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Existence of SRB measures for hyperbolic maps with weak regularity

Houssam Boukhecham

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Abstract

We prove that a C^1 hyperbolic map whose differential is regular enough has an SRB measure. The precise regularity condition is weaker than Hölder and was mentioned by various authors through the development of expanding and uniformly hyperbolic dynamics.

1 Introduction

Let M be a compact Riemannian manifold, $f : M \rightarrow M$ be a C^1 hyperbolic diffeomorphism. Such a map has a lot of invariant measures, however, there is an interesting class called SRB measures (Sinai-Ruelle-Bowen), which generalize the notion of absolutely continuous measure in some sense. If f is $C^{1+\alpha}$, it is a classical result that f has a unique SRB measure (for a survey article see [You02]). In this paper, we will be interested in the existence of an SRB measure in weak regularity. If $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a modulus of continuity, we say that f is $C^{1,\omega}$ if f is C^1 and the modulus of continuity of df is at most a multiple of ω , i.e there is $C > 0$ such that

$$\|df_x - df_y\| \leq C\omega(d(x, y)), \forall x, y \in M.$$

We say that a modulus ω is Dini summable if

$$\int_0^1 \frac{\omega(t)}{t} dt < +\infty. \tag{1}$$

This condition was known since the work of Anosov in the 1967 [Ano67a, Ano67b], but he didn't give it a specific name. Later in the 1975, Walters gave an equivalent formulation of this condition in the case of a symbolic space which is a summability of variation [Wal75]. Then, in the 1994, Gora named this condition Schmitt's condition, and proved that it is the weakest condition that ensures the existence of an absolutely continuous invariant probability measure (ACIP) for a piece-wise expanding maps [Gór94]. Soon after, in the 2000, Li and Zhang named this condition Dini condition, and showed that it is sufficient to

get the convergence of the transfer operator [LZ00]. Separately, in the 2001 Fan and Jiang proved the same results (see [FJ01a]), moreover, they gave a speed of convergence of the transfer operator defined by a potential that has a Dini modulus of continuity ([FJ01b]). (for more classical introduction to transfer operator see [Bal00]).

To summarise, if T is a C^1 expanding map, with derivative that has a modulus that satisfies condition (1), then T has an ACIP. Gora and Schmitt gave an explicit example of a C^1 expanding map that does not admit an ACIP [GS89]. Then, Quass proved that generically C^1 expanding maps of the circle does not have an ACIP [Qua99]. Soon after, Avila and Bochi proved that generically C^1 maps of a smooth compact Riemannian manifold does not have an ACIP [AB06].

A natural question to ask, is whether the condition (1) is sufficient to have an SRB measure for hyperbolic maps, and whether we have the same type result of Quass, Avila and Bochi for hyperbolic maps and SRB measures. The answer to the first question is positive, and it is the main goal of this paper. We don't know yet an answer for the second question. Some examples are constructed in $C^1 \setminus C^{1+\text{Dini}}$ case, (see [Bow75b], exercise 3.2 [Man12]), and in [Qiu11], it is proven that for generic hyperbolic attractor, there is a unique equilibrium measure for the geometric potential, which is physical, but this does not answer the second question either.

Now, assume that f is $C^{1+\text{Dini}}$, meaning that f is C^1 and df has a modulus of continuity that satisfies condition (1). In order to prove the existence of an SRB measure, we start dealing with the modulus of continuity of E^u . Anosov proved in [Ano67b] that if f is $C^{1+\alpha}$ then E^u is Hölder continuous, then he remarked at the end of this paper that if f is $C^{1+\text{Dini}}$ then E^u has a modulus of continuity that satisfies Dini condition. For the sake of completeness, we provide a detailed proof of his remark. Then using this fact, we will prove that we have a distortion lemma for unstable Jacobian of f , which is the main ingredient to prove the existence of an SRB measure. The Dini condition also is sufficient to have the absolute continuity of the holonomy maps (ACH). Once one have that absolute continuity of the holonomy we deduce that the SRB measure is a physical measure.

Another approach to prove the existence of a physical measure, is to use the fact that the hyperbolic map $f : M \rightarrow M$ have a Markov partition (see [Bow75a]), which implies that f is semi-conjugated to a subshift of finite type. More precisely, there are (Σ_A, σ) and a surjective Hölder map $\pi : \Sigma_A \rightarrow M$ such that $\pi \circ \sigma = f \circ \pi$, so if we take a potential with Dini summable modulus $\phi : \Lambda \rightarrow \mathbb{R}$, then $\pi \circ \sigma$ has a Dini summable modulus. To get an equilibrium state for $(\Sigma_A, \sigma, \pi \circ \phi)$ one can reduce the problem to one-sided shift $(\Sigma_A^+, \sigma, \tilde{\phi})$, (where $\tilde{\phi}$ is a potential depending only on the future, and cohomologous to $\pi \circ \phi$ [PP90]) which is an expanding map, then we apply the adapted Ruelle-Perron-Frobenius theorem [FJ01a] to get an equilibrium measure for $(\sigma, \pi \circ \phi)$. We

