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# Space funicular polygons and their applications by Émile Foulon

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**ABSTRACT:** In the thesis he presented for his *agrégation* qualification at the Faculty of Applied Sciences of the University of Liège in 1939, Émile Foulon, a University of Liège Civil Engineer of Construction with a doctorate from the University of Paris, focuses on the transposition to space of the funicular polygons traditionally used in the plane and their applications to the calculation of three-dimensional constructions. Surprisingly, his approach was an original one as apparently no systematic study of space funicular polygons, as he defined them, had been published before. Foulon systematically studies the existence and degrees of freedom of space funicular polygons as a function of the number of forces considered. He further proposes to make use of descriptive geometry and develops practical methods which he applies to spatial structures. This article aims to shed light on Foulon's approaches, their specificities and how they relate to other historical methods of space graphic statics. In particular, Foulon's theoretical and practical motivations will be compared to the radically different approach of Benjamin Mayor. This also explores why Foulon's methods fell into oblivion.

## 1 INTRODUCTION

The transposition of plane funicular polygons to space has apparently not been the subject of a precise and detailed study apart from that which Émile Foulon completed in 1939 as part of his thesis for the *agrégation* qualification, roughly the equivalent of a doctoral thesis, at the Faculty of Applied Sciences of the University of Liège (Foulon 1939). Maurice Lévy (1886) defined space funicular polygons by analogy to plane funicular polygons and set down the conditions that they must verify. He quickly abandoned their study, mainly because they could not be built for any number of fully defined space forces. Benjamin Mayor (1910) considered an analogy linked to the general properties of funicular polygons that should be preserved in space. To this end, he constructed specific concepts and proposed the *funicular chain* notion which differed significantly from that of funicular polygons.

The lack of theoretical and applied studies on space funicular polygons makes the work of Émile Foulon unique and particularly interesting in relation to knowledge of the graphic statics of space. However, despite the originality and potential applications of Émile Foulon's approach, it was not widely disseminated and failed to leave any enduring mark on science. Moreover, little biographical information on its author is available. His publications inform us that he was a civil engineer in construction from the University of Liège, who held a doctorate from the University of Paris (Foulon 1938), and who presented his thesis for the *agrégation* qualification at the Faculty of Applied Sciences, the University of Liège in 1939 (Foulon 1939). His research work, funded by an endowment from the Francqui Foundation, was carried out at the

Swiss Federal Institute of Technology in Zurich, which he describes as "the parent school of graphic statics" and where he worked with Professor F. Stüssi in particular. His thesis for the *agrégation* sets forth a graphical method of composition and decomposition of forces in space based on the traditional knowledge of engineers and requiring few plots compared to other graphical methods. For this, he relies on the use of space funicular polygons, the existence and properties of which he systematically studies according to the number of considered space forces. He also makes use of descriptive geometry to achieve a purely geometric method. This article first discusses how the question of the transposition to space of funicular polygons was approached by Maurice Lévy and Benjamin Mayor. Then, it sets forth Foulon's theoretical and practical approach and addresses the question of its failure to leave an enduring mark.

## 2 THE QUESTION OF TRANSPOSITION OF FUNICULAR POLYGONS FROM PLANE TO SPACE

### 2.1 *Plane funicular polygons*

The construction of plane funicular polygons was introduced by Varignon (1725). The usefulness of the force polygon and funicular polygon concepts for the graphical resolution of static problems was recognized by Carl Culmann (1864). He contributed to disseminating these approaches by formalizing the methods of graphic statics. We shall first briefly present the main elements relating to funicular polygons. A funicular polygon with vertices  $M, a_1, a_2 \dots a_n, N$  (Figure 1,

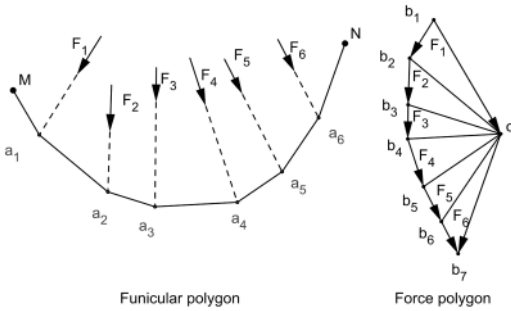


Figure 1. Funicular polygon and force polygon in the plane.

