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# A unified approach of obstructions to small-time local controllability for scalar-input systems

Karine Beauchard\*, Frédéric Marbach\*

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## Abstract

We present a unified approach for determining and proving obstructions to small-time local controllability of scalar-input control systems. Our approach views obstructions to controllability as resulting from interpolation inequalities between the functionals associated with the formal Lie brackets of the system.

Using this approach, we give compact unified proofs of all known necessary conditions, we prove a conjecture of 1986 due to Kawski, and we derive entirely new obstructions. Our work doubles the number of previously-known necessary conditions, all established in the 1980s. In particular, for the third quadratic bracket, we derive a new necessary condition which is complementary to the Agrachev–Gamkrelidze sufficient ones.

We rely on a recent Magnus-type representation formula of the state, a new Hall basis of the free Lie algebra over two generators, an appropriate use of Sussmann’s infinite product to compute the Magnus expansion, and Gagliardo–Nirenberg interpolation inequalities.

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# 1 Introduction

## 1.1 Scalar-input control-affine systems

In this article, we consider an affine control system

$$\dot{x}(t) = f_0(x(t)) + u(t)f_1(x(t)) \quad (1.1)$$

where the state  $x(t)$  lives in  $\mathbb{R}^d$  ( $d \geq 1$ ), the control is a scalar input  $u(t) \in \mathbb{R}$ ,  $f_0$  and  $f_1$  are vector fields on  $\mathbb{R}^d$ , real-analytic on a neighborhood of 0, such that  $f_0(0) = 0$ . These assumptions are valid for the whole article and will not be recalled in the statements. Nevertheless, the analyticity assumption can be removed, and all our results hold assuming only finite regularity on the vector fields, as we prove in Section 10.

For each  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ , there exists a unique maximal mild solution to (1.1) with initial data 0, which we will denote by  $x(\cdot; u)$ . We will consider small enough controls and small enough times so that this solution is defined up to time  $t$ .

## 1.2 Definitions of small-time local controllability

In this article, we study the small-time local controllability of system (1.1) in the sense of Definition 1.1 below, which requires the following notions.

For  $t > 0$  and  $m \in \mathbb{N}$ , we consider the usual Sobolev space  $W^{m, \infty}(0, t)$  equipped with the usual norm  $\|u\|_{W^{m, \infty}} := \|u\|_{L^\infty} + \dots + \|u^{(m)}\|_{L^\infty}$ . For  $j \in \mathbb{N}$ , we define by induction the iterated primitives of  $u$ , denoted  $u_j : (0, t) \rightarrow \mathbb{R}$  and defined by:  $u_0 := u$  and  $u_{j+1}(t) = \int_0^t u_j$ . We let

$$\|u\|_{W^{-1, \infty}} := \|u_1\|_{L^\infty}. \quad (1.2)$$

For scalar-input systems such as (1.1), the  $W^{-1, \infty}$  norm of the control is important because it is an accurate measure of the size of the state (see Lemma 4.3 and [5, Lemma 20]). For  $m \in \mathbb{Z}$ , we use the notation  $\llbracket m, \infty \rrbracket$  for  $[m, \infty) \cap \mathbb{Z}$ .

**Definition 1.1** ( $W^{m, \infty}$ -STLC). *Let  $m \in \llbracket -1, \infty \rrbracket$ . We say that system (1.1) is  $W^{m, \infty}$ -STLC when, for every  $t, \rho > 0$ , there exists  $\delta = \delta(t, \rho) > 0$  such that, for every  $x^* \in B(0, \delta)$ , there exists  $u \in W^{m, \infty}((0, t); \mathbb{R}) \cap L^1((0, t); \mathbb{R})$  with  $\|u\|_{W^{m, \infty}} \leq \rho$ , such that  $x(t; u) = x^*$ .*

Any positive answer to the STLC problem may be thought of as a nonlinear local open mapping theorem, which underlines the deepness and intricacy of this problem, when the inverse mapping theorem (or linear test, see [10, Section 3.1]) cannot be used.

The STLC notions used in the literature usually correspond to what we refer to as  $L^\infty$ -STLC (i.e.  $m = 0$  in Definition 1.1 above), where controls have to be arbitrarily small in  $L^\infty$  norm (see e.g. [10, Definition 3.2] or  $\text{STLC}_\varepsilon$  in [18]). Sometimes (see [31, 32]) authors investigate the  $\rho$ -bounded-STLC:  $\rho > 0$  is fixed and system (1.1) is  $\rho$ -bounded-STLC if, for every  $t > 0$ , there exists  $\delta > 0$  such that, for every  $x^* \in B(0, \delta)$ , there exists  $u \in L^\infty(0, t)$  with  $\|u\|_{L^\infty} \leq \rho$  such that  $x(t; u) = x^*$ .

For any  $m \in \mathbb{N}^*$ ,  $\rho > 0$  and  $t \in (0, 1)$ ,  $\|u\|_{W^{-1, \infty}} \leq t\|u\|_{L^\infty} \leq \|u\|_{W^{m, \infty}}$  thus

$$(W^{m, \infty}\text{-STLC}) \Rightarrow (L^\infty\text{-STLC}) \Rightarrow (\rho\text{-bounded-STLC}) \Rightarrow (W^{-1, \infty}\text{-STLC}), \quad (1.3)$$

where any reciprocal implication is false (see Appendix A.1). See also [7] for a recent comparison of various controllability definitions. The interest of the  $W^{-1, \infty}$ -STLC is that it is equivalent to the small-state small-time local controllability for scalar-input systems (see [5, Section 8.2]).

In the excellent survey [18], Kawski recalls the known necessary conditions (see Theorems 3.1, 3.4 and 3.5 therein) and sufficient conditions (see Theorems 3.6 and 3.7 therein) for  $L^\infty$ -STLC. Then he explains, on clever examples, the obstacles that a more complete theory has to overcome. Kawski's survey is at the root of the present article: our main results are generalizations to any systems, of its observations on particular examples which will be recalled and discussed later in the present article (see Sections 6.1, 7.1, 8.1 and 9.1).

### 1.3 Algebraic notations and Lie brackets

The STLC is closely related to the evaluations at 0 of the iterated Lie brackets of the vector fields  $f_0$  and  $f_1$ . We, therefore, introduce the following definitions and notations.

Let  $X := \{X_0, X_1\}$  be a set of two *non-commutative indeterminates*.

**Definition 1.2** (Free algebra). *We consider  $\mathcal{A}(X)$  the free algebra generated by  $X$  over the field  $\mathbb{R}$ , i.e. the unital associative algebra of polynomials of the indeterminates  $X_0$  and  $X_1$ .*

**Definition 1.3** (Free Lie algebra). *Within  $\mathcal{A}(X)$  one can define the Lie bracket of two elements as  $[a, b] := ab - ba$ . This operation is anti-symmetric and satisfies the Jacobi identity. Let  $\mathcal{L}(X)$  be the free Lie algebra generated by  $X$  over the field  $\mathbb{R}$ , i.e. the smallest linear subspace of  $\mathcal{A}(X)$  containing  $X$  and stable by the Lie bracket  $[\cdot, \cdot]$ .*

**Definition 1.4** (Iterated brackets). *Let  $\text{Br}(X)$  be the free magma over  $X$ , or, more visually, the set of iterated brackets of elements of  $X$ , defined by induction:  $X_0, X_1 \in \text{Br}(X)$  and if  $a, b \in \text{Br}(X)$ , then the ordered pair  $(a, b)$  belongs to  $\text{Br}(X)$ .*

*There is a natural evaluation mapping  $\mathbb{E}$  from  $\text{Br}(X)$  to  $\mathcal{L}(X)$  defined by induction by  $\mathbb{E}(X_i) := X_i$  for  $i = 0, 1$  and  $\mathbb{E}((a, b)) := [\mathbb{E}(a), \mathbb{E}(b)]$ . Through this mapping,  $\text{Br}(X)$  spans  $\mathcal{L}(X)$ .*

**Definition 1.5** (Homogeneous layers within  $\mathcal{L}(X)$ ). *For  $b \in \text{Br}(X)$ ,  $n_0(b)$  (respectively  $n_1(b)$ ) denotes the number of occurrences of the indeterminate  $X_0$  (resp.  $X_1$ ) in  $b$ . For  $A_1, A_0 \subset \mathbb{N}$ ,  $S_{A_1}(X)$  and  $S_{A_1, A_0}(X)$  are the vector subspaces of  $\mathcal{L}(X)$  defined by*

$$S_{A_1}(X) := \text{span}\{\mathbb{E}(b); b \in \text{Br}(X), n_1(b) \in A_1\}, \quad (1.4)$$

$$S_{A_1, A_0}(X) := \text{span}\{\mathbb{E}(b); b \in \text{Br}(X), n_1(b) \in A_1, n_0(b) \in A_0\}. \quad (1.5)$$

For  $i, j \in \mathbb{N}$ , we write<sup>1</sup>  $S_i(X)$  and  $S_{i,j}(X)$  instead of  $S_{\{i\}}(X)$  and  $S_{\{i\}, \{j\}}(X)$ .

**Definition 1.6** (Bracket integration  $b0^\nu$ ). *For  $b \in \text{Br}(X)$  and  $\nu \in \mathbb{N}$ , we use the unconventional short-hand  $b0^\nu$  to denote the right-iterated bracket  $(\cdots (b, X_0), \dots, X_0)$ , where  $X_0$  appears  $\nu$  times.*

**Definition 1.7** (Lie bracket of vector fields). *For smooth vector fields  $f$  and  $g$ , we define*

$$[f, g] := (Dg)f - (Df)g. \quad (1.6)$$

**Definition 1.8** (Evaluated Lie bracket). *Let  $f_0, f_1$  be  $\mathcal{C}^\infty$  vector fields on an open subset  $\Omega$  of  $\mathbb{R}^d$  and  $f = \{f_0, f_1\}$ .*

*For  $B \in \mathcal{L}(X)$ , we define  $f_B := \Lambda(B)$ , where  $\Lambda : \mathcal{L}(X) \rightarrow \mathcal{C}^\infty(\Omega; \mathbb{R}^d)$  is the unique homomorphism of Lie algebras such that  $\Lambda(X_0) = f_0$  and  $\Lambda(X_1) = f_1$ .*

*Overloading this notation, we will write  $f_b$  instead of  $f_{\mathbb{E}(b)}$  when  $b \in \text{Br}(X)$ . The vector field  $f_b$  is obtained by replacing the indeterminates  $X_i$  with the corresponding vector field  $f_i$  in the formal bracket  $b$ . For instance if  $b = (X_1, (X_0, X_1))$  then  $f_b = [f_1, [f_0, f_1]]$  and if  $B = \alpha_1 \mathbb{E}(b_1) + \cdots + \alpha_n \mathbb{E}(b_n) \in \mathcal{L}(X)$  where  $b_1, \dots, b_n \in \text{Br}(X)$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  then  $f_B = \alpha_1 f_{b_1} + \cdots + \alpha_n f_{b_n}$ .*

*Eventually, for a subset  $\mathcal{N}$  of  $\text{Br}(X)$  we use the notation*

$$\mathcal{N}(f)(0) := \text{span}\{f_b(0); b \in \mathcal{N}\} \subset \mathbb{R}^d. \quad (1.7)$$

All the known necessary conditions for STLC are stated in the following way. One focuses on a “bad” bracket  $\mathfrak{b} \in \text{Br}(X)$  and one identifies a subset  $\mathcal{N}$  of  $\text{Br}(X)$  containing all the brackets susceptible to neutralize  $\mathfrak{b}$ . Then the necessary condition for STLC is  $f_{\mathfrak{b}}(0) \in \mathcal{N}(f)(0)$ .

This is linked with Krener’s fundamental result [24, Theorem 1], which states that, if two control systems of the form (1.1) have linearly isomorphic brackets evaluated at 0, then they are diffeomorphic. Thus the entire information about STLC is contained in the subset of  $\mathbb{R}^d$  made of the evaluations at 0 of the Lie brackets of the vector fields  $f_0$  and  $f_1$ .

<sup>1</sup>This choice is different from the usual choice in control theory where, for example, other authors most often write  $S_i(X)$  to denote what we refer to here as  $S_{[1, i]}(X)$ .

## 1.4 A new basis of the free Lie algebra

In this article, we construct a new basis of the free Lie algebra  $\mathcal{L}(X)$ , which is of the form  $E(\mathcal{B}^*)$ , where  $\mathcal{B}^*$  is a Hall set of  $\text{Br}(X)$  (see Definition 2.5). All our results are expressed using this basis.

The main interest of  $\mathcal{B}^*$  is the particular form of the associated coordinates of the second kind, which appear to be very well suited for control results and functional analysis (see Section 3.3). The basis  $\mathcal{B}^*$  answers, in some sense, the first open problem of [21]: “construct a basis for the free Lie algebra such that the corresponding coordinates of the first kind have simple formulas” (see also Remark 3.8 for more details).

In particular, a key point is that  $\mathcal{B}^*$  seems to allow to immediately guess from the structure of a Lie bracket and/or from its associated coordinate of the second kind if this bracket will lead to an obstruction or not. As noted by Kawski in [18, Section 4], such a feature (“splitting” bad and good Lie brackets) is not satisfied by the usual length-compatible or Chen–Fox–Lyndon bases of the free Lie algebra. We plan to investigate further this property of  $\mathcal{B}^*$  in a forthcoming paper.

The first elements of  $\mathcal{B}^*$  are given explicitly in the following statement.

**Proposition 1.9.** *The first  $X_1$ -homogeneous layers  $\mathcal{B}_k^* := \{b \in \mathcal{B}^*; n_1(b) = k\}$  of our basis  $\mathcal{B}^*$  are given by*

$$\mathcal{B}_1^* = \{M_\nu\}, \quad (1.8)$$

$$\mathcal{B}_2^* = \{W_{j,\nu}\}, \quad (1.9)$$

$$\mathcal{B}_3^* = \{P_{j,k,\nu}; j \leq k\}, \quad (1.10)$$

$$\mathcal{B}_4^* = \{Q_{j,k,l,\nu}; j \leq k \leq l\} \cup \{Q_{j,\mu,k,\nu}^\sharp; j < k\} \cup \{Q_{j,\mu,\nu}^\flat\}, \quad (1.11)$$

$$\mathcal{B}_5^* = \{R_{j,k,l,m,\nu}; j \leq k \leq l \leq m\} \cup \{R_{j,k,l,\mu,\nu}^\sharp; j \leq k\}, \quad (1.12)$$

where, implicitly,  $j, k, l, m \in \mathbb{N}^*$ ,  $\mu, \nu \in \mathbb{N}$  and we define (using the notation  $0^\nu$  of Definition 1.6),

$$M_\nu := X_1 0^\nu, \quad (1.13)$$

$$W_{j,\nu} := (M_{j-1}, M_j) 0^\nu, \quad (1.14)$$

$$P_{j,k,\nu} := (M_{k-1}, W_{j,0}) 0^\nu, \quad (1.15)$$

$$Q_{j,k,l,\nu} := (M_{l-1}, P_{j,k,0}) 0^\nu, \quad Q_{j,\mu,k,\nu}^\sharp := (W_{j,\mu}, W_k) 0^\nu, \quad Q_{j,\mu,\nu}^\flat := (W_{j,\mu}, W_{j,\mu+1}) 0^\nu, \quad (1.16)$$

$$R_{j,k,l,m,\nu} := (M_{m-1}, Q_{j,k,l,0}) 0^\nu, \quad R_{j,k,l,\mu,\nu}^\sharp := (W_{l,\mu}, P_{j,k,0}) 0^\nu. \quad (1.17)$$

To lighten the notations,  $W_k$ ,  $P_{j,k}$  and  $Q_{j,k,l}$  will denote  $W_{k,0}$ ,  $P_{j,k,0}$  and  $Q_{j,k,l,0}$ .

We only write explicitly the elements  $\mathcal{B}_{[1,5]}^*$ , because notations for the elements of  $\mathcal{B}_{[6,\infty]}^*$  will not be required in the sequel. Of course, the list could go further, albeit with increasing complexity.

One could also probably extend our construction of  $\mathcal{B}^*$  to the case of control systems with multiple inputs. For such systems, one needs a basis of the free Lie algebra over  $\{X_0, X_1, \dots, X_q\}$ . Many such extended constructions could be proposed and the “correct” one might depend on the intended applications. We discuss some key structural features of  $\mathcal{B}^*$  (which could be preserved with multiple inputs) in Remark 3.5.

## 1.5 Main results: old and new necessary conditions

First, we recover (slightly improved versions of) the necessary conditions for STLC, due to Sussmann [32, Proposition 6.3] (for  $k = 1$ ) and Stefani [31, Theorem 1] (for  $k > 1$ ), concerning the strongest obstruction at each even order of the control, which were historically derived for the stronger  $\rho$ -bounded-STLC notion (recall the implications (1.3)).

**Theorem 1.10.** *If system (1.1) is  $W^{-1,\infty}$ -STLC (or, equivalently, small-state-STLC), then*

$$\forall k \in \mathbb{N}^*, \quad \text{ad}_{f_1}^{2k}(f_0)(0) \in S_{[1,2k-1]}(f)(0). \quad (1.18)$$

Then we prove the following statement, that contains necessary condition for controllability on all the quadratic Lie brackets  $W_k$  for  $k \in \mathbb{N}^*$ .

**Theorem 1.11.** *Let  $m \in \mathbb{N}^*$ . If system (1.1) is  $W^{m,\infty}$ -STLC, then*

$$\forall k \in \mathbb{N}^*, \quad f_{W_k}(0) \in S_{\llbracket 1, \pi(k, m) \rrbracket \setminus \{2\}}(f)(0) \quad (1.19)$$

where

$$\pi(k, m) := 1 + \left\lceil \frac{2k - 2}{m + 1} \right\rceil. \quad (1.20)$$

As particular cases, this result contains necessary conditions on  $W_k$  for

- $W^{2k-3, \infty}$ -STLC:  $f_{W_k}(0) \in S_1(f)(0)$ , which is already proved in [5, Theorem 3],
- $L^\infty$ -STLC:  $f_{W_k}(0) \in S_{\llbracket 1, 2k-1 \rrbracket \setminus \{2\}}(f)(0)$ , which was conjectured in 1986 in [17, p. 63],
- $W^{m, \infty}$ -STLC with  $1 \leq m \leq 2k - 4$ , which is a new result.

Moreover, it would be natural to expect that Theorem 1.11 holds for  $m = -1$  with  $\pi(k, -1) := +\infty$ . We discuss this topic in Section 6.8.

For  $k \in \{2, 3\}$ , a careful analysis allows to refine the necessary condition of Theorem 1.11. In particular, in the case  $m = 0$ , we prove the following results.

**Theorem 1.12.** *If system (1.1) is  $L^\infty$ -STLC, then  $f_{W_j}(0) \in \mathcal{N}_j(f)(0)$  for  $j = 1, 2, 3$ , where*

$$\mathcal{N}_1 := \mathcal{B}_1^*, \quad (1.21)$$

$$\mathcal{N}_2 := \mathcal{N}_1 \cup \{P_{1,1,\nu}; \nu \in \mathbb{N}\}, \quad (1.22)$$

$$\mathcal{N}_3 := \mathcal{N}_2 \cup \{P_{1,l,\nu}, Q_{1,1,1}, Q_{1,1,2,\nu}, Q_{1,0}^b, Q_{1,1}^b, Q_{1,2}^b, R_{1,1,1,1,\nu}, R_{1,1,1,\mu,\nu}^\sharp; l \in \mathbb{N}^*, \mu, \nu \in \mathbb{N}\}. \quad (1.23)$$

The statement concerning  $W_2$  is proved by Kawski in [19, Theorem 1], using the Chen–Fliess expansion and technical results from Stefani [31]. We propose a different strategy, that allows us to obtain similarly the condition concerning  $W_3$ , which is new. Moreover, the lists  $\mathcal{N}_j$  are “minimal” in the sense that, for any bracket in  $\mathcal{N}_j$ , we exhibit a system where it restores STLC when in competition with  $W_j$ . In the hardest case  $j = 3$ , we prove these controllability results using the Agrachev–Gamkrelidze sufficient condition of [1, Theorem 4], illustrating that, the necessary condition  $f_{W_3}(0) \in \mathcal{N}_3(f)(0)$  is somehow complementary to their sufficient conditions theory.

To go beyond quadratic phenomena, we prove the following necessary condition linked with a bracket of the sixth order with respect to the control, which is entirely new.

**Theorem 1.13.** *If system (1.1) is  $L^\infty$ -STLC, then*

$$f_{\text{ad}_{P_{1,1}}^2(X_0)}(0) \in \text{span} \left\{ f_b(0); b \in \mathcal{B}_{\llbracket 1, 7 \rrbracket}^*, b \neq \text{ad}_{P_{1,1}}^2(X_0) \right\}. \quad (1.24)$$

Throughout the paper, we will discuss the optimality of all these necessary conditions by comparing them with the known sufficient conditions, including the ones due to Agrachev and Gamkrelidze [1, Theorem 4] or Krastanov [23, Theorem 2.7]. In particular, let us already point out that, due to their structure, all the brackets involved in the above-mentioned obstructions, namely  $\text{ad}_{X_1}^{2k}(X_0)$ ,  $W_k = \text{ad}_{M_{k-1}}^2(X_0)$  and  $\text{ad}_{P_{1,1}}^2(X_0)$  are indeed always seen as “bad” and required to be compensated by such sufficient conditions (see Appendix A.2).

Eventually, we explain in Section 10 why all these results, derived for analytic vector fields, remain valid without change for  $\mathcal{C}^\infty$  vector fields. More precisely, we show that assuming only finite regularity on  $f_0$  and  $f_1$  is sufficient to preserve the conclusions, provided that one gives the appropriate meaning to the evaluations at 0 of the considered brackets (the brackets themselves being undefined elsewhere).

## 1.6 Heuristic of the unified approach for obstructions to STLC

Our theorems are of the form

$$W^{m,\infty}\text{-STLC} \quad \Rightarrow \quad f_{\mathfrak{b}}(0) \in \mathcal{N}(f)(0) \quad (1.25)$$

where  $m \in \llbracket -1, \infty \llbracket$ ,  $\mathfrak{b} \in \mathcal{B}^*$  and  $\mathcal{N}$  is a subset of  $\mathcal{B}^*$ . Our strategy consists in proving a drift of  $x(t; u)$ , along  $f_{\mathfrak{b}}(0)$  when

$$f_{\mathfrak{b}}(0) \notin \mathcal{N}(f)(0) \quad (1.26)$$

in the sense of Definition 1.15 below, which requires the following notion.

**Definition 1.14** (Component along a vector parallel to a subspace). *Let  $N$  be a vector subspace of  $\mathbb{R}^d$  and  $e \in \mathbb{R}^d \setminus N$ . We say that a linear form  $\mathbb{P} : \mathbb{R}^d \rightarrow \mathbb{R}$  is a component along  $e$  parallel to  $N$  when  $\mathbb{P}e = 1$  and  $N \subset \ker \mathbb{P}$ .*

**Definition 1.15** (Drift). *Let  $\mathfrak{b} \in \mathcal{B}^*$ ,  $\mathcal{N} \subset \text{Br}(X)$  and  $m \in \llbracket -1, \infty \llbracket$ . We say that system (1.1) has a drift along  $f_{\mathfrak{b}}(0)$ , parallel to  $\mathcal{N}(f)(0)$ , in the regime  $t \rightarrow 0$  and  $\|u\|_{W^{m,\infty}} \rightarrow 0$  when there exists  $C > 0$ ,  $\beta > 1$  such that, for every  $\varepsilon > 0$ , there exists  $t^* = t^*(\varepsilon) > 0$  such that, for every  $t \in (0, t^*)$ , there exists  $\rho = \rho(\varepsilon, t) > 0$  such that, for every  $u \in W^{m,\infty}((0, t); \mathbb{R}) \cap L^1((0, t); \mathbb{R})$  with  $\|u\|_{W^{m,\infty}} < \rho$ ,*

$$\mathbb{P}x(t; u) \geq (1 - \varepsilon)\xi_{\mathfrak{b}}(t, u) - C|x(t; u)|^\beta, \quad (1.27)$$

where  $\mathbb{P}$  gives a component along  $f_{\mathfrak{b}}(0)$  parallel to  $\mathcal{N}(f)(0)$  and  $(\xi_{\mathfrak{b}})_{\mathfrak{b} \in \mathcal{B}^*}$  are the coordinates of the second type associated with  $\mathcal{B}^*$  (see Definition 2.9 and Proposition 3.7).

When  $\xi_{\mathfrak{b}}(t, \cdot)$  takes only non-negative values, estimate (1.27) prevents  $x(t; u)$  from reaching targets of the form  $x^* = -af_{\mathfrak{b}}(0)$  with  $0 < a \ll 1$ , because this would entail  $-a = \mathbb{P}x^* \geq -C|x^*|^\beta = -Ca^\beta$ , which fails for  $a$  small enough, since  $\beta > 1$ . Thus, estimate (1.27) falsifies  $W^{m,\infty}$ -STLC. It also proves that the unreachable set contains locally a half-space because  $\beta > 1$ .

**Remark 1.16.** *We will use the terminology “in the regime  $t \rightarrow 0$  and  $\|u\|_{W^{m,\infty}} \rightarrow 0$ ” in the sense given in Definition 1.15. We will sometimes use the slightly different terminology “in the regime  $(t, \|u\|_{W^{m,\infty}}) \rightarrow 0$ ” to mean that the smallness assumption on  $u$  does not depend on  $t$ , i.e. in the quantification above,  $\rho = \rho(\varepsilon)$  does not depend on  $t$ .*

The starting point of our strategy is an approximate representation formula, explained in Section 4.3, that holds in the regime  $(t, \|u\|_{W^{-1,\infty}}) \rightarrow 0$ ,

$$x(t; u) = \mathcal{Z}_M(t, f, u)(0) + O\left(\|u\|_{W^{-1,\infty}}^{M+1} + |x(t; u)|^{1+\frac{1}{M}}\right), \quad (1.28)$$

where, for every  $M \in \mathbb{N}^*$ ,  $\mathcal{Z}_M(t, f, u)$  is an analytic vector field that belongs to  $S_{\llbracket 1, M \rrbracket}(f)$ . By applying the CBHD-formula to Sussmann’s infinite product representation formula, we obtain an expression of  $\mathcal{Z}_M(T, f, u)$  of the form

$$\mathcal{Z}_M(T, f, u) = \sum_{\mathfrak{b} \in \mathcal{B}_{\llbracket 1, M \rrbracket}^*} \xi_{\mathfrak{b}}(t, u)f_{\mathfrak{b}} + \text{some cross products}, \quad (1.29)$$

where the functionals  $\xi_{\mathfrak{b}}(t, u)$  are the coordinates of the second type associated with  $\mathcal{B}^*$  (see Definition 2.9 and Proposition 3.7). They do not depend on  $f$  and are explicitly given by an induction relation on  $\mathcal{B}^*$ .

We consider a bracket  $\mathfrak{b} \in \mathcal{B}^*$  for which  $\xi_{\mathfrak{b}}(t, \cdot)$  is positive-definite (i.e. for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R}) \setminus \{0\}$ ,  $\xi_{\mathfrak{b}}(t, u) > 0$ ) and a subset  $\mathcal{N}$  of  $\mathcal{B}^*$ . We assume (1.26) which allows to consider  $\mathbb{P} : \mathbb{R}^d \rightarrow \mathbb{R}$ , a component along  $f_{\mathfrak{b}}(0)$  parallel to  $\mathcal{N}(f)(0)$ . Then, for every  $M \in \mathbb{N}^*$ ,

$$\begin{aligned} \mathbb{P}x(t; u) = \xi_{\mathfrak{b}}(t, u) + \sum_{\mathfrak{b} \in \mathcal{B}_{\llbracket 1, M \rrbracket}^* \setminus (\mathcal{N} \cup \{\mathfrak{b}\})} \xi_{\mathfrak{b}}(t, u)\mathbb{P}f_{\mathfrak{b}}(0) + \mathbb{P}(\text{cross products}) \\ + O\left(\|u\|_{W^{-1,\infty}}^{M+1} + |x(t; u)|^{1+\frac{1}{M}}\right). \end{aligned} \quad (1.30)$$

Now, we work in the regime  $(t, \|u\|_{W^{m,\infty}}) \rightarrow 0$  for some given  $m \in \llbracket -1, \infty \llbracket$ :

1. we choose  $M$  such that  $\|u\|_{W^{-1,\infty}}^{M+1} = o(\xi_b(t, u))$  in the regime  $(t, \|u\|_{W^{m,\infty}}) \rightarrow 0$ ,
2. we choose  $\mathcal{N}$  as the set of  $b \in \mathcal{B}^* \setminus \{\mathfrak{b}\}$  such that  $\xi_b \neq o(\xi_b)$  in the regime  $(t, \|u\|_{W^{m,\infty}}) \rightarrow 0$ ,
3. we prove bounds on the cross products, of the form: cross products =  $o(\xi_b(t, u) + |x(t; u)|)$

and we conclude that

$$\mathbb{P}x(t; u) = \xi_b(t, u) + o(|x(t; u)| + \xi_b(t, u)). \quad (1.31)$$

In the above method, all three steps rely on various interpolation inequalities.

The third step is the hardest, and is not guaranteed to work systematically. To bound the cross products, we first prove that the assumption (1.26) implies vectorial relations among other elements  $f_b(0)$  for  $b \in \mathcal{B}^*$ . Then, we prove that these vectorial relations entail what we call “closed-loop estimates”, i.e. that some coordinates  $\xi_b(t, u)$  for some particular  $b \in \mathcal{B}^*$  involved in the cross products can be estimated from  $|x(t; u)|$  and higher-order terms involving the control. This is a key argument of the method.

## 1.7 On the invariance by change of coordinates and feedbacks

As can be seen by inspection of Definition 1.1 (see also [5, Lemma 16] for the case  $m > 0$ ), two important classes of transformations preserve small-time local controllability.

First, real-analytic changes of coordinates in the state space. As recalled in Section 1.3, stating conditions involving only Lie brackets evaluated at 0 automatically guarantees that our conditions are coordinate-invariant, since the Lie brackets evaluated at 0 in the new coordinates are linearly isomorphic to the ones in the old coordinates (see [24, Theorem 1] or also [3, Section 8.2]).

Second, static-state feedbacks (so particular classes of changes of coordinates for the couple  $(x, u)$  which preserve the control-affine nature of the system), or even changes of time-scale (see [16, Weak feedback equivalence]). As criticized by Lewis in [25], methods based on the identification of vector fields with a free Lie algebra generally don’t embed the invariance under such feedback transformations. To preserve such an invariance, other approaches are necessary, such as Agrachev and Gamkrelidze’s “control of diffeomorphisms” [1, 2] or Lewis’ “tautological systems” [26, 27].

While some of the above necessary conditions are definitely not feedback-invariant (and we plan to study this matter further in a forthcoming work), they still provide computationally checkable necessary conditions, and provide a nice counterpoint to sufficient conditions such as [1, Theorem 4] or [23, Theorem 2.7]. Moreover, some of them are feedback-invariant. Indeed, [8, Theorem 1] implies that Theorem 1.10 and Theorem 1.12 for  $j = 1, 2$  are feedback-invariant.

## 1.8 Structure of the article

First, in Section 2, we recall the fundamental notion of formal differential equations set in the algebra of formal series over  $X$ , which allows to model control systems of the form (1.1) in a way which is independent of  $f_0$  and  $f_1$ . We give two important expansions of its solutions: *Sussmann’s infinite product* based on coordinates of the second kind within a Hall basis, and our *Magnus in the interaction picture expansion*. We explain how coordinates of the second kind can be used to compute coefficients of the Magnus-type formula.

In Section 3, we introduce a new Hall set  $\mathcal{B}^*$  over  $\{X_0, X_1\}$  which yields a Hall basis of  $\mathcal{L}(X)$  particularly well adapted to control problems, and provide explicit expressions and estimates of these up to the fifth order in the control.

In Section 4, we explain how the formal results of Section 2 translate to system (1.1) driven by analytic vector fields. We state key propositions which will be used as black-boxes in the proofs of the main obstruction results.



We then turn to the proof of the main obstruction results. Each necessary condition for controllability stated in the introduction is derived as a consequence of a more precise drift statement.

In Section 5, we prove Theorem 1.10. In Section 6, we prove Theorem 1.11. In Section 7, we prove Theorem 1.12 for the case  $j = 2$ . In Section 8, we prove Theorem 1.12 for the case  $j = 3$ . In Section 9, we prove Theorem 1.13.

Eventually, in Section 10, we remove the analyticity assumption used throughout the paper.

## 2 Tools from formal power series

In Section 2.1, we introduce the formal differential equation (2.1) whose solution  $x(t)$ , is a formal power series.

In Section 2.2, we recall the well-known notions of Hall sets and Hall bases, which yield bases of  $\mathcal{L}(X)$ , with which one can express the solutions to (2.1).

In Section 2.3, we present an expansion due to Sussmann for the formal power series  $x(t)$  as an (infinite) product of exponentials of the members of a Hall basis, multiplied by coefficients that have simple expressions as iterated integrals, called *coordinates of the second kind*.

In Section 2.4, we recall a Magnus-type formula for the solution  $x(t)$ , called “Magnus expansion in the interaction picture”. It expresses the formal power series  $x(t)$  as the product of the exponential of  $tX_0$  with the exponential of a formal Lie series  $\mathcal{Z}_\infty(t, X, u) \in \widehat{\mathcal{L}}(X)$ . This formal Lie series  $\mathcal{Z}_\infty(t, X, u)$  can be expanded on any basis of the free Lie algebra  $\mathcal{L}(X)$ ; the associated coordinates are called *coordinates of the pseudo first kind*. We give an expression of  $\mathcal{Z}_\infty(t, X, u)$  in terms of coordinates of the second kind associated with a Hall set.

### 2.1 The formal differential equation

Fundamental in this project is the use of the formal differential equation

$$\begin{cases} \dot{x}(t) = x(t)(X_0 + u(t)X_1), \\ x(0) = 1. \end{cases} \quad (2.1)$$

The goal of this section is to define its solutions. This requires the following notions.

**Definition 2.1** (Graded algebra). *The free associative algebra  $\mathcal{A}(X)$  (see Definition 1.2) can be seen as a graded algebra:*

$$\mathcal{A}(X) = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n(X), \quad (2.2)$$

where  $\mathcal{A}_n(X)$  is the finite-dimensional  $\mathbb{R}$ -vector space spanned by monomials of degree  $n$  over  $X$ . In particular  $\mathcal{A}_0(X) = \mathbb{R}$  and  $\mathcal{A}_1(X) = \text{span}_{\mathbb{R}}(X)$ .

**Definition 2.2** (Formal series). *We consider the (unital associative) algebra  $\widehat{\mathcal{A}}(X)$  of formal series generated by  $\mathcal{A}(X)$ . An element  $a \in \widehat{\mathcal{A}}(X)$  is a sequence  $a = (a_n)_{n \in \mathbb{N}}$  written  $a = \sum_{n \in \mathbb{N}} a_n$ , where  $a_n \in \mathcal{A}_n(X)$  with, in particular,  $a_0 \in \mathbb{R}$  being its constant term.*

*We also define the Lie algebra of formal Lie series  $\widehat{\mathcal{L}}(X)$  as the Lie algebra of formal power series  $a \in \widehat{\mathcal{A}}(X)$  for which  $a_n \in \mathcal{L}(X)$  for each  $n \in \mathbb{N}$ .*

Within the realm of formal series, one can define the operators  $\exp$  and  $\log$ . For instance, for  $a \in \widehat{\mathcal{A}}(X)$  with  $a_0 = 0$ ,  $\exp(a) := \sum_{k=0}^{\infty} \frac{a^k}{k!}$  is a well-defined formal series.

The solutions to (2.1) are defined in the following way.

**Definition 2.3** (Solution to a formal ODE). *Let  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ . The solution to the formal ODE (2.1) is the formal-series valued function  $x : [0, t] \rightarrow \widehat{\mathcal{A}}(X)$ , whose homogeneous components  $x_n : \mathbb{R}_+ \rightarrow \mathcal{A}_n(X)$  are the unique continuous functions that satisfy, for every  $s \geq 0$ ,  $x_0(s) = 1$  and, for every  $n \in \mathbb{N}^*$ ,*

$$x_n(s) = \int_0^s x_{n-1}(s')(X_0 + u(s')X_1) ds'. \quad (2.3)$$

## 2.2 Hall sets and bases

We recall the notion of Hall sets and Hall bases. For more details on these bases of  $\mathcal{L}(X)$ , we refer to [9], [29, Chapter 4] or [35, Chapter 1].

**Definition 2.4** (Length, left and right factors). *For  $b \in \text{Br}(X)$ ,  $|b|$  denotes the length of  $b$ . If  $|b| > 1$ ,  $b$  can be written in a unique way as  $b = (b_1, b_2)$ , with  $b_1, b_2 \in \text{Br}(X)$ . We use the notations  $\lambda(b) = b_1$  and  $\mu(b) = b_2$ , which define maps  $\lambda, \mu : \text{Br}(X) \setminus X \rightarrow \text{Br}(X)$ .*

**Definition 2.5** (Hall set). *A Hall set is a subset  $\mathcal{B}$  of  $\text{Br}(X)$ , totally ordered by a relation  $<$  and such that*

- $X \subset \mathcal{B}$ ,
- for  $b = (b_1, b_2) \in \text{Br}(X)$ ,  $b \in \mathcal{B}$  iff  $b_1, b_2 \in \mathcal{B}$ ,  $b_1 < b_2$  and either  $b_2 \in X$  or  $\lambda(b_2) \leq b_1$ ,
- for every  $b_1, b_2 \in \mathcal{B}$  such that  $(b_1, b_2) \in \mathcal{B}$ , one has  $b_1 < (b_1, b_2)$ .