push this measure by π to get an equilibrium measure for (f, ϕ) . Finally, if we consider the geometric potential $\phi^{(u)} = -\log J^u f = -\log \det df|_{E^u}$, then $\phi^{(u)}$ has a Dini modulus of continuity provided that f is $C^{1+\text{Dini}}$ (see next section), and the equilibrium measure $\mu_{\phi^{(u)}}$ of this potential is the physical measure. [Bow75a, BR75]

To sum-up, we will prove that

Theorem 1.1. *Let $f : M \rightarrow M$ be a $C^{1+\text{Dini}}$ hyperbolic diffeomorphism, then the geometric potential $\phi^{(u)} = -\log df|_{E^u}$ has a Dini modulus of continuity. In particular we have*

- i. f admits an invariant SRB measure,
- ii. the local holonomy maps are absolutely continuous,
- iii. if f is transitive, then the SRB measure is ergodic and physical, and it is the unique equilibrium measure for the geometric potential.

2 Preliminaries and notations

2.1 Uniform hyperbolicity

Let U be an open subset of M and $f : U \rightarrow M$ a C^1 diffeomorphism. An invariant set $\Lambda \subset U$ is called hyperbolic if there are some $C > 0$ and $\lambda \in (0, 1)$ such that for all $x \in \Lambda$ we have a splitting of $T_x M = E_x^u \oplus E_x^s$ which is f invariant, i.e $df_x(E_x^u) = E_{f_x}^u$ and $df_x(E_x^s) = E_{f_x}^s$ and such that

$$\|df^n(v)\| \leq C\lambda^n \|v\|, \forall n \in \mathbb{N}, v \in E_x^s, \quad (2)$$

$$\|df^{-n}(v)\| \leq C\lambda^n \|v\|, \forall n \in \mathbb{N}, v \in E_x^u. \quad (3)$$

In the definition, we didn't assume any continuity on E^s and E^u . In fact it is not hard to prove the continuity of E^u and E^s starting from the given definition. One can also assume that $C = 1$ by considering another equivalent Riemannian metric on M , and possibly alternating the value of λ (see Proposition 5.2.2 of [BS02]).

Some example of uniformly hyperbolic maps are: Arnold cat map, the Horseshoe, toral hyperbolic linear automorphism, the Smale solenoid, and in the case where $\Lambda = M$, the diffeomorphism f is called an Anosov diffeomorphism.

A classical approach to deal with uniformly hyperbolic maps, is to consider the space of continuous (resp. bounded) sections $\sigma : \Lambda \rightarrow T\Lambda$, which is a Banach space, and once we have the previous definition, we can write this Banach space as the direct sum of two closed subspaces, corresponding to sections with value on E^u or E^s . Once we do this, we have a natural linear action of f on that Banach space which preserves the closed subspaces. This approach helps us prove a lot of result like shadowing lemma, local stability, etc.

In general it is hard to check uniform hyperbolicity using this definition (for instance we don't know E^u and E^s), to deal with this difficulty we study cones instead of linear subspaces.

2.2 Hyperbolicity via cone techniques

Let $x \in M$ and E a linear subspace of $T_x M$, define the cone centered at E by

$$K_\alpha^E(x) = \{v \in T_x M : \|v_2\| \leq \alpha \|v_1\| \text{ where } v = v_1 + v_2 \text{ and } v_1 \in E, v_2 \in E^\perp\}.$$

For a hyperbolic map f , $K_\alpha^{E^u}$ (resp $K_\alpha^{E^s}$) is called unstable (resp stable) cone field. We say that it has a small angle if α is small.

A cone field K on M is said to be invariant by f if for all $x \in M$

$$df_x(K(x)) \subset \text{int}(K(fx)) \cup \{0\}.$$

Proposition 2.1. (*Proposition 5.4.3 [BS02]*) *Let Λ be a compact invariant set of $f : U \rightarrow M$. Suppose that there is $\alpha > 0$ and for every $x \in \Lambda$ there are continuous subspaces \tilde{E}^s and $\tilde{E}^u(x)$ such that $\tilde{E}^s(x) \oplus \tilde{E}^u(x) = T_x M$, and the cone $K_\alpha^{\tilde{E}^u}(x)$ and $K_\alpha^{\tilde{E}^s}(x)$ are f invariant and $\|df_x v\| < \|v\|$ for non zero $v \in K_\alpha^{\tilde{E}^s}(x)$, and $\|df_x^{-1} v\| < \|v\|$ for non-zero $v \in K_\alpha^{\tilde{E}^u}(x)$. Then Λ is a hyperbolic set of f .*