left) can be defined as the equilibrium shape taken by a wire fixed at its two ends (M and N) to which  $n$  forces are applied ( $F_1, F_2, \dots, F_n$ ) whose lines of action, directions and magnitudes are known. The equilibrium of each vertex  $a_i$  of the polygon results in the equilibrium of three forces: the force  $F_i$ , and the tensions in the two adjacent sides. The balance on either side of the funicular polygon requires that the two forces that apply to the sides are directly opposed. This is reflected in the force polygon (Figure 1, right) by the decomposition of each force from a single point  $o$ , called the pole. A funicular polygon can be constructed by choosing any point  $a_1$  on the line of action of  $F_1$ , which corresponds to one degree of freedom, and any pole in the plane of forces, which corresponds to 2 degrees of freedom. Plane funicular polygons therefore have 3 degrees of freedom for a fully defined force system. That is to say that one can construct 3 infinities of funicular polygons for a given system of forces. A major advantage of funicular polygons is that they allow a system of any  $n$  forces in the plane to be reduced to two forces.

These two forces are those applied to the outermost sides of the funicular polygon and correspond to the vectors  $b_1o$  and  $ob_n$  in the force polygon.

Depending on the case, the two forces can in turn be reduced to a non-zero resulting force or to a non-zero resulting torque, or else they can be in equilibrium. The funicular polygon concept can be extended to space if one is interested in the equilibrium shape taken by a wire fixed at its ends to which are applied any  $n$  forces of space ( $F_1, F_2, \dots, F_n$ ). It is just such a conceptual extension that Maurice Lévy introduced.

## 2.2 Definition of space funicular polygons and the conclusions of Maurice Lévy

Maurice Lévy (1886) addresses the question of funicular polygons in space by first giving their definition by analogy to funiculars in the plane, when considering any space forces.

As in the case of a plane, each vertex of the funicular polygon is subjected to three forces in equilibrium. Moreover, Lévy noticed that the three concurrent forces of space in equilibrium are necessarily coplanar. By also considering the equilibrium on each side of the funicular polygon, he deduced the following two conditions (Lévy 1886, 429):

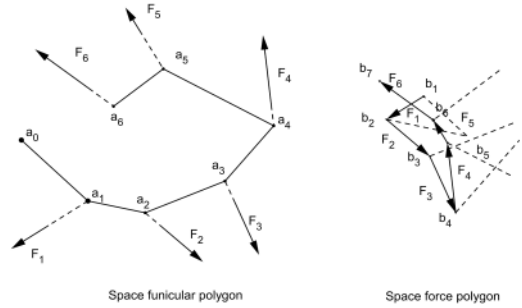


Figure 2. Space funicular polygon geometrically constructed for a given system of space forces and space force polygon incompatible with the equilibrium of the sides.

“Thus, it is necessary: 1° that each force be situated in the plane of the two sides of the polygon which are adjacent to it;

2° that the two components of the given forces, directed along each of the intermediate sides of the polygon, be equal and opposite.”

Lévy assumes that the forces  $F_1$  to  $F_n$  of space, as well as the first side  $a_0 a_1$  of the funicular polygon (Figure 2, left), are known. The first condition makes it possible to successively construct  $a_2, a_3 \dots a_n$  only by intersections of planes and lines. Therefore, vertex  $a_2$  is the point of intersection of the plane containing  $F_1$  and  $a_0 a_1$  with the line of action of  $F_2$ . Likewise, vertex  $a_3$  is the point of intersection of the plane containing  $F_2$  and  $a_1 a_2$  with the line of action of  $F_3$ .

Thus, if one knows a side of the funicular polygon, the condition of coplanarity of the sides adjacent to the vertices of the funicular polygon give a geometric means of constructing the funicular polygon from successive intersections of these planes with the action lines of the following forces. Note that in the case of a plane, the first condition is always true and does not allow for this construction, which leaves the possibility of building the funicular polygon by imposing the balance on its sides.

It then remains to check the equilibrium condition on each side of the funicular polygon (second condition set forth by Lévy). This condition implies, as in the case of a plane, that two consecutive forces must be decomposed according to their common adjacent side in such a way that the forces are balanced.

