existence of a stabilizing dynamic output-feedback controller and a Zames-Falb multiplier certifying the closed-loop stability. The matrices of the controller realization as well as of the Zames-Falb multiplier appear affinely in the conditions, being dealt with as optimization variables. Furthermore, no line search is required to enforce the $\mathscr{L}_1$ norm constraint. An iterative algorithm is constructed to solve the problem through semidefinite programming, providing dynamic controllers of any given order. The controller can take into account both the output of the linear part of the system and of the nonlinearity. Numerical examples illustrate the results.

Keywords: Continuous-time Lur'e systems, absolute stability, Zames-Falb multipliers, linear matrix inequalities, dynamic output-feedback.

1. INTRODUCTION

Lur'e systems describe linear time-invariant plants in feedback with a static nonlinear function. The stability analysis of these systems can be carried out by studying properties of the interconnection considering simple representations of the nonlinearity, as for instance with quadratic inequalities that describe sector and slope restricted functions. Since the stability analysis treats classes of nonlinear functions lying in a sector, it is known as the *absolute stability* problem. While encompassing classes of nonlinear functions, these stability analysis methods are powerful since they consider the LTI system data to conclude on stability of the interconnection.

When certified, the stability of a Lur'e system is robust since it holds for any nonlinearites within the considered class. For Lur'e systems the robustness is not only a rigorous mathematical property for the solutions of the system, it is also of paramount importance in practice since the nonlinear elements in the engineering systems modeled by the interconnection are not precisely known.

Classical results, such as the circle and Popov criteria (Khalil, 2002), consider sector limitations, while more recent approaches also take into account slope bounded nonlinearities. Lyapunov functions (LF) (Gonzaga et al., 2012; Park, 1997, 2002) and Zames-Falb (ZF) multipliers (Turner et al., 2009; Ahmad et al., 2013; Haddad and Bernstein, 1994; Carrasco et al., 2020; Turner and Drummond, 2020) can be considered as the two main strategies to deal with absolute stability.

Multipliers are transfer functions used in loop transformations for frequency domain stability analysis. The absolute stability problem can be converted into the search of the so-called ZF multipliers (Zames and Falb, 1968; O'Shea, 1967). These multipliers have to satisfy a constraint on the $\mathscr{L}_1$ norm and preserve the positive realness of the overall system (Carrasco et al., 2016; Turner et al., 2009, 2012). The ZF multipliers are thus certificates for stability and the problem of computing the parameters of classes of ZF multipliers can be formulated as semi-definite programs (Chang et al., 2012; Boyd et al., 1994; Carrasco et al., 2020; Turner and Drummond, 2020).

For the feedback control gain synthesis, there exist some LF based approaches for state feedback control of Lur'e systems (Castelan et al., 2008; Louis et al., 2015). On the other hand, ZF-based conditions to handle control design prob-

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lems seem more difficult to formulate. Recently, ZF multiplier based conditions for the static output feedback stabilization of discrete-time Lur'e systems have been proposed in Bertolin et al. (2022). In this paper, similar conditions are proposed for continuous-time Lur'e systems.

The aim of this paper is to propose a convex-based procedure for the design of fixed-order dynamic controllers for continuous-time Lur'e systems. Sufficient LMI conditions are given for the absolute stability of the closed-loop Lur'e system in terms of the existence of ZF multipliers. As an important improvement with respect to the results in Turner et al. (2009), any given order can be considered for the multipliers. Moreover, distinct Lyapunov matrices for the $\mathscr{L}_1$ and real positivity constraints are considered (instead of a common one) and the need of performing a scalar search is eliminated. The state-space realization of the ZF multiplier as well as the matrices of the dynamic controller appear in an affine manner in the conditions. Then, using a relaxation in the stability of the linear part of the system, an iterative algorithm is proposed, solving a convex optimization problem at each iteration. The performance of the method is demonstrated by means of numerical examples borrowed from the literature.

Notation: For a symmetric matrix, $A > 0$ ($A < 0$) means that A is positive (negative) definite. For matrices or vectors $(^T)$ indicates the transpose, $\mathrm{He}(A) = A + A^T$ and $diag(A_1, \ldots, A_n)$ represents the block diagonal matrix formed by the square matrices (or elements) A_1, ..., A_n. The symbol $\star$ represents a term induced by symmetry in a square matrix. The identity and the zero matrices are denoted, respectively, by I and 0. Throughout the text the dimensions of the matrices may be omitted for simplicity (being inferred from the context).

2. PROBLEM DEFINITION AND PRELIMINARIES

Consider the continuous-time nonlinear Lur'e system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_\phi \phi(z(t)) + B_u u(t) \\ z(t) &= C_z x(t) + D_\phi \phi(z(t)) + D_u u(t) \\ y(t) &= C_y x(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $\phi \in \mathbb{R}$ is the nonlinear input, $u \in \mathbb{R}^{n_u}$ is the control input, $z \in \mathbb{R}$ is the output that goes through the nonlinearity and $y \in \mathbb{R}^{n_y}$ is the measured output. Matrices A, B_ϕ, B_u, C_z, C_y, D_ϕ and D_u are real and have appropriate dimensions.

