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RIBBONS IN GARSIDE MONOIDS

FRANÇOIS DIGNE AND JEAN MICHEL

ABSTRACT. We expound the properties of ribbons in a setting which is general enough to encompass spherical Artin monoids and dual braid monoids of well-generated complex reflection groups. We generalize to our setting results on parabolic subgroups of spherical Artin group of Godelle [4], Cumplido [2] and others.

This short note expounds the properties of ribbons in a setting (Assumptions 3 and 24) which is general enough to encompass spherical Artin monoids and dual braid monoids of well-generated complex reflection groups. We show in particular that this approach can recover, and generalize to our setting, results on parabolic subgroups of spherical Artin group of Godelle [4], Cumplido [2] and others. We have strived for the text to be self-contained apart from basic results on Garside theory for which we refer to [5]. The assumptions under which we study ribbons are slightly stronger than those in [5, VIII §1.4] but allow us to develop them much further.

Let M be Garside monoid; that is, M is Noetherian, left- and right-cancellative, any pair of elements have left- and right-gcd's and lcm's, and there exists a Garside element Δ , an element whose left- and right-divisors coincide and generate M (see [5, I 2.1]).

We recall that a factor is any left-divisor of a right-divisor (or equivalently a right-divisor of a left-divisor) and that Noetherian means there is no infinite sequence where each term is a proper factor of the previous one.

Definition 1. *A standard parabolic submonoid P of M is a submonoid closed by factors such that any element of M has a maximal left-divisor (the left P -head) in P and a maximal right-divisor (the right P -head) in P .*

This is the particular case, for a monoid, of Definition [5, 1.30 VII] which is in the context of a Garside category.

We denote by $H_P(g)$ the left P -head of $g \in M$ and we define $T_P(g)$ (the "P-tail") by $g = H_P(g)T_P(g)$. We say that g is P -reduced if $H_P(g) = 1$ or equivalently $g = T_P(g)$.

Any intersection of standard parabolic submonoids is a standard parabolic submonoid [5, VII 1.35]. It is clear that any standard parabolic submonoid of a standard parabolic submonoid is a standard parabolic submonoid.

Since M is Noetherian, it is generated by the set S of its atoms (elements which have no proper nontrivial divisors) and a standard parabolic submonoid being closed by factors is also generated by its atoms.

Lemma 2. *A standard parabolic submonoid P has a Garside element Δ_P given by $\Delta_P = H_P(\Delta)$.*

Proof. First P is generated by the left-divisors of Δ_P since it is generated by its atoms which are left-divisors of Δ hence of Δ_P . Now the right- and left- divisors of Δ are the same, thus $\Delta \succcurlyeq \Delta_P$. Hence $H_P^{\succ}(\Delta) \succcurlyeq \Delta_P$, if $H_P^{\succ}(g)$ denotes the right P -head of g . Exchanging right and left, that is H_P and H_P^{\prec} , we get symmetrically $H_P^{\prec}(\Delta) \preccurlyeq \Delta_P$. Hence Δ_P is a factor of itself. It cannot be a strict factor by Noetherianity, thus $\Delta_P = H_P^{\prec}(\Delta)$. The left-divisors of Δ_P are left, thus right-divisors of Δ , and since they are in P they are right-divisors of $H_P^{\prec}(\Delta) = \Delta_P$. Conversely right-divisors of Δ_P are right, thus left-divisors of Δ and being in P are left-divisors of Δ_P . \square

Assumption 3. *We assume that for any subset $S_1 \subset S$, the right-lcm of S_1 is the Garside element given by Lemma 2 of the smallest standard parabolic submonoid containing S_1 .*

In the following we assume that Assumption 3 holds. It clearly holds in a spherical Artin monoid.

Proposition 4. *The dual braid monoid M of a well generated complex reflection group W satisfies Assumption 3.*

Proof. From the fact that any decomposition of Δ is obtained from one of them by the Hurwitz action (see [1, Proposition 7.6 and Proposition 8.5]), it follows that the same holds for any simple. It follows from that that any factor of a simple w is a left-divisor of w . It follows that any simple w is a common right multiple of the atoms which left-divide it. If the right-lcm of these atoms was a strict divisor of w , there would be an atom s whose square is a factor of w , thus of Δ . This does not exist, see [6, Property (iii) of Proposition 2.1 (M2)].