In the general instance, Lévy points out that there is no reason why this should be the case. If this were the case, there would exist, as for a plane, a pole where the lines parallel to the sides of the funicular polygon would converge, passing through the ends of the force vectors of the space force polygon, thus forming what Foulon calls the force pyramid (see 3.1). Figure 2 illustrates the construction of a space funicular polygon when a first side is known and taking into account only the first condition. We observe in this figure, on the right, the space force polygon amounting to the sum of the forces  $F_1 \dots F_n$  and we observe that the lines parallel to the sides of the funicular polygon, shown as dotted lines, do not converge.

Lévy concludes that, in the general case, the conditions of equilibrium on each side are not satisfied and the generality of the construction of funicular polygons is not found in space. He does not pursue the study of space funicular polygons any further. Foulon for his part contests Lévy's conclusion. He points out that the hypothesis of fixing the first side  $a_0 a_1$  is arbitrarily restrictive. He also indicates that, even in this case, if one limits oneself to a set of two space forces, a funicular polygon does exist. In his thesis, he aims in particular to systematically study the conditions of existence of these funicular polygons and to deduce their practical uses. To this end, he considers the conditions of planarity and equilibrium simultaneously in relation to the number of forces in space.

### 2.3 Reduction of a system of forces in space to two equivalent forces

If we manage to determine a funicular polygon for a system of forces in space, we immediately obtain two equivalent forces by considering the forces applied to the outermost sides. One of the consequences of the construction of funicular polygons is thus to determine two forces equivalent to a system of forces in space. Part 4 details the approach proposed by Foulon using funicular polygons. A different approach is proposed by Lévy with the funicular pyramid method.

### 2.4 The funicular pyramid according to Maurice Lévy

Lévy considers the case of any number of forces in space and proposes a method for reducing this system to two forces, one of which passes through an arbitrary point  $O$ .

This allows him to set forth the following theorem: "Forces distributed in any way in space can be reduced to two, one of which passes through an arbitrary point" (Lévy 1886, 433).

The method by which he proves his theorem involves the decomposition of each force  $F_i$  into three concurrent forces at point  $a_i$  in the line of action of  $F_i$ , one of which passes through  $O$ . Figure 3 details the construction in the case of six forces  $F_1$  to  $F_6$  after Lévy (1886, 430).

Force  $F_1$  is broken down into three forces by a first known force in magnitude, line of action and direction, and passing through fixed  $a_1$ . The second force goes through  $a_1$  and  $O$ . The third force is based on the line of action of  $F_2$ . Lévy also proves that such a decomposition is always possible. Gradually, we arrive at the decomposition of  $F_6$  whose last force is based on the line  $a_0 a_1$ , which simultaneously defines  $a_0$ , which was not yet fixed. Thus, the forces  $F_1$  to  $F_6$  are reduced to the sum of the forces passing through  $O$  and to the two outermost forces passing through  $a_0$  which in turn are reduced to a single force.

This method does not make use of funicular polygons but does show that the decomposition of each force  $F_i$  into three concurrent forces, instead of two, makes it possible to carry out a single construction to

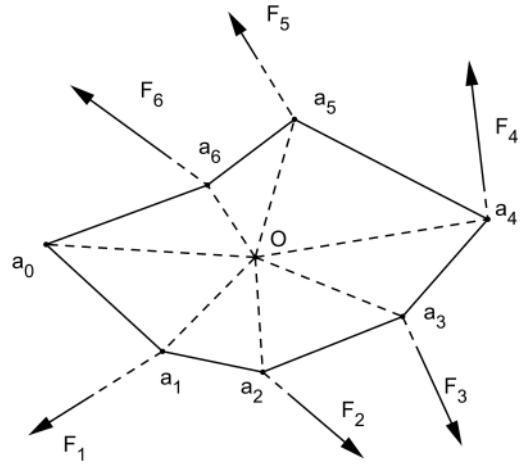


Figure 3. Funicular pyramid after Lévy (1886).

reduce the system to two forces, one of which passes through a fixed point  $O$ .

### 2.5 The Benjamin Mayor approach

Benjamin Mayor (1866–1936) aspired to propose a method that could generalize the methods of graphic statics developed in the plane so they could be applied as much as possible to space. A description of these approaches is presented by Ciblac (2018). Mayor chooses to rule out the use of descriptive geometry and rely instead on ruled geometry, in particular on the notion of linear complex, to take advantage of the dualistic character associated with lines in space.