The time-invariant nonlinearity $\phi : \mathbb{R} \to \mathbb{R}$, with $\phi(0) = 0$, is odd and verifies the slope bound condition given by

$$0 \leq \big(\phi(\hat{z}) - \phi(z)\big)/(\hat{z} - z) \leq \Lambda \tag{2}$$

for all z, $\hat{z} \in \mathbb{R}$, $\hat{z} \neq z$, and $\phi(\hat{z})$, $\phi(z) \in [0, \Lambda]$ where $\Lambda \in \mathbb{R}$ is a given positive scalar. Furthermore, it is assumed that the Lur'e system satisfies the well possedness condition given by

$$(1 + \Lambda D_\phi) > 0. \tag{3}$$

The objective of this work is to design a stabilizing fixed-order dynamic output-feedback controller for system (1) with state-space realization given by

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t) + B_{c\phi} \phi(z(t)) \\ u(t) &= C_c x(t) + D_c y(t) + D_{c\phi} \phi(z(t)) \end{aligned} \tag{4}$$

where $x_c \in \mathbb{R}^{n_c}$ is the state and $A_c, B_c, B_{c\phi}, C_c, D_c$ and $D_{c\phi}$ are real matrices of appropriate dimensions. Connecting the controller (4) with the plant (1) provides the following closed-loop dynamics

$$G_{cl} = \begin{cases} \dot{\overline{x}}(t) = A_{cl}\overline{x}(t) + B_{cl}\phi(z(t)), \\ z(t) = C_{cl}\overline{x}(t) + D_{cl}\phi(z(t)), \end{cases}$$

where $\overline{x} = \begin{bmatrix} x^T & x_c^T \end{bmatrix}^T$ and

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} B_\phi + B_u D_{c\phi} \\ B_{c\phi} \end{bmatrix}, \\ C_{cl}^T &= \begin{bmatrix} C_z + D_u D_c C_y \\ D_u C_c \end{bmatrix}, \quad D_{cl} = D_\phi + D_u D_{c\phi}. \end{aligned} \tag{5}$$

The computation of the matrices of the controller can be done based on the stability analysis of system (5), which can be investigated through the existence of a ZF multiplier (Zames and Falb, 1968) $M(s) = 1 + H(s)$ such that

$$\mathrm{Re}\{M(j\omega)\tilde{G}_{cl}(j\omega)\} > 0, \ \forall \omega \in \mathbb{R} \tag{6}$$

where $\tilde{G}_{cl}(j\omega) = (1 + \Lambda G_{cl}(j\omega))$, and $H(s)$ is a rational strictly proper transfer function. The state-space realization for the multiplier $M(s)$ is given by

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + B_m r(t) \\ y_m(t) &= C_m x_m(t) + r(t), \end{aligned} \tag{7}$$

where $x_m \in \mathbb{R}^{n_M}$, $r \in \mathbb{R}$ and $y_m \in \mathbb{R}$, with real matrices A_m, B_m and C_m of appropriate dimensions. The state space realization of $M(s)\tilde{G}_{cl}(s)$, is given by

$$\begin{aligned} A_I &= \begin{bmatrix} A_{cl} & 0 \\ B_m \Lambda C_{cl} & A_m \end{bmatrix}, \quad B_I = \begin{bmatrix} B_{cl} \\ B_m(1 + \Lambda D_{cl}) \end{bmatrix}, \\ C_I &= [\Lambda C_{cl} \ \ C_m], \quad D_I = (1 + \Lambda D_{cl}). \end{aligned}$$

The stability of the closed-loop Lur'e system (5) can be assessed by searching for a ZF multiplier $M(s)$ satisfying, simultaneously, two sufficient conditions. The first one is the positive realness of $M(s)\tilde{G}_{cl}(s)$, as presented in (6). The second one is to assure that $\|H(s)\|_1 < 1$, i.e., the $\mathscr{L}_1$ norm of $H(s)$ must be smaller than one (Turner et al., 2009). Since the computation of the $\mathscr{L}_1$ norm through convex programming does not seem to be simple, upper bounds (such as the $*$-norm) obtained by LMI conditions combined with a scalar search can be used instead.

For this purpose, consider a linear continuous-time system $\mathscr{S}$ with a state space realization given by (A, B, C, D). The following lemmas provide LMI conditions for positive realness and an upper bound (i.e., the *-norm) to the $\mathscr{L}_1$ norm.

