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The flow theorem for big order generators

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We give an extension of the stochastic flow theorem for big order generators written under Hoermander's form

Keywords: Big order generators. Flow theorem

1. Introduction

Dedicated to Professor Volovich for his birthday

Let us consider on \mathbb{R}^d m vector fields X_i without divergence with bounded derivatives at each order which generate the space uniformly. Let us introduce the operator

$$L = -(-1)^k \sum X_i^{2k} \quad (1)$$

It is an essentially selfadjoint negative operator which generates a contraction semi-group on $L^2(dx)$ and by ellipticity a semi-group P_t on $C_b(\mathbb{R}^d)$, the space of bounded continuous functions on \mathbb{R}^d endowed with the uniform topology.

Let us consider $\mathbb{R}^d \times \mathbb{G}_l^d$, where \mathbb{G}_l^d is the group of invertible matrices on \mathbb{R}^d . On this enlarged space with generic element (x, g) , we consider

$$X^{tot} = \left(X, \frac{\partial X}{\partial x} g \right) \quad (2)$$

The vector fields are considered here as section of the tangent space of the enlarged space. Let us recall that the Lie algebra of the linear group \mathbb{G}_l^d is constituted by matrices. Let us consider the enlarged vector fields as one-order differential operators. We can consider the enlarged operator (which is not elliptic in the extra-variables):

$$L^{tot} = -(-1)^k \sum (X_i^{tot})^{2k} \quad (3)$$

2

It is a negative symmetric operator densely defined on $L^2(dx \otimes dg)$ where dg is the Haar measure on the group \mathbb{G}_t^d . It has therefore by abstract theory a positive self-adjoint extension. This generates by abstract theory a contraction semi-group P_t^{tot} on $L^2(dx \otimes dg)$.

If $h(x)$ is a one form on \mathbb{R}^d , we define $h^{tot}(x, g) = h(x).g$ which is a function from the enlarged space into \mathbb{R}^d . We can extend diagonally the semi group P_t^{tot} such that it takes its values in \mathbb{R}^d . We have

Theorem 1.1. (*Blagoveshchenskii -Freidlin-Malliavin*) *Let us suppose that f and df are smooth bounded*

$$(dP_t f)^{tot} = P_t^{tot}(df^{tot}) \quad (4)$$

Remark. Let us give the heuristic considerations which lead to this theorem. Over \mathbb{R}^m , we consider the operator

$$L^f = -(-1)^k \sum_I \frac{\partial^{2k}}{\partial y_i^{2k}} \quad (5)$$

where $y = (y_1, \dots, y_m)$ is the generic element of \mathbb{R}^m .

This operator generates a semi-group P_t^f on $C_b(\mathbb{R}^m)$. Associated to it, there is a formal pathintegral $dQ^f(w.)$ on a "set" of paths $w_t = (w_{1,t}, \dots, w_{m,t})$ starting from on such that if f belongs to $C_b(\mathbb{R}^m)$ we have "formally"

$$P_t^f f(y) = \int f(w_t + y) dQ^f(w.) \quad (6)$$

We introduce the "stochastic" differential equation issued from x :

$$dx_t(x) = \sum X_i(x_t(x)) dw_{i,t} \quad (7)$$

such that formally we should get

$$P_t f(x) = \int f(x_t(x)) dQ^f(w.) \quad (8)$$

where f belongs to $C_b(\mathbb{R}^d)$ This heuristic considerations lead to the the theorem. We refer to [1] for a rigorous approach.

In the case of diffusion, these considerations are not formal at all. It is the theory of Stratonovitch differential equations in Itô Calculus [2]. But equation (7) has a solution in this case only almost surely defined. Blagoveshchenskii-Freidlin [3] have done pioneering works in the derivation in the starting point of the solution almost surely defined, which were later considerably extended. See the expository paper Meyer [4] and the book of

Kunita [5] for that. We refer to the review papers [6] and [7] for stochastic analysis for non-markovian semi-groups.

In the first part of this paper, we give the algebraic manipulations which lead to the theorem. In the second part, we solve the problem that we consider a **linear** test function in the extra-variable which is not bounded by using a Volterra expansion for the semi group in the extra variables. In order to show that the Volterra expansion converges, we use rough estimates of the heat kernel associated to the original operator.

