



HAL
open science

On the coset graph construction of distance-regular graphs *

Minjia Shi, Denis Krotov, Patrick Solé

► **To cite this version:**

Minjia Shi, Denis Krotov, Patrick Solé. On the coset graph construction of distance-regular graphs *. Discrete Mathematics, 2022. hal-03684611

HAL Id: hal-03684611

<https://hal.science/hal-03684611v1>

Submitted on 1 Jun 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the coset graph construction of distance-regular graphs*

Minjia Shi[†], Denis S. Krotov[‡], Patrick Solé[§]

Abstract

We show that no more new distance-regular graphs in the tables of the book of (Brouwer, Cohen, Neumaier, 1989) can be produced by using the coset graph of additive completely regular codes over finite fields.

Keywords: completely regular codes, distance-regular graphs

AMS Classification (MSC 2010): Primary 94B05 Secondary 05E30

1. Introduction

Since the times of Delsarte [?], an important tool to construct distance-regular graphs has been the coset graph of a completely regular code. Many examples can be found in Chapter 11 of [?], involving notably the Golay codes and the Kasami codes (for a new description of these codes see e.g. [?]). More recently, a new distance-regular graph related to the dodecacode, an important additive \mathbb{F}_4 -code of length 12 was constructed in [?]. This solved both a thirty year old problem in graph theory [?, Ch. 14], and a forty-five year old problem in coding theory [?].

*The work of M. J. Shi is supported by the National Natural Science Foundation of China (12071001) and the Excellent Youth Foundation of Natural Science Foundation of Anhui Province (1808085J20); the work of D. S. Krotov is supported within the framework of the state contract of the Sobolev Institute of Mathematics (FWNF-2022-0017).

[†]Key Laboratory of Intelligent Computing and Signal Processing, Ministry of Education, School of Mathematical Sciences, Anhui University, Hefei, 230601, China, smjwcl.good@163.com

[‡]Sobolev Institute of Mathematics, Novosibirsk 630090, Russia, krotov@math.nsc.ru

[§]I2M (CNRS, Centrale Marseille, University of Aix-Marseille), Marseilles, France, sole@enst.fr

In the present note, we find out which graphs with the parameters in the tables of [?, Ch. 14] can be realized as coset graphs of completely regular additive codes over a finite field. A trivial necessary condition is that the number of vertices should be a power of a prime. More elaborate necessary conditions include a refinement of the Lloyd theorem for equitable partitions (Lemma ??), and a divisibility condition bearing on the entries of the intersection array (Corollary ??). When all these techniques are exhausted, we can eliminate the remaining parameters by showing that the needed code does not exist, either by a direct combinatorial argument, or by long electronic computations using the dedicated software [?] or [?]. We conclude that no more additive codes are left to be found satisfying the said condition.

The material is arranged as follows. The next section lists the parameters we are considering. Section ?? collects basic facts and notations. Section ?? gives the two theoretical tools mentioned above. Sections ?? to ?? study the codes remaining after use of these two tools. Section ?? concludes the article.

2. Parameters

The only “unsolved” parameters of distance-regular graphs with prime-power order in the [BCN] table [?, Ch. 14] are the following (the most of the cited updates are mentioned in [?, Ch. “Tables”]):

- $v = 2187 = 3^7$, $\{140, 126, 20, 1; 1, 10, 126, 140\}$ (Section ??);
- $v = 243 = 3^5$, $\{22, 16, 5; \underline{1, 2, 20}\}$; a graph does not exist [?];
- $v = 256 = 2^8$, $\{25, 24, 3; \underline{1, 3, 20}\}$ (Corollary ??);
- $v = 729 = 3^6$, $\{26, 24, 19; \underline{1, 3, 8}\}$ (Corollary ??);
- $v = 1024 = 2^{10}$, $\{31, 30, 17; 1, 2, 15\}$; a graph exists (Kasami graph [?, Thm. 11.2.1 (13)], $q = j = 2$).
- $v = 625 = 5^4$, $\{32, 28, 9; \underline{1, 2, 28}\}$ (Corollary ??);
- $v = 1024 = 2^{10}$, $\{33, 30, 15; 1, 2, 15\}$; a graph exists [?];
- $v = 625 = 5^4$, $\{36, 28, 4; 1, 2, 24\}$ (Lemma ??);
- $v = 343 = 7^3$, $\{42, 30, 12; 1, 6, 28\}$ (Section ??);
- $v = 243 = 3^5$, $\{44, 36, 5; \underline{1, 9, 40}\}$ (Corollary ??);

