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Analytical method for the evaluation of singular integrals arising from boundary element method in electromagnetism

Soumaya Oueslati ^{*}, Imen Balloumi [†], Christian Daveau [‡] and Abdessatar Khelifi [§]

Abstract

In the framework solving electromagnetic scattering problems with a Leontovich impedance boundary condition (LIBC), the numerical resolution requires the evaluation of singular integrals appearing in the discretization of the variational formulation. Our main interest is to precisely evaluate these integrals. Thus, we propose an analytic method to approximate it. The performance of this method will be evaluated by calculating the radar cross section (RCS). Then, we compare RCS to the Mie series solution for the unit sphere in various configurations.

Key words: Maxwell equations, boundary integral equations, singular integral, homogeneous functions.
AMS Subject Classification:

1 Introduction

Integral equation methods are an alternative for solving linear partial differential equations. It consists in transforming the partial differential equation posed in a domain, into an integral equation posed on the boundary. It is in particular a method adapted for diffraction problems that are posed in an unbounded domain (see [6] [3]) and for which the consideration of the radiation condition is essential in order to insure a unique solution. We are interested in solving time-harmonic scattering problem by the integral method and we will take the constant impedance condition, known as standard or Leontovich impedance boundary condition (LIBC) [14]. This approximation does not depend on incident angle at all. It has been extensively used to simplify the formulation involved in the solution of complicated electromagnetic scattering problems. It was recognized that this type of boundary condition can be advantageously used to get a more tractable problem in numerous complex situations of electromagnetic scattering computations (see for example [1] [7] [8]). Once the boundary problem has been converted into a boundary integral equation, it is necessary to discretize the boundary of the domain using plan polygons. Here we deal with plan triangles and Rao-Wilton-Glisson basis functions.

In this paper, we are particularly interested in the singular integrals resulting from the discretization of the variational problem of Maxwell 3-D equation. This singular character of the integrand makes the precise evaluation of these integrals difficult. It has been the subject of a large number of publications. M. Lenoire and N. Salles have developed in [11] [12] a new approach to evaluate singular integrals. It allows to evaluate analytically and with great precision, singular integrals in the variational case. We adopt the same approach to evaluate a singular integral coming from a double layer operator.

To better understand this singular integrand problem, in the variational case, it should be noted that the domain of integration for these integrals with "problem", is the product of two elements of the mesh, precisely the product of two triangles. When the two triangles are distant, it is possible to evaluate the integral with standard numerical integration methods. But, when the intersection of the two triangles is not empty (adjacent triangles and triangles with a common vertex), it requires separate treatment and this is a field of research still very active.

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The approach developed in this work, makes it possible to reduce a 4-D integral to a linear combination of mono-dimensional integrals whose integrand is regular. It is then possible to numerically evaluate these 1-D integrals, but also explicitly. This method relies on several formulae that make it possible to reduce the dimension of the integration domain from n to $n - 1$. So, repeating the same process, three times using these formulas, we obtain regular 1-D integrals. These reduction formulae are based on the homogeneity of the integrand; therefore, there are two conditions to check. The integrand must be homogeneous and we must use flat polygons to discretize the boundary of the domain in order to maintain the homogeneity of the integrand during the successive stages of the reduction process.

2 Scattering problem

We consider the scattering problem of electromagnetic waves (\mathbf{E}, \mathbf{H}) by a perfect conducting body with a complex coating. We refer by Ω_- to the region of space embodying the scatterer, with a Lipschitz-continuous boundary denoted by Γ and we refer by \mathbf{n} is the exterior unit normal to Γ , pointing to the exterior. Waves propagation medium is characterized by two physical quantities ϵ (electrical permittivity) and μ (magnetic permeability). The propagation of this wave is described in $\Omega_+ = \mathbb{R}^3 \setminus \overline{\Omega_-}$ by the following Maxwell equations:

$$(1) \quad \begin{cases} \text{rot} \mathbf{E} + i\omega\mu\mathbf{H} = 0, \\ \text{rot} \mathbf{H} - i\omega\epsilon\mathbf{E} = 0. \end{cases}$$

In order to ensure a unique solution of boundary value problem, it is necessary to apply the following boundary condition

$$(2) \quad \mathbf{E}_{tg} - Z(\mathbf{n} \times \mathbf{H}) = 0 \quad \text{on } \Gamma.$$

Where Z is the impedance operator that depends on incident angle, subscript tg denotes tangent component on the surface defined as:

$$\mathbf{E}_{tg} = \mathbf{n} \times (\mathbf{E} \times \mathbf{n}).$$

Generally, we take the constant impedance operator, known as standard or Leontovich impedance boundary conditions $Z = \text{const}$ [15, 16, 17].

We should notice that asymptotic behavior of the fields (\mathbf{E}, \mathbf{H}) is depicted by the Silver-Müller radiation condition:

$$(3) \quad \lim_{r \rightarrow \infty} r(\mathbf{E} \times \mathbf{n}_r + \mathbf{H}) = 0.$$

where $r = |\mathbf{x}|$ and $\mathbf{n}_r = \frac{\mathbf{x}}{|\mathbf{x}|}$, $\mathbf{x} \in \mathbb{R}^3$. We introduce current densities \mathbf{J} and \mathbf{M} on the boundary Γ as follows

$$\mathbf{M} = \mathbf{E} \times \mathbf{n}, \quad \mathbf{J} = \mathbf{n} \times \mathbf{H}.$$

We determine total electromagnetic fields (\mathbf{E}, \mathbf{H}) in Ω_+ as:

$$(4) \quad \begin{cases} \mathbf{E} = \mathbf{E}^{inc} + \mathbf{E}^{sc} \text{ in } \Omega_+, \\ \mathbf{H} = \mathbf{H}^{inc} + \mathbf{H}^{sc} \text{ in } \Omega_+. \end{cases}$$

Subscripts inc and sc characterize incident and scattered fields, respectively. Afterwards, using the Stratton-Chu formulation we can give the variational formulation of the problem. For all $\Psi = (\Psi_J, \Psi_M)$, find (\mathbf{J}, \mathbf{M}) such that:

$$(5) \quad \langle Z_0(B - S)\mathbf{J}, \Psi_J \rangle + \langle (P + Q)\mathbf{M}, \Psi_J \rangle = \langle E^{inc}, \Psi_J \rangle,$$

$$(6) \quad - \langle (P + Q)\mathbf{J}, \Psi_M \rangle + \langle \frac{1}{Z_0}(B - S)\mathbf{M}, \Psi_M \rangle = \langle H^{inc}, \Psi_M \rangle.$$

Where

$$(7) \quad \langle (B - S)\phi, \psi \rangle = i \iint_{\Gamma} k G \phi \cdot \psi - \frac{1}{k} G \nabla_y \cdot \phi \nabla_x \cdot \psi \quad dy dx,$$

$$(8) \quad \langle (P + Q)\phi, \psi \rangle = \frac{1}{2} \int_{\Gamma} \psi \cdot (\mathbf{n} \times \phi) \quad dx + \iint_{\Gamma} (\psi \times \phi) \cdot \nabla_x G \quad dy dx,$$

$$(9) \quad \langle \phi, \psi \rangle = \int_{\Gamma} \mathbf{A} \cdot \psi \quad dx,$$

with $G(x, y) = \frac{e^{-ik\|x-y\|}}{4\pi\|x-y\|}$ is the Green kernel in three dimensions.

For a numerical resolution of the problem we will adopt the finite element method of boundary. We assume that the initial boundary problem is defined in a 3-D domain. The boundary Γ of the domain is therefore a surface

and can be discretized using triangles. We approach the surface of the obstacle by a surface Γ_h composed of finite number of two dimensional elements. These elements are triangular facets denoted by T_i for $i = 1$ to N_T :

$$\Gamma_h = \bigcup_{i=1}^{N_T} T_i.$$

We will call it an initial mesh (or original mesh). We denote by N_e the total number of edges of the mesh component Γ_h . The discretization of unknowns \mathbf{J} and \mathbf{M} should verify a condition - flow conservation of these currents. One way to ensure this is to use the basis functions of Rao-Wilton-Glisson introduced below.

We introduce local numbering of a triangle T . The vertices $(a_j^T)_{j=1,3}$ are arranged in clockwise order. Triangle edges are numbered so that the edge T_j^T connects vertices a_j^T and a_{j+1}^T .

Moreover, we give an orientation ν_n to each edge n . Consider the two triangles sharing this edge. We note T_n^+ the triangle so that the direction of the edge n coincides with the forward direction (locally defined) of this triangle. For the other triangle, which we will denote T_n^- , the direction of the edge coincides with the indirect sense.

Each basis function is associated with an edge and ensures the conservation of flux through this edge. If we denote $|T|$ the area of a triangle T , the n^{th} basis function is defined as follows:

Definition 2.1. *If n is the i^{th} local edge of triangle T_n^+ and the j^{th} of triangle T_n^- then*

$$(10) \quad \mathbf{f}_n(x) = \begin{cases} \frac{l_n}{2|T_n^+|} (x - a_{i-1}^+) & \text{if } x \in T_n^+ \\ \frac{l_n}{2|T_n^-|} (a_{j-1}^- - x) & \text{if } x \in T_n^- \\ 0 & \text{if } x \notin T_n^+ \cup T_n^- \end{cases}$$

The density is proportional to $\nabla_\Gamma \cdot \mathbf{f}_n$, where

$$(11) \quad \nabla_\Gamma \cdot \mathbf{f}_n(x) = \begin{cases} +\frac{l_n}{|T_n^+|} & \text{on } T_n^+ \\ -\frac{l_n}{|T_n^-|} & \text{on } T_n^- \\ 0 & \text{elsewhere.} \end{cases}$$

We decompose the electric and magnetic currents:

$$\mathbf{J}(y) = \sum_{i=1}^{N_e} J_i \mathbf{f}_i(y), \quad \mathbf{M}(y) = \sum_{i=1}^{N_e} M_i \mathbf{f}_i(y).$$

Considering to these assumptions we are looking for approximated solution of the problem (5-6). We introduce the following system

$$(12) \quad A^h(U_h, \Psi_h) = \sum_{i=1}^{N_e} \langle \mathbf{E}^{inc}, \mathbf{f}_i \rangle + \sum_{i=1}^{N_e} \langle \mathbf{H}^{inc}, \mathbf{f}_i \rangle,$$

where

$$\begin{aligned} A^h(U_h, \Psi_h) &= \sum_{i,j=1}^{N_e} \langle Z_0(B-S)\mathbf{f}_j, \mathbf{f}_i \rangle + J_j + Z_0^{-1} \sum_{i,j=1}^{N_e} \langle (B-S)\mathbf{f}_j, \mathbf{f}_i \rangle M_j \\ &+ \sum_{i,j=1}^{N_e} \langle Q\mathbf{f}_j, \mathbf{f}_i \rangle M_j - \sum_{i,j=1}^{N_e} \langle Q\mathbf{f}_j, \mathbf{f}_i \rangle J_j + \frac{a_0}{2} \sum_{i,j=1}^{N_e} \langle \mathbf{f}_j, \mathbf{f}_i \rangle J_j + \frac{1}{2a_0} \sum_{i,j=1}^{N_e} \langle \mathbf{f}_j, \mathbf{f}_i \rangle M_j. \end{aligned}$$

Since the basis function \mathbf{f}_i is defined on two triangles, namely adjacent to the i^{th} edge, T_i^+ and T_i^- . The same for the basis function \mathbf{f}_j , they are defined on two triangles, T_j^+ and T_j^- .

We define the matrix elements

$$(B-S)_{i,j} = i \int_{T_i^+ \cup T_i^-} \int_{T_j^+ \cup T_j^-} k G(x,y) \mathbf{f}_j(y) \cdot \mathbf{f}_i(x) - \frac{1}{k} G(x,y) (\nabla_\Gamma \cdot \mathbf{f}_i(x)) (\nabla_\Gamma \cdot \mathbf{f}_j(y)) \, dx dy.$$

Analogically

$$Q_{i,j} = -i \int_{T_i^+ \cup T_i^-} \int_{T_j^+ \cup T_j^-} [\mathbf{f}_i(x) \times \mathbf{f}_j(y)] \cdot \nabla_\Gamma \cdot G(x,y) \, dy dx.$$

with $G(x,y) = \frac{e^{-ik\|x-y\|}}{4\pi\|x-y\|} = -\frac{1}{4\pi\|x-y\|} + H(\|x-y\|)$ is the Green kernel in three dimensions and

$$\nabla_x G(x,y) = -\frac{(1+ik\|x-y\|)}{4\pi\|x-y\|^3} e^{-ik\|x-y\|} (\mathbf{x}-\mathbf{y}) = -\frac{(x-y)}{4\pi\|x-y\|^3} - \frac{k^2(x-y)}{8\pi\|x-y\|} - (x-y)R(\|x-y\|).$$

where H and R are regular analytic functions.

We then, notice that there are singular terms that appear. The regular integrals $\mathcal{R}_{i,j}$ can be calculated numerically with the usual known methods such as the Gauss method.

We denote by S the triangle T_i^+ and by T the triangle T_j^+ . We evaluate the operators $Q_{i,j}$ and $(B - S)_{i,j}$ on $S \times T$ and it will be the same for other combinations of triangles.