2.3 Local stable and unstable manifolds:

Define the local stable and unstable manifolds for $x \in \Lambda$ by

$$\begin{aligned} W_\epsilon^s(x) &= \{y \in M \mid d(f^k x, f^k y) \leq \epsilon, \forall k \geq 0\}, \\ W_\epsilon^u(x) &= \{y \in M \mid d(f^{-k} x, f^{-k} y) \leq \epsilon, \forall k \geq 0\}. \end{aligned}$$

The definition of stable and unstable manifolds is dynamic, and the theorem of Perron-Hadamard proves that $W^u(x)$ and $W^s(x)$ are immersed submanifolds (see theorem 6.2.8 [KH97]).

Stable and unstable manifold are locally uniformly transversal, i.e there is sufficiently small $\epsilon > 0$, such that for $x, y \in M$ with $d(x, y) < \epsilon$, the set $W_\epsilon^s(x) \cap W_\epsilon^u(y)$ has a unique point. We denote $[x, y]$ this unique point, and it is called the Bowen bracket of x and y . A rectangle R is a subset of M such that for all $x, y \in R$, we have $[x, y], [y, x] \in R$.

2.4 SRB versus Physical

In this paper, we follow the convention used by [You02] to define SRB and physical measures, an example of a physical measure that is not SRB was given by Bowen, we don't know if there is an example of hyperbolic map that has an SRB but not physical measure.

Definition 2.2. *An invariant measure μ of a hyperbolic diffeomorphism f is called an SRB measure, if the disintegration of μ along unstable leaves have absolutely continuous conditional measures.*

Definition 2.3. An f invariant measure μ is called a physical measure if the basin $B(\mu) = \left\{ x \in M \mid \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} \rightarrow \mu \right\}$ has a positive measure with respect to the Lebesgue measure.

2.5 Modulus of continuity

In this subsection, we give a formal definition of the summability condition introduced in the abstract. We start by recalling the classical definition of a modulus of continuity.

Definition 2.4. A modulus of continuity is a continuous, increasing and concave map $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that $\omega(0) = 0$.

We say that the modulus ω is Dini summable if

$$\int_0^1 \frac{\omega(t)}{t} dt < +\infty.$$

For instance, for any $\alpha \in (0, 1)$, the map $\omega(t) = t^\alpha$ is a modulus of continuity which is Dini summable, it corresponds to Hölder regularity.

The following proposition gives an equivalent condition on a modulus ω to be Dini summable.

Proposition 2.5. The following conditions are equivalent:

- ω is Dini summable.
- $\forall c \in (0, 1)$ and $t \geq 0$, $\sum_{k=0}^{+\infty} \omega(c^k t) < +\infty$
- $\exists c \in (0, 1)$ and $t > 0$, $\sum_{k=0}^{+\infty} \omega(c^k t) < +\infty$

Proof. Since ω is concave, the map $t \mapsto \frac{\omega(t)}{t}$ is decreasing, hence we have the following inequalities for all n and small t :

$$\sum_{k=0}^{n-1} (c^k - c^{k+1}) \frac{\omega(c^k t)}{c^k t} \leq \int_{c^n t}^t \frac{\omega(x)}{x} dx \leq \sum_{k=0}^{n-1} (c^k - c^{k+1}) \frac{\omega(c^{k+1} t)}{c^{k+1} t}. \quad (4)$$

We deduce the proposition from these inequalities. □

Let ω be a Dini summable modulus, and for $c \in (0, 1)$ define

$$\tilde{\omega}_c(t) = \sum_{k=0}^{+\infty} \omega(c^k t), \forall t \geq 0. \quad (5)$$

It follows immediately that $\tilde{\omega}_c$ is a modulus of continuity.

Definition 2.6. If ω is a Dini summable modulus, we denote $\tilde{\omega}$ the modulus defined by

$$\tilde{\omega}(t) = \int_0^t \frac{\omega(s)}{s} ds.$$

Using the previous proposition, the modulus $\tilde{\omega}$ and ω_c are equivalent for any $c \in (0, 1)$ i.e there is $C > 1$ such that:

$$C^{-1}\tilde{\omega}_c \leq \tilde{\omega} \leq C\tilde{\omega}_c.$$

Remark 2.7. The Dini summability condition might seem artificial, but it becomes more natural once we see in [FJ01a] how it is used to prove Ruelle theorem for transfer operator.

