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Continuous-time stochastic analysis of rumor spreading with multiple operations

François Castella * Bruno Sericola † Emmanuelle Anceaume ‡ Yves Mocquard §

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Abstract

In this paper, we analyze a new asynchronous rumor spreading protocol to deliver a rumor to all the nodes of a large-scale distributed network. This protocol relies on successive pull operations involving k different nodes, with $k \geq 2$, and called k-pull operations. Specifically during a k-pull operation, an uninformed node a contacts $k-1$ other nodes at random in the network, and if at least one of them knows the rumor, then node a learns it. We perform a detailed study in continuous-time of the total time $\Theta_{k,n}$ needed for all the n nodes to learn the rumor. These results extend those obtained in a previous paper which dealt with the discrete-time case. We obtain the mean value, the variance and the distribution of $\Theta_{k,n}$ together with their asymptotic behavior when the number of nodes n tends to infinity.

keywords. Rumor spreading time; k-pull protocol; Poisson Process; Markov chain; Asymptotic analysis.

1 Introduction

Randomized rumor spreading or gossiping is an important communication mechanism that allows the dissemination of information in large-scale and open networks. A large-scale and open network comprises a collection of sequential computing entities (*e.g.*, processes, processors, nodes, agents, peers) that join and leave the system at any time, and communicate with one another by exchanging messages. Randomized rumor spreading was initially proposed by Deemers et al. [6] for the update of a database replicated at different sites, and has then been adopted in many applications due to its robustness and simplicity. In contrast to reliable communication broadcasts which must provide agreement on the broadcast value with possibly additional ordering guarantees on the delivery of updates from different sources, a randomized rumor spreading procedure provides reliability only with some probability. A randomized spreading rumor protocol describes the rules required for one or more pieces of information known by an arbitrary node in the network (we call such a node an *informed* node) to be spread to all the nodes of the network. The push and pull protocols are the basic operations used by the nodes to propagate an information over the entire network [6, 8]. With the push operation, an informed node contacts some randomly chosen node in the system, and gives it the rumor while with the pull operation, an uninformed node contacts some random node and asks for the rumor. Note that in both cases the contacted node may already know the rumor or not. The same node can perform both operations according to whether it knows or not the rumor, which corresponds to the push-pull protocol, or performs only one, either a pull or push operation, which corresponds to the pull or push protocols respectively. One of the important questions raised by these protocols is the spreading time, that is the time needed for the rumor to be known by all the nodes of the network.

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To answer such a question, one first needs to specify how synchronized nodes are, or in other words whether we suppose that all the nodes of the system act in a synchronous way or not. In the former case, the system model is synchronous while in the latter case it is asynchronous. The most studied one is the synchronous model. This model assumes that all the nodes of the network act synchronously, which allows the algorithms designed in this model to divide time in synchronized rounds. During each synchronized round, each node a of the network selects at random one of its neighbors b and either sends the rumor to b if a knows it (push operation) or gets the rumor from b if b knows the rumor (pull operation). In this model, the spreading time of a rumor is defined as the number of synchronous rounds necessary for all the nodes to know the rumor. When the underlying graph is complete, it has been shown by Frieze [11] that the number of rounds divided by $\log_2(n)$ converges in probability to $1 + \ln(2)$ when the number n of nodes in the graph tends to infinity. Further results have been established (see for example [20, 16] and the references therein), the most recent ones resulting from the observation that the rumor spreading time is closely related to the conductance of the graph of the network, see [12]. Investigations have also been done in different topologies of the network as in [2, 5, 10, 18], in the presence of link or nodes failures as in [9], in dynamic graphs as in [3], and in general graphs in terms of vertex expansion [13]. Another alternative consists in letting the nodes make more than one call during the push or pull operations [19]. The authors show that the push-pull protocol takes $O(\log n/\log\log n)$ rounds in expectation if the number of neighbors of a node is chosen independently according to a power law distribution with exponent $\beta \in (2, 3)$.

In large scale networks, that is in networks involving several thousands of nodes, assuming that all nodes act synchronously is a very strong assumption. Thus several authors, including [14, 17, 1, 7, 21], suppose an asynchronous model, that is a model in which nodes asynchronously trigger operations with randomly chosen nodes in the system, either to push, pull or push-pull information. The asynchronous gossip protocol is usually modeled by a time-continuous stochastic (Markovian) process [14, 17, 1, 7, 21]. This type of stochastic processes belongs to the death process category, which has many applications in demography, queuing theory, performance engineering, epidemiology, biology and many other distributed applications. For instance, in [4], an analysis of the SI (Susceptible-Infected) model –corresponding to an asynchronous push-pull model– allows us in some cases to explicit the state probabilities by using the Laplace transform on the Kolmogorov forward equation. However, these techniques prove ineffective when the transition rate is non-linear (Laplace transform inversion becomes a tricky exercise). Most of the rumor spreading protocols studied in the asynchronous models rely either on the push/pull operations or on the push operations. Indeed, pushing the information allows us to initiate the rumor very quickly but then struggles to reach the few uninformed nodes. In contrast, the pull algorithm attracted very little attention because this operation was long considered inefficient to spread a rumor within a large scale network [23]. It is actually very useful in systems fighting against message saturation (see for instance [25]). The ineffectiveness of the pull protocol stems from the fact that it takes some time before the rumor reaches a phase of exponential growth.

The objective of this paper is to further develop this line of inquiry by studying the k-pull protocol in the continuous-time case. This protocol counterbalances the slow initiation of pull-based rumour spreading protocols by increasing the chances of learning the rumor with each operation. A local clock following an exponential distribution with rate λ is associated with each uninformed node of the system. Each time the clock of an uniformed node rings, this node contacts $k-1$, with $k \geq 2$ distinct nodes, chosen at random uniformly among the $n-1$ other nodes. If at least one of these contacted nodes knows the rumor, the initiator of the k-pull operation learns the rumor and clears its clock.

The remainder of the paper is organized as follows. In Section 2, we present the asynchronous k-pull protocol and introduce the random variable $\Theta_{k,n}$which represents the total amount of time needed for all the nodes to know the rumor. We prove in Section 3 that the mean number of k-pull operations needed to inform all the n nodes of the system, assuming that a single node initially knows the rumor, that is $\mathbb{E}(\Theta_{k,n})$, is equivalent to $k\ln(n)/(k-1)\lambda$ when the number of nodes n in the system tends to infinity. We also show that the limiting variance of $\Theta_{k,n}$ is equal to $(1+1/(k-1)^2)\pi^2/6\lambda^2$ when n tends infinity. The distribution of the rumor spreading time $\Theta_{k,n}$ is analyzed in Section 4. We provide explicit limiting distributions of $\Theta_{k,n} - \mathbb{E}(\Theta_{k,n})$ and $\Theta_{k,n} - k\ln(n)/(k-1)\lambda$ when n tends to infinity.

2 The model

We recall in the next subsection the discrete-time model of the k-pull rumor spreading. This model, which has been analyzed in [22], is important because the continuous-time model of the k-pull rumor spreading is based on the discrete-time model, even if the analysis is, as we will see, much more complicated and needs more refined results in continuous-time.

2.1 The discrete-time model

We consider a complete network of size n in which each node may be asked for a piece of information (pull event). The algorithm starts with a single node informed of the rumor. At each discrete time $m \geq 1$, a single uninformed node s contacts $k-1$ distinct nodes, chosen at random uniformly among the $n-1$ other nodes. If at least one of these $k-1$ contacted nodes knows the rumor then node s learns it. Otherwise nothing happens. This is a k-pull operation.

