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Low-power peaking-free high-gain observers for nonlinear systems

Daniele Astolfi^a, Lorenzo Marconi^b and Andrew Teel^c

Abstract—In this note the low-power observer structure presented in [1] is modified to avoid peaking of the observer state while preserving its good sensitivity properties to measurement noise and the implementation advantages in terms of power-gain with respect to standard high-gain observers.

I. INTRODUCTION

High-gain observers have been extensively used in nonlinear control (see, for instance, the survey [8] and the references therein) for their *tunability* property, namely the fact that the rate of convergence of the observer can be tuned. This important feature is motivated by the use of observers in output feedback control and it has been proved (see, among the others, the milestones [3], [12]) that this tunability property plays a key role in establishing a nonlinear separation principle. However, the standard high-gain paradigm has many problems when used in real applications. Mainly:

- 1) the maximum gain to implement is increasing polynomially with the system dimension n , *i.e.* we need to implement a term ℓ^n , where ℓ is the high-gain parameter. When nonlinear systems are considered, the value of the gain has to be taken large enough to dominate the nonlinear terms and, therefore, if the Lipschitz constant is very large, or the system dimension is high, the term ℓ^n can be incredibly large;
- 2) high-gain observers are typically characterized by high sensitivity to high-frequency measurement noise by thus making their use practically impossible in a realistic noisy environment;
- 3) during the transient the variables present a peaking phenomenon which grows polynomially in ℓ , *i.e.* the value of the variables have an order of magnitude $O(\ell^{n-1})$.

Many works have been published trying to overcome some of the above points. Among the others, the works [2], [9], [10] presented different solutions to deal with the peaking phenomenon. Nevertheless it is worth noting that these techniques are quite complicated to implement. More recently, the work presented in [11] solves the peaking problem with a “nested-saturation” design.

A recent publication [1] presented a new “low-power” high-gain observer substantially overtaking the first two

problems. In particular the structure of the new observers is implemented by only using ℓ and ℓ^2 no matter the dimension n of the system is. Furthermore, the relative degree between the measure and the estimates of the new structure is higher than the ones of classical high-gain observers, resulting in a remarkable improvement of the sensitivity to high-frequency noise. The only price to pay is that the dimension of the new observer is $2n - 2$ and no more n . Furthermore, it can be easily proved that during the transient the values of the variables peak with an order of magnitude $O(\ell^{n-1})$.

The aim of this work is to present a modified version of the observer [1] which solves the three problems presented above and which is easy to implement and tune. The observer [1] is modified by adding saturations at various steps. The new structure preserves the benefits of the low-power observer presented in [1]. Furthermore, the values of the internal variables present a peaking whose order of magnitude is only $O(\ell)$. This, from a practical point of view, is acceptable because the gains ℓ and ℓ^2 need to be implemented and therefore internal variables of order $O(\ell)$ are numerically *compatible*. We observe that there are common aspects between the proposed observer design and the design in [11], though the analysis in [11] is complicated by the nested saturation structure. Furthermore, while the work in [11] addresses the problem of peaking only indirectly (indeed, it was not the focus of [11]), in this work we provide a rigorous peaking analysis.

A simulation result completes the presentation of the new observer, confirming the benefits of the proposed structure with respect to the observer proposed in [1] and with respect to standard high-gain observers. Finally a new constructive Lemma (different from the one proposed in [1]) for the choice of the parameters is given in the Appendix.

Notation

We denote with \mathbb{R} the set of real numbers. We denote with \mathbb{N} , the set of positive integer *i.e.* $\mathbb{N} = \{0, 1, \dots, \infty\}$. Throughout the text, the variables $i \in \mathbb{N}$, $j \in \mathbb{N}$ are used as positive integer indexes. The matrices $A \in \mathbb{R}^{2 \times 2}$, $B \in \mathbb{R}^{2 \times 1}$, $B_i \in \mathbb{R}^{i \times 1}$, $C \in \mathbb{R}^{1 \times 2}$ and $D_2(\ell) \in \mathbb{R}^{2 \times 2}$ are defined as

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C := (1 \quad 0),$$

$$B_i := (0 \quad 0 \quad \dots \quad 0 \quad 1)^\top, \quad D_2(\ell) := \begin{pmatrix} \ell & 0 \\ 0 & \ell^2 \end{pmatrix},$$

where $\ell \in \mathbb{R}$. Given the coefficients $(k_{i1}, k_{i2}) \in \mathbb{R}^2$, $i > 0$ we define the following matrices $K_i \in \mathbb{R}^{2 \times 1}$, $E_i \in \mathbb{R}^{2 \times 2}$, $Q_i \in \mathbb{R}^{2 \times 2}$ and $N \in \mathbb{R}^{2 \times 2}$ as

$$K_i := \begin{pmatrix} k_{i1} \\ k_{i2} \end{pmatrix}, \quad E_i := A - K_i C = \begin{pmatrix} -k_{i1} & 1 \\ -k_{i2} & 0 \end{pmatrix},$$