Examples:

- For $\alpha \in (0, 1]$, $\omega(t) = t^\alpha$ is Dini summable, and $\tilde{\omega}(t) = \frac{1}{\alpha}\omega(t)$.
- The modulus $\omega_{\beta \log}(t) = \frac{1}{\left(\log(\frac{1}{t})\right)^\beta}$ is Dini summable if and only if $\beta > 1$.
In this example ω is defined only for small t , then we extend it by an affine map.

Definition 2.8. Let X, Y be two metric spaces, and ω a modulus of continuity, we say that a map $f : X \rightarrow Y$ is $C^{0,\omega}$ if there is a $C > 0$ such that:

$$d(f(x), f(y)) \leq C\omega(d(x, y)), \quad \forall x, y \in X. \quad (6)$$

- If $\omega(t) = t$, then $C^{0,\omega}$ is the set of Lipschitz maps.
- If $\omega(t) = t^\alpha$, where $0 < \alpha < 1$, then $C^{0,\omega}$ is the set of Hölder maps with exponent α .

Given a continuous map $g : M \rightarrow M$ of a compact manifold, a natural way to define the modulus of continuity of g would be to take:

$$\tilde{\omega}_g(t) = \sup_{\substack{x, y \in M \\ d(x, y) \leq t}} d(gx, gy). \quad (7)$$

We can assume that the modulus of continuity is concave by taking

$$\omega_g = \inf\{h \mid h \text{ continuous, concave and increasing and } h \geq \tilde{\omega}_g\}. \quad (8)$$

It is clear that ω_g is a modulus of continuity, and it satisfies the inequality (6) with constant $C = 1$.

3 Modulus of continuity of unstable distribution

Anosov proved that the stable and unstable distribution of a $C^{1+\alpha}$ hyperbolic diffeomorphism are Hölder continuous. The regularity of the stable and unstable distribution is optimal in the sense that; even if f is analytic, the stable and unstable distribution are only Hölder continuous in general (examples are given in the famous paper of Anosov [Ano67a]). At the end of the paper [Ano67b], he remarked that if f is $C^{1,\omega}$, and ω satisfies Dini condition, then stable and unstable distributions have a modulus of continuity that satisfies Dini condition.

It is easy to prove that the stable E^s and unstable E^u distributions are continuous (see [BS02]). In this section, we will recall the proof of the Hölder regularity of E^u , and give details of the computation in the weaker regularity. Fix a small $\epsilon > 0$, and approach E^s (resp E^u) by a smooth distribution \tilde{E}^s (resp \tilde{E}^u) (in other word $\|E^* - \tilde{E}^*\| < \epsilon$). Consider a distribution Y close to \tilde{E}^u , we can view this distribution as a map F defined on M with values on $L(\tilde{E}^u, \tilde{E}^s)$ as follows

$$F(x) : \tilde{E}^u \rightarrow \tilde{E}^s, \quad (9)$$

where $F(x)$ is the linear map whose graph is $Y(x)$. Let \mathcal{S} the space of such maps. For $F \in \mathcal{S}$ let $\|F\| = \max_{x \in M} \|F(x)\|$.

Let A, B, C, D the maps defined on M as follows; for $x \in M$, we have $df_x = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}$, (written with respect to the splitting $\tilde{E}^s \oplus \tilde{E}^u$).

The diffeomorphism f acts on \mathcal{S} by pushing forward the distribution. If $F \in \mathcal{S}$, let $\mathcal{L}F$ denote the map associated to the push-forward of F . Then we have

$$\mathcal{L}F(fx) = [A(x)F(x) + B(x)][C(x)F(x) + D(x)]^{-1}. \quad (10)$$

Notice that we have $\|A(x)\|, \|D^{-1}(x)\| < \lambda + \epsilon$, $\|C(x)\|, \|D(x)\| < \epsilon$ where λ is the hyperbolicity constant of f , we have also

$$\|(C(x)F(x) + D(x))^{-1}\| \leq \lambda + 4\epsilon,$$

and one can check that if $\|F\| \leq 1$ then $\|\mathcal{L}F\| < 1$, furthermore, if we denote \mathcal{S}_1 the space of distributions with norm less or equal 1, then $\mathcal{L} : \mathcal{S}_1 \rightarrow \mathcal{S}_1$ is compact operator (see points (7) and (9) in [Ano67b]).

Claim 3.1 ([Ano67b]). *Let D be a distribution on the Riemannian manifold M , then there is a connection on the bundle E (with base M and fiber D) such that local parallel transport along geodesic (with respect to the Riemannian metric on M) is an isometry.*

This parallel transport will give us a natural way to define modulus of continuity.