The main interest of Hall sets is that their images by  $\mathbb{E}$  yield algebraic bases of  $\mathcal{L}(X)$ , called Hall bases, as proved in [35, Corollary 1.1, Proposition 1.1 and Theorem 1.1].

**Theorem 2.6** (Viennot). *Let  $\mathcal{B} \subset \text{Br}(X)$  be a Hall set. Then  $\mathbb{E}(\mathcal{B})$  is a basis of  $\mathcal{L}(X)$ .*

**Remark 2.7.** *Historically, Hall sets were introduced by Marshall Hall in [12], based on ideas of Philip Hall in [13]. In his historical narrower definition, the third condition in Definition 2.5 was replaced by the stronger condition: for every  $b_1, b_2 \in \mathcal{B}$ ,  $b_1 < b_2 \Rightarrow |b_1| \leq |b_2|$ .*

*Two famous families of Hall sets are the Chen–Fox–Lyndon ones (see [35, p. 15-16]) whose order stems from the lexicographic order on words and the historical length-compatible Hall sets, for which  $b_1 < b_2 \Rightarrow |b_1| \leq |b_2|$ . Other examples, such as the Spitzer–Foata basis are studied in [4] and [35, Chapter 1].*

**Definition 2.8** (Support). *Let  $\mathcal{B}$  be a Hall set of  $\text{Br}(X)$  and  $a \in \mathcal{L}(X)$ . For  $b \in \mathcal{B}$ , we denote by  $\langle a, b \rangle_{\mathcal{B}}$  the coefficient of  $\mathbb{E}(b)$  in the expansion of  $a$  on the basis  $\mathbb{E}(\mathcal{B})$ . We define*

$$\text{supp}_{\mathcal{B}}(a) := \{b \in \mathcal{B}; \langle a, b \rangle_{\mathcal{B}} \neq 0\}. \quad (2.4)$$

*If  $A \subset \mathcal{L}(X)$ , we denote by  $\text{supp}_{\mathcal{B}}(A) := \cup_{a \in A} \text{supp}_{\mathcal{B}}(a)$ . We drop the subscripts  $\mathcal{B}$  when there is no possible confusion on which basis is used.*

## 2.3 Sussmann’s infinite product

In this section, we present an expansion for the formal power series  $x(t)$  solution to (2.1) as a product of exponentials of the members of a Hall set, multiplied by coefficients that have simple expressions as iterated integrals, called *coordinates of the second kind*. This infinite product is an extension to all Hall bases of Sussmann’s infinite product on length-compatible Hall bases [33], suggested in [20] also proved in [3, Section 2.5].

**Definition 2.9.** *Let  $\mathcal{B} \subset \text{Br}(X)$  be a Hall set. The coordinates of the second kind associated with  $\mathcal{B}$  is the unique family  $(\xi_b)_{b \in \mathcal{B}}$  of functionals  $\mathbb{R}_+ \times L_{\text{loc}}^1(\mathbb{R}_+; \mathbb{R}) \rightarrow \mathbb{R}$  defined by induction in the following way: for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$*

- $\xi_{X_0}(t, u) := t$  and  $\xi_{X_1}(t, u) := \int_0^t u = u_1(t)$ ,
- for  $b \in \mathcal{B} \setminus X$ , there exists a unique couple  $(b_1, b_2)$  of elements of  $\mathcal{B}$  such that  $b_1 < b_2$  and a unique maximal integer  $m \in \mathbb{N}^*$  such that  $b = \text{ad}_{b_1}^m(b_2)$  and then

$$\xi_b(t, u) := \frac{1}{m!} \int_0^t \xi_{b_1}^m(s, u) \dot{\xi}_{b_2}(s, u) ds. \quad (2.5)$$

**Theorem 2.10.** Let  $\mathcal{B} \subset \text{Br}(X)$  be a Hall set,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ . The solution to the formal differential equation (2.1) satisfies,

$$x(t) = \prod_{b \in \mathcal{B}}^{\leftarrow} e^{\xi_b(t, u)E(b)}. \quad (2.6)$$

**Remark 2.11.** In (2.6), the right-hand side is an infinite oriented product, indexed by elements of  $\mathcal{B}$  which are increasing towards the left (see [3, Section 2.5] for more precise definitions).

## 2.4 Magnus formula in interaction picture

The following result is proved in [3, Section 2.4].

**Theorem 2.12.** For  $t \in \mathbb{R}_+$ , the solution  $x$  to (2.1) satisfies

$$x(t) = \exp(tX_0) \exp(\mathcal{Z}_\infty(t, X, u)) \quad (2.7)$$

where  $\mathcal{Z}_\infty(t, X, u) \in \widehat{\mathcal{L}}(X)$ . Moreover, if  $\mathcal{B} \subset \text{Br}(X)$  is a Hall set, there exists a unique family  $(\eta_b)_{b \in \mathcal{B}}$  of functionals  $\mathbb{R}_+ \times L_{\text{loc}}^1(\mathbb{R}_+) \rightarrow \mathbb{R}$ , called coordinates of the pseudo first kind associated with  $\mathcal{B}$ , such that, for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,

$$\mathcal{Z}_\infty(t, X, u) = \sum_{b \in \mathcal{B}} \eta_b(t, u)E(b), \quad (2.8)$$

with, in particular,  $\eta_{X_0}(t, u) = 0$  and  $\eta_{X_1}(t, u) = u_1(t)$ .

Since we will work with truncated version of this expansion, we also introduce, for  $M \in \mathbb{N}^*$ , the notation  $\mathcal{Z}_M(t, X, u)$  to denote the canonical projection of  $\mathcal{Z}_\infty(t, X, u)$  onto  $\mathcal{A}_1(X) \oplus \dots \oplus \mathcal{A}_M(X)$ , so that one has

$$\mathcal{Z}_M(t, X, u) = \sum_{n_1(b) \leq M} \eta_b(t, u)E(b). \quad (2.9)$$

One of the points of interest of the infinite product expansion is to provide an expression of  $\mathcal{Z}_\infty(t, X, u)$  and its coordinates of the pseudo-first kind in terms of the coordinates of the second kind.

**Proposition 2.13.** There exists a family of elements  $F_{q,h}(Y_1, \dots, Y_q) \in \mathcal{L}(\{Y_1, \dots, Y_q\})$  for  $q \in \mathbb{N}^*$  and  $h \in (\mathbb{N}^*)^q$ , such that,  $F_{q,h}(Y_1, \dots, Y_q)$  is of degree  $h_i$  with respect to  $Y_i$  for each  $i \in \llbracket 1, q \rrbracket$  and, for every Hall set  $\mathcal{B} \subset \text{Br}(X)$  with  $X_0$  as maximal element,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,

$$\mathcal{Z}_\infty(t, X, u) = \sum_{\substack{q \in \mathbb{N}^*, h \in (\mathbb{N}^*)^q \\ b_1 > \dots > b_q \in \mathcal{B} \setminus \{X_0\}}} \xi_{b_1}^{h_1}(t, u) \dots \xi_{b_q}^{h_q}(t, u) F_{q,h}(b_1, \dots, b_q). \quad (2.10)$$

Equivalently, for every  $b \in \mathcal{B}$ , one has

$$\eta_b(t, u) = \xi_b(t, u) + \sum_{\substack{q \geq 2, h \in (\mathbb{N}^*)^q \\ b_1 > \dots > b_q \in \mathcal{B} \setminus \{X_0\}}} \xi_{b_1}^{h_1}(t, u) \dots \xi_{b_q}^{h_q}(t, u) \langle F_{q,h}(b_1, \dots, b_q), b \rangle, \quad (2.11)$$

where, for  $a \in \mathcal{L}(X)$ ,  $\langle a, b \rangle$  denotes the coefficient of  $b$  in the expansion of  $a$  on  $\mathcal{B}$ .

*Proof.* We deduce from Theorems 2.10 and 2.12 and the maximality of  $X_0$  that

$$e^{\mathcal{Z}_\infty(t, X, u)} = \overleftarrow{\prod}_{b \in \mathcal{B} \setminus \{X_0\}} e^{\xi_b(t, u)E(b)}. \quad (2.12)$$

Then (2.10) and (2.11) follow from the multivariate CBHD formula [3, Proposition 34]<sup>2</sup>.  $\square$

**Remark 2.14.** *The elements  $F_{q,h}$  are deeply linked with the CBHD formula and can be iteratively computed from its usual two-variables coefficients. At the first orders, one has for example  $F_{1,(1)}(Y_1) = Y_1$  and  $F_{2,(1,1)}(Y_1, Y_2) = \frac{1}{2}[Y_1, Y_2]$ .*

Equality (2.11) leads to the idea that, in some sense, at “leading order”, one has  $\eta_b \approx \xi_b$ , provided that one can estimate the appropriate cross products of the right-hand side.

**Definition 2.15** ( $\mathcal{F}$ ). *Given  $q \geq 2$  and  $b_1, \dots, b_q \in \text{Br}(X)$ , we define  $\mathcal{F}(b_1, \dots, b_q)$  as the vector subspace of  $\mathcal{L}(X)$  spanned by Lie brackets of  $b_1, \dots, b_q$  involving each of these elements exactly once. For example*

$$\mathcal{F}(b_1, b_2) = \mathbb{R}[b_1, b_2], \quad (2.13)$$

$$\mathcal{F}(b_1, b_2, b_3) = \mathbb{R}[b_1, [b_2, b_3]] + \mathbb{R}[[b_1, b_2], b_3], \quad (2.14)$$

thanks to Jacobi’s identity  $[b_2, [b_1, b_3]] = [[b_2, b_1], b_3] + [b_1, [b_2, b_3]]$ .

**Proposition 2.16.** *Let  $\mathcal{B}$  be a Hall set with  $X_0$  maximal. Let  $b \in \mathcal{B}$ . There exists  $C > 0$  such that the following property holds. Assume that there exists  $\Xi : \mathbb{R}_+^* \times L_{\text{loc}}^1(\mathbb{R}_+) \rightarrow \mathbb{R}_+$  such that, for all  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}$  such that  $b \in \text{supp } \mathcal{F}(b_1, \dots, b_q)$ , for every  $u \in L_{\text{loc}}^1(\mathbb{R}_+)$  and  $t > 0$ ,*

$$|\xi_{b_1}(t, u) \cdots \xi_{b_q}(t, u)| \leq \Xi(t, u). \quad (2.15)$$

Then, for every  $u \in L_{\text{loc}}^1(\mathbb{R}_+)$  and  $t > 0$ ,

$$|\eta_b(t, u) - \xi_b(t, u)| \leq C\Xi(t, u). \quad (2.16)$$

*Proof.* This is a straightforward consequence of (2.11) and the fact that the sum in the right-hand side of this equality is finite. Indeed,  $\langle F_{q,h}(b_1, \dots, b_q), b \rangle \neq 0$  implies in particular that  $h_1|b_1| + \dots + h_q|b_q| = |b|$ , so there is a finite number of possibilities for  $q$ ,  $h$  and the  $b_i$ .  $\square$

### 3 A new Hall basis of the free Lie algebra

In this section, we define our new basis of the free Lie algebra over two generators  $\{X_0, X_1\}$ , designed for applications to control theory, and compute some of its elements.

In Section 3.1, we introduce our definition of a new Hall set, which we call  $\mathcal{B}^*$  and motivate its interest for control problems. In Section 3.2, we give an exhaustive description of the elements of  $\mathcal{B}^*$  involving  $X_1$  at most 5 times. In Section 3.3, we compute the associated coordinates of the second kind, while in Section 3.4, we provide estimates of these coordinates.

#### 3.1 Definition of $\mathcal{B}^*$ and first properties

The main result of this paragraph is Theorem 3.4 which states the existence and uniqueness of our basis  $\mathcal{B}^*$ . We start by introducing some notations and definitions which will make the presentation more meaningful.

First, we define by induction a subset  $G$  of  $\text{Br}(X)$  by requiring that,  $X_0, X_1 \in G$  and, for every  $a, b \in G$  with  $a \neq X_0$ ,  $(a, b) \in G$ . Heuristically,  $G$  is the subset of  $b \in \text{Br}(X)$  for which  $X_0$  is never the left factor of any sub-bracket within  $b$ . This leads to the following definition.

<sup>2</sup>Technically, this proposition is stated for finite products. Nevertheless, one can use the graded structure of  $\hat{\mathcal{A}}(X)$  to reduce the proof to this finite setting.

**Definition 3.1** (Germ). For any  $b \in G \setminus \{X_0\}$ , there exists a unique couple  $(b^*, \nu_b) \in G \times \mathbb{N}$  such that  $b = b^* 0^{\nu_b}$ , with  $b^* = X_1$  or  $b^* = (b_1, b_2)$  with  $b_1 \neq X_0$  and  $b_2 \neq X_0$ , where the notation  $0^\nu$  is introduced in Definition 1.6. We call  $b^*$  the germ of  $b$  and we say that  $b$  is a germ when  $b = b^*$  (i.e.  $\nu_b = 0$ ). Let  $G^*$  be the subset of  $G$  made of germs.

**Example 3.2.** Let  $b := ((X_1, X_0), X_0) = X_1 0^2$ . The germ of  $b$  is  $X_1$  and  $\nu_b = 2$ . Hence  $b \notin G$ . However  $c := (X_1, (X_1, X_0))$  is a germ and  $c \in G$ .

**Definition 3.3** (Order for  $\mathcal{B}^*$ ). We endow  $G$  with the following total order.

(B0)  $X_0$  is the maximal element.

(B1) for  $a, b \in G \setminus \{X_0\}$ ,  $a < b$  if and only if  $a^* < b^*$  or  $a^* = b^*$  and  $\nu_a < \nu_b$ .

(B2) for  $a^*, b^* \in G^*$ ,  $a^* < b^*$  if and only if

- either  $n_1(a^*) < n_1(b^*)$ ,
- or  $n_1(a^*) = n_1(b^*)$  and  $\lambda(a^*) < \lambda(b^*)$ ,
- or  $n_1(a^*) = n_1(b^*)$  and  $\lambda(a^*) = \lambda(b^*)$  and  $\mu(a^*) < \mu(b^*)$ .

In other words,  $X_1$  is minimal,  $X_0$  is maximal and, on  $G \setminus X$ , the order is the lexicographic order on the quadruple  $b \mapsto (n_1(b^*), \lambda(b^*), \mu(b^*), \nu_b)$ .

**Theorem 3.4.** There exists a unique Hall set  $\mathcal{B}^* \subset G \subset \text{Br}(X)$  associated with Definition 3.3.

*Proof.* By [4, Lemma 1.37], it suffices to check that  $<$  is a Hall order on  $G$ , i.e. a total order such that, for every  $(a, b) \in G \setminus X$ ,  $a \in G$  and  $a < (a, b)$ .

**Step 1.** We prove that, for every  $a, b \in G$ , if neither  $a < b$  nor  $b < a$  holds, then  $a = b$ . By contradiction, let  $a$  and  $b$  be a pair, of minimal total length  $|a| + |b|$ , such that  $a \neq b$ , and neither  $a < b$  nor  $b < a$ . By (B0),  $a \neq X_0$  and  $b \neq X_0$ . By (B1),  $a^* \neq b^*$  (otherwise  $\nu_a = \nu_b$  so  $a = b$ ). By (B2),  $n_1(a^*) = n_1(b^*)$  and,

- either  $\lambda(a^*) \neq \lambda(b^*)$ , and these two brackets are an incomparable pair of shorter total length,
- or  $\lambda(a^*) = \lambda(b^*)$ , but then  $\mu(a^*) \neq \mu(b^*)$  is an incomparable pair of shorter total length.

**Step 2.** We prove that, for every  $(a, b) \in G \setminus X$ ,  $a \in G$  and  $a < (a, b)$ . Let  $(a, b) \in G \setminus X$ . Then  $a \in G$  by construction of  $G$  by induction. If  $b = X_0$  then  $a < (a, b)$  by (B1). If  $b \neq X_0$ , then  $n_1(a) < n_1((a, b))$  so  $a < (a, b)$  by (B2).  $\square$

**Remark 3.5.** In  $\mathcal{B}^*$ ,  $X_0$  is maximal. This is similar to the fact that  $X_0$  would be maximal in the Chen–Fox–Lyndon basis associated with the order  $X_1 < X_0$  on  $X$ . So  $\mathcal{B}^*$  shares some properties of this basis (for example, the fact that  $b \in \mathcal{B}^* \Rightarrow \forall \nu \in \mathbb{N}, b 0^\nu \in \mathcal{B}^*$ ).

If  $b \in \mathcal{B}^*$  is a germ, then, by (B2),  $\mu(b) < b$ , because  $n_1(\mu(b)) < n_1(b)$ . This is similar to the situation in length-compatible Hall sets where one always has  $\mu(b) < b$  because  $|\mu(b)| < |b|$ . In the Chen–Fox–Lyndon basis however, one has  $b < \mu(b)$ . So  $\mathcal{B}^*$  shares some properties of length-compatible Hall sets.

By analogy with (1.4) and (1.5), for  $A_1, A_0 \subset \mathbb{N}$ , we will also adopt the notations

$$\mathcal{B}_{A_1}^* := \{b \in \mathcal{B}^*; n_1(b) \in A_1\} \quad \text{and} \quad \mathcal{B}_{A_1, A_0}^* := \{b \in \mathcal{B}^*; n_1(b) \in A_1, n_0(b) \in A_0\}. \quad (3.1)$$

### 3.2 Elements of $\mathcal{B}^*$ up to the fifth order

The goal of this section is to prove Proposition 1.9, i.e. to determine the germs of  $\mathcal{B}_{[1,5]}^*$ . If  $b^*$  is such a germ, then, by Definition 2.5, for every  $\nu \in \mathbb{N}$ ,  $b^*0^\nu \in \mathcal{B}^*$  and, by (B1), for every  $\nu_1 < \nu_2 \in \mathbb{N}$  then  $b^*0^{\nu_1} < b^*0^{\nu_2}$ .

*Proof of Proposition 1.9.*  $X_1$  is the only possible germ in  $\mathcal{B}_1^*$ , which proves (1.8). Moreover, the sequence  $(M_\nu)_{\nu \in \mathbb{N}}$  is increasing

$$\forall \nu \leq \nu', \quad M_\nu \leq M_{\nu'}. \quad (3.2)$$

By Definition 2.5, any germ of  $\mathcal{B}_{[2,5]}^*$  is of the form  $(a, b)$  where  $a, b \in \mathcal{B}_{[1,4]}^*$ , and  $\lambda(b) \leq a < b$ . By (B1), this implies that either  $a^* = b^*$  and then  $b = (a, X_0)$  so  $(a, b) = \text{ad}_a^2(X_0)$ , or  $a^* < b^*$  and then  $b = b^*$  and  $n_1(a) \leq n_1(b)$ . We proceed by increasing degree in  $X_1$ .

- *Germs of  $\mathcal{B}_2^*$ :* By Definition 2.5, for every  $j \in \mathbb{N}^*$ ,  $W_{j,0}$  belongs to  $\mathcal{B}^*$ . Indeed  $W_{j,0} = \text{ad}_{M_{j-1}}^2(X_0)$  and  $M_{j-1} < X_0$  by (B0). These are the only elements of  $\mathcal{B}_2^*$  that one may construct by bracketing two elements of  $\mathcal{B}_1^*$ . Moreover, by (B2),  $W_{j,0} < W_{k,0}$  when  $j < k$ , thus, by (B1),

$$\forall j < k \in \mathbb{N}^*, \forall \mu \in \mathbb{N}, \quad W_{j,\mu} < W_{k,0}. \quad (3.3)$$

- *Germs of  $\mathcal{B}_3^*$ :* By Definition 2.5, for  $j \leq k \in \mathbb{N}^*$ ,

$$P_{j,k,0} = (M_{k-1}, W_{j,0}) = (M_{k-1}, (M_{j-1}, M_j)) \quad (3.4)$$

belongs to  $\mathcal{B}^*$ . Indeed,  $M_{j-1} \leq M_{k-1} < W_{j,0}$  by (3.2) and (B2) because  $n_1(M_{k-1}) < n_1(W_{j,0})$ . These are the only elements of  $\mathcal{B}_3^*$  that one may construct by bracketing an element of  $\mathcal{B}_1^*$  with an element of  $\mathcal{B}_2^*$ .

- *Germs of  $\mathcal{B}_4^*$  in  $(\mathcal{B}_1^*, \mathcal{B}_3^*)$ :* By Definition 2.5, for  $j \leq k \leq l \in \mathbb{N}^*$ ,

$$Q_{j,k,l,0} = (M_{l-1}, P_{j,k,0}) = (M_{l-1}, (M_{k-1}, W_{j,0})) \quad (3.5)$$

belongs to  $\mathcal{B}^*$  because  $W_{j,\mu} < X_0$  by (B0). These are the only elements of  $\mathcal{B}_4^*$  that one may construct by bracketing an element of  $\mathcal{B}_1^*$  with an element of  $\mathcal{B}_3^*$ .

- *Germs of  $\mathcal{B}_4^*$  in  $(\mathcal{B}_2^*, \mathcal{B}_3^*)$ :* By Definition 2.5, for  $j < k \in \mathbb{N}^*$  and  $\mu \in \mathbb{N}$ ,

$$Q_{j,\mu,k,0}^\sharp = (W_{j,\mu}, W_{k,0}) = (W_{j,\mu}, (M_{k-1}, M_k)) \quad (3.6)$$

belongs to  $\mathcal{B}^*$ . Indeed,  $M_{k-1} < W_{j,\mu} < W_{k,0}$  by (B2) and (3.3). These are the only elements of  $\mathcal{B}_4^*$  that one may construct by bracketing two elements of  $\mathcal{B}_2^*$  having different germs.

For  $j \in \mathbb{N}^*$  and  $\mu \in \mathbb{N}$ ,

$$Q_{j,\mu,0}^\flat = (W_{j,\mu}, W_{j,\mu+1}) = \text{ad}_{W_{j,\mu}}^2(X_0) \quad (3.7)$$

belongs to  $\mathcal{B}^*$ . Indeed, by (B0),  $W_{j,\mu} < X_0$ . These are the only elements of  $\mathcal{B}_4^*$  that one may construct by bracketing two elements of  $\mathcal{B}_2^*$  having the same germ.

- *Germs of  $\mathcal{B}_5^*$  in  $(\mathcal{B}_1^*, \mathcal{B}_4^*)$ :* By Definition 2.5, for  $j \leq k \leq l \leq m \in \mathbb{N}^*$ ,

$$R_{j,k,l,m,0} = (M_{m-1}, Q_{j,k,l,0}) = (M_{m-1}, (M_{l-1}, P_{j,k,0})) \quad (3.8)$$

belongs to  $\mathcal{B}^*$ . Indeed,  $M_{l-1} \leq M_{m-1} < Q_{j,k,l,0}$  by (3.2) and (B2). These are the only elements of  $\mathcal{B}_5^*$  that one may construct by bracketing an element of  $\mathcal{B}_1^*$  with an element of  $\mathcal{B}_4^*$ .

- *Germs of  $\mathcal{B}_5^*$  in  $(\mathcal{B}_2^*, \mathcal{B}_3^*)$ :* By Definition 2.5, for  $j, k, l \in \mathbb{N}^*$  such that  $j \leq k$  and  $\mu \in \mathbb{N}$ ,

$$R_{j,k,l,\mu,0}^\sharp = (W_{l,\mu}, P_{j,k,0}) = (W_{l,\mu}, (M_{k-1}, W_j)) \quad (3.9)$$

belongs to  $\mathcal{B}^*$ . Indeed,  $M_{k-1} < W_{l,\mu} < P_{j,k,0}$  by (B2). These are the only elements of  $\mathcal{B}_5^*$  one may construct by bracketing an element of  $\mathcal{B}_2^*$  with an element of  $\mathcal{B}_3^*$ .

This concludes the proof.  $\square$

### 3.3 Expressions of coordinates of the second kind up to the fifth order

In this paragraph, we give explicit expressions of the coordinates of the second kind, as defined in Definition 2.9 associated with the elements of  $\mathcal{B}^*$  up to the fifth order in the control introduced in Section 3.2. We start with the following lemma, which helps in visualizing the coordinates of the second kind associated with the elements of  $\mathcal{B}_{[1,5]}^*$  listed in Proposition 1.9.

**Lemma 3.6.** *For  $b \in \mathcal{B}^*$  and  $\nu \in \mathbb{N}$ ,*

$$\xi_{b0^\nu}(t, u) = \int_0^t \frac{(t-s)^\nu}{\nu!} \dot{\xi}_b(s, u) ds. \quad (3.10)$$

*Proof.* By induction on  $\nu$ , this follows from Definition 2.9 and the fact that  $\mathcal{B}^*$  satisfies (B0).  $\square$

**Proposition 3.7.** *For every  $j \leq k \leq l \leq m \in \mathbb{N}^*$ ,  $\mu, \nu \in \mathbb{N}$ , we have*

$$\xi_{M_\nu}(t, u) = \int_0^t \frac{(t-s)^\nu}{\nu!} u(s) ds = u_{\nu+1}(t), \quad (3.11)$$

$$\xi_{W_{j,\nu}}(t, u) = \frac{1}{2} \int_0^t \frac{(t-s)^\nu}{\nu!} u_j^2(s) ds, \quad (3.12)$$

$$\xi_{P_{j,k,\nu}}(t, u) = \alpha_{j,k} \int_0^t \frac{(t-s)^\nu}{\nu!} u_k(s) u_j^2(s) ds, \quad (3.13)$$

$$\xi_{Q_{j,k,l,\nu}}(t, u) = \beta_{j,k,l} \int_0^t \frac{(t-s)^\nu}{\nu!} u_l(s) u_k(s) u_j^2(s) ds, \quad (3.14)$$

$$\xi_{Q_{j,\mu,\nu}^\sharp}(t, u) = \frac{1}{8} \int_0^t \frac{(t-s)^\nu}{\nu!} \left( \int_0^s \frac{(s-s')^\mu}{\mu!} u_j^2(s') ds' \right)^2 ds, \quad (3.15)$$

$$\xi_{Q_{j,\mu,k,\nu}^\sharp}(t, u) = \frac{1}{4} \int_0^t \frac{(t-s)^\nu}{\nu!} \left( \int_0^s \frac{(s-s')^\mu}{\mu!} u_j^2(s') ds' \right) u_k^2(s) ds, \quad (3.16)$$

$$\xi_{R_{j,k,l,m,\nu}}(t, u) = \gamma_{j,k,l,m} \int_0^t \frac{(t-s)^\nu}{\nu!} u_m(s) u_l(s) u_k(s) u_j^2(s) ds, \quad (3.17)$$

$$\xi_{R_{j,k,l,\mu,\nu}^\sharp}(t, u) = \frac{\alpha_{j,k}}{2} \int_0^t \frac{(t-s)^\nu}{\nu!} \left( \int_0^s \frac{(s-s')^\mu}{\mu!} u_l^2(s') ds' \right) u_k(s) u_j(s)^2 ds, \quad (3.18)$$

where  $j < k$  in (3.16) (only), and the coefficients are given by

$$\alpha_{j,k} = \frac{1}{2!} \delta_{j < k} + \frac{1}{3!} \delta_{j=k}, \quad (3.19)$$

$$\beta_{j,k,l} = \alpha_{j,k} \delta_{k < l} + \frac{1}{(2!)^2} \delta_{j < k=l} + \frac{1}{4!} \delta_{j=k=l}, \quad (3.20)$$

$$\gamma_{j,k,l,m} = \beta_{j,k,l} \delta_{l < m} + \frac{1}{5!} \delta_{j=k=l=m} + \frac{1}{(2!)^2} \delta_{j < k < l=m} + \frac{1}{2!3!} (\delta_{j < k=l=m} + \delta_{j=k < l=m}). \quad (3.21)$$

*Proof.* It follows from the application of Definition 2.9 to the elements of Proposition 1.9.  $\square$

**Remark 3.8.** *These simple explicit expressions of the coordinates of the second kind associated with  $\mathcal{B}^*$ , together with formula (2.11), prove that we have constructed a basis of the free Lie algebra  $\mathcal{L}(X)$  in which  $\mathcal{Z}_\infty(t, X, u)$  has, in some sense, a simple expression. This is closely related to the first open problem of [21]: “construct a basis for the free Lie algebra such that the corresponding coordinates of the first kind have simple formulas”.*

*We observe in particular that, for every  $k \in \mathbb{N}^*$ , the quadratic form  $\xi_{W_k}$  is positive-definite: this is a key point for Theorem 1.11. The positivity of the values of  $\xi_{Q_{j,k,k}}$  is a key point for the quartic necessary conditions which we intend to study in a forthcoming work. Finally, one may expect that for any germ  $\mathfrak{b} \in \mathcal{B}^*$  such that  $\xi_{\mathfrak{b}}$  is positive-definite, a necessary condition for STLC of the form (1.25) holds.*

### 3.4 Estimates on the coordinates of the second kind up to the fifth order

We start with a rough estimate valid for all brackets of  $\mathcal{B}^* \setminus X$ , which will be mainly used to prove convergence of the considered series. This statement follows from [3, Lemma 156] and is thus valid within any Hall set such that  $X_1 < X_0$ . For self-containedness, we give a direct proof in the case of  $\mathcal{B}^*$  in Appendix A.3.

**Proposition 3.9.** *For every  $k \in \mathbb{N}^*$ , there exists  $c = c(k) > 0$  such that, for every  $b \in \mathcal{B}^* \setminus \{X_1\}$  with  $n_1(b) = k$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,*

$$|\xi_b(t, u)| \leq \frac{(ct)^{|b|}}{|b|!} t^{-(1+k)} \|u_1\|_{L^k}^k. \quad (3.22)$$

To prove our obstruction results, we need more accurate estimates on the coordinates of the second type associated with  $\mathcal{B}_{[[1,5]]}^*$ , in terms of Sobolev norms of primitives of the control. This is the goal the following statement, proved in Appendix A.4

**Proposition 3.10.** *The following bounds hold.*

1. Let  $p \in [1, \infty]$  and  $j_0 \in \mathbb{N}^*$ . There exists  $c > 0$  such that, for every  $j \geq j_0$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,  $\ell := |M_j| \geq j_0 + 1$  and

$$|\xi_{M_j}(t, u)| \leq \frac{(ct)^\ell}{\ell!} t^{-(j_0+1)} t^{1-\frac{1}{p}} \|u_{j_0}\|_{L^p}. \quad (3.23)$$

2. Let  $p \in [1, \infty]$  and  $j_0 \in \mathbb{N}^*$ . There exists  $c > 0$  such that, for every  $j \geq j_0$ ,  $\nu \geq 0$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,  $\ell := |W_{j,\nu}| \geq 2j_0 + 1$  and

$$|\xi_{W_{j,\nu}}(t, u)| \leq \frac{(ct)^\ell}{\ell!} t^{-(2j_0+1)} t^{1-\frac{1}{p}} \|u_{j_0}\|_{L^{2p}}^2. \quad (3.24)$$

3. Let  $p_1, p_2 \in [1, \infty]$  such that  $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$  and  $j_0, k_0 \in \mathbb{N}^*$ . There exists  $c > 0$  such that, for every  $j \geq j_0$ ,  $k \geq k_0$  with  $j \leq k$ ,  $\nu \geq 0$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,  $\ell := |P_{j,k,\nu}| \geq 2j_0 + k_0 + 1$  and

$$|\xi_{P_{j,k,\nu}}(t, u)| \leq \frac{(ct)^\ell}{\ell!} t^{-(2j_0+k_0+1)} t^{1-\frac{1}{p_1}-\frac{1}{p_2}} \|u_{j_0}\|_{L^{2p_1}}^2 \|u_{k_0}\|_{L^{p_2}}. \quad (3.25)$$

4. Let  $p_1, p_2, p_3 \in [1, \infty]$  such that  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \leq 1$  and  $j_0, k_0, l_0 \in \mathbb{N}^*$ . There exists  $c > 0$  such that, for every  $j \geq j_0$ ,  $k \geq k_0$ ,  $l \geq l_0$  with  $j \leq k \leq l$ ,  $\nu \geq 0$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,  $\ell := |Q_{j,k,l,\nu}| \geq 2j_0 + k_0 + l_0 + 1$  and

$$|\xi_{Q_{j,k,l,\nu}}(t, u)| \leq \frac{(ct)^\ell}{\ell!} t^{-(2j_0+k_0+l_0+1)} t^{1-\frac{1}{p_1}-\frac{1}{p_2}-\frac{1}{p_3}} \|u_{j_0}\|_{L^{2p_1}}^2 \|u_{k_0}\|_{L^{p_2}} \|u_{l_0}\|_{L^{p_3}}. \quad (3.26)$$

5. Let  $p \in [1, \infty]$  and  $j_0 \in \mathbb{N}^*$ . There exists  $c > 0$  such that, for every  $j \geq j_0$ ,  $\mu, \nu \in \mathbb{N}$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,  $\ell := |Q_{j,\mu,\nu}^\flat| \geq 4j_0 + 3$  and

$$|\xi_{Q_{j,\mu,\nu}^\flat}(t, u)| \leq \frac{(ct)^\ell}{\ell!} t^{-(4j_0+3)} t^{3-\frac{2}{p}} \|u_{j_0}\|_{L^{2p}}^4. \quad (3.27)$$

6. Let  $p_1, p_2 \in [1, \infty]$  and  $j_0, k_0 \in \mathbb{N}^*$ . There exists  $c > 0$  such that, for every  $j \geq j_0$ ,  $k \geq k_0$ , with  $j < k$ ,  $\mu, \nu \geq 0$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,  $\ell := |Q_{j,\mu,k,\nu}^\sharp| \geq 2j_0 + 2k_0 + 2$  and

$$|\xi_{Q_{j,\mu,k,\nu}^\sharp}(t, u)| \leq \frac{(ct)^\ell}{\ell!} t^{-(2j_0+2k_0+2)} t^{2-\frac{1}{p_1}-\frac{1}{p_2}} \|u_{j_0}\|_{L^{2p_1}}^2 \|u_{k_0}\|_{L^{2p_2}}^2. \quad (3.28)$$



7. Let  $p_1, p_2, p_3, p_4 \in [1, \infty]$  such that  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} \leq 1$  and  $j_0, k_0, l_0, m_0 \in \mathbb{N}^*$ . There exists  $c > 0$  such that, for every  $j \geq j_0$ ,  $k \geq k_0$ ,  $l \geq l_0$ ,  $m \geq m_0$  with  $j \leq k \leq l \leq m$ ,  $\nu \geq 0$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,  $\ell := |R_{j,k,l,m,\nu}| \geq 2j_0 + k_0 + l_0 + m_0 + 1$  and

$$|\xi_{R_{j,k,l,m,\nu}}(t, u)| \leq \frac{(ct)^\ell}{\ell!} t^{-(2j_0+k_0+l_0+m_0+1)} t^{1-\frac{1}{p_1}-\frac{1}{p_2}-\frac{1}{p_3}-\frac{1}{p_4}} \times \|u_{j_0}\|_{L^{2p_1}}^2 \|u_{k_0}\|_{L^{p_2}} \|u_{l_0}\|_{L^{p_3}} \|u_{m_0}\|_{L^{p_4}}. \quad (3.29)$$

8. Let  $p, p_1, p_2 \in [1, \infty]$  such that  $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$  and  $j_0, k_0, l_0 \in \mathbb{N}^*$ . There exists  $c > 0$  such that, for every  $j \geq j_0$ ,  $k \geq k_0$ ,  $l \geq l_0$ , with  $j \leq k$ ,  $\mu, \nu \geq 0$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,  $\ell := |R_{j,k,l,\mu,\nu}^\sharp| \geq 2j_0 + k_0 + 2l_0 + 2$  and

$$|\xi_{R_{j,k,l,\mu,\nu}^\sharp}(t, u)| \leq \frac{(ct)^\ell}{\ell!} t^{-(2j_0+k_0+2l_0+2)} t^{2-\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2}} \|u_{j_0}\|_{L^{2p_1}}^2 \|u_{k_0}\|_{L^{p_2}} \|u_{l_0}\|_{L^{2p}}^2. \quad (3.30)$$

## 4 Toolbox for our approach to obstructions

In this section, we gather results of various nature as a toolbox for the sequel.

First, we recall elementary definitions and estimates for analytic vector fields in Section 4.1 and introduce in Section 4.2 a notation  $O(\cdot)$  which will be used heavily throughout the paper.

Then, we state in Section 4.3 the counterpart for system (1.1) of the formal expansion (2.7) and give in Section 4.4 a sufficient condition to replace, in some sense, the coordinates of the pseudo-first kind by those of the second kind in (2.8). We show nevertheless in Section 4.5 that this simplification is not always valid.

Eventually, we recall in Section 4.6 the Gagliardo–Nirenberg interpolation inequalities, and state straight-forward consequences of the Jacobi identity in Section 4.7.