To analyze this k-pull model, we introduce the discrete-time stochastic process $Y = \{Y_m,\ m \geq 0\}$ where Y_m represents the number of informed nodes at time m. The stochastic process Y is a discrete-time homogeneous Markov chain with n states where states $1, \ldots, n-1$ are transient and state n is absorbing. From the description of the protocol, we deduce that when the Markov chain Y is in state i at time m, then at time $m+1$, either it remains in state i if none of the $k-1$ chosen nodes were informed of the rumor, or it transits to state $i+1$ if at least one of the $k-1$ chosen nodes were informed of the rumor. We denote by P the transition probability matrix of Markov chain Y. The non zero entries of matrix P are thus $P_{i,i}$ and $P_{i,i+1}$, for any $i = 1, \ldots, n-1$. Obviously, we get, for any $i = 1, \ldots, n-1$, $P_{i,i} = 1 - P_{i,i+1}$, where

$$P_{i,i} = \begin{cases} \dfrac{\binom{n-1-i}{k-1}}{\binom{n-1}{k-1}} & \text{if } i \leq n-k \\ \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, given that $Y_t = i$, *i.e.*, when i nodes are informed of the rumor at time t, we have $Y_{t+1} = i$ if and only if, at time $t+1$, the set of $k-1$ chosen nodes (*i.e.*, $k-1$ among $n-1$) must be chosen among the $n-1-i$ non informed nodes.

We denoted in [22] by $T_{k,n}$ the random variable defined by

$$T_{k,n} = \inf\{t \geq 0 \mid Y_m = n\}$$

which represents the spreading time, that is the total number of k-pull operations needed for all the nodes in the network to know the rumor.

The spreading time distribution can thus be expressed as a sum of independent random variables $S_{k,n}(i)$, where $S_{k,n}(i)$ is the sojourn time of Markov chain Y in state i. For all $i = 1, \ldots, n-k$, $S_{k,n}(i)$ follows a geometric distribution with parameter $p_{k,n}(i) = 1 - P_{i,i}$ and $S_{k,n}(i) = 1$, for $i = n-k+1, \ldots, n-1$. Thus $T_{k,n}$ verifies

$$T_{k,n} = \sum_{i=1}^{n-1} S_{k,n}(i) = k - 1 + \sum_{i=1}^{n-k} S_{k,n}(i).$$

Observe that

$$p_{k,n}(i) = \begin{cases} 1 - \displaystyle\prod_{h=1}^{k-1} \left(1 - \frac{i}{n-h}\right) & \text{if } 1 \leq i \leq n-k \\ \\ 1 & \text{if } n-k+1 \leq i \leq n-1. \end{cases} \tag{1}$$

2.2 The continuous-time model

Based on this discrete-time model, a local clock following an exponential distribution with rate λ is associated with each uniformed node of the system. Each time the clock of an uniformed node s rings, this node contacts $k-1$, with $k \geq 2$, distinct nodes, chosen at random uniformly among the $n-1$ other nodes. If at least one of these contacted nodes knows the rumor, node s learns it and clears its clock (*i.e.*, s remains contactable but does not contact other nodes). As in the discrete-time case, we suppose that the k-pull operation, *i.e.*, the time for a node to contact $k-1$ other nodes and to receive their response, is instantaneous.

Since the clock of an uninformed node rings after a time that is exponentially distributed with rate λ, we naturally introduce the continuous-time Markov chain $Z = \{Z_t,\ t \geq 0\}$, where Z_t represents the number of informed nodes at continuous-time $t \geq 0$. Specifically the transitions of Z occur at successive instants $\tau_0 = 0$, τ_1, ..., where the $\tau_i - \tau_{i-1}$, $i \geq 1$, are independent and exponentially distributed with rate $(n-i)\lambda p_{k,n}(i)$, and the $p_{k,n}(i)$ are given by (1). Hence the global clock of the process rings according to an exponential distribution whose rate is proportional to the amount of uniformed nodes. Note that a jump of process Z corresponds to a state change of process Y.

Observe also that the continuous-time model of the rumor spreading corresponds to a physical time, that is the total amount of time needed for all the n nodes to learn the rumor, while the discrete-time model stands for the total number of k-pull operations needed for all the n nodes to learn the rumor.

We denote by $\Theta_{k,n}$ the random variable defined by

$$\Theta_{k,n} = \inf\{t \geq 0 \mid Z_t = n\}$$

which represents the continuous-time model *i.e.*, the total amount of time needed for all the nodes to know the rumor. The spreading time $\Theta_{k,n}$ can thus be expressed as a sum of independent and exponentially distributed random variables. Specifically, introducing the notation $U_{k,n}(i) = \tau_i - \tau_{i-1}$, for $i \geq 1$ with $\tau_0 = 0$, we have

$$\Theta_{k,n} = \sum_{i=1}^{n-1} U_{k,n}(i), \tag{2}$$

where $U_{k,n}(1), \ldots, U_{k,n}(n-1)$ are independent and $U_{k,n}(i)$ is exponentially distributed with rate $(n-i)\lambda p_{k,n}(i)$.

The authors of [22] used two technical lemmas to analyze the moments and the distribution of the rumor spreading time in the discrete-time case. These lemmas allowed them to provide lower and upper bounds for the probabilities $p_{k,n}(i)$. These bounds are sufficiently precise to deal with the sum of geometric random variables with parameters $p_{k,n}(i)$, but they are not precise enough to deal with the sum of exponential random variables with rates $(n-i)p_{k,n}(i)$ as in the continuous-time case. We thus consider here a different method to analyze the problem.

3 Moments of the rumor spreading time

We analyze in this section the first two moments of the rumor spreading time by using appropriate lower and upper bounds. Observe that the bounds obtained for $p_{k,n}(i)$ in [22] do not allow us to obtain the limiting behavior of these first two moments in the continuous-time case. The following technical lemma is used to obtain their asymptotic behavior.

Lemma 3.1 *Let g be a C^∞ (infinitely differentiable) function on interval* $[0,1]$.

$$\textit{If } g(0) \neq 0 \textit{ then } \sum_{i=1}^{n} \frac{1}{i} g\left(\frac{i}{n}\right) \underset{n \longrightarrow \infty}{\sim} g(0)\ln(n).$$

$$\textit{If } g(1) \neq 0 \textit{ then } \sum_{i=1}^{n} \frac{1}{i} g\left(1 - \frac{i}{n}\right) \underset{n \longrightarrow \infty}{\sim} g(1)\ln(n).$$

Proof. Function g being C^∞ on interval $[0,1]$, it is Lipschitz and so, for all $x, y \in [0,1]$, we have

$$|g(x) - g(y)| \leq C|x-y|, \text{ where } C = \max_{z\in[0,1]} |g'(z)|$$

and g' is the derivative of function g. We then have, by taking $x = i/n$ and $y = 0$,

$$\left|\sum_{i=1}^{n} \frac{1}{i} g\left(\frac{i}{n}\right) - g(0)\sum_{i=1}^{n}\frac{1}{i}\right| \leq \sum_{i=1}^{n}\frac{1}{i}\left|g\left(\frac{i}{n}\right) - g(0)\right| \leq C\sum_{i=1}^{n}\frac{1}{i}\frac{i}{n} = C.$$

This, together with the fact that $\sum_{i=1}^{n} 1/i \underset{n\longrightarrow\infty}{\sim} \ln(n)$, implies that

$$\sum_{i=1}^{n}\frac{1}{i}g\left(\frac{i}{n}\right) \underset{n\longrightarrow\infty}{\sim} g(0)\sum_{i=1}^{n}\frac{1}{i} \underset{n\longrightarrow\infty}{\sim} g(0)\ln(n),$$

which completes the proof of the first equivalence. The same steps with $x = 1 - i/n$ and $y = 1$ lead to the second equivalent. ■