3.1 Modulus of continuity

Take a quadruplet (R^p, R^q, P, Q) , where R^p and R^q are two smooth distributions, and $P, Q : M \rightarrow M$ are smooth maps. Consider a map ϕ defined on M

to linear maps as follows

$$\phi(x) : R_{P_x}^p \rightarrow R_{Q_x}^q.$$

Let $\prod_{x',x}^p : R_{x'}^p \rightarrow R_x^p$ be the parallel transport of the connection associated to R^p given in the claim 3.1, then, we define the modulus of continuity of ϕ as follows

$$\omega_\phi(r) = \sup_{x,x' \in M, d(x,x') \leq r} \left| \Pi_{Q_{x'}, Q_x}^q \phi(x') \Pi_{P_x, P_{x'}}^p - \phi(x) \right|. \quad (11)$$

Using the definition, we get (whenever it make sense) that

$$\omega_{\psi\phi}(r) \leq \|\psi\| \omega_\phi(r) + \|\phi\| \omega_\psi(r). \quad (12)$$

$$\omega_{\phi^{-1}}(r) \leq \|\phi^{-1}\|^2 \omega_\phi(r). \quad (13)$$

$$\omega_{\phi+\psi}(r) \leq \omega_\phi(r) + \omega_\psi(r). \quad (14)$$

$$\omega_{\phi \circ g}(r) \leq \omega_\phi(\omega_g(r)). \quad (15)$$

3.2 Sketch of the proof of Hölder regularity

Using the previous inequalities and (10), we get

$$\omega_{\mathcal{L}F \circ f}(r) \leq \lambda \omega_F(r) + 4Kr, \quad (16)$$

where K is a Lipschitz constant for A, B, C, D meaning

$$\omega_A(r) \leq Kr, \text{ etc}, \quad (17)$$

and F represent a smooth distribution sufficiently close to E^u . Then we deduce that

$$\omega_{\mathcal{L}F}(r) \leq \tilde{\lambda} \omega_F(Lr) + 4KLr, \quad (18)$$

where L is a Lipschitz constant for f^{-1} which is bigger than 1, and $\tilde{\lambda} \in (\lambda, 1)$. Taking a small enough $\beta > 0$ and a well chosen constant C_f , and choosing Hölder map F with constant M and exponent β we get from (18) that $\omega_{\mathcal{L}F} \leq Mr^\beta$, in other word, \mathcal{L} preserves the space of Hölder maps with constant M and exponent β . Now using the fact that

$$\lim_{n \rightarrow +\infty} \|\mathcal{L}^n F - F_0\| = 0, \quad (19)$$

where F_0 is the map associated to the distribution E^u , we deduce that the distribution E^u is β Hölder. (considering f^{-1}, E^s is also β Hölder).

Remarks 3.2. *The connections used to define the modulus of continuity are not unique, but if we take another connections, they give a modulus of continuity equivalent to the previous one.*

3.3 Proof of theorem 1.1

Now, assume that f is $C^{1,\omega}$, where ω is a modulus of continuity that satisfies Dini condition, and we will prove using the same inequalities that ω_{F_0} satisfies Dini condition. Since Df has modulus ω , then instead of (17) we have

$$\omega_A(r), \omega_B(r), \omega_C(r), \omega_D(r) \leq \omega(r),$$

so we have

$$\omega_{\mathcal{L}F \circ f}(r) \leq \tilde{\lambda} \omega_F(r) + \omega(r),$$

which gives

$$\omega_{\mathcal{L}F}(r) \leq \tilde{\lambda} \omega_F(Lr) + \omega(Lr),$$

in particular, for all $n \in \mathbb{N}^*$ we have

$$\omega_{\mathcal{L}^n F}(r) \leq \tilde{\lambda} \omega_{\mathcal{L}^{n-1} F}(Lr) + \omega(Lr).$$

Taking n to $+\infty$ we get

$$\omega_{F_0}(r) \leq \tilde{\lambda} \omega_{F_0}(Lr) + \omega(Lr).$$

Using induction, we get for $m \in \mathbb{N}$

$$\omega_{F_0}(r) \leq \tilde{\lambda}^m \omega_{F_0}(L^m r) + \sum_{k=1}^m \tilde{\lambda}^k \omega(L^k r),$$

hence we have

$$\omega_{F_0}\left(\frac{r}{L^m}\right) \leq \tilde{\lambda}^m \omega_{F_0}(r) + \sum_{k=1}^m \tilde{\lambda}^k \omega\left(\frac{r}{L^{m-k}}\right),$$

and since ω satisfies Dini condition, the series with general term

$$\tilde{\lambda}^m \omega_{F_0}(r) + \sum_{k=1}^m \tilde{\lambda}^k \omega\left(\frac{r}{L^{m-k}}\right)$$

is convergent, hence $\sum_m \omega_{F_0}\left(\frac{r}{L^m}\right) < +\infty$, which is equivalent to saying that ω_{F_0} satisfies Dini condition.