- $v = 343 = 7^3$, $\{48, 35, 9; \underline{1}, 7, 40\}$ (Corollary ??);
- $v = 625 = 5^4$, $\{52, 42, 16; 1, 6, 28\}$ (Section ??);
- $v = 729 = 3^6$, $\{56, 42, 20; 1, 6, 28\}$ (Section ??);
- $v = 512 = 2^9$, $\{63, 48, 10; 1, 8, 54\}$ (Lemma ??);
- $v = 729 = 3^6$, $\{80, 63, 11; \underline{1}, 9, 70\}$ (Corollary ??);
- $v = 729 = 3^6$, $\{104, 66, 8; 1, 12, 88\}$; a graph does not exist [?].

In this note we discuss the possibility to construct distance-regular graphs with the parameters above as coset graphs of linear completely regular codes in Hamming graph $H(n, q)$, where q is prime. (Note that the existence of a completely regular code in $H(n, q)$ where $q = p^s$ for prime p implies the existence of a completely regular code in $H(n(q-1)/(p-1), p)$ with the same intersection matrix, because $H(n(q-1)/(p-1), p)$ covers $H(n, q)$, see also Lemma ?? below.)

3. Definitions and basic facts

3.1. Graphs

All graphs in this note are finite, undirected, connected, without loops or multiple edges. In a graph Γ , the *neighborhood* $\Gamma(\bar{v})$ of a vertex \bar{v} is the set of vertices adjacent to \bar{v} . The *degree* of a vertex \bar{v} is the size of $\Gamma(\bar{v})$. A graph is *regular* if every vertex has the same degree. The *i -neighborhood* $\Gamma_i(\bar{v})$ is the set of vertices at geodetic distance i to \bar{v} . The *distance- i* graph of Γ is the graph on the same vertex set where two vertices are adjacent if and only if the geodetic distance between them in Γ is i . A graph is *distance-regular* (DR) if for every pair of vertices \bar{u} and \bar{v} at distance i apart the quantities

$$b_i = |\Gamma_{i+1}(\bar{u}) \cap \Gamma(\bar{v})|, \quad c_i = |\Gamma_{i-1}(\bar{u}) \cap \Gamma(\bar{v})|,$$

which are referred to as the *intersection numbers* of the graph, solely depend on i and not on the special choice of the pair (\bar{u}, \bar{v}) . The *automorphism group* of a graph is the set of permutations of the vertices that preserve adjacency.

Given a group \mathbb{G} and an inversion-closed set G of non-identity elements of \mathbb{G} , the Cayley graph $\text{Cal}(\mathbb{G}, G)$ is the graph on the vertex set \mathbb{G} where two elements \bar{u} and \bar{v} are adjacent if and only if $\bar{u}\bar{v}^{-1} \in G$.

The *Hamming graph* $H(n, q)$ is a distance-regular graph on the set Σ^n of n -words over the alphabet Σ of size q , two vectors being adjacent if they

are at Hamming distance one. We mostly consider the case when q is a prime power, Σ is the finite field \mathbb{F}_q of order q , and Σ^n is an n -dimensional vector space over \mathbb{F}_q . The *weight* of a vertex of $H(n, q)$ is the number of nonzero elements in the corresponding tuple. The *weight distribution* of a set C of vertices of $H(n, q)$ is the sequence $W = (W_0, W_1, \dots, W_d)$, where W_i is the number of words of weight i in C . Sometimes, we will write the weight distribution in the form $\{i_0^{W_{i_0}}, i_1^{W_{i_1}}, \dots, i_k^{W_{i_k}}\}$, where W_{i_0}, \dots, W_{i_k} are the nonzero terms in W ; for example, $(1, 0, 0, 7, 7, 0, 0, 1) = \{0^1, 3^7, 4^7, 7^1\}$.