Lemma 1. (Real positivity (Sun et al., 1994)). System $\mathscr{S}$ is stable and positive real if and only if there exists a positive definite matrix $P = P^T$, such that

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ \star & -D - D^T \end{bmatrix} < 0. \tag{8}$$

Lemma 2. (*-norm (Abedor et al., 1996)). Consider system $\mathscr{S}$ with $D = 0$. Let $\alpha \in (0, \kappa)$, $\kappa = -2\max_i(\mathrm{Re}(\lambda_i(A))$, be a given scalar and $W = W^T > 0$ the solution of the convex optimization problem

$$\min_W \ \mathrm{tr}(W), \quad AW + WA^T + \alpha W + \alpha^{-1} BB^T < 0. \tag{9}$$

Then, system $\mathscr{S}$ is asymptotically stable with norm $\mathscr{L}_1$ bounded by γ, solution of the convex problem (note that W is a known matrix, solution of (9))

$$\min \ \gamma, \quad \gamma W^{-1} > C^T C. \tag{10}$$

The conditions of Lemma 2 are solved in two steps, each one based on LMIs, but note that (9) is only an LMI if the value of α is fixed. Usually, a line search on $\alpha \in (0, \kappa)$ is performed to compute the smallest value of γ limiting the $\mathscr{L}_1$ norm.

The Finsler's Lemma (de Oliveira and Skelton, 2001), reproduced below, is important for the derivation of the proposed conditions.

Lemma 3. (Finsler's lemma). Consider matrices $\mathscr{Q} \in \mathbb{R}^{\ell\times\ell}$ and $\mathscr{B} \in \mathbb{R}^{m\times\ell}$, with $rank(\mathscr{B}) < \ell$ and $\mathscr{B}\mathscr{B}_\perp = 0$. Then, the following conditions are equivalent:

i) $\mathscr{B}_\perp^T \mathscr{Q} \mathscr{B}_\perp < 0$;
ii) $\exists \mathscr{X} \in \mathbb{R}^{\ell\times m}$ such that $\mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}^T\mathscr{X}^T < 0$.

3. MAIN RESULTS

Next theorem presents the main contribution of this paper, that is, LMI conditions assuring the closed-loop stability of the Lur'e system G_{cl} through the existence of a ZF multiplier.

Theorem 1. Let the matrices $\overline{Y}_i$, $\overline{X}_i$, $i = 1,\dots,4$, with $\overline{Y}_4$ and $\overline{X}_4$ of full rank and the nonnegative integers n_M and n_c be given. If there exist matrices Y_i, X_i $i = 1,\dots,4$, A_m, B_m, C_m, A_c, B_c, $B_{c\phi}$, C_c, D_c, $D_{c\phi}$, positive definite matrices $P = P^T$ and $S = S^T$, and positive scalars α and $\gamma \le 1$ such that

$$\begin{bmatrix} S & C_m^T \\ \star & \gamma I \end{bmatrix} > 0, \quad \mathscr{Q}_1 + \mathrm{He}\left(\underbrace{\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} \overline{Y}_1 & \overline{Y}_2 & \overline{Y}_3 & \overline{Y}_4 \end{bmatrix}}_{\overline{Y}} \right) < 0, \quad (11)$$

$$\mathscr{Q}_2 + \mathrm{He}\left(\underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \overline{X}_1 & \overline{X}_2 & \overline{X}_3 & \overline{X}_4 \end{bmatrix}}_{\overline{X}} \right) < 0, \quad (12)$$

where $\tilde{A}_m = A_m + \frac{1}{2}\alpha I$,

$$\mathscr{Q}_1 = \begin{bmatrix} 0 & S & 0 & \tilde{A}_m^T \\ \star & 0 & 0 & -I \\ \star & \star & -\alpha I & -B_m^T \\ \star & \star & \star & 0 \end{bmatrix}, \ \mathscr{Q}_2 = \begin{bmatrix} 0 & P & -C_I^T & A_I^T \\ \star & 0 & 0 & -I \\ \star & \star & -2D_I & B_I^T \\ \star & \star & \star & 0 \end{bmatrix}$$

are verified, then $M(s)$ is a Zames-Falb multiplier (with realization given by A_m, B_m and C_m) certifying the stability of the closed-loop system G_{cl}.

Proof. Observing that the second inequality in (11) is in the form *ii)* of Lemma 3, one has the following equivalent condition

$$\begin{bmatrix} 0 & S & 0 \\ \star & 0 & 0 \\ \star & \star & -\alpha I \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} (\overline{Y}_4^{-1}\overline{Y}_1)^T \\ (\overline{Y}_4^{-1}\overline{Y}_2)^T \\ (\overline{Y}_4^{-1}\overline{Y}_3)^T \end{bmatrix} \begin{bmatrix} A_m^T + \frac{1}{2}\alpha I \\ -I \\ -B_m^T \end{bmatrix}^T \right) < 0, \quad (13)$$

where

$$\overline{Y}_\perp = \left[\begin{array}{ccc} I & & \\ \hline -\overline{Y}_4^{-1}\overline{Y}_1 & -\overline{Y}_4^{-1}\overline{Y}_2 & -\overline{Y}_4^{-1}\overline{Y}_3 \end{array}\right]$$

has been used as a basis for the null space of $\overline{Y}$ (observe that $\overline{Y}_4$ is full rank by hypothesis).