The method thus developed, called "graphic statics of the systems of space", consists in transforming a spatial problem into a plane problem that can be treated by the conventional methods of graphic statics. Mayor introduces the notion of action complex of a system of forces in space as follows: "A system of forces, acting on a rigid solid, is completely defined by the linear complex formed by its straight lines of zero moment and by the magnitude of its general resultant. This complex, which can play the same role as the line of action of a force belonging to a planar system, will be called, for this reason, the *complex of action* of the considered system" (Mayor 1896). The lines of zero moment must be understood as being the axes with respect to which the sum of the moments of the forces of the system is zero. Mayor then introduces the notion of a funicular chain relating to a system of forces using action complexes. He shows that these funicular chains have geometric and mechanical properties which correspond exactly to those possessed by funicular polygons.

This approach contains the major drawback, emphasized by Foulon, of taking recourse to the theory of ruled geometry, which is not part of the theoretical background taught to engineers, thereby making it difficult to use. Émile Foulon also considered

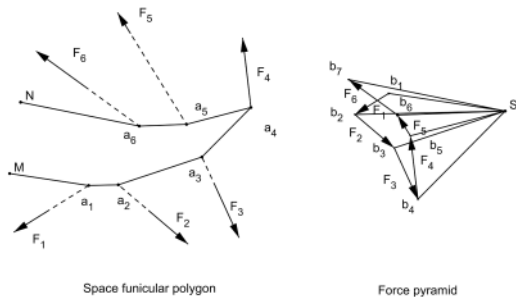


Figure 4. Space funicular polygon, force pyramid and pole S.

that rejecting descriptive geometry deprived users of powerful methods for solving spatial problems.

### 3 DEFINITION AND EXISTENCE OF SPACE FUNICULAR POLYGONS

#### 3.1 Definition of the space funicular polygons and force pyramid

Foulon (1939, 13–16) reworks Lévy’s definition (see 2.2), laying down the following two conditions:

- First condition: “Two successive sides of the funicular polygon must be in the same plane as the force that they intercept and must intersect on it.”
- Second condition: “The components, according to any intermediate side, of the two forces on which it rests at its ends, must be equal and opposite.”

He deduces the following theorem:

“For a funicular polygon to exist between a system of given forces in space, it is necessary and sufficient that we can find a continuous broken line whose vertices lie on the lines of action of the given forces and such that if, through the vertices of a polygon of these forces, we lead the rays parallel to its respective sides, these rays all converge at the same point, which is the apex of the corresponding force pyramid. The vertex S of the force pyramid will, in the continuation, often be called pole S of the corresponding funicular polygon.”

Figure 4 represents a funicular polygon assumed to exist for 6 forces of space with the force pyramid indicating satisfaction of the second condition. The conditions of existence of the space funicular polygons are presented below.

#### 3.2 Demonstration of the existence of space funicular polygons

In his study, Foulon first presents the demonstrations regarding the existence and the number of degrees of freedom of funicular polygons depending on the number of forces. To do so, he relies on the consequences of the two conditions that define them. He begins with the case of three forces and proves that there is a double infinity of space funicular polygons relating to three forces in space corresponding to two

Table 1. Number of degrees of freedom of funicular polygons depending on the number of space forces given.

Number of space forces	1	2	3	4	5	6	7	8
Number of degrees of freedom	4	3	2	1	0	-1	-2	-3

degrees of freedom (Foulon 1939, 19). He also determines analytically and geometrically the locations of corresponding poles and proves that these are quadric-type ruled surfaces. In the case of four forces of space (Foulon 1939, 45), he proves that there is an infinity of funicular polygons and that the location of the corresponding poles is the intersection curve of two quadrics. He deduces that, unlike plane funicular polygons, the existence of space funicular polygons is linked to the number of forces considered and that the addition of a force reduces their degrees of freedom by one. He establishes that in the general case of space forces arbitrarily defined in magnitude, line of action and direction, the degrees of freedom are as given in Table 1. By naming  $n$ , the number of space forces completely determined by the line of action, direction and magnitude, and  $p$ , the number of degrees of freedom of funicular polygons that can be constructed by considering these forces, we reach the following equation:

$$p = 5 - n \quad (1)$$

The previous results prove it is possible to determine funicular polygons in an infinite number for fewer than five forces and in a finite number for five forces. Beyond five forces,  $p$  being negative, it is not possible to construct a funicular polygon if all the forces are arbitrarily chosen. For this to be possible, the forces must then meet as many conditions as there are negative degrees of freedom.