Thus any simple w is the left-lcm of the set S_w of atoms which left divide it. The submonoid P_w generated by these atoms is standard parabolic since it is clearly stable by factor, and every element has a P -head : to see this it is sufficient to check it for a simple v by [5, VII, 1.25], in which case the left-gcd of v and w is the head.

Finally, for any set S_1 of atoms, the right-lcm w of this set defines thus a parabolic P_w which is clearly the minimal parabolic subgroup containing S_1 . \square

If $b, g \in M$ and $g \preccurlyeq bg$ we say that the conjugate b^g , equal to $g^{-1}bg$, is defined. For a set $P \subset M$ we say that P^g is defined if p^g is defined for any $p \in P$.

Definition 5. *For P a standard parabolic submonoid we call P -ribbon a P -reduced element $g \in M$ such that P^g is defined. We denote such a ribbon by $P \xrightarrow{g}$.*

Let P be a standard parabolic submonoid with Garside element Δ_P and s be an atom of M which is not in P ; we define $P^{(s)}$ as the smallest standard parabolic submonoid containing P and s . Since $\Delta_{P^{(s)}}$ is left-divisible by Δ_P the following makes sense:

Definition 6. *For P a standard parabolic submonoid and $s \in S, s \notin P$, we define $v_{s,P}$ by $\Delta_{P^{(s)}} = \Delta_P v_{s,P}$.*

Recall that conjugation by Δ is an automorphism of M (see for example [5, V 2.17]).

Lemma 7. *For P a standard parabolic submonoid and for $s \in S, s \notin P$ the element $v_{s,P}$ is a P -ribbon.*

Proof. We have $\Delta_P = H_P(\Delta_{P^{(s)}})$ since any divisor in P of $\Delta_{P^{(s)}}$ divides Δ hence divides Δ_P . Hence $v_{s,P}$ is P -reduced. And $P^{v_{s,P}}$ is defined since the conjugation by $v_{s,P}$ is the composition of the inverse of the conjugation by Δ_P , which is an automorphism of P , and the conjugation by $\Delta_{P^{(s)}}$, which is an automorphism of $P^{(s)}$. \square

Lemma 8. *If g is a P -ribbon and $s \in S$ left-divides g then $v_{s,P} \preceq g$.*

Proof. We have $\Delta_P g = g \Delta_P^g$. Thus $s \preceq \Delta_P g$, thus $\Delta_{P^{(s)}} \preceq \Delta_P g$ by Assumption 3, thus $v_{s,P} = \Delta_P^{-1} \Delta_{P^{(s)}} \preceq g$. \square

Note that the converse of this lemma, that is $s \preceq v_{s,P}$, holds in an Artin monoid but not necessarily in a dual braid monoid, see Example 13.

Proposition 9. *Let P be a standard parabolic submonoid; then*

- *If $P \xrightarrow{g}$ is a ribbon then P^g is a standard parabolic submonoid, and $x \mapsto x^g$ is a monoid isomorphism $P \xrightarrow{\sim} P^g$.*
- *A P -ribbon is a product of elements $v_{s,P'}$ where P' are standard parabolic submonoids conjugate to P .*
- *If $P \xrightarrow{g}$ is a ribbon then the conjugate by g of the atoms of P are the atoms of P^g .*
- *If $P \xrightarrow{g}$ is a ribbon then $\Delta_P^g = \Delta_{P^g}$.*

Proof. Let s be an atom dividing g , then $v_{s,P}$ conjugates P on a standard parabolic submonoid since the conjugation by $\Delta_{P^{(s)}}$ is an automorphism of $P^{(s)}$, hence conjugates P on a standard parabolic submonoid of $P^{(s)}$ thus of M , and the conjugation by Δ_P conjugates P on itself. If $g_1 = v_{s,P}^{-1} g$, it follows that $P^{v_{s,P}} \xrightarrow{g_1}$ is a ribbon (g_1 is $P^{v_{s,P}}$ -reduced since for an atom s if $s^{v_{s,P}} \preceq g_1$ then $s \preceq g$). By Noetherian induction we get the first two items.