On each triangle, the current is written as a linear combination of three RWG basis functions associated to three edges of a triangle. Now we come to the integrals developed in this paper. So we have to evaluate three type of singular integrals

$$(13) \quad \mathcal{D}_{i,j} = \int_{S \times T} (x - a_i) \cdot \frac{(x - y)}{\|x - y\|^{\xi+1}} \times (y - b_j) \, dxdy; \, i, j = 1, 2, 3; \, \xi = 0, 2$$

$$(14) \quad \mathcal{S}_{i,j} = \int_{S \times T} \frac{(x - a_i) \cdot (y - b_j)}{\|x - y\|} \, dxdy; \, i, j = 1, 2, 3.$$

$$(15) \quad \mathcal{M}_{i,j} = \int_{S \times T} \frac{1}{\|x - y\|} \, dxdy; \, i, j = 1, 2, 3.$$

In this work, we adopt a purely analytic method for the evaluation of the singular integrals. It was developed in [11] [12]. In these works, the term (15) was evaluated and studied theoretically.

3 Definitions Tools and Notations

We present some notations and we give tools that we will need in our calculation.

3.1 Notations

Let the triangles S β_j , $i, j = 1, 2, 3$ su the length of the ε $(b_{i+2} - b_{i+1}) = (b$ β_j .

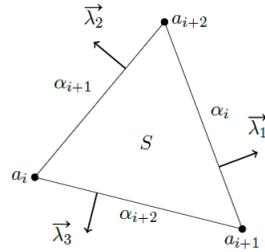
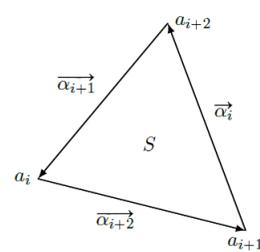


Figure 1: Definition of Triangle S

by its vertices a_i and b_j , i, j $\circ a_i$ respectively b_j (see Fig and $\vec{\beta}_j$ be the vectors $(a_{i+}$ note by λ_i and ϑ_j the outsi



tively $|\beta_j|$ tively

Figure 2: Orientation of the triangle S

Note $p_i(x)$ the $\|x - p_i(x)\|$ the dis is the length of the which involves the Figure 5).

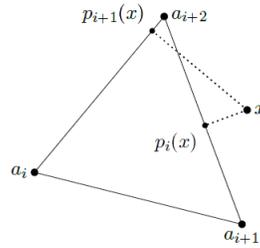


Figure 3: Projection

Figure 3) of a point x on the side α_i of α_i . When $x = a_i$, we note that $p_i = j$ ll also use the distance signed from a gle on α_i . This distance is defined b

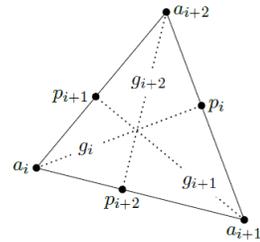


Figure 4: Projections and distances

We use notations for the triangle T similar to those introduced for the triangle S . The orthogonal projection of y on the support of β_j , similar to $p_i(x)$, is denoted by $q_j(y)$ and the distance from y to the support of β_j is denoted by $d_j(y) = \|y - q_j(y)\|$. We define the signed distance $\delta_j(y)$ from y to the support of β_j by $\delta_j(y) = (q_j(y) - y) \cdot \vec{\vartheta}_j$. Finally, $d_j = d_j(b_j)$ is the equivalent of g_i , namely the length of the height from b_j to β_j .

On side α (resp β) the abscissa s (resp t) is defined with respect to an origin o_α (resp o_β) and a unitary direction vector $\frac{\vec{\alpha}}{|\alpha|}$ (resp $\frac{\vec{\beta}}{|\beta|}$).

The abscissas of the ends a^\pm and b^\pm are respectively denoted by s^\pm and t^\pm .

$$s^k = (a^k - o | \frac{\vec{\alpha}}{|\alpha|}) \quad t^l = (b^l - o | \frac{\vec{\beta}}{|\beta|}).$$

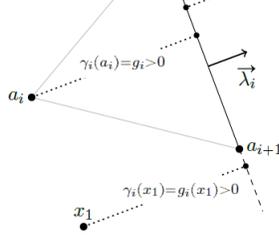


Figure 5: Distances signes

3.2 Reduction process

The reduction process is based on several simplification formulas presented in section . These formulas reduce the size of the domain integration. They can be used when the integrand is homogeneous, or positively homogeneous.

Definition 3.1. A function is said to be homogeneous of degree q , or q -homogeneous, with respect to the variable z

$$(16) \quad f(\lambda z) = \lambda^q f(z).$$

Definition 3.2. A function is said to be q -homogeneous with parameter if it is an homogeneous function with respect to a pair of variables (z, h) , where z is the integration variable and h a real parameter

$$(17) \quad f(\lambda z, \lambda h) = \lambda^q f(z, h).$$

Theorem 1. Let $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ a positively homogeneous function of degree q with $q + n \neq 0$

$$(18) \quad \int_{\Omega} f(z) dz = \frac{1}{q+n} \int_{\partial\Omega} (\vec{z} | \vec{\nu}) f(z) ds_z,$$

where $\vec{\nu}$ is the exterior normal to $\partial\Omega$ and $(\vec{z} | \vec{\nu})$ is the scalar inner product between z and $\vec{\nu}$.

Theorem 2. Let $g : \Omega \subset \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$, be a homogeneous function of degree q , referred to as a homogeneous function with parameter. Then, provided that

$$(19) \quad \left(\frac{h}{h_0}\right)^{q+n} \int_{\Omega} g(z, h_0) ds_z \rightarrow 0 \text{ quand } h_0 \rightarrow \text{sgn}(h)\infty,$$

we get

$$(20) \quad \int_{\Omega} g(z, h) dz = \int_{\partial\Omega} (\vec{z} | \vec{\nu}) F(z, h) ds_z,$$

where $F(z, h) = h^{q+n} \int_h^{\text{sgn}(h)\infty} \frac{g(z, u)}{u^{q+n+1}} du$.

Theorem 3. Let $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ a positively homogeneous function of degree q with $q = -n$. We introduce an arbitrary homogeneous positive function $\theta : \Omega \rightarrow \mathbb{R}$ of degree $p > 0$. Then we have

$$(21) \quad \int_{\Omega} f(z) dz = \frac{1}{p} \int_{\partial\Omega} (\vec{z} | \vec{\nu}) f(z) \ln(\theta(z)) ds_z.$$

These formulae were proved in ([11]).

In the following we present a method for the explicit evaluation of singular integrals. It is built on a recursive reduction of a 4-dimension integral into a linear combination of $(m-1)$ -dimensional integrals. It leads to a linear combination of 1-dimensional regular integrals. The method we have used requires to distinguish between several geometric configurations and bases to the choice of the origin during the steps of reduction of the dimension of the integration domain. This technique is assured the homogeneity of the singular part of the Green function.

3.3 Application of the reduction formula

We propose to evaluate the integral

$$(22) \quad I(x, y) = \int_{S \times \beta_j} \frac{x-y}{\|x-y\|^3} dx dy,$$

where S is a triangle, x a point in the triangle S , β_j is a segment in the same plan as S and y is a point in β_j . Let $b_j^+ = b_{j+1}$ and $b_j^- = b_{j+2}$ be the two extremities of β . The reduction process can be applied, provided that one takes some common point to β_j and S as origin, so that the function $f(z) = \frac{x-y}{\|x-y\|^3}$ is homogeneous on $S \times \beta_j$ with $q = 2$, $n = 3$. The origin can be taken anywhere on the support of β_j , but a wise choice is to choose one of the ends of β_j as origin. Indeed, some of the distances being canceled, one will have a simplification. We choose for origin the point b^- . Formula (18) provides

$$(23) \quad I(x, y) = \frac{1}{5} \int_{\partial(S \times \beta_j)} ((x, y) | \vec{\nu}) \frac{x-y}{\|x-y\|^3} \partial(x, y),$$

where ν is the outward normal at the border of the edge of $S \times \beta_j$ and $\partial(x, y)$ is the surface element along $\partial(S \times \beta_j)$. We have $\partial(S \times \beta_j) = (\partial S \times \beta_j) \cup (S \times \partial\beta_j)$.

$$((x, y) | \vec{\nu})_{b_j^+ \times S} = |\beta_j|; \quad ((x, y) | \vec{\nu})_{\alpha_i \times \beta_j} = \gamma_i(b_j^-)$$

where $|\beta_j|$ denotes the length of the segment β_j . Then, we get

$$(24) \quad I(x, y) = \frac{|\beta_j|}{5} \int_S \frac{x-b_j^+}{\|x-b_j^+\|^3} ds_x + \sum_{i=1}^3 \frac{\gamma_i(b_j^-)}{5} \int_{\partial(\alpha_i \times \beta_j)} \frac{x-y}{\|x-y\|^3} ds_x ds_y.$$

So we get a linear combinaison of 2-D integrals.

4 Calculation of the singular integral in the forme double layer operator

Before explicitly evaluating the integral defined in (13), we will rewrite the RWG basis functions as a combination of the barycentric basis functions which allowed to transform the integral into integrals simpler to study. We introduce the basis functions ϕ_i and ϕ_j associated with the vertices of the triangle S respectively T . We use their properties (see appendix 6.1) above to determine $\theta, \kappa, \zeta, \rho, \eta$ and μ verifying

$$(25) \quad \begin{aligned} (x - a_i) &= \theta\phi_i(x) + \kappa\phi_{i+1}(x) + \zeta\phi_{i+2}(x), \\ (y - b_j) &= \rho\psi_j(y) + \eta\psi_{j+1}(y) + \mu\psi_{j+2}(y). \end{aligned}$$

We obtain

$$(26) \quad \begin{aligned} (x - a_i) &= (a_{i+1} - a_i)\phi_{i+1}(x) + (a_{i+2} - a_i)\phi_{i+2}(x), \\ (y - b_j) &= (a_{j+1} - a_j)\psi_{j+1}(y) + (a_{j+2} - a_j)\psi_{j+2}(y). \end{aligned}$$

We are interested in calculating the singular integral (13). Using the basis functions as defined in (26), this integral becomes

$$(27) \quad \begin{aligned} \mathcal{D}_{i,j} &= \int_{S \times T} (\kappa \times \eta) \cdot (\phi_{i+1}(x)\psi_{j+1}(y)) \frac{(x-y)}{\|x-y\|^{\xi+1}} dx dy \\ &+ \int_{S \times T} (\kappa \times \mu) \cdot (\phi_{i+1}(x)\psi_{j+2}(y)) \frac{(x-y)}{\|x-y\|^{\xi+1}} dx dy \\ &+ \int_{S \times T} (\zeta \times \eta) \cdot (\phi_{i+2}(x)\psi_{j+1}(y)) \frac{(x-y)}{\|x-y\|^{\xi+1}} dx dy \\ &+ \int_{S \times T} (\zeta \times \mu) \cdot (\phi_{i+2}(x)\psi_{j+2}(y)) \frac{(x-y)}{\|x-y\|^{\xi+1}} dx dy. \end{aligned}$$

So we have to evaluate four integrals of the same type:

$$(28) \quad \mathcal{J}^{i,j} = \int_{S \times T} (\phi_i(x)\psi_j(y)) \frac{(x-y)}{\|x-y\|^{\xi+1}} dx dy; \xi = 0, 2; i, j = 1, 2, 3.$$

There is two different geometric configurations. Here, we deal with the case of adjacent triangles. We detail the formula for the evaluation of the expressions (28). In the case triangles have a common vertex, there no real additional difficulty. We can apply the same techniques.

Since the basis functions is affine, the integrant is no longer homogeneous, so we are led to decompose these into a linear part and a constant part by making a change of origin:

$$(29) \quad \begin{aligned} \phi_i(x) &= 1 + (\overrightarrow{o - a_i} | \vec{e}_i) + (\overrightarrow{x - o} | \vec{e}_i) = \phi_i(o) + (\underline{x} | \vec{e}_i); i = 1, 2, 3, \\ \psi_j(y) &= 1 + (\overrightarrow{o - b_j} | \vec{l}_j) + (\overrightarrow{y - o} | \vec{l}_j) = \psi_j(o) + (\underline{y} | \vec{l}_j); j = 1, 2, 3. \end{aligned}$$

where o is the new origin, $\underline{x} = x - o$ and $\underline{y} = y - o$. More generally, we note $\underline{\alpha}$ and \underline{S} the segment α and the triangle S after this change of origin. According to 29 we constat that the integrals $\mathcal{J}^{i,j}$ must be decomposed into four integrals whose integrands are homogeneous $\mathcal{J}^{i,j}$

$$(30) \quad \begin{aligned} \mathcal{J}^{i,j} &= \int_{S \times T} \frac{(\underline{x} | \vec{e}_i)(\underline{y} | \vec{l}_j)(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} dx dy + \phi_i(o) \int_{S \times T} \frac{(\underline{x} | \vec{e}_i)(\underline{y} | \vec{l}_j)(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} dx dy \\ &+ \psi_j(o) \int_{S \times T} \frac{(\underline{x} | \vec{e}_i)(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} dx dy + \psi_j(o)\phi_i(o) \int_{S \times T} \frac{(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} dx dy. \end{aligned}$$

Once this decomposition has taken place, we obtain homogeneous integrands we can then apply the reduction method intermediate of the formulas (18), (20) or (21).