4 Distortion lemma

In this section, we prove that the Dini condition is sufficient to have a distortion lemma for unstable Jacobian, which is the main ingredient to prove the existence of an SRB measure.

Lemma 4.1 (Distortion lemma). *Given any small enough open set U , there exist $C_d > 0$ such that if W_ϵ^u is a piece of unstable manifold in U , then $\forall x, y \in W_\epsilon^u$ and $n > 0$ we have*

$$C_d^{-1} \leq \frac{|\det df^{-n}(x)|_{E_x^u}|}{|\det df^{-n}(y)|_{E_y^u}|} \leq C_d. \quad (20)$$

Proof. Using the fact that E^u has a modulus ω^u we have

$$\left| \frac{\det df_{|E_{\tilde{x}}^u}^{-1}}{\det df_{|E_{\tilde{y}}^u}^{-1}} - 1 \right| \leq \omega^u(d(\tilde{x}, \tilde{y}))$$

we deduce that

$$\begin{aligned} \left| \frac{\det df_{|E_x^u}^{-n}}{\det df_{|E_y^u}^{-n}} \right| &= \left| \prod_{k=0}^{n-1} \frac{\det df_{|E_{f^{-k}x}^u}^{-1}}{\det df_{|E_{f^{-k}y}^u}^{-1}} \right| \leq \prod_{k=0}^{n-1} (1 + \omega^u(d(f^{-k}x, f^{-k}y))) \\ &\leq \prod_{k=0}^{+\infty} (1 + \omega^u(\lambda^k d(x, y))), \end{aligned}$$

and since ω^u satisfies Dini condition, we deduce that there is $C_d = C_d(\omega^u, \lambda)$ such that

$$\prod_{k=0}^{+\infty} (1 + \omega^u(\lambda^k d(x, y))) \leq C_d,$$

which finishes the proof. \square

Remark 4.2. *We prove in the same way that if $y \in W_\epsilon^s(x)$, then there is a constant C_d such that for all $n \in \mathbb{N}$ we have*

$$C_d^{-1} \leq \frac{J^u f^n(x)}{J^u f^n(y)} \leq C_d, \quad (21)$$

where we recall that $J^u f^n(x) = \det df^n(x)|_{E_x^u}$.

4.1 Proof of (i) in theorem 1.1

Consider a small piece of unstable manifold L , and let m_L be the normalized Lebesgue measure on L . Define a sequence of measures μ_n by

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k m_L.$$

Let μ be a weak* limit of $(\mu_n)_n$, and fix a rectangle R of small diameter. Let ρ_n be the density of $f_*^n m_L$ with respect to the volume of $f^n(L)$. Using lemma

4.1 we deduce that there is $C_d > 0$ that does not depend on n , such that for any $x, y \in R \cap W^u(x)$ we have

$$C^{-1} \leq \frac{\rho_n(x)}{\rho_n(y)} \leq C,$$

which implies in particular that

$$C^{-1} \leq \frac{\frac{1}{n} \sum_{k=0}^{n-1} \rho_k(x)}{\frac{1}{n} \sum_{k=0}^{n-1} \rho_k(y)} \leq C.$$

We disintegrate the measures μ_n in the rectangle R along unstable leaves, and let μ_n^s be the transversal measure of the disintegration. By the previous inequality we deduce that

$$\mu_n^s \times C^{-1}m \leq \mu_n \leq \mu_n^s \times Cm, \quad (22)$$

where m here denote the normalized Lebesgue measure along local unstable leaves in R . Up to passing to a subsequence of (μ_n) we can assume that μ_n^s converges to a measure μ^s , and since the stable holonomy is continuous, the measure μ^s is the transversal measure of μ . We deduce that

$$C^{-1}\mu^s \times m \leq \mu \leq C\mu^s \times m,$$

hence μ is an SRB measure (see [HY95]).

Remark 4.3. *The measure μ is supported on $\Omega(f)$ the set of non wandering points of f ($x \in \Omega(f)$ if and only if for all small enough neighborhood U of x there is $n > 0$ such that $f^n U \cap U \neq \emptyset$.) It is not known whether $\Omega(f) = M$ if f is Anosov.*

5 Proof of (ii) in theorem 1.1

Given an Anosov diffeomorphism, we have two foliations, stable \mathcal{F}^s and unstable \mathcal{F}^u foliation. Consider two local transversal τ_1, τ_2 to \mathcal{F}^s , then when it make sense, we call a holonomy map h_{τ_1, τ_2} the map defined from τ_1 to τ_2 , that associate for each element in τ_1 an element in τ_2 by sliding along stable leaf, in other word $h_{\tau_1, \tau_2}(x) = W_{\text{loc}}^s(x) \cap \tau_2$ (We define similarly the holonomy defined by sliding along unstable leaves). It is know that for a C^1 Anosov map, the holonomy is Hölder continuous. In this section, we will be interested in the absolute continuity, because provided that we have it, we deduce that the SRB measure is physical.