2. Algebraic proof of the theorem

Lemma 2.1. *Let Y_1, \dots, Y_l be a collection of vector fields on \mathbb{R}^d . We associate the collection of vector fields $Y_1^{tot}, \dots, Y_l^{tot}$ on the enlarged space. Then*

$$(dY_1 \dots Y_l f)^{tot} = Y_1^{tot} \dots Y_l^{tot} (df)^{tot} \quad (9)$$

Proof of the lemma: We proceed by induction on l . We have

$$\begin{aligned} (dY f)^{tot}(x, g) &= \langle dY f(x), g \rangle = \\ &= \langle \nabla^2 f(x), Y, g \rangle + \langle \nabla f \frac{\partial}{\partial x} Y, g \rangle = Y^{tot} (df^{tot})(x, g) \end{aligned} \quad (10)$$

because $\nabla^2 f$ is symmetric. Therefore

$$\langle dY_1 \dots Y_l f, g \rangle = Y_1^{tot} (dY_2 \dots Y_l)^{tot}(x, g) = Y_1^{tot} \dots Y_l^{tot} (df^{tot})(x, g) \quad (11)$$

Algebraic proof of the theorem We have

$$\frac{\partial}{\partial t} P_t f = L P_t f \quad (12)$$

such that

$$\frac{\partial}{\partial t} (dP_t) f^{tot} = dL(P_t) f^{tot} = L^{tot} (dP_t f)^{tot} \quad (13)$$

In the expression $dP_t f$ we take only derivatives in the initial space. Moreover

$$\frac{\partial}{\partial t} (P_t^{tot} (df^{tot})) = L^{tot} (P_t^{tot} (df^{tot})) \quad (14)$$

It is the same parabolic equations with the same starting condition $df(x) \cdot g = df^{tot}(x, g)$.

4

3. Estimates: the Volterra expansion

Let us write

$$L^{tot} = L + R \quad (15)$$

where R is a differential operator in x of order strictly smaller than $2k$ in the original variables and a differential operator which is a polynomial in the $A_i g$ where A_i constitute a basis in the matrice in \mathbb{R}^d independent of x .

Basic fact P_t has an heat kernel $p_t(x, y)$

$$P_t f(x) = \int_{\mathbb{R}^d} p_t(x, y) f(y) dy \quad (16)$$

such that

$$|R_x^1 p_t(x, y)| < \frac{C}{t^{1-1/2k}} \frac{1}{t^{d/2k}} \exp\left[-\frac{|y-x|}{t^{1/k}}\right] \quad (17)$$

if R^1 is a differential operator with bounded component of order strictly smaller than $2k$.

The Volterra expansion of P_t^{tot} if it converges reads

$$P_t^{tot}(df^{tot}) = P_t(df^{tot}) + \sum_n \int_{t > s_1 > \dots > s_n > 0} P_{t-s_1} R P_{s_1-s_2} \dots R P_{s_n} df^{tot} ds_1 \dots ds_n \quad (18)$$

Let us suppose by induction on n that

$$R P_{t_1} R P_{t_2} \dots R_{t_n} df^{tot} = R_{t_1, \dots, t_n} g \quad (19)$$

where R_{t_1, \dots, t_n} is a one form with supremum norm smaller than $\frac{C^n}{\prod t_i^{1-1/2k}} \|df\|_\infty$. ($\|df\|_\infty$ denotes the supremum norm of df).

It is still true at the order $n+1$ by the basic fact. In order to show that the Volterra expansion converges, it remains to study if $\alpha = -1 + 1/2k$

$$I_n(t) = \int_{0 < s_1 < \dots < s_n < t} s_1^\alpha (s_2 - s_1)^\alpha \dots (t - s_n)^\alpha ds_1 \dots ds_n = C_n t^{n/2k} \quad (20)$$

where

$$C_n = \int_{0 < s_1 < \dots < s_n < 1} s_1^\alpha (s_2 - s_1)^\alpha \dots (1 - s_n)^\alpha ds_1 \dots ds_n = C_{n-1} \int_{0 < s < 1} s^{n/2k} (1-s)^\alpha ds = C_{n-1} \alpha_n \quad (21)$$

But

$$\alpha_n \leq n^{-\alpha} \int_{0 < s < 1/n} s^{n/2k} ds + (1/n)^{1+\alpha} \leq Cn^{-1/2k} \quad (22)$$

Therefore by induction

$$C_n \leq C^n (n!)^{-1/2k} \quad (23)$$

and the Volterra expansion converges.