3.2. Equitable partitions, completely regular codes

A partition (P_0, \dots, P_r) of the vertex set of a graph Γ is called an *equitable partition* (also known as regular partition, partition design, perfect coloring) if there are constants S_{ij} such that

$$|\Gamma(\bar{v}) \cap P_j| = S_{ij} \quad \text{for every } \bar{v} \in P_i, \quad i, j = 0, \dots, r.$$

The numbers S_{ij} are referred to as the intersection numbers, and the matrix $(S_{ij})_{i,j=0}^r$ as the *intersection* (or quotient) *matrix*. If the intersection matrix is tridiagonal, then P_0 is called a *completely regular code* of covering radius r and *intersection array* $(S_{01}, S_{12}, \dots, S_{r-1,r}; S_{10}, S_{21}, \dots, S_{r,r-1})$. That is to say, a set of vertices C is a completely regular code if the distance partition $(C = C^{(0)}, C^{(1)}, \dots, C^{(r)})$ with respect to C is equitable. As was proven by Neumaier [?], for a distance-regular graph, this definition is equivalent to the original Delsarte definition [?]: a code C is completely regular if the outer distance distribution $(|C \cap \Gamma_i(\bar{v})|)_{i=0,1,2,\dots}$ of C with respect to a vertex \bar{v} depends only on the distance from \bar{v} to C .

3.3. Linear and additive codes

A linear code (that is, a linear subspace of \mathbb{F}_q^n) of dimension k and minimum distance d in $H(n, q)$ is called an $[n, k, d]_q$ code. The duality is understood with respect to the standard inner product.

Linear codes are special cases of the additive codes, which are, by definition, the subgroups of the additive group of \mathbb{F}_q^n . A *coset* of an additive code C is any translate of C by a constant vector. A *coset leader* is any coset element that minimizes the weight. The *weight of a coset* is the weight of any of its leaders. The *coset graph* Γ_C of an additive code C is defined on the cosets of C , two cosets being adjacent if they differ in a coset of weight one.

Lemma 1 (see, e.g., [?]). *An additive code with distance at least 3 is completely regular with intersection array $\{b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho\}$ if and only if the coset graph is distance-regular with intersection numbers $b_0, \dots, b_{\rho-1}, c_1, \dots, c_\rho$.*

On the other hand,

Lemma 2 (see [?, Theorem 11.1.10]). *If a distance-regular graph of diameter at least 3 is a Cayley graph on an elementary abelian p -group, then it is the coset graph of a linear completely regular 1-code over \mathbb{F}_p .*

So, the existence of a distance-regular graph of diameter at least 3 that is a Cayley graph on an elementary abelian p -group is equivalent to the existence of a linear completely regular 1-code in a Hamming graph over \mathbb{F}_p with the same intersection array.

4. Weight distributions

Lemma 3 ([?],[?]). *If P is an equitable partition of the Hamming graph $H(n, q)$ with quotient matrix S , then P is an equitable partition of the distance- w graph of $H(n, q)$ with quotient matrix $S^{(w)} := K_w(K_1^{-1}(S))$, where*

$$K_k(x) := K_k(x; n, q) := \sum_{j=0}^k (-1)^j (q-1)^{k-j} \binom{x}{j} \binom{n-x}{k-j}$$

is the Krawtchouk polynomial. In particular, $K_w(K_1^{-1}(S))$ is integer. (The same is true for any distance-regular graph, with the corresponding P -polynomial.)

The sequence of matrices $(S^{(w)})_{w=0}^n$ can be called the generalized weight distribution of the equitable partition. Indeed, if the all-zero word belongs to P_i , then the number of vertices of weight w in P_j is precisely the (i, j) th element of the matrix $S^{(w)}$. The following corollary shows that for some putative intersection arrays we can easily, without any calculations, see that $S^{(w)}$ is not integer for some w .