4.1 Three-dimensional reduction

We are dealing with the case of triangles with an edge in common, so there are two common vertices. For the first reduction, we take for origin one of the common vertices $o = a_i = b_j$. Let $a_{i+1} = b_{j+1}$ be the second common vertex and a_{i+2}, b_{j+2} be the remaining vertices. We propose to calculate the integral (28). It will be wise to choose the vertex $a_i = b_j$, as origin o . In this case the basis functions are written

$$(31) \quad \phi_i(x) = 1 + (\underline{x} | \vec{e}_i), \quad \psi_j(y) = 1 + (\underline{y} | \vec{l}_j)$$

The four integrands are homogeneous of degrees $q = 1 - \xi, q = 2 - \xi, q = 1 - \xi$ and $q = -\xi$ with $n = 4$. We apply the formula (18), it gives the following result

$$\begin{aligned} \mathcal{J}^{i,j} &= \sum_{i=1}^3 \frac{\gamma_i(a_i)}{5 - \xi} \int_{\underline{\alpha_i} \times \underline{T}} \frac{(\underline{y} | \vec{l}_j)(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} ds_x dy + \sum_{j=1}^3 \frac{\delta_j(b_j)}{5 - \xi} \int_{\underline{S} \times \underline{\beta_j}} \frac{(\underline{y} | \vec{l}_j)(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} dx ds_y \\ &+ \sum_{i=1}^3 \frac{\gamma_i(a_i)}{6 - \xi} \int_{\underline{\alpha_i} \times \underline{T}} \frac{(\underline{x} | \vec{e}_i)(\underline{y} | \vec{l}_j)(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} ds_x dy + \sum_{j=1}^3 \frac{\delta_j(b_j)}{6 - \xi} \int_{\underline{S} \times \underline{\beta_j}} \frac{(\underline{x} | \vec{e}_i)(\underline{y} | \vec{l}_j)(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} dx ds_y \\ &+ \sum_{i=1}^3 \frac{\gamma_i(a_i)}{5 - \xi} \int_{\underline{\alpha_i} \times \underline{T}} \frac{(\underline{x} | \vec{e}_i)(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} ds_x dy + \sum_{j=1}^3 \frac{\delta_j(b_j)}{5 - \xi} \int_{\underline{S} \times \underline{\beta_j}} \frac{(\underline{x} | \vec{e}_i)(\underline{x} - \underline{y})}{\|\underline{x} - \underline{y}\|^{1+\xi}} dx ds_y \end{aligned}$$

$$(32) \quad + \sum_{i=1}^3 \frac{\gamma_i(a_i)}{4-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{x-y}{\|x-y\|^{1+\xi}} ds_x dy + \sum_{j=1}^3 \frac{\delta_j(b_j)}{4-\xi} \int_{\underline{\beta} \times \beta_j} \frac{x-y}{\|x-y\|^{1+\xi}} dx ds_y.$$

where $\gamma_i(x)$ and $\delta_j(y)$ are the signed distances respectively of x to the support of α_i and of y in support of β_j (see Figure 4).

In particular $\gamma_i(x) = 0$ for all x belongs to the support of α_i . $\gamma_i(a_i) = g_i$ and $\delta_j(b_j) = d_j$ are the lengths of the height from the vertex a_i or b_j to the support of α_i (respectively β_j). However the formula is simplified and we get

$$(33) \quad \begin{aligned} \mathcal{J}^{i,j} &= \frac{g_i}{5-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{(\underline{y}|\vec{l}_j)(x-y)}{\|x-y\|^{1+\xi}} ds_x dy + \frac{d_j}{5-\xi} \int_{\underline{\beta} \times \beta_j} \frac{(\underline{y}|\vec{l}_j)(x-y)}{\|x-y\|^{1+\xi}} dx ds_y \\ &+ \frac{g_i}{6-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{(\underline{x}|\vec{e}_i^*)(\underline{y}|\vec{l}_j)(x-y)}{\|x-y\|^{1+\xi}} ds_x dy + \frac{d_j}{6-\xi} \int_{\underline{\beta} \times \beta_j} \frac{(\underline{x}|\vec{e}_i^*)(\underline{y}|\vec{l}_j)(x-y)}{\|x-y\|^{1+\xi}} dx ds_y \\ &+ \frac{g_i}{5-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{(\underline{x}|\vec{e}_i^*)(x-y)}{\|x-y\|^{1+\xi}} ds_x dy + \frac{d_j}{5-\xi} \int_{\underline{\beta} \times \beta_j} \frac{(\underline{x}|\vec{e}_i^*)(x-y)}{\|x-y\|^{1+\xi}} dx ds_y \\ &+ \frac{g_i}{4-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{x-y}{\|x-y\|^{1+\xi}} ds_x dy + \frac{d_j}{4-\xi} \int_{\underline{\beta} \times \beta_j} \frac{x-y}{\|x-y\|^{1+\xi}} dx ds_y. \end{aligned}$$

We express the results in terms of ϕ_i and ψ_j in order to benefit the best of the properties (94) of the basic functions.

$$(\underline{x}|\vec{e}_i^*) = \phi_i(x) - 1, \quad (\underline{y}|\vec{l}_j) = \psi_j(y) - 1$$

we obtain

$$\begin{aligned} \mathcal{J}^{i,j} &= \frac{g_i}{5-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{\psi_j(y)(x-y)}{\|x-y\|^{1+\xi}} ds_x dy - \frac{g_i}{5-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{(x-y)}{\|x-y\|^{1+\xi}} ds_x dy \\ &+ \frac{d_j}{5-\xi} \int_{\underline{\beta} \times \beta_j} \frac{\psi_j(y)(x-y)}{\|x-y\|^{1+\xi}} dx ds_y - \frac{d_j}{5-\xi} \int_{\underline{\beta} \times \beta_j} \frac{(x-y)}{\|x-y\|^{1+\xi}} dx ds_y + \frac{g_i}{6-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{\phi_i(x)\psi_j(y)(x-y)}{\|x-y\|^{1+\xi}} ds_x dy \\ &- \frac{g_i}{6-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{\phi_i(x)(x-y)}{\|x-y\|^{1+\xi}} ds_x dy - \frac{g_i}{6-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{\psi_j(y)(x-y)}{\|x-y\|^{1+\xi}} ds_x dy + \frac{g_i}{6-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{(x-y)}{\|x-y\|^{1+\xi}} ds_x dy \\ &+ \frac{d_j}{6-\xi} \int_{\underline{\beta} \times \beta_j} \frac{\phi_i(x)\psi_j(y)(x-y)}{\|x-y\|^{1+\xi}} dx ds_y - \frac{d_j}{6-\xi} \int_{\underline{\beta} \times \beta_j} \frac{\phi_i(x)(x-y)}{\|x-y\|^{1+\xi}} dx ds_y - \frac{d_j}{6-\xi} \int_{\underline{\beta} \times \beta_j} \frac{\psi_j(y)(x-y)}{\|x-y\|^{1+\xi}} dx ds_y \\ &+ \frac{d_j}{6-\xi} \int_{\underline{\beta} \times \beta_j} \frac{(x-y)}{\|x-y\|^{1+\xi}} dx ds_y + \frac{g_i}{5-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{\phi_i(x)(x-y)}{\|x-y\|^{1+\xi}} ds_x dy - \frac{g_i}{5-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{(x-y)}{\|x-y\|^{1+\xi}} ds_x dy \\ &+ \frac{d_j}{5-\xi} \int_{\underline{\beta} \times \beta_j} \frac{\phi_i(x)(x-y)}{\|x-y\|^{1+\xi}} dx ds_y - \frac{d_j}{5-\xi} \int_{\underline{\beta} \times \beta_j} \frac{(x-y)}{\|x-y\|^{1+\xi}} dx ds_y + \frac{g_i}{4-\xi} \int_{\underline{\alpha}_i \times \mathcal{T}} \frac{x-y}{\|x-y\|^{1+\xi}} ds_x dy \\ &+ \frac{d_j}{4-\xi} \int_{\underline{\beta} \times \beta_j} \frac{x-y}{\|x-y\|^{1+\xi}} dx ds_y. \end{aligned}$$

There are terms that vanish because of the fact that $\phi_i|_{\alpha_i} = \psi_j|_{\beta_j} = 0$.

We have integrals in three dimensions we reapply the procedure to reduce dimension to two. The integrands are no more homogeneous. It is possible to continue the reduction by taking the second common vertex as the origin $a_2 = b_2 = o$. We reexpress the basis function as in (31) and we obtain

$$\phi_i(x) = \phi_i(a_{i+1}) + (\underline{x}|\vec{e}_i^*) = (\underline{x}|\vec{e}_i^*); \quad \psi_j(x) = \psi_j(b_{j+1}) + (\underline{y}|\vec{l}_j) = (\underline{y}|\vec{l}_j)$$

Finally, we get

$$\begin{aligned} \mathcal{J}^{i,j} &= \left(\frac{|S|}{5-\xi} - \frac{|S|}{6-\xi}\right) \frac{2}{4-\xi} P^{\psi_j}(a_{i+2}, T, \xi) + \left(\frac{g_i}{5-\xi} - \frac{g_i}{6-\xi}\right) \frac{d_j+1}{4-\xi} Q(\alpha, \beta_{j+1}, \xi) \\ &+ \left(\frac{|S|}{6-\xi} - 2\frac{|S|}{5-\xi} + \frac{|S|}{4-\xi}\right) \frac{2}{3-\xi} P(a_{i+2}, T, \xi) + \left(\frac{g_i}{6-\xi} - 2\frac{g_i}{5-\xi} + \frac{g_i}{4-\xi}\right) \frac{d_j+1}{3-\xi} Q(\alpha_i, \beta_{j+1}, \xi) \\ &- \left(\frac{|T|}{5-\xi} - \frac{|T|}{6-\xi}\right) \frac{2}{4-\xi} P^{\phi_i}(b_{j+2}, S, \xi) - \left(\frac{d_1}{5-\xi} - \frac{d_j}{6-\xi}\right) \frac{g_i+1}{4-\xi} Q^1, \phi_i(\beta_j, \alpha_{i+1}, \xi) \\ &- \left(\frac{|S|}{6-\xi} - \frac{|S|}{5-\xi}\right) \frac{2}{4-\xi} P(b_{j+2}, S, \xi) + \left(\frac{d_j}{4-\xi} + \frac{d_j}{6-\xi} - \frac{d_j}{5-\xi}\right) \frac{g_i+1}{4-\xi} Q(\alpha_{i+1}, \beta_j, \xi), \end{aligned}$$

where

$$\begin{aligned} P(a, T, \xi) &= \int_{\mathcal{T}} \frac{(a-y)}{\|a-y\|^{1+\xi}} ds_x ds_y, \quad Q(\alpha_i, \beta_j, \xi) = \int_{\alpha_i \times \beta_j} \frac{(x-y)}{\|x-y\|^{1+\xi}} ds_x ds_y, \\ P^{\psi_q}(a, T, \xi) &= \int_{\mathcal{T}} \frac{\psi_q(y)(a-y)}{\|a-y\|^{1+\xi}} dy, \quad Q^1, \psi_q(\alpha_i, \beta_j, \xi) = \int_{\alpha_i \times \beta_j} \frac{\psi_q(y)(x-y)}{\|x-y\|^{1+\xi}} ds_x ds_y. \end{aligned}$$

We give also the other terms calculated with the same procedure:

$$(34) \quad \begin{aligned} \mathcal{J}^{i,j+1} &= P(a_{i+2}, T, \xi) \left(\frac{2|S|}{(5-\xi)(3-\xi)} - \frac{2|S|}{(5-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(3-\xi)} + \frac{2|S|}{(6-\xi)(4-\xi)} \right) \\ &+ P(b_{j+2}, S, \xi) \left(-\frac{2|T|}{(5-\xi)(3-\xi)} + \frac{2|T|}{(5-\xi)(4-\xi)} + \frac{2|T|}{(6-\xi)(3-\xi)} - \frac{2|T|}{(6-\xi)(4-\xi)} \right) \\ &+ Q(\alpha_i, \beta_{i+1}, \xi) \left(\frac{g_i d_j+1}{(5-\xi)(3-\xi)} - \frac{g_i d_j+1}{(5-\xi)(4-\xi)} - \frac{g_i d_j+1}{(6-\xi)(3-\xi)} + \frac{g_i d_j+1}{(6-\xi)(4-\xi)} \right) \\ &+ Q(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_i+1}{(5-\xi)(3-\xi)} - \frac{d_j g_i+1}{(6-\xi)(3-\xi)} + \frac{d_j g_i+1}{(6-\xi)(4-\xi)} - \frac{d_j g_i+1}{(5-\xi)(4-\xi)} \right) \\ &+ Q^1, \psi_{j+1}(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_i+1}{(5-\xi)(4-\xi)} - \frac{d_j g_i+1}{(6-\xi)(4-\xi)} \right) + P^{\psi_{j+1}}(a_{i+2}, T, \xi) \left(\frac{2|S|}{(5-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(4-\xi)} \right) \\ &- Q^1, \phi_i(\beta_j, \alpha_{i+1}, \xi) \left(\frac{d_j g_i+1}{(6-\xi)(4-\xi)} - \frac{d_j g_i+1}{(6-\xi)(5-\xi)} \right) + P^{\phi_i}(b_{j+2}, S, \xi) \left(-\frac{2|T|}{(6-\xi)(4-\xi)} + \frac{2|T|}{(6-\xi)(5-\xi)} \right) \\ &+ Q^{\phi_i, \psi_{j+1}}(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_i+1}{(6-\xi)(5-\xi)} \right), \end{aligned}$$

where

$$(35) \quad Q^{\phi_p, \psi_q}(\alpha_i, \beta_j, \xi) = \int_{\alpha_i \times \beta_j} \frac{\phi_p(x)\psi_q(y)(y-x)}{\|x-y\|^{1+\xi}} ds_x ds_y,$$

