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ORIGINAL ARTICLE

## Global asymptotic stability for a distributed delay differential-difference system of a Kermack-McKendrick SIR model

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### ABSTRACT

We investigate a system of distributed delay differential-difference equations describing an epidemic model of susceptible, infected, recovered and temporary protected population dynamics. A nonlocal term (distributed delay) appears in this model to describe the temporary protection period of the susceptible individuals. We investigate mathematical properties of the model. We obtain the global asymptotic stability of the two steady states: disease-free and endemic. We construct appropriate Lyapunov functionals where the basic reproduction number appears as a threshold for the global asymptotic behavior of the solution between disease extinction and persistence.

### KEYWORDS

SIR epidemic model; Delay differential-difference system; Lyapunov-Krasovskii functional; Local and global asymptotic stability

### AMS CLASSIFICATION

34K60; 37N25; 92D30

## 1. Introduction

Recently, Adimy *et al.* [1] (see also [2]) proposed and analyzed a new age-structured partial differential system describing the dynamics of an epidemic model of susceptible, infected and recovered individuals. More precisely, they considered an age-structured phase where individuals can be protected from disease for a limited period of time, for example by vaccination or drugs with temporary immunity. In the same way as in their previous works (see [3,4] and subsequent works), this modeling corresponds, in a sense to be defined according to the studied problem, to the fact that there are mainly two phases: an active phase and an inactive one. By integrating the age-structured equation using the method of characteristics, the system can be reduced to distributed delay differential-difference equations. Contrary to the model obtained here, the epidemic system studied in [1] is with a discrete delay. Adimy *et al.* [1], focused on the mathematical analysis of the behavior of the steady states (local and

global asymptotic stability). Our objective in this paper is to further analyze this epidemic model by considering a distribution delay which is more realistic, but also more complicated, than a discrete delay.

The global asymptotic stability of an SIR epidemic model with distributed delay was first studied in [5]. The authors proved the global asymptotic stability of the disease-free steady state when the basic reproduction number  $\mathcal{R}_0 < 1$ . They also gave some sufficient conditions for the global asymptotic stability of the endemic steady-state when  $\mathcal{R}_0 > 1$  (see also [6,7]). In [8], McCluskey proposed a method based on a Lyapunov functional to prove the global asymptotic stability of the endemic steady-state when  $\mathcal{R}_0 > 1$ . His technique has been used to study the global asymptotic stability of a variety of epidemic models such as an SIR model with general nonlinear incidence [9], a multigroup SEIR model [10], and an SIRS model with general nonlinear incidence [11]. The global dynamics and the existence of traveling waves in epidemic models with diffusion and distributed delay have also been studied (see for instance, [12,13]). In most of these studies, the distributed delay was incorporated into the force of infection term.

We consider four compartments of individuals: susceptible  $S(t)$ , infectious  $I(t)$ , recovered  $R(t)$  and protected  $P(t)$ . As in [1], we can consider the density of protected individuals  $p(t, a)$ , where  $a \in (0, \tau)$  is the time since an individual entered this compartment. The length of the protection phase is assumed to be distributed according to a probability density - a probability kernel - denoted by  $g(a)$ , with  $\int_0^\tau g(a)da = 1$ . This last property means that all protected individuals lose their protection after a maximum fixed time  $\tau > 0$ . There is also a minimum period after which an individual is effectively protected. So, we assume that there exists  $\tau_0 \in [0, \tau)$  such that  $g(a) = 0$  for  $a \in [0, \tau_0]$ . If we put  $u(t) := p(0, t)$ , we obtain a Kermack–McKendrick type system composed of delay differential equation and renewal difference one, with distributed delay,  $t > 0$

$$\begin{cases} S'(t) &= \Lambda - (\gamma_S + h)S(t) - \beta S(t)I(t) + (1 - \alpha) \int_0^\tau e^{-\gamma_P a} g(a)u(t - a)da, \\ I'(t) &= -(\gamma_I + \mu)I(t) + \beta S(t)I(t), \\ u(t) &= hS(t) + \alpha \int_0^\tau e^{-\gamma_P a} g(a)u(t - a)da, \end{cases} \quad (1)$$

with initial conditions

$$S(0) = S_0, \quad I(0) = I_0 \quad \text{and} \quad u(t) = \phi(t) \quad \text{for} \quad -\tau \leq t \leq 0. \quad (2)$$

The function  $u$  represents the new protected individuals. The total population of protected individuals is then given by  $P(t) = \int_0^\tau e^{-\gamma_P a} u(t - a)da$ . The recovered individuals that get permanent immunity, satisfy the equation  $R'(t) = -\gamma_R R(t) + \mu I(t)$ ,  $R(0) = R_0$ . All the parameters of the model are constant and nonnegative.  $\Lambda$  is the birth rate in the compartment of the susceptible individuals,  $\gamma_S$ ,  $\gamma_I$ ,  $\gamma_P$  and  $\gamma_R$  are the death rates in each compartment. The parameter  $h$  describes the protection rate of susceptible individuals through for instance vaccination or drugs with temporary immunity. The constant  $\beta$  is the contact rate per infectious individual that results in infection.  $\mu$  is the recovering rate with permanent immunity.  $\alpha \in [0, 1)$  is the specific protection rate with temporary immunity. The integral  $\int_0^\tau e^{-\gamma_P a} g(a)u(t - a)da$  in (1) gives the rate at which individuals (who have survived) leave the protection phase. A fraction  $\alpha$  of these

individuals returns to the protection phase (for example by updating their vaccine). These individuals re-enter the protection phase with a rate  $\alpha \int_0^\tau e^{-\gamma_P a} g(a) u(t-a) da$ , then their age is reset to 0 (because the age here is the time since an individual entered this compartment). The other individuals lose their protection and become susceptible with a rate  $(1-\alpha) \int_0^\tau e^{-\gamma_P a} g(a) u(t-a) da$ . Through this paper we put

$$f(a) = e^{-\gamma_P a} g(a) \quad \text{and} \quad \nu(\tau) := \int_0^\tau f(a) da \leq 1. \quad (3)$$

The quantity  $\nu$  represents the proportion of individuals leaving the protection phase. It also contains the survival rate in this phase. A part of these individuals returns to the protection phase with a rate  $\alpha$  and the other part becomes susceptible again with a rate  $1-\alpha$ . We remark that the equations of  $S$ ,  $I$  and  $u$  are independent on  $R$  and  $P$ . Then, we can focus only on the reduced system without  $R$  and  $P$  components.

In the following sections, we perform a detailed mathematical analysis and obtain global stability properties of the two steady states: disease-free equilibrium and endemic equilibrium. More precisely, we are interested in the validity for our model, of the threshold theorem known in epidemiology and based on the basic reproduction number

$$\mathcal{R}_0 = \frac{\Lambda \beta (1 - \alpha \nu(\tau))}{(\mu + \gamma_I)(\gamma_S + h - (\alpha \gamma_S + h) \nu(\tau))}.$$

Summarizing the main results of this paper, we obtain the following threshold theorem.