Corollary 1. *Assume that we have a completely regular code C with intersection array $\{\beta_0, \dots, \beta_{\rho-1}; \gamma_1, \dots, \gamma_\rho\}$ in a distance-regular graph with intersection array $\{b_0, \dots, b_{D-1}; c_1, \dots, c_D\}$. Then for any appropriate $i \in \{1, \dots, \rho\}$ and $j \in \{0, \dots, \rho - i\}$ the product $\beta_j \beta_{j+1} \dots \beta_{j+i-1}$ of i consequent intersection numbers (similarly, for $\gamma_{j+1} \dots \gamma_{j+i}$) is divisible by $c_1 c_2 \dots c_i$ (for Hamming graphs, by $i!$).*

Proof. We fix any vertex v from $C^{(j)}$ and denote by W_l the set of vertices at distance l from v . We will prove by induction that

$$|W_i \cap C^{(i)}| = \prod_{l=1}^i \frac{\beta_{j+l-1}}{c_l}. \quad (1)$$

For $i = 0$, it is trivially $1 = 1$ (by usual convention, the result of multiplying no factors is 1). For $i > 0$, it is sufficient to show that

$$c_i |W_i \cap C^{(i)}| = \beta_{j+i-1} |W_{i-1} \cap C^{(i-1)}|. \quad (2)$$

By the definition of a completely regular code, the number of edges beginning in $W_{i-1} \cap C^{(i-1)}$ and ending in $C^{(i-1)}$ is $\beta_{j+i-1} |W_{i-1} \cap C^{(i-1)}|$. Since $(C^{(t)})_t$ is a distance partition, all these edges end in W_i .

By the definition of a distance-regular graph, the number of edges beginning in $W_i \cap C^{(i)}$ and ending in W_{i-1} is $c_i |W_i \cap C^{(i)}|$. Since $\{C^{(t)}\}_t$ is a distance partition, all these edges end in $C^{(i-1)}$.

So, by double-counting the edges connecting $W_{i-1} \cap C^{(i-1)}$ and $W_i \cap C^{(i)}$ we get (??). By induction, we have (??). Since the left part of (??) is integer, the claim of the corollary is now obvious. \square

Corollary 2. *In Hamming graphs, there are no completely regular 1-codes with the following putative intersection arrays:*

$$\begin{aligned} &\{22, 16, 5; \underline{1, 2, 20}\}; \{25, 24, 3; \underline{1, 3, 20}\}; \{26, 24, 19; \underline{1, 3, 8}\}; \\ &\{32, 28, 9; \underline{1, 2, 28}\}; \{36, 28, 4; \underline{1, 2, 24}\}; \{44, 36, 5; \underline{1, 9, 40}\}; \\ &\{48, 35, 9; \underline{1, 7, 40}\}; \{63, 48, 10; \underline{1, 8, 54}\}; \{80, 63, 11; \underline{1, 9, 70}\}. \end{aligned}$$

Proof. For all arrays except $\{36, 28, 4; \underline{1, 2, 24}\}$ and $\{63, 48, 10; \underline{1, 8, 54}\}$, the existence of completely regular codes contradicts Corollary ??, which states that $\gamma_1 \gamma_2$ is divisible by 2 (false for $\{25, 24, 3; \underline{1, 3, 20}\}$, $\{26, 24, 19; \underline{1, 3, 8}\}$, $\{44, 36, 5; \underline{1, 9, 40}\}$, $\{48, 35, 9; \underline{1, 7, 40}\}$, and $\{80, 63, 11; \underline{1, 9, 70}\}$) and $\gamma_1 \gamma_2 \gamma_3$ is divisible by 6 (false for $\{22, 16, 5; \underline{1, 2, 20}\}$ and $\{32, 28, 9; \underline{1, 2, 28}\}$).