$$(36) \quad \begin{aligned} \mathcal{J}^{i,j+2} &= P(b_{j+2}, S, \xi) \left(-\frac{2|T|}{(5-\xi)(4-\xi)} + \frac{2|T|}{(6-\xi)(4-\xi)} \right) - P^{\phi_i}(b_{j+2}, S, \xi) \left(\frac{2|T|}{(6-\xi)(5-\xi)} \right) \\ &+ P^{\psi_{j+2}}(a_{i+2}, T, \xi) \left(\frac{2|S|}{(5-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(5-\xi)} \right) + Q^{1, \psi_{j+2}}(\alpha_i, \beta_{i+1}, \xi) \left(\frac{d_{j+1}g_i}{(5-\xi)(4-\xi)} - \frac{d_{j+1}g_i}{(6-\xi)(4-\xi)} \right) \\ &+ Q^{1, \psi_{j+2}}(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_{i+1}}{(5-\xi)(4-\xi)} - \frac{d_j g_{i+1}}{(6-\xi)(4-\xi)} \right) + Q^{\phi_i, \psi_{j+2}}(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_{i+1}}{(6-\xi)(5-\xi)} \right), \end{aligned}$$

$$(37) \quad \begin{aligned} \mathcal{J}^{i+1,j+1} &= P(a_{i+2}, T, \xi) \left(\frac{2|S|}{(6-\xi)(3-\xi)} - \frac{4|S|}{(6-\xi)(4-\xi)} + \frac{2|S|}{(5-\xi)(6-\xi)} \right) + P^{\psi_{j+1}}(a_{i+2}, T, \xi) \left(\frac{2|S|}{(6-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(5-\xi)} \right) \\ &+ Q(\alpha_i, \beta_{i+1}, \xi) \left(\frac{g_i d_{j+1}}{(6-\xi)(3-\xi)} - \frac{2g_i d_{j+1}}{(6-\xi)(4-\xi)} + \frac{g_i d_{j+1}}{(6-\xi)(5-\xi)} \right) + Q^{1, \phi_{i+1}}(\beta_{i+1}, \alpha_i, \xi) \left(-\frac{d_{j+1}g_i}{(6-\xi)(4-\xi)} + \frac{d_{j+1}g_i}{(6-\xi)(5-\xi)} \right) \\ &+ P(b_{j+2}, S, \xi) \left(-\frac{2|T|}{(6-\xi)(3-\xi)} + \frac{4|T|}{(6-\xi)(4-\xi)} - \frac{2|T|}{(6-\xi)(5-\xi)} \right) + P^{\phi_{i+1}}(b_{j+2}, S, \xi) \left(\frac{2|T|}{(6-\xi)(5-\xi)} - \frac{2|T|}{(6-\xi)(4-\xi)} \right) \\ &+ Q(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_{i+1}}{(6-\xi)(3-\xi)} - \frac{2d_j g_{i+1}}{(6-\xi)(4-\xi)} + \frac{d_j g_{i+1}}{(6-\xi)(5-\xi)} \right) + Q^{1, \psi_{j+1}}(\alpha_{i+1}, \beta_j, \xi) \left(-\frac{d_j g_{i+1}}{(6-\xi)(5-\xi)} + \frac{d_j g_{i+1}}{(6-\xi)(4-\xi)} \right), \end{aligned}$$

$$(38) \quad \begin{aligned} \mathcal{J}^{i+1,j+2} &= P(b_{j+2}, S, \xi) \left(\frac{2|T|}{(6-\xi)(5-\xi)} - \frac{2|T|}{(6-\xi)(4-\xi)} \right) + P^{\psi_{j+2}}(a_{i+2}, T, \xi) \left(\frac{2|S|}{(6-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(5-\xi)} \right) \\ &+ P^{\phi_{i+1}}(b_{j+2}, S, \xi) \left(-\frac{2|T|}{(6-\xi)(5-\xi)} \right) + Q^{1, \psi_{j+2}}(\alpha_i, \beta_{i+1}, \xi) \left(\frac{d_{j+1}g_i}{(6-\xi)(4-\xi)} - \frac{d_{j+1}g_i}{(6-\xi)(5-\xi)} \right) \\ &+ Q^{1, \psi_{j+2}}(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_{i+1}}{(6-\xi)(4-\xi)} - \frac{d_j g_{i+1}}{(6-\xi)(5-\xi)} \right) + Q^{\phi_{i+1}, \psi_{j+2}}(\alpha_i, \beta_{i+1}, \xi) \left(\frac{d_{j+1}g_i}{(6-\xi)(5-\xi)} \right), \end{aligned}$$

and

$$(39) \quad \begin{aligned} \mathcal{J}^{i+2,j+2} &= P^{\psi_{j+2}}(a_{i+2}, T, \xi) \left(\frac{2|S|}{(6-\xi)(5-\xi)} \right) + P^{\phi_{i+2}}(b_{j+2}, S, \xi) \left(-\frac{2|T|}{(6-\xi)(5-\xi)} \right) \\ &+ Q^{\phi_{i+2}, \psi_{j+2}}(\alpha_i, \beta_{i+1}, \xi) \left(\frac{d_{j+1}g_i}{(6-\xi)(5-\xi)} \right) + Q^{\phi_{i+2}, \psi_{j+2}}(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_{i+1}}{(6-\xi)(5-\xi)} \right). \end{aligned}$$

We can also give the symmetrical terms:

$$(40) \quad \begin{aligned} \mathcal{J}^{i+1,j} &= P(a_{i+2}, T, \xi) \left(\frac{2|S|}{(5-\xi)(3-\xi)} - \frac{2|S|}{(5-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(3-\xi)} + \frac{2|S|}{(6-\xi)(4-\xi)} \right) \\ &+ P(b_{j+2}, S, \xi) \left(-\frac{2|T|}{(5-\xi)(3-\xi)} + \frac{2|T|}{(5-\xi)(4-\xi)} + \frac{2|T|}{(6-\xi)(3-\xi)} - \frac{2|T|}{(6-\xi)(4-\xi)} \right) \\ &+ Q(\alpha_i, \beta_{i+1}, \xi) \left(\frac{g_i d_{j+1}}{(5-\xi)(3-\xi)} - \frac{g_i d_{j+1}}{(5-\xi)(4-\xi)} - \frac{g_i d_{j+1}}{(6-\xi)(3-\xi)} + \frac{g_i d_{j+1}}{(6-\xi)(4-\xi)} \right) \\ &+ Q(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_{i+1}}{(5-\xi)(3-\xi)} - \frac{d_j g_{i+1}}{(6-\xi)(3-\xi)} + \frac{d_j g_{i+1}}{(6-\xi)(4-\xi)} - \frac{d_j g_{i+1}}{(5-\xi)(4-\xi)} \right) \\ &+ Q^{1, \psi_j}(\alpha_i, \beta_{i+1}, \xi) \left(\frac{d_{j+1}g_i}{(6-\xi)(4-\xi)} - \frac{d_{j+1}g_i}{(6-\xi)(5-\xi)} \right) - Q^{1, \phi_{i+1}}(\beta_{i+1}, \alpha_i, \xi) \left(\frac{d_j g_{i+1}}{(5-\xi)(4-\xi)} - \frac{d_j g_{i+1}}{(6-\xi)(4-\xi)} \right) \\ &+ P^{\psi_j}(a_{i+2}, T, \xi) \left(\frac{2|S|}{(6-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(5-\xi)} \right) + P^{\phi_{i+1}}(b_{j+2}, S, \xi) \left(-\frac{2|T|}{(5-\xi)(4-\xi)} + \frac{2|T|}{(6-\xi)(4-\xi)} \right) \\ &+ Q^{\phi_{i+1}, \psi_j}(\alpha_i, \beta_{i+1}, \xi) \left(\frac{d_{j+1}g_i}{(6-\xi)(5-\xi)} \right), \end{aligned}$$

$$(41) \quad \begin{aligned} \mathcal{J}^{i+2,j} &= P(a_{i+2}, T, \xi) \left(\frac{2|S|}{(5-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(4-\xi)} \right) + P^{\phi_{i+2}}(b_{j+2}, S, \xi) \left(\frac{2|T|}{(6-\xi)(4-\xi)} - \frac{2|T|}{(4-\xi)(4-\xi)} \right) \\ &+ P^{\psi_j}(a_{i+2}, T, \xi) \left(\frac{2|S|}{(5-\xi)(6-\xi)} \right) + Q^{1, \phi_{i+2}}(\beta_j, \alpha_{i+1}, \xi) \left(\frac{d_j g_{i+1}}{(6-\xi)(4-\xi)} - \frac{d_j g_{i+1}}{(5-\xi)(4-\xi)} \right) \\ &+ Q^{1, \phi_{i+2}}(\beta_{i+1}, \alpha_i, \xi) \left(\frac{d_{j+1}g_i}{(6-\xi)(4-\xi)} - \frac{d_{j+1}g_i}{(5-\xi)(4-\xi)} \right) + Q^{\phi_{i+2}, \psi_j}(\alpha_i, \beta_{i+1}, \xi) \left(\frac{d_{j+1}g_i}{(6-\xi)(5-\xi)} \right), \end{aligned}$$

and

$$(42) \quad \begin{aligned} \mathcal{J}^{i+2,j+1} &= P(a_{i+2}, T, \xi) \left(\frac{2|S|}{(6-\xi)(4-\xi)} - \frac{2|S|}{(6-\xi)(5-\xi)} \right) + P^{\phi_{i+2}}(b_{j+2}, S, \xi) \left(\frac{2|T|}{(6-\xi)(5-\xi)} - \frac{2|T|}{(6-\xi)(4-\xi)} \right) \\ &+ P^{\psi_{j+1}}(a_{i+2}, T, \xi) \left(\frac{2|S|}{(6-\xi)(5-\xi)} \right) + Q^{1, \phi_{i+2}}(\beta_{i+1}, \alpha_i, \xi) \left(\frac{d_{j+1}g_i}{(6-\xi)(5-\xi)} - \frac{d_{j+1}g_i}{(6-\xi)(4-\xi)} \right) \\ &+ Q^{1, \phi_{i+2}}(\beta_j, \alpha_{i+1}, \xi) \left(-\frac{d_j g_{i+1}}{(6-\xi)(4-\xi)} + \frac{d_j g_{i+1}}{(6-\xi)(5-\xi)} \right) + Q^{\phi_{i+2}, \psi_{j+1}}(\alpha_{i+1}, \beta_j, \xi) \left(\frac{d_j g_{i+1}}{(6-\xi)(5-\xi)} \right). \end{aligned}$$

4.2 Reduction to mono-dimensional integrals

4.2.1 Calculation of $P(a, T, \xi)$

we will evaluate 43

$$(43) \quad P(a, T, \xi) = \int_T \frac{(a-y)}{\|a-y\|^{1+\xi}} ds_x ds_y,$$

with $a = a_{i+2}$

- Case a don't belongs to the support of T

In this case the reduction formulas are not usable. In order to operate the reduction, it is necessary to break P into two parts.

$$(44) \quad P(a, T, \xi) = \int_T \frac{(a-\hat{a})}{(h^2 + \|\hat{a}-y\|^2)^{\frac{1+\xi}{2}}} ds_x ds_y + \int_T \frac{(\hat{a}-y)}{(h^2 + \|\hat{a}-y\|^2)^{\frac{1+\xi}{2}}} ds_x ds_y$$

where \hat{a} is the projection of a into the support of T and h is the distance $h = |a - \hat{a}|$. Taking \hat{a} as a new origin we obtain homogeneous integrands of degrees $-1 - \xi$ and $-\xi$ with parameter h . We can then apply the formula (20) which gives

$$(45) \quad P(a, T, \xi) = (a - \hat{a}) \sum_{j=1}^3 \delta_j(\hat{a}) S_1(\hat{a}, \beta_j, h, \xi) + \sum_{j=1}^3 \delta_j(\hat{a}) R_2(\hat{a}, \beta_j, h, \xi),$$

where S_1 and R_2 are defined by

$$(46) \quad S_1(a, \beta, h, \xi) = \int_{\beta} h^{1-\xi} \int_h^{+\infty} \frac{du}{u^{2-\xi}(u^2 + \|a - y\|^2)^{\frac{1+\xi}{2}}} ds_y = \int_{\beta} f_1(h, \|a - y\|, \xi),$$

and

$$(47) \quad R_2(a, \beta, h, \xi) = \int_{\beta} (a - y) h^{2-\xi} \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2 + \|a - y\|^2)^{\frac{1+\xi}{2}}} ds_y = \int_{\beta} (a - y) f_2(h, \|a - y\|, \xi),$$

where

$$(48) \quad f_n(h, s, \xi) = h^{n-\xi} \int_h^{+\infty} \frac{du}{u^{n+1-\xi}(u^2 + s^2)^{\frac{1+\xi}{2}}}.$$

$S_1(a, \beta, h, \xi)$ and $R_2(a, \beta, h, \xi)$ are calculated in (6.3) and (6.5).

- Case a belongs to the support of T

Since a belongs to the support of T we can consider a as an origin and get homogeneous integrand with $q = -\xi$ and $n = 2$ but we must distinguish two different cases in order to apply the suitable reduction formula.