3.1 Expected rumor spreading time

Using (1) and (2), the expected rumor spreading time writes

$$\mathbb{E}(\Theta_{k,n}) = \frac{1}{\lambda}\sum_{i=1}^{n-1}\frac{1}{(n-i)p_{k,n}(i)} = \frac{1}{\lambda}\sum_{i=1}^{n-k}\frac{1}{(n-i)p_{k,n}(i)} + \frac{1}{\lambda}\sum_{i=1}^{k-1}\frac{1}{i}. \tag{3}$$

Using the fact that $0 \leq h \leq k$ in Relation (1), we easily get

$$1 - \left(1 - \frac{i}{n}\right)^{k-1} \leq p_{k,n}(i) \leq 1 - \left(1 - \frac{i}{n-k}\right)^{k-1}. \tag{4}$$

Introducing the notation

$$\alpha_n = \sum_{i=1}^{n-k}\frac{1}{(n-i)p_{k,n}(i)},$$

these inequalities lead to

$$\gamma_n \leq \alpha_n \leq \beta_n, \tag{5}$$

where

$$\beta_n = \sum_{i=1}^{n-k}\frac{1}{(n-i)\left[1-\left(1-\frac{i}{n}\right)^{k-1}\right]} \text{ and } \gamma_n = \sum_{i=1}^{n-k}\frac{1}{(n-i)\left[1-\left(1-\frac{i}{n-k}\right)^{k-1}\right]}.$$

In the following two lemmas, we obtain an equivalent for both β_n and γ_n when n tends to infinity.

Lemma 3.2 *For all $k \geq 2$, we have*

$$\beta_n \underset{n\longrightarrow\infty}{\sim} \frac{k\ln(n)}{k-1}.$$

Proof. For all $k \geq 2$ and $y \in \mathbb{R}$, using the identity

$$1 - y^{k-1} = (1-y)\left(1 + y + \cdots + y^{k-2}\right),$$

and introducing the function f_k is defined, for all $x \in [0,1]$ by

$$f_k(x) = \frac{1}{1 + (1-x) + (1-x)^2 + \cdots + (1-x)^{k-2}},$$

we obtain

$$\begin{aligned}
\beta_n &= \sum_{i=1}^{n-k} \frac{f_k\left(\frac{i}{n}\right)}{(n-i)\left[1-\left(1-\frac{i}{n}\right)\right]} = \sum_{i=1}^{n-k} \frac{n}{i(n-i)} f_k\left(\frac{i}{n}\right) \\
&= \sum_{i=1}^{n-k} \left(\frac{1}{i} + \frac{1}{n-i}\right) f_k\left(\frac{i}{n}\right) \qquad (6) \\
&= \sum_{i=1}^{n-k} \frac{1}{i} f_k\left(\frac{i}{n}\right) + \sum_{i=k}^{n-1} \frac{1}{i} f_k\left(1-\frac{i}{n}\right) \\
&= \sum_{i=1}^{n} \frac{1}{i} f_k\left(\frac{i}{n}\right) + \sum_{i=1}^{n} \frac{1}{i} f_k\left(1-\frac{i}{n}\right) \\
&\quad - \sum_{i=n-k+1}^{n} \frac{1}{i} f_k\left(\frac{i}{n}\right) - \sum_{i=1}^{k-1} \frac{1}{i} f_k\left(1-\frac{i}{n}\right) - \frac{1}{n} f_k(0).
\end{aligned}$$

Observing that

$$\lim_{n\longrightarrow\infty} \left[\sum_{i=n-k+1}^{n} \frac{1}{i} f_k\left(\frac{i}{n}\right) + \sum_{i=1}^{k-1} \frac{1}{i} f_k\left(1-\frac{i}{n}\right) + \frac{1}{n} f_k(0)\right] = f_k(1) \sum_{i=1}^{k-1} \frac{1}{i},$$

we obtain

$$\beta_n = \sum_{i=1}^{n} \frac{1}{i} f_k\left(\frac{i}{n}\right) + \sum_{i=1}^{n} \frac{1}{i} f_k\left(1-\frac{i}{n}\right) - f_k(1) \sum_{i=1}^{k-1} \frac{1}{i} + \varepsilon(n), \qquad (7)$$

where $\varepsilon(n)$ is such that $\lim_{n\longrightarrow\infty} \varepsilon(n) = 0$. Since the function f_k is C^∞ on interval $[0,1]$ and verifies $f_k(0) = 1/(k-1) \neq 0$ and $f_k(1) = 1 \neq 0$, k being fixed, we obtain, using Lemma 3.1,

$$\beta_n \underset{n\longrightarrow\infty}{\sim} (f_k(0) + f_k(1)) \ln(n) = \frac{k \ln(n)}{k-1},$$

which completes the proof. ■

We consider now the term γ_n.

Lemma 3.3 *For all $k \geq 2$, we have*

$$\gamma_n \underset{n\longrightarrow\infty}{\sim} \frac{k \ln(n)}{k-1}.$$

Proof. Using the function f_k defined in the proof of Lemma 3.2, we obtain easily

$$\begin{aligned}
\gamma_n &= \sum_{i=1}^{n-k} \frac{f_k\left(\frac{i}{n-k}\right)}{(n-i)\left[1-\left(1-\frac{i}{n-k}\right)\right]} = (n-k) \sum_{i=1}^{n-k} \frac{1}{i(n-i)} f_k\left(\frac{i}{n-k}\right) \\
&= \frac{n-k}{n} \sum_{i=1}^{n-k} \left(\frac{1}{i} + \frac{1}{n-i}\right) f_k\left(\frac{i}{n-k}\right) \qquad (8)
\end{aligned}$$

The function f_k being C^∞ on interval $[0,1]$, defining $C_k = \max_{z\in[0,1]} |f_k'(z)|$ where f_k' is the derivative of f_k, we get, using Relations (6) and (8),

$$\begin{aligned}\left|\gamma_n - \frac{n-k}{n}\beta_n\right| &\leq \frac{n-k}{n}\sum_{i=1}^{n-k}\left(\frac{1}{i}+\frac{1}{n-i}\right)\left|f_k\left(\frac{i}{n-k}\right)-f_k\left(\frac{i}{n}\right)\right| \\ &\leq C_k\frac{n-k}{n}\sum_{i=1}^{n-k}\left(\frac{1}{i}+\frac{1}{n-i}\right)\frac{ik}{n(n-k)} \\ &\leq C_k\frac{k}{n}\left(\sum_{i=1}^{n-k}\frac{1}{i}+\sum_{i=k}^{n-1}\frac{1}{i}\right).\end{aligned}$$

This bound tends to 0 when n tends to infinity. We thus have, from Lemma 3.1,

$$\lim_{n\longrightarrow\infty}\left|\gamma_n - \frac{n-k}{n}\beta_n\right| = 0 \text{ and } \frac{n-k}{n}\beta_n \underset{n\longrightarrow\infty}{\sim} \beta_n \underset{n\longrightarrow\infty}{\sim} \frac{k\ln(n)}{k-1},$$

which completes the proof. ■

Theorem 3.4 *For all $k \geq 2$, we have*

$$\mathbb{E}(\Theta_{k,n}) \underset{n\longrightarrow\infty}{\sim} \frac{k\ln(n)}{(k-1)\lambda}.$$

Proof. Relations (3) and (5) give

$$\frac{\gamma_n}{\lambda} + \frac{1}{\lambda}\sum_{i=1}^{k-1}\frac{1}{i} \leq \mathbb{E}(\Theta_{k,n}) \leq \frac{\beta_n}{\lambda} + \frac{1}{\lambda}\sum_{i=1}^{k-1}\frac{1}{i}.$$

The use of Lemma 3.2 and Lemma 3.3 leads to the desired result. ■

We now show that the following refined analysis of the term β_n leads to a precise description of the asymptotic behavior of the expected rumor spreading time $\mathbb{E}(\Theta_{k,n})$. We need the following lemma in which γ is the Euler-Mascheroni constant given by $\gamma \approx 0.5772156649$.