The third item is an immediate consequence of the first one, and the fourth also by Assumption 3. \square

In view of the third item above, the definition of a P -ribbon could instead of asking that P^g is defined just ask that the conjugate of any atom of P by g is defined.

Lemma 10. *If P is a standard parabolic submonoid, if s is an atom of P and $P \xrightarrow{g}$ is a ribbon, then the right-lcm of s and g is sg .*

Proof. Since g is a P -ribbon it conjugates s into some atom s' . The right-lcm of s and g left-divides $sg = gs'$ and is a strict right-multiple of g (since g is P -reduced), hence is equal to gs' . \square

Lemma 11. *Let $P \xrightarrow{g}$ be a ribbon; for $x \in P$, $y \in M$, it is equivalent that $g \preceq xy$ or that $g \preceq y$.*

Proof. If $y = gy'$, then $xy = gx^g y'$ so that $g \preceq xy$. To prove the converse, by Noetherian induction on x , it is sufficient to prove that if an atom $s \in P$ left-divides gh for some h , then $s^g \preceq h$. By Lemma 10, the right-lcm of s and g is $sg = gs^g$. Thus $s \preceq gh$ is equivalent to $gs^g \preceq gh$ which is finally equivalent to $s^g \preceq h$. \square

Lemma 12. *Let $P \xrightarrow{g}$ be a ribbon and let $h \in M$. Then $T_P(gh) = gT_{P^g}(h)$ and $H_P(gh)^g = H_{P^g}(h)$. In particular if g is a P -ribbon and h is a P^g -ribbon, then gh is a P -ribbon.*

Proof. Let s be an atom of P and set $s' := s^g$. Both formulae clearly follow if we show that it is equivalent that $s \preccurlyeq gh$ or that $s' \preccurlyeq h$. This is true by the proof of Lemma 11. \square

It follows from Lemma 12 and the first item of Proposition 9 that the ribbons form a category. The second item of Proposition 9 shows that the atoms of this category are the $P \xrightarrow{v_{s,P}}$ which are not strict multiple of some other one. This is always the case in a spherical Artin monoid but not in a dual braid monoid as the following example shows.

Example 13. We consider the Coxeter Group of type A_4 identified with the symmetric group on 5 letters. We number the transpositions s_1, \dots, s_{10} in the order $(1, 2), (2, 3), (3, 4), (4, 5), (1, 3), (2, 4), (3, 5), (1, 4), (2, 5), (1, 5)$. We choose as Coxeter element the product $s_1 \cdots s_4$. The atoms of the corresponding dual monoid are in one to one correspondence with the transpositions. We denote the atoms again by s_1, \dots, s_{10} . Let P be the standard parabolic subgroup generated by the atom s_5 ; then $\text{right-lcm}(s_5, s_6) = s_5 s_1 s_3$, thus $v(s_6, P) = s_1 s_3$ and $\text{right-lcm}(s_5, s_1) = s_5 s_1$ so that $v(s_1, P) = s_1$.

Note that s_6 does not divide $v(s_6, P) = s_1 s_3$.

Lemma 14. *If $b, g, g' \in M$ and b^g and $b^{g'}$ are defined, then so are $b^{\text{left-gcd}(g, g')}$ and $b^{\text{right-lcm}(g, g')}$.*

Proof. The conditions for b^g and $b^{g'}$ to be defined are $g \preccurlyeq bg$ and $g' \preccurlyeq bg'$. It follows easily that $\text{left-gcd}(g, g') \preccurlyeq b \text{left-gcd}(g, g')$ and $\text{right-lcm}(g, g') \preccurlyeq b \text{right-lcm}(g, g')$. \square

For $g \in M$ we denote by $H(g)$ the first term of its Garside normal form, equal to $\text{left-gcd}(g, \Delta)$ and define $T(g)$ by $g = H(g)T(g)$.