- * If $n + q = 0$, we apply (21) and we get

$$(49) \quad P(a, T, \xi) = \sum_{j=1}^3 \frac{\delta_j}{2 - \xi} \mathcal{L}(a, \beta_j),$$

where

$$(50) \quad \mathcal{L}(a, \beta_j) = \int_{\beta_j} \frac{a - y}{\|a - y\|^3} \ln(\|a - y\|) ds_y.$$

- * If $n + q \neq 0$, we apply (18) and we obtain

$$(51) \quad P(a, T, \xi) = \sum_{j=1}^3 \frac{\delta_j}{2 - \xi} S(a, \beta, h, \xi),$$

where

$$(52) \quad S(a, \beta, \xi) = \int_{\beta} \frac{a - y}{\|a - y\|^{1+\xi}} ds_y.$$

See appendix (6.2).

4.2.2 Calculation of $Q(\alpha, \beta, \xi)$

Here we deal with

$$(53) \quad Q(\alpha_i, \beta_j, \xi) = \int_{\alpha_i \times \beta_j} \frac{(x - y)}{\|x - y\|^{1+\xi}} ds_x ds_y.$$

We distinguish two cases.

- Case α and β are neither parallel nor secants

First of all, we must find a new origin to ensure homogeneity of the integrand. Let $\check{\alpha}$ be the projection of α on the plane parallel to β going through β and let o be the intersection of $\check{\alpha}$ with β . We will consider o as a new origin. Define $z = \check{\alpha} - a$ and $h = \|z\|$. We express Q as a sum of two integrals whose integrands are homogeneous of degrees $q = -1 - \xi$ and $-\xi$.

$$(54) \quad Q(\alpha, \beta, \xi) = \int_{\check{\alpha} \times \beta} \frac{(x - \check{x})}{(h^2 + \|\check{x} - y\|^2)^{\frac{1+\xi}{2}}} ds_x ds_y + \int_{\check{\alpha} \times \beta} \frac{(\check{x} - y)}{(h^2 + \|\check{x} - y\|^2)^{\frac{1+\xi}{2}}} ds_x ds_y,$$

where \check{x} is a point $\check{\alpha}$. This decomposition allows us to apply the formula (20) and gives

$$(55) \quad \begin{aligned} Q(\alpha, \beta, \xi) &= (x - \check{x}) \left[\sum_{k=\pm} ks^k S_1(\check{\alpha}^k, \beta, h, \xi) + \sum_{l=\pm} lt^l S_1(b^l, \check{\alpha}, h, \xi) \right] \\ &+ \left[\sum_{k=\pm} ks^k R_2(\check{\alpha}^k, \beta, h, \xi) - \sum_{l=\pm} lt^l R_2(a, \beta, h, \xi) \right], \end{aligned}$$

where the abscissae are evaluated by $s^k = (\check{\alpha}^k - o)_{|\check{\alpha}|}$ and $t^l = (b^l - o)_{|\check{\beta}|}$.

- Case α and β are secants

Let $o = \alpha \cap \beta$ be the new origin. Note that can be $o = a_{i+1} = b_{j+1}$ or $o = a_{i+2} = b_{j+2}$. In this case the integrand is homogeneous of degree $q = -\xi$ and $n = 2$. We must distinguish two cases:

- * If $q + n \neq 0$, the formula (18) gives

$$(56) \quad Q(\alpha, \beta, 0) = \sum_{k=\pm} \frac{ks^k}{2} S(a^k, \beta, 0) - \sum_{l=\pm} \frac{lt^l}{2} S(b^l, \alpha, 0).$$

- * If $q + n = 0$, we apply (21) and we get

$$(57) \quad Q(\alpha, \beta, 2) = \sum_{k=\pm} ks^k \mathcal{L}(a^k, \beta) - \sum_{l=\pm} lt^l \mathcal{L}(b^l, \alpha).$$

- Case α and β are parallel

When the supports of α and β are parallel, we must use the second formula (20), called with parameter. This is possible by orthogonally projecting one of the segments (we choose β) on the support of the other (α). Thus, the distance d between the supports of the two segments, appears and plays the role of the parameter. Let $\tilde{\beta}$ and \tilde{y} , be the respective orthogonal projections of β and y on the support of α , $\tilde{z} = \tilde{y} - y$ and $d = \|\tilde{y} - y\|$ the distance from α to the support of β . Then, the integral $Q(\alpha, \beta, \xi)$ becomes

$$(58) \quad Q(\alpha, \beta, \xi) = \tilde{z} \int_{\alpha \times \tilde{\beta}} \frac{1}{(d^2 + \|x - \tilde{y}\|^2)^{\frac{1+\xi}{2}}} ds_x ds_{\tilde{y}} + \int_{\alpha \times \tilde{\beta}} \frac{(x - \tilde{y})}{(d^2 + \|x - \tilde{y}\|^2)^{\frac{1+\xi}{2}}} ds_x ds_{\tilde{y}}.$$

Taking any point of α as a new origine, for example one of the two vertices of the triangle a^- and a^+ , we get two homogeneous integrands with parameter d and degrees $q = -1 - \xi$ respectively $q = -\xi$.

(59)

$$Q(\alpha, \beta, \xi) = \tilde{z} [s^+ \int_{\tilde{\beta}} d^{1-\xi} \int_d^{+\infty} \frac{du}{u^{2-\xi}(u^2 + \|a^+ - \tilde{y}\|^2)^{\frac{1}{2}}} ds_{\tilde{y}} + \sum_{l=\pm} lt^l \int_{\alpha} d^{1-\xi} \int_d^{+\infty} \frac{du}{u^{2-\xi}(u^2 + \|x - \tilde{b}^l\|^2)^{\frac{1}{2}}} ds_x] \\ + s^+ \int_{\tilde{\beta}} d^{2-\xi}(a^+ - y) \int_d^{+\infty} \frac{du}{u^{3-\xi}(u^2 + \|a^+ - y\|^2)^{\frac{1}{2}}} ds_{\tilde{y}} + \sum_{l=\pm} lt^l \int_{\alpha} d^{2-\xi}(x - \tilde{b}^l) \int_d^{+\infty} \frac{du}{u^{3-\xi}(u^2 + \|x - \tilde{b}^l\|^2)^{\frac{1}{2}}} ds_x,$$

which is equivalent to

$$(60) \quad Q(\alpha, \beta, \xi) = \sum_{l=\pm} lt^l [\tilde{z} S_1(\tilde{b}^l, \beta, d, \xi) - R_2(\tilde{b}^l, \beta, h, \xi)] + s^+ [S_1(a^+, \tilde{\beta}, d, \xi) + R_2(a^+, \tilde{\beta}, d, \xi)].$$

4.2.3 Calculation of $P^{\psi_q}(a, T, \xi)$

To calculate the singular integral

$$(61) \quad P^{\psi_q}(a, T, \xi) = \int_T \frac{\psi_q(y)(a - y)}{\|a - y\|^{1+\xi}} dy.$$

we are inspired by the section (4.2.1). There are two different geometrical cases.

- Case a don't belongs to the support of T

Note that $a = a_{i+2}$ is a vertex of S . To maintain the homogeneity of the integrands the origin must belongs to the support of the triangle T . So we consider \hat{a} the projection of a on the support of T (see Figure ??) and we break $P^{\psi_q}(a, T, \xi)$ into two parts

$$(62) \quad P^{\psi_q}(a, T, \xi) = (a - \hat{a}) \int_T \frac{\psi_q(y)}{(h^2 + \|\hat{a} - y\|^2)^{\frac{1+\xi}{2}}} dy + \int_T \frac{\psi_q(y)(\hat{a} - y)}{(h^2 + \|\hat{a} - y\|^2)^{1+\xi}},$$

where $h = |a - \hat{a}|$ and $\xi = 0, 2$. Next, we express $\psi_q(y)$ with the new origin $o = \hat{a}$,

$$\psi_q(y) = \psi_q(\hat{a}) + (y|l_q),$$

which gives

$$(63) \quad P^{\psi_q}(a, T, \xi) = (a - \hat{a}) [\psi_q(\hat{a}) \int_T \frac{1}{(h^2 + \|\hat{a} - y\|^2)^{\frac{1+\xi}{2}}} dy + \int_T \frac{(y|l_q)}{(h^2 + \|\hat{a} - y\|^2)^{\frac{1+\xi}{2}}} dy] \\ + \psi_q(\hat{a}) \int_T \frac{(\hat{a} - y)}{(h^2 + \|\hat{a} - y\|^2)^{\frac{1+\xi}{2}}} dy + \int_T \frac{(y|l_q)(\hat{a} - y)}{(h^2 + \|\hat{a} - y\|^2)^{\frac{1+\xi}{2}}} dy.$$

We notice that the four intergrands are homogeneous with parameter h and degrees (respectively) $q = -1 - \xi$, $q = -\xi$, $q = -\xi$ and $q = 1 - \xi$. We can then apply (20) and we obtain

$$(64) \quad P^{\psi_q}(a, T, \xi) = (a - \hat{a}) \psi_q(\hat{a}) \sum_{j=1}^3 \delta_j(\hat{a}) \int_{\beta_j} h^{1-\xi} \int_h^{+\infty} \frac{du}{u^{2-\xi}(u^2 + \|\hat{a} - y\|^2)^{\frac{1+\xi}{2}}} dy \\ + (a - \hat{a}) \sum_{j=1}^3 \delta_j(\hat{a}) \int_{\beta_j} h^{2-\xi} (y|l_q) \frac{du}{u^{3-\xi}(u^2 + \|\hat{a} - y\|^2)^{\frac{1+\xi}{2}}} dy \\ + \psi_q(\hat{a}) \sum_{j=1}^3 \delta_j(\hat{a}) \int_{\beta_j} h^{2-\xi} (\hat{a} - y) \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2 + \|\hat{a} - y\|^2)^{\frac{1+\xi}{2}}} dy \\ + \sum_{j=1}^3 \delta_j(\hat{a}) \int_{\beta_j} h^{3-\xi} (y|l_q)(\hat{a} - y) \int_h^{+\infty} \frac{du}{u^{4-\xi}(u^2 + \|\hat{a} - y\|^2)^{\frac{1+\xi}{2}}} dy.$$

To simplify the formula, we express the basic functions in this form $(y|l_q) = \psi_q(y) - \psi_q(\hat{a})$ and we delete the terms that vanish, thanks to (94)

$$(65) \quad \begin{aligned} P^{\psi_q}(a, T, \xi) &= (a - \hat{a})\psi_q(\hat{a}) \sum_{j=1}^3 \delta_j(\hat{a}) \int_{\beta_j} f_1(h, \|\hat{a} - y\|, \xi) - f_2(h, \hat{a} - y, \xi) ds_y \\ &+ (a - \hat{a})[\delta_{q+1}(\hat{a}) \int_{\beta_{q+1}} \psi_q(y) f_2(h, \hat{a} - y, \xi) ds_y + \delta_{q+2}(\hat{a}) \int_{\beta_{q+2}} \psi_q(y) f_2(h, \hat{a} - y, \xi) ds_y] \\ &+ \psi_q(\hat{a}) \sum_{j=1}^3 \delta_j(\hat{a}) \int_{\beta_j} (\hat{a} - y)(f_2(h, \|\hat{a} - y\|, \xi) - f_3(h, \hat{a} - y, \xi)) ds_y \\ &+ \delta_{q+1}(\hat{a}) \int_{\beta_{q+1}} \psi_q(y)(\hat{a} - y) f_3(h, \|\hat{a} - y\|, \xi) ds_y + \delta_{q+2}(\hat{a}) \int_{\beta_{q+2}} \psi_q(y)(\hat{a} - y) f_3(h, \hat{a} - y, \xi) ds_y. \end{aligned}$$

otherwise we can write

$$(66) \quad \begin{aligned} P^{\psi_q}(a, T, \xi) &= (a - \hat{a})\psi_q(\hat{a}) \sum_{j=1}^3 \delta_j(\hat{a})(S1(\hat{a}, \beta, h, \xi) - S2(\hat{a}, \beta, h, \xi)) \\ &+ (a - \hat{a})[\delta_{q+1}(\hat{a}) S_2^{\psi_q}(\hat{a}, \beta_{q+1}, h, \xi) + \delta_{q+2}(\hat{a}) S_2^{\psi_q}(\hat{a}, \beta_{q+2}, h, \xi)] \\ &+ \psi_q(\hat{a}) \sum_{j=1}^3 \delta_j(\hat{a}) R_2(\hat{a}, \beta_j, h, \xi) - R_3(\hat{a}, \beta_j, h, \xi) \\ &+ \delta_{q+1}(\hat{a}) D_3^{\psi_q}(\hat{a}, \beta_{q+1}, h, \xi) + \delta_{q+2}(\hat{a}) D_3^{\psi_q}(\hat{a}, \beta_{q+2}, h, \xi), \end{aligned}$$

where

$$(67) \quad D_n^{\psi_q}(a, \beta_p, h, \xi) = \delta_{q+1}(a) \int_{\beta_p} \psi_q(y)(a - y) f_n(h, a - y, \xi) ds_y.$$

- Case a belongs to the support of T

It suffices to consider a as origin to have homogeneity. Since $o = a$ is the new origin, $\psi_q(y)$ can be written

$$\psi_q(y) = \psi_q(a) + (y|l_q).$$

We obtain

$$(68) \quad P^{\psi_q}(a, T, \xi) = \psi_q(a) \int_T \frac{(a - y)}{\|a - y\|^{1+\xi}} dy + \int_T \frac{(y|l_q)(a - y)}{\|a - y\|^{1+\xi}} dy.$$