### 4.1 Analytic estimates for vector fields

For  $a \in \mathbb{N}^*$  and a multi-index  $\alpha = (\alpha^1, \dots, \alpha^a) \in \mathbb{N}^a$ , we use the notations  $|\alpha| := \alpha^1 + \dots + \alpha^a$ ,  $\partial^\alpha := \partial_{x_1}^{\alpha^1} \dots \partial_{x_a}^{\alpha^a}$  and  $\alpha! := \alpha^1! \dots \alpha^a!$ . Then, the following estimate can be proved by iterating  $2^{-(p+q)}(p+q)! \leq p!q! \leq (p+q)!$  for every  $p, q \in \mathbb{N}$ ,

$$\forall a \in \mathbb{N}^*, \forall \alpha = (\alpha^1, \dots, \alpha^a) \in \mathbb{N}^a, \quad 2^{-(a-1)|\alpha|} |\alpha!| \leq \alpha! \leq |\alpha|! \quad (4.1)$$

**Definition 4.1** (Analytic vector fields, analytic norms). *Let  $\delta > 0$  and  $B_\delta$  be the closed ball of radius  $\delta$ , centered at  $0 \in \mathbb{R}^d$ . For  $r > 0$ , we define  $\mathcal{C}^{\omega,r}(B_\delta; \mathbb{R}^d)$  as the subspace of real-analytic vector fields on an open neighborhood of  $B_\delta$ , for which the following norm is finite*

$$\|f\|_r := \sum_{i=1}^b \sum_{\alpha \in \mathbb{N}^d} \frac{r^{|\alpha|}}{\alpha!} \|\partial^\alpha f_i\|_{L^\infty(B_\delta)}. \quad (4.2)$$

For each a real-analytic vector field  $f$  on a neighborhood of  $0$ , there exist  $r, \delta > 0$  such that  $f \in \mathcal{C}^{\omega,r}(B_\delta; \mathbb{R}^d)$  (see [22, Proposition 2.2.10]).

The following classical result is proved, for instance in [3, Lemma 80].

**Lemma 4.2** (Analytic estimate). *Let  $r, \delta > 0$ ,  $r' := r/e$ ,  $f_0, f_1 \in \mathcal{C}^{\omega,r}(B_\delta; \mathbb{R}^d)$  and  $b \in \text{Br}(X)$ . Then,  $f_b \in \mathcal{C}^{\omega,r'}(B_\delta; \mathbb{R}^d)$  and*

$$\|f_b\|_{r'} \leq \frac{r}{9} (|b| - 1)! \left( \frac{9 \|f\|_r}{r} \right)^{|b|}, \quad (4.3)$$

where  $\|f\|_r := \max\{\|f_0\|_r; \|f_1\|_r\}$ .

## 4.2 Regime for the notation $O$

Given two functions  $A(x, u)$  and  $B(x, u)$  of interest, we will write that  $A(x, u) = O(B(x, u))$  when there exists  $C, \rho > 0$  such that, for every  $t \in (0, \rho)$ ,  $u \in L^1((0, t); \mathbb{R})$  with  $\|u\|_{W^{-1, \infty}} \leq \rho$  (recall definition (1.2)), then

$$|A(x(t; u), u)| \leq CB(x(t; u), u). \quad (4.4)$$

Hence this notation refers to the regime  $(t, \|u\|_{W^{-1, \infty}}) \rightarrow 0$ . As examples, one has  $t = O(1)$  and  $\|u_1\| = O(1)$ . A deeper result is the following estimate which states that, for scalar-input systems of the form (1.1), the  $W^{-1, \infty}$  norm of the control is an upper bound for the size of the state.

**Lemma 4.3.** *Let  $f_0, f_1$  be analytic vector fields on a neighborhood of 0 with  $f_0(0) = 0$ . Then*

$$x(t; u) = O(\|u_1\|_{L^\infty}). \quad (4.5)$$

*Proof.* This follows from [3, Proposition 145].  $\square$

## 4.3 A new representation formula for ODEs

We recall the following recent approximate representation formula for the solution to (1.1).

**Theorem 4.4.** *Let  $M \in \mathbb{N}^*$ ,  $\delta, r > 0$  and  $f_0, f_1 \in \mathcal{C}^{\omega, r}(B_\delta; \mathbb{R}^d)$  with  $f_0(0) = 0$ . Then*

$$x(t; u) = \mathcal{Z}_M(t, f, u)(0) + O\left(\|u_1\|_{L^{M+1}}^{M+1} + |x(t; u)|^{1+\frac{1}{M}}\right), \quad (4.6)$$

where

$$\mathcal{Z}_M(t, f, u) = \sum_{b \in \mathcal{B}_{[1, M]}^*} \eta_b(t, u) f_b, \quad (4.7)$$

where this sum converges absolutely in  $\mathcal{C}^{\omega, r'}(B_\delta; \mathbb{R}^d)$  for any  $r' \in [r/e, r)$ .

*Proof.* Equality (4.6) is the third item of [3, Proposition 161]. The absolute convergence in (4.7) is proved in [3, Proposition 103] and relies on the fundamental observation that the structure constants of Hall bases exhibit *asymmetric geometric growth* (see [4, Theorem 1.9]).  $\square$

## 4.4 A black-box estimate of cross products

All our obstruction results are based on considering a component  $\mathbb{P}$  of the state  $x(t; u)$  along  $f_b(0)$  for some “bad” bracket  $b$ , parallel to  $\mathcal{N}(f)(0)$ , where  $\mathcal{N}$  is an appropriate subset of  $\text{Br}(X)$ . By (4.6), we would like to compute  $\mathbb{P}\mathcal{Z}_M(t, f, u)(0)$  and show that it behaves like  $\xi_b(t, u)$ . This requires to bound uniformly the cross products appearing in (2.10). We will rely on the following results, proved in Appendix A.5 and Appendix A.6.

**Lemma 4.5.** *Let  $M \in \mathbb{N}^*$ . Let  $\delta, r > 0$  and  $f_0, f_1 \in \mathcal{C}^{\omega, r}(B_\delta; \mathbb{R}^d)$  with  $f_0(0) = 0$ . There exists  $\rho > 0$  such that, for every  $t \in (0, \rho)$  and  $u \in L^1((0, t); \mathbb{R})$ ,*

$$\mathcal{Z}_M(t, f, u) = \sum_{\substack{q \in \mathbb{N}^*, h \in (\mathbb{N}^*)^q \\ b_1 > \dots > b_q \in \mathcal{B}^* \setminus \{X_0\} \\ h_1 n_1(b_1) + \dots + h_q n_1(b_q) \leq M}} \xi_{b_1}^{h_1}(t, u) \cdots \xi_{b_q}^{h_q}(t, u) f_{F_{q, h}(b_1, \dots, b_q)}, \quad (4.8)$$

where the sum converges absolutely in  $\mathcal{C}^{\omega, r'}(B_\delta; \mathbb{R}^d)$  for every  $r' \in [r/e, r)$ .

**Proposition 4.6.** *Let  $M, L \in \mathbb{N}^*$ . Let  $b \in \mathcal{B}_{[1, M]}^*$  and  $\mathcal{N} \subset \mathcal{B}_{[1, M]}^*$  with  $b \notin \mathcal{N}$ . Assume that there exist  $c > 0$  and  $\Xi : \mathbb{R}_+^* \times L_{\text{loc}}^1(\mathbb{R}_+) \rightarrow \mathbb{R}_+$  with  $\Xi(t, u) = O(1)$  such that, for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,*

- for all  $b \in \mathcal{B}_{\llbracket 1, M \rrbracket}^*$  such that  $b \notin \mathcal{N} \cup \{\mathbf{b}\}$ , there exists  $\sigma \leq L$  such that  $|b| \geq \sigma$  and

$$|\xi_b(t, u)| \leq \frac{(ct)^{|b|}}{|b|!} t^{-\sigma} \Xi(t, u), \quad (4.9)$$

- for all  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}^* \setminus \{X_0\}$  such that  $n_1(b_1) + \dots + n_1(b_q) \leq M$  and  $\text{supp } \mathcal{F}(b_1, \dots, b_q) \not\subset \mathcal{N}$ , there exists  $\sigma_1, \dots, \sigma_q \leq L$  and  $(\alpha_1, \dots, \alpha_q) \in [0, 1]^q$  with  $\alpha := \alpha_1 + \dots + \alpha_q \geq 1$  such that, for each  $i \in \llbracket 1, q \rrbracket$ ,  $|b_i| \geq \sigma_i$  and

$$|\xi_{b_i}(t, u)| \leq \frac{(ct)^{|b_i|}}{|b_i|!} t^{-\sigma_i} (\Xi(t, u))^{\alpha_i}. \quad (4.10)$$

Let  $f_0, f_1$  be analytic vector fields on a neighborhood of 0 with  $f_0(0) = 0$ . If  $f_b(0) \notin \mathcal{N}(f)(0)$  and  $\mathbb{P}$  is a component along  $f_b(0)$  parallel to  $\mathcal{N}(f)(0)$ ,

$$\mathbb{P} \mathcal{Z}_M(t, f, u)(0) = \xi_b(t, u) + O(\Xi(t, u)). \quad (4.11)$$

#### 4.5 Cross products are not negligible in general

The expression (2.11) of  $\eta_b$  as  $\xi_b$  plus a finite sum of cross products leads to the idea that one could maybe replace the coordinates of the pseudo-first kind by those of the second kind in (4.6), by absorbing the difference in the remainder terms which already appear in the right-hand side. One could define

$$\mathcal{Z}_M^{\text{pure}}(t, X, u) := \sum_{b \in \mathcal{B}_{\llbracket 1, M \rrbracket}^*} \xi_b(t, u) \mathbf{E}(b) \quad (4.12)$$

and ask whether, in the regime  $(t, \|u\|_{W^{-1, \infty}}) \rightarrow 0$ ,

$$x(t; u) = \mathcal{Z}_M^{\text{pure}}(t, f, u)(0) + O\left(\|u_1\|_{L^{M+1}}^{M+1} + |x(t; u)|^{1+\frac{1}{M}}\right). \quad (4.13)$$

Such a formula would be very nice to prove positive and negative controllability results.

Estimate (4.13) is satisfied on particular systems for which the cross products are  $o(|x(t; u)|)$ . For instance, for the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = \frac{1}{2}x_1^2, \end{cases} \quad (4.14)$$

we have  $x(t; u) = \mathcal{Z}_2^{\text{pure}}(t, f, u)(0) = u_1(t)e_1 + u_2(t)e_2 + \int_0^t \frac{u_1^2}{2} e_3$  thus the estimate (4.13) is valid with  $M = 2$ . Indeed, the difference  $(\mathcal{Z}_2 - \mathcal{Z}_2^{\text{pure}})(t, f, u)(0)$  is proportional to  $u_1(t)u_2(t)e_3 = o(|x(t; u)|)$ .

Unfortunately, estimate (4.13) is not valid in general. For instance, let us consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 + \frac{1}{2}x_1^2 \\ \dot{x}_3 = x_1x_2. \end{cases} \quad (4.15)$$

One has

$$x(t; u) = u_1(t)e_1 + \left(u_2(t) + \int_0^t \frac{u_1^2}{2}\right) e_2 + \left(\frac{1}{2}u_2(t)^2 + u_2(t) \int_0^t \frac{u_1^2}{2} - \int_0^t u_2 \frac{u_1^2}{2}\right) e_3 \quad (4.16)$$

and for every  $M \geq 3$ ,

$$\begin{aligned} \mathcal{Z}_M(t, f, u)(0) &= \eta_{X_1}(t, u)e_1 + \eta_{(X_1, X_0)}(t, u)e_2 + \eta_{W_1}(t, u)e_2 - \eta_{P_{1,2}}(t, u)e_3 \\ &= u_1(t)e_1 + \left( u_2(t) + \int_0^t \frac{u_1^2}{2} + \frac{1}{2}u_1(t)u_2(t) \right) e_2 \\ &\quad - \left( \int_0^t u_2 \frac{u_1^2}{2} - \frac{1}{2}u_2(t) \int_0^t \frac{u_1^2}{2} \right) e_3. \end{aligned} \quad (4.17)$$

Thus we observe that

$$x(t; u) - \mathcal{Z}_M(t, f, u)(0) = -\frac{1}{2}x_1(t)u_2(t)e_2 + \frac{1}{2}u_2(t)x_2(t)e_3, \quad (4.18)$$

which is indeed  $o(|x(t; u)|)$  in the regime  $(t, \|u\|_{W^{-1, \infty}}) \rightarrow 0$ . However,

$$x(t; u) - \mathcal{Z}_M^{\text{pure}}(t, f, u)(0) = \left( \frac{1}{2}u_2(t)^2 + \frac{3}{4}u_2(t) \int_0^t u_1^2 \right) e_3. \quad (4.19)$$

Thus, for any  $u \in L^1((0, t); \mathbb{R})$  such that  $x_2(t; u) = 0$ ,

$$x(t; u) - \mathcal{Z}_M^{\text{pure}}(t, f, u)(0) = -\frac{1}{4} \left( \int_0^t u_1^2 \right)^2 e_3. \quad (4.20)$$

Since system (4.15) is  $L^\infty$ -STLC (for example thanks to Hermes' sufficient condition of [14, Theorem 3.2]), there exist controls  $u \neq 0$  such that  $x(t; u) = 0$ . For such controls, relation (4.20) falsifies (4.13) for  $M \geq 4$ . It also falsifies the validity of the estimate

$$x(t; u) = \mathcal{Z}_M^{\text{pure}}(t, f, u)(0) + o(|x(t; u)|) \quad (4.21)$$

for nilpotent vector fields such that  $S_{[M+1, \infty]}(f)(0) = \{0\}$ .

## 4.6 Interpolation inequalities

We recall below the Gagliardo–Nirenberg interpolation inequalities (see [11, 28]) used in this article.

**Proposition 4.7.** *Let  $p \in [1, +\infty[$ ,  $q, r, s \in [1, +\infty]$ ,  $0 \leq j < l \in \mathbb{N}$  and  $\alpha \in (0, 1)$  such that*

$$\frac{j}{l} \leq \alpha \quad \text{and} \quad \frac{1}{p} = j + \left( \frac{1}{r} - l \right) \alpha + \frac{1 - \alpha}{q}. \quad (4.22)$$

*There exists  $C > 0$  such that, for every  $t > 0$  and  $\phi \in C^\infty([0, t]; \mathbb{R})$ ,*

$$\|D^j \phi\|_{L^p} \leq C \|D^l \phi\|_{L^r}^\alpha \|\phi\|_{L^q}^{1-\alpha} + Ct^{\frac{1}{p} - j - \frac{1}{s}} \|\phi\|_{L^s}. \quad (4.23)$$

**Remark 4.8.** *For functions on bounded intervals, adding the lower-order term in the right-hand side of (4.23) is mandatory (see [28, item 5, p. 126]). To obtain the dependency of the constant on  $t > 0$ , one uses scaling arguments to work within a fixed domain, say  $[0, 1]$ .*

## 4.7 A consequence of the Jacobi identity

The following straightforward consequences of the Jacobi identity will be useful to compute the expansion of brackets of two elements of  $\mathcal{B}^*$  (see Definition 1.6 for the notation  $0^\nu$ ).

**Lemma 4.9.** *Using the notation  $0^\nu$  of Definition 1.6, the following expansions hold.*

1. For any  $\nu \in \mathbb{N}$  and any  $a, b \in \mathcal{L}(X)$ ,

$$[a, b0^\nu] = \sum_{\nu'=0}^{\nu} \binom{\nu}{\nu'} (-1)^{\nu'} [a0^{\nu'}, b]0^{\nu-\nu'}. \quad (4.24)$$

2. For any  $\nu \in \mathbb{N}^*$ , there exist coefficients  $\alpha_j^\nu \in \mathbb{Z}$  for  $1 \leq 2j+1 \leq \nu$ , such that, for any  $b \in \mathcal{L}(X)$ ,

$$[b, b0^\nu] = \sum_{1 \leq 2j+1 \leq \nu} \alpha_j^\nu [b0^j, b0^{j+1}]0^{\nu-2j-1}. \quad (4.25)$$

*Proof.* The validity of (4.24) for any  $a, b$  can be proved by induction on  $\nu \in \mathbb{N}$ , the heredity relies on the Jacobi identity and the binomial relation  $\binom{\nu-1}{\nu'} + \binom{\nu-1}{\nu'-1} = \binom{\nu}{\nu'}$  for  $\nu' = 1, \dots, \nu-1$ . The validity of (4.25) for any  $b$  can be proved by induction on  $\nu \in \mathbb{N}^*$ ; the Jacobi relation leads to  $\alpha_j^\nu = \alpha_j^{\nu-1} - \alpha_{j-1}^{\nu-2}$ .  $\square$

## 5 Sussmann's and Stefani's obstructions

The goal of this section is to give a new proof of Theorem 1.10, within the framework of the unified approach proposed in this paper, as a consequence of the following more precise statement.

**Theorem 5.1.** *Assume that (1.18) does not hold. Let  $k \in \mathbb{N}^*$  such that*

$$\text{ad}_{f_1}^{2k}(f_0)(0) \notin S_{\llbracket 1, 2k-1 \rrbracket}(f)(0). \quad (5.1)$$

*Then system (1.1) has a drift along  $\text{ad}_{f_1}^{2k}(f_0)(0)$ , parallel to  $S_{\llbracket 1, 2k-1 \rrbracket}(f)(0)$ , of amplitude  $\xi_{\text{ad}_{X_1}^{2k}}(X_0)$ , in the regime  $(t, \|u\|_{W^{-1, \infty}}) \rightarrow 0$ .*

### 5.1 Dominant part of the logarithm

**Lemma 5.2.** *Let  $k \in \mathbb{N}^*$  such that (5.1) holds. Let  $\mathbb{P}$  be a component along  $\text{ad}_{f_1}^{2k}(f_0)(0)$ , parallel to  $S_{\llbracket 1, 2k-1 \rrbracket}(f)(0)$ . Then*

$$\mathbb{P}\mathcal{Z}_{2k}(t, f, u)(0) = \xi_{\text{ad}_{X_1}^{2k}}(X_0)(t, u) + O\left(|u_1(t)|^{2k} + t^{\frac{1}{2k-1}} \|u_1\|_{L^{2k}}^{2k}\right). \quad (5.2)$$

*Proof.* We intend to apply Proposition 4.6 with  $M \leftarrow 2k$ ,  $L \leftarrow 2k+2$ ,  $\mathbf{b} \leftarrow \text{ad}_{X_1}^{2k}(X_0)$  and  $\mathcal{N} \leftarrow \mathcal{B}_{\llbracket 1, 2k-1 \rrbracket}^*$ , so that (5.2) will follow from (4.11), for the appropriate choice of  $\Xi(t, u)$ . Let us check that the required estimates are satisfied.

**Step 1.** *Estimates of other coordinates of the second kind.* Let  $b \in \mathcal{B}_{\llbracket 1, 2k \rrbracket}^*$  such that  $b \notin \mathcal{N} \cup \{\mathbf{b}\}$ .

Since  $\mathcal{N} = \mathcal{B}_{\llbracket 1, 2k-1 \rrbracket}^*$ , one has  $n_1(b) = 2k$  and  $n_0(b) \geq 2$ . Hence  $|b| \geq 2k+2$ . By (3.22) of Proposition 3.9, estimate (4.9) holds with  $\sigma = 2k+2$  and  $\Xi(t, u) = t \|u_1\|_{L^{2k}}^{2k}$ .

**Step 2.** *Estimates of cross products.* Let  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}^* \setminus \{X_0\}$  such that  $n_1(b_1) + \dots + n_1(b_q) \leq 2k$  and  $\text{supp } \mathcal{F}(b_1, \dots, b_q) \not\subset \mathcal{N}$ .

For each  $i \in \llbracket 1, q \rrbracket$ ,

- if  $b_i = X_1$ , then

$$|\xi_{b_i}(t, u)| = |u_1(t)|, \quad (5.3)$$

so (4.10) holds with  $\sigma_i = 1$  and  $\alpha_i = 1/(2k) = n_1(b_i)/2k$  and  $\Xi(t, u) = |u_1(t)|^{2k}$ .

- otherwise,  $|b_i| \geq 1 + n_1(b_i)$  and, by (3.22) of Proposition 3.9 and Hölder's inequality,

$$|\xi_{b_i}(t, u)| \leq \frac{(ct)^{|b_i|}}{|b_i|!} t^{-1-n_1(b_i)} \|u_1\|_{L^{n_1(b_i)}}^{n_1(b_i)} \leq \frac{(ct)^{|b_i|}}{|b_i|!} t^{-\sigma_i} \left( t^{\frac{1}{\alpha_i}-1} \|u_1\|_{L^{2k}}^{2k} \right)^{\alpha_i} \quad (5.4)$$

with  $\sigma_i = 1 + n_1(b_i)$  and  $\alpha_i = n_1(b_i)/(2k)$ . Since  $q \geq 2$ ,  $n_1(b_i) \leq 2k - 1$ . Thus  $\frac{1}{\alpha_i} - 1 \geq \frac{1}{2k-1}$  and, assuming  $t \leq 1$ ,

$$t^{\frac{1}{\alpha_i}-1} \|u_1\|_{L^{2k}}^{2k} \leq t^{\frac{1}{2k-1}} \|u_1\|_{L^{2k}}^{2k}, \quad (5.5)$$

so (4.10) holds with  $\Xi(t, u) = t^{\frac{1}{2k-1}} \|u_1\|_{L^{2k}}^{2k}$ .

Since  $\mathcal{N} = \mathcal{B}_{[1, 2k-1]}^*$ , one has  $n_1(b_1) + \dots + n_1(b_q) = 2k$ . Hence  $\alpha = \alpha_1 + \dots + \alpha_q = 1$ .  $\square$

## 5.2 Vectorial relation

**Lemma 5.3.** *Let  $k \in \mathbb{N}^*$  such that (5.1) holds. Then,  $f_1(0) \neq 0$ .*

*Proof.* By contradiction, if  $f_1(0) = 0$ , since  $f_0(0) = 0$ , all iterated Lie brackets of  $f_0$  and  $f_1$  vanish so  $\text{ad}_{f_1}^{2k}(f_0)(0) = 0 \in S_{[1, 2k-1]}(f)(0) = \{0\}$ .  $\square$

## 5.3 Closed-loop estimate

**Lemma 5.4.** *Assume that  $f_1(0) \neq 0$ . Then,*

$$|u_1(t)| = O(|x(t; u)| + \|u_1\|_{L^1}). \quad (5.6)$$

*Proof.* This estimate is proved in [3, Proposition 162]. For the sake of self-containedness, and as an illustration of the approach used in the following sections, let us give another proof.

Let  $\mathbb{P}$  be a component along  $f_1(0)$ , parallel to the null vector space  $\{0\}$ . By Proposition 4.6 with  $M \leftarrow 1$ ,  $L \leftarrow 2$ ,  $\mathfrak{b} \leftarrow X_1$  and  $\mathcal{N} \leftarrow \emptyset$ , (4.11) entails that

$$\mathbb{P}\mathcal{Z}_1(t, f, u)(0) = u_1(t) + O(\|u_1\|_{L^1}). \quad (5.7)$$

Indeed, on the one hand, for every  $b \in \mathcal{B}_1^* \setminus \{X_1\}$ , by (3.23) with  $(p, j_0) \leftarrow (1, 1)$ , one has  $|b| \geq 2$  and

$$|\xi_b(t, u)| \leq \frac{(ct)^{|b|}}{|b|!} t^{-2} \|u_1\|_{L^1}, \quad (5.8)$$

so (4.9) holds with  $\sigma = 2$  and  $\Xi(t, u) = \|u_1\|_{L^1}$ . On the other hand, we don't need to estimate any cross products because, when  $q \geq 2$  and  $b_1, \dots, b_q \in \mathcal{B}^* \setminus \{X_0\}$ ,  $n_1(b_1) + \dots + n_1(b_q) > 1$ .

By Theorem 4.4 with  $M \leftarrow 1$ ,

$$x(t; u) = \mathcal{Z}_1(t, f, u)(0) + O(\|u_1\|_{L^2}^2 + |x(t; u)|^2). \quad (5.9)$$

Then (5.6) follows from (5.7), (5.9) and the small-state estimate of Lemma 4.3.  $\square$

## 5.4 Interpolation inequality

**Lemma 5.5.** *For  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,*

$$\|u_1\|_{L^{2k+1}}^{2k+1} \leq \|u_1\|_{L^\infty} \|u_1\|_{L^{2k}}^{2k}. \quad (5.10)$$

## 5.5 Proof of the drift

*Proof of Theorem 5.1.* Let  $\mathbb{P}$  be a component along  $\text{ad}_{f_1}^{2k}(f_0)(0)$  parallel to  $S_{\llbracket 1, 2k-1 \rrbracket}(f)(0)$ . By Theorem 4.4,

$$x(t; u) = \mathcal{Z}_{2k}(t, f, u)(0) + O\left(\|u_1\|_{L^{2k+1}}^{2k+1} + |x(t; u)|^{1+\frac{1}{2k}}\right) \quad (5.11)$$

and, by (5.2) and (2.5),

$$\mathbb{P}\mathcal{Z}_{2k}(t, f, u)(0) = \frac{1}{(2k)!} \int_0^t u_1^{2k} + O\left(|u_1(t)|^{2k} + t^{\frac{1}{2k-1}} \|u_1\|_{L^{2k}}^{2k}\right). \quad (5.12)$$

Moreover, by the closed-loop estimate (5.6) and Hölder's inequality,

$$|u_1(t)|^{2k} = O\left(|x(t; u)|^{2k} + t^{2k-1} \|u_1\|_{L^{2k}}^{2k}\right). \quad (5.13)$$

Gathering these equalities and (5.10) yields

$$\mathbb{P}x(t; u) = \int_0^t \frac{u_1^{2k}}{(2k)!} + O\left(\left(t^{\frac{1}{2k-1}} + \|u_1\|_{L^\infty}\right) \int_0^t u_1^{2k} + |x(t; u)|^{1+\frac{1}{2k}}\right). \quad (5.14)$$

This matches Definition 1.15 of a drift along  $\text{ad}_{f_1}^{2k}(f_0)(0)$ , parallel to  $S_{\llbracket 1, 2k-1 \rrbracket}(f)(0)$ , of amplitude  $\xi_{\text{ad}_{X_1}^{2k}(X_0)}$ , in the regime  $(t, \|u\|_{W^{-1, \infty}}) \rightarrow 0$ .  $\square$

## 6 New loose quadratic obstructions, conjectured by Kawski

We prove Theorem 1.11, as a consequence of the following more precise statement.

**Theorem 6.1.** *Let  $m \in \mathbb{N}$  and  $k \in \mathbb{N}^*$ . We assume  $k$  is the smallest integer for which*

$$f_{W_k}(0) \notin S_{\llbracket 1, \pi(k, m) \rrbracket \setminus \{2\}}(f)(0), \quad (6.1)$$

where  $\pi(k, m)$  is defined in (1.20). Then system (1.1) has a drift along  $f_{W_k}(0)$ , parallel to  $S_{\llbracket 1, \pi(k, m) \rrbracket \setminus \{2\}}(f)(0)$ , in the regime  $t \rightarrow 0$  and  $\|u\|_{W^{m, \infty}} \rightarrow 0$ .

When  $m = 0$ , the drift actually holds in the (weaker) regime  $(t, \|u\|_{L^\infty}) \rightarrow 0$  (see Remarks 1.16 and 6.6), where the smallness assumption on  $\|u\|_{L^\infty}$  does not depend on  $t$ .

### 6.1 A previous result on a prototype example

In [18, System (32)], Kawski considers the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dots \\ \dot{x}_k = x_{k-1} \\ \dot{x}_{k+1} = x_k^2 - \lambda x_1^p, \end{cases} \quad (6.2)$$

where  $\lambda > 0$ . Written in the form (1.1), this system satisfies

$$f_{M_{j-1}}(0) = e_j \text{ for } j \in \llbracket 1, k \rrbracket, \quad f_{W_k}(0) = 2e_{k+1}, \quad f_{\text{ad}_{X_1}^p(X_0)}(0) = -\lambda p! e_{k+1} \quad (6.3)$$

and  $f_b(0) = 0$  for any other  $b \in \mathcal{B}^*$ . In [18, Proposition 5.1], Kawski proves that, if  $p \geq 2^{k+1}$  then the system (6.2) is not  $L^\infty$ -STLC. This result can be recovered by applying Theorem 6.1 to system (6.2) with  $m \leftarrow 0$ . Indeed,  $p \geq 2^{k+1} > 2k - 1 = \pi(k, 0)$ .

With respect to this previous result, Theorem 6.1 can be viewed as an improvement in the following directions:

- any perturbation in  $\mathcal{B}_{[p,\infty]}^*$  is allowed (not only  $\text{ad}_{X_1}^p(X_0)$ ),
- as correctly conjectured in [17, section 2.4, p. 63], the critical threshold for  $L^\infty$ -STLC is proved to be  $2k - 1$  (instead of  $2^{k+1} - 1$  obtained in [18, Proposition 5.1]),
- other regularity scales  $W^{m,\infty}$  for  $m > 0$  are included.

## 6.2 Dominant part of the logarithm

**Lemma 6.2.** *Let  $k \in \mathbb{N}^*$ . Assume that  $k$  is the minimal value for which (6.1) holds. Let  $\mathbb{P}$  be a component along  $f_{W_k}(0)$ , parallel to  $S_{[1,\pi(k,m)] \setminus \{2\}}(f)(0)$ . Then*

$$\mathbb{P}\mathcal{Z}_{\pi(k,m)}(t, f, u)(0) = \xi_{W_k}(t, u) + O(|(u_1, \dots, u_k)(t)|^2 + t\|u_k\|_{L^2}^2). \quad (6.4)$$

*Proof.* By minimality of  $k$ , for every  $j \in [1, k - 1]$ ,

$$f_{W_j}(0) \in S_{[1,\pi(j,m)] \setminus \{2\}}(f)(0) \subset S_{[1,\pi(k,m)] \setminus \{2\}}(f)(0), \quad (6.5)$$

since  $\pi(\cdot, m)$  is non-decreasing. Since  $S_{[1,\pi(k,m)] \setminus \{2\}}(X)$  is stable by right bracketing with  $X_0$ , one also has

$$f_{W_{j,\nu}}(0) \in S_{[1,\pi(k,m)] \setminus \{2\}}(f)(0), \quad (6.6)$$

for every  $j \in [1, k - 1]$  and  $\nu \geq 0$ . Hence  $S_{[1,\pi(k,m)] \setminus \{2\}}(f)(0) = \mathcal{N}(f)(0)$  where

$$\mathcal{N} := \mathcal{B}_{[1,\pi(k,m)] \setminus \{2\}}^* \cup \{W_{j,\nu}; j \in [1, k - 1], \nu \in \mathbb{N}\}. \quad (6.7)$$

We intend to apply Proposition 4.6 with  $M \leftarrow \pi(k, m)$ ,  $L \leftarrow 2k + 2$ ,  $\mathfrak{b} \leftarrow W_k$  and  $\mathcal{N}$  as in (6.7), so that (6.4) will follow from (4.11), for the appropriate choice of  $\Xi(t, u)$ . Let us check that the required estimates are satisfied.

**Step 1.** *Estimates of other coordinates of the second kind.* Let  $b \in \mathcal{B}_{[1,\pi(k,m)]}^*$  such that  $b \notin \mathcal{N} \cup \{\mathfrak{b}\}$ .

By definition (6.7), one necessarily has  $n_1(b) = 2$  and  $b = W_{j,\nu}$  with either  $j > k$  or  $(j = k$  and  $\nu \geq 1)$ . By estimate (3.24) with  $(p, j_0) \leftarrow (1, k)$ , (4.9) holds with  $\sigma = 2k + 2$  and

$$\Xi(t, u) := t\|u_k\|_{L^2}^2. \quad (6.8)$$

**Step 2.** *Estimates of other cross products.* Let  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}^*$  such that  $n_1(b_1) + \dots + n_1(b_q) \leq \pi(k, m)$  and  $\text{supp } \mathcal{F}(b_1, \dots, b_q) \not\subset \mathcal{N}$ .

We start with preliminary estimates.

- If  $b_i = M_j$  for some  $j \in [0, k - 1]$ , by (3.11),

$$|\xi_{b_i}(t, u)| = |u_{j+1}(t)| = \frac{t^{|b_i|}}{|b_i|!} t^{-(j+1)} (j+1)! |u_{j+1}(t)| \quad (6.9)$$

so (4.10) holds with  $\sigma_i = j + 1$ ,  $\alpha_i = 1/2$  and  $\Xi(t, u) = |(u_1, \dots, u_k)(t)|^2$ .

- If  $b_i = M_j$  for  $j \geq k$ , by (3.23) (with  $(p, j_0) \leftarrow (2, k)$ ), (4.10) holds with  $\sigma_i = k + 1$ ,  $\alpha_i = 1/2$  and  $\Xi(t, u) = t\|u_k\|_{L^2}^2$ .

Since  $\text{supp } \mathcal{F}(b_1, \dots, b_q) \not\subset \mathcal{N}$ , one has  $q = 2$  and  $b_1, b_2 \in \mathcal{B}_1^*$ . So the previous estimates apply and  $\alpha_1 = \alpha_2 = 1/2$  so  $\alpha_1 + \alpha_2 = 1$ .  $\square$



### 6.3 Vectorial relations

**Lemma 6.3.** *Let  $k \in \mathbb{N}^*$ ,  $\pi : \mathbb{N}^* \rightarrow \mathbb{N}^*$  be a non-decreasing map and  $\vartheta : \mathbb{N}^* \rightarrow \mathbb{N}^*$  be defined by  $\vartheta(k) = \max\{1; \lfloor \frac{\pi(k)}{2} \rfloor\}$ . Assume that  $k$  is the minimal value for which  $f_{W_k}(0) \notin S_{\llbracket 1, \pi(k) \rrbracket \setminus \{2\}}(f)(0)$ . Then,*

1. *the vectors  $f_{M_0}(0), \dots, f_{M_{k-1}}(0)$  are linearly independent,*
2. *if  $\vartheta(k) \geq 2$ , then  $\text{span}\{f_{M_0}(0), \dots, f_{M_{k-1}}(0)\} \cap S_{\llbracket 2, \vartheta(k) \rrbracket}(f)(0) = \{0\}$ .*

*Proof.* Let  $H_0 := f'_0(0)$ . Since  $f_0(0) = 0$ , for any  $b \in \text{Br}(X)$ ,  $f_{(b, X_0)}(0) = H_0 f_b(0)$ . Thus, for each  $A \subset \mathbb{N}$ , the space  $S_A(f)(0)$  is stable by left multiplication by  $H_0$ . In particular, by minimality of  $k$ , for each  $l \in \llbracket 1, k-1 \rrbracket$  and  $\nu \in \mathbb{N}$ ,

$$f_{W_{l,\nu}}(0) = H_0^\nu f_{W_{l,0}}(0) \in S_{\llbracket 1, \pi(l) \rrbracket \setminus \{2\}}(f)(0) \subset S_{\llbracket 1, \pi(k) \rrbracket \setminus \{2\}}(f)(0), \quad (6.10)$$

where the last inclusion results from the monotony of  $\pi$ . Thus,

$$S_{2, \llbracket 1, 2k-2 \rrbracket}(f)(0) \subset S_{\llbracket 1, \pi(k) \rrbracket \setminus \{2\}}(f)(0). \quad (6.11)$$

**Step 1.** *Proof of statement 1.* By contradiction, assume that there exists  $(\beta_0, \dots, \beta_{k-1}) \in \mathbb{R}^k \setminus \{0\}$  such that  $\beta_0 f_{M_0}(0) + \dots + \beta_{k-1} f_{M_{k-1}}(0) = 0$ , i.e.  $f_{B_1}(0) = 0$  where  $B_1 := \beta_{k-1} M_{k-1} + \dots + \beta_0 M_0$ . One may assume that  $\beta_{k-1} \neq 0$ ; otherwise replace  $B_1$  by  $\text{ad}_{X_0}^{k-1-K}(B_1)$  where  $K = \max\{j; \beta_j \neq 0\}$ . By linearity, one may assume  $\beta_{k-1} = 1$ . Then  $f_{B_2}(0) = 0$  where

$$B_2 := \text{ad}_{B_1}^2(X_0) = [M_{k-1} + \dots + \beta_0 M_0, M_k + \dots + \beta_0 M_1] = W_k - B_3, \quad (6.12)$$

where  $B_3 \in S_{2, \llbracket 1, 2k-2 \rrbracket}(X)$ . Finally, by (6.11),  $f_{W_k}(0) = f_{B_3}(0) \in S_{\llbracket 1, \pi(k) \rrbracket \setminus \{2\}}(f)(0)$ , which contradicts (6.1).