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$$Q_i := K_i B^\top = \begin{pmatrix} 0 & k_{i1} \\ 0 & k_{i2} \end{pmatrix}, \quad N := B B^\top = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We define in a recursive manner the matrices M_i as

$$M_1 := E_1, \quad M_i := \begin{pmatrix} M_{i-1} & \bar{N}_i \\ \bar{Q}_i & E_i \end{pmatrix}, \quad i > 1,$$

with $\bar{N}_i := B_{2(i-1)} B^\top$, $\bar{Q}_i := K_i B_{2(i-1)}^\top$. The matrices M_i are characterized by a block-tridiagonal structure

$$M_i := \begin{pmatrix} E_1 & N & 0 & \dots & \dots & 0 \\ Q_2 & E_2 & N & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & Q_j & E_j & N & \ddots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & Q_{i-1} & E_{i-1} & N \\ 0 & \dots & \dots & \dots & 0 & Q_i & E_i \end{pmatrix}. \quad (1)$$

We denote with $M \in \mathbb{R}^{(2n-2) \times (2n-2)}$ the block-tridiagonal matrix defined as $M := M_{n-1}$. We denote with $\Gamma_1 \in \mathbb{R}^{n \times (2n-2)}$ and $\Gamma_2 \in \mathbb{R}^{n \times (2n-2)}$ the following matrices

$$\Gamma_1 := \text{blkdiag} \left(\underbrace{C, \dots, C}_{(n-2) \text{ times}}, I_2 \right),$$

$$\Gamma_2 := \text{blkdiag} \left(I_2, \underbrace{B^T, \dots, B^T}_{(n-2) \text{ times}} \right).$$

For any $s \in \mathbb{R}$, the function sat is the function defined as

$$\text{sat}(s) := \min \left\{ 1, \frac{1}{|s|} \right\} s.$$

We denote with $\|\cdot\|_a$, respectively $\|\cdot\|_\infty$, the asymptotic norm, respectively the infinity norm, of a time-varying signal:

$$\|s\|_a := \limsup_{t \rightarrow \infty} |s(t)|, \quad \|s\|_\infty := \max_{t \geq 0} |s(t)|.$$

II. MAIN RESULTS

In this work we consider systems of the form (known also as *phase-variable representation*, see [6])

$$\begin{aligned} \dot{x}_i &= x_{i+1} & \forall i = 1, \dots, n-1 \\ \dot{x}_n &= \varphi(x, u) \\ y &= x_1 \end{aligned} \quad (2)$$

where $x = (x_1, \dots, x_n) \in X \subseteq \mathbb{R}^n$ is the state and $y \in \mathbb{R}$ is the output. The function φ may depend on some other known and bounded inputs $u \in \mathbb{R}^\nu$. Without loss of generality and with a mild abuse of notation, we will write $\varphi(x)$, omitting the dependence on u . The following assumption is made throughout the paper.

Assumption 1 *The function φ is locally Lipschitz, i.e. for any compact set $\mathcal{C} \subset \mathbb{R}^n$, there exists a $L_\varphi > 0$ such that*

$$|\varphi(x) - \varphi(\hat{x})| \leq L_\varphi |x - \hat{x}|$$

for all $(x, \hat{x}) \in \mathcal{C} \times \mathcal{C}$.

It is worth noting that this is the standard context in which high-gain observer are used (see, for instance [8]). The class of system considered includes all the nonlinear systems which are *uniformly observable* and that can be put in the form (2) by a proper global diffeomorphism (see, for instance, [4], [5], [6]). We further assume that the set X where the state evolves is a compact set of \mathbb{R}^n .

A. The Low-Power High-Gain Observer

The structure of the observer proposed in [1] has the following form

$$\begin{aligned} \dot{\xi}_i &= A \xi_i + N \xi_{i+1} + D_2(\ell) K_i e_i \\ & \quad i = 1, \dots, n-2 \\ \dot{\xi}_{n-1} &= A \xi_{n-1} + B \varphi_s(\hat{x}') + D_2(\ell) K_{(n-1)} e_{n-1} \end{aligned} \quad (3)$$

with $\xi = \text{col}(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{2n-2}$, $\hat{x}' = \Gamma_1 \xi$

$$e_1 = y - C \xi_1, \quad e_i = B^\top \xi_{i-1} - C \xi_i \quad i = 2, \dots, n-1,$$

and $\varphi_s(\cdot)$ is an appropriate saturated version of $\varphi(\cdot)$ that agrees with $\varphi(\cdot)$ on X . The variable \hat{x}' represents an asymptotic estimate of the state x of (2). It is obtained by “extracting” n components from the state ξ according to the matrix Γ_1 defined above. The redundancy of the observer can be used to extract from ξ an extra state estimation that is $\hat{x}'' = \Gamma_2 \xi$.