Integrands are homogeneous of degrees $q = -\xi$ and $q = 1 - \xi$. We apply (18) and we get

$$(69) \quad P^{\psi_q}(a, T, \xi) = \psi_q(a) \sum_{j=1}^3 \frac{\delta_j(a)}{2 - \xi} \int_{\beta_j} \frac{(a - y)}{\|a - y\|^{1+\xi}} ds_y + \sum_{j=1}^3 \frac{\delta_j(a)}{3 - \xi} \int_{\beta_j} \frac{(y|l_q)(a - y)}{\|a - y\|^{1+\xi}} dy.$$

We replace $(y|l_q)$ by $(y|l_q) = \psi_q(y) - \psi_q(a)$, then we get

$$(70) \quad \begin{aligned} P^{\psi_q}(a, T, \xi) &= \psi_q(a) \sum_{j=1}^3 \frac{\delta_j(a)}{2 - \xi} \int_{\beta_j} \frac{(a - y)}{\|a - y\|^{1+\xi}} ds_y \\ &+ \sum_{j=1}^3 \frac{\delta_j(a)}{3 - \xi} \int_{\beta_j} \frac{\psi_q(y)(a - y)}{\|a - y\|^{1+\xi}} ds_y - \psi_q(a) \sum_{j=1}^3 \frac{\delta_j(a)}{3 - \xi} \int_{\beta_j} \frac{(a - y)}{\|a - y\|^{1+\xi}} ds_y, \end{aligned}$$

which is equivalent to

$$(71) \quad P^{\psi_q}(a, T, \xi) = \sum_{j=1}^3 \frac{\delta_j(a)}{3 - \xi} S^{\psi_q}(a, \beta_j, \xi) + \psi_q(a) \sum_{j=1}^3 \left[\frac{\delta_j(a)}{2 - \xi} S(a, \beta_j, \xi) - \frac{\delta_j(a)}{3 - \xi} S(a, \beta_j, \xi) \right].$$

4.2.4 Calculation of $Q^{1, \psi_q}(\alpha, \beta, \xi)$

We want to evaluate the singular integral

$$(72) \quad Q^{1, \psi_q}(\alpha, \beta, \xi) = \int_{\alpha_i \times \beta_j} \frac{\psi_q(y)(x - y)}{\|x - y\|^{1+\xi}} ds_x ds_y,$$

where $\beta \neq \alpha \neq \alpha_{i+2}$, $\beta \neq \beta_{i+2} \neq \beta_q$. We are going to study two different cases.

- Case α and β are neither parallel nor secants

Let $\tilde{\alpha}$ be the projection of α on the plane containing β and parallel to α , \tilde{x} a point in $\tilde{\alpha}$, $h = |\alpha - \tilde{\alpha}|$ and $\tilde{z} = x - \tilde{x}$. We decompose Q^{1, ψ_q} in two parts

$$(73) \quad Q^{1, \psi_q}(\alpha, \beta, \xi) = \tilde{z} \int_{\tilde{\alpha} \times \beta} \frac{\psi_q(y)}{(h^2 + \|\tilde{x} - y\|^2)^{\frac{1+\xi}{2}}} ds_{\tilde{x}} ds_y + \int_{\tilde{\alpha} \times \beta} \frac{\psi_q(y)(\tilde{x} - y)}{(h^2 + \|\tilde{x} - y\|^2)^{\frac{1+\xi}{2}}} ds_{\tilde{x}} ds_y.$$

We will follow the same approach as previously. We consider $o = \tilde{\alpha} \cap \beta$ the new origin and we replace in the equation (73) the basic function $\psi_q(y)$ by the equivalent expression

$$\psi_q(y) = \psi_q(o) + (y|l_q).$$

Once we have homogeneous integrands, the reduction formula (20) gives

$$\begin{aligned} Q^{1, \psi_q}(\alpha, \beta, \xi) &= \psi_q(o) [\sum_{k=\pm} k s^k R_2(\hat{\alpha}^k, \beta, h, \|\hat{\alpha}^k - y\|, \xi) - R_2(b^l, \alpha, h, \|\hat{x} - b^l\|, \xi)] \\ &+ \sum_{k=\pm} k s^k [D_3^{\psi_q}(\hat{\alpha}^k, \beta_j, \|\hat{\alpha}^k - y\|, \xi) - \psi_q(o) R_3(\hat{\alpha}^k, \beta, \|\hat{\alpha}^k - y\|, \xi)] \\ &+ \sum_{l=\pm} l^l [(-\psi_q(b^l) + \psi_q(o)) R_3(b^l, \alpha, \|\hat{x} - b^l\|, \xi)] \\ &+ \tilde{z} \psi_q(o) [\sum_{k=\pm} k s^k S_1(\hat{\alpha}^k, \beta, h, \xi) + \sum_{l=\pm} l^l S_1(b^l, \alpha, h, \xi)] \\ &+ \tilde{z} \sum_{k=\pm} k s^k [S_2^{\psi_q}(\hat{\alpha}^k, \beta, h, \|\hat{\alpha}^k - y\|, \xi) - \psi_q(o) S_2(\hat{\alpha}^k, \beta, h, \|\hat{\alpha}^k - y\|, \xi)] \\ &+ \tilde{z} \sum_{l=\pm} l^l [(\psi_q(b^l) - \psi_q(o)) S_2(b^l, \alpha, h, \|\hat{x} - b^l\|, \xi)]. \end{aligned}$$

- Case α and β are secants

Let $o = \alpha \cap \beta$ be the new origin. The basis function $\psi_q(y)$ can be expressed as following

$$\psi_q(y) = \psi_q(o) + (y|l_q),$$

which gives

$$(74) \quad Q^{1,\psi_q}(\alpha, \beta, \xi) = \psi_q(o) \int_{\alpha \times \beta} \frac{(x-y)}{\|x-y\|^{1+\xi}} ds_x ds_y + \int_{\alpha \times \beta} \frac{(y|l_q)(x-y)}{\|x-y\|^{1+\xi}} ds_x ds_y$$

The integrants are homogeneous of degrees $q = -\xi$ and $q = 1 - \xi$ with $n = 2$.

- * If $q + n \neq 0$ the formula (18) gives

$$(75) \quad \begin{aligned} Q^{1,\psi_q}(\alpha, \beta, \xi) &= \psi_q(o) \left[\sum_{k=\pm} \frac{ks^k}{2} S(a^k, \beta, 0) - \sum_{l=\pm} \frac{lt^l}{2} S(b^l, \alpha, 0) \right] + \sum_{k=\pm} \frac{ks^k}{3} S^{\psi_q(y)}(a^k, \beta, 0) \\ &- \psi_q(o) \sum_{k=\pm} \frac{ks^k}{3} S(a^k, \beta, 0) - \sum_{l=\pm} \frac{lt^l}{3} (\psi_q(b^l) - \psi_q(o)) S(b^l, \alpha, 0), \end{aligned}$$

where

$$(76) \quad S^{\psi_q(y)}(a^k, \beta, 0) = \int_{\beta} \frac{\psi_q(y)(a^k - y)}{\|a^k - y\|} ds_y.$$

- * If $q + n = 0$ it will be appropriate to apply the formula (21), which leads us to this expression

$$(77) \quad \begin{aligned} Q^{1,\psi_q}(\alpha, \beta, \xi) &= \psi_q(o) \left[\sum_{k=\pm} ks^k L(a^k, \beta) - \sum_{l=\pm} lt^l L(b^l, \alpha) \right] + \sum_{k=\pm} ks^k L^{\psi_q}(a^k, \beta) \\ &- \sum_{k=\pm} ks^k \psi_q(o) L(a^k, \beta) - \sum_{l=\pm} lt^l (\psi_q(b^l) - \psi_q(o)) L(b^l, \alpha), \end{aligned}$$

where

$$(78) \quad L^{\psi_q}(a^k, \beta) = \int_{\beta} \frac{\psi_q(y)(a^k - y)}{\|a^k - y\|^3} ds_y.$$

4.2.5 Calculation of $Q^{\phi_p, \psi_q}(\alpha, \beta, \xi)$

Here we deal with the singular integral

$$(79) \quad Q^{\phi_p, \psi_q}(\alpha, \beta, \xi) = \int_{\alpha \times \beta} \frac{\phi_p(x) \psi_q(y) (y-x)}{\|x-y\|^{1+\xi}} ds_x ds_y.$$

- Intersecting triangles

We project α on the plan parallel to α and going through β . Let $\hat{\alpha}$ be the projection of α and \hat{x} the projection of x . Then, we provide an equivalent expression of the basis functions:

$$(80) \quad \phi_p(x) = 1 + \mu \frac{(x - a_p | \hat{\alpha})}{|\alpha|^2} = 1 + \mu \frac{(\hat{x} - \hat{a}_p | \hat{\alpha})}{|\alpha|^2},$$

$$(81) \quad \psi_q(y) = 1 + \nu \frac{(y - b_q | \vec{\beta})}{|\beta|^2},$$

where $\mu = \pm 1$ ($\mu = 1$ if $a_p = a^+$ and $\mu = -1$ if $a_p = a^-$). Likewise, $\nu = \pm 1$ ($\nu = 1$ if $b_p = b^+$ and $\nu = -1$ if $b_p = b^-$). We choose the intersection of the supports of $\hat{\alpha}$ and β , as origin and we note it o . we obtain :

$$(82) \quad \begin{aligned} \phi_p(x) &= 1 + \mu \frac{(o - \hat{a}_p | \hat{\alpha})}{|\alpha|^2} + \mu \frac{(\hat{x} - o | \hat{\alpha})}{|\alpha|^2} \\ &= C_p + \mu \frac{(\hat{x} - o)}{\hat{\alpha}}, \end{aligned}$$

$$(83) \quad \begin{aligned} \psi_q(y) &= 1 + \nu \frac{(o - b_q | \vec{\beta})}{|\beta|^2} + \nu \frac{(y - o | \vec{\beta})}{|\beta|^2} \\ &= C_q + \mu \frac{(y - o | \vec{\beta})}{|\beta|^2} \end{aligned}$$

where $C_p = 1 + \mu \frac{(o - \hat{a}_p | \hat{\alpha})}{|\alpha|^2}$ and $C_q = 1 + \nu \frac{(o - b_q | \vec{\beta})}{|\beta|^2}$. Then the expression of Q^{ϕ_p, ψ_q} becomes

$$\begin{aligned}
(84) \quad Q^{\phi_p, \psi_q}(\alpha, \beta, \xi) &= C_q C_q \int_{\hat{\alpha} \times \beta} \frac{(\hat{x} - y)}{(h^2 + \|\hat{x} - y\|^2)^{\frac{1+\xi}{2}}} ds_{\hat{x}} ds_y + \frac{C_p \nu}{|\beta|^2} \int_{\hat{\alpha} \times \beta} \frac{(y - o|\vec{\beta})(\hat{x} - y)}{(h^2 + \|\hat{x} - y\|^2)^{\frac{1+\xi}{2}}} ds_{\hat{x}} ds_y \\
&+ \frac{C_q \mu}{|\alpha|^2} \int_{\hat{\alpha} \times \beta} \frac{(\hat{x} - o|\vec{\alpha})(\hat{x} - y)}{(h^2 + \|\hat{x} - y\|^2)^{\frac{1+\xi}{2}}} ds_{\hat{x}} ds_y + \frac{\mu \mu}{|\hat{\alpha}|^2 |\beta|^2} \int_{\hat{\alpha} \times \beta} \frac{(y - o|\vec{\beta})(\hat{x} - o|\vec{\alpha})(\hat{x} - y)}{(h^2 + \|\hat{x} - y\|^2)^{\frac{1+\xi}{2}}} ds_{\hat{x}} ds_y.
\end{aligned}$$

In the formula below the integrands are homogeneous with variable h and of degrees $q = -\xi$, $q = 1 - \xi$, $1 - \xi$ and $2 - \xi$. Formula (20) gives