**Theorem 1.1.** - *If  $\mathcal{R}_0 < 1$ , then the disease-free equilibrium of System (1) is the only equilibrium and it is globally asymptotically stable.*  
- *If  $\mathcal{R}_0 > 1$ , then a unique endemic equilibrium of System (1) exists and it is globally asymptotically stable.*

This result supports the claim that the basic reproduction number  $\mathcal{R}_0$  of our model serves as a threshold parameter that determines the outcome of the initial outbreak.

## 2. Preliminaries

Let us introduce  $C := C([- \tau, 0], \mathbb{R})$ , the space of continuous functions on  $[- \tau, 0]$  and  $C^+ := C([- \tau, 0], \mathbb{R}^+)$ , the set of nonnegative continuous functions on  $[- \tau, 0]$ . Throughout this paper, we assume  $S_0 \geq 0$ ,  $I_0 \geq 0$ ,  $\phi \in C^+$ . The existence and uniqueness of nonnegative solutions of (1)-(2) can be obtained as for neutral differential equations with delay (see [14]). Consider the auxiliary linear homogeneous difference equation

$$u(t) = \mathcal{D}(u_t), \quad t \geq 0, \quad (4)$$

where the history function  $u_t \in C$  is defined for  $t \geq 0$  and  $u \in C([- \tau, +\infty), \mathbb{R})$ , by  $u_t(\theta) = u(t + \theta)$  for  $\theta \in [- \tau, 0]$ . The linear operator  $\mathcal{D}: C([- \tau, 0], \mathbb{R}) \rightarrow \mathbb{R}$  is given by  $\mathcal{D}(\psi) = \alpha \int_0^\tau f(a) \psi(-a) da$ ,  $\psi \in C$ . Remark that Condition (3) implies that

$$\|\mathcal{D}\| := \sup_{\|\psi\| \leq 1} |\mathcal{D}(\psi)| < 1, \quad (5)$$

with  $\|\psi\| = \sup_{\theta \in [-\tau, 0]} |\psi(\theta)|$ . Condition (5) says that the zero solution of the linear difference equation (4) is globally asymptotically stable [15]. From the last equation of (1), we can write

$$u(t) - \mathcal{D}(u_t) = hS(t), \quad t > 0. \quad (6)$$

Using Eq. (6), Condition (5) and ([16], Lemma 3.5), we have the following result.

**Lemma 2.1.** *Let  $(S(t), u(t))$  be a sub-solution of System (1). Then,  $\lim_{t \rightarrow +\infty} u(t) > 0$  if and only if  $\lim_{t \rightarrow +\infty} S(t) > 0$ .*

We now consider the following fundamental auxiliary system, for  $t > 0$ ,

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - (\gamma_S + h)S(t) + (1 - \alpha) \int_0^\tau f(a)u(t-a)da, \\ u(t) = hS(t) + \alpha \int_0^\tau f(a)u(t-a)da, \\ S(0) = S_0, \quad u(s) = \phi(s), \quad \text{for } -\tau \leq s \leq 0. \end{cases} \quad (7)$$

The system (7) has a unique steady state

$$(S^0, u^0) = \left( \frac{\Lambda(1 - \alpha\nu(\tau))}{\gamma_S + h - (\alpha\gamma_S + h)\nu(\tau)}, \frac{\Lambda h}{\gamma_S + h - (\alpha\gamma_S + h)\nu(\tau)} \right). \quad (8)$$

We have the following result.

**Theorem 2.2.** *The steady state  $(S^0, u^0)$  of System (7) is globally asymptotically stable.*

**Proof.** We put, for  $t > 0$ ,  $\hat{S}(t) = S(t) - S^0$  and  $\hat{u}(t) = u(t) - u^0$ . Then, we get the linear differential-difference system

$$\begin{cases} \hat{S}'(t) = -(\gamma_S + h)\hat{S}(t) + (1 - \alpha) \int_0^\tau f(a)\hat{u}(t-a)da, \\ \hat{u}(t) = h\hat{S}(t) + \alpha \int_0^\tau f(a)\hat{u}(t-a)da. \end{cases} \quad (9)$$

Let us consider the following Lyapunov functional  $L^0 : \mathbb{R}^+ \times C^+ \rightarrow \mathbb{R}^+$  defined by

$$L^0(S_0, \phi) = \frac{S_0^2}{2} + \frac{1}{\nu(\tau)} \frac{\gamma_S(1 - (\alpha\nu(\tau))^2) + h(1 - \alpha(\nu(\tau))^2)}{2h^2} \int_0^\tau f(a) \int_{t-a}^t \phi^2(\theta) d\theta da.$$

We set  $\eta_1(s) = s^2/2$  and  $\eta_2(s) = ((1/2) + \tau\vartheta)s^2$ , with

$$\vartheta := \frac{\gamma_S(1 - (\alpha\nu(\tau))^2) + h(1 - \alpha(\nu(\tau))^2)}{2h^2}. \quad (10)$$

Then, this functional satisfies the inequalities  $\eta_1(S_0) \leq L^0(S_0, \phi) \leq \eta_2(\|(S_0, \phi)\|)$ . We can notice that System (9) is input-to-state stable (see [3,17]). More precisely, there exist constants  $C > 0$  and  $\sigma > 0$  such that the solution  $(\hat{S}, \hat{u})$  of (9) satisfies  $|\hat{u}(t)| \leq$

$C \left[ \|\phi\| e^{-\sigma t} + \sup_{0 \leq s \leq t} |\hat{S}(s)| \right]$ . The above estimation is an immediate consequence of ([15], Theorem 3.5, page 275). By differentiating the functional  $t \mapsto L^0(\hat{S}(t), \hat{u}_t)$  along the solution  $(\hat{S}, \hat{u})$  of the system (9), we obtain

$$\begin{aligned} \frac{d}{dt} L^0(\hat{S}, \hat{u}_t) &= \hat{S}(t) \hat{S}'(t) + \frac{\vartheta}{\nu(\tau)} \int_0^\tau f(a) [\hat{u}^2(t) - \hat{u}^2(t-a)] da, \\ &= -(\gamma_S + h - \vartheta h^2) \hat{S}^2(t) + \hat{S}(t) \int_0^\tau f(a) \hat{u}(t-a) da [(1-\alpha) + 2\vartheta h \alpha] \\ &\quad - \frac{\vartheta}{\nu(\tau)} \int_0^\tau f(a) \hat{u}^2(t-a) da + \vartheta \alpha^2 \left[ \int_0^\tau f(a) \hat{u}(t-a) da \right]^2. \end{aligned}$$