**Step 2.** *Proof of statement 2.* By contradiction, assume that  $\vartheta(k) \geq 2$  and that there exists  $B \in S_{\llbracket 2, \vartheta(k) \rrbracket}(X)$  and  $(\gamma_0, \dots, \gamma_{k-1}) \in \mathbb{R}^k \setminus \{0\}$  such that  $f_{B_4}(0) = 0$  where  $B_4 := \gamma_{k-1} M_{k-1} + \dots + \gamma_0 M_0 + B$ . One may assume  $\gamma_{k-1} = 1$ ; otherwise, replace  $B_4$  by  $\text{ad}_{X_0}^{k-1-K}(B_4)$  where  $K = \max\{j; \gamma_j \neq 0\}$  and renormalize. Then  $f_{B_5}(0) = 0$  where

$$\begin{aligned} B_5 &:= \text{ad}_{B_4}^2(X_0) = [M_{k-1} + \dots + \gamma_0 M_0 + B, M_k + \dots + \gamma_0 M_1 + [B, X_0]] \\ &\in W_k + S_{2, \llbracket 1, 2k-2 \rrbracket}(X) + S_{\llbracket 3, 2\vartheta(k) \rrbracket}(X). \end{aligned} \quad (6.13)$$

This fact and (6.11) contradict (6.1) because  $2\vartheta(k) \leq \pi(k)$ .  $\square$

### 6.4 Closed-loop estimate

**Lemma 6.4.** *Let  $k \in \mathbb{N}^*$ ,  $\pi : \mathbb{N}^* \rightarrow \mathbb{N}^*$  be a non-decreasing map and  $\vartheta : \mathbb{N}^* \rightarrow \mathbb{N}^*$  be defined by  $\vartheta(k) = \max\{1; \lfloor \frac{\pi(k)}{2} \rfloor\}$ . Assume that  $k$  is the minimal value for which  $f_{W_k}(0) \notin S_{\llbracket 1, \pi(k) \rrbracket \setminus \{2\}}(f)(0)$ . Then,*

$$|(u_1, \dots, u_k)(t)| = O\left(|x(t; u)| + \|u_1\|_{L^{\vartheta(k)+1}}^{\vartheta(k)+1} + t^{\frac{1}{2}} \|u_k\|_{L^2}\right). \quad (6.14)$$

*Proof.* By Theorem 4.4 with  $M \leftarrow \vartheta(k)$ ,

$$x(t; u) = \mathcal{Z}_{\vartheta(k)}(t, f, u)(0) + O\left(\|u_1\|_{L^{\vartheta(k)+1}}^{\vartheta(k)+1} + |x(t; u)|^{1+\frac{1}{\vartheta(k)}}\right). \quad (6.15)$$

Let  $i \in \llbracket 0, k-1 \rrbracket$ . By Lemma 6.3, we can consider  $\mathbb{P}$ , a component along  $f_{M_i}(0)$ , parallel to  $\mathcal{N}(f)(0)$  where  $\mathcal{N} := (\{M_0, \dots, M_{k-1}\} \setminus M_i) \cup \mathcal{B}_{\llbracket 2, \vartheta(k) \rrbracket}^*$ . We intend to apply Proposition 4.6 with

$M \leftarrow \vartheta(k)$ ,  $L \leftarrow k + 1$ ,  $\mathbf{b} \leftarrow M_i$  and  $\mathcal{N}$  as above, so that (4.11), for the appropriate choice of  $\Xi(t, u)$ , will yield

$$\mathbb{P}\mathcal{Z}_{\vartheta(k)}(t, f, u)(0) = u_{i+1}(t) + O\left(t^{\frac{1}{2}}\|u_k\|_{L^2}\right). \quad (6.16)$$

Then, combining (6.15) and (6.16) concludes the proof of (6.14). Let us check that the required estimates are satisfied.

**Step 1. Estimates of other coordinates of the second kind.** Let  $b \in \mathcal{B}_{[1, \vartheta(k)]}^*$  such that  $b \notin \mathcal{N} \cup \{\mathbf{b}\}$ .

By choice of  $\mathcal{N}$ , one has necessarily  $n_1(b) = 1$ . Then  $b = M_j$  for  $j \geq k$ . Thus, by (3.23) (with  $(p, j_0) \leftarrow (2, k)$ ),  $|b| \geq k + 1$  and (4.9) holds with  $\sigma = k + 1$  and  $\Xi(t, u) := t^{\frac{1}{2}}\|u_k\|_{L^2}$ .

**Step 2. Estimates of cross products.** Let  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}^* \setminus \{X_0\}$  such that  $n_1(b_1) + \dots + n_1(b_q) \leq \vartheta(k)$  and  $\text{supp } \mathcal{F}(b_1, \dots, b_q) \not\subset \mathcal{N}$ .

By construction of  $\mathcal{N}$ , there is no such cross product term.  $\square$

## 6.5 Interpolation inequality

**Lemma 6.5.** *There exists  $C > 0$  such that, for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,*

$$\|u_1\|_{L^{\frac{\pi(k,m)+1}{\pi(k,m)+1}}}^{\pi(k,m)+1} \leq C \left( \|u_1\|_{L^\infty}^{\pi(k,m)+1-p} \|u\|_{W^{m,\infty}}^p + t^{\pi(k,m)+1-2k} \|u\|_{L^\infty}^{\pi(k,m)-1} \right) \|u_k\|_{L^2}^2, \quad (6.17)$$

where  $p := (2m + 2k)/(m + 1)$  satisfies  $p \leq \pi(k, m) + 1$ .

*Proof.* By Proposition 4.7 with  $\phi \leftarrow u_k$ ,  $(p, q, r, s) \leftarrow (p, 2, \infty, 2)$ ,  $(j, l) \leftarrow (k - 1, m + k)$ ,  $\alpha \leftarrow (p - 2)/p$ , we obtain

$$\|u_1\|_{L^p}^p \leq C \|u^{(m)}\|_{L^\infty}^{p-2} \|u_k\|_{L^2}^2 + Ct^{1-pk+\frac{p}{2}} \|u_k\|_{L^2}^p. \quad (6.18)$$

By Hölder's inequality,

$$\|u_k\|_{L^2}^{p-2} \leq t^{(\frac{1}{2}+k)(p-2)} \|u\|_{L^\infty}^{p-2}. \quad (6.19)$$

Moreover, by (1.20),

$$\pi(k, m) + 1 \geq \frac{2k + m - 1}{m + 1} + 1 = \frac{2k + 2m}{m + 1} = p, \quad (6.20)$$

and this concludes the proof of (6.17), writing

$$\|u_1\|_{L^{\frac{\pi(k,m)+1}{\pi(k,m)+1}}}^{\pi(k,m)+1} \leq \|u_1\|_{L^\infty}^{\pi(k,m)+1-p} \|u_1\|_{L^p}^p \quad (6.21)$$

and  $\|u_1\|_{L^\infty} \leq t\|u\|_{L^\infty}$ .  $\square$

## 6.6 Proof of the drift for $m \geq 0$

*Proof of Theorem 6.1.* Let  $\mathbb{P}$  be a component along  $f_{W_k}(0)$  parallel to  $S_{[1, \pi(k,m)] \setminus \{2\}}(f)(0)$ . Let  $M := \pi(k, m)$ . Let  $\vartheta := \max\{1; \lfloor \frac{\pi(k,m)}{2} \rfloor\}$ . By Theorem 4.4,

$$x(t; u) = \mathcal{Z}_M(t, f, u)(0) + O\left(\|u_1\|_{L^{M+1}}^{M+1} + |x(t; u)|^{1+\frac{1}{M}}\right), \quad (6.22)$$

where, by (6.4) and (3.12),

$$\mathbb{P}\mathcal{Z}_M(t, f, u)(0) = \frac{1}{2} \int_0^t u_k^2 + O(|(u_1, \dots, u_k)(t)|^2 + t\|u_k\|_{L^2}^2). \quad (6.23)$$

Moreover, by the closed-loop estimate (6.14),

$$|(u_1, \dots, u_k)(t)|^2 = O(|x(t; u)|^2 + \|u_1\|_{L^{\vartheta+1}}^{2\vartheta+2} + t\|u_k\|_{L^2}^2). \quad (6.24)$$

By definition of  $\vartheta$ , one has  $2(\vartheta + 1) \geq \pi(k, m) + 1$ . Hence, in particular,

$$\|u_1\|_{L^{\vartheta+1}}^{2\vartheta+2} = O\left(\|u_1\|_{L^{M+1}}^{M+1}\right). \quad (6.25)$$

Gathering these equalities and the interpolation estimate (6.17) yields

$$\mathbb{P}x(t; u) = \frac{1}{2} \int_0^t u_k^2 + O\left(\left(t + (1 + t^{\pi(k, m) + 1 - 2k})\|u\|_{W^{m, \infty}}^{\pi(k, m) - 1}\right)\|u_k\|_{L^2}^2 + |x(t; u)|^{1 + \frac{1}{M}}\right). \quad (6.26)$$

This implies, in the sense of Definition 1.15, a drift along  $f_{W_k}(0)$ , parallel to  $S_{\llbracket 1, \pi(k, m) \rrbracket \setminus \{2\}}(f)(0)$ , of amplitude  $\xi_{W_k}$ , in the regime  $t \rightarrow 0$  and  $\|u\|_{W^{m, \infty}} \rightarrow 0$  (the smallness assumption on the control depends on the final time; see Remark 1.16).  $\square$

**Remark 6.6.** *In the previous proof, when  $m = 0$ , one has  $\pi(k, m) + 1 - 2k = 0$ . Thus, in this case, the smallness assumption on the control does not depend on the final time. The drift along  $f_{W_k}(0)$  then holds in the regime  $(t, \|u\|_{L^\infty}) \rightarrow 0$  (see Remark 1.16).*

*When  $m > 0$ , the dependence on time of the smallness assumption on the control stems from the second term in the right-hand side of the Gagliardo–Nirenberg inequality of Proposition 4.7. For appropriate classes of functions, for instance  $\phi \in W_0^{m, \infty}$ , the Gagliardo–Nirenberg inequality holds without this second term. Thus, for controls  $u \in W_0^{m, \infty}$ , the argument above proves a drift in the regime  $(t, \|u\|_{W_0^{m, \infty}}) \rightarrow 0$ .*

## 6.7 Optimality of the functional framework

We illustrate the optimality of the functional framework given in Theorem 6.1 by an example in the case  $m = 0$  and  $k = 2$ . In this case, the condition  $f_{W_2}(0) \in S_{\{1, 3\}}(f)(0)$  is necessary for  $L^\infty$ -STLC, but not for a different small-time local controllability notion involving large enough controls in  $L^\infty$  (instead of arbitrarily small controls in  $L^\infty$ ), called  $\rho$ -bounded-SLTC in Section 1.2. In this sense, our result is optimal.

To prove this claim, let us consider the following system (introduced in [30, Example 5.2]):

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2^2 - x_1^4. \end{cases} \quad (6.27)$$

Written in the form (1.1), this system satisfies

$$f_{M_0} = e_1, \quad f_{M_1}(0) = e_2, \quad f_{W_2}(0) = 2e_3, \quad f_{Q_{1,1,1}}(0) = -24e_3 \quad (6.28)$$

and  $f_b(0) = 0$  for any  $b \in \mathcal{B}^* \setminus \{M_0, M_1, W_2, Q_{1,1,1}\}$ . Thus  $f_{W_1}(0) \in \mathcal{S}_1(f)(0)$  and  $f_{W_2}(0) \notin \mathcal{S}_{\{1, 3\}}(f)(0)$ . By Theorem 1.11, this system is not  $L^\infty$ -STLC, i.e. locally controllable in small time with -small controls. By Theorem 6.1, solutions associated to controls small in  $L^\infty$  cannot reach in small time targets of the form  $-\beta e_3$  with  $\beta > 0$ .

In [18, Example 5.1 and p. 452], Kawski claims that this system is STLC with controls large enough in  $L^\infty$ . This can also be deduced from the arguments given in [30, Example 5.2] by a scaling argument. Let us indeed construct explicit controls (large in  $L^\infty$ ) achieving a motion along  $-e_3$ . Let  $\varphi \in \mathcal{C}_c^\infty((0, 1); \mathbb{R}) \setminus \{0\}$  and  $A > 0$  large enough such that

$$C := - \int_0^1 \varphi^2 + A^2 \int_0^1 (\varphi')^4 > 0. \quad (6.29)$$

Let  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$  be defined by  $u(s) := A\varphi''(s/t)$ . Then  $u_1(s) = At\varphi'(s/t)$  and  $u_2(s) = At^2\varphi(s/t)$ . Thus

$$x_3(t) = \int_0^t u_2^2 - u_1^4 = \int_0^t \left( \left( At^2 \varphi' \left( \frac{s}{t} \right) \right)^2 - \left( At \varphi' \left( \frac{s}{t} \right) \right)^4 \right) ds = -t^5 A^2 C. \quad (6.30)$$

Therefore  $x(t; u) = -t^5 A^2 C e_3$ , so we have indeed achieved a motion along  $-e_3$ . Standard arguments using either tangent vectors or power series expansions (see e.g. [18, Appendix] or [10, Section 8.1]) then allow to prove that there exists  $\rho > 0$  large enough such that (6.27) is indeed  $\rho$ -bounded-STLC.

## 6.8 An extension to the case $m = -1$

As mentioned in Section 1.5, it would be natural to expect that Theorem 1.11 holds in the case  $m = -1$  with  $\pi(k, -1) := +\infty$ . This would correspond to the heuristic that, for  $k \in \mathbb{N}^*$ , the bracket  $W_k$  is “bad” even for  $W^{-1, \infty}$ -STLC, i.e. that it has to be neutralized by some other bracket (here, such a statement would entail that it should be neutralized by a bracket outside of  $\mathcal{B}_2^*$ ).

Up to our knowledge, this is an open problem. We give here a partial result in this direction, under an extra nilpotency assumption. Indeed, we prove that, if  $f_1$  is semi-nilpotent with respect to  $f_0$  (see below) and system (1.1) is  $W^{-1, \infty}$ -STLC then

$$\forall k \in \mathbb{N}^*, \quad f_{W_k}(0) \in S_{[1, \infty[\setminus\{2\}}(f)(0). \quad (6.31)$$

**Definition 6.7** (Semi-nilpotent family of vector fields). *Let  $\Omega$  be an open subset of  $\mathbb{R}^d$ ,  $f_0, f_1 \in \mathcal{C}^\infty(\Omega; \mathbb{R}^d)$  and  $M \in \mathbb{N}^*$ . We say that the vector field  $f_1$  is semi-nilpotent of index  $M$  with respect to  $f_0$  when*

$$\forall b \in \text{Br}(X), n_1(b) = M \Rightarrow f_b = 0 \text{ on } \Omega \quad (6.32)$$

(every bracket of  $f_0$  and  $f_1$  involving  $M$  occurrences of  $f_1$  vanishes identically on  $\Omega$ ) and  $M$  is the smallest positive integer for which this property holds.

**Theorem 6.8.** *Assume that  $f_1$  is semi-nilpotent with respect to  $f_0$ . If system (1.1) is  $W^{-1, \infty}$ -STLC, then, for every  $k \in \mathbb{N}^*$ ,*

$$f_{W_k}(0) \in S_{[1, \infty[\setminus\{2\}}(f)(0). \quad (6.33)$$

The proof follows the same steps as the proof of Theorem 6.1. A key point is that the truncated formula (4.6) is replaced with the following one.

**Proposition 6.9.** *Let  $f_0, f_1$  be analytic vector fields on a neighborhood of 0 with  $f_0(0) = 0$ . Assume that  $f_1$  is semi-nilpotent of index  $M \in \mathbb{N}^*$  with respect to  $f_0$ . Then*

$$x(t; u) = \mathcal{Z}_M(t, f, u)(0) + O(\|u_1\|_{L^\infty} |x(t; u)|). \quad (6.34)$$

*Proof.* This follows from the third item of [3, Corollary 163].  $\square$

**Lemma 6.10.** *Let  $k \in \mathbb{N}^*$  be the minimal value for which (6.33) fails. Then,*

1. the vectors  $f_{M_0}(0), \dots, f_{M_{k-1}}(0)$  are linearly independent,
2.  $\text{span}\{f_{M_0}(0), \dots, f_{M_{k-1}}(0)\} \cap S_{[2, \infty[[f](0) = \{0\}$ .

*Proof.* The proof is exactly the same as in Lemma 6.3, with  $\pi(k) = \vartheta(k) = +\infty$ .  $\square$

**Lemma 6.11.** *Let  $k \in \mathbb{N}^*$ . Assume that  $f_1$  is semi-nilpotent with respect to  $f_0$  and that  $k$  is the minimal value for which (6.33) fails. Then,*

$$|(u_1, \dots, u_k)(t)| = O\left(|x(t; u)| + t^{\frac{1}{2}} \|u_k\|_{L^2}\right). \quad (6.35)$$

*Proof.* The proof is performed along the same lines as in Lemma 6.4. Instead of  $M = \vartheta(k)$ , one uses  $M$  such that  $f_1$  is semi-nilpotent of index  $(M + 1)$  with respect to  $f_0$ . One replaces (6.15) by (6.34) and concludes as previously.  $\square$

*Proof of Theorem 6.8.* Let  $\mathbb{P}$  be a component along  $f_{W_k}(0)$  parallel to  $S_{[1,\infty]\setminus\{2\}}(f)(0)$ . Let  $M \in \mathbb{N}^*$  be such that  $f_1$  is semi-nilpotent of index  $(M+1)$  with respect to  $f_0$  (see Definition 6.7). By Proposition 6.9,

$$x(t; u) = \mathcal{Z}_M(t, f, u)(0) + O(\|u_1\|_{L^\infty} |x(t; u)|), \quad (6.36)$$

where, by (6.4) and (3.12),

$$\mathbb{P}\mathcal{Z}_M(t, f, u)(0) = \frac{1}{2} \int_0^t u_k^2 + O(|(u_1, \dots, u_k)(t)|^2 + t\|u_k\|_{L^2}^2). \quad (6.37)$$

Moreover, by the closed-loop estimate (6.35),

$$|(u_1, \dots, u_k)(t)|^2 = O(|x(t; u)|^2 + t\|u_k\|_{L^2}^2). \quad (6.38)$$

Gathering these equalities yields

$$\mathbb{P}x(t; u) = \frac{1}{2} \int_0^t u_k^2 + O(t\|u_k\|_{L^2}^2 + \|u_1\|_{L^\infty} |x(t; u)|), \quad (6.39)$$

which prevents from reaching target states of the form  $x^* = -\delta f_{W_k}(0)$  for  $\delta > 0$  small enough.  $\square$

**Remark 6.12.** Equation (6.39) almost entails that there is “a drift along  $f_{W_k}(0)$  parallel to  $(f)(0)$ , in the regime  $(t, \|u\|_{W^{-1,\infty}}) \rightarrow 0$ ” in the sense of Definition 1.15. However, unlike the other situations in this paper, we have not been able to obtain an error term of the form  $|x(t; u)|^\beta$  for some  $\beta > 1$ . Here,  $\|u_1\|_{L^\infty} |x(t; u)|$  is only  $o(|x(t; u)|)$  and it is not possible to use Young’s inequality as in the other cases since one cannot absorb  $\|u_1\|_{L^\infty}$  (event at a high power) by interpolation. Hence, in the limit  $\|u_1\|_{L^\infty} \rightarrow 0$ , the “ultimately unreachable” set contains a half-space. But, for fixed  $\|u_1\|_{L^\infty}$  (even small) our proof does entail that the unreachable set contains locally a half-space. Nevertheless, this prevents controllability as stated in Theorem 6.8.

## 7 Kawski’s refined $W_2$ obstruction

The goal of this section is to prove the case  $j = 2$  in Theorem 1.12, as a consequence of the following more precise statement.

**Theorem 7.1.** *Assume that  $f_{W_1}(0) \in \mathcal{N}_1(f)(0)$  and  $f_{W_2}(0) \notin \mathcal{N}_2(f)(0)$ . Then, system (1.1) has a drift along  $f_{W_2}(0)$ , parallel to  $\mathcal{N}_2(f)(0)$ , in the regime  $(t, \|u\|_{L^\infty}) \rightarrow 0$ .*

### 7.1 Limiting examples

Let us illustrate that the set  $\mathcal{N}_2$  defined in (1.22) of brackets which can compensate  $W_2$  must include  $P_{1,1,\nu}$  for every  $\nu \in \mathbb{N}$ . As an illustration, we prove the following controllability results using classical sufficient conditions due to Sussmann or Bianchini and Stefani because they are simpler to apply. Nevertheless, the same results would follow from Agrachev and Gamkrelidze conditions as in Section 8.1.

*Limiting example for  $P_{1,1,0}$ .* Consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2^2 + x_1^3. \end{cases} \quad (7.1)$$

Written in the form (1.1), this system satisfies

$$f_{M_0}(0) = e_1, \quad f_{M_1}(0) = e_2, \quad f_{P_{1,1,0}}(0) = 6e_3, \quad f_{W_2}(0) = 2e_3 \quad (7.2)$$

and  $f_b(0) = 0$  for any  $b \in \mathcal{B}^* \setminus \{M_0, M_1, W_2, P_{1,1,0}\}$ .

This system was proposed by Jakubczyk and is known to be  $L^\infty$ -STLC since<sup>3</sup> [32, p. 711-712]. It also satisfies Sussmann's  $\mathcal{S}(\theta)$  condition (see [34, Theorem 7.3] or [10, Theorem 3.29]) for any  $\theta > 1/2$  (see also [5, Section 2.4.1] for a short direct proof).

*Limiting example for  $P_{1,1,\nu}$ .* Let  $\nu \in \mathbb{N}^*$ . We consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^3 \\ \dot{x}_{3+i} = x_{3+i-1} & \text{for } i = 1, \dots, \nu - 1 \\ \dot{x}_{3+\nu} = x_2^2 + x_{3+\nu-1}. \end{cases} \quad (7.3)$$

Written in the form (1.1), this system satisfies

$$f_{M_0}(0) = e_1, \quad f_{M_1}(0) = e_2, \quad f_{P_{1,1,\mu}}(0) = 3!e_{3+\mu} \text{ for } \mu = 0, \dots, \nu, \quad f_{W_2}(0) = 2e_{3+\nu} \quad (7.4)$$

and  $f_b(0) = 0$  for any  $b \in \mathcal{B}^* \setminus \{M_0, M_1, W_2, P_{1,1,\mu}; \mu \in \llbracket 0, \nu \rrbracket\}$ .

For  $\nu = 3$ , this system corresponds to [15, Example 2.4]. To prove that it is  $L^\infty$ -STLC, the key point is to prove that  $\pm e_3 = \pm 6f_{P_{1,1,0}}(0)$  are tangent vectors. Then, the  $L^\infty$ -STLC follows from the elementary remark that, if, for some  $b \in \mathcal{B}^*$ ,  $\pm f_b(0)$  are tangent vectors, then so are  $\pm f_{(b, X_0)}(0)$  (see [15, Theorem 6] or [6, claim P2]). As in the case  $\nu = 0$ , the fact that  $\pm e_3$  are tangent vectors can be proved using oscillating controls or Sussmann's  $\mathcal{S}(\theta)$  condition with  $\theta > 1/2$  as reformulated in [6, Theorem 2] by Bianchini and Stefani.

*A non-controllable example involving  $Q_{1,1,1}$ .* In Section 6.7, we recalled that system (6.27) is small-time locally controllable with large enough controls in  $L^\infty$ , but not  $L^\infty$ -STLC in the sense of Definition 1.1. For this system, one has  $6f_{W_2}(0) = -f_{Q_{1,1,1}}(0)$  (and  $Q_{1,1,1}$  is the only bracket "compensating"  $W_2$ ). But  $Q_{1,1,1}$  does not belong to the set  $\mathcal{N}_2$  defined in (1.22) of brackets which can compensate  $W_2$  for  $L^\infty$ -STLC. Hence, the fact that (6.27) is not  $L^\infty$ -STLC can be seen as an application of the case  $j = 2$  of Theorem 1.12.

## 7.2 Dominant part of the logarithm

**Lemma 7.2.** *Assume that  $f_{W_1}(0) \in \mathcal{N}_1(f)(0)$  and  $f_{W_2}(0) \notin \mathcal{N}_2(f)(0)$ . Let  $\mathbb{P}$  be a component along  $f_{W_2}(0)$ , parallel to  $\mathcal{N}_2(f)(0)$ . Then*

$$\mathbb{P}\mathcal{Z}_3(t, f, u)(0) = \xi_{W_2}(t, u) + O\left(|(u_1, u_2)(t)|^2 + t\|u_2\|_{L^2}^2 + \|u_1\|_{L^4}^2\|u_2\|_{L^2} + \|u_1\|_{L^4}^4\right). \quad (7.5)$$

*Proof.* By assumption,  $f_{W_1}(0) \in \mathcal{N}_1(f)(0)$ . Since  $\mathcal{N}_1$  is stable by right bracketing with  $X_0$ ,  $f_{W_{1,\nu}}(0) \in \mathcal{N}_1(f)(0)$  for every  $\nu \geq 0$ . Thus, since  $\mathcal{N}_1 \subset \mathcal{N}_2$ ,  $\mathcal{N}_2(f)(0) = \mathcal{N}(f)(0)$ , where  $\mathcal{N}$  is defined as

$$\mathcal{N} := \mathcal{N}_2 \cup \{W_{1,\nu}; \nu \in \mathbb{N}\}, \quad (7.6)$$

where  $\mathcal{N}_2$  is defined in (1.22). By assumption,  $f_{W_2}(0) \notin \mathcal{N}_2(f)(0) = \mathcal{N}(f)(0)$ .

We intend to apply Proposition 4.6 with  $M \leftarrow 3$ ,  $L \leftarrow 6$ ,  $\mathfrak{b} \leftarrow W_2$  and  $\mathcal{N}$  as in (7.6), so that (7.5) will follow from (4.11), for the appropriate choice of  $\Xi(t, u)$ . Let us check that the required estimates are satisfied.

**Step 1.** *Estimates of other coordinates of the second kind.* Let  $b \in \mathcal{B}_{\llbracket 1,3 \rrbracket}^*$  such that  $b \notin \mathcal{N} \cup \{\mathfrak{b}\}$ .

We investigate the different possibilities depending on  $n_1(b)$ .

- One cannot have  $n_1(b) = 1$  since  $\mathcal{B}_1^* \subset \mathcal{N}_2$ .

<sup>3</sup>Sussmann's initial proof involves controls with  $\|u\|_{L^\infty} \leq 1$ , but extends easily to any bound on  $u$ .

- If  $n_1(b) = 2$ , by (1.9) and (7.6), one has  $b = W_{j,\nu}$  with either  $(j \geq 3)$  or  $(j = 2 \text{ and } \nu \geq 1)$ . Thus  $|b| \geq 6$ . By estimate (3.24) with  $(p, j_0) \leftarrow (1, 2)$ , (4.9) holds with  $\sigma = 6$  and

$$\Xi(t, u) := t \|u_2\|_{L^2}^2. \quad (7.7)$$

- If  $n_1(b) = 3$ , by (1.10) and (1.22),  $b = P_{j,k,\nu}$  with  $k \geq 2$ . Thus  $|b| \geq 5$ . By estimate (3.25) with  $(p_1, p_2, j_0, k_0) \leftarrow (2, 2, 1, 2)$ , (4.9) holds with  $\sigma = 5$  and

$$\Xi(t, u) := \|u_1\|_{L^4}^2 \|u_2\|_{L^2}. \quad (7.8)$$

**Step 2. Estimates of cross products.** Let  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}^* \setminus \{X_0\}$  such that  $n_1(b_1) + \dots + n_1(b_q) \leq 3$  and  $\text{supp } \mathcal{F}(b_1, \dots, b_q) \not\subset \mathcal{N}$ .

We start with preliminary estimates.

- If  $b_i = M_j$  for some  $j \in \llbracket 0, 1 \rrbracket$ , by (3.11),

$$|\xi_{b_i}(t, u)| = |u_{j+1}(t)| = \frac{t^{|b_i|}}{|b_i|!} t^{-(j+1)} (j+1)! |u_{j+1}(t)| \quad (7.9)$$

so (4.10) holds with  $\sigma_i = j+1$ ,  $\alpha_i = 1/2$  and  $\Xi(t, u) = |(u_1, u_2)(t)|^2$ .

- If  $b_i = M_j$  for  $j \geq 2$ , by (3.23) (with  $(p, j_0) \leftarrow (2, 2)$ ), (4.10) holds with  $\sigma_i = 3$ ,  $\alpha_i = 1/2$  and  $\Xi(t, u) = t \|u_2\|_{L^2}^2$ .
- By (3.22), for each  $b_i \in \mathcal{B}_2^*$ , (4.10) holds with  $\sigma_i = 3$ ,  $\alpha_i = 1/2$  and  $\Xi(t, u) = t \|u_1\|_{L^4}^4$ .

Since  $n_1(b_1) + \dots + n_1(b_q) \leq 3$  and  $q \geq 2$ , all the  $b_i$  belong to  $\mathcal{B}_{\llbracket 1, 2 \rrbracket}^*$ . Thanks to the preliminary estimates,  $\alpha = q/2 \geq 1$ .  $\square$

### 7.3 Vectorial relation

**Lemma 7.3.** *Assume that  $f_{W_1}(0) \in \mathcal{N}_1(f)(0)$  and  $f_{W_2}(0) \notin \mathcal{N}_2(f)(0)$ . Then, the vectors  $f_{M_0}(0)$  and  $f_{M_1}(0)$  are linearly independent.*

*Proof.* This statement is implied by the case  $k = 2$  and  $\pi(k) = 2$  in Lemma 6.3.  $\square$

### 7.4 Closed-loop estimate

**Lemma 7.4.** *Assume that  $f_{M_0}(0)$  and  $f_{M_1}(0)$  are linearly independent. Then,*

$$|(u_1, u_2)(t)| = O\left(|x(t; u)| + \|u_1\|_{L^2}^2 + t^{\frac{1}{2}} \|u_2\|_{L^2}\right). \quad (7.10)$$

*Proof.* This statement is implied by the case  $k = 2$  and  $\pi(k) = 2$  in Lemma 6.4.  $\square$

### 7.5 Interpolation inequality

**Lemma 7.5.** *There exists  $C > 0$  such that, for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,*

$$\|u_1\|_{L^4}^4 \leq C \|u\|_{L^\infty}^2 \|u_2\|_{L^2}^2. \quad (7.11)$$

*Proof.* Inequality (7.11) follows from Proposition 4.7 with  $\phi \leftarrow u_2$ ,  $(p, q, r, s) \leftarrow (4, 2, \infty, 2)$ ,  $(j, l) \leftarrow (1, 2)$ ,  $\alpha \leftarrow 1/2$ . The lower-order term in (4.23) is absorbed using the estimate  $\|u_2\|_{L^2} \leq t^{\frac{5}{2}} \|u\|_{L^\infty}$ , which stems from Hölder's inequality and the equality  $u_2(t) = \int_0^t (t-s)u(s) ds$ .  $\square$

## 7.6 Proof of the drift

*Proof of Theorem 7.1.* Let  $\mathbb{P}$  be a component along  $f_{W_2}(0)$  parallel to  $\mathcal{N}_2(f)(0)$ . By Theorem 4.4 with  $M \leftarrow 3$ ,

$$x(t; u) = \mathcal{Z}_3(t, f, u)(0) + O\left(\|u_1\|_{L^4}^4 + |x(t; u)|^{1+\frac{1}{3}}\right), \quad (7.12)$$

where, by (7.5) and (3.12),

$$\mathbb{P}\mathcal{Z}_3(t, f, u)(0) = \frac{1}{2} \int_0^t u_2^2 + O\left(|(u_1, u_2)(t)|^2 + t\|u_2\|_{L^2}^2 + \|u_1\|_{L^4}^2 \|u_2\|_{L^2} + \|u_1\|_{L^4}^4\right). \quad (7.13)$$

Moreover, by the closed-loop estimate (7.10),

$$|(u_1, u_2)(t)|^2 = O\left(|x(t; u)|^2 + \|u_1\|_{L^4}^4 + t\|u_2\|_{L^2}^2\right). \quad (7.14)$$

Gathering these equalities and the interpolation estimate (7.11) yields

$$\mathbb{P}x(t; u) = \frac{1}{2} \int_0^t u_2^2 + O\left((t + \|u\|_{L^\infty}) \int_0^t u_2^2 + |x(t; u)|^{1+\frac{1}{3}}\right). \quad (7.15)$$

This implies, in the sense of Definition 1.15, a drift along  $f_{W_2}(0)$ , parallel to  $\mathcal{N}_2(f)(0)$ , of amplitude  $\xi_{W_2}$ , in the regime  $(t, \|u\|_{L^\infty}) \rightarrow 0$ .  $\square$

## 8 New refined $W_3$ obstruction

The goal of this section is to prove the case  $j = 3$  of Theorem 1.12, as a consequence of the following more precise statement.

**Theorem 8.1.** *Assume that  $f_{W_1}(0) \in \mathcal{N}_1(f)(0)$ ,  $f_{W_2}(0) \in \mathcal{N}_2(f)(0)$  and  $f_{W_3}(0) \notin \mathcal{N}_3(f)(0)$ . Then there exist a linear form  $\mathbb{P}_{W_3} : \mathbb{R}^d \rightarrow \mathbb{R}$  giving a component along  $f_{W_3}(0)$ , another linear form  $\mathbb{P} : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $C > 0$ ,  $\beta > 1$  such that, for every  $\varepsilon > 0$ , there exists  $t^* = t^*(\varepsilon)$ ,  $\rho = \rho(\varepsilon) > 0$  such that for every  $t \in (0, t^*)$ ,  $u \in L^\infty((0, t), \mathbb{R})$  with  $\|u\|_{L^\infty} < \rho$ ,*

$$(\mathbb{P}_{W_3} + t\mathbb{P})x(t, u) \geq (1 - \varepsilon)\xi_{W_3}(t, u) - C|x(t, u)|^\beta. \quad (8.1)$$

The conclusion (8.1) is not exactly a drift along  $f_{W_3}(0)$ , parallel to  $\mathcal{N}_3(f)(0)$ , in the regime  $(t, \|u\|_{L^\infty}) \rightarrow 0$ , in the sense of Definition 1.15, because the left hand side of the inequality involves a linear form  $\mathbb{P}$  that may not give a component along  $f_{W_3}(0)$ . Nevertheless this result is still an obstruction to  $L^\infty$ -STLC. Indeed, one may assume  $\mathbb{P}_{W_3}$  and  $\mathbb{P}$  are linearly independent, then by considering  $e \in \mathbb{R}^d$  such that  $\mathbb{P}_{W_3}e = 1$  and  $\mathbb{P}e = 0$ , estimate (8.1) prevents  $x(t, u)$  from reaching targets of the form  $x^* = -ae$  with  $0 < a \ll 1$ , because this would entail  $-a = (\mathbb{P}_{W_3} + t\mathbb{P})x(t, u) \geq -C|x^*|^\beta = -Ca^\beta$ , which fails for  $a$  small enough, because  $\beta > 1$ .

The proof of Theorem 8.1 is a slight variation of the unified approach as presented in Section 1.6, in which closed-loop estimates are used not only for cross products of coordinates of the second kind, but also for some coordinates of the second kind.

### 8.1 Limiting examples

We illustrate that the set  $\mathcal{N}_3$  defined in (1.23) of brackets which can compensate  $W_3$  is “minimal” in the following sense: for each bracket  $b$  of  $\mathcal{N}_3$ , we construct an example of an  $L^\infty$ -STLC system for which there is a competition between  $W_3$  and  $b$ . One has

$$\mathcal{N}_3 = \{M_\nu, P_{1,l,\nu}, Q_{1,1,2,\nu}, R_{1,1,1,1,\nu}, R_{1,1,1,\mu,\nu}^\sharp; l \in \mathbb{N}^*, \mu, \nu \in \mathbb{N}\} \cup \{Q_{1,1,1}, Q_{1,0}^\flat, Q_{1,1}^\flat, Q_{1,2}^\flat\}. \quad (8.2)$$

The brackets of the first list can be considered as “good”, and those of the second list as “bad” in senses detailed below. We treat both lists separately.



### 8.1.1 Good-bad competitions

We consider the first list of (8.2). We skip the case of the  $M_\nu$  since it is clear by the linear test that any system with  $f_{W_3}(0) \in S_1(f)(0)$  is  $L^\infty$ -STLC. For all the other brackets, we will prove the  $L^\infty$ -STLC property thanks to Agrachev and Gamkrelidze's sufficient condition [1, Theorem 4], of which we now recall a version well-suited to our setting.

**Theorem 8.2.** *Let  $\sigma \in [0, 1]$ ,  $r \geq 0$  and  $\Pi^1 \subset \text{Br}(X)$  such that  $\Pi^1$  generates a Lie algebra  $\text{Lie}(\Pi^1) \subset \mathcal{L}(X)$  with the following properties:*

- $\Pi^1$  is a set of free generators of  $\text{Lie}(\Pi^1)$ ,
- for each  $b \in \text{Br}(X)$  with  $n_1(b)$  even and  $n_0(b)$  odd,  $\mathbb{E}(b) \in \text{Lie}(\Pi^1)$ .

For  $k \in \mathbb{N}^*$  let  $\Pi^{k+1} := [\Pi^1, \Pi^k]$  and  $\Pi^\infty := \cup_{k \in \mathbb{N}^*} \Pi^k$ . For  $k \in \mathbb{N}^*$  and  $\pi \in \Pi^k$ , let  $\omega(\pi) := |\pi| - \sigma k$ . Suppose that, for all  $k \in \mathbb{N}$ , and every  $\pi \in \Pi^{2k+1}$  with  $n_1(\pi)$  even and  $n_0(\pi)$  odd, and  $\omega(\pi) \leq r$ ,

$$f_\pi(0) \in \text{span}\{f_{\pi'}(0); \pi' \in \Pi^\infty, \omega(\pi') < \omega(\pi)\}. \quad (8.3)$$

Assume moreover that

$$\mathbb{R}^d = S_1(f)(0) + \text{span}\{f_\pi(0); \pi \in \Pi^\infty, \omega(\pi) \leq r\}. \quad (8.4)$$

Then, the system is  $L^\infty$ -STLC.