The following theorem, taken from [1], shows that the observer (3) recovers the same asymptotic properties for the two estimates \hat{x}' and \hat{x}'' of the “standard” high-gain observer. In the statement of the theorem we let

$$\hat{\mathbf{x}} = \text{col}(\hat{x}', \hat{x}''), \quad \mathbf{x} = \text{col}(x, x).$$

Theorem 1 *Consider system (2) and the observer (3) with the coefficients $(k_{i1} k_{i2})$ fixed so that the matrix M defined in (1) is Hurwitz (see Lemma 1 in the Appendix). Let $\varphi_s(\cdot)$ be any bounded function that agrees with $\varphi(\cdot)$ on X . Then there exist $c_1 > 0$, $c_2 > 0$ and $\underline{\ell} \geq 1$ such that for any $\ell \geq \underline{\ell}$ and for any $\xi(0) \in \mathbb{R}^{2n-2}$ the following bound holds*

$$|\hat{\mathbf{x}}(t) - \mathbf{x}(t)| \leq c_1 \ell^{n-1} \exp(-c_2 \ell t) |\hat{\mathbf{x}}(0) - \mathbf{x}(0)|, \quad (4)$$

for all $t \geq 0$ such that $x(t) \in X$.

The previous observer is proved to work in output feedback stabilization and principle of separations as shown for instance in [13]. Furthermore, in [1] the observer (3) has been shown also to improve the sensitivity to high-frequency noise with respect to standard implementation.

B. Saturated low-power high-gain observer

Having a look to the proof of Theorem 1 presented in [1], it is not so difficult to see that during the transient we have that the variables ξ_{i1} “peak” as $O(\ell^{i-1})$ whereas the variables ξ_{i2} “peak” as $O(\ell^i)$. This phenomenon can be prevented by modifying the low-power high-gain observer (3) as follows

$$\dot{\xi}_i = A \xi_i + B \eta_{i+1} + D_2(\ell) K_i (\eta_{i-1} - C \xi_i) \quad (5)$$

for $i = 1, \dots, n-1$, with $\xi = \text{col}(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{2n-2}$, $\hat{x}' = \Gamma_1 \xi$, $\hat{x}'' = \Gamma_2 \xi$,

$$\eta_0 = y$$

$$\eta_i = L_{i+1} \text{sat} \left(\frac{B^\top \xi_i}{L_{i+1}} \right), \quad i = 1, \dots, n-1, \quad (6)$$

$$\eta_n = L_{n+1} \text{sat} \left(\frac{\varphi_s(\hat{x}')}{L_{n+1}} \right),$$

and where the saturations level L_i are defined as

$$L_i := r_i + \varrho,$$

with ϱ a small positive real number and where we define the values $r_i > 0$, $i = 1, \dots, n+1$ as

$$\begin{aligned} r_i &:= \max_{x \in X} |x_i|, \quad \forall i = 1, \dots, n, \\ r_{n+1} &:= \max_{x \in X} |\varphi(x)|. \end{aligned} \quad (7)$$

It can be proved that the observer (5) has the same asymptotic properties of the observer (3) and moreover that the peaking of the variables ξ_{i1} does not grow with ℓ , whereas the variables ξ_{i2} peak as $O(\ell)$, with thus evident benefits in term of implementation. Moreover the variables \hat{x}'_i for $i = 1, \dots, n-1$ present a peaking phenomenon which does not grow with ℓ , with only the n -th estimate of \hat{x} affected by a peaking of order $O(\ell)$.

The tuning of the parameters k_{i1} and k_{i2} , $i = 1, \dots, n-1$, must be done according to the forthcoming definition. With respect to the choice required in Theorem 1 in which the parameters were just asked to make the matrix M Hurwitz (see Lemma 1), in this case the presence of saturation functions asks the parameters also fulfill additional ‘‘common-Lyapunov’’ conditions.

Definition 1 *The coefficients (k_{i1}, k_{i2}) , $i = 1, \dots, n-1$, are said to be admissible if, for any $i = 2, \dots, n-1$, there exists a $P_i = P_i^\top > 0$ satisfying*

$$P_i M_i + M_i^\top P_i < 0, \quad P_i F_i + F_i^\top P_i < 0, \quad (8)$$

where

$$M_i = \begin{pmatrix} M_{i-1} & \bar{N}_i \\ \bar{Q}_i & E_i \end{pmatrix}, \quad F_i := \begin{pmatrix} M_{i-1} & 0 \\ \bar{Q}_i & E_i \end{pmatrix}.$$

The set of admissible coefficients is denoted by \mathcal{G}_a .

According to Lemma 2 in the Appendix the set \mathcal{G}_a is non-empty. Therein, moreover, a constructive design procedure for the coefficients k_{i1}, k_{i2} , $i = 1, \dots, n-1$, is given.