For $\{36, 28, 4; \underline{1, 2, 24}\}$, since 36 is the degree $(q-1)n$ of the Hamming graph $H(n, q)$, $n \geq 3$, we have $q \in \{2, 3, 4, 5, 7, 10, 13\}$. In all the cases, direct calculations show that $K_3(K_1^{-1}(S))$ is not integer, where $K_w(\cdot) = K_w(\cdot; n, q)$ and S is the quotient matrix corresponding to the array $\{36, 28, 4; \underline{1, 2, 24}\}$.

For $\{63, 48, 10; \underline{1, 8, 54}\}$, the proof is similar. \square

5. $\{42, 30, 12; \underline{1, 6, 28}\}$, $q = 7$

A putative distance-regular graph with intersection array $\{42, 30, 12; \underline{1, 6, 28}\}$ has order $343 = 7^3$. If such a graph can be realized as the coset graph

of a linear completely regular code in a Hamming graph, then it is a 7-ary code of length $7 = 42/(6 - 1)$, dimension $4 = 7 - 3$, distance 3, and covering radius 3. The weight distributions of the code and of its dual are $(1, 0, 0, 42, 42, 630, 840, 846)$ and $(1, 0, 0, 0, 42, 0, 210, 90)$, respectively.

Proposition 1. *If q is an odd prime power larger than 4, then there are no q -ary codes of length 7, dimension 3, and non-zero weights 4, 6, 7.*

Proof. Assume that such a code C exists. W.l.o.g., it has a generator matrix

$$\begin{pmatrix} 1 & 0 & 0 & ? & ? & ? & ? \\ 0 & 1 & 0 & ? & ? & ? & ? \\ 0 & 0 & 1 & ? & ? & ? & ? \end{pmatrix}$$

We note that the weight of each row can only be 4, and two rows do not have two common zero positions (otherwise, for $q > 4$, they can be combined in a weight-5 codeword contradicting the hypothesis). So, w.l.o.g., we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & a & 0 & b & c \\ 0 & 0 & 1 & x & y & 0 & z \end{pmatrix}.$$

If $b \neq c$, then there is a weight-5 linear combination of the first two rows, a contradiction. So, $b = c$; similarly, $y = z$; and, if we assume w.l.o.g. that $a = b$, we also see $x = y$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & a & 0 & a & a \\ 0 & 0 & 1 & x & x & 0 & x \end{pmatrix}.$$

Multiplying rows by coefficients, we get

$$\begin{pmatrix} -1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & a^{-1} & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & x^{-1} & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Now, the sum of the rows $(-1, a^{-1}, x^{-1}, 2, 0, 0, 1)$ has weight 5, a contradiction. \square

Corollary 3. *There are no linear completely regular codes with intersection array $\{42, 30, 12; 1, 6, 28\}$ in $H(7, 7)$.*

6. $\{52,42,16;1,6,28\}$, $q = 5$

A putative distance-regular graph with intersection array $\{52, 42, 16; 1, 6, 28\}$ has order $625 = 5^4$. If such a graph can be realized as the coset graph of a linear completely regular code in a Hamming graph, then it is a 5-ary code of length $13 = 52/(5 - 1)$, dimension $9 = 13 - 4$, distance 3, and covering radius 3. The weight distributions of the code and of its dual are $(1, 0, 0, 52, 260, 2028, 10660, 48620, 128076, 305240, 479596, 521040, 350480, 107072)$ and $(1, 0, 0, 0, 0, 0, 52, 0, 0, 364, 0, 208, 0) = \{0^1, 7^{52}, 10^{364}, 12^{208}\}$, respectively.