To apply Theorem 8.2, the key point is thus to find a set  $\Pi^1$  and a parameter  $\sigma \in [0, 1]$  such that the “good” brackets that one intends to use have a smaller weight  $\omega$  than the “bad” ones. All the following examples will be handled with  $\sigma = 1$  and the following choice of  $\Pi^1$ :

$$\Pi^1 := \{\text{ad}_{M_2}^{i_2} \text{ad}_{M_1}^{i_1} \text{ad}_{X_1}^{i_0}(X_0); i_0, i_1, i_2 \in \mathbb{N}, (i_0, i_1) \notin \{1\} \times \mathbb{N}^*, (i_0, i_1, i_2) \notin \{0\} \times \{1\} \times \mathbb{N}^*\}. \quad (8.5)$$

By the elimination theorem [35, Proposition 1.1],  $\Pi^1$  is a set of free generators of  $\text{Lie}(\Pi^1)$  and  $\mathcal{L}(X) = \mathbb{R}X_1 \oplus \mathbb{R}M_1 \oplus \mathbb{R}M_2 \oplus \text{Lie}(\Pi^1)$ . In particular,  $\text{Lie}(\Pi^1)$  contains  $\mathbb{E}(b)$  for every  $b \in \text{Br}(X)$  of type (even, odd). We compute the associated weights for all brackets of interest with  $\sigma = 1$ .

- Since  $W_3 = \text{ad}_{M_2}^2(X_0) \in \Pi^1$ ,  $\omega(W_3) = |W_3| - 1 = 6$ .
- For  $l \in \{1, 2, 3\}$  and  $\nu \in \mathbb{N}$ ,  $P_{1,l,\nu} = \text{ad}_{M_{l-1}} \text{ad}_{M_0}^2(X_0)0^\nu \in \Pi^{1+\nu}$ , so  $\omega(P_{1,l,\nu}) = 2 + l \leq 5$ .
- For  $l \geq 4$  and  $\nu \in \mathbb{N}$ ,  $P_{1,l,\nu} = (\text{ad}_{M_2}(X_0)0^{l-3}, \text{ad}_{M_0}^2(X_0)0^\nu) \in \Pi^{l-1+\nu}$ , so  $\omega(P_{1,l,\nu}) = 4$ .
- For  $\nu \in \mathbb{N}$ ,  $Q_{1,1,2,\nu} = \text{ad}_{M_1} \text{ad}_{M_0}^3(X_0)0^\nu \in \Pi^{1+\nu}$ , so  $\omega(Q_{1,1,2,\nu}) = 5$ .
- For  $\nu \in \mathbb{N}$ ,  $R_{1,1,1,1,\nu} = \text{ad}_{M_0}^5(X_0)0^\nu \in \Pi^{1+\nu}$ , so  $\omega(R_{1,1,1,1,\nu}) = 5$ .
- For  $\mu, \nu \in \mathbb{N}$ ,  $R_{1,1,1,\mu,\nu}^\sharp = (\text{ad}_{M_0}^2(X_0)0^\mu, \text{ad}_{M_0}^3(X_0)0^\nu) \in \Pi^{2+\mu+\nu}$ , so  $\omega(R_{1,1,1,\mu,\nu}^\sharp) = 5$ .

Hence, all these brackets have a smaller weight than  $W_3$ . Moreover, Theorem 8.2 does not require for them to be compensated. Indeed, the  $P$ s and  $R$ s have an odd  $n_1$ . Moreover,  $n_0(Q_{1,1,2,\nu}) = 2 + \nu$  and  $Q_{1,1,2,\nu} \in \Pi^{1+\nu}$  so  $Q_{1,1,2,\nu}$  is never simultaneously of type (even, odd) and inside  $\Pi^{2k+1}$ .

We now give the limiting examples of which the above argument proves the  $L^\infty$ -STLC. These examples prove that the first list of (8.2) is minimal.

**Limiting example for  $P_{1,l,\nu}$  with  $l \in \{1, 2, 3\}$ .** Let  $\nu \in \mathbb{N}$ . If  $\nu = 0$ , we consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_3^2 + x_1^2 x_l \end{cases} \quad (8.6)$$

otherwise we consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_1^2 x_l \\ \dot{x}_{4+\mu} = x_{4+\mu-1} & \text{for } \mu = 1, \dots, \nu - 1 \\ \dot{x}_{4+\nu} = x_3^2 + x_{4+\nu-1}. \end{cases} \quad (8.7)$$

Written in the form (1.1), these systems satisfy

$$f_{M_{i-1}}(0) = e_i \text{ for } i \in \llbracket 1, 3 \rrbracket, \quad f_{P_{1,l,\mu}}(0) = ce_{4+\mu} \text{ for } \mu \in \llbracket 0, \nu \rrbracket, \quad f_{W_3}(0) = 2e_{4+\nu}, \quad (8.8)$$

where  $c = 6$  if  $l = 1$  and  $c = 2$  otherwise, and  $f_b(0) = 0$  for any other  $b \in \mathcal{B}^*$ .

**Limiting example for  $P_{1,l,\nu}$  for  $l \geq 4$ .** Let  $\nu \in \mathbb{N}$ . If  $\nu = 0$ , consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_i = x_{i-1} & \text{for } i = 2, \dots, l \\ \dot{x}_{l+1} = x_3^2 + x_1^2 x_l \end{cases} \quad (8.9)$$

otherwise we consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_i = x_{i-1} & \text{for } i = 2, \dots, l \\ \dot{x}_{l+1} = x_1^2 x_l \\ \dot{x}_{l+1+\mu} = x_{l+\mu} & \text{for } \mu = 1, \dots, \nu - 1 \\ \dot{x}_{l+1+\nu} = x_3^2 + x_{l+\nu}. \end{cases} \quad (8.10)$$

Written in the form (1.1), these systems satisfy

$$f_{M_{i-1}}(0) = e_i \text{ for } i \in \llbracket 1, l \rrbracket, \quad f_{P_{1,l,\mu}}(0) = ce_{l+1+\mu} \text{ for } \mu \in \llbracket 0, \nu \rrbracket, \quad f_{W_3}(0) = 2e_{l+1+\nu}, \quad (8.11)$$

where  $c = 6$  if  $l = 1$  and  $c = 2$  otherwise, and  $f_b(0) = 0$  for any other  $b \in \mathcal{B}^*$ .

**Limiting example for  $Q_{1,1,2,\nu}$ .** Let  $\nu \in \mathbb{N}$ . If  $\nu = 0$ , consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_3^2 + x_1^3 x_2 \end{cases} \quad (8.12)$$

otherwise we consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_1^3 x_2 \\ \dot{x}_{4+\mu} = x_{4+\mu-1} \\ \dot{x}_{4+\nu} = x_3^2 + x_{4+\nu-1}. \end{cases} \quad \text{for } \mu = 1, \dots, \nu - 1 \quad (8.13)$$

Written in the form (1.1), these systems satisfy

$$f_{M_{i-1}}(0) = e_i \text{ for } i = 1, 2, 3 \quad f_{Q_{1,1,2,\mu}} = 2e_{4+\mu} \text{ for } \mu = 0, \dots, \nu \quad f_{W_3}(0) = 2e_{4+\nu}, \quad (8.14)$$

and  $f_b(0) = 0$  for any other  $b \in \mathcal{B}^*$ .

**Limiting example for  $R_{1,1,1,1,\nu}$ .** Let  $\nu \in \mathbb{N}$ . If  $\nu = 0$ , we consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_3^2 + x_1^5 \end{cases} \quad (8.15)$$

otherwise we consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_1^5 \\ \dot{x}_{4+\mu} = x_{4+\mu-1} \\ \dot{x}_{4+\nu} = x_3^2 + x_{4+\nu-1}. \end{cases} \quad \text{for } \mu = 1, \dots, \nu - 1 \quad (8.16)$$

Written in the form (1.1), these systems satisfy

$$f_{M_{i-1}}(0) = e_i \text{ for } i = 1, 2, 3, \quad f_{R_{1,1,1,1,\mu}}(0) = 5!e_{4+\mu} \text{ for } \mu \in \llbracket 0, \nu \rrbracket, \quad f_{W_3}(0) = 2e_{4+\nu} \quad (8.17)$$

and  $f_b(0) = 0$  for any other  $b \in \mathcal{B}^*$ .

**Limiting example for  $R_{1,1,1,\mu,\nu}^\sharp$ .** Let  $\mu, \nu \in \mathbb{N}$ . If  $\nu = 0$ , we consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_1^3 \\ \dot{x}_{4+\mu'} = x_{4+\mu'-1} \\ \dot{x}_{5+\mu} = x_3^2 + x_1^2 x_{4+\mu} \end{cases} \quad \text{for } \mu' = 1, \dots, \mu \quad (8.18)$$

otherwise we consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_1^3 \\ \dot{x}_{4+\mu'} = x_{4+\mu'-1} & \text{for } \mu' = 1, \dots, \mu \\ \dot{x}_{5+\mu} = x_1^2 x_{4+\mu} \\ \dot{x}_{5+\mu+\nu'} = x_{5+\mu+\nu'-1} & \text{for } \nu' = 1, \dots, \nu - 1 \\ \dot{x}_{5+\mu+\nu} = x_3^2 + x_{5+\mu+\nu-1}. \end{cases} \quad (8.19)$$

Written in the form (1.1), this system satisfies

$$\begin{aligned} f_{M_{i-1}}(0) &= e_i \text{ for } i = 1, 2, 3, & f_{P_{1,1,\mu'}}(0) &= 6e_{4+\mu'} \text{ for } \mu' \in \llbracket 0, \mu \rrbracket, \\ f_{R_{1,1,1,\mu,\nu'}^\sharp}(0) &= -12(-1)^\mu e_{5+\mu+\nu'} \text{ for } \nu' \in \llbracket 0, \nu \rrbracket, & f_{W_3}(0) &= 2e_{5+\mu+\nu} \end{aligned} \quad (8.20)$$

and  $f_b(0) = 0$  for any other  $b \in \mathcal{B}^*$ .

### 8.1.2 Bad-bad competitions

The second list of (8.2) consists of brackets which are associated with signed coordinates of the second-kind. Hence, they restore controllability in competition with  $W_3$  only in situations where both signed terms push the state in opposite directions. Such “bad-bad” competitions (see e.g. [18, Section 5] for an introduction) are not handled by classical sufficient conditions such as [1, Theorem 4]. We present the straightforward case of  $Q_{1,1,1}$  here, and postpone the examples involving  $Q_{1,0}^b, Q_{1,1}^b$  and  $Q_{1,2}^b$ , for which the proofs are more intricate, to Appendix A.7.

**Limiting example for  $Q_{1,1,1}$ .** We consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_3^2 - x_1^4. \end{cases} \quad (8.21)$$

Written in the form (1.1), this system satisfies

$$f_{M_0}(0) = e_1, \quad f_{M_1}(0) = e_2, \quad f_{M_2}(0) = e_3, \quad f_{W_3}(0) = 2e_4, \quad f_{Q_{1,1,1}}(0) = -24e_4 \quad (8.22)$$

and  $f_b(0) = 0$  for any  $b \in \mathcal{B}^* \setminus \{M_0, M_1, M_2, W_3, Q_{1,1,1}\}$ . In [18, Example 5.2], Kawski proves that system (8.21) is  $L^\infty$ -STLC. This can also be proved using oscillating controls as in Section 6.7.

## 8.2 Dominant part of the logarithm

The following lemma is a little intricate due to the fact that the list (1.23) is minimal. If one only wishes to prove an easier version with  $\mathcal{N}_3 \leftarrow \mathcal{N}_3 \cup \{Q_{1,1,1,\nu}^b, Q_{1,\mu,\nu}^b\}$ , the proof could be shorter.

**Lemma 8.3.** *Let  $\mathcal{N}'_3 := \mathcal{N}_3 \setminus \{Q_{1,0}^b, Q_{1,1}^b, Q_{1,2}^b\}$  i.e.,*

$$\mathcal{N}'_3 = \{M_\nu, P_{1,l,\nu}, Q_{1,1,1}, Q_{1,1,2,\nu}, R_{1,1,1,1,\nu}, R_{1,1,1,\mu,\nu}^\sharp; l \in \mathbb{N}^*, \mu, \nu \in \mathbb{N}\}, \quad (8.23)$$

$\mathcal{N}_3'' := \mathcal{N}_3' \setminus \{Q_{1,1,1}\}$  and  $\mathbb{P}_{W_3} : \mathbb{R}^d \rightarrow \mathbb{R}$  be a component along  $f_{W_3}(0)$  parallel to  $\mathcal{N}_3'(f)(0)$ . Under the assumptions of Theorem 8.1,

$$\begin{aligned} \mathbb{P}_{W_3} \mathcal{Z}_5(t, f, u)(0) &= \xi_{W_3}(t, u) + O\left(t\xi_{W_3}(t, u) + \|u_2\|_{L^3}^3 + \|u_1\|_{L^6}^3 \|u_3\|_{L^2} + \|u_1\|_{L^6}^6 \right. \\ &\quad \left. + |(u_1, u_2, u_3)(t)|^2 + a\xi_{W_1}^2(t, u) + a't\xi_{Q_{1,1,1}}(t, u)\right). \end{aligned} \quad (8.24)$$

where

- $a = 1$  if  $f_{W_1}(0) \neq 0$  and  $a = 0$  otherwise,
- $a' = 1$  if  $f_{Q_{1,1,1}}(0) \notin \mathcal{N}_3''(f)(0)$  and  $a' = 0$  otherwise.

*Proof.* For  $i \in \{1, 2\}$ ,  $f_{W_i}(0) \in \mathcal{N}_i(f)(0)$  and  $\mathcal{N}_i$  is stable by right bracketing with  $X_0$  thus  $f_{W_{i,\nu}}(0) \in \mathcal{N}_i(f)(0)$  for every  $\nu \geq 0$ . Since  $\mathcal{N}_1 \subset \mathcal{N}_2 \subset \mathcal{N}_3'$  then  $f_{W_{i,\nu}}(0) \in \mathcal{N}_3'(f)(0)$  for every  $i \in \{1, 2\}$  and  $\nu \in \mathbb{N}$ .

When  $a = 0$ , i.e.  $f_{W_1}(0) = 0$ , then  $f_{Q_{1,\mu,\nu}}^b(0) = 0$  for every  $\mu, \nu \in \mathbb{N}$  because it is an iterated bracket of the vector fields  $f_{W_1}$  and  $f_0$  that vanish at 0, see (1.16).

When  $a' = 0$ , i.e.  $f_{Q_{1,1,1}}(0) \in \mathcal{N}_3''(f)(0)$  then for every  $\nu \in \mathbb{N}$ ,  $f_{Q_{1,1,1,\nu}}(0) \in \mathcal{N}_3''(f)(0)$  because  $\mathcal{N}_3''$  is stable by right bracketing with  $X_0$ , thus  $f_{Q_{1,1,1,\nu}}(0) \in \mathcal{N}_3'(f)(0)$ .

These remarks prove that  $\mathcal{N}_3'(f)(0) = \mathcal{N}(f)(0)$  where

$$\mathcal{N} = \begin{cases} \mathcal{N}_3' \cup \{W_{1,\nu}, W_{2,\nu}, Q_{1,1,1,\nu}, Q_{1,\mu,\nu}^b; \mu, \nu \in \mathbb{N}\} & \text{when } (a, a') = (0, 0), \\ \mathcal{N}_3' \cup \{W_{1,\nu}, W_{2,\nu}, Q_{1,\mu,\nu}^b; \mu, \nu \in \mathbb{N}\} & \text{when } (a, a') = (0, 1), \\ \mathcal{N}_3' \cup \{W_{1,\nu}, W_{2,\nu}, Q_{1,1,1,\nu}; \mu, \nu \in \mathbb{N}\} & \text{when } (a, a') = (1, 0), \\ \mathcal{N}_3' \cup \{W_{1,\nu}, W_{2,\nu}; \mu, \nu \in \mathbb{N}\} & \text{when } (a, a') = (1, 1). \end{cases} \quad (8.25)$$

By assumption,  $f_{W_3}(0) \notin \mathcal{N}_3(f)(0)$ , so  $f_{W_3}(0) \notin \mathcal{N}(f)(0)$ .

We intend to apply Proposition 4.6 with  $M \leftarrow 5$ ,  $L \leftarrow 11$ ,  $\mathbf{b} \leftarrow W_3$  and  $\mathcal{N}$  as in (8.25), so that (8.24) will follow from (4.11), for the appropriate choice of  $\Xi(t, u)$  (corresponding to the quantities within the  $O(\cdot)$  in (8.24)). Let us check that the required estimates are satisfied.

**Step 1.** *Estimates of other coordinates of the second kind.* Let  $b \in \mathcal{B}_{[[1,5]]}^*$  such that  $b \notin \mathcal{N} \cup \{\mathbf{b}\}$ .

We investigate the different possibilities depending on  $n_1(b)$ .

- One cannot have  $n_1(b) = 1$  since  $\mathcal{B}_1^* \subset \mathcal{N}_3' \subset \mathcal{N}$ .
- If  $n_1(b) = 2$ , by (1.9) and (8.25), one has  $b = W_{j,\nu}$  with either  $(j \geq 4)$  or  $(j = 3 \text{ and } \nu \geq 1)$ . Thus  $|b| \geq 8$ . By estimate (3.24) with  $(p, j_0) \leftarrow (1, 3)$ , (4.9) holds with  $\sigma = 8$  and

$$\Xi(t, u) := t\|u_3\|_{L^2}^2. \quad (8.26)$$

- If  $n_1(b) = 3$ , by (1.10) and (8.25),  $b = P_{j,l,\nu}$  with  $2 \leq j \leq l$ . Thus  $|b| \geq 7$ . By estimate (3.25) with  $(p_1, p_2, j_0, k_0) \leftarrow (3/2, 3, 2, 2)$ , (4.9) holds with  $\sigma = 7$  and

$$\Xi(t, u) := \|u_2\|_{L^3}^3. \quad (8.27)$$

- If  $n_1(b) = 4$ , by (1.11) and (8.25), we are in one of the following cases.

- $b = Q_{1,1,1,\nu}$  with  $\nu \geq 1$  and  $a' = 1$  thus  $|b| \geq 6$  and by estimate (3.26) with  $(p_1, p_2, p_3, j_0, k_0, l_0) \leftarrow (2, 4, 4, 1, 1, 1)$ , (4.9) holds with  $\sigma = 6$  and

$$\Xi(t, u) := a't\|u_1\|_{L^4}^4 = a'4!t\xi_{Q_{1,1,1}}(t, u). \quad (8.28)$$

- $b = Q_{1,1,l,\nu}$  with  $l \geq 3$ , thus  $|b| \geq 7$  and, by estimate (3.26) with  $(p_1, p_2, p_3, j_0, k_0, l_0) \leftarrow (3, 6, 2, 1, 1, 3)$ , (4.9) holds with  $\sigma = 7$  and

$$\Xi(t, u) := \|u_1\|_{L^6}^3 \|u_3\|_{L^2}. \quad (8.29)$$

- $b = Q_{j,k,l,\nu}$  with  $2 \leq k$ , thus  $|b| \geq 7$  and, by estimate (3.26) with  $(p_1, p_2, p_3, j_0, k_0, l_0) \leftarrow (3, 3, 3, 1, 2, 2)$ , (4.9) holds with  $\sigma = 7$  and

$$\Xi(t, u) := \|u_1\|_{L^6}^2 \|u_2\|_{L^3}^2. \quad (8.30)$$

- $b = Q_{j,\mu,k,\nu}^\sharp$ , thus  $|b| \geq 8$  and, by estimate (3.28) with  $(p_1, p_2, j_0, k_0) \leftarrow (3, 3/2, 1, 2)$ , (4.9) holds with  $\sigma = 8$  and

$$\Xi(t, u) := t \|u_1\|_{L^6}^2 \|u_2\|_{L^3}^2. \quad (8.31)$$

- $b = Q_{j,\mu,\nu}^b$  and  $a = 1$  thus  $|b| \geq 8$  and, by estimate (3.27) with  $(p, j_0) \leftarrow (1, 1)$ , (4.9) holds with  $\sigma = 7$  and

$$\Xi(t, u) := at \|u_1\|_{L^2}^4 = 4at \xi_{W_1}^2(t, u). \quad (8.32)$$

- If  $n_1(b) = 5$ , by (1.12) and (1.23), we are in one of the following cases.

- $b = R_{j,k,l,m,\nu}$  with  $m \geq 2$ , thus  $|b| \geq 7$  and, by estimate (3.29) with  $(p_1, p_2, p_3, p_4, j_0, k_0, l_0, m_0) \leftarrow (3, 6, 6, 3, 1, 1, 1, 2)$ , (4.9) holds with  $\sigma = 7$  and

$$\Xi(t, u) := \|u_1\|_{L^6}^4 \|u_2\|_{L^3}. \quad (8.33)$$

- $b = R_{j,k,l,\mu,\nu}^\sharp$  with  $l \geq 2$ , thus  $|b| \geq 9$  and, by estimate (3.30) with  $(p, p_1, p_2, j_0, k_0, l_0) \leftarrow (3/2, 3, 6, 1, 1, 2)$ , (4.9) holds with  $\sigma = 9$  and

$$\Xi(t, u) := t^{\frac{5}{6}} \|u_1\|_{L^6}^3 \|u_2\|_{L^3}^2. \quad (8.34)$$

**Step 2. Estimates of cross products.** Let  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}^* \setminus \{X_0\}$  such that  $n_1(b_1) + \dots + n_1(b_q) \leq 5$  and  $\text{supp } \mathcal{F}(b_1, \dots, b_q) \not\subset \mathcal{N}$ .

We start with preliminary estimates.

- By (3.22), for each  $b_i \in \mathcal{B}^*$  with  $n_1(b_i) \leq 5$ , (4.10) holds with  $\sigma_i = n_1(b_i) + 1$ ,  $\alpha_i = n_1(b_i)/6$  and  $\Xi(t, u) = t^{6/n_1(b_i)-1} \|u_1\|_{L^6}^6$ .
- If  $b_i = M_j$  for some  $j \in \llbracket 0, 2 \rrbracket$ , by (3.11),

$$|\xi_{b_i}(t, u)| = |u_{j+1}(t)| = \frac{t^{|b_i|}}{|b_i|!} t^{-(j+1)} (j+1)! |u_{j+1}(t)| \quad (8.35)$$

so (4.10) holds with  $\sigma_i = j + 1$ ,  $\alpha_i = 1/2$  and  $\Xi(t, u) = |(u_1, u_2, u_3)(t)|^2$ .

- If  $b_i = M_j$  for  $j \geq 3$ , by (3.23) with  $(p, j_0) \leftarrow (2, 3)$ , (4.10) holds with  $\sigma_i = 4$ ,  $\alpha_i = 1/2$  and  $\Xi(t, u) = t \|u_3\|_{L^2}^2$ .

We now consider the different possibilities, based on the condition  $n_1(b_1) + \dots + n_1(b_q) \leq 5$ .

- *Case: at least two  $b_i \in \mathcal{B}_1^*$ .* Then, by the preliminary steps,  $\alpha \geq 1/2 + 1/2 = 1$ .
- *Case:  $q = 3$ ,  $b_1, b_2 \in \mathcal{B}_2^*$ ,  $b_3 \in \mathcal{B}_1^*$ .* Then, by the preliminary steps,  $\alpha = 1/3 + 1/3 + 1/2 > 1$ .
- *Case:  $q = 2$ ,  $b_1 \in \mathcal{B}_{[3,4]}^*$ ,  $b_2 \in \mathcal{B}_1^*$ .* Then, by the preliminary steps,  $\alpha = n_1(b_1)/6 + 1/2 \geq 1$ .

- *Case:*  $q = 2$ ,  $b_1 \in \mathcal{B}_2^*$ ,  $b_2 \in \mathcal{B}_1^*$ . Say  $b_1 = W_{j,\nu}$  and  $b_2 = M_{k-1}$ . One cannot have  $j = 1$  because, by (4.24),  $\text{supp}[b_1, b_2]$  would be contained in  $\{P_{1,k',\nu'}, k' \geq 1, \nu' \geq 0\} \subset \mathcal{N}$ . So  $j \geq 2$ . Then, by (3.24) with  $(p, j_0) \leftarrow (3/2, 2)$ , (4.10) holds for  $b_1$  with  $\sigma_1 = 5$ ,  $\alpha_1 = 2/3$  and  $\Xi(t, u) = t^{1/2}\|u_2\|_{L^3}^3$ . By the preliminary steps,  $\alpha_1 + \alpha_2 = 2/3 + 1/2 > 1$ .
- *Case:*  $q = 2$ ,  $b_1, b_2 \in \mathcal{B}_2^*$ . Say  $b_1 = W_{j,\nu}$  and  $b_2 = W_{j',\nu'}$ .
  - If  $j = j' = 1$ , one cannot have  $a = 0$  because, by (4.25),  $\text{supp}[b_1, b_2]$  would be contained in  $\{Q_{1,\mu',\nu'}^b; \mu', \nu' \in \mathbb{N}\} \subset \mathcal{N}$ , see (8.25). So  $a = 1$ . By (3.24) with  $(p, j_0) \leftarrow (1, 1)$ , (4.10) holds for  $\sigma_1 = \sigma_2 = 3$  and  $\Xi(t, u) = a\|u_1\|_{L^2}^2 = 4a\xi_{W_1}^2(t, u)$ .
  - If  $j \geq 2$ , then, by (3.24) with  $(p, j_0) \leftarrow (3/2, 2)$ , (4.10) holds for  $b_1$  with  $\sigma_1 = 5$ ,  $\alpha_1 = 2/3$  and  $\Xi(t, u) = t^{1/2}\|u_2\|_{L^3}^3$ . By the preliminary steps,  $\alpha_1 + \alpha_2 = 2/3 + 1/3 = 1$ .
- *Case:*  $q = 2$ ,  $b_1 \in \mathcal{B}_3^*$ ,  $b_2 \in \mathcal{B}_2^*$ . Say  $b_1 = P_{j,k,\nu}$  and  $b_2 = W_{l,\mu}$ . One cannot have  $j = k = l = 1$  because, by (4.25),  $\text{supp}[b_1, b_2]$  would be contained in  $\{R_{1,1,1,\mu',\nu'}^\sharp; \mu', \nu' \in \mathbb{N}\} \subset \mathcal{N}$ . Thus  $l \geq 2$  or  $k \geq 2$ .
  - If  $l \geq 2$ , using (3.24) with  $(p, j_0) \leftarrow (3/2, 2)$ , (4.10) holds for  $b_2$  with  $\sigma_2 = 5$ ,  $\Xi(t, u) = t^{1/2}\|u_2\|_{L^3}^3$  and  $\alpha_2 = 2/3$ . By the preliminary step  $\alpha_1 + \alpha_2 = 1/2 + 2/3 > 1$ .
  - If  $k \geq 2$ , then using (3.25) with  $(p_1, p_2, j_0, k_0) \leftarrow (3, 3, 1, 2)$ , (4.10) holds for  $b_1$  with  $\sigma_1 = 5$ ,  $\alpha_1 = 2/3$  and  $\Xi(t, u) = t^{1/2}\|u_1\|_{L^6}^3\|u_2\|_{L^3}^{3/2}$ . By the preliminary step  $\alpha_1 + \alpha_2 = 2/3 + 1/2 > 1$ .  $\square$

### 8.3 Vectorial relations

**Lemma 8.4.** *Under the assumptions of Theorem 8.1,*

1. *the vectors  $f_{M_0}(0), f_{M_1}(0), f_{M_2}(0)$  are linearly independent,*
2. *if  $f_{W_1}(0) \neq 0$  then  $f_{W_1}(0) \notin \text{span}\{f_{M_0}(0), f_{M_1}(0), f_{M_2}(0)\}$ .*

*Proof.* We have  $S_{2,[[1,4]]}(f)(0) \subset \mathcal{N}_2(f)(0)$  because  $S_{2,[[1,4]]}(X) = \text{span}\{W_{j,\nu}; 2j + \nu - 1 \leq 4\}$ ,  $f_{W_1}(0), f_{W_2}(0) \in \mathcal{N}_2(f)(0)$  and  $\mathcal{N}_2$  is stable by right bracketing with  $X_0$ . Thus, since  $\mathcal{N}_2 \subset \mathcal{N}_3$ ,

$$S_{2,[[1,4]]}(f)(0) \subset \mathcal{N}_3(f)(0). \quad (8.36)$$

**Step 1.** *Proof of item 1.* We assume there exists  $(\beta_0, \beta_1, \beta_2) \in \mathbb{R}^3 \setminus \{0\}$  such that  $f_{B_1}(0) = 0$  where  $B_1 = \beta_2 M_2 + \beta_1 M_1 + \beta_0 M_0$ . One may assume that  $\beta_2 = 1$ ; otherwise consider  $[B_1, X_0]$  or  $[[B_1, X_0], X_0]$  and renormalize. Then  $f_{B_2}(0) = 0$  where

$$B_2 = \text{ad}_{B_1}^2(X_0) = [M_2 + \beta_1 M_1 + \beta_0 M_0, M_3 + \beta_1 M_2 + \beta_0 M_1] \in W_3 + S_{2,[[1,4]]}(X) \quad (8.37)$$

and (8.36) leads to a contradiction with the assumption  $f_{W_3}(0) \notin \mathcal{N}_3(f)(0)$ .

**Step 2.** *Proof of item 2.* Proceeding by contradiction, we assume that there exists  $(\gamma_0, \gamma_1, \gamma_2) \in \mathbb{R}^3 \setminus \{0\}$  such that  $f_{B_4}(0) = 0$  where  $B_4 = \gamma_2 M_2 + \gamma_1 M_1 + \gamma_0 M_0 + W_1$ . Let  $\kappa = \max\{j \in \{0, 1, 2\}; \gamma_j \neq 0\}$ . Then  $f_{B_5}(0) = 0$  where  $B_5 = [B_4 0^{2-\kappa}, B_4 0^{3-\kappa}]$  i.e.

$$B_5 = [\gamma_\kappa M_2 + \cdots + \gamma_0 M_{2-\kappa} + W_{1,2-\kappa}, \gamma_\kappa M_3 + \cdots + \gamma_0 M_{3-\kappa} + W_{1,3-\kappa}] = \gamma_\kappa^2 W_3 + Q_{1,2-\kappa}^b + B_6 + B_7 \quad (8.38)$$

where  $B_6 \in \text{span}\{[M_l, W_{1,\nu}]; l \in \mathbb{N}, \nu \in \mathbb{N}\}$  and  $B_7 \in S_{2,[[1,4]]}(X)$ . By (4.24),  $\text{supp } B_6 \subset \{P_{1,l,\nu}; l \in \mathbb{N}^*, \nu \in \mathbb{N}\} \subset \mathcal{N}_3$ . Together with (8.36), this leads to a contradiction.  $\square$

## 8.4 Closed-loop estimates

**Lemma 8.5.** *Under the assumptions of Theorem 8.1,*

$$|(u_1, u_2, u_3)(t)| = O\left(|x(t; u)| + \|u_1\|_{L^3}^3 + \|u_2\|_{L^2}^2 + t^{\frac{1}{2}}\|u_3\|_{L^2}\right) \quad (8.39)$$

and if  $f_{W_1}(0) \neq 0$  then

$$\xi_{W_1}(t, u) = O\left(|x(t; u)| + \|u_1\|_{L^3}^3 + \|u_2\|_{L^2}^2 + t^{\frac{1}{2}}\|u_3\|_{L^2}\right). \quad (8.40)$$

*Proof.* By Theorem 4.4 with  $M \leftarrow 2$ ,

$$x(t; u) = \mathcal{Z}_2(t, f, u)(0) + O\left(\|u_1\|_{L^3}^3 + |x(t; u)|^{1+\frac{1}{2}}\right). \quad (8.41)$$

**First case:**  $f_{W_1}(0) = 0$ . Then  $f_{W_{1,\nu}}(0) = 0$  for every  $\nu \in \mathbb{N}$  because it is an iterated bracket of the vector fields  $f_{W_1}$  and  $f_0$  that vanish at 0. Let  $i \in \llbracket 0, 2 \rrbracket$  and  $\mathbb{P}$  be a component along  $f_{M_i}(0)$  parallel to  $\overline{\mathcal{N}}(f)(0)$  where  $\overline{\mathcal{N}} = \{M_0, M_1, M_2\} \setminus \{M_i\}$ . We have  $\overline{\mathcal{N}}(f)(0) = \mathcal{N}(f)(0)$  for

$$\mathcal{N} = \{M_0, M_1, M_2, W_{1,\nu}; \nu \in \mathbb{N}\} \setminus \{M_i\} \quad (8.42)$$

because  $f_{W_{1,\nu}}(0) = 0$  for every  $\nu \in \mathbb{N}$ . We intend to apply Proposition 4.6 with  $M \leftarrow 2$ ,  $L \leftarrow 5$ ,  $\mathfrak{b} \leftarrow M_i$  and  $\mathcal{N}$  as above, so that (4.11), for the appropriate choice of  $\Xi(t, u)$ , will yield

$$\mathbb{P}\mathcal{Z}_2(t, f, u)(0) = u_{i+1}(t) + O\left(|(u_1, u_2, u_3)(t)|^2 + t^{\frac{1}{2}}\|u_3\|_{L^2} + \|u_2\|_{L^2}^2\right). \quad (8.43)$$

Then, combining (8.41) and (8.43) concludes the proof of (8.39). Let us check that the required estimates are satisfied.

**Step 1.** *Estimates of other coordinates of the second kind.* Let  $b \in \mathcal{B}_{\llbracket 1, 2 \rrbracket}^*$  such that  $b \notin \mathcal{N} \cup \{\mathfrak{b}\}$ .

- If  $n_1(b) = 1$ , then by (1.8) and (8.42),  $b = M_j$  for  $j \geq 3$ . Thus  $|b| \geq 4$ . By (3.23) with  $(p, j_0) \leftarrow (2, 3)$ , (4.9) holds with  $\sigma = 4$  and  $\Xi(t, u) := t^{\frac{1}{2}}\|u_3\|_{L^2}$ .
- If  $n_1(b) = 2$ , by (1.9) and (8.42),  $b = W_{j,\nu}$  with  $j \geq 2$ . Thus  $|b| \geq 5$ . By (3.24) with  $(p, j_0) \leftarrow (1, 2)$ , (4.9) holds with  $\sigma = 5$  and  $\Xi(t, u) := \|u_2\|_{L^2}^2$ .

**Step 2.** *Estimates of cross products.* Let  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}^* \setminus \{X_0\}$  such that  $n_1(b_1) + \dots + n_1(b_q) \leq 2$  and  $\text{supp } \mathcal{F}(b_1, \dots, b_q) \not\subset \mathcal{N}$ .

Thus  $q = 2$  and  $b_1 = M_{j_1}$ ,  $b_2 = M_{j_2}$  for some  $j_1, j_2 \in \mathbb{N}$ . By the preliminary estimates of Step 2 of the proof of Lemma 8.3,  $b_1$  and  $b_2$  satisfy (4.10) with  $\Xi(t, u) = |(u_1, u_2, u_3)(t)|^2 + t\|u_3\|_{L^2}^2$  and  $\alpha_1 = \alpha_2 = 1/2$ .

**Second case:**  $f_{W_1}(0) \neq 0$ . First, we apply Proposition 4.6 with  $M \leftarrow 2$ ,  $L \leftarrow 5$ ,  $\mathfrak{b} \leftarrow W_1$  and  $\mathcal{N} = \{M_0, M_1, M_2\}$  so that (4.11), for the appropriate choice of  $\Xi$ , will yield

$$\mathbb{P}\mathcal{Z}_2(t, f, u)(0) = \xi_{W_1}(t, u) + O\left(t\xi_{W_1}(t, u) + t^{\frac{1}{2}}\|u_3\|_{L^2} + \|u_2\|_{L^2}^2 + |(u_1, u_2, u_3)(t)|^2\right) \quad (8.44)$$

where  $\mathbb{P}$  is a component along  $f_{W_1}(0)$  parallel to  $\mathcal{N}(f)(0)$ . The only difference in the estimates, with respect to the first case above, concerns the estimate of coordinates of the second kind associated with  $b \in \mathcal{B}_2^*$  such that  $b \notin \mathcal{N} \cup \{\mathfrak{b}\}$ : then  $b = W_{j,\nu}$  with  $(j, \nu) \neq (1, 0)$  thus  $|b| \geq 4$ . By estimate (3.24) with  $(p, j_0) \leftarrow (1, 1)$ , (4.9) holds with  $\sigma = 4$  and  $\Xi(t, u) := t\|u_1\|_{L^2}^2 = 2t\xi_{W_1}(t, u)$ .