Theorem 2 *Consider system (2) and the observer (5) and let the coefficients (k_{i1}, k_{i2}) be admissible according to Definition 1. Then, for any compact sets $X \subset \mathbb{R}^n$ and $\Xi \subset \mathbb{R}^{2n-2}$ and for any $\varrho > 0$, there exist $\mu_1 > 0$, $\mu_2 > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $\underline{\ell} \geq 1$ such that, for any $\ell > \underline{\ell}$ and for any $(x(0), \xi(0)) \in X \times \Xi$, the following bounds hold*

$$\begin{aligned} &|\hat{\mathbf{x}}(t) - \mathbf{x}(t)| \leq \\ &\min \left\{ \mu_1 \ell^{n-1} \exp(-\mu_2 \ell t) |\hat{\mathbf{x}}(0) - \mathbf{x}(0)|, \delta_1 \ell \right\}, \quad (9) \end{aligned}$$

$$|\hat{x}'_i(t)| \leq \delta_2 \quad i = 1, \dots, n-1, \quad (10)$$

for all $t \geq 0$ and as long as $x(t) \in X$.

By going throughout the proof of the theorem, it is easy to see that the observer (5) asymptotically coincides with the observer (3), namely the steady-state behaviour is not affected by the saturations. As a consequence the new design does not alter the sensitivity property of the observer (3).

III. PROOF OF THEOREM 2

The proof is divided in two parts. First we prove the asymptotic convergence of the observer. In this proof we get a conservative bound. Then, we make a more detailed analysis to show that this bound can be refined and that, indeed, the state variables are ‘‘peaking-free’’. In the following we define v_i , $i = 1, \dots, n-1$ as

$$\begin{aligned} v_i &:= \eta_{i+1} - x_{i+2} \quad \forall i = 1, \dots, n-2 \\ v_{n-1} &:= \eta_n - \varphi(x). \end{aligned}$$

Note that

$$|\eta_i(t)|_\infty \leq L_{i+1}, \quad |v_i(t)|_\infty \leq r_{i+2} + L_{i+2}. \quad (11)$$

Furthermore, let the variables $\tilde{\xi}_i$, ζ_i and ε_i be defined as

$$\begin{aligned} \tilde{\xi}_i &:= \xi_i - \text{col}(x_i, x_{i+1}), \\ \zeta_i &:= \ell D_2(\ell)^{-1} \tilde{\xi}_i, \\ \varepsilon_i &:= \ell^{2-i} D_2(\ell)^{-1} \tilde{\xi}_i. \end{aligned} \quad (12)$$

Finally, let $R_i > 0$, $i = 1, \dots, n$ be the numbers such that

$$|\tilde{\xi}_i(0)| \leq R_i, \quad \forall x \in X, \forall \xi \in \Xi. \quad (13)$$

A. Exit from saturation and convergence

a) *Case $i = 1$* : Consider the first variable ξ_1 . Its dynamic is given by

$$\dot{\xi}_1 = A \xi_1 + B \eta_2 + D_2(\ell) K_1 (y - C \xi_1).$$

By using the change of coordinates (12) the latter is transformed into

$$\dot{\varepsilon}_1 = \ell E_1 \varepsilon_1 + \ell^{-1} B v_1$$

Let $\underline{\ell}_1 \geq 1$ be chosen according to Lemma 4 with $\kappa = 1$ for some $T_1 > 0$ and $\varepsilon_1 < \varrho$. We obtain

$$|\varepsilon_1(t)| \leq \varepsilon_1 \quad \forall t \geq T_1,$$

for all $\ell \geq \underline{\ell}_1$. Furthermore, by recalling (for any $\ell \geq 1$)

$$|\xi_{12}| \leq |\xi_{12} - x_2| + |x_2| \leq \ell |\varepsilon_{12}| + |x_2| \leq \ell |\varepsilon_1| + |x_2|$$

we get

$$|\xi_{12}(t)| \leq \ell |\varepsilon_1| + |x_2| \leq \varepsilon_1 + r_2 \leq L_2$$

for any $\ell \geq \underline{\ell}_1$ and for all $t \geq T_1$. With this in mind, we have that $\eta_1 = \xi_{12}$ for all $t \geq T_1$.

b) *General Case* $i > 1$: After time $T_{i-1} > 0$ we get $\eta_j = \xi_{j2}$ for $j = 1, \dots, i-1$. So consider the cascade system

$$\begin{aligned} \dot{\xi}_1 &= A\xi_1 + N\xi_{j+1} + D_2(\ell)K_1(y - C\xi_1) \\ &\vdots \\ \dot{\xi}_{i-1} &= A\xi_{i-1} + B\eta_i + D_2(\ell)K_{i-1}(B^\top\xi_{i-2} - C\xi_{i-1}) \\ \dot{\xi}_i &= A\xi_i + B\eta_{i+1} + D_2(\ell)K_{i-1}(B^\top\xi_{i-1} - C\xi_i) \end{aligned}$$