Proposition 15. *If $P \xrightarrow{g}$ is a ribbon, so are all the terms of its Garside normal form.*

Proof. It is sufficient to prove that $P \xrightarrow{H(g)}$ and $P^{H(g)} \xrightarrow{T(g)}$ are ribbons.

For the first fact, we have $H_P(H(g)) = 1$ since $H_P(g) = 1$, and $P^{H(g)}$ is defined by Lemma 14, since $H(g) = \text{left-gcd}(g, \Delta)$.

For the second fact, since $(P^{H(g)})^{T(g)} = P^g$ is defined, it is sufficient to show that $T(g)$ is $P^{H(g)}$ -reduced. By Lemma 12 with $H(g), T(g)$ for g, h , we get that $H_{P^{H(g)}}(T(g)) = H_P(H(g)T(g))^{H(g)} = 1$. \square

Proposition 16. *If $P \xrightarrow{g}$ and $P \xrightarrow{g'}$ are in the ribbon category, then also the map $P \xrightarrow{\text{left-gcd}(g, g')}$*

Proof. It is clear that $H_P(\text{left-gcd}(g, g')) = 1$, and $P^{\text{left-gcd}(g, g')}$ is defined by lemma 14. \square

Proposition 17. *If $P \xrightarrow{g}$ and $P \xrightarrow{g'}$ are in the ribbon category, then also the map $P \xrightarrow{\text{right-lcm}(g, g')}$*

Proof. It follows from Lemma 14 that $P^{\text{right-lcm}(g,g')}$ is defined. It remains to show that $k = \text{right-lcm}(g,g')$ is P -reduced. Note that Lemma 12 implies that if $P \xrightarrow{g}$ is in the ribbon category and $g \preceq k$ then $g \preceq T_P(k)$. It follows that g and g' left-divide $T_P(k)$, thus $k = \text{right-lcm}(g,g')$ left-divides $T_P(k)$ which proves that k is P -reduced. \square

Proposition 18. *For P a standard parabolic submonoid, let $g \in M$ be P -reduced. Then there is a unique maximal left-divisor of g which is a P -ribbon. If we denote by $R_P(g)$ this left-divisor, then $R_P(g)^{-1}g$ is $P^{R_P(g)}$ -reduced and there is equivalence between:*

- (i) $R_P(g) = 1$.
- (ii) Any atom which left-divides $\Delta_P g$ is in P .

Proof. The existence of $R_P(g)$ is a consequence of the fact that the ribbon category is stable by right-lcms. The fact that $R_P(g)^{-1}g$ is $P^{R_P(g)}$ -reduced is an immediate consequence of Lemma 12.

We finally prove the equivalence of (i) and (ii) by observing that $R_P(g) \neq 1$ is equivalent to the existence of an atom $s \notin P$ such that $v_{s,P} \preceq g$, which is in turn equivalent to $\Delta_{P(s)} \preceq \Delta_P g$ which is equivalent to s left-dividing $\Delta_P g$. \square

Proposition 19. *Let P be standard parabolic submonoid of M , and let $b \in P$ and $g \in M$ be such that g is P -reduced and b^g is defined. Assume further that $\Delta_P \preceq b^i$ for some integer i . Then g is a P -ribbon.*

Proof. The fact that b^g is defined implies that $(b^i)^g$ is defined, thus replacing b by b^i we may assume that $\Delta_P \preceq b$; let us thus write $b = \Delta_P v$.

The assumption that b^g is defined can be written $g \preceq \Delta_P v g$. If s is an atom left-dividing g it follows, using Assumption 3, that $\text{right-lcm}(\Delta_P, s) = \Delta_{P(s)} \preceq \Delta_P v g$, whence $v_{s,P} \preceq v g$. By Lemma 11 it follows that $v_{s,P} \preceq T_P(v g) = g$.