We deduce from (8.41) and (8.44) that

$$\xi_{W_1}(t, u) = O\left(|x(t, u)| + \|u_1\|_{L^3}^3 + t^{\frac{1}{2}}\|u_3\|_{L^2} + \|u_2\|_{L^2}^2 + |(u_1, u_2, u_3)(t)|^2\right). \quad (8.45)$$



By Theorem 4.4 with  $M \leftarrow 1$ ,

$$x(t; u) = \mathcal{Z}_1(t, f, u)(0) + O(\|u_1\|_{L^2}^2 + |x(t; u)|^2). \quad (8.46)$$

Let  $i \in \llbracket 0, 2 \rrbracket$ . By applying Proposition 4.6 with  $M \leftarrow 1$ ,  $L \leftarrow 4$ ,  $\mathbf{b} \leftarrow M_i$  and  $\mathcal{N} = \{M_0, M_1, M_2\} \setminus \{M_i\}$ , and using (3.23) with  $(p, j_0) \leftarrow (2, 3)$  to prove the only required estimate, we obtain

$$\mathbb{P}\mathcal{Z}_1(t, f, u)(0) = u_{i+1}(t) + O\left(t^{\frac{1}{2}}\|u_3\|_{L^2}\right). \quad (8.47)$$

We deduce from (8.46) and (8.47) that

$$|(u_1, u_2, u_3)(t)| = O\left(|x(t, u)| + t^{\frac{1}{2}}\|u_3\|_{L^2} + \|u_1\|_{L^2}^2\right). \quad (8.48)$$

By incorporating (8.45) into (8.48) thanks to  $\xi_{W_1}(t, u) = 2\|u_1\|_{L^2}^2$  one proves (8.39). And by incorporating (8.39) into (8.45) one proves (8.41).  $\square$

**Lemma 8.6.** *Under the assumptions of Theorem 8.1, if  $f_{Q_{1,1,1}}(0) \notin \mathcal{N}_3''(f)(0)$*

$$\mathbb{P}_{Q_1}x(t, u) = \xi_{Q_{1,1,1}}(t, u) + O\left(t\xi_{Q_{1,1,1}}(t, u) + \xi_{W_3}(t, u) + \|u_2\|_{L^3}^3 + |x(t, u)|^{\frac{5}{4}}\right) \quad (8.49)$$

where  $\mathbb{P}_{Q_1}$  is a component along  $f_{Q_{1,1,1}}(0)$  parallel to  $(\mathcal{N}_3'' \cap \mathcal{B}_{\llbracket 1, 4 \rrbracket}^*)(f)(0)$ .

*Proof.* By Theorem 4.4 with  $M \leftarrow 4$ ,

$$x(t; u) = \mathcal{Z}_4(t, f, u)(0) + O\left(\|u_1\|_{L^5}^5 + |x(t; u)|^{1+\frac{1}{5}}\right). \quad (8.50)$$

For  $i \in \{1, 2\}$ ,  $f_{W_i}(0) \in \mathcal{N}_i(f)(0)$  and  $\mathcal{N}_i$  is stable by right bracketing with  $X_0$  thus, for every  $\nu \in \mathbb{N}$ ,  $f_{W_{i,\nu}}(0) \in \mathcal{N}_i(f)(0) \subset (\mathcal{N}_3'' \cap \mathcal{B}_{\llbracket 1, 4 \rrbracket}^*)(f)(0)$ . Thus  $(\mathcal{N}_3'' \cap \mathcal{B}_{\llbracket 1, 4 \rrbracket}^*)(f)(0) = \mathcal{N}(f)(0)$  where  $\mathcal{N} = (\mathcal{N}_3'' \cap \mathcal{B}_{\llbracket 1, 4 \rrbracket}^*) \cup \{W_{1,\nu}, W_{2,\nu}; \nu \in \mathbb{N}\}$  i.e.

$$\mathcal{N} = \{M_\nu, W_{1,\nu}W_{2,\nu}, P_{1,l,\nu}, Q_{1,1,2,\nu}; l \in \mathbb{N}^*, \mu, \nu \in \mathbb{N}\}. \quad (8.51)$$

We intend to apply Proposition 4.6 with  $M \leftarrow 4$ ,  $L \leftarrow 11$ ,  $\mathbf{b} \leftarrow Q_{1,1,1}$  and  $\mathcal{N}$  as in (8.51), so that (4.11), for the appropriate choice of  $\Xi(t, u)$ , will yield

$$\begin{aligned} \mathbb{P}_{Q_1}\mathcal{Z}_4(t, f, u)(0) &= \xi_{Q_{1,1,1}}(t, u) + O\left(t\xi_{Q_{1,1,1}}(t, u) + \xi_{W_3}(t, u) + \|u_2\|_{L^3}^3 \right. \\ &\quad \left. + \|u_1\|_{L^6}^3\|u_3\|_{L^2} + \|u_1\|_{L^6}^6 + |(u_1, u_2, u_3)(t)|^2\right). \end{aligned} \quad (8.52)$$

All the needed estimates are contained in the proof of Lemma 8.3, except for

- $b \in \mathcal{B}_2^*$ , i.e.  $b = W_{j,\nu}$  with  $j \geq 3$ , for which  $|b| \geq 7$  and (3.24) with  $(p, j_0) \leftarrow (1, 3)$  proves that (4.9) holds with  $\sigma = 7$  and  $\Xi(t, u) := \|u_3\|_{L^2}^2 = 2\xi_{W_3}(t, u)$ ,
- $b = Q_{j,\mu,\nu}^b$  for which  $|b| \geq 7$  and (3.27) with  $(p, j_0) \leftarrow (2, 1)$  proves that (4.9) holds with  $\sigma = 7$  and  $\Xi(t, u) := t^2\|u_1\|_{L^4}^4 = 4!t^2\xi_{Q_{1,1,1}}(t, u)$ ,
- $q = 2$ ,  $b_1, b_2 \in \mathcal{B}_2^*$  for which  $|b_i| \geq 3$  and (3.24) with  $(p, j_0) \leftarrow (2, 1)$  proves that (4.10) holds for  $\sigma_i = 3$ ,  $\alpha_i = 1/2$  and  $\Xi(t, u) = t\|u_1\|_{L^4}^4 = 4t\xi_{Q_{1,1,1}}(t, u)$ .

Then, using  $\|u_1\|_{L^6}^6 = O(t^2\|u_1\|_{L^4}^4)$ , we deduce from (8.52) that

$$\mathbb{P}_{Q_1}\mathcal{Z}_4(t, f, u)(0) = \xi_{Q_{1,1,1}}(t, u) + O\left(t\xi_{Q_{1,1,1}}(t, u) + \xi_{W_3}(t, u) + \|u_2\|_{L^3}^3 + |(u_1, u_2, u_3)(t)|^2\right). \quad (8.53)$$

Then, by combining (8.50), (8.53) and  $\|u_1\|_{L^5}^5 = O(t\|u_1\|_{L^4}^4)$  we obtain

$$\mathbb{P}_{Q_1}x(t, u) = \xi_{Q_{1,1,1}}(t, u) + O\left(t\xi_{Q_{1,1,1}}(t, u) + \xi_{W_3}(t, u) + \|u_2\|_{L^3}^3 + |(u_1, u_2, u_3)(t)|^2 + |x(t, u)|^{\frac{5}{4}}\right). \quad (8.54)$$

Finally by incorporating (8.39) in the previous estimate we obtain (8.49).  $\square$

## 8.5 Interpolation inequalities

**Lemma 8.7.** *There exists  $C > 0$  such that, for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,*

$$\|u_2\|_{L^3}^3 \leq C \|u\|_{L^\infty} \|u_3\|_{L^2}^2, \quad (8.55)$$

$$\|u_1\|_{L^6}^6 \leq C \|u\|_{L^\infty}^4 \|u_3\|_{L^2}^2. \quad (8.56)$$

*Proof.* Inequality (8.55) follows from Proposition 4.7 with  $\phi \leftarrow u_3$ ,  $(p, q, r, s) \leftarrow (3, 2, \infty, 2)$ ,  $(j, l) \leftarrow (1, 3)$ ,  $\alpha \leftarrow 1/3$ . Similarly, (8.56) follows from Proposition 4.7 with  $\phi \leftarrow u_3$ ,  $(p, q, r, s) \leftarrow (6, 2, \infty, 2)$ ,  $(j, l) \leftarrow (2, 3)$ ,  $\alpha \leftarrow 2/3$ . In both cases, the lower-order term in (4.23) is absorbed using the estimate  $\|u_3\|_{L^2} \leq t^{\frac{7}{2}} \|u\|_{L^\infty}$ , which stems from Hölder's inequality and the equality  $u_3(t) = \frac{1}{2} \int_0^t (t-s)^2 u(s) ds$ .  $\square$

## 8.6 Proof of the drift

*Proof of Theorem 8.1.* In this proof, to lighten the notations, we write  $x$  and  $\xi_b$  instead of  $x(t; u)$  and  $\xi_b(t, u)$ . By Theorem 4.4 with  $M \leftarrow 5$ ,

$$x = \mathcal{Z}_5(t, f, u)(0) + O\left(\|u_1\|_{L^6}^6 + |x|^{1+\frac{1}{5}}\right). \quad (8.57)$$

We deduce from (8.57), (8.24), (8.39) and (8.40) that

$$\mathbb{P}_{W_3} x = \xi_{W_3} + O\left(t\xi_{W_3} + \|u_2\|_{L^3}^3 + \|u_1\|_{L^6}^3 \|u_3\|_{L^2} + \|u_1\|_{L^6}^6 + a't\xi_{Q_{1,1,1}} + |x|^{\frac{6}{5}}\right). \quad (8.58)$$

We deduce from the above estimate (see Section 4.2) the existence of  $C, \rho > 0$  such that, for every  $t \in (0, \rho)$ ,  $u \in L^1((0, t); \mathbb{R})$  with  $\|u\|_{W^{-1, \infty}} \leq \rho$ ,

$$\mathbb{P}_{W_3} x + Ca't\xi_{Q_{1,1,1}} \geq (1 - Ct)\xi_{W_3} - C\left(\|u_2\|_{L^3}^3 + \|u_1\|_{L^6}^3 \|u_3\|_{L^2} + \|u_1\|_{L^6}^6 + |x|^{\frac{6}{5}}\right) \quad (8.59)$$

where, when  $a' \neq 0$ , by (8.49),

$$\xi_{Q_{1,1,1}} \leq 2\mathbb{P}_{Q_1} x + C\left(\xi_{W_3} + \|u_2\|_{L^3}^3 + |x|^{\frac{5}{4}}\right). \quad (8.60)$$

Thus

$$\mathbb{P}_{W_3} x + 2Ca't\mathbb{P}_{Q_1} x \geq (1 - 2Ct)\xi_{W_3} - 2C\left(\|u_2\|_{L^3}^3 + \|u_1\|_{L^6}^3 \|u_3\|_{L^2} + \|u_1\|_{L^6}^6 + |x|^{\frac{6}{5}}\right). \quad (8.61)$$

Gathering this inequality and the interpolation estimates (8.55) and (8.56) yields

$$(\mathbb{P}_{W_3} + 2Ca't\mathbb{P}_{Q_1})x \geq \frac{1}{2} \int_0^t u_3^2 - C' \left( (t + \|u\|_{L^\infty}) \int_0^t u_3^2 + |x|^{\frac{6}{5}} \right) \quad (8.62)$$

for some constant  $C'$  independent of  $t$  and  $u$ .  $\square$

## 9 New obstruction of the sixth order

The goal of this section is to prove Theorem 1.13, as a consequence of the following more precise statement. In this section, we use the short-hand notation  $D$  for the following bracket of  $\mathcal{B}_6^*$ :

$$D := \text{ad}_{P_{1,1}}^2(X_0) \quad (9.1)$$

and we introduce

$$\mathcal{N}_D := \mathcal{B}_{[1,7]}^* \setminus \{D\}. \quad (9.2)$$

**Theorem 9.1.** *Assume that  $f_D(0) \notin \mathcal{N}_D(f)(0)$ . Then system (1.1) has a drift along  $f_D(0)$ , parallel to  $\mathcal{N}_D(f)(0)$ , in the regime  $(t, \|u\|_{L^\infty}) \rightarrow 0$ .*

## 9.1 Limiting examples

Let us give an example motivating the threshold 7 for this loose necessary condition. In [18, Example 6.1], Kawski considers the systems

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^3 \\ \dot{x}_4 = x_3^2 - x_2^p \end{cases} \quad (9.3)$$

for  $p \in \{7, 8\}$ . Written in the form (1.1), these systems satisfy

$$f_{M_0}(0) = e_1, \quad f_{M_1}(0) = e_2, \quad f_{P_{1,1}}(0) = 6e_3, \quad f_D(0) = 72e_4, \quad f_{\text{ad}_{M_1}^p(X_0)}(0) = -p!e_4 \quad (9.4)$$

and  $f_b(0) = 0$  for all  $b \in \mathcal{B}^* \setminus \{M_0, M_1, P_{1,1}, D, \text{ad}_{M_1}^p(X_0)\}$ . Thus, they feature a competition between  $D$  and  $\text{ad}_{M_1}^p(X_0)$ .

Kawski proves that this system is  $L^\infty$ -STLC for  $p = 7$  (see [18, Claim 6.1]) but not  $L^\infty$ -STLC for  $p = 8$  (see [18, Claim 6.3]). This both motivates and is consistent with Theorem 1.13, which can be seen as a generalization of Kawski's negative claim.

**Remark 9.2.** *Theorem 1.13 is a “loose” condition, in the sense that we have not attempted to separate, within  $\mathcal{B}_6^*$  and  $\mathcal{B}_7^*$ , which brackets can or cannot compensate for the drift. It is possible that our method could also be used to perform such a distinction.*

*An interesting example is studied by Kawski in [18, Example 5.3];*

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^3 \\ \dot{x}_4 = x_3^2 - x_2^2 x_1^4, \end{cases} \quad (9.5)$$

which exhibits in  $\mathcal{B}^*$  a competition between  $D$  and  $\text{ad}_{M_1}^2 \text{ad}_{X_1}^4(X_0)$ . Kawski proves that this systems is  $L^\infty$ -STLC.

Conversely, the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_1^3 \\ \dot{x}_4 = x_3^2 + x_3 x_1^4 \end{cases} \quad (9.6)$$

exhibits in  $\mathcal{B}^*$  a competition between  $D$  and  $\text{ad}_{P_{1,1}} \text{ad}_{X_1}^4(X_0)$  because  $f_{\text{ad}_{P_{1,1}} \text{ad}_{X_1}^4(X_0)}(0) = 144e_4$ . Using the estimates of the next paragraphs, one can prove that this system is not  $L^\infty$ -STLC. This hints towards the fact that it is not necessary to include the bracket  $\text{ad}_{P_{1,1}} \text{ad}_{X_1}^4(X_0)$  (of  $\mathcal{B}_7^*$ ) in the list of brackets which can compensate  $D$ .

## 9.2 Algebraic preliminaries

To lighten the proof of the following paragraph, we start with algebraic preliminaries concerning the expansions on  $\mathcal{B}^*$  of some brackets of order 6, linked with cross products along  $D$ . We use the trailing zero notation of Definition 1.6 and compute the expansions of the considered brackets on  $\mathcal{B}^*$  using Jacobi's identity as many times as necessary (see [4, Section 2.1] for an exposition and a more theoretical point of view on the classical recursive decomposition algorithm on Hall bases).

For  $B \in \mathcal{L}(X)$ ,  $\langle B, D \rangle$  denotes the coefficient of  $D$  in the expansion of  $B$  on  $\mathcal{B}^*$ .

### 9.2.1 Brackets of two elements

**Lemma 9.3.** *Let  $a < b \in \mathcal{B}_3^*$  such that  $\langle [a, b], D \rangle \neq 0$ . Then  $a = P_{1,1}$  and  $b = P_{1,1}0$ .*

*Proof.* First  $n_0(a) + n_0(b) = n_0(D) = 3$ . Thus  $a = P_{1,1}$  and  $b \in \{P_{1,1}0, P_{1,2}\}$ . Since  $(P_{1,1}, P_{1,2}) \in \mathcal{B}^* \setminus \{D\}$ , the conclusion follows.  $\square$

**Lemma 9.4.** *Let  $a \in \mathcal{B}_2^*$  and  $b \in \mathcal{B}_4^*$ . Then  $\langle [a, b], D \rangle = 0$ .*

*Proof.* First  $n_0(a) + n_0(b) = n_0(D) = 3$ . Since  $n_0(b) \geq 1$ ,  $n_0(a) \in \llbracket 0, 2 \rrbracket$  so  $a \in \{W_1, W_10\}$ .

- Case  $a = W_10$ . Then  $b = \text{ad}_{X_1}^4(X_0)$  and  $[a, b] = [W_10, \text{ad}_{X_1}^4(X_0)]$ , which is in  $\mathcal{B}^* \setminus \{D\}$ .

- Case  $a = W_1$ . Then either,

- $b = \text{ad}_{X_1}^4(X_0)0$  and

$$[a, b] = [W_1, \text{ad}_{X_1}^4(X_0)]0 - [W_10, \text{ad}_{X_1}^4(X_0)], \quad (9.7)$$

both terms being in  $\mathcal{B}^* \setminus \{D\}$ .

- $b = (M_1, \text{ad}_{X_1}^3(X_0))$  and  $[a, b] = [W_1, [M_1, \text{ad}_{X_1}^3(X_0)]]$ , which is in  $\mathcal{B}^* \setminus \{D\}$ .

Hence, in all cases  $\langle [a, b], D \rangle = 0$ .  $\square$

**Lemma 9.5.** *Let  $a \in \mathcal{B}_1^*$  and  $b \in \mathcal{B}_5^*$ , such that  $\langle [a, b], D \rangle \neq 0$ . Then  $a = X_1$  and  $b = R_{1,1,1,1,1}^\sharp$ .*

*Proof.* First  $n_0(a) + n_0(b) = n_0(D) = 3$ . Since  $n_0(b) \geq 1$ ,  $n_0(a) \in \llbracket 0, 2 \rrbracket$  so  $a \in \{X_1, M_1, M_2\}$ .

- Case  $a = M_2$ . Then  $b = \text{ad}_{X_1}^5(X_0)$  and  $[a, b] = [M_2, \text{ad}_{X_1}^5(X_0)]$ , which is in  $\mathcal{B}^* \setminus \{D\}$ .

- Case  $a = M_1$ . Then either,

- $b = \text{ad}_{X_1}^5(X_0)0$  and

$$[a, b] = [M_1, \text{ad}_{X_1}^5(X_0)]0 - [M_2, \text{ad}_{X_1}^5(X_0)], \quad (9.8)$$

both terms being in  $\mathcal{B}^* \setminus \{D\}$ .

- $b = (M_1, \text{ad}_{X_1}^4(X_0))$  and  $[a, b] = \text{ad}_{M_1}^2 \text{ad}_{X_1}^4(X_0)$ , which is in  $\mathcal{B}^* \setminus \{D\}$ .

- $b = (W_1, \text{ad}_{X_1}^3(X_0))$  and

$$[a, b] = [W_1, [M_1, P_{1,1}]] - [P_{1,1}, P_{1,2}], \quad (9.9)$$

both terms being in  $\mathcal{B}^* \setminus \{D\}$ .

- Case  $a = X_1$ . Then either,

- $b = \text{ad}_{X_1}^5(X_0)0^2$  and

$$[a, b] = \text{ad}_{X_1}^6(X_0)0^2 - 2[M_1, \text{ad}_{X_1}^5(X_0)] + [M_2, \text{ad}_{X_1}^5(X_0)], \quad (9.10)$$

all terms being in  $\mathcal{B}^* \setminus \{D\}$ .

- $b = (M_1, \text{ad}_{X_1}^4(X_0))0$  and

$$[a, b] = [W_1, \text{ad}_{X_1}^4(X_0)]0 + [M_1, \text{ad}_{X_1}^5(X_0)]0 - \text{ad}_{M_1}^2 \text{ad}_{X_1}^4(X_0), \quad (9.11)$$

all terms being in  $\mathcal{B}^* \setminus \{D\}$ .

–  $b = \text{ad}_{M_1}^2 \text{ad}_{X_1}^3(X_0)$  and

$$[a, b] = 2[W_1, [M_1, P_{1,1}]] + \text{ad}_{M_1}^2 \text{ad}_{X_1}^4(X_0) - [P_{1,1}, P_{1,2}], \quad (9.12)$$

all terms being in  $\mathcal{B}^* \setminus \{D\}$ .

–  $b = (M_2, \text{ad}_{X_1}^4(X_0))$  and

$$[a, b] = [W_1 0, \text{ad}_{X_1}^4(X_0)] + [M_2, \text{ad}_{X_1}^5(X_0)], \quad (9.13)$$

both terms being in  $\mathcal{B}^* \setminus \{D\}$ .

–  $b = (W_1, \text{ad}_{X_1}^3(X_0))0$  and

$$[a, b] = [W_1, \text{ad}_{X_1}^4(X_0)]0 - [W_1, [M_1, P_{1,1}]] + [P_{1,1}, P_{1,2}] \quad (9.14)$$

all terms being in  $\mathcal{B}^* \setminus \{D\}$ .

–  $b = (W_1 0, \text{ad}_{X_1}^3(X_0))$  and

$$[a, b] = -D + [P_{1,1}, P_{1,2}] + [W_1 0, \text{ad}_{X_1}^4(X_0)], \quad (9.15)$$

so  $\langle [a, b], D \rangle = -1$ .

–  $b = (W_1, (M_1, W_1))$  and

$$[a, b] = [P_{1,1}, P_{1,2}] + [W_1, [M_1, \text{ad}_{X_1}^3(X_0)]] \quad (9.16)$$

both terms being in  $\mathcal{B}^* \setminus \{D\}$ .

Hence, the only case where  $\langle [a, b], D \rangle = -1 \neq 0$  is  $a = X_1$  and  $b = (W_1 0, \text{ad}_{X_1}^3(X_0)) = R_{1,1,1,1}^\sharp$ .  $\square$

## 9.2.2 Brackets of three elements

**Lemma 9.6.** *For every  $a, b, c \in \mathcal{B}_2^*$ ,  $\langle [a, [b, c]], D \rangle = 0$ .*

*Proof.* By contradiction, assume that  $\langle [a, [b, c]], D \rangle \neq 0$ . Then  $n_0(a) + n_0(b) + n_0(c) = 3$ . Thus  $a = b = c = W_1$ , so  $[a, [b, c]] = 0$ .  $\square$

**Lemma 9.7.** *Let  $a \in \mathcal{B}_1^*$ ,  $b \in \mathcal{B}_2^*$ ,  $c \in \mathcal{B}_3^*$  such that  $\langle [a, [b, c]], D \rangle \neq 0$  or  $\langle [[a, b], c], D \rangle \neq 0$ . Then  $a = X_1$  and, either  $(b = W_1 0 \text{ and } c = P_{1,1})$  or  $(b = W_1 \text{ and } c = P_{1,1} 0)$ .*

*Proof.* First  $n_0(a) + n_0(b) + n_0(c) = 3$ .

**Step 1.** *First form:  $[a, [b, c]]$ .*

- Case  $a = M_1$ . Then  $b = W_1$ ,  $c = P_{1,1}$  and

$$[a, [b, c]] = -[P_{1,1}, P_{1,2}] + [W_1, [M_1, P_{1,1}]], \quad (9.17)$$

both terms being in  $\mathcal{B}^* \setminus \{D\}$ .

- Case  $a = X_1$ .

– Case  $b = W_1 0$ . Then  $c = P_{1,1}$  and  $\langle [a, [b, c]], D \rangle = -1$ .

– Case  $b = W_1$ . Then either,

\*  $c = P_{1,1} 0$  and  $\langle [a, [b, c]], D \rangle = +1$ .

\*  $c = P_{1,2}$  and  $[a, [b, c]] = [W_1, [M_1, P_{1,1}]] + [P_{1,1}, P_{1,2}]$  both terms being in  $\mathcal{B}^* \setminus \{D\}$ .

**Step 2.** *Second form:*  $[[a, b], c]$ .

- Case  $a = M_1$ . Then  $b = W_1$  and  $c = P_{1,1}$  and  $[[a, b], c] = -[P_{1,1}, P_{1,2}]$  which is in  $\mathcal{B}^* \setminus \{D\}$ .
- Case  $a = X_1$ .
  - Case  $b = W_1 0$ . Then  $c = P_{1,1}$  and  $\langle [[a, b], c], D \rangle = -1$ .
  - Case  $b = W_1$ . Then either
    - \*  $c = P_{1,1} 0$  and  $\langle [[a, b], c], D \rangle = 1$ .
    - \*  $c = P_{1,2}$  and  $[[a, b], c] = [P_{1,1}, P_{1,2}]$ , which is in  $\mathcal{B}^* \setminus \{D\}$ .

This concludes the case disjunction. □

**Lemma 9.8.** *Let  $a, b \in \mathcal{B}_1^*$  and  $c \in \mathcal{B}_4^*$  such that  $\langle [a, [b, c]], D \rangle \neq 0$ , or  $\langle [[a, b], c] \rangle \neq 0$ . Then  $a = b = X_1$ .*

*Proof.* First  $n_0(a) + n_0(b) + n_0(c) = 3$ .

**Step 1.** *First form:*  $[a, [b, c]]$  with  $a \leq b$ .

- Case  $a = b = M_1$ . Then  $c = \text{ad}_{X_1}^4(X_0)$  and  $[a, [b, c]] = \text{ad}_{M_1}^2 \text{ad}_{X_1}^4(X_0)$ , which is in  $\mathcal{B}^* \setminus \{D\}$ .
- Case  $a = X_1, b = M_2$ . Then  $c = \text{ad}_{X_1}^4(X_0)$  and

$$[a, [b, c]] = [W_1 0, \text{ad}_{X_1}^4(X_0)] + [M_2, \text{ad}_{X_1}^5(X_0)], \quad (9.18)$$

both terms being in  $\mathcal{B}^* \setminus \{D\}$ .

- Case  $a = X_1, b = M_1$ . Then either,

- $c = \text{ad}_{X_1}^4(X_0) 0$  and

$$\begin{aligned} [a, [b, c]] &= [W_1, \text{ad}_{X_1}^4(X_0) 0] - [W_1 0, \text{ad}_{X_1}^4(X_0)] - [M_2, \text{ad}_{X_1}^5(X_0)] \\ &\quad - \text{ad}_{M_1}^2 \text{ad}_{X_1}^4(X_0) + [M_1, \text{ad}_{X_1}^5(X_0) 0], \end{aligned} \quad (9.19)$$

all terms being in  $\mathcal{B}^* \setminus \{D\}$ .

- $c = (M_1, \text{ad}_{X_1}^3(X_0))$  and

$$[a, [b, c]] = -[P_{1,1}, P_{1,2}] + 2[W_1, [M_1, \text{ad}_{X_1}^3(X_0)]] + \text{ad}_{M_1}^2 \text{ad}_{X_1}^4(X_0), \quad (9.20)$$

all terms being in  $\mathcal{B}^* \setminus \{D\}$ .

- Case  $a = b = X_1$ . One may have  $\langle [a, [b, c]], D \rangle \neq 0$ . Since the conclusion of the lemma does not concern  $c$ , we do not need to study all possible cases.

Thus, the only case leading to a (possibly) nonzero value of  $\langle [a, [b, c]], D \rangle$  is  $a = b = X_1$ .

**Step 2.** *Second form:*  $[[a, b], c]$  with  $a < b$ . Since  $n_0(a) + n_0(b) \leq 2$ ,  $a = X_1$  and  $b = M_1$ . Thus  $[a, b] = W_1$ . By Lemma 9.4,  $\langle [W_1, c], D \rangle = 0$ .

**Step 3.** *Third form:*  $[a, [b, c]]$  with  $a > b$ . Then  $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$  so the conclusions of the previous forms apply. □

### 9.3 Dominant part of the logarithm

**Lemma 9.9.** *Assume that  $f_D(0) \notin \mathcal{N}_D(f)(0)$ . Let  $\mathbb{P}$  be a component along  $f_D(0)$  parallel to  $\mathcal{N}_D(f)(0)$ . Then*

$$\begin{aligned} \mathbb{P}\mathcal{Z}_7(t, f, u)(0) &= \xi_D(t, u) + O\left(|u_1(t)|^4 + |\xi_{P_{1,1}}(t, u)|^2 + |\xi_{P_{1,1}0}(t, u)|^2 \right. \\ &\quad \left. + |u_1(t)\xi_{R_{1,1,1,1}^\sharp}(t, u)| + \|u_1\|_{L^8}^8\right). \end{aligned} \quad (9.21)$$

*Proof.* We start with a preliminary estimate. By (3.22) and Hölder's inequality, there exists  $c > 0$  such that, for every  $t \leq 1$ ,  $u \in L^1((0, t); \mathbb{R})$  and  $b \in \mathcal{B}_{[1,6]}^* \setminus \{X_1\}$ ,

$$|\xi_b(t, u)| \leq c\|u_1\|_{L^8}^{n_1(b)}. \quad (9.22)$$

By (4.7) and definition of  $\mathbb{P}$ ,

$$\mathbb{P}\mathcal{Z}_7(t, f, u)(0) = \eta_D(t, u). \quad (9.23)$$

To apply Proposition 2.16, let us prove that, for every  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}^*$  such that  $D \in \text{supp } \mathcal{F}(b_1, \dots, b_q)$ , for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ , the estimate (2.15) holds, for an appropriate choice of  $\Xi$ . We split cases depending on  $q$ .

**Step 1.** *Case  $q = 2$ .*

- Case  $n_1(b_1) = 5$  and  $n_1(b_2) = 1$ . By Lemma 9.5,  $b_1 = R_{1,1,1,1}^\sharp$  and  $b_2 = X_1$  so (2.15) holds with  $\Xi(t, u) := |u_1(t)\xi_{R_{1,1,1,1}^\sharp}(t, u)|$ .
- Case  $n_1(b_1) = 4$  and  $n_1(b_2) = 2$ . By Lemma 9.4,  $D \notin \text{supp } \mathcal{F}(b_1, b_2)$  in this case.
- Case  $n_1(b_1) = 3$  and  $n_1(b_2) = 3$ . By Lemma 9.3,  $b_1 = P_{1,1}0$  and  $b_2 = P_{1,1}$  so (2.15) holds with  $\Xi(t, u) := |\xi_{P_{1,1}}(t, u)\xi_{P_{1,1}0}(t, u)|$ .

**Step 2.** *Case  $q = 3$ .*

- Case  $n_1(b_1) = 4$ ,  $n_1(b_2) = 1$ ,  $n_1(b_3) = 1$ . By Lemma 9.8,  $b_2 = b_3 = X_1$ . Hence, using (9.22), (2.15) holds with  $\Xi(t, u) := c|u_1(t)|^2\|u_1\|_{L^8}^4$ .
- Case  $n_1(b_1) = 3$ ,  $n_1(b_2) = 2$ ,  $n_1(b_3) = 1$ . By Lemma 9.7,  $b_3 = X_1$  and  $b_1 \in \{P_{1,1}, P_{1,1}0\}$ . Hence, using (9.22), (2.15) holds with  $\Xi(t, u) := c|u_1(t)|\|u_1\|_{L^8}^2(|\xi_{P_{1,1}}(t, u)| + |\xi_{P_{1,1}0}(t, u)|)$ .
- Case  $n_1(b_1) = 2$ ,  $n_1(b_2) = 2$ ,  $n_1(b_3) = 2$ . By Lemma 9.6,  $D \notin \text{supp } \mathcal{F}(b_1, b_2, b_3)$  in this case.

**Step 3.** *Case  $q = 4$ .*

- Case  $n_1(b_1) = 3$ ,  $n_1(b_2) = 1$ ,  $n_1(b_3) = 1$ ,  $n_1(b_4) = 1$ . Counting the occurrences of  $X_0$  and using (9.22) implies that either,
  - $b_3 = b_4 = X_1$ , and (2.15) holds with  $\Xi(t, u) := c\|u_1\|_{L^8}^4|u_1(t)|^2$ .
  - $b_1 = P_{1,1}$ ,  $b_2 = b_3 = M_1$  and  $b_4 = X_1$ , and thus (2.15) holds with  $\Xi(t, u) := |\xi_{P_{1,1}}(t, u)|\|u_1\|_{L^8}^2|u_1(t)|$ .
- Case  $n_1(b_1) = 2$ ,  $n_1(b_2) = 2$ ,  $n_1(b_3) = 1$ ,  $n_1(b_4) = 1$ . Counting the occurrences of  $X_0$  and using (9.22) implies that either,
  - $b_1 = b_2 = W_1$ ,  $b_3 = M_1$  and  $b_4 = X_1$  and  $D \notin \text{supp } \mathcal{F}(b_1, b_2, b_3, b_4)$ . Indeed, a non-zero bracket of  $W_1$ ,  $W_1$ ,  $M_1$  and  $X_1$  is either a bracket over  $(M_1, W_1)$  and  $(X_1, W_1)$  or over  $(X_1, W_1)$  and  $(M_1, W_1)$ . But such brackets have a vanishing coefficient along  $D$  by Lemma 9.7.

–  $b_1 = W_1 0$ ,  $b_2 = W_1$ ,  $b_3 = b_4 = X_1$  and (2.15) holds with  $\Xi(t, u) := \|u_1\|_{L^8}^4 |u_1(t)|^2$ .

**Step 4.** *Case  $q \in \{5, 6\}$ .* Counting the occurrences of  $X_0$  implies that  $b_{q-1} = b_q = X_1$ . Using (9.22) implies that (2.15) holds with  $\Xi(t, u) := (1+c)^4 |u_1(t)|^k \|u_1\|_{L^8}^{6-k}$  for some  $k \in \llbracket 2, 5 \rrbracket$ .

**Step 5.** *Conclusion.* Gathering the previous estimates and using Young's inequality proves (9.21).  $\square$

## 9.4 Vectorial relations

**Lemma 9.10.** *Assume that  $f_D(0) \notin \mathcal{N}_D(f)(0)$ . Then*

1.  $f_{X_1}(0) \notin \text{span}\{f_b(0); b \in \mathcal{B}_1^* \setminus \{X_1\}\}$ ,
2.  $f_{P_{1,1}}(0) \notin \text{span}\{f_b(0); b \in \mathcal{B}_{\llbracket 1, 3 \rrbracket}^* \setminus \{P_{1,1}\}\}$ .

*Proof.* We proceed by contradiction.

**Step 1.** *First statement.* Assume that  $f_{X_1}(0) = \sum_{j \geq 1} \alpha_j f_{M_j}(0)$  where  $\alpha_j \in \mathbb{R}$  and the sum is finite. Hence  $f_{B_1}(0) = 0$  where  $B_1 := X_1 - \sum_{j \geq 1} \alpha_j M_j \in S_1(X)$ . Let  $B_2 := \text{ad}_{\text{ad}_{B_1}^3(X_0)}(X_0)$ . Then  $f_{B_2}(0) = 0$ . Moreover, by definition of  $B_1$  and  $B_2$ , one checks that  $B_2 = D + B_3$  where  $B_3 \in \text{span}\{b \in \mathcal{B}_6^*; n_0(b) \geq 4\}$ . The equality  $f_D(0) = -f_{B_3}(0)$  contradicts the assumption on  $f_D(0)$ .

**Step 2.** *Second statement.* Assume that there exists  $B_0 \in \text{span}\{b \in \mathcal{B}_{\llbracket 1, 3 \rrbracket}^*; n_1(b) < 3 \text{ or } n_0(b) > 1\}$  such that  $f_{P_{1,1}}(0) = f_{B_0}(0)$ . Let  $B_1 := P_{1,1} - B_0$  so that  $f_{B_1}(0) = 0$ . Then  $f_{B_2}(0) = 0$  where  $B_2 := \text{ad}_{B_1}^2(X_0)$ . Thus  $f_D(0) = f_{B_3}(0)$  where  $B_3 \in \text{span}\{b \in \mathcal{B}_{\llbracket 1, 6 \rrbracket}^*; n_1(b) \leq 5 \text{ or } n_0(b) \geq 4\}$ , which contradicts the assumption on  $f_D(0)$ .  $\square$

## 9.5 Closed-loop estimates

**Lemma 9.11.** *Assume that  $f_D(0) \notin \mathcal{N}_D(f)(0)$ . Then*

$$|u_1(t)| = O(|x(t; u)| + \|u_1\|_{L^2}^2), \quad (9.24)$$

$$|\xi_{P_{1,1}}(t, u)| = O(|x(t; u)| + \|u_1\|_{L^4}^4). \quad (9.25)$$

*Proof.* We rely on Lemma 9.10.

**Step 1.** *First estimate.* By Theorem 4.4 with  $M \leftarrow 1$ ,

$$x(t; u) = \mathcal{Z}_1(t, f, u)(0) + O(\|u_1\|_{L^2}^2 + |x(t; u)|^{1+1}). \quad (9.26)$$

By Lemma 9.10, we can consider  $\mathbb{P}$ , a component along  $f_1(0)$ , parallel to  $\mathcal{N}(f)(0)$  where  $\mathcal{N} := \mathcal{B}_1^* \setminus \{X_1\}$ . Hence  $\mathbb{P}\mathcal{Z}_1(t, f, u)(0) = u_1(t)$ . Thus (9.26) yields (9.24).

**Step 2.** *Second estimate.* By Theorem 4.4 with  $M \leftarrow 3$ ,

$$x(t; u) = \mathcal{Z}_3(t, f, u)(0) + O\left(\|u_1\|_{L^4}^4 + |x(t; u)|^{1+\frac{1}{3}}\right). \quad (9.27)$$

By Lemma 9.10, we can consider  $\mathbb{P}$ , a component along  $f_{P_{1,1}}(0)$ , parallel to  $\mathcal{N}(f)(0)$  where  $\mathcal{N} := \mathcal{B}_{\llbracket 1, 3 \rrbracket}^* \setminus \{P_{1,1}\}$ . By (4.7),

$$\mathbb{P}\mathcal{Z}_3(t, f, u)(0) = \eta_{P_{1,1}}(t, u). \quad (9.28)$$

We apply Proposition 2.16 (see below) to obtain

$$\eta_{P_{1,1}}(t, u) = \xi_{P_{1,1}}(t, u) + O(|u_1(t)| \|u_1\|_{L^2}^2 + |u_1(t)|^2 \|u_1\|_{L^1}). \quad (9.29)$$

Then (9.27), (9.28) and (9.29), combined with the previous estimate (9.24), yield (9.25).