$$\begin{aligned}
(85) \quad Q^{\phi_p, \psi_q}(\alpha, \beta, \xi) &= C_p C_q [\sum_{k=\pm} k \hat{s}^k \int_{\beta} (\hat{a}^k - y) h^{2-\xi} \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2 + \|\hat{a}^k - y\|^2)^{\frac{1+\xi}{2}}} ds_y \\
&+ \sum_{l=\pm} l \hat{t}^l \int_{\hat{\alpha}} (\hat{x} - b^l) h^{2-\xi} \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2 + \|\hat{x} - b^l\|^2)^{\frac{1+\xi}{2}}} ds_x] \\
&+ \frac{C_p \nu}{|\beta|^2} [\sum_{k=\pm} k \hat{s}^k \int_{\beta} (y - o|\vec{\beta})(\hat{a}^k - y) h^{3-\xi} \int_h^{+\infty} \frac{du}{u^{4-\xi}(u^2 + \|\hat{a}^k - y\|^2)^{\frac{1+\xi}{2}}} ds_y \\
&+ \sum_{l=\pm} l \hat{t}^l \int_{\hat{\alpha}} (b^l - o|\vec{\alpha})(\hat{x} - b^l) h^{3-\xi} \int_h^{+\infty} \frac{du}{u^{4-\xi}(u^2 + \|\hat{x} - b^l\|^2)^{\frac{1+\xi}{2}}} ds_x] \\
&+ \frac{C_q \mu}{|\alpha|^2} [\sum_{k=\pm} k \hat{s}^k \int_{\beta} (\hat{a}^k - o|\vec{\alpha})(\hat{a}^k - y) h^{3-\xi} \int_h^{+\infty} \frac{du}{u^{4-\xi}(u^2 + \|\hat{a}^k - y\|^2)^{\frac{1+\xi}{2}}} ds_y \\
&+ \sum_{l=\pm} l \hat{t}^l \int_{\hat{\alpha}} (\hat{x} - o|\vec{\alpha})(\hat{x} - b^l) h^{3-\xi} \int_h^{+\infty} \frac{du}{u^{4-\xi}(u^2 + \|\hat{x} - b^l\|^2)^{\frac{1+\xi}{2}}} ds_x] \\
&+ \frac{\nu \mu}{|\alpha|^2 |\beta|^2} [\sum_{k=\pm} k \hat{s}^k \int_{\beta} (\hat{a}^k - o|\vec{\alpha})(y - o|\vec{\beta})(\hat{a}^k - y) h^{4-\xi} \int_h^{+\infty} \frac{du}{u^{5-\xi}(u^2 + \|\hat{a}^k - y\|^2)^{\frac{1+\xi}{2}}} ds_y \\
&+ \sum_{l=\pm} l \hat{t}^l \int_{\hat{\alpha}} (\hat{x} - o|\vec{\alpha})(b^l - o|\vec{\alpha})(\hat{x} - b^l) h^{4-\xi} \int_h^{+\infty} \frac{du}{u^{5-\xi}(u^2 + \|\hat{x} - b^l\|^2)^{\frac{1+\xi}{2}}} ds_x],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
(86) \quad Q^{\phi_p, \psi_q}(\alpha, \beta, \xi) &= C_p C_q [\sum_{k=\pm} k \hat{s}^k R_2(\hat{a}^k, \beta, h, \xi) \xi - \sum_{l=\pm} l \hat{t}^l R_2(b^l, \hat{\alpha}, h, \xi)] \\
&+ \sum_{k=\pm} k \hat{s}^k [\frac{C_q \mu}{|\alpha|^2} (\hat{a}^k - o|\vec{\alpha}) R_3(\hat{a}^k, \beta, h, \xi) + \frac{C_p \nu}{|\beta|^2} M_3(\hat{a}^k, o, \beta, h, \xi)] \\
&+ \sum_{l=\pm} l \hat{t}^l [-\frac{C_p \nu}{|\beta|^2} (b^l - o|\vec{\beta}) R_3(b^l, \hat{\alpha}, h, \xi) - \frac{C_q \mu}{|\alpha|^2} M_3(b^l, o, \alpha, h, \xi)] \\
&+ \frac{\nu \mu}{|\alpha|^2 |\beta|^2} [\sum_{k=\pm} k \hat{s}^k (\hat{a}^k - o|\vec{\alpha}) M_4(\hat{a}^k, o, \beta, h, \xi) - \sum_{l=\pm} l \hat{t}^l (b^l - o|\vec{\beta}) M_4(b^l, o, \alpha, h, \xi)].
\end{aligned}$$

where

$$(87) \quad M_3(\hat{a}^k, o, \beta, h, \xi) = \int_{\beta} (\hat{a}^k - o|\vec{\alpha})(\hat{a}^k - y) f_3(h, \|\hat{a}^k - y\|, \xi) ds_y.$$

$$(88) \quad M_4(\hat{a}^k, o, \beta, h, \xi) = \int_{\beta} (a - o|\vec{\alpha})(a - y) f_4(h, \|\hat{a}^k - y\|, \xi) ds_y.$$

- Coplanar triangles

As we saw in section (4.2.2), we need to distinguish two geometrical configurations for the supports of the sides α and β .

Two intersecting segments

When the supports of α and β are intersecting, we choose the intersection of the supports, denoted o , as the origin. Then we use the same decompositions as (82) and (83).

$$(89) \quad \phi_p(y) = C_p + \mu \frac{(x - o|\vec{\alpha})}{|\alpha|^2} \quad \psi_q(y) = C_q + \mu \frac{(y - o|\vec{\beta})}{|\beta|^2}.$$

In case $\xi - 2 \neq 0$:

If we combine (89) with (18) we obtain

$$\begin{aligned}
(90) \quad Q^{\phi_p, \psi_q}(\alpha, \beta, 0) &= \frac{C_p C_q}{2} [\sum_{k=\pm} k \hat{s}^k S(a^k, \beta, 0) - \sum_{l=\pm} l \hat{t}^l S(b^l, \alpha, 0)] \\
&+ \frac{C_p \mu}{3|\beta|^2} [\sum_{k=\pm} k \hat{s}^k N(a^k, o, \beta, 0) - \sum_{l=\pm} l \hat{t}^l (b^l - o|\vec{\beta}) S(b^l, \alpha, 0)] \\
&+ \frac{C_q \nu}{3|\alpha|^2} [\sum_{k=\pm} k \hat{s}^k (a^k - o|\vec{\alpha}) S(a^k, \beta, 0) - \sum_{l=\pm} l \hat{t}^l N(b^k, o, \alpha, 0)] \\
&+ \frac{\nu \mu}{4|\alpha|^2 |\beta|^2} \sum_{k=\pm} k \hat{s}^k [(a^k - o|\vec{\alpha}) N(a^k, o, \beta, 0) - \sum_{l=\pm} l \hat{t}^l (b^l - o|\vec{\beta}) N(b^k, o, \alpha, 0)].
\end{aligned}$$

where

$$(91) \quad N(a^k, o, \beta, \xi) = \int_{\beta} \frac{(y - o|\vec{\beta})(a^k - y)}{\|a^k - y\|^{1+\xi}} ds_y.$$

5 Numerical application

5.1 Calculation of the Radar Cross Section

In radiation theory it's well known that the energy intercepted by an object can be reflected, transmitted or absorbed through the target. We can assume that most of the energy is reflected. The spatial distribution of this energy depends on the size, shape and composition of the target, and on the frequency and nature of the incident wave. This distribution of energy is called scattering, and the target itself is often referred to as a scatterer. The radar cross section (RCS) of the body is a measure of the energy scattered in a particular direction for a given illumination [15].

Bistatic scattering is the name given to the situation when the scattering direction is not back toward the source of the radiation. If \mathbf{E} and \mathbf{H} represent fields scattered by an object illuminated by incident plane wave \mathbf{E}^{inc} traveling in the direction of the unit vector \mathbf{k} , the *bistatic radar cross section* in the observation direction \mathbf{r} is

$$\sigma(\mathbf{r}, \mathbf{k}) = \lim_{r \rightarrow \infty} 4\pi r^2 \frac{|\mathbf{E}|^2}{|\mathbf{E}^{inc}|^2}.$$

This cross section is defined as the area through which an incident plane wave carries sufficient power to produce, by omnidirectional radiation, the same scattered power density as that observed in a given far field direction. The *monostatic radar cross section* is defined as the radar cross section observed in the back scattering direction, $\sigma(-\mathbf{k}, \mathbf{k})$.

Radar cross section is the measure of a target's ability to reflect radar signals in the direction of the radar receiver, i.e. it is a measure of the ratio of backscatter power per steradian (unit solid angle) in the direction of the radar (from the target) to the power density that is intercepted by the target.

The radar cross section of a target can be viewed as a comparison of the strength of the reflected signal from a target to the reflected signal from a perfectly smooth sphere.

The units for RCS are square meters. As RCS can span a wide range of values, a logarithmic decibel scale is also used with a typical reference value σ_{ref} equal to $1m^2$:

$$(92) \quad \sigma_{dBm^2} = 10 \log_{10}\left(\frac{\sigma}{\sigma_{ref}}\right)$$

5.2 Numerical results

To illustrate our approach, we compute the RCS of a unit sphere coated with a thin dielectric layer. The radius of the inner conductor is denoted by r and the thickness of the coating by d . The material properties are described by a Leontovich Impedance $Z = const.$

$$Z = z_0 \sqrt{\frac{\mu_r}{\epsilon_r}} \tan(\sqrt{\mu_r \epsilon_r} k_0 d)$$

where μ_r and ϵ_r denote the material properties. We test it on a sphere with radius $1m$ and incident wave at $f = 0.14313GHz$.

In figure (6), we compare our code results in different configurations. The continuous line is the Mie Series, the red line is UCP code1, the blue line is UCP code2, and the black line is UCP code3.

In the second test (figure 7), we use the same frequency and sphere radius, but with a different coating thickness $d = 0.0125m$. The red line is the Mie Series, the blue line is UCP code1, the black line is UCP code2, and the green line is UCP code3.

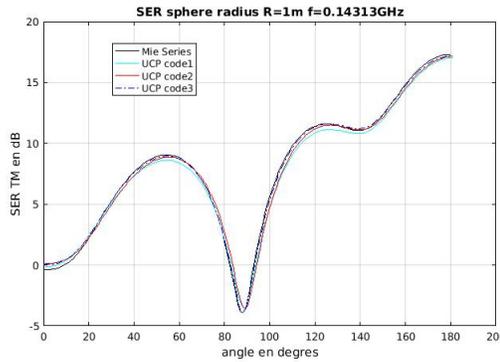


Figure 6: Bistatic RCS for a coated unit sphere, when $d = 0.05$, $r = 0.95$, $\mu_r = 1.0$, $\epsilon_r = 2$ and $\omega = 0.9GHz$

the coating thickness is $d = 0.0125m$.

In figure 7, we use the same frequency and sphere radius, but with a different coating thickness $d = 0.0125m$. The red line is the Mie Series, the blue line is UCP code1, the black line is UCP code2, and the green line is UCP code3.

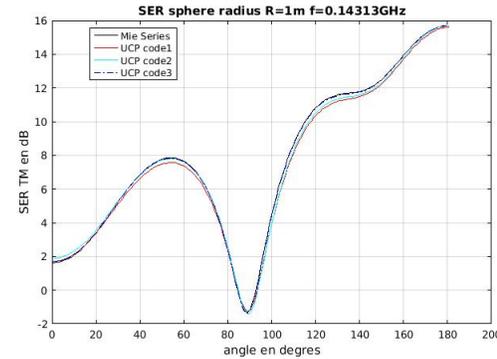


Figure 7: Bistatic RCS for a coated unit sphere, when $d = 0.0125$, $\mu_r = 1.0$, $\epsilon_r = 2$ and $\omega = 0.9GHz$

In figure 8, we consider a coating thickness $d = 0.125$, a frequency $f = 0.14313GHz$, the same constant for the magnetic permeability $\mu_r = 1.0$ and electrical permittivity $\epsilon_r = 2073.42$. In figure (8), we plot the exact RCS calculated by Mie Series (continuous red line) and the RCS calculated with our code (dashed black line).

6 Appendix

6.1 Barycentric basis function

We begin by presenting the basic functions used and some of their properties that will simplify calculations in the next section. Let S be the triangle of vertices a_i and side α_i . We introduce the basic functions ϕ_p associated with

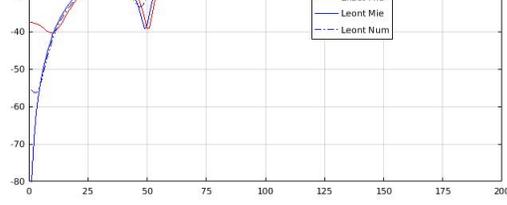


Figure 8: Bistattis RCS for a coated unit sphere

the vertices of the triangle S :

$$(93) \quad \phi_i(x) = 1 + (\overrightarrow{x - a_i} | \vec{e}_i^{\rightarrow}); i = 1, 2, 3$$

where

$$\vec{e}_i^{\rightarrow} = \mu_1 \overrightarrow{\alpha_{i+1}} + \mu_2 \overrightarrow{\alpha_{i+2}}$$

with

$$\mu_1 = \frac{\|\overrightarrow{\alpha_{i+2}}\|^2 + (\overrightarrow{\alpha_{i+1}} | \overrightarrow{\alpha_{i+2}})}{\|\overrightarrow{\alpha_{i+2}}\|^2 \|\overrightarrow{\alpha_{i+1}}\|^2 - (\overrightarrow{\alpha_{i+1}} | \overrightarrow{\alpha_{i+2}})^2}$$

$$\mu_2 = -\frac{\|\overrightarrow{\alpha_{i+1}}\|^2 + (\overrightarrow{\alpha_{i+1}} | \overrightarrow{\alpha_{i+2}})}{\|\overrightarrow{\alpha_{i+2}}\|^2 \|\overrightarrow{\alpha_{i+1}}\|^2 - (\overrightarrow{\alpha_{i+1}} | \overrightarrow{\alpha_{i+2}})^2}$$

The basis function ϕ_i verify $\phi_i(a_i) = 1$, $\phi_i(a_{i+1}) = \phi_i(a_{i+2}) = 0$ besides

$$(94) \quad \phi_i|_{\alpha_i} = 0, \phi_i(x)|_{\alpha_{i+1}} = 1 - \frac{x - a_i}{|\alpha_{i+1}|}; \phi_i(x)|_{\alpha_{i+2}} = 1 - \frac{x - a_i}{|\alpha_{i+2}|}.$$

6.2 Calcul of $S(a, \beta, \xi)$

$$(95) \quad S(a, \beta, \xi) = \int_{\beta} \frac{a - y}{\|a - y\|^{1+\xi}} ds_y$$

Let $q(a)$ be the projection of a on the support of β and $d = \|q(a) - a\|$. Then

$$(96) \quad S(a, \beta, \xi) = (a - q(a)) \int_{t^-}^{t^+} \frac{1}{(d^2 + s^2)^{\frac{1+\xi}{2}}} ds + \int_{t^-}^{t^+} \frac{s}{(d^2 + s^2)^{\frac{1+\xi}{2}}} ds$$

where s is the abscissa on β which will be calculated as follows $s = y - q(a)$ and t^{\pm} where t^{\pm} are abscisses of the extremities of beta with $q(a)$ origin and $d = q(a) - a$.