Lemma 3.5 *Let g be a C^∞ function on interval $[0,1]$.*

If $g(0) \neq 0$ then $\displaystyle\sum_{i=1}^{n}\frac{1}{i}g\left(\frac{i}{n}\right) = g(0)\ln(n) + \gamma g(0) + \int_0^1 \frac{g(x)-g(0)}{x}dx + \varepsilon(n).$

If $g(1) \neq 0$ then $\displaystyle\sum_{i=1}^{n}\frac{1}{i}g\left(1-\frac{i}{n}\right) = g(1)\ln(n) + \gamma g(1) + \int_0^1 \frac{g(1-x)-g(1)}{x}dx + \varepsilon(n),$

where $\varepsilon(n)$ is such that $\lim_{n\longrightarrow\infty}\varepsilon(n) = 0$.

Proof. Using the integral form of the remainder for the Taylor series of function g, we get

$$g(x) - g(0) = xG(x), \text{ where } G(x) = \int_0^1 g'(tx)dt,$$

function g' being the derivative of g. We then have

$$\sum_{i=1}^{n}\frac{1}{i}g\left(\frac{i}{n}\right) - g(0)\sum_{i=1}^{n}\frac{1}{i} = \sum_{i=1}^{n}\frac{1}{i}\frac{i}{n}G\left(\frac{i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}G\left(\frac{i}{n}\right) \underset{n\to\infty}{\longrightarrow} \int_0^1 G(x)dx,$$

since the last term is a Riemann sum. We then use the following well-known development of the harmonic sum

$$\sum_{i=1}^{n} \frac{1}{i} = \ln(n) + \gamma + \varepsilon(n),$$

where $\varepsilon(n)$ is such that $\lim_{n\longrightarrow\infty} \varepsilon(n) = 0$. In the same way, we have

$$g(1-x) - g(1) = -xG(x), \text{ where } G(x) = \int_0^1 g'(1-tx)dt$$

and we obtain

$$\sum_{i=1}^{n} \frac{1}{i} g\left(1 - \frac{i}{n}\right) - g(1) \sum_{i=1}^{n} \frac{1}{i} = -\sum_{i=1}^{n} \frac{1}{i}\frac{i}{n} G\left(\frac{i}{n}\right) = -\frac{1}{n}\sum_{i=1}^{n} G\left(\frac{i}{n}\right) \underset{n\to\infty}{\longrightarrow} -\int_0^1 G(x)dx.$$

Again the development of the harmonic sum completes the proof. ■

The following theorem gives a more precise description of the asymptotic behavior of β_n.

Lemma 3.6 *For all $k \geq 2$, we have*

$$\beta_n = \frac{k\ln(n)}{k-1} + \frac{\gamma k}{k-1} - \sum_{i=1}^{k-1} \frac{1}{i} - \frac{\ln(k-1)}{k-1} + \varepsilon(n),$$

where $\varepsilon(n)$ is such that $\lim_{n\longrightarrow\infty} \varepsilon(n) = 0$.

Proof. Since $f_k(0) = 1/(k-1)$ and $f_k(1) = 1$, Relation (7) and Lemma 3.5 give

$$\beta_n = \frac{k\ln(n)}{k-1} + \frac{\gamma k}{k-1} - \sum_{i=1}^{k-1} \frac{1}{i} + \int_0^1 \left(\frac{f_k(1-x) + f_k(x)}{x} - \frac{k}{(k-1)x}\right) dx + \varepsilon(n).$$

We need to compute the quantity

$$I := \int_0^1 \left(\frac{f_k(1-x) + f_k(x)}{x} - \frac{k}{(k-1)x}\right) dx.$$

Coming back to the definition of f_k, we introduce the polynomial

$$q_k(x) = 1 + x + \cdots + x^k,$$

so that $f_k(x) = 1/q_{k-2}(1-x)$ and $f_k(1-x) = 1/q_{k-2}(x)$. On top of that, we observe that $k/(k-1) = 1 + 1/(k-1) = f_k(0) + f_k(1)$. These two observations lead to

$$I = \int_0^1 \frac{1}{x}\left(\frac{1}{q_{k-2}(x)} - \frac{1}{q_{k-2}(0)} + \frac{1}{q_{k-2}(1-x)} - \frac{1}{q_{k-2}(1)}\right) dx,$$

and the variable change $x := 1-x$ to deal with the second difference leads to

$$I = \int_0^1 \left[\frac{1}{x}\left(\frac{1}{q_{k-2}(x)} - \frac{1}{q_{k-2}(0)}\right) + \frac{1}{1-x}\left(\frac{1}{q_{k-2}(x)} - \frac{1}{q_{k-2}(1)}\right)\right] dx.$$

Now the whole point in order to compute this integral is to factorize x in the first difference $1/q_{k-2}(x) - 1/q_{k-2}(0)$, and to factorize $1-x$ in the second difference $1/q_{k-2}(x) - 1/q_{k-2}(1)$, so as to remove the apparent singularities and to recover computable quantities.

Concerning the first difference, we observe that

$$q_{k-2}(0) - q_{k-2}(x) = -x - x^2 - \cdots - x^{k-2} = -xq_{k-3}(x),$$

which gives

$$\frac{1}{x}\left(\frac{1}{q_{k-2}(x)} - \frac{1}{q_{k-2}(0)}\right) = -\frac{q_{k-3}(x)}{q_{k-2}(0)q_{k-2}(x)} = -\frac{q_{k-3}(x)}{q_{k-2}(x)}.$$

The other difference requires slightly more attention. We claim that the following formula holds

$$q_{k-2}(1) - q_{k-2}(x) = (1-x)\left[(k-2)q_{k-3}(x) - xq'_{k-3}(x)\right]. \tag{9}$$

Admitting the result for the time being, we obtain

$$\begin{aligned}\frac{1}{1-x}\left(\frac{1}{q_{k-2}(x)} - \frac{1}{q_{k-2}(1)}\right) &= \frac{(k-2)q_{k-3}(x) - xq'_{k-3}(x)}{q_{k-2}(1)q_{k-2}(x)} \\ &= \frac{(k-2)q_{k-3}(x) - xq'_{k-3}(x)}{(k-1)q_{k-2}(x)}.\end{aligned}$$

Eventually, this provides

$$\begin{aligned}I &= \int_0^1 \left[-\frac{q_{k-3}(x)}{q_{k-2}(x)} + \frac{(k-2)q_{k-3}(x) - xq'_{k-3}(x)}{(k-1)q_{k-2}(x)}\right] dx \\ &= -\frac{1}{k-1}\int_0^1 \frac{q_{k-3}(x) + xq'_{k-3}(x)}{q_{k-2}(x)} dx \\ &= -\frac{1}{k-1}\int_0^1 \frac{q_{k-3}(x) + xq'_{k-3}(x)}{xq_{k-3}(x) + 1} dx,\end{aligned}$$

where the last equality comes from the observation $q_{k-2}(x) = xq_{k-3}(x) + 1$. As a final result, we recover, since $(xq_{k-3}(x) + 1)' = xq'_{k-3}(x) + q_{k-3}(x)$, the value

$$I = -\frac{\ln\left(p_{k-3}(1)\right)}{k-1} = -\frac{\ln(k-1)}{k-1}.$$

There remains to prove formula (9). The formula can easily be proved using a recursion procedure. Alternatively, one may write, using the fact that $1 - x^j = (1-x)\sum_{\ell=0}^{j-1} x^\ell$, the relations

$$\begin{aligned}q_{k-2}(1) - q_{k-2}(x) &= \sum_{j=1}^{k-2}\left(1 - x^j\right) = (1-x)\sum_{j=1}^{k-2}\sum_{\ell=0}^{j-1} x^\ell = (1-x)\sum_{\ell=0}^{k-3}\sum_{j=\ell+1}^{k-2} x^\ell \\ &= (1-x)\sum_{\ell=0}^{k-3}(k-2-\ell)x^\ell = (1-x)\left[(k-2)q_{k-3}(x) - xq'_{k-3}(x)\right],\end{aligned}$$

which completes the proof. ■

The following theorem will be used in Corollary 4.4 to get an asymptotic behavior of the distribution of $\Theta_{k,n}$.