By using the change of coordinates (12) the latter is transformed into

$$\begin{aligned} \begin{pmatrix} \dot{\varepsilon}_{[1,i-1]} \\ \dot{\varepsilon}_i \end{pmatrix} &= \ell \begin{pmatrix} M_{i-1} & 0 \\ Q_i & E_i \end{pmatrix} \begin{pmatrix} \varepsilon_{[1,i-1]} \\ \varepsilon_i \end{pmatrix} \\ &\quad + \frac{1}{\ell^i} \begin{pmatrix} B_{2(i-1)} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \ell v_{i-1}(t) \\ v_i(t) \end{pmatrix} \end{aligned}$$

with the notation $\varepsilon_{[1,i-1]} = (\varepsilon_1, \dots, \varepsilon_{i-1})^\top$. Note that

$$\begin{aligned} v_{i-1}(t) &= \eta_i - x_{i+1} \\ &= L_{i+1} \text{sat} \left(\frac{\ell^i \varepsilon_{i2} + x_{i+1}}{L_{i+1}} \right) - x_{i+1} \end{aligned}$$

By the Lipschitz mean-value theorem, for each t sufficiently large, there exists a continuous function $\rho(t) \in [0, 1]$ such that

$$v_{i-1}(t) = \rho(t) \ell^i \varepsilon_{i2}.$$

As a consequence we get

$$\begin{pmatrix} \dot{\varepsilon}_{[1,i-1]} \\ \dot{\varepsilon}_i \end{pmatrix} = \ell \Lambda_i(t) \begin{pmatrix} \varepsilon_{[1,i-1]} \\ \varepsilon_i \end{pmatrix} + \frac{1}{\ell^i} \begin{pmatrix} 0 \\ B \end{pmatrix} v_i(t)$$

where $\Lambda_i(t)$ is defined as in (16) in the Appendix. Let $\underline{\ell}_i \geq \underline{\ell}_{i-1}$ be chosen according to Lemma 3 and 4 with $\kappa = i$ and some $T_i > T_{i-1}$ and $\epsilon_i < \varrho$. We obtain

$$|\ell^i \varepsilon_{[1,i]}(t)| \leq \epsilon_i \quad \forall t \geq T_i,$$

Furthermore, by recalling (for any $\ell \geq 1$)

$$\begin{aligned} |\xi_{i2}| &\leq |\xi_{i2} - x_{i+1}| + |x_{i+1}| \leq \ell^i |\varepsilon_{i2}| + |x_{i+1}| \\ &\leq \ell^i |\varepsilon_{[1,i]}| + |x_{i+1}| \end{aligned}$$

we get

$$|\xi_{i2}(t)| \leq \ell^i |\varepsilon_{[1,i]}| + |x_{i+1}| \leq \epsilon_i + r_{i+1} \leq L_{i+1}$$

for any $\ell \geq \underline{\ell}_i$ and for all $t \geq T_i$. This prove that for $t \geq T_i$ also $\eta_i = \xi_{i2}$.

c) *Asymptotic Convergence*: Finally, after the $(n-1)$ -th step we recover the observer (3). By direct application of Theorem 1 we thus have that if ℓ is taken sufficiently large ($\ell \geq \underline{\ell}_n$) then the following estimate holds

$$|\hat{\mathbf{x}}(t) - \mathbf{x}(t)| \leq \mu_1 \ell^{n-1} \exp(-\mu_2 \ell t) |\hat{\mathbf{x}}(0) - \mathbf{x}(0)| \quad (14)$$

for some strictly positive real numbers μ_1, μ_2 and for all $t \geq T_{n-1}$. Note that the estimate bound ℓ^{n-1} is too conservative. Indeed, it can be proved that during the transient the peaking depends only on ℓ and not on ℓ^{n-1} , as shown in the next part of the proof.

B. Bound Estimate

Consider the ξ_i dynamic. By using the change of coordinates (12), it is transformed into

$$\dot{\zeta}_i = \ell E_i \zeta_i + \ell^{-1} v_i + \ell K_{i-1} \eta_i$$

(except for the ζ_1 for which the term η_1 is not present). Now consider the Lyapunov function V_i defined as $V_i := \eta_i^\top S_i \eta_i$ with S_i given by $S_i E_i + E_i^\top S_i = -I$, we get

$$\dot{V}_i \leq -\ell |\zeta_i|^2 + 2|\zeta_i| |S_i| (\ell^{-1} v_i + \ell |K_{i-1}| \eta_{i-1}).$$