We conclude by Noetherian induction on g , since replacing simultaneously g by $v_{s,P}^{-1}g$, P by $P^{v_{s,P}}$ and b by $b^{v_{s,P}}$ all the assumptions remain. \square

We conjecture that the assumption in Proposition 19 that $\Delta_P \preceq b^i$ for some i can be replaced by the assumption that P is the smallest standard parabolic submonoid containing b . For spherical Artin monoids, this is a result of the beautiful paper [3]. If we could prove that conjecture, we could extend all results of [3] to our setting.

We now give a version of Proposition 19 where g is not P -reduced.

Proposition 20. *Let P be a standard parabolic submonoid of M and let $b \in P$, $g \in M$ be such that $b^g \in M$. Assume that for some $i > 0$ we have $\Delta_P \preceq b^i$. Then $T_P(g)$ is a P -ribbon.*

Proof. Let us prove first that $b^{H_P(g)} \in P$, that is $H_P(g) \preceq b H_P(g)$. Indeed, from $g \preceq b g$ we get $H_P(g) \preceq H_P(b g) = b H_P(g)$. Since $T_P(g)$ is P -reduced we can now apply Proposition 19 with b replaced by $b^{H_P(g)}$ and g replaced by $T_P(g)$ and we get the result. \square

Lemma 21. *Let P and Q be standard parabolic submonoids of M and $g \in M$, $k > 0$ be such that $(\Delta_P^k)^g \in Q$; then $T_P(g)$ is a P -ribbon and $P^{T_P(g)} \subset Q$. If in addition $(\Delta_P^k)^g = \Delta_Q^k$ then $P^{T_P(g)} = Q$.*

Proof. Proposition 20 applied with $b = \Delta_P^k$ shows that $T_P(g)$ is a P -ribbon. Replacing Δ_P^k by some power, we can assume that $k > l_\Delta(H_P(g))$, where l_Δ is the Garside length, that is the number of factors in a Garside normal form. We then have $\Delta_P \preccurlyeq (\Delta_P^k)^{H_P(g)} \in P$. Now any atom of P left-divides $(\Delta_P^k)^{H_P(g)}$ and is thus conjugate to an element of Q by $T_P(g)$, hence $P^{T_P(g)} \subset Q$.

If $(\Delta_P^k)^g = \Delta_Q^k$, let $Q' = P^{T_P(g)}$. We have $\Delta_Q^k = (\Delta_P^k)^g = ((\Delta_P^k)^{H_P(g)})^{T_P(g)} \subset Q'$. Hence the divisors of Δ_Q in particular all the atoms of Q are in Q' , so that $Q = Q'$. \square

GARSDIE GROUPS

We now denote by G the group of fractions of the Garside monoid M (which exists since M is an Ore monoid, see [5, 2.32, V and 3.11, II]).

We call standard parabolic subgroup G_P the group of fractions of a standard parabolic submonoid P .

Definition 22. *We say that $p^{-1}q$ is a left reduced fraction for $b \in G$ if $b = p^{-1}q$ with $p, q \in M$ and the left-gcd of p and q is trivial.*

Symmetrically there are right reduced fractions pq^{-1} . By “reduced fraction” we will mean left reduced fraction.

The reduced fraction for an element is unique; more precisely if $p^{-1}q$ is reduced and $p^{-1}q = p'^{-1}q'$ there exists d such that $p' = dp$ and $q' = dq$ (see [5, 3.11, II]).

By [5, II, 3.18] G_P is a subgroup of G and $G_P \cap M = P$. If an element of G_P has $p^{-1}q$ as its reduced fraction in G , then $p, q \in P$ since there is a reduced fraction $a^{-1}b$ in P and $a \succcurlyeq p$ and $b \succcurlyeq q$. It follows that if P and Q are standard parabolic submonoids and $P \subsetneq Q$ then $G_P \subsetneq G_Q$.

Lemma 23. *Let $b \in G_P$ and $g \in M$ be such that $b^g \in M$. Then $b^{H_P(g)} \in P$.*

Proof. Let $b' = b^g$, and write $g = H_P(g)T_P(g)$. Let $b^{H_P(g)} = pq^{-1}$ where the right-hand side is a right reduced fraction in P , that is $p, q \in P$. From the equality of the two fractions $T_P(g)b'T_P(g)^{-1} = pq^{-1}$ we deduce $q \preccurlyeq T_P(g)$ which implies $q = 1$ since $T_P(g)$ is P -reduced. \square

Assumption 24. *We assume that the automorphism induced by the conjugation by Δ is of finite order, or equivalently that some power of Δ is central.*

From now on we assume that Assumption 24 holds. It holds in spherical Artin monoids and in the dual braid monoids of well generated complex reflection groups.