Theorem 3.7 *For all $k \geq 2$, we have*

$$\mathbb{E}(\Theta_{k,n}) = \frac{1}{\lambda}\left(\frac{k\ln(n)}{k-1} + \frac{\gamma k}{k-1} - \frac{\ln(k-1)}{k-1}\right) + \varepsilon(n),$$

where $\varepsilon(n)$ is such that $\lim_{n\longrightarrow\infty}\varepsilon(n) = 0$.

Proof. Relations (3) and (5) give

$$\frac{\gamma_n}{\lambda} + \frac{1}{\lambda}\sum_{i=1}^{k-1}\frac{1}{i} \leq \mathbb{E}(\Theta_{k,n}) \leq \frac{\beta_n}{\lambda} + \frac{1}{\lambda}\sum_{i=1}^{k-1}\frac{1}{i}.$$

Using Lemma 3.6 and the fact that $\lim_{n\longrightarrow\infty}(\gamma_n - \beta_n) = 0$ (see the end of the proof of Lemma 3.3), we easily get the desired result. ■

3.2 Variance of the rumor spreading time

The following lemma is needed to obtain the limiting value of the variance of $\Theta_{k,n}$ when n tends to infinity.

Lemma 3.8 *Let g be a C^∞ function on interval $[0,1]$.*

$$\textit{If } g(0) \neq 0 \textit{ then } \lim_{n\longrightarrow\infty}\sum_{i=1}^{n}\frac{1}{i^2}g\left(\frac{i}{n}\right) = g(0)\frac{\pi^2}{6}.$$

$$\textit{If } g(1) \neq 0 \textit{ then } \lim_{n\longrightarrow\infty}\sum_{i=1}^{n}\frac{1}{i^2}g\left(1-\frac{i}{n}\right) = g(1)\frac{\pi^2}{6}.$$

Proof. Function g being C^∞ on interval $[0,1]$, it is Lipschitz and so, for all $x, y \in [0,1]$, we have

$$|g(x) - g(y)| \leq C|x-y|, \text{ where } C = \max_{z\in[0,1]}|g'(z)|$$

and g' is the derivative of function g. We then have, by taking $x = i/n$ and $y = 0$,

$$\left|\sum_{i=1}^{n}\frac{1}{i^2}g\left(\frac{i}{n}\right) - g(0)\sum_{i=1}^{n}\frac{1}{i^2}\right| \leq \sum_{i=1}^{n}\frac{1}{i^2}\left|g\left(\frac{i}{n}\right) - g(0)\right| \leq C\frac{1}{n}\sum_{i=1}^{n}\frac{1}{i} \underset{n\longrightarrow\infty}{\sim} C\frac{\ln(n)}{n}.$$

This means that

$$\lim_{n\longrightarrow\infty}\sum_{i=1}^{n}\frac{1}{i^2}g\left(\frac{i}{n}\right) = g(0)\sum_{i=1}^{\infty}\frac{1}{i^2} = g(0)\frac{\pi^2}{6}.$$

The same steps with $x = 1 - i/n$ and $y = 1$ lead to the second result. ■

Using (1) and (2), the variance of the rumor spreading time writes

$$\mathbb{V}(\Theta_{k,n}) = \frac{1}{\lambda^2}\sum_{i=1}^{n-1}\frac{1}{(n-i)^2 p_{k,n}^2(i)} = \frac{1}{\lambda^2}\sum_{i=1}^{n-k}\frac{1}{(n-i)p_{k,n}^2(i)} + \frac{1}{\lambda^2}\sum_{i=1}^{k-1}\frac{1}{i^2}. \tag{10}$$

As we did for the expected rumor spreading time and introducing the notation

$$\delta_n = \sum_{i=1}^{n-k}\frac{1}{(n-i)^2 p_{k,n}^2(i)},$$

we obtain

$$\eta_n \leq \delta_n \leq \zeta_n, \tag{11}$$

where

$$\zeta_n = \sum_{i=1}^{n-k}\frac{1}{(n-i)^2\left[1-\left(1-\frac{i}{n}\right)^{k-1}\right]^2},\ \eta_n = \sum_{i=1}^{n-k}\frac{1}{(n-i)^2\left[1-\left(1-\frac{i}{n-k}\right)^{k-1}\right]^2}.$$

In the following two lemmas, we obtain the limiting value of both ζ_n and η_n when n tends to infinity.

Lemma 3.9 *For all $k \geq 2$, we have*

$$\lim_{n\longrightarrow\infty} \zeta_n = \left(1 + \frac{1}{(k-1)^2}\right)\frac{\pi^2}{6} - \sum_{i=1}^{k-1}\frac{1}{i^2}.$$

Proof. Using again the function f_k introduced in the proof of Lemma 3.2, we obtain

$$\zeta_n = \sum_{i=1}^{n-k} \frac{f_k^2\left(\frac{i}{n}\right)}{(n-i)^2\left[1-\left(1-\frac{i}{n}\right)\right]^2} = \sum_{i=1}^{n-k}\frac{n^2}{i^2(n-i)^2} f_k^2\left(\frac{i}{n}\right). \tag{12}$$

Observing that

$$\frac{n^2}{i^2(n-i)^2} = \frac{1}{(n-i)^2} + \frac{1}{i^2} + \frac{2}{n}\left(\frac{1}{i} + \frac{1}{n-i}\right), \tag{13}$$

we have

$$\zeta_n = \sum_{i=1}^{n-k}\frac{1}{(n-i)^2} f_k^2\left(\frac{i}{n}\right) + \sum_{i=1}^{n-k}\frac{1}{i^2} f_k^2\left(\frac{i}{n}\right) + \frac{2}{n}\sum_{i=1}^{n-k}\left(\frac{1}{i}+\frac{1}{n-i}\right) f_k^2\left(\frac{i}{n}\right).$$

We denote respectively by $\zeta_{n,1}$, $\zeta_{n,2}$ and $\zeta_{n,3}$ these three sums. Concerning $\zeta_{n,3}$, we have

$$\sum_{i=1}^{n-k}\left(\frac{1}{i}+\frac{1}{n-i}\right) f_k^2\left(\frac{i}{n}\right) = \sum_{i=1}^{n-k}\frac{1}{i} f_k^2\left(\frac{i}{n}\right) + \sum_{i=k}^{n-1}\frac{1}{i} f_k^2\left(1-\frac{i}{n}\right).$$