Moreover, Jensen's integral inequality implies that

$$\left[ \frac{1}{\nu(\tau)} \int_0^\tau f(a) \hat{u}(t-a) da \right]^2 \leq \frac{1}{\nu(\tau)} \int_0^\tau f(a) \hat{u}^2(t-a) da.$$

Then, we get

$$\begin{aligned} \frac{d}{dt} L^0(\hat{S}, \hat{u}_t) &\leq -(\gamma_S + h - \vartheta h^2) \hat{S}^2(t) + \hat{S}(t) \int_0^\tau f(a) \hat{u}(t-a) da [(1-\alpha) + 2\vartheta h \alpha] \\ &\quad - \frac{\vartheta}{(\nu(\tau))^2} \left[ \int_0^\tau f(a) \hat{u}(t-a) da \right]^2 + \vartheta \alpha^2 \left[ \int_0^\tau f(a) \hat{u}(t-a) da \right]^2. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{d}{dt} L^0(\hat{S}, \hat{u}_t) &\leq -(\gamma_S + h - \vartheta h^2) \hat{S}^2(t) + [(1-\alpha) + 2\vartheta h \alpha] \int_0^\tau f(a) \hat{u}(t-a) da \hat{S}(t) \\ &\quad - \frac{\vartheta}{(\nu(\tau))^2} [1 - \alpha^2 (\nu(\tau))^2] \left[ \int_0^\tau f(a) \hat{u}(t-a) da \right]^2. \end{aligned}$$

We set

$$a = \gamma_S + h - \vartheta h^2, \quad b = [(1-\alpha) + 2\vartheta h \alpha], \quad c = \frac{\vartheta}{(\nu(\tau))^2} [1 - \alpha^2 (\nu(\tau))^2]. \quad (11)$$

Then, the derivative of  $t \mapsto L^0(\hat{S}(t), \hat{u}_t)$  along the solution is given by

$$\begin{aligned} \frac{d}{dt} L^0(\hat{S}(t), \hat{u}_t) &= -a \hat{S}^2(t) + b \hat{S}(t) \int_0^\tau f(a) \hat{u}(t-a) da - c \left[ \int_0^\tau f(a) \hat{u}(t-a) da \right]^2, \\ &\leq -c \left[ \left( \int_0^\tau f(a) \hat{u}(t-a) da - \frac{b}{2c} \hat{S}(t) \right)^2 + \frac{4ac - b^2}{4c^2} \hat{S}^2(t) \right]. \end{aligned}$$

Remember that  $\alpha \in [0, 1)$ . Then by using (3) and (10), we can show that  $4ac - b^2 > 0$ . This is equivalent to

$$\begin{aligned} &[(1-\alpha)\nu(\tau) + 2\vartheta h \alpha \nu(\tau)]^2 - 4(\gamma_S + h - \vartheta h^2)\vartheta [1 - \alpha^2 (\nu(\tau))^2] \\ &= 4h^2\vartheta^2 - 4(\gamma_S(1 - \alpha^2 (\nu(\tau))^2) + h(1 - \alpha(\nu(\tau))^2))\vartheta + (1 - \alpha)^2 (\nu(\tau))^2 < 0. \end{aligned}$$

Since  $c > 0$ , we obtain

$$\frac{d}{dt}L^0(\hat{S}(t), \hat{u}_t) \leq \frac{b^2 - 4ac}{4c}\hat{S}^2(t) = -\varrho\hat{S}^2(t),$$

with  $\varrho := (4ac - b^2)/4c > 0$ . Hence  $(0, 0)$  is a globally asymptotically stable steady state of (9) (see, [3,17]). This completes the proof of Theorem 2.2.  $\square$

Back to System (1). We investigate a fundamental result about the boundedness of solutions of (1).

**Proposition 2.3.** *Suppose that  $\tau_0 > 0$ . All the solutions of System (1) are bounded.*

**Proof.** Let  $(S, I, u)$  be the solution of (1) associated to the initial condition  $(S_0, I_0, \phi) \in \mathbb{R}^+ \times \mathbb{R}^+ \times C^+$ . We remark that the system (7),

$$\begin{cases} \frac{dS_b(t)}{dt} = \Lambda - (\gamma_S + h)S_b(t) + (1 - \alpha) \int_0^\tau f(a)u_b(t-a)da, \\ u_b(t) = hS_b(t) + \alpha \int_0^\tau f(a)u_b(t-a)da, \\ S_b(0) = S_0, \quad u_b(s) = \phi(s), \quad \text{for } -\tau \leq s \leq 0, \end{cases}$$

represents an upper bound of System (1). Moreover, it is shown in Theorem 2.2 that  $S_b(t)$  converges to a finite constant as  $t$  tends to infinity. This implies that  $S_b$  is bounded.

We denote  $\tilde{S} := S_b - S$  and  $\tilde{u} := u_b - u$ . Clearly, we have, for  $t > 0$ ,

$$\begin{cases} \frac{d\tilde{S}(t)}{dt} \geq -(\gamma_S + h)\tilde{S}(t) + (1 - \alpha) \int_0^\tau f(a)\tilde{u}(t-a)da, \\ \tilde{u}(t) = h\tilde{S}(t) + \alpha \int_0^\tau f(a)\tilde{u}(t-a)da, \\ \tilde{S}(0) = 0, \quad \tilde{u}(s) = 0, \quad \text{for } -\tau \leq s \leq 0. \end{cases}$$

Suppose by contradiction that there exist  $t_1 \in \mathbb{R}^+$  and  $0 < \epsilon \ll \tau_0$  such that

- (i)  $\tilde{S}(t) \geq 0$  for  $t \in [0, t_1)$ ,  $\tilde{S}(t_1) = 0$  and  $\tilde{S}(t) < 0$  for  $t \in (t_1, t_1 + \epsilon]$ , or
- (ii)  $\tilde{u}(t) \geq 0$  for  $t \in [0, t_1)$ ,  $\tilde{u}(t_1) = 0$  and  $\tilde{u}(t) < 0$  for  $t \in (t_1, t_1 + \epsilon]$ .