Proposition 2. *There is no 5-ary code of length 13, dimension 4, and non-zero weights 7, 10, 12.*

Proof. Let G be a 4×13 generator matrix of a putative code with considered parameters. The columns of G form a set S of 13 points of the vector space $\mathbb{F}_5^4 = \text{GF}(5)^4$. Any (3-dimensional) hyperplane $H \ni \bar{0}$ in \mathbb{F}_5^4 is determined by its dual vector \bar{h} : $H = \{\bar{x} \in \mathbb{F}_5^4 : \bar{x} \cdot \bar{h} = 0\}$. Since $\bar{h} \cdot G$ is a codeword of weight 7, 10, or 12, any such hyperplane intersects with S in $6 = 13 - 7$, $3 = 13 - 10$, or $1 = 13 - 12$ points. Now, consider a (2-dimensional) plane P containing at least two points of S . It is included in exactly six 3-dimensional hyperplanes, call them H_i , $i = 0, 1, 2, 3, 4, 5$. In the rest of the proof, we will show that any intersection numbers of S with P and H_i , $i = 0, 1, 2, 3, 4, 5$, are contradictory. Denote $a := |S \cap P|$, $a > 1$, and $b_i := |S \cap H_i|$, $b_i \in \{3, 6\}$. Now we have $13 = |S| = a + \sum_{i=0}^5 (b_i - a) = \sum_{i=0}^5 b_i - 5a = 6 \cdot 3 + k \cdot 3 - 5a$, where k is the number of i in $\{0, 1, 2, 3, 4, 5\}$ such that $b_i = 6$. We derive $k = 5(a - 1)/3$. Since k is integer, we have $a = 4$ and $k = 5$. However, $a = 4$ also implies that b_i cannot be 3 and hence $k = 6$, a contradiction. \square

Corollary 4. *There are no linear completely regular codes with intersection array $\{52, 42, 16; 1, 6, 28\}$ in $H(13, 5)$.*

7. $\{56,42,20;1,6,28\}$, $q = 3$

A putative distance-regular graph with intersection array $\{56, 42, 20; 1, 6, 28\}$ has order $729 = 3^6$. If such a graph can be realized as the coset graph of a linear completely regular code in a Hamming graph, then it is a 3-ary code of length $28 = 56/(3 - 1)$, dimension $22 = 28 - 6$, distance 3, and covering radius 3. The weight distributions of the dual code is $\{0^1, 12^{56}, 18^{392}, 21^{280}\}$.

Proposition 3. *There is no 3-ary code of length 28, dimension 6, and non-zero weights 12, 18, 21.*

We currently do not have a theoretical proof of the last proposition. The nonexistence of a code with the specified parameters was checked using the software [?] (1 hour of computation) and [?] (10 sec. of computation).

Corollary 5. *There are no linear completely regular codes with intersection array $\{56, 42, 20; 1, 6, 28\}$ in $H(28, 3)$.*

The following straightforward corollary of the fact above has an easy computer-free proof.

Corollary 6. *There are no linear completely regular codes with intersection array $\{56, 42, 20; 1, 6, 28\}$ in $H(7, 9)$.*

Proof. If such a code exists, then its dual weight distribution is $\{0^1, 4^{56}, 6^{392}, 7^{280}\}$, which is impossible by Proposition ?? \square

8. $\{140, 126, 20, 1; 1, 10, 126, 140\}$, $q = 3$

The only putative distance-regular graph in the [BCN] table with prime-power order and diameter more than 3 has order $2187 = 3^7$ and intersection array $\{140, 126, 20, 1; 1, 10, 126, 140\}$.

The corresponding completely regular code C in $H(70, 3)$ has intersection matrix

$$\begin{pmatrix} 0 & 140 & 0 & 0 & 0 \\ 1 & 13 & 126 & 0 & 0 \\ 0 & 10 & 110 & 20 & 0 \\ 0 & 0 & 126 & 13 & 1 \\ 0 & 0 & 0 & 140 & 0 \end{pmatrix}.$$

It is easy to see that $C \cup C^{(4)}$ is a completely regular code with intersection matrix

$$\begin{pmatrix} 0 & 140 & 0 \\ 1 & 13 & 126 \\ 0 & 30 & 110 \end{pmatrix}.$$

Since $C^{(4)}$ consists of 3 cosets of C , the code $C \cup C^{(4)}$ is linear; its coset graph is a strongly regular graph whose parameters are also questionable, according to [?].