Then, standard Lyapunov arguments show that there exist positive constants a_{ij} , $j = 1, \dots, 4$, such that, if ℓ is taken sufficiently large, we get

$$|\zeta_i(t)| \leq a_{i1} \exp(-a_{i2} \ell t) |\zeta_i(0)| + \frac{a_{i3}}{\ell^2} v_i + a_{i4} |K_{i-1}| \eta_{i-1}$$

for any $i = 1, \dots, n-1$. Furthermore, by recalling (for any $\ell \geq 1$)

$$|\tilde{\xi}_{i1}| \leq |\zeta_i| \leq |\tilde{\xi}_i|, \quad \frac{1}{\ell} |\tilde{\xi}_{i2}| \leq |\zeta_i| \leq |\tilde{\xi}_i|,$$

we get

$$\begin{aligned} |\tilde{\xi}_{i1}(t)| &\leq a_{i1} e^{-a_{i2} \ell t} |\tilde{\xi}_i(0)| + \ell^{-2} a_{i3} v_i + a_{i4} |K_{i-1}| \eta_{i-1}, \\ |\tilde{\xi}_{i2}(t)| &\leq a_{i1} \ell e^{-a_{i2} \ell t} |\tilde{\xi}_i(0)| + \ell^{-1} a_{i3} v_i + a_{i4} \ell |K_{i-1}| \eta_{i-1}, \end{aligned}$$

for all $t \geq 0$. By using $|\hat{\mathbf{x}} - \mathbf{x}| \leq \sum_{i=1}^{n-1} |\tilde{\xi}_{i1}| + |\tilde{\xi}_{i2}|$ and by defining δ_1 as

$$\delta_1 := \sum_{i=1}^{n-1} 2a_{i1} R_i + 2a_{i3} (r_{i+2} + L_{i+2}) + a_{i4} |K_{i-1}| L_i$$

with R_i defined in (13), we get $|\hat{\mathbf{x}}(t) - \mathbf{x}(t)| \leq \ell \delta_1$ for all $t \geq 0$. By combining the latter with (14) we finally get (9). The bound (10) follows by by noting

$$|\hat{x}_i| \leq |\xi_{i1} - x_i| + |x_i|, \quad i = 1, \dots, n-1,$$

and by setting

$$\delta_2 := \max_{i \in [1, n-1]} \left\{ r_i + a_{i1} R_i + a_{i3} (L_{i+2} + r_{i+2}) + a_{i4} L_i |K_{i-1}| \right\}.$$

IV. SIMULATIONS

We consider as example a forced Van Der Pol oscillator, described by

$$\ddot{w} + b(w^2 - 1)\dot{w} + w = \alpha \cos(\omega t + \phi)$$

where $b > 0$. The system can be described in the *phase-variable representation* (2) with $n = 4$,

$$\begin{aligned} \varphi(x) &= -6bx_1 x_2 x_3 - 2bx_2^3 - x_3 - b(x_1^2 - 1)x_4 \\ &\quad - \omega^2(x_3 + b(x_1^2 - 1)x_2 + x_1) \end{aligned}$$

and initial conditions

$$\begin{aligned} x_1(0) &= w(0), \\ x_2(0) &= \dot{w}(0), \\ x_3(0) &= -b(w(0)^2 - 1)\dot{w}(0) - w(0) + \alpha \cos(\phi), \\ x_4(0) &= -2bw(0)\dot{w}(0)^2 - \dot{w}(0) \\ &\quad - b(w(0)^2 - 1)x_3(0) + \alpha \omega \sin(\phi). \end{aligned}$$

TABLE I: Features of the low-power high-gain observer (3).

	$\ell = 5$	$\ell = 10$	$\ell = 100$	$\ell = 1000$
$T_{0.01}$	4.9898	1.8382	0.2544	0.0399
$\ \tilde{x}'_1\ _\infty$	1	1	1	1
$\ \tilde{x}''_2\ _\infty$	7.5	14.4	139	$1.4 \cdot 10^3$
$\ \tilde{x}'_2\ _\infty$	7	13.6	133	$1.3 \cdot 10^3$
$\ \tilde{x}''_3\ _\infty$	16.5	70	$7.2 \cdot 10^3$	$7.2 \cdot 10^5$
$\ \tilde{x}'_3\ _\infty$	14.1	60	$6.4 \cdot 10^3$	$6.4 \cdot 10^5$
$\ \tilde{x}'_4\ _\infty$	16.2	126	$1.3 \cdot 10^5$	$1.3 \cdot 10^8$

By taking $\omega = 2$ and $\alpha \leq 1$, it can be verified that, for any $\phi \in \mathbb{R}$, the solutions of the systems are ultimately bounded. Taking $b = 0.01$, a numerical simulation can be used to show that the solutions satisfy

$$\|x_1\|_a < 2.2, \quad \|x_2\|_a < 2.6, \quad \|x_3\|_a < 3, \\ \|x_4\|_a < 4.6, \quad \|\varphi(x)\|_a < 6.9.$$