We distinguish three cases:

- (1) Assume that (i) holds. If  $\tilde{u}(t) \geq 0$  and  $\tilde{u}(t) \not\equiv 0$  on  $[0, t_1 + \epsilon]$ , then we have  $\tilde{S}'(t_1) \geq (1 - \alpha) \int_0^\tau f(a)\tilde{u}(t_1 - a)da > 0$ . This gives a contradiction. If  $\tilde{u}(t) = 0$  on  $[0, t_1 + \epsilon]$ , then there exists a  $\tilde{t} \in (t_1, t_1 + \epsilon]$  such that  $\tilde{S}(\tilde{t}), \tilde{S}'(\tilde{t}) < 0$  and we get, for  $t_1 - \tau < \tilde{t} - a \leq t_1 + \epsilon - \tau_0$ ,  $\tilde{S}'(\tilde{t}) \geq -(\gamma_S + h)\tilde{S}(\tilde{t}) + (1 - \alpha) \int_{\tau_0}^\tau f(a)\tilde{u}(\tilde{t} - a)da > 0$ . This gives also a contradiction.
- (2) Assume that (ii) holds and  $\tilde{S}(t) \geq 0$  for all  $t \in [0, t_1 + \epsilon]$ . By using the equation of  $u$ , we get  $\tilde{u}(t_1 + \epsilon) = h\tilde{S}(t_1 + \epsilon) + \alpha \int_{\tau_0}^\tau f(a)\tilde{u}(t_1 + \epsilon - a)da \geq 0$ . This contradicts the fact that  $\tilde{u}(t_1 + \epsilon) < 0$ .
- (3) Assume that (i) and (ii) hold at the same time. We have  $\tilde{S}'(t_1) \geq (1 - \alpha) \int_0^\tau f(a)\tilde{u}(t_1 - a)da \geq 0$ . If  $\tilde{S}'(t_1) > 0$ , we get directly a contradiction and

if  $\tilde{S}'(t_1) = 0$ , then we can proceed as (i). There exists a  $\check{t} \in (t_1, t_1 + \epsilon]$  such that  $\tilde{S}(\check{t}), \tilde{S}'(\check{t}) < 0$  and we get, for  $t_1 - \tau < \check{t} - a \leq t_1 + \epsilon - \tau_0$ ,  $\tilde{S}'(\check{t}) \geq -(\gamma_S + h)\tilde{S}(\check{t}) + (1 - \alpha) \int_{\tau_0}^{\tau} f(a)\tilde{u}(\check{t} - a)da > 0$ . This leads to a contradiction with  $\tilde{S}'(\check{t}) < 0$ .

Finally, we conclude that  $S(t) \leq S_b(t)$  and  $u(t) \leq u_b(t)$ .

As a consequence by this comparison,  $S$  is also bounded. Then, there exists  $S^* > 0$  such that  $S(t) < S^*$ ,  $t > 0$ . On another side, we have (see Theorem 3.5 - page 275 in [15]),  $|u(t)| \leq C [\|\phi\|e^{-\alpha t} + h \sup_{0 \leq s \leq t} |S(s)|]$ ,  $t > 0$ , with  $\alpha > 0$ ,  $C > 0$  and  $\|\phi\| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|$ . Then,

$$|u(t)| \leq C [\|\phi\|e^{-\alpha t} + S^*]. \quad (12)$$

This implies that  $u$  is bounded. Consequently, there exists  $u^* > 0$  such that  $u(t) < u^*$ , for all  $t > 0$ .

Let focus now on the component  $I$ . By summing the equations of  $S$  and  $I$ , we get, for  $t > 0$ ,  $(S + I)'(t) < \Lambda - (\gamma_S + h)S(t) - (\gamma_I + \mu)I(t) + (1 - \alpha)\nu(\tau)u^*$ . Thus, we obtain  $(S + I)'(t) < \Lambda + (1 - \alpha)\nu(\tau)u^* - \omega(S + I)(t)$ , with  $\omega := \min\{\gamma_S + h, \gamma_I + \mu\}$ . As a consequence,  $I$  is also bounded.  $\square$

### 3. The basic reproduction number and existence of steady states

The basic reproduction number,  $\mathcal{R}_0$ , is used to measure the potential for disease transmission. It is the average number of secondary infections produced by an infected individual in a population completely susceptible. By dividing the equation of  $I(t)$ , in the system (1), by  $(\mu + \gamma_I)I$  we get  $I'(t)/[(\mu + \gamma_I)I(t)] = -1 + \beta S(t)/(\mu + \gamma_I)$ . The fraction  $\beta/(\mu + \gamma_I)$  can be interpreted as the number of contacts per infected individuals during their infectious period that lead to the transmission of the disease. If  $\beta S(t)/(\mu + \gamma_I) > 1$ , the disease persists, otherwise, it disappears. Then, the basic reproduction number of the disease is defined by  $\mathcal{R}_0 := \beta S^0/(\mu + \gamma_I)$ , where  $S^0$  is the disease-free steady state of the susceptible population.

Next, we establish the existence of the steady states of System (1). We will see that the condition of existence of an endemic equilibrium is equivalent to  $\mathcal{R}_0 > 1$ . Let  $(S^*, I^*, u^*)$  be a steady state of (1). Then, we have  $u^* = hS^*/[1 - \alpha\nu(\tau)]$ , and  $(\beta S^* - \mu - \gamma_I)I^* = 0$ . We distinguish two cases  $I^* = 0$  or  $S^* = (\mu + \gamma_I)/\beta$ . Suppose that  $I^* = 0$ . Then, since  $\gamma_S + h - (\alpha\gamma_S + h)\nu(\tau) > 0$ ,

$$S^* = \frac{\Lambda(1 - \alpha\nu(\tau))}{\gamma_S + h - (\alpha\gamma_S + h)\nu(\tau)} \quad \text{and} \quad u^* = \frac{\Lambda h}{\gamma_S + h - (\alpha\gamma_S + h)\nu(\tau)}.$$

We obtain the disease-free steady state

$$\begin{aligned} (S^*, I^*, u^*) &:= (S^0, 0, u^0), \\ &= \left( \frac{\Lambda(1 - \alpha\nu(\tau))}{\gamma_S + h - (\alpha\gamma_S + h)\nu(\tau)}, 0, \frac{\Lambda h}{\gamma_S + h - (\alpha\gamma_S + h)\nu(\tau)} \right). \end{aligned}$$



It describes the disappearance of the epidemic. Then, we can write

$$\mathcal{R}_0 := \frac{\beta S^0}{\mu + \gamma_I} = \frac{\Lambda \beta (1 - \alpha \nu(\tau))}{(\mu + \gamma_I)(\gamma_S + h - (\alpha \gamma_S + h) \nu(\tau))}.$$

Suppose now that  $I^* > 0$ . Then,  $S^* = (\mu + \gamma_I)/\beta$ . We have also  $u^* = h(\mu + \gamma_I)/[\beta(1 - \alpha \nu(\tau))] > 0$ . and

$$I^* = \frac{\Lambda - (\gamma_S + h)S^* + (1 - \alpha)\nu(\tau)u^*}{\beta S^*} = \frac{\Lambda}{\mu + \gamma_I} - \frac{\gamma_S + h - (\alpha \gamma_S + h)\nu(\tau)}{\beta(1 - \alpha \nu(\tau))}.$$

So, the existence of an endemic steady state is equivalent to  $\mathcal{R}_0 > 1$ . That is equivalent to

$$\frac{\Lambda}{\mu + \gamma_I} > \frac{\gamma_S + h - (\alpha \gamma_S + h)\nu(\tau)}{\beta(1 - \alpha \nu(\tau))}. \quad (13)$$

We set

$$\begin{aligned} (S^*, I^*, u^*) &:= (\bar{S}, \bar{I}, \bar{u}), \\ &= \left( \frac{\mu + \gamma_I}{\beta}, \frac{\Lambda}{\mu + \gamma_I} - \frac{\gamma_S + h - (\alpha \gamma_S + h)\nu(\tau)}{\beta(1 - \alpha \nu(\tau))}, \frac{h(\mu + \gamma_I)}{\beta(1 - \alpha \nu(\tau))} \right). \end{aligned} \quad (14)$$

We summarize the existence of the two steady states in the following result.