For simplicity, we will consider local unstable manifold as transversal, and prove that if f is $C^{1+\text{Dini}}$, then the holonomy is absolutely continuous. Many authors proved the absolute continuity of the holonomy in various settings (Anosov [Ano67a] for C^2 hyperbolic maps, Mané [Man12] for $C^{1+\alpha}$), the proof of Mané

can be adapted for $C^{1+\text{Dini}}$ maps. There is also a proof by Abdenur and Viana for $C^{1+\alpha}$ partially hyperbolic diffeomorphism [FH19] theorem B.7.6, and the proof we will give here is inspired by their proof. In [BR75], Bowen and Ruelle proved a weaker version of ACH, which they called volume lemma, that gives an estimation of the volume of a dynamical ball, and is very important to prove the existence of a physical measure using symbolic dynamics approach.

The proof is as follows: we prove that the holonomy does not change a lot the volume of unstable dynamical balls of the transversal, then using the Besicovitch covering argument, we will deduce the absolute continuity.

Let $x \in M$, and define the unstable dynamical ball $B_n^u(x, \epsilon)$ to be the ball in $W^u(x)$ of center x and radius ϵ for the distance $d_n(y, z) = d(f^n y, f^n z)$, where d is the Riemannian distance on $W^u(f^n x)$.

Lemma 5.1 (Volume lemma). *There is $C_\epsilon > 0$, such that for all $x \in M$ and $n \geq 0$*

$$\frac{C_\epsilon^{-1}}{J^u f^n(x)} \leq m(B_n^u(x, \epsilon)) \leq \frac{C_\epsilon}{J^u f^n(x)}, \quad (23)$$

where m denote the volume measure on $W_\epsilon^u(x)$.

Proof. Since W_ϵ^u are C^1 embedded manifold, we have for sufficiently small ϵ and for all x

$$m(B^u(x, \epsilon)) \sim \epsilon^{\dim E^u}. \quad (24)$$

Let $x \in M$, and consider the diffeomorphism $f : B_n^u(x, \epsilon) \rightarrow B^u(f^n x, \epsilon)$. Applying the change of variable formulas we get

$$m(B^u(f^n x, \epsilon)) = \int_{B_n^u(x, \epsilon)} J^u f^n(y) dm(y).$$

By distortion lemma 4.1, there is C_d such that for all $y \in B_n^u(x, \epsilon)$ we have

$$C_d^{-1} J^u f^n(y) \leq J^u f^n(x) \leq C_d J^u f^n(y),$$

integrating with respect to y we get

$$C_d^{-1} \epsilon^{\dim E^u} \leq J^u f^n(x) m(B_n^u(x, \epsilon)) \leq C_d \epsilon^{\dim E^u}, \quad (25)$$

which finishes the proof of the lemma. \square

Consider $\tau_1 = W_\epsilon^u(x_1), \tau_2 = W_\epsilon^u(x_2)$, where $x_2 \in W_\epsilon^s(x_1)$, and denote by $h = h_{\tau_1, \tau_2}$ the holonomy from τ_1 to τ_2 .

Lemma 5.2. *There is a constant $C'_\epsilon > 0$ such that for any $x \in \tau_1$ and $n \geq 0$ we have*

$$C'_\epsilon{}^{-1} m(B_n^u(x, \epsilon)) \leq m(h(B_n^u(x, \epsilon))) \leq C'_\epsilon m(B_n^u(x, \epsilon)). \quad (26)$$

Proof. Let $x \in \tau_1$, $y \in B_n^u(x, \epsilon)$ and $x' = h(x)$, $y' = h(y)$. Since the holonomy is α -Hölder for some $\alpha > 0$, and constant $C = C(h, \epsilon)$ we have

$$d_n(x', y') \leq C d_n(x, y)^\alpha,$$

hence $h(B_n^u(x, \epsilon)) \subset B_n^u(x', C\epsilon^\alpha)$, which implies that

$$\begin{aligned} m\left(h(B_n^u(x, \epsilon))\right) &\leq m(B_n^u(x', C\epsilon^\alpha)) \\ &\leq \frac{C_\epsilon^\alpha}{J^u f^n(x')} \\ &\leq \frac{C_\epsilon^\alpha C_d}{J^u f^n(x)} \\ &\leq C_\epsilon C_\alpha C_d \cdot m(B_n^u(x, \epsilon)), \end{aligned}$$

where the constant C_d is the constant coming from lemma 4.1, C_ϵ and C_α are given by lemma 5.1. We do the same argument for $h^{-1} = h_{\tau_2, \tau_1}$, to deduce the other inequality, which finishes the proof. \square

Since the constant C'_ϵ does not depend on n , and that f restricted to τ_1 is expanding, we can make the dynamical balls sufficiently small by taking n big enough, so the set \mathcal{B} of dynamical balls in τ_1 generates the open sets of τ_1 . To prove that the holonomy is absolutely continuous, it will be enough to prove that the volume of small balls does not distort much. To do this we will need the Besicovitch covering lemma.