Condition (13) is again in form *ii)* of Lemma 3 with $\mathscr{B}_1 = \begin{bmatrix} A_m + \frac{1}{2}\alpha I & -I & -B_m \end{bmatrix}$. Adopting the following basis for the null space of $\mathscr{B}_1$

$$\mathscr{B}_{1\perp} = \begin{bmatrix} I & 0 \\ A_m + \frac{1}{2}\alpha I & -B_m \\ 0 & I \end{bmatrix},$$

then condition (13) is equivalent to

$$\begin{bmatrix} SA_m + A_m^T S + \alpha S & -SB_m \\ \star & -\alpha I \end{bmatrix} < 0, \quad (14)$$

which applying the Schur complement and a congruence transformation using the inverse of matrix S, yields

$$A_m S^{-1} + S^{-1}A_m^T + \alpha S^{-1} + \alpha^{-1} B_m B_m^T < 0,$$

which is equivalent to (9) with $W = S^{-1}$. Moreover, the first inequality in (11) also assures (10) and, as $\gamma \le 1$, matrices A_m, B_m and C_m are the state-space realization of the ZF multiplier with $\mathscr{L}_1$ norm bounded by one.

Since $\overline{X}_4$ is full rank by assumption, consider

$$\mathscr{B}_{2\perp} = \left[\begin{array}{ccc} I & & \\ \hline -\overline{X}_4^{-1}\overline{X}_1 & -\overline{X}_4^{-1}\overline{X}_2 & -\overline{X}_4^{-1}\overline{X}_3 \end{array}\right]$$

as basis for the null space of $\overline{X}$, that is, $\overline{X}\mathscr{B}_{2\perp} = 0$. Using Lemma 3 (Finsler), condition (12) is equivalent to $\mathscr{B}_{2\perp}^T \mathscr{Q}_2 \mathscr{B}_{2\perp} < 0$, that can be rewritten as

$$\begin{bmatrix} 0 & P & -C_I^T \\ \star & 0 & 0 \\ \star & \star & -2D_I \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} (\overline{X}_4^{-1}\overline{X}_1)^T \\ (\overline{X}_4^{-1}\overline{X}_2)^T \\ (\overline{X}_4^{-1}\overline{X}_3)^T \end{bmatrix} \begin{bmatrix} A_I^T \\ -I \\ B_I^T \end{bmatrix}^T \right) < 0. \quad (15)$$

The above inequality is, again, in the form *ii)* of Lemma 3. Computing a basis for the null space of $[A_I \ -I \ B_I]$, with some manipulation, one gets a condition that is equivalent to (8), proving the stability of the Lur'e system in closed-loop.

The conditions of Theorem 1 have some appealing aspects: first, differently of Lemma 2, α appears affinely in condition (11), and, therefore, it is not necessary to perform one-dimensional searches (α is an optimization variable). Note that as A_m is an optimization variable, it is not possible to establish a value for κ (and consequently a bound for α) in Lemma 2. This is certainly an issue for the approaches that perform a linear search on α, as in Turner et al. (2009) and Carrasco et al. (2012). On the other hand, Theorem 1, as in Turner and Drummond (2020), does not present this inconvenient. The second and more important advantage is the fact that the matrices of the closed-loop system appear affinely in the conditions. Consequently, the matrices of the dynamic controller can also be dealt with as optimization variables (no need of change of variables, as usual in the literature). Note that the order of the controller can be arbitrarily chosen and particular structures for the matrices, as in the case of decentralized control, can be imposed straightforwardly.

On the other hand, Theorem 1 has an adverse characteristic. The proposed conditions are LMIs only if the initialization matrices Y_i and X_i are given and B_m is fixed. This last problem disappears if $D_u = 0$ or if a realization differently from (7) is adopted for the multiplier. For instance, a controllable canonical form with $B_m = [0 \ \cdots \ 1]^T$. Regarding the first issue, the following theorem establishes one possible initialization through the introduction of a relaxation scalar parameter r.

Theorem 2. Let $\overline{Y} = [I \ I \ 0 \ -I]$, $\overline{X} = [I \ \nu I \ 0 \ -I]$, with $0 < \nu < 4(1 + \Lambda D_\phi)(B_\phi^T B_\phi)^{-1}$, and

$$\overline{A}_{cl} = A_{cl} - rI, \quad \overline{A}_m = A_m - rI \quad (16)$$

where r is a positive scalar. Then, the conditions of Theorem 1 with $\overline{A}_{cl}$, $\overline{A}_m$ as (16) always present a feasible solution with a finite value of r.