**Theorem 3.1.** *Assume that (13) holds. Then, System (1) has two distinct steady states: a disease-free steady state  $(S^0, 0, u^0)$ , which is given by (3), and an endemic steady state  $(\bar{S}, \bar{I}, \bar{u})$ , which is given by (14). If (13) does not hold, then  $(S^0, 0, u^0)$  is the only steady state.*

## 4. Stability of the disease-free steady state

### 4.1. Local asymptotic stability of the disease-free steady state

We recall that a steady state of (1) is locally asymptotically stable if all roots of the associated characteristic equation have negative real parts, and unstable if a root with positive real part exists (see [3] and [18] for more details).

The linearized system of (1) about the equilibrium  $(S^0, 0, u^0)$  is

$$\begin{cases} S'(t) &= -(\gamma_S + h)S(t) - \beta S^0 I(t) + (1 - \alpha) \int_0^\tau f(a)u(t - a)da, \\ I'(t) &= -(\gamma_I + \mu)I(t) + \beta S^0 I(t), \\ u(t) &= hS(t) + \alpha \int_0^\tau f(a)u(t - a)da, \end{cases}$$

and the characteristic equation is given by (see [3] for more details)

$$\Delta_0(\lambda) = (\lambda + \mu + \gamma_I - \beta S^0) \times \left[ \lambda + \gamma_S + h - [\alpha(\lambda + \gamma_S + h) + h(1 - \alpha)] \int_0^\tau f(a)e^{-\lambda a} da \right] = 0. \quad (15)$$

The instability of the disease-free steady state is directly obtained from the above equation.

**Proposition 4.1.** *Assume that  $\mathcal{R}_0 > 1$ . Then, the steady state  $(S^0, 0, u^0)$  is unstable.*

**Proof.** Assume that  $\mathcal{R}_0 > 1$ . From the characteristic equation (15), there exists a positive real root of (15). In fact, we have the following eigenvalue:  $\lambda = -\mu - \gamma_I + \beta S^0 = (\mu + \gamma_I)(\mathcal{R}_0 - 1) > 0$ . Then, the steady state  $(S^0, 0, u^0)$  is unstable.  $\square$

Now, we consider the case  $\mathcal{R}_0 < 1$ . Then,  $\lambda = (\mu + \gamma_I)(\mathcal{R}_0 - 1) < 0$ , and the local asymptotic stability of  $(S^0, 0, u^0)$  is determined by the sign of the real part of  $\lambda \in \mathbb{C}$  satisfying (see [3] and [18]),

$$D_0(\lambda) = \lambda + \gamma_S + h - [\alpha(\lambda + \gamma_S + h) + h(1 - \alpha)] \int_0^\tau f(a)e^{-\lambda a} da = 0. \quad (16)$$

We have the following theorem.

**Theorem 4.2.** *Assume that  $\mathcal{R}_0 < 1$ . Then, the steady state  $(S^0, 0, u^0)$  of System (1) is locally asymptotically stable.*

**Proof.** The equation (16) can be rewritten as

$$\lambda + \gamma_S + h = [\alpha(\lambda + \gamma_S + h) + h(1 - \alpha)] \int_0^\tau f(a)e^{-\lambda a} da. \quad (17)$$

Suppose that  $\lambda = \mu + i\omega$  ( $\mu, \omega \in \mathbb{R}$ ) and  $\mu \geq 0$ . Note that

$$\left| \int_0^\tau f(a)e^{-\lambda a} da \right| \leq \int_0^\tau f(a)e^{-\mu a} |e^{-i\omega a}| da = \int_0^\tau f(a)e^{-\mu a} da \leq 1.$$

Hence, by taking the modulus of (17), we have  $|\lambda + \gamma_S + h| \leq |\alpha(\lambda + \gamma_S) + h|$ . Taking square of both sides, we get

$$(\lambda + \gamma_S + h)(\bar{\lambda} + \gamma_S + h) \leq [\alpha(\lambda + \gamma_S) + h][\alpha(\bar{\lambda} + \gamma_S) + h].$$

Arranging this inequality, we obtain

$$(1 - \alpha^2)|\lambda + \gamma_S|^2 + (1 - \alpha)h(\lambda + \bar{\lambda} + 2\gamma_S) \leq 0.$$

However, since  $0 \leq \alpha < 1$  and  $\lambda + \bar{\lambda} = 2\mu \geq 0$ , we have

$$(1 - \alpha^2)|\lambda + \gamma_S|^2 + (1 - \alpha)h(\lambda + \bar{\lambda} + 2\gamma_S) > 0,$$

which is a contradiction. Hence,  $\mu < 0$ . We conclude that  $(S^0, 0, u^0)$  is locally asymptotically stable.  $\square$

#### 4.2. Global asymptotic stability of the disease-free steady state

Let  $\epsilon > 0$  and consider the set

$$\Omega_\epsilon := \left\{ (S, I, u) \in \mathbb{R}^+ \times \mathbb{R}^+ \times C^+ : 0 \leq S \leq S^0 + \epsilon \text{ and } 0 \leq u(s) \leq u^0 + \epsilon, \text{ for all } s \in [-\tau, 0] \right\}.$$

**Lemma 4.3.** *Suppose that  $\tau_0 > 0$ . For any sufficiently small  $\epsilon > 0$ , the subset  $\Omega_\epsilon$  of  $\mathbb{R}^+ \times \mathbb{R}^+ \times C^+$  is a global attractor for System (1).*

**Proof.** The solutions of (1) satisfy, for all  $t > 0$ ,

$$\begin{cases} S'(t) & \leq \Lambda - (\gamma_S + h)S(t) + (1 - \alpha) \int_0^\tau f(a)u(t-a)da, \\ u(t) & = hS(t) + \alpha \int_0^\tau f(a)u(t-a)da. \end{cases}$$

By the comparison principle (see the proof of Proposition 2.3), we have  $S(t) \leq S_b(t)$  and  $u(t) \leq u_b(t)$  for all  $t > 0$ , where  $(S_b, u_b)$  is the solution of the system (7). Theorem 2.2 shows that  $S_b(t) \rightarrow S^0$  and  $u_b(t) \rightarrow u^0$  as  $t \rightarrow +\infty$ . This convergence implies that  $\Omega_\epsilon$  is a global attractor for the system (1) in  $\mathbb{R}^+ \times \mathbb{R}^+ \times C^+$ . This completes the proof.  $\square$

Thanks to Lemma 4.3, we can restrict the global stability analysis of the disease-free steady state of (1) to the set  $\Omega_\epsilon$ .