Applying Lemma 3.1 with function $g = f_k^2$, we obtain

$$\sum_{i=1}^{n-k}\frac{1}{i} f_k^2\left(\frac{i}{n}\right) + \sum_{i=k}^{n-1}\frac{1}{i} f_k^2\left(1-\frac{i}{n}\right) \underset{n\longrightarrow\infty}{\sim} \left(f_k^2(0) + f_k^2(1)\right)\ln(n) = \left(1+\frac{1}{(k-1)^2}\right)\ln(n)$$

which implies that

$$\lim_{n\longrightarrow\infty}\zeta_{n,3} = \lim_{n\longrightarrow\infty} 2\left(1+\frac{1}{(k-1)^2}\right)\frac{\ln(n)}{n} = 0.$$

Concerning $\zeta_{n,2}$, since we have

$$\zeta_{n,2} = \sum_{i=1}^{n-k}\frac{1}{i^2} f_k^2\left(\frac{i}{n}\right),$$

applying Lemma 3.8 with function $g = f_k^2$, we obtain

$$\lim_{n\longrightarrow\infty}\zeta_{n,2} = f_k^2(0)\frac{\pi^2}{6} = \frac{1}{(k-1)^2}\frac{\pi^2}{6}.$$

Finally, for term $\zeta_{n,1}$, we have

$$\zeta_{n,1} = \sum_{i=1}^{n-k}\frac{1}{(n-i)^2} f_k^2\left(\frac{i}{n}\right) = \sum_{i=k}^{n-1}\frac{1}{i^2} f_k^2\left(1-\frac{i}{n}\right)$$

Applying again Lemma 3.8 with function $g = f_k^2$, we obtain

$$\lim_{n\longrightarrow\infty}\zeta_{n,1} = f_k^2(1)\sum_{i=k}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{6} - \sum_{i=1}^{k-1}\frac{1}{i^2}.$$

Putting these three limits together leads to

$$\lim_{n\longrightarrow\infty}\zeta_n=\left(1+\frac{1}{(k-1)^2}\right)\frac{\pi^2}{6}-\sum_{i=1}^{k-1}\frac{1}{i^2},$$

which completes the proof. ■

We analyze now the limiting value of η_n when n tends to infinity.

Lemma 3.10 *For all $k\geq 2$, we have*

$$\lim_{n\longrightarrow\infty}\eta_n=\left(1+\frac{1}{(k-1)^2}\right)\frac{\pi^2}{6}-\sum_{i=1}^{k-1}\frac{1}{i^2}.$$

Proof. Using again the function f_k introduced in the proof of Lemma 3.2, we obtain

$$\eta_n=\sum_{i=1}^{n-k}\frac{f_k^2\left(\dfrac{i}{n-k}\right)}{(n-i)^2\left[1-\left(1-\dfrac{i}{n-k}\right)\right]^2}=\frac{(n-k)^2}{n^2}\sum_{i=1}^{n-k}\frac{n^2}{i^2(n-i)^2}f_k^2\left(\frac{i}{n-k}\right). \tag{14}$$

The function f_k being Lipschitz on interval $[0,1]$, defining $C_k=\max_{z\in[0,1]}|(f_k^2)'(z)|$ where $(f_k^2)'$ is the derivative of f_k^2, we get, using Relations (12) and (14),

$$\begin{aligned}\left|\eta_n-\frac{(n-k)^2}{n^2}\zeta_n\right|&\leq\frac{(n-k)^2}{n^2}\sum_{i=1}^{n-k}\frac{n^2}{i^2(n-i)^2}\left|f_k^2\left(\frac{i}{n-k}\right)-f_k\left(\frac{i}{n}\right)\right|\\&\leq C_k\frac{(n-k)^2}{n^2}\sum_{i=1}^{n-k}\frac{n^2}{i^2(n-i)^2}\frac{ik}{n(n-k)}\\&\leq C_k\frac{k(n-k)}{n^2}\sum_{i=1}^{n-k}\frac{n^2}{i^2(n-i)^2}.\end{aligned}$$

Using Relation (13), we have

$$\lim_{n\longrightarrow\infty}\sum_{i=1}^{n-k}\frac{n^2}{i^2(n-i)^2}=\sum_{i=1}^{\infty}\frac{1}{i^2}+\sum_{i=k}^{\infty}\frac{1}{i^2}=\frac{\pi^2}{3}-\sum_{i=1}^{k-1}\frac{1}{i^2}.$$

It follows that

$$\lim_{n\longrightarrow\infty}\left|\eta_n-\frac{(n-k)^2}{n^2}\zeta_n\right|=0.$$

This result together with the result of Lemma 3.9 leads to

$$\lim_{n\longrightarrow\infty}\eta_n=\lim_{n\longrightarrow\infty}\zeta_n=\left(1+\frac{1}{(k-1)^2}\right)\frac{\pi^2}{6}-\sum_{i=1}^{k-1}\frac{1}{i^2},$$

which completes the proof.

Theorem 3.11 *For all $k\geq 2$, we have*

$$\lim_{n\longrightarrow\infty}\mathbb{V}(\Theta_{k,n})=\left(1+\frac{1}{(k-1)^2}\right)\frac{\pi^2}{6\lambda^2}.$$

Proof. Relations (10) and (11) give

$$\frac{\eta_n}{\lambda^2}+\frac{1}{\lambda^2}\sum_{i=1}^{k-1}\frac{1}{i^2}\leq\mathbb{V}(\Theta_{k,n})\leq\frac{\zeta_n}{\lambda^2}+\frac{1}{\lambda^2}\sum_{i=1}^{k-1}\frac{1}{i^2}.$$

The use of Lemma 3.9 and Lemma 3.10 leads to the desired result.

4 Distribution of the rumor spreading time

This section provides explicit limiting distributions such as $\Theta_{k,n} - \mathbb{E}(\Theta_{k,n})$ when n tends to infinity. We introduce the notation $\mu_{k,n}(i) = \lambda(n-i)p_{k,n}(i)$. Recall that $U_{k,n}(i)$ is exponentially distributed with rate $\mu_{k,n}(i)$ and that

$$\Theta_{k,n} = \sum_{i=1}^{n-1} U_{k,n}(i).$$

The main result of this section is Theorem 4.2 whose proof needs the following lemma.

Lemma 4.1 *For all $k \geq 2$, we have*

$$\lim_{m\longrightarrow\infty} \limsup_{n\longrightarrow\infty} \sum_{i=m}^{n} \frac{1}{(2n+1-i)^2 p_{k,2n+1}^2(i)} = 0.$$

and

$$\lim_{m\longrightarrow\infty} \limsup_{n\longrightarrow\infty} \sum_{i=m}^{n} \frac{1}{i^2 p_{k,2n+1}^2(2n+1-i)} = 0.$$

Proof. Introducing the notation

$$\Delta_{m,n}(k) = \sum_{i=m}^{n} \frac{1}{(2n+1-i)^2 \left[1 - \left(1 - \frac{i}{2n+1}\right)^{k-1}\right]^2}$$

we obtain, using inequality (4),

$$\sum_{i=m}^{n} \frac{1}{(2n+1-i)^2 p_{k,2n+1}^2(i)} \leq \Delta_{m,n}(k).$$

Using the function f_k defined in the proof of Lemma 3.2 we write

$$\Delta_{n,m}(k) = \sum_{i=m}^{n} \frac{(2n+1)^2}{(2n+1-i)^2 i^2} f_k^2\left(\frac{i}{2n+1}\right).$$

Observing that function f_k is increasing on interval $[0,1]$ and that $f_k(1) = 1$, we obtain using Relation (13), with $2n+1$ instead n,

$$\begin{aligned}
\Delta_{n,m}(k) &\leq \sum_{i=m}^{n} \frac{(2n+1)^2}{(2n+1-i)^2 i^2} \\
&= \sum_{i=m}^{n} \frac{1}{(2n+1-i)^2} + \sum_{i=m}^{n} \frac{1}{i^2} + \frac{2}{2n+1} \sum_{i=m}^{n} \left(\frac{1}{i} + \frac{1}{2n+1-i}\right) \\
&= \sum_{i=n+1}^{2n+1-m} \frac{1}{i^2} + \sum_{i=m}^{n} \frac{1}{i^2} + \frac{2}{2n+1} \left(\sum_{i=m}^{n} \frac{1}{i} + \sum_{i=n+1}^{2n+1-m} \frac{1}{i}\right) \\
&= \sum_{i=m}^{2n+1-m} \frac{1}{i^2} + \frac{2}{2n+1} \left(\sum_{i=m}^{2n+1-m} \frac{1}{i}\right) \leq \sum_{i=m}^{2n+1-m} \frac{1}{i^2} + \frac{2(1+\ln(2n+1-m)}{2n+1}.
\end{aligned}$$

The $\lim_{m\longrightarrow\infty} \limsup_{n\longrightarrow\infty}$ of both terms is 0 because $\sum_i 1/i^2$ is a converging series. This proves the first relation.