Lemma 5.3. *There is $C_l = C_l(\epsilon)$ such that for any ball B in τ_1 , we have*

$$C_l^{-1} m(B) \leq m(h(B)) \leq C_l m(B). \quad (27)$$

Proof. Let B a ball of radius $\delta < \epsilon$ in τ_1 , and cover it by dynamical balls of depth n , i.e $B \subset \bigcup_{i \in I} B_n^u(x_i, \epsilon)$, and $x_i \in B$. By definition of dynamical balls we have

$$f^n(B) \subset \bigcup_{i \in I} B(f^n x_i, \epsilon), \quad (28)$$

applying Besicovitch covering lemma to (28), we can find a subsequence $(x_i^j)_{i,j}$ of $(x_i)_i$ such that $B \subset \bigcup_{i,j} B_n^u(x_i^j, \epsilon)$, the j varies in $\{1, 2, \dots, K\}$, where K is universal and depends only on the dimension of E^u , and for $j \neq j'$, we have for all i, i'

$$\overline{B_n^u(x_i^j, \epsilon)} \cap \overline{B_n^u(x_{i'}^{j'}, \epsilon)} = \emptyset.$$

We deduce that

$$m(B) \leq m\left(\bigcup_{i,j} B_n^u(x_i^j, \epsilon)\right) \leq K \sum_{i,j} m(B_n^u(x_i^j, \epsilon)), \quad (29)$$

and

$$m(h(B)) \leq m\left(\bigcup_{i,j} h(B_n^u(x_i^j, \epsilon))\right) \leq K \sum_{i,j} m\left(h(B_n^u(x_i^j, \epsilon))\right). \quad (30)$$

The element of the sequence (x_i^j) that lies in $\partial B_{(\lambda-\epsilon)^n}$ (the $(\lambda-\epsilon)^n$ neighborhood of the boundary of B), form a set Z_n that does not contribute much to the volume of $\bigcup_{i,j} B_n^u(x_i^j, \epsilon)$, in other word $m\left(\bigcup_{x_i^j \in Z_n} B_n^u(x_i^j, \epsilon)\right) = \epsilon_n$, and $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

So we deduce that

$$\frac{1}{K} \sum_{i,j} m(B_n^u(x_i^j, \epsilon)) - K \sum_{x_i^j \in Z_n} m(B_n^u(x_i^j, \epsilon)) \leq m(B). \quad (31)$$

Using lemma 5.2, we get also

$$\frac{1}{K \cdot C'_\epsilon} m\left(h(B_n^u(x_i^j))\right) - K \cdot C'_\epsilon \sum_{x_i^j \in Z_n} m\left(h(B_n^u(x_i^j))\right) \leq m(h(B)). \quad (32)$$

Using lemma 5.2 again, and (29) and (32) we deduce that

$$m(h(B)) \geq \frac{1}{(KC'_\epsilon)^2} m(B) - (KC'_\epsilon)^2 \epsilon_n, \quad (33)$$

taking n to infinity finishes the proof. \square

5.1 Proof of (iii) in 1.1

We proved that for a $C^{1+\text{Dini}}$ hyperbolic map f that the holonomy is absolutely continuous, and that f has an SRB measure μ . In this section we give the proof of the physicality of the SRB measure μ . In fact, in our case, the basin of the SRB measure μ has a full measure with respect to the Lebesgue measure.

Since the holonomy is absolutely continuous, we can apply Hopf argument to prove that the SRB measure μ is ergodic. Indeed, let $h \in L^2(M, \mu)$ be f invariant, then in a set of full measure with respect to μ , h is constant along local stable and unstable manifolds. Using the fact that μ is SRB measure, and the absolute continuity of the local holonomy maps, we deduce that h is constant, hence the SRB measure is ergodic. Now, by the Birkhoff ergodic theorem the basin $B(\mu)$ has a full measure with respect to μ , and since μ is an SRB, we can find a local unstable leaf L , such that almost every point in L with respect to the volume of L is in the basin of μ . Notice that $x \in B(\mu)$ implies $W^s(x) \subset B(\mu)$, using the absolute continuity of the holonomy maps again, we deduce that $B(\mu)$ has a positive measure with respect to the Lebesgue measure of M . Using symbolic dynamic approach, we can prove that the equilibrium measure is unique and is physical, hence it coincide with μ .

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