**Theorem 4.4.** *Suppose that  $\tau_0 > 0$ . Assume that  $\mathcal{R}_0 < 1$ . Then, the disease-free steady state  $(S^0, 0, u^0)$  of System (1) is globally asymptotically stable.*

**Proof.** It suffices to consider the solutions in  $\Omega_\epsilon$  for any sufficiently small  $\epsilon > 0$ . We then have, for  $t > 0$ ,

$$I'(t) \leq -(\gamma_I + \mu)I(t) + \beta(S^0 + \epsilon)I(t) = -(\gamma_I + \mu) \left( 1 - \frac{\beta(S^0 + \epsilon)}{\mu + \gamma_I} \right) I(t).$$

Since  $R_0 < 1$ , we can choose  $\epsilon > 0$  such that the right-hand side of the above inequality is negative. This implies that  $\lim_{t \rightarrow +\infty} I(t) = 0$ .

From the above result, we see that for any  $\epsilon > 0$ , there exists a  $T_\epsilon > 0$  such that  $I(t) \leq \epsilon$  for all  $t \geq T_\epsilon$ . We then have, for  $t > T_\epsilon$

$$\begin{cases} S'(t) & \geq \Lambda - (\gamma_S + h)S(t) - \epsilon\beta S(t) + (1 - \alpha) \int_0^\tau f(a)u(t-a)da, \\ u(t) & = hS(t) + \alpha \int_0^\tau f(a)u(t-a)da. \end{cases}$$

Consequently, we have  $S(t) \geq S_\epsilon(t)$  and  $u(t) \geq u_\epsilon(t)$  for all  $t \geq T_\epsilon$ , where  $(S_\epsilon, u_\epsilon)$  is

the solution of the following problem

$$\begin{cases} \frac{dS_\epsilon(t)}{dt} = \Lambda - (\gamma_S + h)S_\epsilon(t) - \epsilon\beta S_\epsilon(t) + (1 - \alpha) \int_0^\tau f(a)u(t-a)da, \\ u_\epsilon(t) = hS_\epsilon(t) + \alpha \int_0^\tau f(a)u(t-a)da, \\ S_\epsilon(0) = S_0, \quad u_\epsilon(s) = \phi(s), \quad \text{for } -\tau \leq s \leq 0. \end{cases} \quad (18)$$

As in the proof of Theorem 2.2, we can show that  $S_\epsilon(t) \rightarrow S_\epsilon^0$  and  $u_\epsilon(t) \rightarrow u_\epsilon^0$  as  $t \rightarrow +\infty$ , where  $(S_\epsilon^0, u_\epsilon^0)$  is the steady state of (18). Then, there exists a  $\tilde{T}_\epsilon > T_\epsilon > 0$  such that, for  $t \geq \tilde{T}_\epsilon$ ,  $S_\epsilon^0 - \epsilon \leq S(t) \leq S^0 + \epsilon$  and  $u_\epsilon^0 - \epsilon \leq u(t) \leq u^0 + \epsilon$ . Since  $\epsilon > 0$  is arbitrary,  $S_\epsilon^0 \rightarrow S^0$  and  $u_\epsilon^0 \rightarrow u^0$  as  $\epsilon \rightarrow 0$ , we have that  $\lim_{t \rightarrow +\infty} S(t) = S^0$  and  $\lim_{t \rightarrow +\infty} u(t) = u^0$ . Recalling from Theorem 4.2 that  $(S^0, 0, u^0)$  is locally asymptotically stable. Then, it is globally asymptotically stable. This completes the proof.  $\square$

## 5. Stability of the endemic steady state

### 5.1. Local asymptotic stability of the endemic steady state

We now show the local asymptotic stability of the endemic steady state  $(\bar{S}, \bar{I}, \bar{u})$  for  $\mathcal{R}_0 > 1$ . The linearized system of (1) around  $(\bar{S}, \bar{I}, \bar{u})$  is given by

$$\begin{cases} S'(t) = -(\gamma_S + h)S(t) - \beta\bar{I}S(t) - \beta\bar{S}I(t) + (1 - \alpha) \int_0^\tau f(a)u(t-a)da, \\ I'(t) = \beta\bar{I}S(t), \\ u(t) = hS(t) + \alpha \int_0^\tau f(a)u(t-a)da. \end{cases}$$

Substituting  $S(t) = Xe^{\lambda t}$ ,  $I(t) = Ye^{\lambda t}$  and  $u(t) = Ze^{\lambda t}$ , and dividing them by  $e^{\lambda t}$ , we obtain

$$\begin{cases} \lambda X = -(\gamma_S + h)X - \beta\bar{I}X - \beta\bar{S}Y + (1 - \alpha) \left( \int_0^\tau f(a)e^{-\lambda a} da \right) Z, \\ \lambda Y = \beta\bar{I}X, \\ Z = hX + \alpha \left( \int_0^\tau f(a)e^{-\lambda a} da \right) Z. \end{cases}$$

From the third equation, we get  $Z = hX \left( 1 - \alpha \int_0^\tau f(a)e^{-\lambda a} da \right)^{-1}$  and injecting it into the first equation, we obtain

$$\begin{cases} [(\lambda + \gamma_S + h + \beta\bar{I})X + \beta\bar{S}Y] \left( 1 - \alpha \int_0^\tau f(a)e^{-\lambda a} da \right) \\ - (1 - \alpha)hX \int_0^\tau f(a)e^{-\lambda a} da = 0, \\ -\beta\bar{I}X + \lambda Y = 0. \end{cases}$$

Thus, the characteristic equation is given by

$$[\lambda^2 + (k_1 + h)\lambda + k_2] \left(1 - \alpha \int_0^\tau f(a)e^{-\lambda a} da\right) - (1 - \alpha)h\lambda \int_0^\tau f(a)e^{-\lambda a} da = 0,$$

where  $k_1 := \gamma_S + \beta\bar{I}$  and  $k_2 := \beta^2\bar{S}\bar{I}$ .