Let us check the required conditions to obtain (9.29). Let  $q \geq 2$ ,  $b_1 \geq \dots \geq b_q \in \mathcal{B}^*$  such that  $P_{1,1} \in \text{supp } \mathcal{F}(b_1, \dots, b_q)$ . Since  $n_1(P_{1,1}) = 3$  and  $n_0(P_{1,1}) = 1$ , the only possibilities are



- $q = 2$ ,  $b_1 = W_1$ ,  $b_2 = X_1$ , in which case

$$|\xi_{b_1}(t, u)\xi_{b_2}(t, u)| = |u_1(t)| \int_0^t \frac{u_1^2}{2} \leq |u_1(t)| \|u_1\|_{L^2}^2. \quad (9.30)$$

- $q = 3$ ,  $b_1 = M_1$ ,  $b_2 = b_3 = X_1$ , in which case

$$|\xi_{b_1}(t, u)\xi_{b_2}(t, u)\xi_{b_3}(t, u)| = |u_1(t)|^2 |u_2(t)| \leq |u_1(t)|^2 \|u_1\|_{L^1}. \quad (9.31)$$

This concludes the proof of (9.29) by Proposition 2.16.  $\square$

## 9.6 Interpolation inequalities

**Lemma 9.12.** *There exists  $C > 0$  such that, for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,*

$$\|u_1\|_{L^8}^8 \leq Ct|u_1(t)|^8 + C\|u\|_{L^\infty}^2 \xi_D(t, u), \quad (9.32)$$

$$|\xi_{P_{1,1,0}}(t, u)|^2 \leq 2t\xi_D(t, u), \quad (9.33)$$

$$|\xi_{R_{1,1,1,1}^\sharp}(t, u)| \leq Ct\|u_1\|_{L^2}^2 |\xi_{P_{1,1}}(t, u)| + Ct^{\frac{1}{2}} \|u_1\|_{L^2}^2 \xi_D(t, u)^{\frac{1}{2}}. \quad (9.34)$$

*Proof. Step 1. First estimate.* By integration by parts,

$$\int_0^t u_1^8 = u_1^5(t) \int_0^t u_1^3 - 5 \int_0^t u(s)u_1^4(s) \left( \int_0^s u_1^3 \right) ds. \quad (9.35)$$

By Cauchy–Schwarz and Hölder inequalities and (2.5), we obtain

$$\|u_1\|_{L^8}^8 \leq t^{\frac{5}{8}} |u_1(t)|^5 \|u_1\|_{L^8}^3 + 30\sqrt{2} \|u\|_{L^\infty} \|u_1\|_{L^8}^4 \xi_D(t, u)^{\frac{1}{2}}, \quad (9.36)$$

which proves (9.32) using Young’s inequality.

*Step 2. Second estimate.* By (2.5),  $\xi_{P_{1,1,0}} = \int \xi_{P_{1,1}}$  and  $\xi_D = \frac{1}{2} \int \xi_{P_{1,1}}^2$  so (9.33) follows directly from the Cauchy–Schwarz inequality.

*Step 3. Third estimate.* By (2.5) and since  $R_{1,1,1,1}^\sharp = (W_1 0, P_{1,1})$ , integration by parts yields

$$\xi_{R_{1,1,1,1}^\sharp}(t, u) = \int_0^t \xi_{W_1 0} \dot{\xi}_{P_{1,1}} = \xi_{W_1 0}(t) \xi_{P_{1,1}}(t) - \int_0^t \xi_{W_1} \xi_{P_{1,1}}. \quad (9.37)$$

Then (9.34) follows by the Cauchy–Schwarz inequality and the estimates  $\xi_{W_1}(0)(t) \leq t\|u_1\|_{L^2}^2$  and  $\xi_{W_1}(t) \leq \|u_1\|_{L^2}^2$ .  $\square$

## 9.7 Proof of the drift

*Proof of Theorem 9.1.* Let  $\mathbb{P}$  be a component along  $f_D(0)$  parallel to  $\mathcal{N}_D(f)(0)$ . By Theorem 4.4 with  $M \leftarrow 7$ ,

$$x(t; u) = \mathcal{Z}_7(t, f, u)(0) + O\left(\|u_1\|_{L^8}^8 + |x(t; u)|^{1+\frac{1}{7}}\right), \quad (9.38)$$

where  $\mathbb{P}\mathcal{Z}_7(t, f, u)(0)$  satisfies (9.21). Combining the closed-loop estimate (9.24) and the interpolation estimate (9.32), one obtains

$$\|u_1\|_{L^8}^8 = O\left(|x(t; u)|^8 + \|u\|_{L^\infty}^2 \xi_D(t, u)\right). \quad (9.39)$$

Substituting in the closed-loop estimate (9.24) yields

$$|u_1(t)|^4 = O\left(|x(t; u)|^4 + \|u\|_{L^\infty}^2 \xi_D(t, u)\right) \quad (9.40)$$

and in the closed-loop estimate (9.25) yields

$$|\xi_{P_{1,1}}(t, u)|^2 = O(|x(t; u)|^2 + \|u\|_{L^\infty}^2 \xi_D(t, u)). \quad (9.41)$$

Eventually, using (9.34) and Young's inequality,

$$\begin{aligned} |u_1(t) \xi_{R_{1,1,1,1}^\sharp}(t, u)| &= O(|\xi_{P_{1,1}}(t, u)|^2 + \|u_1\|_{L^2}^8 + |u_1(t)|^4 + t \xi_D(t, u)) \\ &= O(|x(t; u)|^2 + (t + \|u\|_{L^\infty}^2) \xi_D(t, u)). \end{aligned} \quad (9.42)$$

Gathering all these equalities in (9.21) and the interpolation estimate (9.33) yields

$$\mathbb{P}x(t; u) = \xi_D(t, u) + O\left((t + \|u\|_{L^\infty}^2) \xi_D(t, u) + |x(t; u)|^{1+\frac{1}{2}}\right). \quad (9.43)$$

This implies, in the sense of Definition 1.15, a drift along  $f_D(0)$ , parallel to  $\mathcal{N}_D(f)(0)$ , of amplitude  $\xi_D$ , in the regime  $(t, \|u\|_{L^\infty}) \rightarrow 0$ .  $\square$

## 10 Obstructions without analyticity

Except for this section, all our paper is written with an analyticity assumption on the vector fields  $f_0$  and  $f_1$ . This allows to work with convergent series. However, as announced in the introduction, the obstruction mechanisms on which our necessary conditions for controllability rely are sufficiently robust to absorb an approximation scheme for non-analytic vector fields.

Let  $\delta > 0$ . For smooth vector fields  $f_0$  and  $f_1$  in  $\mathcal{C}^\infty(B_\delta; \mathbb{R}^d)$ , one can still define all Lie brackets  $f_b \in \mathcal{C}^\infty(B_\delta; \mathbb{R}^d)$  for  $b \in \text{Br}(X)$ . The arguments of the next paragraphs will prove that all the statements of Section 1.5 remain true without any change under this (weaker) regularity setting.

Furthermore, even in a finite regularity setting, one can give a sense to some Lie brackets, once evaluated at zero. This stems from the equilibrium assumption  $f_0(0) = 0$ . More precisely, the value of  $f_b(0)$  only depends on the coefficients of the Taylor expansion at 0 of  $f_0$  up to order  $n_1(b)$  and of  $f_1$  up to order  $n_1(b) - 1$  (see Lemma 10.6 below). This leads to the following definition.

**Definition 10.1.** *Let  $M \in \mathbb{N}^*$ ,  $\delta > 0$ ,  $f_0 \in \mathcal{C}^M(B_\delta; \mathbb{R}^d)$  with  $f_0(0) = 0$  and  $f_1 \in \mathcal{C}^{M-1}(B_\delta; \mathbb{R}^d)$ . Let  $\hat{f}_0 := \mathbb{T}_M f_0$  (respectively  $\hat{f}_1 := \mathbb{T}_{M-1} f_1$ ) be the truncated Taylor series at 0 of  $f_0$  (resp.  $f_1$ ) of order  $M$  (resp.  $M - 1$ ). For  $b \in \text{Br}(X)$  with  $n_1(b) \in \llbracket 1, M \rrbracket$ , we define  $f_b(0) := \hat{f}_b(0)$ .*

With this notation, we will prove that the following corollaries of the main theorems of Section 1.5 hold. As a rule of thumb, the theorems continue to hold as soon as the vector fields have enough regularity for the involved Lie brackets to be defined as above. More rigorously, we assume one extra derivative to be able to estimate the truncation error properly (see Lemma 10.7).

We make the blanket hypothesis that  $f_0(0) = 0$ .

**Corollary 10.2.** *Let  $M \in \mathbb{N}^*$ . Assume that  $f_0 \in \mathcal{C}^{M+1}$ ,  $f_1 \in \mathcal{C}^M$ . If system (1.1) is  $W^{-1, \infty}$ -STLC, then, for every  $k \in \mathbb{N}^*$  such that  $2k \leq M$ ,*

$$ad_{f_1}^{2k}(f_0)(0) \in S_{\llbracket 1, 2k-1 \rrbracket}(f)(0). \quad (10.1)$$

**Corollary 10.3.** *Let  $M \in \mathbb{N}^*$ . Assume that  $f_0 \in \mathcal{C}^{M+1}$ ,  $f_1 \in \mathcal{C}^M$ . Let  $m \in \mathbb{N}^*$ . If system (1.1) is  $W^{m, \infty}$ -STLC, then, for every  $k \in \mathbb{N}^*$  such that  $\pi(k, m) \leq M$ ,*

$$f_{W_k}(0) \in S_{\llbracket 1, \pi(k, m) \rrbracket \setminus 2}(f)(0), \quad (10.2)$$

where  $\pi(k, m)$  is defined in (1.20).

**Corollary 10.4.** *Assume that system (1.1) is  $L^\infty$ -STLC. If  $f_0 \in \mathcal{C}^4$  and  $f_1 \in \mathcal{C}^3$ , then  $f_{W_2}(0) \in \mathcal{N}_2(f)(0)$  (see (1.22)). If  $f_0 \in \mathcal{C}^6$  and  $f_2 \in \mathcal{C}^5$ , then  $f_{W_3}(0) \in \mathcal{N}_3(f)(0)$  (see (1.23)).*

**Corollary 10.5.** *Assume that  $f_0 \in \mathcal{C}^8$  and  $f_1 \in \mathcal{C}^7$ . Then Theorem 1.13 holds.*

All these corollaries follow from the main theorems and the approximation result Lemma 10.7. One writes  $x \approx \hat{x}$ , where  $\hat{x}$  is the solution to a system driven by the truncated Taylor expansions of  $f_0$  and  $f_1$ . For the  $\hat{x}$  system, one can apply the drift results of the previous sections. Since the truncation error is of the same size (or smaller than) as the error terms which were already absorbed by the drift, the drift conclusion remains true on the state  $x$ .

## 10.1 Brackets at zero only depend on low-order Taylor coefficients

**Lemma 10.6.** *Let  $M \in \mathbb{N}^*$ ,  $\delta > 0$ ,  $f_0 \in \mathcal{C}^\infty(B_\delta; \mathbb{R}^d)$  with  $f_0(0) = 0$  and  $f_1 \in \mathcal{C}^\infty(B_\delta; \mathbb{R}^d)$ . Let  $\hat{f}_0 := \mathbb{T}_M f_0$  (respectively  $\hat{f}_1 := \mathbb{T}_{M-1} f_1$ ) be the truncated Taylor series at 0 of  $f_0$  (resp.  $f_1$ ) of order  $M$  (resp.  $M-1$ ). For all  $b \in \text{Br}(X)$  with  $n_1(b) \leq M$ ,  $f_b(0) = \hat{f}_b(0)$ .*

*Proof. Step 1. Notations and preliminary remarks.* As in [5, Section 3.1], for two vector fields  $g, h \in \mathcal{C}^\infty(B_\delta; \mathbb{R}^d)$  and  $k \in \mathbb{N}^*$ , we write  $g =_{[k]} h$  when the Taylor expansions of  $g$  and  $h$  at 0 are equal up to order  $k-1$ . When  $k \geq 2$ ,  $g =_{[k]} \hat{g}$  and  $h =_{[k]} \hat{h}$ , straightforward computations prove that  $[g, h] =_{[k-1]} [\hat{g}, \hat{h}]$ . When  $k \geq 1$ ,  $g(0) = 0$ ,  $g =_{[k+1]} \hat{g}$  and  $h =_{[k]} \hat{h}$ , straightforward computations prove that  $[g, h] =_{[k]} [\hat{g}, \hat{h}]$ , so that there is “no loss of derivative” in this weak sense.

*Step 2. Computation of brackets.* We now proceed by induction on  $n_1(b) \in \llbracket 1, M \rrbracket$ , proving that, for every  $b \in \text{Br}(X)$  with  $1 \leq n_1(b) \leq M$ ,  $f_b =_{[M+1-n_1(b)]} \hat{f}_b$ .

When  $n_1(b) = 1$ , by symmetry, we can assume that  $b = X_1 0^\nu$  for some  $\nu \in \mathbb{N}$ . Since  $f_1 =_{[M]} \hat{f}_1$ , iterating the previous remarks yields  $f_{X_1 0^\nu} =_{[M]} \hat{f}_{X_1 0^\nu}$ , which gives the initialization.

Now let  $b \in \text{Br}(X)$ . By symmetry, we can assume that  $b = (b_1, b_2) 0^\nu$  for some  $\nu \in \mathbb{N}$ , with  $b_1, b_2 \neq X_0$ . By the induction hypothesis  $f_{b_1} =_{[M+1-n_1(b_1)]} \hat{f}_{b_1}$  and  $f_{b_2} =_{[M+1-n_1(b_2)]} \hat{f}_{b_2}$ . Hence, by the preliminary remark,  $f_{(b_1, b_2)} =_{[M+1-n]} \hat{f}_{(b_1, b_2)}$  with  $n := 1 + \max n_1(b_1), n_1(b_2) \leq n_1(b)$ . And by the preliminary remark, bracketing with  $f_0$  preserves this approximation level, so we have proved that  $f_b =_{[M+1-n_1(b)]} \hat{f}_b$ .

*Step 3. Evaluation at zero.* When  $b = X_0$ ,  $f_0(0) = \hat{f}_0(0)$ . When  $b \in \text{Br}(X)$  with  $1 \leq n_1(b) \leq M$ , we have proved that  $f_b =_{[M+1-n_1(b)]} \hat{f}_b$  so  $f_b =_{[1]} \hat{f}_b$  and thus  $f_b(0) = \hat{f}_b(0)$ .  $\square$

## 10.2 Estimate of the approximation error

**Lemma 10.7.** *Let  $M \in \mathbb{N}^*$ ,  $\delta > 0$ ,  $f_0 \in \mathcal{C}^{M+1}(B_\delta; \mathbb{R}^d)$  with  $f_0(0) = 0$  and  $f_1 \in \mathcal{C}^M(B_\delta; \mathbb{R}^d)$ . Let  $\hat{f}_0 := \mathbb{T}_M f_0$  (respectively  $\hat{f}_1 := \mathbb{T}_{M-1} f_1$ ) be the truncated Taylor series at 0 of  $f_0$  (resp.  $f_1$ ) of order  $M$  (resp.  $M-1$ ). Then*

$$x(t; u) - \hat{x}(t; u) = O\left(\|u_1\|_{L^{M+1}}^{M+1} + |u_1(t)|^{M+1}\right), \quad (10.3)$$

where  $\hat{x}(t; u)$  denotes the solution with initial data 0 to

$$\dot{\hat{x}} = \hat{f}_0(\hat{x}) + u(t) \hat{f}_1(\hat{x}). \quad (10.4)$$

*Proof.* Such an estimate is straightforward to prove when the right-hand side of (10.3) is replaced by  $\|u\|_{L^1}^{M+1}$ . To obtain an estimate involving only  $u_1$ , we need to consider an appropriate “auxiliary system” as in [3, Section 7] or [5, Section 6.3].

*Step 1. Computations on the auxiliary system.* Let  $\Phi_1$  denote the flow of  $f_1$ , which is well-defined locally. We then introduce

$$y(t; u) := \Phi_1(-u_1(t), x(t; u)). \quad (10.5)$$

This new unknown satisfies  $y(0; u) = 0$  and

$$\dot{y} = (\Phi_1(-u_1(t))_* f_0)(\hat{y}), \quad (10.6)$$

where  $\Phi_1(-u_1(t))_* f_0$  is the push-forward of the vector field  $f_0$  by the diffeomorphism  $\Phi_1(-u_1(t), \cdot)$ . In particular, for  $v \in \mathbb{R}$  and  $p \in \mathbb{R}^d$  small enough, (see e.g. [3, equation (169)], albeit with swapped indexes),

$$(\Phi_1(-v)_* f_0)(p) = \sum_{k=0}^{M-1} \frac{v^k}{k!} \text{ad}_{f_1}^k(f_0)(p) + \int_0^v \frac{(v-v')^{M-1}}{(M-1)!} \left( \Phi_1(-v')_* \text{ad}_{f_1}^M(f_0) \right)(p) dv'. \quad (10.7)$$

By Lemma 10.8 (with  $k \leftarrow M$ ) and Lemma 10.9 (with  $g \leftarrow \text{ad}_{f_1}^M(f_0)$  and  $\nu \leftarrow 0$ ),

$$\text{ad}_{f_1}^M(f_0)(p) = \text{ad}_{f_1}^M(f_0)(0) + \underset{|p| \rightarrow 0}{O}(|p|). \quad (10.8)$$

Moreover, since  $f_1 \in \mathcal{C}^1$ ,

$$\Phi_1(v, p) = 0 + \underset{v \rightarrow 0, |p| \rightarrow 0}{O}(|v| + |p|) \quad (10.9)$$

and

$$(\partial_p \Phi_1(v, p))^{-1} = \text{Id} + \underset{v \rightarrow 0, |p| \rightarrow 0}{O}(|v| + |p|). \quad (10.10)$$

Thus, combining the last three estimates proves that, for  $|v'| \leq |v|$

$$\left( \Phi_1(-v')_* \text{ad}_{f_1}^M(f_0) \right)(p) = \text{ad}_{f_1}^M(f_0)(0) + \underset{v \rightarrow 0, |p| \rightarrow 0}{O}(|v| + |p|). \quad (10.11)$$

Substituting in (10.7) and using Young's inequality proves that

$$(\Phi_1(-v)_* f_0)(p) = \sum_{k=0}^{M-1} \frac{v^k}{k!} \text{ad}_{f_1}^k(f_0)(p) + \frac{v^M}{M!} \text{ad}_{f_1}^M(f_0)(0) + \underset{v \rightarrow 0, |p| \rightarrow 0}{O}(|v|^{M+1} + |p|^{M+1}). \quad (10.12)$$

**Step 2. Grönwall estimate for the auxiliary systems.** We introduce similarly  $\hat{y}(t; u)$  using  $\hat{\Phi}_1$  (the flow of  $\hat{f}_1$ ) and  $\hat{f}_0$ . Then the counterpart for (10.12) holds, *mutatis mutandis*, since  $\hat{f}_0$  and  $\hat{f}_1$  are smooth. Using these estimates, one obtains

$$\begin{aligned} \dot{y} - \dot{\hat{y}} &= \sum_{k=0}^{M-1} \frac{u_1^k(t)}{k!} \left( \text{ad}_{f_1}^k(f_0)(y) - \text{ad}_{\hat{f}_1}^k(\hat{f}_0)(\hat{y}) \right) + \frac{u_1^M(t)}{M!} \left( \text{ad}_{f_1}^M(f_0)(0) - \text{ad}_{\hat{f}_1}^M(\hat{f}_0)(0) \right) \\ &\quad + O(|u_1(t)|^{M+1} + |y|^{M+1} + |\hat{y}|^{M+1}). \end{aligned} \quad (10.13)$$

For  $k = 0$ , since  $f_0 \in \mathcal{C}^{M+1}$  and  $\hat{f}_0 = \mathbb{T}_M f_0$ ,

$$f_0(y) - \hat{f}_0(\hat{y}) = f_0(y) - \hat{f}_0(y) + \hat{f}_0(y) - \hat{f}_0(\hat{y}) = O(|y|^{M+1} + |y - \hat{y}|). \quad (10.14)$$

For  $k \in \llbracket 1, M-1 \rrbracket$ , one has

$$\begin{aligned} \text{ad}_{f_1}^k(f_0)(y) - \text{ad}_{\hat{f}_1}^k(\hat{f}_0)(\hat{y}) &= \text{ad}_{f_1}^k(f_0)(y) - \text{ad}_{\hat{f}_1}^k(\hat{f}_0)(y) + \text{ad}_{\hat{f}_1}^k(\hat{f}_0)(y) - \text{ad}_{\hat{f}_1}^k(\hat{f}_0)(\hat{y}) \\ &= O(|y|^{M+1-k} + |y - \hat{y}|), \end{aligned} \quad (10.15)$$

where we used the estimate

$$\text{ad}_{f_1}^k(f_0)(p) = (\mathbb{T}_{M-k} \text{ad}_{f_1}^k(f_0))(p) + O(|p|^{M+1-k}), \quad (10.16)$$

which follows from Lemma 10.8 and Lemma 10.9 (with  $g \leftarrow \text{ad}_{f_1}^k(f_0)$  and  $\nu \leftarrow M - k$ ). Eventually, we obtain

$$\dot{y} - \dot{\hat{y}} = O(|y - \hat{y}| + |u_1(t)|^{M+1} + |y|^{M+1} + |\hat{y}|^{M+1}). \quad (10.17)$$

Moreover, from classical estimates  $|y| = O(\|u_1\|_{L^1})$  and  $|\hat{y}| = O(\|u_1\|_{L^1})$  (see e.g. [5, Lemma 9]). Thanks to Grönwall's lemma, we conclude

$$y(t; u) - \hat{y}(t; u) = O(\|u_1\|_{L^{M+1}}^{M+1}). \quad (10.18)$$

**Step 3. Conclusion.** First, using similar estimates as above, one proves that

$$\Phi_1(v, p) - \hat{\Phi}_1(v, p) = \underset{v \rightarrow 0, |p| \rightarrow 0}{O}(|p|^{M+1} + |v|^{M+1}). \quad (10.19)$$

(For example, one can bound the difference between the trajectories to  $\dot{z} = f_1(z)$ ,  $z(0) = p$  and  $\dot{\hat{z}} = \hat{f}_1(\hat{z})$ ,  $\hat{z}(0) = p$ , at time  $v$ , using a Grönwall estimate, then apply Young's inequality). Therefore, we obtain

$$\begin{aligned} x - \hat{x} &= \Phi_1(u_1(t), y) - \Phi_1(u_1(t), \hat{y}) + \Phi_1(u_1(t), \hat{y}) - \hat{\Phi}_1(u_1(t), \hat{y}) \\ &= O(|y - \hat{y}| + |\hat{y}|^{M+1} + |u_1(t)|^{M+1}), \end{aligned} \quad (10.20)$$

where we can use again the estimate  $\hat{y} = O(\|u_1\|_{L^1})$ , which concludes the proof of (10.3).  $\square$

**Lemma 10.8.** *Let  $M \in \mathbb{N}^*$  and  $\delta > 0$ . Let  $f_0 \in \mathcal{C}^{M+1}(B_\delta; \mathbb{R}^d)$  and  $f_1 \in \mathcal{C}^M(B_\delta; \mathbb{R}^d)$ . For each  $k \in \llbracket 1, M \rrbracket$ , there exists  $h_k \in \mathcal{C}^{M+1-k}(B_\delta; \mathbb{R}^d)$  such that*

$$\text{ad}_{f_1}^k(f_0) = -D^k f_1 \cdot (f_0, f_1, \dots, f_1) + h_k. \quad (10.21)$$

*Proof.* For  $k = 1$ , this holds with  $h_1 := Df_0 \cdot f_1 \in \mathcal{C}^M$ . Then the general formula follows by induction on  $k$ .  $\square$

**Lemma 10.9.** *Let  $\nu \in \mathbb{N}$  and  $\delta > 0$ . Assume that  $g \in \mathcal{C}^\nu(B_\delta; \mathbb{R}^d)$  is of the form  $g = Af_0 + h$  where  $A \in \mathcal{C}^\nu(B_\delta; \mathcal{M}_d(\mathbb{R}))$  and  $h \in \mathcal{C}^{\nu+1}(B_\delta; \mathbb{R}^d)$ . Then, if  $f_0(0) = 0$  and  $f_0 \in \mathcal{C}^{\nu+1}(B_\delta; \mathbb{R}^d)$ ,*

$$g(p) = (\mathbb{T}_\nu g)(p) + \underset{p \rightarrow 0}{O}(|p|^{\nu+1}), \quad (10.22)$$

where  $\mathbb{T}_\nu g$  denotes the truncated Taylor series at 0 of  $g$ .

*Proof.* The claimed estimate is straightforward when  $g \in \mathcal{C}^{\nu+1}$ . In particular, by linearity, one can assume that  $h = 0$ . When  $\nu = 0$ ,  $A \in \mathcal{C}^0$  so is locally bounded, and, since  $f_0 \in \mathcal{C}^1$  with  $f_0(0) = 0$ ,  $f_0(p) = \underset{p \rightarrow 0}{O}(|p|)$  and  $g(p) = A(p)f_0(p) = \underset{p \rightarrow 0}{O}(|p|)$ . Then, one proceeds by induction. Assuming Lemma 10.9 holds for some  $\nu \in \mathbb{N}$ , let us prove it at step  $\nu + 1$ . Using Taylor's formula

$$g(p) = g(0) + \int_0^1 (Dg(sp))p \, ds. \quad (10.23)$$

Moreover,  $Dg = (DA)f_0 + A(Df_0)$ , where  $DA \in \mathcal{C}^\nu$ ,  $f_0 \in \mathcal{C}^{\nu+2}$ ,  $ADf_0 \in \mathcal{C}^{\nu+1}$ . In particular, the induction assumption applies and

$$Dg(sp) = (\mathbb{T}_\nu(Dg))(sp) + \underset{p \rightarrow 0}{O}(|sp|^{\nu+1}). \quad (10.24)$$

Combining both equalities yields

$$g(p) = g(0) + \int_0^1 (\mathbb{T}_\nu(Dg))(sp)p \, ds + \underset{p \rightarrow 0}{O}(|p|^{\nu+2}) = (\mathbb{T}_{\nu+1}g)(p) + \underset{p \rightarrow 0}{O}(|p|^{\nu+2}), \quad (10.25)$$

which concludes the proof.  $\square$

## A Proofs of technical results and estimates

### A.1 On the differences between STLC definitions

In this paragraph, we prove the claim made in Section 1.2 concerning the invalidity of the three reciprocal implications in (1.3). For  $k \in \{3, 4, 5\}$ , consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2^2 - x_1^k. \end{cases} \quad (\text{A.1})$$

**Proposition A.1.** *The following results hold for system (A.1):*

- For  $k = 3$ , it is  $L^\infty$ -STLC but not  $W^{1,\infty}$ -STLC.
- For  $k = 4$ , it is not  $L^\infty$ -STLC, but there exists  $\rho > 0$  such that it is  $\rho$ -bounded-STLC.
- For  $k = 5$ , it is  $W^{-1,\infty}$ -STLC, but not  $\rho$ -bounded-STLC for any  $\rho > 0$ .

*Proof. Step 1. Case  $k = 3$ .* With  $k = 3$ , the  $L^\infty$ -STLC of system (A.1) follows from the same arguments as for system (7.1) (with the opposite sign), studied in Section 7.1. The fact that it is not  $W^{1,\infty}$ -STLC follows from Theorem 1.11 with  $m = 1$  and  $k = 2$  (or [5, Theorem 3]), which states that  $f_{W_3}(0) \in S_1(f)(0)$  is a necessary condition of  $W^{1,\infty}$ -STLC (see also [5, Section 2.3.3]).

*Step 2. Case  $k = 4$ .* Both claims are proved in Section 6.7 concerning system (6.27).

*Step 3. Case  $k = 5$ .* The fact that system (A.1) is  $W^{-1,\infty}$ -STLC for  $k = 5$  follows from Sussmann's  $\mathcal{S}(\theta)$  condition<sup>4</sup> (see [34, Theorem 7.3] or [10, Theorem 3.29]) with  $\theta > 3/2$  (here  $f_{W_2}(0)$  of type (2,3) is compensated by  $f_{R_{1,1,1,1,0}}(0)$  of type (5,1)). It can also be proved explicitly using oscillating controls as in [5, Section 2.4.1] or Section 6.7.

Let  $\rho > 0$ . Let us prove that (A.1) is not  $\rho$ -bounded-STLC. Let  $u \in L^\infty((0, T); \mathbb{R})$  with  $\|u\|_{L^\infty} \leq \rho$  such that  $x_1(T; u) = 0$ ,

$$\int_0^T u_1^4 = -3 \int_0^T uu_1^2u_2 \leq 3\rho \left( \int_0^T u_1^4 \int_0^T u_2^2 \right)^{\frac{1}{2}}. \quad (\text{A.2})$$

Thus

$$\left| \int_0^T u_1^5 \right| \leq \|u_1\|_{L^\infty} \int_0^T u_1^4 \leq \rho T \|u_1\|_{L^4}^4 \leq 9\rho^3 T \int_0^T u_2^2. \quad (\text{A.3})$$

In particular, if  $T \leq 1/(9\rho^3)$ , one has  $x_3(T; u) \geq 0$  and the system is not controllable.  $\square$

### A.2 On the structure of “bad” brackets

In this paragraph, we prove the claim made in Section 1.5 that, due to the structure of the brackets  $\text{ad}_{X_1}^{2k}(X_0)$ ,  $W_k$  and  $D := \text{ad}_{P_{1,1}}^2(X_0)$ , they are always required to be compensated by the Agrachev–Gamkrelidze sufficient condition of Theorem 8.2. Since they are of type (even, odd), this follows from the following claims.

**Lemma A.2.** *Let  $\Pi^1 \subset \text{Br}(X)$  as in Theorem 8.2. Then  $X_0 \in \Pi^1$ .*

**Lemma A.3.** *Let  $\Pi^1 \subset \text{Br}(X)$  as in Theorem 8.2. Let  $k \in \mathbb{N}^*$ . Then*

$$\mathbb{E}(\text{ad}_{X_1}^{2k}(X_0)) \notin \text{span}\{\mathbb{E}(\pi); \pi \in \Pi^{\text{even}}\} \quad \text{in } \mathcal{L}(X). \quad (\text{A.4})$$

<sup>4</sup>Sussmann's initial proof requires  $\theta \leq 1$ , but, if one is only interested in  $W^{-1,\infty}$ -STLC, it is possible to work with any  $\theta \in [0, +\infty)$ .

*Proof.* By contradiction, assume that there exist  $r \in \mathbb{N}^*$ ,  $\pi_1, \dots, \pi_r \in \Pi^{\text{even}}$  and  $\alpha_1, \dots, \alpha_r \in \mathbb{R}$  such that  $\mathbb{E}(\text{ad}_{X_1}^{2k}(X_0)) = \alpha_1 \mathbb{E}(\pi_1) + \dots + \alpha_r \mathbb{E}(\pi_r)$  in  $\mathcal{L}(X)$ . Thus, there exists  $\pi = \pi_j \in \Pi^{\text{even}}$  such that  $\text{ad}_{X_1}^{2k}(X_0) \in \text{supp}_{\mathcal{B}^*} \mathbb{E}(\pi)$ . Since  $n_0(\pi) = 1$  and  $n_0(\pi) = 2k$ ,  $\mathbb{E}(\pi) = \pm \mathbb{E}(\text{ad}_{X_1}^{2k}(X_0))$ . Since  $\mathbb{E}(\pi) \neq 0$  and  $n_0(\pi) = 1$ , one has  $\lambda(\pi) = X_1$  or  $\mu(\pi) = X_1$ . Thus  $X_1 \in \Pi^1$ , and  $\text{ad}_{X_1}^{2k}(X_0) \in \Pi^{1+2k} \subset \Pi^{\text{odd}}$ , which contradicts the initial assumption since  $\Pi^1$  freely generates  $\text{Lie}(\Pi^1)$ .  $\square$

**Lemma A.4.** *Let  $\Pi^1 \subset \text{Br}(X)$  as in Theorem 8.2. Let  $k \in \mathbb{N}^*$ . Then*

$$\mathbb{E}(W_k) \notin \text{span}\{\mathbb{E}(\pi); \pi \in \Pi^{\text{even}}\} \quad \text{in } \mathcal{L}(X). \quad (\text{A.5})$$

*Proof.* By contradiction, let  $r \in \mathbb{N}^*$ ,  $\pi_1, \dots, \pi_r \in \Pi^{\text{even}}$  and  $\alpha_1, \dots, \alpha_r \in \mathbb{R}$  such that  $\mathbb{E}(W_k) = \alpha_1 \mathbb{E}(\pi_1) + \dots + \alpha_r \mathbb{E}(\pi_r)$  in  $\mathcal{L}(X)$ . Thus, there exists  $\pi = \pi_j \in \Pi^{\text{even}}$  such that  $W_k \in \text{supp}_{\mathcal{B}^*} \mathbb{E}(\pi)$ . Since  $W_k$  is a germ (see Definition 3.1), one cannot have  $\lambda(\pi) = X_0$  or  $\mu(\pi) = X_0$ . Thus  $n_1(\lambda(\pi)) = n_1(\mu(\pi)) = 1$ . This implies the existence of a unique  $l \in \llbracket 0, k-1 \rrbracket$  and  $\pi^* \in \Pi^1$  such that  $\mathbb{E}(\pi^*) = \pm \mathbb{E}(M_l)$ . Thus  $W_k \in \text{span}\{\mathbb{E}(\pi); \pi \in \Pi^{1+2(k-l)} \subset \Pi^{\text{odd}}\}$ , which contradicts the initial assumption since  $\Pi^1$  freely generates  $\text{Lie}(\Pi^1)$ .  $\square$

**Lemma A.5.** *Let  $\Pi^1 \subset \text{Br}(X)$  as in Theorem 8.2. Then*

$$\mathbb{E}(D) \notin \text{span}\{\mathbb{E}(\pi); \pi \in \Pi^{\text{even}}\} \quad \text{in } \mathcal{L}(X). \quad (\text{A.6})$$

*Proof.* By contradiction, let  $r \in \mathbb{N}^*$ ,  $\pi_1, \dots, \pi_r \in \Pi^{\text{even}}$  and  $\alpha_1, \dots, \alpha_r \in \mathbb{R}$  such that  $\mathbb{E}(D) = \alpha_1 \mathbb{E}(\pi_1) + \dots + \alpha_r \mathbb{E}(\pi_r)$  in  $\mathcal{L}(X)$ . Thus, there exists  $\pi = \pi_j \in \Pi^{\text{even}}$  such that  $D \in \text{supp}_{\mathcal{B}^*} \mathbb{E}(\pi)$ . Thus  $n_1(\pi) = 6$  and  $n_0(\pi) = 3$ . We write  $\pi = (a, b)$ . Since  $D$  is a germ (see Definition 3.1) and  $\mathcal{B}^*$  is stable by bracketting with  $X_0$ , one cannot have  $a = X_0$  or  $b = X_0$ . By symmetry, one can assume  $1 \leq n_1(a) \leq n_1(b) \leq 5$ .

**Step 1.** *Case  $n_1(a) = 1$  and  $n_1(b) = 5$ .* If  $n_0(a) \geq 1$ , up to decomposing  $a$  and  $b$  on  $\mathcal{B}^*$ , by Lemma 9.5,  $D \notin \text{supp}_{\mathcal{B}^*} \mathbb{E}((a, b))$ , which contradicts  $D \in \text{supp}_{\mathcal{B}^*} \mathbb{E}(\pi)$ . Thus  $a = X_1 \in \Pi^1$  and  $\Pi^1 = \{X_0, X_1\}$ . Hence  $D \in \text{span}\{\mathbb{E}(\pi); \pi \in \Pi^9 \subset \Pi^{\text{odd}}\}$  which contradicts the initial assumption since  $\Pi^1$  freely generates  $\text{Lie}(\Pi^1)$ .

**Step 2.** *Case  $n_1(a) = 2$  and  $n_1(b) = 4$ .* Then, up to decomposing  $a$  and  $b$  on  $\mathcal{B}^*$ , by Lemma 9.4,  $D \notin \text{supp}_{\mathcal{B}^*} \mathbb{E}((a, b))$ , which contradicts  $D \in \text{supp}_{\mathcal{B}^*} \mathbb{E}(\pi)$ . Hence this case doesn't happen.