In the following we consider any compact set X of initial conditions satisfying (see also (7))

$$r_1 < 2.2, \quad r_2 < 2.6, \quad r_3 < 3, \quad r_4 < 4.6, \quad r_5 < 6.9.$$

By following the procedure indicated by Lemma 1, we chose the coefficients of the observer as

$$K_1 = \begin{pmatrix} 3 \\ 6.4 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 3 \\ 2.131 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 3 \\ 0.7095 \end{pmatrix},$$

such that the poles of M_3 are in $(-2, -1.8, -1.6, -1.4, -1.2, -1)$. It is easy to verify that they are also admissible, according to Definition 1. The saturations level are chosen as

$$L_1 = 2.5, \quad L_2 = 3, \quad L_3 = 3.5, \quad L_4 = 5, \quad L_5 = 7.5.$$

In the simulations we compared the observer (3) with the new observer (5) for different choices of ℓ . The initial conditions of the forced Van Der Pol are chosen as $w(0) = 1$, $\dot{w}(0) = -1$, $\gamma = 1$, $\phi = 0$, that is $x(0) = (1, -1, 0, 0.98)$, whereas the two observers are initialized in the origin. The tables show the maximum peaking values in the error coordinates and the time needed to convergence to an error sufficiently small, *i.e.* the time T_ϵ such that

$$|\mathbf{x}(t) - \hat{\mathbf{x}}(t)| < \epsilon \quad \forall t \geq T_\epsilon.$$

In the tables we denote

$$\tilde{x}'_i = \hat{x}'_i - x_i, \quad \tilde{x}''_i = \hat{x}''_i - x_i.$$

Note that $\tilde{x}'_1 = \tilde{x}''_1$ and $\tilde{x}'_4 = \tilde{x}''_4$. The big numerical improvement during the transient, given by the saturations in the proposed design (5), it is evident from the simulation, especially for the variables \tilde{x}'_3 , \tilde{x}''_3 and \tilde{x}'_4 . The Table II confirms the theoretical result of Theorem 2. In particular the variables \tilde{x}'_1 , \tilde{x}'_2 and \tilde{x}'_3 are always bounded with a bound independent of the high-gain parameter ℓ . Finally, it can be noticed from the simulations that the when the high-gain parameter is small, the speed of convergence of the novel observer (5) is worsen with respect to the standard low-power high-gain observer (3), whereas when the high-gain parameter is high the rate of convergence is improved. This

TABLE II: Features of the low-power peaking-free high-gain observer (5).

	$\ell = 5$	$\ell = 10$	$\ell = 100$	$\ell = 1000$
$T_{0.01}$	5.6072	2.0222	0.2364	0.0312
$\ \tilde{x}'_1\ _\infty$	1	1	1	1
$\ \tilde{x}''_2\ _\infty$	7.2	13.4	127	1300
$\ \tilde{x}'_2\ _\infty$	4.3	4.4	4.4	4.4
$\ \tilde{x}''_3\ _\infty$	9.3	21	261	2600
$\ \tilde{x}'_3\ _\infty$	3.7	3.9	3.8	3.8
$\ \tilde{x}'_4\ _\infty$	4.7	9.4	92	900

phenomenon is caused by the effect of the saturations which prevent the state of the observer to have very large transients.

V. CONCLUSION

By following the ideas in [1], we proposed a modified version of the low-power high-gain observer introduced therein. The new structure is designed by adding saturations at each block. The new design preserves the same benefits of the observer proposed in [1] and overtakes the peaking phenomenon. The proposed observer has a dimension which is $2n - 2$ (where n is the system dimension) with a high-gain parameter which grows only up to power 2 (whereas in standard high-gain observers the parameter is powered up to n) and the peaking phenomenon exhibits with a growth $O(\ell)$ on the last component only (and not $O(\ell^{n-1})$ as in the observer of [1]), giving evident benefits with respect to the standard implementations of high-gain observers. A new constructive Lemma for the choice of the parameter is given in the appendix. The new procedure may result easier with respect to the one proposed in [1]. Due to the asymptotic tunable convergence property (9), the novel proposed observer can be used in any output feedback design where a standard high-gain observer applies (see, for instance, [12] and [13]).

APPENDIX

The eigenvalues of M_i , $i > 0$, can be arbitrarily assigned by appropriately choosing the coefficients (k_{j1}, k_{j2}) , $j = 1, \dots, i$, as claimed in the next lemma (taken off-the-shelf from the Lemma 1 published in [1] by setting $i = n - 1$).

Lemma 1 *Let*

$$\mathcal{P}_{M_i}(\lambda) = \lambda^{2i} + m_1 \lambda^{2i-1} + \dots + m_{2i-1} \lambda + m_{2i}$$

be an arbitrary Hurwitz polynomial. There exists a choice of k_{j1}, k_{j2} , $j = 1, \dots, i$, such that the characteristic polynomial of M_i coincides with $\mathcal{P}_{M_i}(\lambda)$.

Lemma 2 *The set \mathcal{G}_a is non-empty.*

Proof: We show constructively, though conservatively, a recursive procedure to assign the coefficients k_{i1}, k_{i2} .