**Theorem 5.1.** *Assume that  $\mathcal{R}_0 > 1$ . Then, the steady state  $(\bar{S}, \bar{I}, \bar{u})$  of System (1) is locally asymptotically stable.*

**Proof.** The characteristic equation can be rewritten as

$$\lambda^2 + (k_1 + h)\lambda + k_2 = [\alpha(\lambda^2 + k_1\lambda + k_2) + h\lambda] \int_0^\tau f(a)e^{-\lambda a} da. \quad (19)$$

Suppose that  $\lambda = \mu + i\omega$  ( $\mu, \omega \in \mathbb{R}$ ) and  $\mu \geq 0$ . Note that  $|\int_0^\tau f(a)e^{-\lambda a} da| \leq 1$ . By taking the modulus of (19), we have

$$|\lambda^2 + (k_1 + h)\lambda + k_2| \leq |\alpha(\lambda^2 + k_1\lambda + k_2) + h\lambda|.$$

Taking square of both sides, we get

$$\begin{aligned} & (\lambda^2 + k_1\lambda + k_2 + h\lambda) \left(\bar{\lambda}^2 + k_1\bar{\lambda} + k_2 + h\bar{\lambda}\right) \\ & \leq [\alpha(\lambda^2 + k_1\lambda + k_2) + h\lambda] \left[\alpha(\bar{\lambda}^2 + k_1\bar{\lambda} + k_2) + h\bar{\lambda}\right]. \end{aligned}$$

Arranging this inequality, we obtain

$$(1 - \alpha^2)|\lambda^2 + k_1\lambda + k_2|^2 + (1 - \alpha)h [ (|\lambda|^2 + k_2)(\lambda + \bar{\lambda}) + 2k_1|\lambda|^2 ] \leq 0.$$

However, since  $0 \leq \alpha < 1$  and  $\lambda + \bar{\lambda} = 2\mu \geq 0$ , we have

$$(1 - \alpha^2)|\lambda^2 + k_1\lambda + k_2|^2 + (1 - \alpha)h [ (|\lambda|^2 + k_2)(\lambda + \bar{\lambda}) + 2k_1|\lambda|^2 ] > 0.$$

This gives a contradiction. Therefore,  $\mu < 0$  and the steady state  $(\bar{S}, \bar{I}, \bar{u})$  is locally asymptotically stable.  $\square$

## 5.2. Global asymptotic stability of the endemic steady state

This section provides the global asymptotic stability of the unique endemic steady state  $(\bar{S}, \bar{I}, \bar{u})$  given by (14). First, we notice that the component  $I$  of System (1) is persistent for  $\mathcal{R}_0 > 1$ , it remains above a positive level  $\hat{\epsilon}$  for all sufficiently large times. Indeed, we consider the following sub-system (without difference equation), for  $t > 0$

$$\begin{cases} S'(t) = \Lambda - (\gamma_S + h)S(t) - \beta S(t)I(t), \\ I'(t) = -(\gamma_I + \mu)I(t) + \beta S(t)I(t), \\ S(0) = S_0, \quad I(0) = I_0. \end{cases}$$

It is a classical SIR system and we know that it is persistent (see for instance Theorem 3.7.1 and 3.7.2 of [19], for a detailed proof). In fact, we have  $I'(t) = \beta S^0 I(t) [S(t)/S^0 - 1/\mathcal{R}_0]$ . Formally, if  $S$  is sufficiently close to  $S^0$ , then  $I'(t) > 0$ . This implies that  $I(t)$  increases. By the comparison principle, we can conclude that the solution  $I$  of System (1) satisfies  $\liminf_{t \rightarrow +\infty} I(t) > \hat{\epsilon} > 0$ . Let  $(\bar{S}, \bar{I}, \bar{u})$  be the endemic steady state with  $\bar{S} > 0$ ,  $\bar{I} > 0$  and  $\bar{u} > 0$ . We can write the system (1) for the variables  $\tilde{S}(t) := S(t) - \bar{S}$  and  $\tilde{u}(t) := u(t) - \bar{u}$  and by using the fact that  $\beta \bar{S} = \mu + \gamma_I$  as

$$\begin{cases} \tilde{S}'(t) = -(\gamma_S + h)\tilde{S}(t) - \beta \tilde{S}(t)I(t) - \beta \bar{S}I(t) + \beta \bar{S}\bar{I} + (1 - \alpha) \int_0^\tau f(a)\tilde{u}(t-a)da, \\ I'(t) = -(\gamma_I + \mu)I(t) + \beta \tilde{S}(t)I(t) + \beta \bar{S}I = \beta \tilde{S}(t)I(t), \\ \tilde{u}(t) = h\tilde{S}(t) + \alpha \int_0^\tau f(a)\tilde{u}(t-a)da. \end{cases} \quad (20)$$

**Theorem 5.2.** *Assume that  $\mathcal{R}_0 > 1$ . Then, the steady state  $(\bar{S}, \bar{I}, \bar{u})$  of System (1) is globally asymptotically stable.*

**Proof.** Let us consider the following Lyapunov functional  $L^e: \mathbb{R}^+ \times \mathbb{R}^+ \times C([-\tau, 0], \mathbb{R}^+) \rightarrow \mathbb{R}^+$  defined by

$$\begin{aligned} L^e(S_0, I_0, \phi) &= \frac{S_0^2}{2} + \bar{S} \left( I_0 - \bar{I} - \bar{I} \ln \frac{I_0}{\bar{I}} \right) \\ &\quad + \frac{1}{\nu(\tau)} \frac{\gamma_S(1 - (\alpha\nu(\tau))^2) + h(1 - \alpha(\nu(\tau))^2)}{2h^2} \int_0^\tau f(a) \int_{t-a}^t \phi^2(\theta) d\theta da. \end{aligned}$$

The function  $g(x) = x - \bar{x} - \bar{x} \ln(x/\bar{x})$  for  $x > 0$ , satisfies  $g(x) \geq 0$  for all  $x > 0$  and  $g(x) = 0$  if and only if  $x = \bar{x}$ . Then,  $L^e(S_0, I_0, u_0) = 0$  if and only if  $(S_0, I_0, u_0) = (0, \bar{I}, 0)$ . Recall that the constants  $a$ ,  $b$  and  $c$  are given in (11). The derivative of  $t \mapsto L^e(\tilde{S}(t), I(t), \tilde{u}_t)$  along the solution trajectory is given by

$$\begin{aligned} \frac{d}{dt} L^e(\tilde{S}(t), I(t), \tilde{u}_t) &= -a\tilde{S}^2(t) + b\tilde{S}(t) \int_0^\tau f(a)\tilde{u}(t-a)da \\ &\quad - c \left[ \int_0^\tau f(a)\tilde{u}(t-a)da \right]^2 - \beta I(t)\tilde{S}^2(t), \\ &\leq -c \left[ \left( \int_0^\tau f(a)\tilde{u}(t-a)da - \frac{b}{2c}\tilde{S}(t) \right)^2 + \frac{4ac - b^2}{4c^2}\tilde{S}^2(t) \right]. \end{aligned}$$

Thus, we obtain

$$\frac{d}{dt} L^e(\tilde{S}(t), I(t), \tilde{u}_t) \leq \frac{b^2 - 4ac}{4c} \tilde{S}^2(t) =: -\varrho \tilde{S}^2(t). \quad (21)$$