Concerning the second relation, introducing the notation

$$\Lambda_{m,n}(k) = \sum_{i=m}^{n} \frac{1}{i^2 \left[1 - \left(1 - \frac{2n+1-i}{2n+1}\right)^{k-1}\right]^2}$$

we obtain in the same way, using inequality (4),

$$\sum_{i=m}^{n} \frac{1}{i^2 p_{k,2n+1}^2 (2n+1-i)} \leq \Lambda_{m,n}(k)$$

and, using function f_k,

$$\Lambda_{n,m}(k) = \sum_{i=m}^{n} \frac{(2n+1)^2}{(2n+1-i)^2 i^2} f_k^2 \left(1 - \frac{i}{2n+1}\right).$$

As we did for term $\Delta_{n,m}(k)$, we have

$$\Lambda_{m,n}(k) \leq \sum_{i=m}^{n} \frac{(2n+1)^2}{(2n+1-i)^2 i^2},$$

which in turn leads to the same result. We are now able to prove the following theorem.

Theorem 4.2 *Let $(Z_i)_{i\geq 1}$ be a sequence of i.i.d. random variables exponentially distributed with rate 1 and let W be defined by*

$$W = \sum_{i=1}^{\infty} \frac{Z_i - 1}{i}.$$

We then have

$$\Theta_{k,n} - \mathbb{E}(\Theta_{k,n}) \xrightarrow{\mathcal{L}} \frac{1}{(k-1)\lambda} W^{(1)} + \frac{1}{\lambda} W^{(2)} \text{ as } n \longrightarrow \infty$$

where $W^{(1)}$ and $W^{(2)}$ are i.i.d. with the same distribution as W.

Proof. Recalling that $\mu_{k,n}(i) = \lambda(n-i)p_{k,n}(i)$, we have, for all $n \geq 2$, $i \in \{1, \ldots, n-1\}$ and $x \geq 0$,

$$\mathbb{P}\{\mu_{k,n}(i)U_{k,n}(i) > x\} = \mathbb{P}\{U_{k,n}(i) > x/\mu_{k,n}(i)\} = e^{-x}.$$

Thus if Z_i is exponentially distributed with rate 1, we have $\mu_{k,n}(i)U_{k,n}(i) \stackrel{\mathcal{L}}{=} Z_i$. Moreover since the $(U_{k,n}(i))_{i\in\{1,\ldots,n-1\}}$ are independent, the $(Z_i)_{i\geq 1}$ are also independent.

Observing now that for each fixed i and k, we have, from Relation (1),

$$p_{k,n}(i) = i \sum_{h=1}^{k-1} \frac{1}{n-h} + o(1/n),$$

we obtain

$$\lim_{n\longrightarrow\infty} \mu_{k,n}(i) = \lambda \lim_{n\longrightarrow\infty} (n-i)p_{k,n}(i) = (k-1)\lambda i.$$

From Relation (1), we also trivially get

$$\lim_{n\longrightarrow\infty} \mu_{k,n}(n-i) = \lambda i \lim_{n\longrightarrow\infty} p_{k,n}(n-i) = \lambda i.$$

Defining $R_{k,n}(i) = U_{k,n}(i) - \mathbb{E}(U_{k,n}(i))$ we obtain, since $\mathbb{E}(U_{k,n}(i)) = 1/\mu_{k,n}(i)$,

$$R_{k,n}(i) = U_{k,n}(i) - \mathbb{E}(U_{k,n}(i)) = \frac{\mu_{k,n}(i)U_{k,n}(i) - 1}{\mu_{k,n}(i)} \xrightarrow{\mathcal{L}} \frac{Z_i - 1}{(k-1)\lambda i} \text{ as } n \longrightarrow \infty. \tag{15}$$

In the same way, we get

$$R_{k,n}(n-i) = \frac{\mu_{k,n}(n-i)U_{k,n}(n-i)-1}{\mu_{k,n}(n-i)} \xrightarrow{\mathcal{L}} \frac{Z_i-1}{\lambda i} \text{ as } n \longrightarrow \infty. \tag{16}$$

Suppose that n is odd, *i.e.*, that $n = 2\ell+1$. Defining

$$V_{k,\ell} = \sum_{i=1}^{\ell} R_{k,2\ell+1}(i) \text{ and } \overline{V}_{k,\ell} = \sum_{i=1}^{\ell} R_{k,2\ell+1}(2\ell+1-i),$$

we have

$$\begin{aligned}\Theta_{k,2\ell+1} - \mathbb{E}(\Theta_{k,2\ell+1}) &= \sum_{i=1}^{2\ell} [U_{k,2\ell+1}(i) - \mathbb{E}(U_{k,2\ell+1}(i))] \\ &= \sum_{i=1}^{2\ell} R_{k,2\ell+1}(i) = V_{k,\ell} + \overline{V}_{k,\ell}. \end{aligned} \tag{17}$$

Observe that the random variables $V_{k,\ell}$ and $\overline{V}_{k,\ell}$ are independent.

The rest of the proof consists in checking the hypothesis of the principle of accompanying laws of Theorem 3.1.14 of [24]. We introduce the notation

$$V_{k,\ell,m} = \sum_{i=1}^{m-1} R_{k,2\ell+1}(i) \text{ and } \overline{V}_{k,\ell,m} = \sum_{i=1}^{m-1} R_{k,2\ell+1}(2\ell+1-i).$$

Using the fact that $\mathbb{E}(R_{k,n}(i)) = 0$ and that the $R_{k,n}(i)$ are independent, we have

$$\begin{aligned}\mathbb{E}\left((V_{k,\ell} - V_{k,\ell,m})^2\right) &= \mathbb{E}\left(\left[\sum_{i=m}^{\ell} R_{k,2\ell+1}(i)\right]^2\right) = \mathbb{V}\left(\sum_{i=m}^{\ell} R_{k,2\ell+1}(i)\right) \\ &= \sum_{i=m}^{\ell} \mathbb{V}(R_{k,2\ell+1}(i)) = \sum_{i=m}^{\ell} \mathbb{V}(U_{k,2\ell+1}(i)) = \sum_{i=m}^{\ell} \frac{1}{\mu_{k,2\ell+1}^2(i)} \\ &= \frac{1}{\lambda^2} \sum_{i=m}^{\ell} \frac{1}{(2\ell+1-i)^2 p_{k,2\ell+1}^2(i)}\end{aligned}$$

and, in the same way,

$$\mathbb{E}\left((\overline{V}_{k,\ell} - \overline{V}_{k,\ell,m})^2\right) = \sum_{i=m}^{\ell} \frac{1}{\mu_{k,2\ell+1}^2(2\ell+1-i)} = \frac{1}{\lambda^2} \sum_{i=m}^{\ell} \frac{1}{i^2 p_{k,2\ell+1}^2(2\ell+1-i)}$$