Proposition 25. *For two standard parabolic submonoids P and Q and $g \in G$ the three following properties are equivalent*

- (i) $G_P^g \subset G_Q$
- (ii) *There exists $k > 0$ such that $(\Delta_P^k)^g \in G_Q$.*
- (iii) *There exist $p \in G_P$ and a central power Δ^i of Δ such that $P \xrightarrow{pg\Delta^i}$ is a ribbon such that $P^{pg} \subset Q$.*

Proof. Clearly (iii) implies (i) and (i) implies (ii). We prove that (ii) implies (iii). Let $u = (\Delta_P^k)^g \in G_Q$. Multiplying g^{-1} by some central power Δ^i of Δ we get $h \in M$ such that $u^h = \Delta_P^k$. Then by the Lemma 23 we have $u^{H_Q(h)} \in Q$. We thus have $(\Delta_P^k)^{T_Q(h)^{-1}} \in Q$. Multiplying $T_Q(h)^{-1}$ by a central power Δ^j of Δ we get $m \in M$

such that $(\Delta_P^k)^m \in Q$. By Lemma 21 $P \xrightarrow{T_P(m)}$ is a ribbon. Now $h \in g^{-1}\Delta^i$, thus $T_Q(h) \in Qg^{-1}\Delta^i$, whence $m \in gG_Q\Delta^{j-i}$ and $T_P(m)$ is in $PgG_Q\Delta^{j-i} = G_Pg\Delta^{j-i}$, the last equality since g conjugates G_Q to G_P . \square

Proposition 26. *Assume Assumption 24. For two standard parabolic submonoids P, Q we have equivalence between:*

- (i) *There is $g \in G$ such that $G_P^g \cap G_Q$ is of finite index in G_P^g and in G_Q .*
- (ii) *There is $g \in G$ such that $G_P^g = G_Q$.*
- (iii) *There exists an integer $k > 0$ and $g \in G$ such that $(\Delta_P^k)^g = \Delta_Q^k$.*
- (iv) *There exists a ribbon $P \xrightarrow{g}$ such that $P^g = Q$.*

Proof. It is clear that g satisfying (iv) satisfies (iii). If g satisfies (iii), we may assume that $g \in M$ up to multiplying g by some central power of Δ . Then Lemma 21 shows that $T_P(g)$ satisfies (iv). Thus (iv) and (iii) are equivalent.

It is clear that g satisfying (iv) satisfies (ii), and that g satisfying (ii) satisfies (i). It is thus enough to show that (i) implies (iv).

So we assume (i). Since $G_P^g \cap G_Q$ is of finite index in G_P , there is some positive power k such that $(\Delta_P^k)^g \in G_Q$. It follows by Proposition 25 that there exists $g' \in M$ of the form $g\Delta^i q$ with Δ^i a central power and $q \in G_Q$ such that $P \xrightarrow{g'}$ is a ribbon (and $P^{g'} \subset Q$).

The element g' satisfies clearly the same finite index assumptions as g . Thus there exists k such that $G_P^{g'} \ni \Delta_Q^k$. Let $u \in G_P$ be such that $u^{g'} = \Delta_Q^k$. By Lemma 23 we have $u^{H_P(g')} \in P$, but $H_P(g') = 1$ thus $P^{g'} \ni \Delta_Q^k$. This implies that $P^{g'} = Q$ since $P^{g'}$ is a standard parabolic submonoid, thus contains all divisors of its element Δ_Q^k , that is all atoms of Q . \square

For a similar proposition for spherical Artin groups see [4, Théorème 0.1].