It is shown in the proof of theorem 2.2 that  $c > 0$  and  $\varrho > 0$ . Then, the functional  $t \mapsto L^e(\tilde{S}(t), I(t), \tilde{u}_t)$  is nonincreasing and we have

$$L^e(\tilde{S}(t), I(t), \tilde{u}_t) \xrightarrow[t \rightarrow +\infty]{s \geq 0} \inf L^e(\tilde{S}(s), I(s), \tilde{u}_s) =: L^* \in \mathbb{R}^+.$$

By integration, (21) implies that

$$\varrho \int_0^t \tilde{S}^2(s) ds \leq L^e(\tilde{S}(0), I(0), \tilde{u}_0) - L^e(\tilde{S}(t), I(t), \tilde{u}_t). \quad (22)$$

The both sides of the inequality (22) are nondecreasing functions. Then, the limits exist and satisfy

$$\lim_{t \rightarrow +\infty} \int_0^t \tilde{S}^2(s) ds \leq \frac{1}{\varrho} \left[ L^e(\tilde{S}(0), I(0), \tilde{u}_0) - L^* \right].$$

As the function  $\tilde{S}'(t)$  is uniformly bounded and  $\tilde{S}(t)$  is uniformly continuous. Then, the Barbalat's Lemma [20] applied to the function  $t \mapsto \int_0^t \tilde{S}^2(s) ds$ , shows that  $\lim_{t \rightarrow +\infty} \tilde{S}(t) = 0$ . The result in Lemma 3.5 of [16] implies that  $\lim_{t \rightarrow +\infty} \tilde{u}(t) = 0$ . Then, the expression of the functional  $L^e$  implies that  $\lim_{t \rightarrow +\infty} g(I(t)) = L^*/\bar{S}$ . Furthermore, the function  $\tilde{S}(t)$  is bounded and differentiable, then the fluctuations Lemma implies that there exists a sequence  $t_k \rightarrow +\infty$  such that  $\lim_{k \rightarrow +\infty} \tilde{S}'(t_k) = 0$ . Then, the first equation of (20), implies that  $\lim_{k \rightarrow +\infty} I(t_k) = \bar{I}$ . The continuity of the function  $g$  gives  $\lim_{k \rightarrow +\infty} g(I(t_k)) = g(\bar{I}) = 0$ . Then,  $L^* = 0$ . From the properties of the function  $g$ , we conclude that  $\lim_{t \rightarrow +\infty} I(t) = \bar{I}$ . This prove the global asymptotic stability of  $(\bar{S}, \bar{I}, \bar{u})$ .  $\square$

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## References

- [1] Adimy M, Chekroun A, Ferreira CP. Global dynamics of a differential-difference system: a case of Kermack-McKendrick SIR model with age-structured protection phase. *Mathematical Biosciences and Engineering*. 2020;17(2):1329. Available from: <https://www.aimspress.com/article/10.3934/mbe.2020067>.
- [2] Adimy M, Chekroun A, Kuniya T. Traveling waves of a differential-difference diffusive Kermack-McKendrick epidemic model with age-structured protection phase. *Journal of Mathematical Analysis and Applications*. 2022;505(1):125464. Available from: <https://www.sciencedirect.com/science/article/pii/S0022247X21005436>.
- [3] Adimy M, Chekroun A, Touaoula TM. Age-structured and delay differential-difference model of hematopoietic stem cell dynamics. *Discrete and Continuous Dynamical Systems - Series B*. 2015;20(9):2765–2791.

- [4] Adimy M, Chekroun A, Touaoula TM. Global asymptotic stability for an age-structured model of hematopoietic stem cell dynamics. *Applicable Analysis*. 2017;96:429–440. Available from: <http://dx.doi.org/10.1080/00036811.2016.1139698>.
- [5] Beretta E, Takeuchi Y. Global stability of an SIR epidemic model with time delays. *Journal of Mathematical Biology*. 1995;33:250–260.
- [6] Beretta E, Hara T, Ma W, et al. Global asymptotic stability of an SIR epidemic model with distributed time delay. *Nonlinear Analysis*. 2001;47:4107–4115.
- [7] Takeuchi Y, Ma W, Beretta E. Global asymptotic properties of a delay SIR epidemic model with finite incubation times. *Nonlinear Analysis*. 2000;42:931–947.
- [8] McCluskey CC. Complete global stability for an SIR epidemic model with delay - distributed or discrete. *Nonlinear Analysis: Real World Applications*. 2010;11:55–59.
- [9] McCluskey CC. Global stability of an SIR epidemic model with delay and general nonlinear incidence. *Mathematical Biosciences and Engineering*. 2010;7:837–850.
- [10] Li MY, Shuai Z, Wang C. Global stability of multi-group epidemic models with distributed delays. *Journal of Mathematical Analysis and Applications*. 2010;361:38–47.
- [11] Enatsu Y, Nakata Y, Muroya Y. Lyapunov functional techniques for the global stability analysis of a delayed sirs epidemic model. *Nonlinear Analysis: Real World Applications*. 2012;13:2120–2133.
- [12] Chekroun A, Kuniya T. An infection age-space structured SIR epidemic model with Neumann boundary condition. *Applicable Analysis*. 2020;99(11):1972–1985.
- [13] Bai Z, Zhang S. Traveling waves of a diffusive SIR epidemic model with a class of nonlinear incidence rates and distributed delay. *Communications in Nonlinear Science and Numerical Simulation*. 2015;22:1370–1381.
- [14] Hale JK, Cruz MA. Existence, uniqueness and continuous dependence for hereditary systems. *Annali di Matematica Pura ed Applicata*. 1970;85(1):63–81.
- [15] Hale JK, Verduyn Lunel SM. *Introduction to functional differential equations*. Springer; 1993.
- [16] Cruz MA, Hale JK. Stability of functional differential equations of neutral type. *Journal of Differential Equations*. 1970;7(2):334 – 355.
- [17] Gu K, Liu Y. Lyapunov-Krasovskii functional for uniform stability of coupled differential-functional equations. *Automatica*. 2009;45(3):798–804.
- [18] Diekmann O, Getto P, Nakata Y. On the characteristic equation  $\lambda = \alpha_1 + (\alpha_2 + \alpha_3\lambda)e^{-\lambda}$  and its use in the context of a cell population model. *Journal of Mathematical Biology*. 2015;72:877–908.
- [19] Li MY. *An Introduction to Mathematical Modeling of Infectious Diseases*. Springer International Publishing; 2018. Mathematics of Planet Earth, Mathematics of Planet Earth Collection; Available from: <https://www.springer.com/gp/book/9783319721217>.
- [20] Hou M, Duan G, Guo M. New versions of Barbalat's lemma with applications. *Journal of Control Theory and Applications*. 2010;8(4):545–547.