#### 3.3 Space funicular polygons relating to one and two arbitrary space forces.

In the case of a single force  $F_1$  of space, the four degrees of freedom correspond for example to the possibility of decomposing this force into two concurrent forces using pole O freely chosen in space (three degrees of freedom) and a freely chosen intersection point  $a_1$  on the line of action of  $F_1$  (one degree of freedom).

The three degrees of freedom relating to the construction of a funicular polygon for any two space forces  $F_1$  and  $F_2$  to correspond for example to the possibility of freely choosing points  $a_1$  and  $a_2$  on their lines of action and of freely choosing pole O on a parallel to  $a_1a_2$  passing through  $b_2$ . Thus, for  $a_1$ ,  $a_2$  and O fixed in this way, the three degrees of freedom are blocked and a single funicular polygon can be constructed.

Figure 5 shows the construction of the funicular polygon according to these assumptions in descriptive

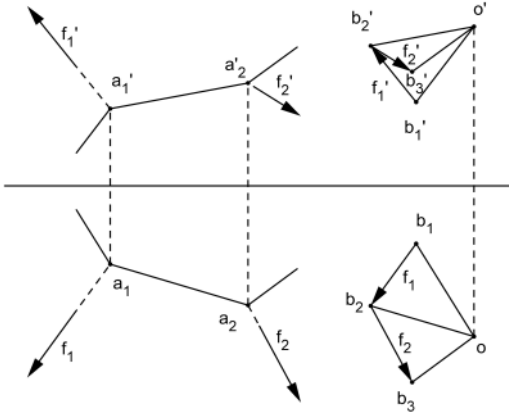


Figure 5. Descriptive geometry construction of a funicular polygon for two forces of space (left) and a force pyramid (right).

geometry. The funicular polygon is shown on the left. By convention, forces  $F_1$  and  $F_2$  are represented in horizontal projection by  $f_1$  and  $f_2$  and in frontal projection by  $f_1'$  and  $f_2'$ . The two projections of the force pyramid are shown on the right. Constructing the force pyramid makes it possible to deduce the outermost sides of the funicular polygon and the two space forces that are applied to it, which are vectorially equal to  $\mathbf{b}_1\mathbf{O}$  and  $\mathbf{O}\mathbf{b}_2$ . These two forces are statically equivalent to forces  $F_1$  and  $F_2$ .

A different way to fix a funicular polygon, blocking three degrees of freedom, is to choose any pole  $O$  in space to build the force pyramid. In this case, points  $a_1$  and  $a_2$  and the last sides are deduced from the funicular pyramid and from the first condition.

Another consequence of the existence of three degrees of freedom for building the funicular polygon in the case of two space forces is that one can also arbitrarily give oneself the first side of the funicular polygon, that is to say, any straight line passing through any point  $a_1$  of the line of action of  $F_1$ .

Thus, in particular, the last side of the funicular polygon is uniquely defined. Conversely, if we take the last side of the funicular polygon, passing through point  $a_2$  of  $F_2$ , the first side is uniquely defined. Foulon deduces from this that the two outermost forces correspond to the combined forces used by Mayor, among others (Foulon 1939, 80).

He also notices that the line  $a_1a_2$  is a line of zero moment of the force system. It therefore belongs to the action complex defined by Mayor in his theory of funicular chains. He thus links the construction of the funicular polygon in the case of two forces to the concepts of conjugate forces and action complex of Mayor's theory. He also proposes adapting Mayor's method of calculation of reticulated structures. To do so, Foulon utilizes space funicular polygons and descriptive geometry in order to make Mayor's method purely graphic and also to avoid any errors in signs arising due to recourse to analytical calculations.