Proof. First, fixing $Y_1 = Y_2 = -\frac{1}{2}rI$, $Y_3 = 0$, $Y_4 = \frac{1}{2}rI$ and $S = rI$ in condition (11) and applying a Schur complement, one has

$$\begin{bmatrix} \alpha I & B_m^T \\ \star & -r^{-1}((A_m + \frac{\alpha}{2}I)(A_m^T + \frac{\alpha}{2}I) + I) + 2I \end{bmatrix} > 0.$$

Considering r sufficiently large, $B_m = 0$ one has $\alpha I > 0$, which is always true. Similarly, the choice $C_m = 0$ simplifies the first condition in (11) to $diag(rI, \gamma I) > 0$, which is always verified.

Finally, considering $X_1 = -X_4 = -\frac{1}{2}rI$, $X_2 = -\frac{\nu}{2}rI$ ($\nu > 0$), $X_3 = 0$, and also $P = \nu rI$, one has (after a Schur complement)

$$\begin{bmatrix} 2D_I - \frac{1}{r}C_I C_I^T & -B_I^T + \frac{1}{r}C_I A_I^T \\ \star & -\frac{1}{r}(A_I A_I^T + \frac{1}{\nu^2}I) + \frac{2}{\nu}I \end{bmatrix} > 0.$$

Considering r sufficiently large, $B_{c\phi} = 0$ and $D_{c\phi} = 0$ leads to

$$diag\Big(\begin{bmatrix} 2(1+\Lambda D_\phi) & -B_\phi^T \\ \star & \frac{2}{\nu} \end{bmatrix}, \frac{2}{\nu}, \frac{2}{\nu} \Big) > 0$$

The two lower diagonal elements are verified (since $\nu > 0$) and, from the first block, with a Schur complement, one has $4(1+\Lambda D_\phi) - \nu B_\phi^T B_\phi > 0$ that, with $(1+\Lambda D_\phi) > 0$ due to assumption (3), is satisfied with $0 < \nu < 4(1+\Lambda D_\phi)(B_\phi^T B_\phi)^{-1}$.

The scalar r introduced by Theorem 2 can be seen as a relaxation factor in matrices A_{cl} and A_m, ensuring that the conditions of Theorem 1 always produce a feasible solution if the initializations indicated by Theorem 2 are adopted. Furthermore, as r appears affinely in the condition, it can be minimized as an objective function. Clearly, for the stabilization of the original Lur'e system (1) a solution with $r = 0$ must be obtained. If this is not the case, the conditions of Theorem 1 can be tested again with the new initial conditions $\overline{Y} = Y^T$ and $\overline{X} = X^T$. This is possible because conditions (11) and (12) have the structure $\mathscr{Q} + \mathrm{He}(\mathscr{X}\mathscr{B}) < 0$. Thus, as $\mathrm{He}(\mathscr{X}\mathscr{B}) = \mathrm{He}(\mathscr{B}^T\mathscr{X}^T)$, the choice $\mathscr{B} = \mathscr{X}^T$ guarantees a feasible solution in the next evaluation, assuring a value of r equal or smaller than the previous one. This motivates the iterative procedure described in Algorithm 1. The parameter it_{max} sets a stop criterion and $\overline{Y}$ and $\overline{X}$ are the initial matrices as given in Theorem 1. The value of r is minimized at each iteration (denoted r_k) and if $r_k \leq 0$ then the closed-loop stability of the Lur'e system is certified and a dynamic stabilizing controller is provided. Otherwise, nothing can be concluded. Note that there is no guarantee that the algorithm converges to a negative or null value of r. The convergence depends on the initial choices of $\overline{Y}$ and $\overline{X}$.

Algorithm 1: Iterative procedure for control design.

Input parameters: it_{max}, $\overline{Y} = [\overline{Y}_1\ \overline{Y}_2\ \overline{Y}_3\ \overline{Y}_4]$ and $\overline{X} = [\overline{X}_1\ \overline{X}_2\ \overline{X}_3\ \overline{X}_4]$, $k \leftarrow 0$;
Make the changes of variables as in (16);
while $k < it_{max}$ **do**
  $k = k + 1$;
  minimize r_k **subject to** (11) and (12);
  if $r_k \leq 0$ **then**
    return r_k *and the matrices of* (4)*;*
  end
  $\overline{Y} = Y^T, \overline{X} = X^T$;
end

Note that when Algorithm 1 terminates with success (i.e., $r \leq 0$), the closed-loop dynamic matrices A_{cl} and A_m have all eigenvalues placed at the left-hand side of r, what is enough to assure stability, as demonstrated next.

Theorem 3. If the conditions of Theorem 1, with $\overline{A}_{cl}$ and $\overline{A}_m$ as in (16), are feasible for some $\hat{r} < 0$, then the conditions (14) and (8) remains feasible for any $r > \hat{r}$.