Corollary 27. *It is equivalent for standard parabolic submonoids P and Q that $g \in G$ conjugates G_P onto G_Q , or that it conjugates some central power Δ_P^k to Δ_Q^k . If $g \in M$ is P -reduced it is equivalent that it conjugates Δ_P to Δ_Q or that it conjugates P to Q .*

In particular if G_P is conjugate to G_Q and Δ_P^k is the smallest power central in P then Δ_Q^k is the smallest power central in Q .

Proof. We remark that in the proof (iii) \Rightarrow (iv) \Rightarrow (ii) of Proposition 26 the element obtained differs from g by a central power of Δ and an element of P , which gives that if g conjugates Δ_P^k to Δ_Q^k it conjugates P to Q .

In the proof of (ii) \Rightarrow (iii) of Proposition 26 the element obtained is in $G_Pg\Delta^iG_Q$ for some central power Δ^i . If Δ_P^k is central this element has the same effect as g on it. This proves the reverse implication.

The second sentence results from Lemma 21. \square

We call parabolic subgroups the conjugates of the standard parabolic subgroups. Note that the notion of parabolic subgroup of G depends on the Garside monoid M ; in particular in a spherical Artin monoid the parabolic subgroups for the ordinary Garside structure are not the same as those for the dual Garside structure.

Definition 28. *If $K = G_P^g$ is a parabolic subgroup, we denote by z_K the element $(\Delta_P^k)^g$ where Δ_P^k is the smallest central power of Δ_P .*

The notation z_K makes sense thanks to the following proposition:

Proposition 29. *Let K, K' be parabolic subgroups of G .*

- z_K depends only on K .
- It is equivalent that $g \in G$ conjugates K to K' or z_K to $z_{K'}$.

Proof. For the first item, let $K = G_P^g$ and $K = G_{P'}^{g'}$ two ways of conjugating K to a standard parabolic subgroup. This defines two candidates for z_K which are $(\Delta_P^k)^g$ and $(\Delta_{P'}^k)^{g'}$. But these two elements are equal by Corollary 27 since $g'g^{-1}$ conjugates P' to P , thus $\Delta_{P'}^k$ to Δ_P^k .

For the second item, let $K = G_P^g$ and $K' = G_{P'}^{g'}$ be ways of conjugating K, K' to standard parabolic subgroups. Then $z_K = (\Delta_P^k)^g$ and $z_{K'} = (\Delta_{P'}^k)^{g'}$. Now $K^x = K'$ for some $x \in G$ is equivalent to $G_P^{g^xg'^{-1}} = G_{P'}$ and we conclude again by Corollary 27. \square

Proposition 30. *Let $K = G_P^b$ be a parabolic subgroup of G , where P is a standard parabolic submonoid and $b \in M$. Define b' by $H_P(b)R_P(T_P(b))b' = b$ and Q by $Q = P^{R_P(T_P(b))}$. Then $b'^{-1}\Delta_Q^k b'$ is the reduced fraction of z_K , where Δ_Q^k is the smallest central power of Δ_Q in Q .*

Proof. We first remark that by definition we have $(\Delta_P^k)^b = z_K$. We may clearly replace b in this equality by $T_P(b)$. Let $c = R_P(T_P(b))$; we have $(\Delta_P^k)^c = \Delta_Q^k$. We thus get $z_K = b'^{-1}\Delta_Q^k b'$. We claim this is a reduced fraction. Indeed by construction $R_Q(b') = 1$ thus by Proposition 18(ii) any atom left-dividing $\Delta_Q b'$ is in Q , thus the same is true for $\Delta_Q^k b'$ by induction, using that for $k > 1$ one has $H(\Delta_Q^k b') = H(\Delta_Q H(\Delta_Q^{k-1} b'))$. Since b' is Q -reduced, the fraction is reduced. \square

Note that b' above is minimal such that $b'K$ is standard, that is any $u \in M$ such that uK is standard is a left multiple of b' , and G_Q is a “canonical” standard parabolic subgroup conjugate to K . For spherical Artin groups the proposition is [2, Theorem 3] and the element b' is called a minimal standardizer.

An immediate consequence of Proposition 30 is that it is equivalent that $z_K \in M$ or that K is standard.

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