**Step 3.** *Case  $n_1(a) = n_1(b) = 3$ .* By symmetry, one can assume that  $n_0(a) = 1$  and  $n_0(b) = 2$ . Thus  $\mathbb{E}(a) = \pm \mathbb{E}(P_{1,1})$  and  $\mathbb{E}(b) = \pm \mathbb{E}(P_{1,2})$  or  $\mathbb{E}(b) = \pm \mathbb{E}((P_{1,1}, X_0))$ . Since  $(P_{1,1}, P_{1,2}) \in \mathcal{B}^*$  and  $D \in \text{supp}_{\mathcal{B}^*} \mathbb{E}(\pi)$ , we are in the second case, i.e.  $\pi = (P_{1,1}, (P_{1,1}, X_0))$  (or a permutation thereof). In particular  $\mathbb{E}(\pi) = \pm \mathbb{E}(D)$ . Thus  $P_{1,1} = \lambda(\pi)$  (or a permutation thereof) belongs to  $\Pi^1$ , and thus  $\pi \in \Pi^3$ , which contradicts the initial assumption.  $\square$

### A.3 Universal rough estimate for coordinates of the second kind

*Proof of Proposition 3.9.* The proof is by induction on  $k \in \mathbb{N}^*$ .

**Step 1.** *Case  $k = 1$ .* Then  $b = X_1 0^\nu$  for some  $\nu \geq 1$  and  $|b| = \nu + 1$ . Thus, for every  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,

$$|\xi_b(t, u)| = \left| \int_0^t \frac{(t-s)^{\nu-1}}{(\nu-1)!} u_1(s) \, ds \right| \leq \frac{t^{\nu-1}}{(\nu-1)!} \|u_1\|_{L^1} \leq 2 \frac{(2t)^{\nu+1}}{(\nu+1)!} t^{-2} \|u_1\|_{L^1}, \quad (\text{A.7})$$

which gives the conclusion with  $c(1) := 4$ .

**Step 2.** *Case  $k \geq 2$ .* To simplify notations, we write  $c$  instead of  $c(k-1)$  and, without loss of generality, we assume that  $1 \leq c(1) \leq \dots \leq c(k-1) = c$ . Let  $b \in \mathcal{B}^* \setminus \{X_1\}$  with  $n_1(b) = k$ . Then  $b = b^* 0^\nu$  for some  $\nu \geq 0$  and there exists  $j \in \mathbb{N}^*$ ,  $m_1, \dots, m_j \in \mathbb{N}^*$ ,  $m \in \mathbb{N}$  and  $b_1 > \dots >$

$b_j > X_1 \in \mathcal{B}_{[[1, k-1]]}^*$  such that  $b^* = \text{ad}_{b_1}^{m_1} \dots \text{ad}_{b_j}^{m_j} \text{ad}_{X_1}^m(X_0)$ . In particular,  $k = n_1(b) = n_1(b^*) = m_1 n_1(b_1) + \dots + m_j n_1(b_j) + m$  and  $|b| = m_1 |b_1| + \dots + m_j |b_j| + m + \nu + 1$ .

First, for each  $i \in [[1, j]]$ , using the induction assumption and Hölder's inequality,

$$|\xi_{b_i}(t, u)| \leq \frac{(ct)^{|b_i|}}{|b_i|!} t^{-n_1(b_i)(1+\frac{1}{k})} \|u_1\|_{L^k}^{n_1(b_i)}. \quad (\text{A.8})$$

Thus, by (2.5),

$$\begin{aligned} |\xi_{b^*}(t, u)| &= \left| \int_0^t \frac{\xi_{b_1}^{m_1}(s, u)}{m_1!} \dots \frac{\xi_{b_j}^{m_j}(s, u)}{m_j!} \frac{u_1^m(s)}{m!} ds \right| \\ &\leq \frac{(ct)^{1+m+\sum m_i |b_i|}}{m! |b_1|^{m_1} \dots |b_j|^{m_j}} t^{-(m+\sum m_i n_1(b_i))(1+\frac{1}{k})} \|u_1\|_{L^k}^{m+\sum m_i n_1(b_i)} \\ &\leq \frac{(2^{k+1}ct)^{|b|}}{|b|!} t^{-(1+k)} \|u_1\|_{L^k}^k, \end{aligned} \quad (\text{A.9})$$

where we used  $\|u_1\|_{L^m}^m \leq t^{1+m} t^{-m(1+\frac{1}{k})} \|u_1\|_{L^k}^m$  and

$$|b|! = \left( 1 + m + \sum_{i=1}^j m_i |b_i| \right)! \leq 2^{(\sum m_i + 2) - 1} 1! m! \prod_{i=1}^j |b_i|^{m_i} \quad (\text{A.10})$$

which follows from (4.1) and the estimate  $m_1 + \dots + m_j \leq k$ .

Finally, if  $\nu \geq 1$ , using Lemma 3.6 and (4.1),

$$\begin{aligned} |\xi_b(t, u)| &= \left| \int_0^t \frac{(t-s)^{\nu-1}}{(\nu-1)!} \xi_{b^*}(s, u) ds \right| \\ &\leq \frac{t^\nu (2^{k+1}ct)^{|b^*|}}{\nu! |b^*|!} t^{-(1+k)} \|u_1\|_{L^k}^k \\ &\leq \frac{(2^{k+2}ct)^{|b|}}{|b|!} t^{-(1+k)} \|u_1\|_{L^k}^k, \end{aligned} \quad (\text{A.11})$$

which gives the conclusion with  $c(k) := 2^{k+2}c$ .  $\square$

#### A.4 Precise estimates of coordinates up to the fifth order

We start with an elementary estimate.

**Lemma A.6.** *For every  $p \in [1, \infty]$ ,  $j_0 \leq j \in \mathbb{N}^*$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ ,*

$$\|u_j\|_{L^p} \leq \frac{t^{j-j_0}}{(j-j_0)!} \|u_{j_0}\|_{L^p}. \quad (\text{A.12})$$

*Proof.* One can assume  $j > j_0$ . By definition,  $u_j$  is the  $(j-j_0)$ -th primitive of  $u_{j_0}$  vanishing iteratively at zero, i.e.

$$u_j(s) = \int_0^s \frac{(s-s')^{j-j_0-1}}{(j-j_0-1)!} u_{j_0}(s') ds'. \quad (\text{A.13})$$

Thus  $u_j = g_{j-j_0-1} * \bar{u}_{j_0}$ , where  $\bar{u}_{j_0}$  is the extension of  $u_{j_0}$  from  $(0, t)$  to  $\mathbb{R}$  by zero and  $g_\nu(s) := s^\nu / \nu!$  for  $s \in (0, t)$  and 0 elsewhere, so that  $\|g_\nu\|_{L^1} = t^{\nu+1} / (\nu+1)!$ . Hence, (A.12) follows from Young's convolution inequality.  $\square$

This leads to the following estimates.



*Proof of Proposition 3.10.* We prove the bounds one by one.

1. By (3.11), Hölder's inequality, (A.12) and (4.1),

$$\begin{aligned}
|\xi_{M_j}(t, u)| &\leq \|u_j\|_{L^1} \leq t^{1-\frac{1}{p}} \|u_j\|_{L^p} \\
&\leq \frac{t^{j-j_0}}{(j-j_0)!} t^{1-\frac{1}{p}} \|u_{j_0}\|_{L^p} \\
&\leq (j_0+1)! \frac{(2t)^{j+1}}{(j+1)!} t^{-(j_0+1)} t^{1-\frac{1}{p}} \|u_{j_0}\|_{L^p},
\end{aligned} \tag{A.14}$$

which proves (3.23) with  $c := 2(j_0+1)!$  since  $|M_j| = j+1$ .

2. By (3.12), Hölder's inequality, (A.12) and (4.1),

$$\begin{aligned}
|\xi_{W_{j,\nu}}(t, u)| &\leq \frac{t^\nu}{\nu!} t^{1-\frac{1}{p}} \|u_j\|_{L^{2p}}^2 \leq \frac{t^\nu}{\nu!} \frac{t^{2(j-j_0)}}{(j-j_0)!^2} t^{1-\frac{1}{p}} \|u_{j_0}\|_{L^{2p}}^2 \\
&\leq (2j_0+1)! \frac{(2^3 t)^{2j+\nu+1}}{(2j+\nu+1)!} t^{-(2j_0+1)} t^{1-\frac{1}{p}} \|u_{j_0}\|_{L^{2p}}^2,
\end{aligned} \tag{A.15}$$

which proves (3.24) with  $c := 2^2(2j_0+1)!$  since  $|W_{j,\nu}| = 2j+\nu+1$ .

3. For (3.25), we proceed as in the second item, starting from (3.13).
4. For (3.26), we proceed as in the second item, starting from (3.14).
5. By (3.15) and (3.24), there exists  $c_2 > 0$  such that

$$\begin{aligned}
|\xi_{Q_{j,\mu,\nu}^b}(t, u)| &= \frac{1}{2} \left| \int_0^t \frac{(t-s)^\nu}{\nu!} \xi_{W_{j,\mu}}^2(s, u) ds \right| \\
&\leq \frac{t^{\nu+1}}{(\nu+1)!} \left( \frac{(c_2 t)^{|W_{j,\mu}|}}{|W_{j,\mu}|!} t^{-(2j_0+1)} t^{1-\frac{1}{p}} \|u_{j_0}\|_{L^{2p}}^2 \right)^2 \\
&\leq \frac{(2^2 c_2 t)^{2|W_{j,\mu}|+\nu+1}}{(2|W_{j,\mu}|+\nu+1)!} t^{-(4j_0+3)} t^{3-\frac{2}{p}} \|u_{j_0}\|_{L^{2p}}^4
\end{aligned} \tag{A.16}$$

using (4.1), which proves (3.27) with  $c := 2^2 c_2$  since  $|Q_{j,\mu,\nu}^b| = 2|W_{j,\mu}| + \nu + 1$ .

6. By (3.16), (A.12) and (3.24), there exists  $c_2 > 0$  such that

$$\begin{aligned}
|\xi_{Q_{j,\mu,k,\nu}^{\sharp}}(t, u)| &= \frac{1}{2} \left| \int_0^t \frac{(t-s)^\nu}{\nu!} \xi_{W_{j,\mu}}(s, u) u_k^2(s) ds \right| \\
&\leq \frac{t^\nu}{\nu!} \left( \frac{(c_2 t)^{|W_{j,\mu}|}}{|W_{j,\mu}|!} t^{-(2j_0+1)} t^{1-\frac{1}{p_1}} \|u_{j_0}\|_{L^{2p_1}}^2 \right) t^{1-\frac{1}{p_2}} \left( \frac{t^{k-k_0}}{(k-k_0)!} \|u_{k_0}\|_{L^{2p_2}} \right)^2 \\
&\leq (2k_0+1)! \frac{(2^4 c_2 t)^{|W_{j,\mu}|+2k+\nu+1}}{(|W_{j,\mu}|+2k+\nu+1)!} \\
&\quad \times t^{-(2j_0+2k_0+2)} t^{2-\frac{1}{p_1}-\frac{1}{p_2}} \|u_{j_0}\|_{L^{2p_1}}^2 \|u_{k_0}\|_{L^{2p_2}}^2
\end{aligned} \tag{A.17}$$

using (4.1), which proves (3.27) with  $c := 2^4 c_2 (2k_0+1)!$  since  $|Q_{j,\mu,k,\nu}^{\sharp}| = 2|W_{j,\mu}| + 2k + \nu + 1$ .

7. For (3.29), we proceed as in the second item, starting from (3.17).

8. By (3.18), Hölder's inequality, (A.12), and (3.24), there exists  $c_2 > 0$  such that

$$\begin{aligned}
|\xi_{R_{j,k,l,\mu,\nu}^\sharp}(t, u)| &= \alpha_{j,k} \left| \int_0^t \frac{(t-s)^\nu}{\nu!} \xi_{W_{l,\mu}}(s, u) u_k(s) u_j^2(s) ds \right| \\
&\leq \frac{t^\nu}{\nu!} \left( \frac{(c_2 t)^{|W_{l,\mu}|}}{|W_{l,\mu}|!} t^{-(2l_0+1)} t^{1-\frac{1}{p}} \|u_{l_0}\|_{L^{2p}}^2 \right) t^{1-\frac{1}{p_1}-\frac{1}{p_2}} \\
&\quad \times \left( \frac{t^{j-j_0}}{(j-j_0)!} \|u_{j_0}\|_{L^{2p_1}} \right)^2 \frac{t^{k-k_0}}{(k-k_0)!} \|u_{k_0}\|_{L^{p_2}} \quad (\text{A.18}) \\
&\leq (2j_0 + k_0 + 1)! \frac{(2^4 c_2) t^{|W_{l,\mu}|+2j+k+\nu+1}}{(|W_{l,\mu}| + 2j + k + \nu + 1)!} t^{-(2l_0+2j_0+k_0+2)} \\
&\quad \times t^{2-\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2}} \|u_{l_0}\|_{L^{2p}}^2 \|u_{j_0}\|_{L^{2p_1}}^2 \|u_{k_0}\|_{L^{p_2}}
\end{aligned}$$

using (4.1), which proves (3.30) with  $c := 2^4 c_2 (2j_0 + k_0 + 1)!$  since  $|R_{j,k,l,\mu,\nu}^\sharp| = |W_{l,\mu}| + 2j + k\nu + 1$ .  $\square$

## A.5 Expression of the logarithm from coordinates of the second kind

*Proof of Lemma 4.5.* For  $M, \ell \in \mathbb{N}^*$ ,  $t > 0$  and  $u \in L^1((0, t); \mathbb{R})$ , let

$$\mathcal{Z}_M^\ell(t, X, u) := \sum_{\substack{b \in \mathcal{B}^* \\ n_1(b) \leq M, |b| \leq \ell}} \eta_b(t, u) \mathbb{E}(b). \quad (\text{A.19})$$

In particular, the sum in the right-hand side is finite since  $|\{b \in \mathcal{B}^*; |b| \leq \ell\}| \leq 2^\ell$ . By identification in (2.10), one has

$$\mathcal{Z}_M^\ell(t, X, u) = \sum_{\substack{q \in \mathbb{N}^*, h \in (\mathbb{N}^*)^q \\ b_1 > \dots > b_q \in \mathcal{B}^* \setminus \{X_0\} \\ h_1 n_1(b_1) + \dots + h_q n_1(b_q) \leq M \\ h_1 |b_1| + \dots + h_q |b_q| \leq \ell}} \xi_{b_1}^{h_1}(t, u) \cdots \xi_{b_q}^{h_q}(t, u) F_{q,h}(b_1, \dots, b_q). \quad (\text{A.20})$$

The sum in the right-hand side is finite since the constraints imply that  $q \leq M$ ,  $h_1, \dots, h_q \leq M$  and  $|b_1|, \dots, |b_q| \leq \ell$ . Applying the homomorphism of Lie algebras which sends  $X_0$  to  $f_0$  and  $X_1$  to  $f_1$ , we therefore obtain that

$$\sum_{\substack{b \in \mathcal{B}^* \\ n_1(b) \leq M, |b| \leq \ell}} \eta_b(t, u) f_b = \sum_{\substack{q \in \mathbb{N}^*, h \in (\mathbb{N}^*)^q \\ b_1 > \dots > b_q \in \mathcal{B}^* \setminus \{X_0\} \\ h_1 n_1(b_1) + \dots + h_q n_1(b_q) \leq M \\ h_1 |b_1| + \dots + h_q |b_q| \leq \ell}} \xi_{b_1}^{h_1}(t, u) \cdots \xi_{b_q}^{h_q}(t, u) f_{F_{q,h}(b_1, \dots, b_q)}. \quad (\text{A.21})$$

Moreover, by Theorem 4.4, the sum (4.7) converges absolutely in  $\mathcal{C}^{\omega, r'}$  for every  $r' \in [r/e, r)$ . In particular,  $\mathcal{Z}_M^\ell(t, f, u) \rightarrow \mathcal{Z}_M(t, f, u)$  in  $\mathcal{C}^{\omega, r'}$  as  $\ell \rightarrow +\infty$ . Hence, to obtain (4.8), it is sufficient to prove that its right-hand side converges absolutely in  $\mathcal{C}^{\omega, r'}$ .

For each  $q \in \mathbb{N}^*$  and  $h \in (\mathbb{N}^*)^q$ , there exists a finite subset  $\mathcal{A} \subset \text{Br}(\{Y_1, \dots, Y_q\})$  and coefficients  $(\alpha_a)_{a \in \mathcal{A}}$  such that  $F_{q,h}(Y_1, \dots, Y_q) = \sum \alpha_a \mathbb{E}(a)$ . Let  $\|F_{q,h}\| := \sum |\alpha_a|$ . Since the set of considered  $q$  and  $h$  is finite,  $\|F_{q,h}\| \leq C_F$  for some uniform bound  $C_F > 0$ .

Then, by Lemma 4.2,

$$\| \|f_{F_{q,h}(b_1, \dots, b_q)} \| \|_{r'} \leq C_F \frac{r}{9} (\ell - 1)! \left( \frac{9 \| \|f \| \|_{r'} \|}{r} \right)^\ell \quad (\text{A.22})$$

where  $\ell := h_1|b_1| + \dots + h_q|b_q|$ . Moreover, by Proposition 3.9, there exists  $c = c(M) > 0$  such that, for every  $b \in \mathcal{B}^* \setminus \{X_0\}$ ,

$$|\xi_b(t, u)| \leq \frac{1}{|b|!} (ct)^{n_0(b)} (c\|u_1\|_{L^\infty})^{n_1(b)}. \quad (\text{A.23})$$

Thus, by (4.1),

$$|\xi_{b_1}^{h_1}(t, u) \cdots \xi_{b_q}^{h_q}(t, u)| \leq \frac{(2^M)^\ell}{\ell!} (ct)^{N_0} (c\|u_1\|_{L^\infty})^{N_1} \quad (\text{A.24})$$

where  $\ell := h_1|b_1| + \dots + h_q|b_q|$  and  $N_i := h_1 n_i(b_1) + \dots + h_q n_i(b_q)$ .

Hence,

$$\begin{aligned} & \sum_{\substack{q \in \mathbb{N}^*, h \in (\mathbb{N}^*)^q \\ b_1 > \dots > b_q \in \mathcal{B}^* \setminus \{X_0\} \\ h_1 n_1(b_1) + \dots + h_q n_1(b_q) \leq M}} \left\| \xi_{b_1}^{h_1}(t, u) \cdots \xi_{b_q}^{h_q}(t, u) f_{F_{q,h}(b_1, \dots, b_q)} \right\|_r \\ & \leq \sum_{\substack{q \in \mathbb{N}^*, h \in (\mathbb{N}^*)^q \\ b_1 > \dots > b_q \in \mathcal{B}^* \setminus \{X_0\} \\ h_1 n_1(b_1) + \dots + h_q n_1(b_q) \leq M}} \frac{r C_F}{9} (2^M)^\ell (ct)^{N_0} (c\|u_1\|_{L^\infty})^{N_1} \left( \frac{9 \|f\|_r}{r} \right)^\ell \end{aligned} \quad (\text{A.25})$$

Since there are at most  $M$  choices for  $q$ ,  $M^M$  choices for  $h$  and  $(2^\ell)^M$  choices for the  $b_i$ , the sum is bounded by

$$\sum_{N_1=1}^M \sum_{\ell=N_1}^{+\infty} M M^M (2^M)^\ell \frac{r C_F}{9} (2^M)^\ell (ct)^{\ell-N_1} (c\|u_1\|_{L^\infty})^{N_1} \left( \frac{9 \|f\|_r}{r} \right)^\ell \quad (\text{A.26})$$

and this sum is finite as soon as  $4^M 9ct \|f\|_r < r$ .  $\square$

## A.6 Black-box result for the dominant part of the logarithm

*Proof of Proposition 4.6.* Let  $r, \delta > 0$  such that  $f_0, f_1 \in \mathcal{C}^{\omega, r}(B_\delta; \mathbb{R}^d)$ . The proof relies on the formula (4.8) of Lemma 4.5. Through the component  $\mathbb{P}$  along  $f_b(0)$  parallel to  $\mathcal{N}(f)(0)$ , all the terms on which we have not made any assumption vanish. It thus suffices to check that the estimates for the remaining ones can indeed be summed. First, for  $q = 1$ , using (4.9) and (4.3),

$$\begin{aligned} \sum_{\substack{b \in \mathcal{B}_{[1, M]}^* \setminus \mathcal{N} \\ b \neq \mathbf{b}}} |\xi_b(t, u) \mathbb{P} f_b(0)| & \leq \sum_{\sigma=1}^L \sum_{\ell=\sigma}^{+\infty} \sum_{\substack{b \in \mathcal{B}_{[1, M]}^* \setminus \mathcal{N} \\ b \neq \mathbf{b}, |b|=\ell}} \frac{(ct)^\ell}{\ell!} t^{-\sigma} \Xi(t, u) \frac{r}{9} (\ell-1)! \left( \frac{9 \|f\|_r}{r} \right)^\ell \\ & \leq \frac{r}{9} \Xi(t, u) \sum_{\sigma=1}^L \sum_{\ell=\sigma}^{+\infty} \left( \frac{18ct \|f\|_r}{r} \right)^\ell t^{-\sigma} \end{aligned} \quad (\text{A.27})$$

which converges provided that  $18ct \|f\|_r < r$ , and is then bounded by  $C\Xi(t, u)$  for an appropriate constant depending on  $r, c$  and  $\|f\|_r$ .

For  $q \geq 2$ , we proceed similarly, using (A.22) and (4.10).  $\square$

## A.7 Bad-bad limiting examples involving $W_3$ versus $Q^b$

As announced in Section 8.1.2, in this paragraph, we give examples illustrating that one must include  $Q_{1,0}^b, Q_{1,1}^b$  and  $Q_{1,2}^b$  in the list  $\mathcal{N}_3^b$  of (1.23).

**Limiting example for  $Q_{1,0}^b$ .** Consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 + x_1^2 \\ \dot{x}_4 = x_3 \\ \dot{x}_5 = x_3^2 + 2x_1^2x_4. \end{cases} \quad (\text{A.28})$$

Written in the form (1.1), this system satisfies

$$\begin{aligned} f_{M_{i-1}}(0) &= e_i \text{ for } i \in \llbracket 1, 4 \rrbracket, & f_{W_1}(0) &= 2e_3, & f_{W_{1,1}}(0) &= 2e_4, \\ f_{Q_{1,0}^b}(0) &= -8e_5, & f_{W_3}(0) &= 2e_5 \end{aligned} \quad (\text{A.29})$$

and  $f_b(0) = 0$  for any other  $b \in \mathcal{B}^*$ .

**Proposition A.7.** *System (A.28) is  $L^\infty$ -STLC (but not  $W^{1,\infty}$ -STLC).*

*Proof.* By Theorem 1.11, if a system is  $W^{1,\infty}$ -STLC, then  $f_{W_3}(0) \in S_3(f)(0)$ . Since this condition is not satisfied by system (A.28), it is not  $W^{1,\infty}$ -STLC. We now prove that it is  $L^\infty$ -STLC.

**Step 1.** *Computation of the state.* Let us fix  $T > 0$ . Explicit integrations lead to  $x_1(T) = u_1(T)$ ,  $x_2(T) = u_2(T)$  and

$$\begin{aligned} x_3(T) &= u_3(T) + \int_0^T u_1^2(t) dt, \\ x_4(T) &= u_4(T) + \int_0^T (T-t)u_1^2(t) dt, \\ x_5(T) &= \int_0^T u_3^2(t) dt - \int_0^T \left( \int_0^t u_1^2(s) ds \right)^2 dt + 2x_4(T) \int_0^T u_1^2(t) dt. \end{aligned} \quad (\text{A.30})$$

**Step 2.** *Motions in the linear directions.* Let  $i \in \llbracket 1, 4 \rrbracket$ . By the usual linear theory, there exists  $\bar{u}^i \in L^\infty((0, T); \mathbb{R})$  such that, for  $a \in [-1, 1]$ ,  $x(T; a\bar{u}^i) = ae_i + O(a^2)$ .

**Step 3.** *Motion in the easy quadratic direction  $+e_5$ .* Take a non-zero function  $\chi \in C_c^\infty((0, T); \mathbb{R})$ , normalized such that  $\|\chi^{(1)}\|_{L^2} = 1$  and define, for  $a \in [0, 1]$ ,

$$\mathcal{U}_{e_5}(a) := a\chi^{(4)} - a^2 \left( \int_0^T (\chi^{(3)})^2(t) dt \right) \bar{u}^3 - a^2 \left( \int_0^T (T-t)(\chi^{(3)})^2(t) dt \right) \bar{u}^4. \quad (\text{A.31})$$

Using (A.30), one checks that  $x(T, \mathcal{U}_{e_5}(a)) = a^2e_5 + O(a^3)$ .

**Step 4.** *Motion in the difficult quartic direction  $-e_5$ .* In order to benefit from the quartic term, we use oscillating controls. Let  $\phi \in C^\infty(\mathbb{R}; \mathbb{R})$  be a fixed non-zero  $T$ -periodic function with  $\phi(0) = \phi'(0) = \phi''(0) = 0$ ,  $\langle \phi \rangle = 0$  and  $\langle \phi^2 \rangle = \frac{1}{4}$ . Let  $\chi \in C_c^\infty((0, T]; \mathbb{R})$  such that  $\chi'(T) = \chi''(T) = 0$  to be chosen later. For  $a \in [0, 1]$  small enough, we use controls of the form

$$u_3(t) = a^4\chi(t) (1 + \phi((T-t)/a)). \quad (\text{A.32})$$

In particular,  $u_i(0) = u_i(T) = 0$  for  $i = 1, 2$ . Moreover, the map  $(a, \chi) \rightarrow u$  is continuous from  $[0, 1] \times C^3([0, T]; \mathbb{R})$  to  $L^\infty((0, T); \mathbb{R})$  and

$$\|u\|_{L^\infty} = O(a\|\chi\|_{C^3}). \quad (\text{A.33})$$

Differentiating (A.32) twice yields

$$u_1(t) = a^4 \chi''(t) (1 + \phi((T-t)/a)) - 2a^3 \chi'(t) \phi'((T-t)/a) + a^2 \chi(t) \phi''((T-t)/a). \quad (\text{A.34})$$

Hence, heuristically,  $u_1 \approx a^2 \chi \phi''$ . Substituting in (A.30) and using a Riemann–Lebesgue type argument (see Lemma A.10 in Appendix A.8 below) yields

$$x_3(T) = a^4 (G_3(\chi) + F_3(a, \chi)), \quad G_3(\chi) := \chi(T) + \langle (\phi'')^2 \rangle \int_0^T \chi^2 \quad (\text{A.35})$$

$$x_4(T) = a^4 (G_4(\chi) + F_4(a, \chi)), \quad G_4(\chi) := \int_0^T \chi + \langle (\phi'')^2 \rangle \int_0^T (T-t) \chi^2(t) dt \quad (\text{A.36})$$

where  $F_3$  and  $F_4$  are  $C^1$  maps on  $[0, 1] \times C_c^5((0, T]; \mathbb{R})$  with  $F_3(0, \cdot) = F_4(0, \cdot) = 0$ . Similar arguments prove that

$$\begin{aligned} x_5(T) &= a^8 (G_5(\chi) + F_5(a, \chi)) + 2x_4(T) \int_0^T u_1^2, \\ G_5(\chi) &:= \langle (1 + \phi)^2 \rangle \int_0^T \chi^2 - \langle (\phi'')^2 \rangle^2 \int_0^T \left( \int_0^t \chi^2 \right)^2 dt \end{aligned} \quad (\text{A.37})$$

where  $F_5$  is a  $C^1$  map on  $[0, 1] \times C_c^5((0, T]; \mathbb{R})$  with  $F_5(0, \cdot) = 0$ .

We now prove that there exists  $\chi^*$  such that  $G_3(\chi^*) = G_4(\chi^*) = 0$ ,  $G_5(\chi^*) < 0$ , and  $\chi \mapsto (G_3(\chi), G_4(\chi))$  is locally onto. More precisely, given  $0 < \tau \ll 1$ , let  $\bar{\chi} \in C_c^\infty((0, T); [0, 1])$  with  $\bar{\chi} \equiv 1$  on  $[2\tau, T - 2\tau]$ ,  $\text{supp } \bar{\chi} \subset [\tau, T - \tau]$ , and  $\bar{h} \in C_c^\infty([0, 1]; [0, 1])$  with  $\bar{h}'$  compactly supported in  $(0, 1)$  and  $\bar{h}(0) = 1$ . We look for  $\chi$  under the form

$$\chi(t) = c_0 \bar{\chi}(t) + c_1 \bar{h}((T-t)/\tau) \quad (\text{A.38})$$

Let  $\Lambda := \langle (\phi'')^2 \rangle$ . Using the assumptions on  $\phi$ ,

$$\begin{aligned} G_3(\chi) &= c_1 + \Lambda c_0^2 T + O(\tau), \\ G_4(\chi) &= c_0 T + \Lambda c_0^2 \frac{T^2}{2} + O(\tau), \\ G_5(\chi) &= \frac{5}{4} c_0^2 T - \Lambda^2 c_0^4 \frac{T^3}{3} + O(\tau). \end{aligned} \quad (\text{A.39})$$

At  $\tau = 0$ , one checks that, for  $\bar{c}_0 = -2/(\Lambda T)$  and  $\bar{c}_1 = -\Lambda \bar{c}_0 T$ ,  $G_3 = G_4 = 0$  and  $G_5 < 0$ . By the implicit function theorem, there exists  $(c_0^*, c_1^*, \tau^*)$  with  $\tau^* > 0$ ,  $c_0^* \approx \bar{c}_0$  and  $c_1^* \approx \bar{c}_1$  such that  $G_3(\chi^*) = G_4(\chi^*) = 0$  and  $G_5(\chi^*) < 0$ . Moreover, since  $\partial_{c_1}(G_3, G_4)|_{\bar{c}_0, \bar{c}_1, \tau} = (1, 0) + O(\tau)$  and  $\partial_{c_0}(G_3, G_4)|_{\bar{c}_0, \bar{c}_1, \tau} = (0, -T)$ , the  $C^1$  map  $(c_0, c_1) \mapsto (G_3, G_4)(\chi_{c_0, c_1, \tau^*})$  vanishes at  $(c_0^*, c_1^*)$  and its differential is onto.

By the implicit function theorem, there thus exists a  $C^1$  map  $a \mapsto \chi_a$  from  $[0, 1]$  to  $C_c^5((0, T]; \mathbb{R})$  such that  $G_3(\chi_a) + F_3(a, \chi_a) = G_4(\chi_a) + F_4(a, \chi_a) = 0$ , with  $G_5(\chi_0) < 0$ . Thus  $x(T) = a^8 (G_5(\chi_a) + F_5(a, \chi_a)) e_5 = a^8 G_5(\chi_0) e_5 + O(a^9)$ , so one can move in the direction  $-e_5$ .

**Step 5. Conclusion.** Standard arguments using either tangent vectors or power series expansions (see e.g. [18, Appendix] or [10, Section 8.1]) entail that (A.28) is  $L^\infty$ -STLC.  $\square$

**Limiting example for  $Q_{1,1}^b$ .** Consider the system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 + x_1^2 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_3 \\ \dot{x}_5 = x_4 \\ \dot{x}_6 = x_3^2 - 2x_1^2x_5. \end{cases} \quad (\text{A.40})$$

Written in the form (1.1), this system satisfies

$$\begin{aligned} f_{M_{i-1}}(0) &= e_i \text{ for } i \in \llbracket 1, 5 \rrbracket, & f_{W_{1,\nu}}(0) &= 2e_{2+\nu} \text{ for } \nu \in \llbracket 0, 3 \rrbracket, \\ f_{Q_{1,1}^b}(0) &= -8e_6, & f_{W_3}(0) &= 2e_6 \end{aligned} \quad (\text{A.41})$$

and  $f_b(0) = 0$  for any other  $b \in \mathcal{B}^*$ .

**Lemma A.8.** *System (A.40) is  $L^\infty$ -STLC.*

*Proof.* Let us perform the (nonlinear) static-state-feedback transformation  $v(t) := u(t)(1+2x_1(t))$ . Then system (A.40) is mapped to a system which satisfies

$$\begin{aligned} f_{M_{i-1}}(0) &= e_i \text{ for } i \in \llbracket 1, 5 \rrbracket, & f_{W_3}(0) &= 2e_6, & f_{P_{1,5}} &= -4e_6, \\ f_{(M_4, \text{ad}_{X_1}^k(X_0))}(0) &= c_k e_6 \text{ for } k \geq 3 \end{aligned} \quad (\text{A.42})$$

for some constant  $c_k \in \mathbb{R}$  and  $f_b(0) = 0$  for every other  $b \in \mathcal{B}^*$ . In particular, as (8.9) with  $l = 5$ , it mainly features a competition between  $W_3$  and  $P_{1,5,0}$ , which was proved to be  $L^\infty$ -STLC in Section 8.1 using Theorem 8.2. Let us check that we can indeed ignore the added brackets. For  $k \geq 3$ , using the set  $\Pi^1$  of (8.5), one has

$$(M_4, \text{ad}_{X_1}^k(X_0)) = ((\text{ad}_{M_2}(X_0), X_0), \text{ad}_{X_1}^k(X_0)) \in \Pi^3. \quad (\text{A.43})$$

Thus  $\omega((M_4, \text{ad}_{X_1}^k(X_0))) = |(M_4, \text{ad}_{X_1}^k(X_0))| - 3 = 3 + k \geq 6 > 5 = \omega(P_{1,5})$ . Hence, the system after feedback is  $L^\infty$ -STLC, so (A.40) is  $L^\infty$ -STLC.  $\square$

**Limiting example for  $Q_{1,2}^b$ .** Consider the system

$$\begin{cases} \dot{x}_1 = u + x_1^2 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_3 \\ \dot{x}_5 = x_4 \\ \dot{x}_6 = x_5 \\ \dot{x}_7 = x_3^2 + 2x_1^2x_6. \end{cases} \quad (\text{A.44})$$

Unlike the previous examples, this system is not nilpotent. Nevertheless, written in the form (1.1), it satisfies

$$\begin{aligned} f_{W_3}(0) &= 2e_7, & f_{Q_{1,2}^b}(0) &= -8e_7, \\ \forall b \in \mathcal{N}_3 \setminus \{Q_{1,2}^b\}, & f_b(0) \in \mathbb{R}e_1 + \dots + \mathbb{R}e_6. \end{aligned} \quad (\text{A.45})$$

Hence, along the line  $e_7$ , there is only a competition between  $W_3$ ,  $Q_{1,2}^b$ , and brackets outside of  $\mathcal{N}_3$ , so that yield negligible contributions with respect to  $\int u_3^2$ . In particular, it satisfies

$$f_{W_3}(0) \notin (\mathcal{N}_3 \setminus \{Q_{1,2}^b\})(f)(0). \quad (\text{A.46})$$

**Lemma A.9.** *System (A.44) is  $L^\infty$ -STLC.*

*Proof.* Let us perform the (nonlinear) static-state-feedback transformation  $v(t) := u(t) + x_1^2(t)$ . Then system (A.44) is mapped to example (8.9) with  $l = 6$ , featuring a competition between  $W_3$  and  $P_{1,6,0}$ , which was proved to be  $L^\infty$ -STLC in Section 8.1. Hence (A.44) is  $L^\infty$ -STLC.  $\square$

## A.8 A Riemann-Lebesgue type lemma

**Lemma A.10.** *Let  $T > 0$  and  $\mathcal{V} := \{h \in C^3([0, T]; \mathbb{R}); h(0) = h'(0) = h''(0) = 0\}$  endowed with its usual topology. Let  $\theta \in C^0(\mathbb{R}; \mathbb{R})$  be a  $T$ -periodic function. The map*

$$F : \begin{cases} (0, 1] \times \mathcal{V} & \rightarrow \mathbb{R}, \\ (\tau, h) & \mapsto \int_0^T h(t)\theta((T-t)/\tau) dt \end{cases} \quad (\text{A.47})$$

*admits a  $C^1$  extension to  $[0, 1] \times \mathcal{V}$  which satisfies  $F(0, h) = (1/T) \int_0^T \theta \int_0^T h$  for every  $h \in \mathcal{V}$ .*

*Proof.* Let  $\theta_0 := \theta$ . For  $k \in \mathbb{N}$ , we set  $\theta_{k+1}(t) := (t/T) \int_0^T \theta_k - \int_0^t \theta_k$ . In particular, for all  $k \in \mathbb{N}$ ,  $\theta_{k+1}(0) = 0$ ,  $\theta_{k+1}$  is  $T$ -periodic and  $\theta'_{k+1} = \langle \theta_k \rangle - \theta_k$ . Integrating by parts in (A.47) yields

$$\begin{aligned} F(\tau, h) &= \langle \theta \rangle \int_0^T h + \tau [h\theta_1((T-t)/\tau)]_0^T - \tau \int_0^T h'(t)\theta_1((T-t)/\tau) dt \\ &= \langle \theta \rangle \int_0^T h - \tau \int_0^T h'(t)\theta_1((T-t)/\tau) dt. \end{aligned} \quad (\text{A.48})$$

Hence  $F$  admits a continuous extension setting  $F(0, h) = \langle \theta \rangle \int_0^T h$ . Repeating the process twice,

$$F(\tau, h) = \langle \theta \rangle \int_0^T h - \tau \langle \theta_1 \rangle h(T) + \tau^2 \langle \theta_2 \rangle h'(T) - \tau^3 \int_0^T h'''(t)\theta_3((T-t)/\tau) dt. \quad (\text{A.49})$$

This development with respect to  $\tau$  can be used to prove that  $F$  enjoys  $C^1$  regularity up to  $\tau = 0$ , since the  $\tau^3$  factor allows to absorb the derivative with respect to  $\tau$  of the last integral as  $\tau \rightarrow 0$ .  $\square$

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