Proof. Consider that the conditions of Theorem 1 are verified, with $A_{cl} - \hat{r}I$ and $A_m - \hat{r}I$, for a given $\hat{r}$. Then the conditions (14) and (8), with A_{cl} and A_m given in (16), are also verified, as presented bellow, respectively,

$$\begin{bmatrix} A_m^T S + SA_m + \alpha S & SB_m \\ \star & -\alpha I \end{bmatrix} - \hat{r}\begin{bmatrix} 2S & 0 \\ \star & 0 \end{bmatrix} < 0,$$

$$\begin{bmatrix} A_I^T P + PA_I & PB_I - C_I^T \\ \star & -2D_I \end{bmatrix} - \hat{r}\begin{bmatrix} 2P & 0 \\ \star & 0 \end{bmatrix} < 0.$$

Clearly, the conditions are also verified for any $r > \hat{r}$.

The result of Theorem 3 assures that a controller and a multiplier computed through Algorithm 1 with a negative value of r also assures the stability of the original closed-loop system.

4. INCREASING THE ORDER OF THE CONTROLLER

An important feature when designing controllers of a given order is to guarantee that the increase of the order cannot provide worse results (in terms of some criterion). For instance, if for a given order n_c, Algorithm 1 assures closed-loop stability for a certain slope limit, then the order $n_c + 1$ should provide an equal or better slope bound. As presented, Algorithm 1 does not have this property.

As another contribution of the paper, a new initialization is proposed for Theorem 1 when searching for a controller of order n_c, whenever a solution of order $n_c - 1$ is available. For this, consider the parallel connection between two controllers (i.e., both receive as inputs y and ϕ from the system and provide the sum of the two control signals as output), where the first one is a known controller of order n_c that assures a certain slope Λ for the closed-loop system. The strategy relies on fixing the structure of the second controller (of order p) such that the synthesis condition of Theorem 1, feasible for n_c, remains feasible for $n_c + p$. For instance, consider $A_c = -I_p$, and all the other matrices equal to zero for the second controller, producing the following closed-loop matrices associated to the resulting controller of order $n_c + p$

$$\tilde{A}_{cl} = \begin{bmatrix} -I & 0 \\ 0 & A_{cl} \end{bmatrix}, \tilde{B}_{cl} = \begin{bmatrix} 0 \\ B_{cl} \end{bmatrix}, \tilde{C}_{cl}^T = \begin{bmatrix} 0 \\ C_{cl}^T \end{bmatrix}, \tilde{D}_{cl} = D_{cl},$$

$$\tilde{A}_I = \begin{bmatrix} -I & 0 \\ 0 & A_I \end{bmatrix}, \tilde{B}_I = \begin{bmatrix} 0 \\ B_I \end{bmatrix}, \tilde{C}_I^T = \begin{bmatrix} 0 \\ C_I^T \end{bmatrix}^T, \tilde{D}_I = D_I. \quad (17)$$

The next theorem shows that condition (15), feasible whenever a solution has been obtained from Theorem 1, can be made feasible with the matrices in (17) and particular choices of the decision variables of Theorem 1. This is the first step to construct a feasible solution to Theorem 1 for $n_c + p$.

Theorem 4. Let the matrices A_{cl}, B_{cl}, C_{cl}, D_{cl}, A_m, B_m and C_m (also A_I, B_I, C_I and D_I) and scalar α be solutions of Theorem 1 with order n_c. Then, the following LMI

$$\begin{bmatrix} 0 & \tilde{P} & -\tilde{C}_I^T \\ \star & 0 & 0 \\ \star & \star & -(\tilde{D}_I^T + \tilde{D}_I) \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} \tilde{X}_1^T \\ \tilde{X}_2^T \\ \tilde{X}_3^T \end{bmatrix} \begin{bmatrix} \tilde{A}_I^T \\ -I \\ \tilde{B}_I^T \end{bmatrix}^T \right) < 0, \quad (18)$$

is guaranteedly feasible with a controller of order $n_c + p$ with the closed-loop matrices $\tilde{A}_I$, $\tilde{B}_I$, $\tilde{C}_I$ and $\tilde{D}_I$ given in (17).