The dual code is a ternary two-weight code with weight distribution $\{0^1, 45^{588}, 54^{140}\}$. However, a $[70, 6, 45]_3$ code does not exist [?].

Corollary 7. *There are no linear completely regular codes with intersection array $\{140, 126, 20, 1; 1, 10, 126, 140\}$ or $\{140, 126; 1, 30\}$ in $H(70, 3)$.*

9. Conclusion

We have shown that no distance-regular graphs with an unsolved intersection array from the table [?, Ch. 14] can be constructed as a Cayley graph on an elementary abelian group, or, equivalently, as the coset graph of a linear completely regular code.

Finally, we observe that the existence of unrestricted (not necessarily linear) completely regular codes with the parameters considered in Section- s ??-?? and the existence of few-distance codes with dual parameters remain unsolved.

Acknowledgments

The authors thank Sasha Kurz for helpful discussions and computer help.

References

- [1] L. A. Bassalygo, G. V. Zaitsev, and V. A. Zinoviev. On uniformly packed codes. *Probl. Inf. Transm.*, 10(1):6–9, 1974. Translated from *Probl. Peredachi Inf.*, 10(1): 9-14, 1974.
- [2] I. Bouyukliev and S. Bouyuklieva. QextNewEdition. <http://www.moi.math.bas.bg/moiuser/~data/Software/QextNewEdition>.
- [3] A. E. Brouwer. Parameters of strongly regular graphs. <https://www.win.tue.nl/~aeb/graphs/srg/srgtab.html>.
- [4] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs*. Springer-Verlag, Berlin, 1989. DOI: 10.1007/978-3-642-74341-2.
- [5] P. Delsarte. *An Algebraic Approach to Association Schemes of Coding Theory*, volume 10 of *Philips Res. Rep., Supplement*. N.V. Philips' Gloeilampenfabrieken, Eindhoven, 1973.
- [6] D. S. Krotov. On weight distributions of perfect colorings and completely regular codes. *Des. Codes Cryptography*, 61(3):315–329, 2011. DOI: 10.1007/s10623-010-9479-4.
- [7] S. Kurz. LINCOTE — computer classification of linear codes. E-print 1912.09357, arXiv.org, 2019.

- [8] W. J. Martin. *Completely Regular Subsets*. PhD thesis, University of Waterloo, Waterloo, Ontario, Canada, 1992. Online: <http://users.wpi.edu/~martin/RESEARCH/THESIS/>.
- [9] T. Maruta. The nonexistence of some ternary linear codes of dimension 6. *Discrete Math.*, 288(1-3):125–133, Nov. 2004. DOI: 10.1016/j.disc.2004.07.003.
- [10] A. Neumaier. Completely regular codes. *Discrete Math.*, 106-107:353–360, Sept. 1992. DOI: 10.1016/0012-365X(92)90565-W.
- [11] M. Shi, D. S. Krotov, and P. Solé. A new distance-regular graph of diameter 3 on 1024 vertices. *Des. Codes Cryptography*, 2019. DOI: 10.1007/s10623-019-00609-w.
- [12] M. Shi, D. S. Krotov, and P. Solé. A new approach to the Kasami codes of type 2. *IEEE Trans. Inf. Theory*, 66(4):2456–2465, Apr. 2020. DOI: 10.1109/TIT.2019.2949609.
- [13] S. Sumalroj and C. Worawannotai. The nonexistence of a distance-regular graph with intersection array $\{22, 16, 5; 1, 2, 20\}$. *Electr. J. Comb.*, 24(1):#P1.32(1–10), 2016. DOI: 10.37236/5486.
- [14] M. Urlep. Triple intersection numbers of Q -polynomial distance-regular graphs. *Eur. J. Comb.*, 33(6):1246–1252, Aug. 2012. DOI: 10.1016/j.ejc.2012.02.005.
- [15] E. R. van Dam, J. H. Koolen, and H. Tanaka. Distance-regular graphs. *Electr. J. Comb.*, Dynamic Surveys(#DS22), Apr. 2016.