Proof of the basic fact

This estimate is more or less classic in analysis (See [8] in the measurable case). We will give a simple proof based on the Greiner rescaling [9] and the Davies gauge transform.

We do on \mathbb{R}^d the change of variable $y \rightarrow \frac{x}{t^{1/2k}}$ (We operate in $x = 0$) such the vector $t^{1/2k} X(y)$ are transformed in $X^t(x)$ and tL is transformed in L^t which is an operator uniformly elliptic and which don't degenerate when $t \rightarrow 0$. By elliptic theory, L^t generates a semi-group P_s^t and P_1^t has an heat-kernel $p_1^t(x', y)$ which behaves well when $t \rightarrow 0$. We have

$$\int_{\mathbb{R}^d} p_1^t(0, y) f(y) dy = \int_{\mathbb{R}^d} p_t(0, y) f\left(\frac{y}{t^{1/2k}}\right) dy \quad (24)$$

such that

$$p_t(0, y) = t^{-d/2k} p_1^t\left(0, \frac{y}{t^{1/2k}}\right) \quad (25)$$

It remains to show that

$$|\nabla_y^l p_1^t(0, y)| \leq C \exp[-C|y|] \quad (26)$$

We do for that the Davies gauge transform associated to $\Psi^i(f)(y) = \exp[Cy^i]f(y)$. We get

$$\tilde{L}^t = (\Psi^i)^{-1} L^t \Psi^i = L^t + \tilde{L}_1^t \quad (27)$$

where \tilde{L}_1^t is a differential operator of order strictly smaller than $2k$, with bounded derivative at each order. \tilde{L}^t generates a semi group \tilde{P}_s^t which has an heat-kernel with bounded derivatives at each order $\tilde{P}_s^t(x, y)$ and

$$\tilde{p}_1^t(0, y) = p_1^t(0, y) \exp[Cy^i] \quad (28)$$

Therefore the result.

Remark We correct an error in the estimate of [10].

References

1. R. Léandre, *Toward a Wong-Zakai approximation for big order generators*, In *Symmetry in ordinary and partial differential equations and applications*, ed C. Vetro . *Symmetry*, **12** (2020).
2. N .Ikeda and S. Watanabe S., *Stochastic differential equations and diffusion processes. Sec edition.*(North-Holland, Amsterdam.(1989).
3. Y. Blagoveshchenkii and M.Freidlin., Certain properties of diffusion processes depending on a parameter, *Sov. Math. Dokl.*, **2**, 633, (1961).
4. P.A. Meyer , *Flot d'une équation différentielle stochastique* In *Séminaire de Probabilités XV. Université de Strasbourg*, eds P.A. Meyer and al .Lectures Notes in Maths., Vol. 850. (Springer: 1981), pp 103–117.
5. H. Kunita , *Stochastic flows and jump diffusions* Probability theory and stochastic modelling. (Springer: Singapore., 2019).
6. R. Léandre, Stochastic analysis for a non-Markovian generator: an introduction. *Russ. J. Math. Phys.* **22** , 39 (2015).
7. R. Léandre, *Bismut's way of the Malliavin Calculus for non-Markovian semi-groups: an introduction*, in *Analysis of pseudo-differential operators*, eds M.W. Wong and al. (Springer: Cham, 2019), pp. 157–179.
8. E.B. Davies ,Uniformly elliptic operators with measurable coefficients. *Journ. Funct. Ana.*, **132** 141 (1995).
9. P .Greiner , An asymptotic expansion for the heat equation. *Arch.Rat.Mech.Ana.*, **41**, 163 (1971).
10. R. Léandre , The stochastic flow theorem for an operator of order four. In *Geometric science of information*, eds F. Barbaresco and al. Lectures Notes in Computer Sciences, Vol 8085 (Springer: 2013) , pp.497–501.