Proof. Condition (18) is nothing but condition (15), which is feasible whenever Theorem 1 has a solution. Consider the choices

$$\tilde{P} = \begin{bmatrix} \beta I & 0 \\ 0 & P \end{bmatrix}, \tilde{X}_1 = \begin{bmatrix} \chi_1 I & 0 \\ 0 & X_1 \end{bmatrix}, \tilde{X}_2 = \begin{bmatrix} \chi_2 I & 0 \\ 0 & X_2 \end{bmatrix}, \tilde{X}_3 = [0\ X_3].$$

where P, X_1, X_2 and X_3 are solutions of Theorem 1 for order n_c and β, χ_1 and χ_2 are scalars. Applying congruence transformations in (18) to interchange columns and rows, one has

$$diag(\Omega, \Xi) + diag(\Upsilon_Y, \Upsilon_X) < 0 \quad (19)$$

with

$$\Omega = \begin{bmatrix} 0 & P & -C_I^T \\ \star & 0 & 0 \\ \star & \star & -2D_I \end{bmatrix}, \Xi = \begin{bmatrix} 0 & \beta I \\ \star & 0 \end{bmatrix}, \Upsilon_X = \begin{bmatrix} -2\chi_1 I & -\chi_1 I - \chi_2 I \\ \star & -2\chi_2 I \end{bmatrix},$$

$$\Upsilon_X = \begin{bmatrix} A_I^T X_1^T + X_1 A_I & -X_1 + A_I^T X_2^T & X_1 B_I + A_I X_3^T \\ \star & -X_2 - X_2^T & X_2 B_I - X_3^T \\ \star & \star & X_3 B_I + B_I^T X_3^T \end{bmatrix},$$

As can be seen, condition (19) has a block diagonal structure, being feasible if the diagonal blocks are feasible, that is $\Omega + \Upsilon_Y < 0$ (feasible by hypothesis, from the existence of a controller of order n_c) and $\Xi + \Upsilon_X < 0$ that, considering $\chi_1 = \chi_2 = \beta/2$, leads to $diag(-\beta I, -\beta I) < 0$, verified $\forall \beta > 0$.

The results of Theorem 4 allow to compute a feasible solution for condition (15) constructed in terms of a known controller of order n_c increased of p units. As a consequence, following the proof of Theorem 1, the choice

$$\overline{X} = [\tilde{X}^T - I] \tag{20}$$

guarantees that condition (12) is feasible where $\tilde{X}$ is a solution for (15). Note that the increase of the order of the controller does not affect the LMIs (11). Thus, $\overline{Y}$ can be considered as Y^T, solution associated to the order n_c.

As a final remark in the search for dynamic controllers using Algorithm 1, it must be noted that the relaxation $A_I - rI$ tends to provide an undesirable effect in the matrices of the controller, that is, A_c diagonal and $B_{cy} = B_{c\phi} = C_c = 0$ for $n_c > 0$. Although this phenomenon is not an issue in principle, it has been observed in the numerical experiments that the results tend to be conservative. The same behavior can be noted in the matrices of the multiplier. To remedy this problem, the following representation for the augmented state vector is proposed

$$\begin{bmatrix} \hat{x} \\ \hat{x}_c \\ \hat{x}_m \end{bmatrix} = \begin{bmatrix} I_{n_x} & 0 & 0 \\ R_1 & -I_{n_c} & 0 \\ R_2 & 0 & -I_{n_M} \end{bmatrix} \begin{bmatrix} x \\ x_c \\ x_m \end{bmatrix} \tag{21}$$

$$R_1 = \begin{bmatrix} I_{n_c} & 0_{n_c \times (n_x - n_c)} \end{bmatrix}, \quad R_2 = \begin{bmatrix} I_{n_M} & 0_{n_M \times (n_x - n_M)} \end{bmatrix}.$$

5. EXAMPLES

Numerical examples are presented to illustrate the performance of the proposed conditions. For analysis purposes, the technique presented in Algorithm 1 (named Alg1) with $it_{max} = 50$ and $\nu = D_I (B_\phi^T B_\phi)^{-1}$ is compared with time and frequency domain-based approaches. Note that, although based on the conditions from Turner et al. (2009), Alg1 can possibly provide better analysis results than Turner et al. (2009) in the case of full order multipliers because Theorem 1 uses two distinct matrices, one to cope with the positive realness (P) and one to address the bound on the $\mathscr{L}_1$ norm (S). For each model, the aim is to find the maximum value of Λ such that the system is stable, and this search is performed by a bisection procedure with precision given by 10^{-3}, in systems 1 to 4, and for 5 and 6 by 10^{-4}. Regarding the order of the ZF multiplier, $n_M = n_x$ has been fixed in all tests, for simplicity, but note that the proposed approach can deal with any given order n_M. Whenever $D_u \neq 0$, a controllable canonical form realization is chosen for the ZF multiplier. The implementation of the methods (Turner et al., 2009, 2012; Carrasco et al., 2014) follows the steps: *i-* A bisection is used to find the maximum value of Λ, considering 100 equally spaced points for $\alpha \in [0, 100]$. *ii-* For the value of α that provided the best Λ in previous step (denoted by $\alpha^\star$), a new bisection is performed, this time considering 2000 equally spaced points in the interval $\alpha \in [0.75\alpha^\star, 1.25\alpha^\star]$, that is, in the neighborhood of $\alpha^\star$. The analysis conditions from Turner and Drummond (2020), based on externally positive ZF multipliers (i.e., do not require linear searches) are included in the comparisons. The LMIs are programmed with the parser Yalmip (Löfberg, 2004) and solved using Mosek (Andersen and Andersen, 2000) (version 9.2.47) in a computer equipped with a Core i7 processor, running Windows 11, Matlab 9.10 (R2021a) 64 bits.