6.3 Calcul of S_1, S_2 and S_3

- $S_1(a, \beta, h, \xi)$

$$(97) \quad S_1(a, \beta, h, \xi) = \int_{\beta} f_1(h, \|a - y\|, \xi) ds_y$$

$$= \int_{\beta} h^{1-\xi} \int_h^{+\infty} \frac{du}{u^{2-\xi}(u^2 + \|a - y\|^2)^{\frac{1+\xi}{2}}} ds_y$$

where $f_n(h, s, \xi)$ were be defined in 48. Let $q(a)$ be the projection of a into the support of β , then

$$(98) \quad S_1(a, \beta, h, \xi) = \int_{t^-}^{t^+} h^{1-\xi} \int_h^{+\infty} \frac{du}{u^{2-\xi}(u^2 + d^2 + s^2)^{\frac{1+\xi}{2}}} ds_y$$

where t^{\pm} are abscisses of the extremities of beta with $q(a)$ origin and $d = q(a) - a$.
if $\xi = 0$

$$(99) \quad S_1(a, \beta, h, 0) = \int_{t^-}^{t^+} h^1 \int_h^{+\infty} \frac{du}{u^2(u^2 + d^2 + s^2)^{\frac{1}{2}}} ds_y$$

$$= \int_{t^-}^{t^+} \frac{\sqrt{h^2 + d^2 + s^2} - h}{d^2 + s^2} ds$$

$$= \int_{t^-}^{t^+} \frac{1}{\sqrt{h^2 + d^2 + s^2} - h} ds$$

$$(100) \quad S_1(a, \beta, h, 0) = \left[\frac{h}{s} - \frac{\sqrt{h^2 + s^2}}{s} + \lg(s + \sqrt{h^2 + s^2}) \right]_{t^-}^{t^+}$$

for $\xi = 2$

$$\begin{aligned}
 S_1(a, \beta, h, 2) &= \int_{t^-}^{t^+} h^{-1} \int_h^{+\infty} \frac{du}{(u^2 + d^2 + s^2)^{\frac{3}{2}}} ds_y \\
 &= \int_{t^-}^{t^+} \frac{1}{h(d^2 + s^2)} \left(1 - \frac{h}{\sqrt{h^2 + d^2 + s^2}}\right) ds \\
 (101) \quad &= \left[\frac{1}{dh} \arctan\left(\frac{s}{d}\right) - \frac{1}{dh} \arctan\left(\frac{hs}{d\sqrt{h^2 + d^2 + s^2}}\right) \right]_{t^-}^{t^+}
 \end{aligned}$$

• $S_2(a, \beta, h, \xi)$

$$\begin{aligned}
 S_2(a, \beta, h, \xi) &= \int_{\beta} f_2(h, \|a - y\|, \xi) ds_y \\
 (102) \quad &= \int_{\beta} h^{2-\xi} \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2 + \|a - y\|^2)^{\frac{1+\xi}{2}}} ds_y
 \end{aligned}$$

$$(103) \quad S_2(a, \beta, h, \xi) = \int_{t^-}^{t^+} h^{2-\xi} \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2 + d^2 + s^2)^{\frac{1+\xi}{2}}} ds$$

if $\xi = 0$

$$\begin{aligned}
 S_2(a, \beta, h, 0) &= \int_{t^-}^{t^+} h^2 \int_h^{+\infty} \frac{du}{u^3(u^2 + d^2 + s^2)^{\frac{1}{2}}} ds \\
 (104) \quad &= \left[\frac{h^2 + d^2}{2d^2} \operatorname{arcsinh}\left(\frac{s}{\sqrt{h^2 + d^2}}\right) - \frac{sh^2}{2d^2\sqrt{s^2 + d^2}} \operatorname{arcsinh}\left(\frac{\sqrt{d^2 + s^2}}{h}\right) \right]_{t^-}^{t^+}
 \end{aligned}$$

for $\xi = 2$

$$\begin{aligned}
 S_2(a, \beta, h, 2) &= \int_{t^-}^{t^+} \int_h^{+\infty} \frac{du}{u(u^2 + d^2 + s^2)^{\frac{3}{2}}} ds \\
 &= \int_{t^-}^{t^+} -\frac{1}{(d^2 + s^2)\sqrt{h^2 + d^2 + s^2}} + \frac{1}{(d^2 + s^2)^{\frac{3}{2}}} \operatorname{arcsinh}\left(\frac{\sqrt{d^2 + s^2}}{h}\right) ds \\
 (105) \quad &= \left[\frac{s}{d^2(\sqrt{d^2 + s^2})} \operatorname{arcsinh}\left(\frac{\sqrt{d^2 + s^2}}{h}\right) - \frac{1}{d^2} \log(s + \sqrt{d^2 + h^2 + s^2}) \right]_{t^-}^{t^+}
 \end{aligned}$$

• $S_3(a, \beta, h, \xi)$

$$\begin{aligned}
 S_3(a, \beta, h, \xi) &= \int_{\beta} f_3(h, \|a - y\|, \xi) ds_y \\
 (106) \quad S_3(a, \beta, h, \xi) &= \int_{t^-}^{t^+} h^{3-\xi} \int_h^{+\infty} \frac{du}{u^{4-\xi}(u^2 + d^2 + s^2)^{\frac{1+\xi}{2}}} ds
 \end{aligned}$$

if $\xi = 0$

$$\begin{aligned}
 S_3(a, \beta, h, 0) &= \int_{t^-}^{t^+} \frac{2h^3 + (\sqrt{d^2 + h^2 + s^2})(d^2 + s^2 - 2h^2)}{3(d^2 + s^2)^2} ds \\
 &= \frac{1}{3} \left[\frac{h^3 s - h^2 s \sqrt{d^2 + h^2 + s^2}}{d^2(d^2 + s^2)} + \frac{h^3 \arctan\left(\frac{s}{d}\right)}{d^3} \right. \\
 (107) \quad &\quad \left. - \frac{h^3}{d^3} \arctan\left(\frac{hs}{d\sqrt{d^2 + h^2 + s^2}}\right) + \lg(s + \sqrt{d^2 + h^2 + s^2}) \right]
 \end{aligned}$$

if $\xi = 2$

$$\begin{aligned}
 S_3(a, \beta, h, 2) &= \int_{t^-}^{t^+} \frac{d^2 + s^2 + 2h^2(h - \sqrt{d^2 + h^2 + s^2})}{(d^2 + s^2)^2 \sqrt{d^2 + h^2 + s^2}} ds \\
 (108) \quad &= \left[\frac{s(-h + s\sqrt{d^2 + h^2 + s^2})}{d^2(d^2 + s^2)} - \frac{h}{d^3} \left(\arctan\left(\frac{s}{d}\right) - \arctan\left(\frac{hs}{d\sqrt{d^2 + h^2 + s^2}}\right) \right) \right]
 \end{aligned}$$

6.4 Calculation of T_1, T_2 and T_3

- $T_1(a, \beta, h, \xi)$

$$(109) \quad T_1(a, \beta, h, \xi) = \int_{t^-}^{t^+} s f_1(h, \sqrt{d^2 + s^2}, \xi) ds$$

if $\xi = 0$

$$T_1(a, \beta, h, 0) = \int_{t^-}^{t^+} sh \int_h^{+\infty} \frac{du}{u^2(u^2 + ||a - y||^2)^{\frac{1}{2}}} ds$$

if $\xi = 2$

$$T_1(a, \beta, h, 2) = \int_{t^-}^{t^+} sh^{-1} \int_h^{+\infty} \frac{du}{(u^2 + ||a - y||^2)^{\frac{3}{2}}} ds$$

- $T_2(a, \beta, h, \xi)$

$$(110) \quad T_2(a, \beta, h, \xi) = \int_{t^-}^{t^+} s f_2(h, \sqrt{d^2 + s^2}, \xi) ds$$

if $\xi = 0$

$$\begin{aligned} T_1(a, \beta, h, 0) &= \int_{t^-}^{t^+} sh^2 \int_h^{+\infty} \frac{du}{u^3(u^2 + ||a - y||^2)^{\frac{1}{2}}} ds_y \\ &= \int_{t^-}^{t^+} \frac{s\sqrt{d^2 + h^2 + s^2}}{2(d^2 + s^2)} - \frac{sh^2}{2(d^2 + s^2)^{\frac{3}{2}}} \operatorname{arcsinh}\left(\frac{\sqrt{d^2 + s^2}}{h}\right) ds \end{aligned}$$

$$(111) \quad = \left[\frac{\sqrt{d^2 + h^2 + s^2}}{2} + \frac{h^2}{2\sqrt{d^2 + s^2}} \operatorname{arcsinh}\left(\frac{\sqrt{d^2 + s^2}}{h}\right) \right]_{t^-}^{t^+}$$

if $\xi = 2$

$$\begin{aligned} T_1(a, \beta, h, 2) &= \int_{t^-}^{t^+} s \int_h^{+\infty} \frac{du}{u(u^2 + ||a - y||^2)^{\frac{3}{2}}} ds \\ &= \int_{t^-}^{t^+} -\frac{s}{(d^2 + s^2)\sqrt{d^2 + h^2 + s^2}} + \frac{s}{(d^2 + s^2)^{\frac{3}{2}}} \operatorname{arcsinh}\left(\frac{\sqrt{d^2 + s^2}}{h}\right) ds \end{aligned}$$

$$(112) \quad \left[-\frac{1}{\sqrt{d^2 + s^2}} \operatorname{arcsinh}\left(\frac{\sqrt{d^2 + s^2}}{h}\right) \right]_{t^-}^{t^+}$$

- $T_3(a, \beta, h, \xi)$

$$(113) \quad T_3(a, \beta, h, \xi) = \int_{t^-}^{t^+} s f_3(h, \sqrt{d^2 + s^2}, \xi) ds$$

if $\xi = 0$

$$T_3(a, \beta, h, 0) = \int_{t^-}^{t^+} \frac{s}{3(d^2 + s^2)^2} (\sqrt{d^2 + h^2 + s^2}(d^2 + s^2 - 2h^2) + 2h^3) ds$$

$$(114) \quad \left[\frac{-h^3 + (d^2 + h^2 + s^2)^{\frac{3}{2}}}{3(d^2 + s^2)} \right]_{t^-}^{t^+}$$

if $\xi = 2$

$$T_3(a, \beta, h, 2) = \int_{t^-}^{t^+} \frac{sd^2 + s^3 + 2sh(h - \sqrt{d^2 + h^2 + s^2})}{\sqrt{d^2 + h^2 + s^2}(d^2 + s^2)^2}$$

$$(115) \quad \left[\frac{h - \sqrt{d^2 + h^2 + s^2}}{d^2 + s^2} \right]_{t^-}^{t^+}$$

6.5 Calcul of R_1 , R_2 and R_3

- $R_1(a, \beta, h, \xi)$

$$(116) \quad \begin{aligned} R_1(a, \beta, h, \xi) &= \int_{\beta} (a-y) f_1(h, \|a-y\|, \xi) \\ &= \int_{\beta} (a-y) h^{1-\xi} \int_h^{+\infty} \frac{du}{u^{2-\xi}(u^2+\|a-y\|^2)^{\frac{1+\xi}{2}}} ds_y \end{aligned}$$

- $R_2(a, \beta, h, \xi)$

$$(117) \quad \begin{aligned} R_2(a, \beta, h, \xi) &= \int_{\beta} (a-y) f_2(h, \|a-y\|, \xi) ds_y \\ &= (a-q(a)) \int_{\beta} f_2(h, \|a-y\|, \xi) ds_y + \int_{\beta} (q(a)-y) f_2(h, \|a-y\|, \xi) ds_y \\ &= (a-q(a)) \int_{t^-}^{t^+} h^{2-\xi} \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2+d^2+s^2)^{\frac{1+\xi}{2}}} ds_y - \frac{\vec{\beta}}{|\beta|} \int_{t^-}^{t^+} sh^{2-\xi} \int_h^{+\infty} \frac{du}{u^{3-\xi}(u^2+d^2+s^2)^{\frac{1+\xi}{2}}} ds_y \end{aligned}$$

wich is equivalent to

$$(118) \quad R_2(a, \beta, h, \xi) = (a-q(a))S_2(a, \beta, h, \xi) - \frac{\vec{\beta}}{|\beta|}T_2(a, \beta, h, \xi)$$

- $R_3(a, \beta, h, \xi)$

$$(119) \quad \begin{aligned} R_3(a, \beta, h, \xi) &= \int_{\beta} (a-y) f_3(h, \|a-y\|, \xi) ds_y \\ &= (a-q(a)) \int_{\beta} f_3(h, \|a-y\|, \xi) ds_y + \int_{\beta} (q(a)-y) f_3(h, \|a-y\|, \xi) ds_y \\ &= (a-q(a)) \int_{t^-}^{t^+} h^{3-\xi} \int_h^{+\infty} \frac{du}{u^{4-\xi}(u^2+d^2+s^2)^{\frac{1+\xi}{2}}} ds - \frac{\vec{\beta}}{|\beta|} \int_{t^-}^{t^+} sh^{3-\xi} \int_h^{+\infty} \frac{du}{u^{4-\xi}(u^2+d^2+s^2)^{\frac{1+\xi}{2}}} ds \end{aligned}$$

wich is equivalent to

$$(120) \quad R_3(a, \beta, h, \xi) = (a-q(a))S_3(a, \beta, h, \xi) - \frac{\vec{\beta}}{|\beta|}T_3(a, \beta, h, \xi)$$

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