Using Lemma 4.1, we have

$$\lim_{m\longrightarrow\infty} \limsup_{\ell\longrightarrow\infty} \mathbb{E}((V_{k,\ell} - V_{k,\ell,m})^2) = \lim_{m\longrightarrow\infty} \limsup_{\ell\longrightarrow\infty} \mathbb{E}((\overline{V}_{k,\ell} - \overline{V}_{k,\ell,m})^2) = 0.$$

Using now the Markov inequality, we obtain, for all $\varepsilon > 0$,

$$\mathbb{P}\{|V_{k,\ell} - V_{k,\ell,m}| \geq \varepsilon\} = \mathbb{P}\{(V_{k,\ell} - V_{k,\ell,m})^2 \geq \varepsilon^2\} \leq \frac{\mathbb{E}((V_{k,\ell} - V_{k,\ell,m})^2)}{\varepsilon^2}$$

and

$$\mathbb{P}\{\left|\overline{V}_{k,\ell} - \overline{V}_{k,\ell,m}\right| \geq \varepsilon\} = \mathbb{P}\{(\overline{V}_{k,\ell} - \overline{V}_{k,\ell,m})^2 \geq \varepsilon^2\} \leq \frac{\mathbb{E}((\overline{V}_{k,\ell} - \overline{V}_{k,\ell,m})^2)}{\varepsilon^2}.$$

Putting together these results, we deduce that for all $\varepsilon > 0$, we have

$$\lim_{m\longrightarrow\infty} \limsup_{\ell\longrightarrow\infty} \mathbb{P}\{|V_{k,\ell} - V_{k,\ell,m}| \geq \varepsilon\} = \lim_{m\longrightarrow\infty} \limsup_{\ell\longrightarrow\infty} \mathbb{P}\{\left|\overline{V}_{k,\ell} - \overline{V}_{k,\ell,m}\right| \geq \varepsilon\} = 0. \tag{18}$$

Let us introduce the notation

$$W_{k,m} = \frac{1}{(k-1)\lambda} \sum_{i=1}^{m-1} \frac{Z_i - 1}{i} \text{ and } \overline{W}_{k,m} = \frac{1}{\lambda} \sum_{i=1}^{m-1} \frac{Z_i - 1}{i}.$$

Using (15) and (16) and the fact that the $R_{k,n}(i)$ are independent, we have

$$V_{k,\ell,m} \xrightarrow{\mathcal{L}} W_{k,m} \text{ and } \overline{V}_{k,\ell,m} \xrightarrow{\mathcal{L}} \overline{W}_{k,m} \text{ as } \ell \longrightarrow \infty. \tag{19}$$

The hypothesis of the principle of accompanying laws of Theorem 3.1.14 of [24] are properties (18) and (19). We can thus conclude that

$$V_{k,\ell} \xrightarrow{\mathcal{L}} \frac{1}{(k-1)\lambda} W \text{ and } \overline{V}_{k,\ell} \xrightarrow{\mathcal{L}} \frac{1}{\lambda} W \text{ as } \ell \longrightarrow \infty.$$

This means, from Relation (17), that

$$\Theta_{k,2\ell+1} - \mathbb{E}(\Theta_{k,2\ell+1}) \xrightarrow{\mathcal{L}} \frac{1}{(k-1)\lambda} W^{(1)} + \frac{1}{\lambda} W^{(2)} \text{ as } \ell \longrightarrow \infty,$$

where $W^{(1)}$ and $W^{(2)}$ are independent and identically distributed as W. The same reasoning applies in the case where $n = 2\ell$.

Corollary 4.3 *For all $x \in \mathbb{R}$ and $k \geq 2$, we have*

$$\lim_{n\longrightarrow\infty} \mathbb{P}\left\{\Theta_{k,n} - \mathbb{E}(\Theta_{k,n}) \leq x\right\} = \int_0^\infty \exp\left(-t - t^{-(k-1)} e^{-(k-1)\lambda x - k\gamma}\right) dt.$$

Proof. L. Gordon has proved in [15] that

$$-\gamma + \sum_{i=1}^{+\infty} \frac{1 - Z_i}{i} \overset{\mathcal{L}}{=} \ln(Z_1),$$

where (Z_i) are i.i.d. exponential with rate 1. Thus, by definition of W in Theorem 4.2, we have

$$W \overset{\mathcal{L}}{=} -\gamma - \ln(Z_1).$$

Introducing $W^{(1)} \overset{\mathcal{L}}{=} -\gamma - \ln(Z_1)$ and $W^{(2)} \overset{\mathcal{L}}{=} -\gamma - \ln(Z_2)$, we obtain from Theorem 4.2, for all $x \in \mathbb{R}$,

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \mathbb{P}\left\{\Theta_{k,n} - \mathbb{E}(\Theta_{k,n}) > x\right\} &= \mathbb{P}\left\{\frac{1}{(k-1)\lambda} W^{(1)} + \frac{1}{\lambda} W^{(2)} > x\right\} \\
&= \mathbb{P}\left\{-\frac{1}{\lambda}\left(\frac{k\gamma}{k-1} + \frac{\ln(Z_1)}{k-1} + \ln(Z_2)\right) \leq x\right\} \\
&= \mathbb{P}\left\{\frac{\ln(Z_1)}{k-1} + \ln(Z_2) \geq -\lambda x - \frac{k\gamma}{k-1}\right\} \\
&= \mathbb{P}\left\{Z_1^{1/(k-1)} Z_2 \geq e^{-\lambda x - k\gamma/(k-1)}\right\} \\
&= \int_0^\infty \mathbb{P}\left\{Z_1 \geq t^{-(k-1)} e^{-(k-1)\lambda x - k\gamma}\right\} e^{-t} dt \\
&= \int_0^\infty \exp\left(-t - t^{-(k-1)} e^{-(k-1)\lambda x - k\gamma}\right) dt,
\end{aligned}$$

which completes the proof.

Corollary 4.4 *For all $x \in \mathbb{R}$ and $k \geq 2$, we have*

$$\lim_{n \longrightarrow \infty} \mathbb{P}\left\{\Theta_{k,n} - \frac{k\ln(n)}{(k-1)\lambda} \leq x\right\} = \int_0^\infty \exp\left(-t - \frac{t^{-(k-1)}e^{-(k-1)\lambda x}}{k-1}\right) dt$$

Proof. Observe first that we have

$$\mathbb{P}\left\{\Theta_{k,n} - \frac{k\ln(n)}{(k-1)\lambda} \leq x\right\} = \mathbb{P}\left\{\Theta_{k,n} - \mathbb{E}(\Theta_{k,n}) \leq x + \frac{k\ln(n)}{(k-1)\lambda} - \mathbb{E}(\Theta_{k,n})\right\}.$$

From Theorem 3.7, we have

$$\lim_{n \longrightarrow \infty} \left(\frac{k\ln(n)}{(k-1)\lambda} - \mathbb{E}(\Theta_{k,n})\right) = -\frac{\gamma k - \ln(k-1)}{(k-1)\lambda}.$$

It follows from Corollary 4.3 that

$$\lim_{n \longrightarrow \infty} \mathbb{P}\left\{\Theta_{k,n} - \frac{k\ln(n)}{(k-1)\lambda} \leq x\right\} = \int_0^\infty \exp\left(-t - \frac{t^{-(k-1)}e^{-(k-1)\lambda x}}{k-1}\right) dt.$$

which completes the proof.

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