Analysis: In this experiment the six systems investigated in Turner et al. (2009) are tested. The objective is to compute the maximum Λ such that the systems remain stable. For this purpose, Alg1 is compared with other methods from the literature: the Lur'e-Postnikov-type of function proposed by Park (2002) (Par02), Circle approach, Popov (Park, 1997), causal (Turner et al., 2009, 2012) (TKP09,TKP12) and anticausal (Carrasco et al., 2014) (C14) ZF multipliers, and also Turner and Drummond (2020). The results are presented in Table 1 where the sub indices c and ac associated to Alg1 indicate causal and anticausal multipliers (since the conditions of Theorem 1 only hold for causal multipliers, the anticausal ones are obtained through the transformation indicated in Carrasco et al. (2012)).

As can be seen, Alg1 provided slope values equal to (or lesser than) the ones computed through the other methods. Note that the algorithm could be initialized taking into account, for instance, a ZF multiplier provided by any other method, and then iterate in the search for larger slope values. Regarding only the ZF approaches (Turner et al., 2009, 2012; Carrasco et al., 2014), note that Alg1, although iterative, does not require a line search on α, which can be advantageous in terms of computational time. Overall, the results obtained by Alg1 can be considered good since the main interest is not to perform stability analysis, but control synthesis, as investigated in the next experiment.

Control Design: In this experiment, Alg1 is used to compute dynamic output feedback controllers for the six systems of Turner et al. (2009) considering $B_u^T = [1 \ \dots \ 1]$, $D_u = 1$ and canonical realizations for the transfer functions associated to the ZF multipliers.

The strategy is to start with the initial condition given in Theorem 2 with $\nu = D_I (B_\phi^T B_\phi)^{-1}$, for $n_c = 0$ (static output feedback control), and and then to use the result of Theorem 4 with $p = 1$ and $\overline{X}$ given in (20), to design controllers with $n_c = 1, 2$. The results (in terms of maximum values obtained for Λ) are shown in Table 1. As can be seen, the controllers provide larger bounds for Λ (when compared with the ones obtained in the stability analysis) and, as n_c increases, the results cannot be worse (as assured by the conditions of Theorem 4 and the initial condition given in (20)). The similarity transformation in (21) plays an important role when searching for dynamic controllers that provide better results in terms of Λ.

6. CONCLUSIONS

LMI-based conditions for fixed order dynamic output-feedback control of continuous-time Lur'e systems with slope bounded nonlinearities certified through ZF multipliers were presented. As illustrated by the numerical examples, the control law can take into account both the measured output of the linear part and the output of the nonlinearity, providing larger values for the slope bound of the allowable nonlinearities. The proposed method eliminated the scalar search usually needed to assure

Table 1. Maximum Λ obtained by Alg1 using causal (*cs*) or anticausal (*ac*) multipliers, circle, Popov, Par02, TKP09, TKP12, C14 and TD20 in stability analysis and in control design (by Alg1 with $n_c = (0,1,2)$). The symbol '–' is used for no feasible solution and 100+ means that values of Λ larger than 100 have been obtained. The mean time (*s*) for each analysis method is given in the last row.

	Analysis									Control
System	Circle	Popov	TKP09	TKP12	Par02	C14	TD20	Alg1$_{cs}$	Alg1$_{acs}$	Alg1
1	1.243	1.763	2.428	3.274	4.589	4.021	3.3050	2.628	4.589	(100+, 100+, 100+)
2	0.764	1.082	1.089	1.089	1.089	1.089	1.089	1.089	1.075	(1.111, 1.839, 2.765)
3	0.326	0.384	0.706	0.760	0.788	1.025	1.232	0.402	0.322	(1.612, 3.939, 5.772)
4	0.308	0.364	0.853	1.079	0.708	0.674	0.976	0.285	0.283	(1.022, 2.643, 5.349)
5	–	0.001	0.001	0.003	0.001	0.0008	0.001	–	–	(–, 9.537, 12.146)
6	–	0.001	0.0008	0.003	0.001	0.001	–	–	–	(–, 6.149, 7.194)
	0.17	0.19	77.91	183.93	0.12	69.61	0.12	22.27	22.02	

$\mathscr{L}_1$ norm bounds, dealing with conditions that are affine on the variables of interest (matrices of the dynamic controller and of the ZF multipliers). Future work on this research topic would be the inclusion of Popov multipliers and performance indices (such as the $\mathscr{L}_2$ gain) in the conditions for control design.

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