Step $i = 1$) Let $k_{11} > 0$ and $k_{12} > 0$ be any (positive) real numbers.

Step $i > 1$) Let $k_{i1} = k_{(i-1)1}$, and let $k_{i2} > 0$ be chosen such that $k_{i2} < k_{i1}/\gamma_{i-1}$, with γ_{i-1} defined in the forthcoming equation (15). \triangleleft

To verify that this choice satisfies the Definition 1 let consider first the case $i = 1$. By choosing $k_{11} > 0$ and $k_{12} > 0$ the matrix E_1 is Hurwitz, and so is M_1 . Now consider the case $i > 1$ and the following two systems

$$\begin{cases} \dot{x}_{i-1} = M_{i-1}x_{i-1} + B_{2(i-1)}u_{i-1} \\ y_{i-1} = B_{2(i-1)}^\top x_{i-1} \\ \dot{x}_i = E_i x_i + K_i u_i \\ y_i = B^\top x_i \end{cases}$$

and let denote $\bar{x}_i = (x_{i-1}, x_i)$. Let assume the matrix M_{i-1} is Hurwitz and note that the matrix E_i is Hurwitz for any choice of positive real numbers k_{i1}, k_{i2} . Let $H_{i-1}(s), H_i(s)$ be the transfer function of the two systems

$$\begin{aligned} H_{i-1}(s) &:= B_{2(i-1)}^\top (sI_{2(i-1)} - M_{i-1})^{-1} B_{2(i-1)}, \\ H_i(s) &:= B^\top (sI_2 - E_i)^{-1} K_i. \end{aligned}$$

and let γ_{i-1}, γ_i be the real numbers defined as

$$\begin{aligned} \gamma_{i-1} &:= \max_{s \in \mathbb{R}} |H_{i-1}(s)|, \\ \gamma_i &:= \max_{s \in \mathbb{R}} |H_i(s)|. \end{aligned} \quad (15)$$

It is easy to verify that

$$H_i(s) = \frac{sk_{i2}}{s^2 + k_{i1}s + k_{i2}}, \quad \gamma_i = \frac{k_{i2}}{k_{i1}}.$$

By interconnecting the two subsystems with $u_{i-1} = 0$ and $u_i = y_{-1}$ we obtain a closed loop system described by

$$\dot{\bar{x}}_i = F_i \bar{x}_i,$$

whereas, by interconnecting the two subsystems with $u_{i-1} = y_i$ and $u_i = y_{-1}$, we obtain a closed loop system described by

$$\dot{\bar{x}}_i = M_i \bar{x}_i.$$

The choice of k_{i2} of the Step i satisfies the small-gain theorem (see, for instance, [7])

$$\gamma_{i-1} \cdot \gamma_i < 1$$

and therefore the interconnection (x_{i-1}, x_i) is stable in both cases. We conclude that F_i and M_i are Hurwitz and that there exists a common Lyapunov function satisfying (8). ■

Note that, by using MATLAB, the values of γ_i can be easily calculated with the command `getPeakGain`.

Lemma 3 *Let the coefficients k_{j1}, k_{j2} , $j = 1, \dots, i$ be admissible and let the matrix $\Lambda_i(t)$ be defined as*

$$\Lambda_i(t) := \begin{pmatrix} M_{i-1} & \rho(t)\bar{N}_{i-1} \\ \bar{Q}_i & E_i \end{pmatrix} \quad (16)$$

with $\rho(t) \in [0, 1]$ any continuous function. Then the origin of the system

$$\dot{x} = \Lambda_i(t)x \quad (17)$$

is globally exponentially stable.

Proof: When $\rho(t) \equiv 0$ we get $\Lambda_i = F_i$ whereas when $\rho(t) \equiv 1$ we get $\Lambda_i = M_i$. As a consequence, by using the P_i claimed in the Definition 1 we get the inequality

$$P_i \Lambda_i(t) + \Lambda_i(t)^\top P_i < 0$$

is satisfied for all $\rho(t) \in [0, 1]$ and therefore the claim follows trivially. ■

Lemma 4 *Consider the system*

$$\dot{x} = \ell \Lambda_i(t)x + \ell^{-\kappa} v_i(t)$$

where $x \in \mathbb{R}^{2i}$, $\kappa \geq 0$ and $v_i(t)$ satisfying $|v_i(t)|_\infty \leq \bar{v}_i$ for some $\bar{v}_i > 0$. Suppose the origin of the system (17) is globally asymptotically stable and locally exponentially stable. Then, for any compact set $X \subset \mathbb{R}^{2i}$, for any $T > 0$ and for any $\epsilon > 0$, there exist a $\ell^* \geq 1$ such that

$$|\ell^\kappa x(t)| \leq \epsilon, \quad \forall t \geq T$$

for any $\ell > \ell^*$ and for any initial condition $x(0) \in X$.

Proof: Trivial, by applying standard ISS-Lyapunov arguments. ■

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