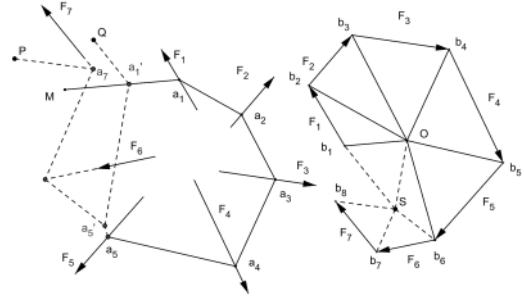


Figure 6. Reduction to two forces of a system of seven forces of space using two space funicular polygons.

#### 4 REDUCTION OF A SPACE FORCES SYSTEM TO TWO EQUIVALENT FORCES

Whenever it is possible to determine a funicular polygon for a set of forces, we deduce two statically equivalent forces. Émile Foulon proposes an iterative method for reducing a system of  $n$  space forces ( $n > 5$ ) to two equivalent forces. To this end, we begin by reducing at most five of these forces to two equivalent forces. We then consider these two forces and the remaining forces. This allows us to decrease the total number of forces that are to be reduced.

The same procedure is then applied as many times as necessary until two forces are obtained. Figure 6 illustrates the example of seven space forces (Foulon 1939, 70). It first considers the funicular polygon associated with the forces  $F_1$  to  $F_5$  according to pole  $O$  and deduces from it two resulting forces  $\mathbf{b}_1\mathbf{O}$  and  $\mathbf{O}\mathbf{b}_5$  on the sides  $Ma_1$  and  $a_5a_5'$ . These two forces, associated with the remaining forces  $F_6$  and  $F_7$ , form a set of four forces for which a funicular polygon is built with the pole  $S$ . The overall system is reduced to the forces  $\mathbf{b}_8\mathbf{S}$  and  $\mathbf{S}\mathbf{b}_1$  acting on the sides  $Qa_1'$  and  $Pa_7$  of the last funicular polygon.

#### 5 APPLICATION TO STRUCTURAL CALCULATIONS

##### 5.1 Seeking reactions in support of rigid structures

Foulon presents a systematic approach for determining the reactions of rigid structures based on the types of bonds and their combinatorics in order to treat the externally isostatic and rigid constructions for which the reactions can be obtained graphically.

He considers three types of links according to the fixity of the support point (type a, three link bars) or its mobility on a line (type b, two link bars) or a surface (type c, one link bar). Externally, isostatic and rigid constructions can be supported according to one of the six general arrangements described in Table 2.

These arrangements are not sufficient to ensure the complete connection of the rigid solid under any load. Conditions on the positions of the bars must be given in order to ensure complete bonding. Foulon

Table 2. Arrangements according to the number of types of joints

Number of the arrangement	Number of types of joints		
	a	b	c
1	1	1	1
2	1	0	3
3	0	3	0
4	0	2	2
5	0	1	4
6	0	0	6

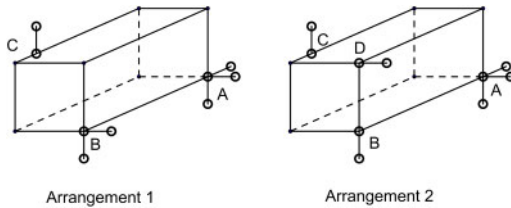


Figure 7. Examples of cases of arrangements 1 and 2.

systematically details these provisions as well as the restrictions concerning the relative positions of the bars.

By way of illustration, Figure 7 shows, in axonometry, examples of connections of a rigid solid in the cases of arrangements 1 and 2. On the left, arrangement 1 with, respectively in A, B and C, connections of type a, b and c. On the right, arrangement 2 with, in A, a type a connection, and in B, C and D, a type c connection. For each of the arrangements, Foulon proposes the use of space funicular polygons in order to determine the reactions in the supports as a function of the number  $f$  of external forces applied to it and the number  $a$  of supports. He first determines the number  $s$  of degrees of freedom of the funicular polygons that can then be used for the graphic determination of the reactions according to  $f$  and  $a$ . Equation (2) provides this relation (Foulon 1939, 103):

$$s = 6 - (f + a) \quad (2)$$

Foulon gives the proof of this equation as follows. The number  $a$  of supports corresponds to the number of external forces of connections, and the number  $f + a$  corresponds to the total number of forces relating to the funicular polygon. From equation (1) we deduce:

$$p = 5 - (f + a) \quad (3)$$

In all cases, the  $a$  forces of connection must satisfy six equilibrium conditions which correspond to six additional degrees of freedom of the funicular polygons. Foulon deduces the number of degrees of freedom  $l$  of the funicular polygons between the  $f$  given forces and the  $a$  reactions that are sought:

$$l = p + 6 \quad (4)$$

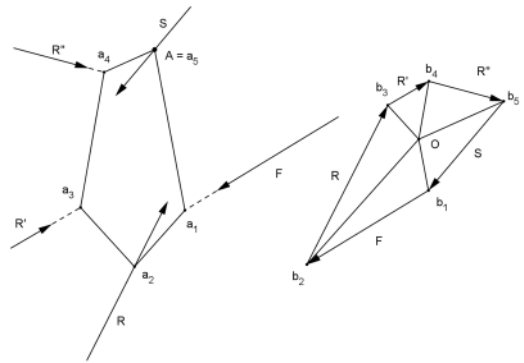


Figure 8. Solving the problem of determining the reactions for the second arrangement of supports in the case of a force.

To these conditions are added the equilibrium conditions resulting in the closure of the funicular polygon (four degrees of freedom) and the force pyramid (one degree of freedom), totalling five degrees of freedom to be removed. Thus, we obtain the following relation which, by considering (3) and (4), allows us to deduce the relation (2) yielded by the following calculation:

$$s = l - 5 = p + 1 = 6 - (f + a) \quad (5)$$

### 5.2 Example of resolution for the second arrangement of the supports

The second arrangement of supports considers a rigid construction which has a fixed support and three supports whose reactions have given lines of action. The reactions are called  $R, R', R''$  and  $S$ . The first three have their given line of action, the last passes through a fixed point A. In this case,  $a = 4$  hence, according to (2), the number  $s$  of funicular polygons making it possible to determine the reactions is equal to  $2 - f$ . In the case of a single force  $F, f = 1$  and therefore  $s = 1$ , there is therefore a degree of freedom and therefore an infinity of funicular polygons giving the four reactions to be determined.

Figure 8, on the left, illustrates the construction of one of these funicular polygons passing through A, inspired by a figure of Foulon's (1939, 124). Point  $a_2$  is built at the intersection of plane (A, F) and line of action of  $R$ . Point  $a_4$  is at the intersection of plane ( $a_2, R'$ ) and the line of action of  $R''$ . Point  $a_3$  is at the intersection of the plane (A,  $R''$ ) and the line of action of  $R'$ . The point  $a_1$  is at the intersection of the plane ( $a_3, R$ ) and the line of action of  $F$ . The closed funicular polygon (A,  $a_1, a_2, a_3, a_4$ ) is thus fully determined. Figure 9, on the right, shows the construction of the force polygon and the force pyramid. The vector  $b_1 b_2$  relating to the force  $F$  is given. The pole O is constructed at the intersection of the parallels to  $Aa_1$  and  $a_1 a_2$  passing respectively through  $b_1$  and  $b_2$ . The sides  $Ob_3, Ob_4$  and  $Ob_5$  of the funicular pyramid are parallel to the sides  $a_2 a_3, a_3 a_4$  and  $a_4 A$  of the funicular polygon. We deduce the vectors  $R = b_2 b_3, R' = b_3 b_4, R'' = b_4 b_5$  from the reactions that are sought by successively taking the

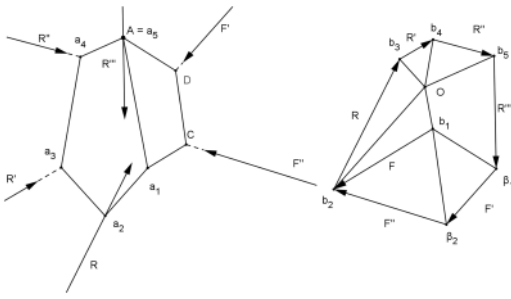


Figure 9. Resolution of the problem of determining the reactions for the second arrangement of the supports in the case of two forces.

parallels to the lines of action of the reactions  $R$ ,  $R'$  and  $R''$  passing through the vertices of the force polygon. Lastly, the force vector  $S = b_5b_1$  of the force  $S$  is determined by the closure of the force polygon. We finally deduce the line of action of  $S$  passing through  $A$ .

Figure 9 also illustrates the case of two forces  $F'$  and  $F''$ , and therefore with  $s = 0$ , which means that only one funicular polygon can be used. A solution proposed by Foulon consists of using a funicular polygon relating to two space forces  $F'$  and  $F''$ , one of the two outermost sides of which passes through  $A$  and a point  $D$  of the line of action of  $F'$  and the other,  $Ca_1$ , does not rest on  $R$ . Point  $C$  is first constructed at the intersection of the plane ( $A F'$ ) and the line of action of  $F''$ . Point  $D$  is taken anywhere on the line of action of  $F'$ . The force polygon  $\beta_1\beta_2b_2$  is first constructed knowing  $F'$  and  $F''$ . The pole  $b_1$  of the funicular polygon relative to  $F'$  and  $F''$  is deduced from the  $AD$  and  $DC$  sides of the funicular polygon.  $F'$  and  $F''$  are equivalent to the forces  $\beta_1b_1$  applied in  $A$  and  $b_1b_2 = F$  applied in  $a_1$ .

The problem is thus reduced to the previous problem. The reactions are then  $R$ ,  $R'$ ,  $R''$  and finally  $R'''$ , the reaction passing through  $A$ .

The case of three forces  $F'$ ,  $F''$   $F'''$  is resolved by Foulon by considering an auxiliary funicular polygon relating to  $F'$  and  $F''$ , and that is then reduced to the case of a single resulting force  $F$  of  $F'''$  and of  $c'$ . In the case of more than three forces, Foulon proposes methods using auxiliary funicular polygons.

### 5.3 Resolution by descriptive geometry

The methods presented above convey the feasibility of determining the reactions of structures using funicular polygons. Figures 8 and 9 show only a projection of the forces considered without putting into practice their spatial construction. Their significance is therefore purely illustrative. From a practical and graphically computational point of view, Foulon shows how a spatial problem can be completely determined by the use of descriptive geometry. He relies in particular on conventional methods of determining intersections of planes and lines used in the construction of space funicular polygons. The construction procedure is the same but it is entirely defined in space thanks to the two correlated projections.

### 5.4 Application to the calculation of reticulated frameworks in space

Descriptive geometry was used in the calculation of isostatic reticulated spatial structures by Föppl (1900) but without applying the concept of space funicular polygons. Émile Foulon provides concrete applications of his approach involving space funicular polygons and descriptive geometry through studies of a triangulated pylon, a triangulated dome and a bridge.

## 6 A METHOD THAT LEFT NO ENDURING LEGACY

Émile Foulon's work on space graphic statics was not widely disseminated. Apart from his *agrégation* thesis, we found traces of only one publication in a journal (Foulon 1940). It seems that no further work by Foulon about space funicular polygons was ever published. Nor does it appear that this approach was widely taught or included in textbooks. Moreover, this research has hardly ever been cited in the scientific literature. Corentin Fivet (2013) does, however, cite Foulon's work in his doctoral thesis on the development of a numerical approach to space graphic statics, which allowed us to discover this source. The non-dissemination of the approach by its author probably suffices to explain its lack of enduring impact. The competition of alternative graphic methods and analytical approaches can also explain the lack of interest in the developments he contributed. Nevertheless, the theoretical and practical contributions seem sufficiently important for his approach to have aroused the interest of the scientific community. We can also hypothesize that Mayor's approach, little used but cited by Pirard (1967) in his reference work on graphic statics, might have been enhanced by the adaptation proposed by Émile Foulon, had it been disseminated.

## 7 CONCLUSION

The study of space funicular polygons by Émile Foulon constitutes an original development of graphic statics of space, bringing into play knowledge of plane graphic statics and descriptive geometry that were still in use by construction professionals in the mid-twentieth century. Space funicular polygons possess properties that are not as general as those of plane funicular polygons, but the former type of polygon can be applied iteratively to solve spatial problems. The proposed methods thus allow the composition and decomposition of forces in space as well as the fully graphic resolution of calculations of isostatic reticulated spatial structures. Émile Foulon emphasizes the reduction of the number of graphical plots resulting from the use of these methods compared to other spatial methods. If this research work left no lasting mark on scientific history, it may not have been due solely to its author's failure to disseminate it but also perhaps



because of the existence of alternative graphic methods and a context in which graphic statics methods were beginning to decline in importance.

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