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# Elementary remarks about Pisano periods.

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## Abstract

In this short note, we reprove in a very elementary way some known facts about Pisano periods as well as some considerations about the link between Pisano periods and the order of roots of the characteristic equation. The technics only requires a small background in ring theory (merely the definition of a commutative ring). The tools set here can be reused for all linear recurrences with quadratic non-constant characteristic equation.

**Keywords:** Pisano periods, ring extensions, order.

## 1 Introduction

Let  $R$  be a commutative ring. The Fibonacci sequence within  $R$  is defined by the following recurrence.

$$F_0 = 0_R, F_1 = 1_R, F_{n+2} = F_n + F_{n+1} \quad (1)$$

It can be easily checked that this sequence is periodic iff the characteristic of the ring  $R$  is positive<sup>1</sup>.

The period of  $F_n$  (called Pisano period) is exactly the order of the invertible matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{M}(2, R) \quad (2)$$

For introducing matters with respect to Pisano periods, see [2].

The aim of this short note is to write down some elementary facts about these periods and how, according to cases, it is exactly related to orders of some units in  $R$ .

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<sup>1</sup>Although all the Fibonacci sequence lives in the subring  $\mathbb{Z}.1_R (\simeq \mathbb{Z}/n\mathbb{Z}$  where  $n$  is the characteristic of  $R$ ), it be more compact (and a bit more elegant, due to the possible adjunctions) to consider the sequence within an arbitrary ring  $R$ .

## 2 Setting and results.

We discuss around the possibility of solving the equation  $X^2 - X - 1 = 0$  within  $R$ .

Throughout the paper, we suppose that  $1_R \neq 0_R$  (i.e.  $R$  is not the null ring) and that  $5 = 5 \cdot 1_R$  is a unit in  $R$ . We will call (E) the preceding equation i.e.

$$x^2 - x - 1 = 0 \quad (\text{E})$$

and make use of  $\sigma$  the automorphism of  $R[x]$  sending  $x \mapsto 1 - x$ .

### First case: $x^2 - x - 1 = 0$ has (at least) one root

We have the following lemma

**Lemma 2.1.** *We suppose that the equation (E) has at least one root in  $R$  (call it  $r$  and set  $s := 1 - r$ ), then*

1.  $s$  is a root of (E)
2.  $r$  and  $s$  are units in  $R$
3.  $r - s$  is a square root of 5 and therefore also a unit in  $R$
4.  $F_n = \frac{r^n - s^n}{r - s}$  (called Binet formula in ([2], see also Max Alekseyev's answer in [3]).
5. The following matrix

$$\begin{pmatrix} 1 & 1 \\ r & s \end{pmatrix} \quad (3)$$

is a unit in  $\mathcal{M}(2, R)$  and

6. We have in  $\mathcal{M}(2, R)$

$$\begin{pmatrix} 1 & 1 \\ r & s \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ r & s \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \quad (4)$$

7. Then the period of  $F_n$  within  $R$  is infinite iff  $r$  or  $s$  is of infinite order.

*Proof.* 1) Remarking that  $\sigma(x^2 - x - 1) = x^2 - x - 1$  we get the result.

2)  $r(-s) = r(r - 1) = r^2 - r = 1$  then  $r^{-1} = -s$  and similarly  $s^{-1} = -r$

3)

$$(r - s)^2 = r^2 - 2rs + s^2 \underset{rs=-1}{=} r^2 + s^2 + 2 \underset{\substack{r^2=r+1 \\ s^2=s+1 \\ r+s=1}}{=} 5$$

Now, if  $y^2 = 5$  and as 5 is a unit, we have  $y \cdot (y/5) = y^2 \cdot 5^{-1} = 1$  which proves that  $\pm(r - s)$  are units in  $R$ .

4)  $(r^n)$  and  $(s^n)$  are sequences satisfying  $x_{n+2} = x_n + x_{n+1}$  and the same holds for all their linear combinations in particular

$$t_n = \frac{r^n - s^n}{r - s}$$

it suffices now to check that  $t_i = F_i$  for  $i = 0, 1$ . QED

5) One has

$$\begin{vmatrix} 1 & 1 \\ r & s \end{vmatrix} = s - r \quad (5)$$

which is a unit from point (3). Therefore

$$\begin{pmatrix} 1 & 1 \\ r & s \end{pmatrix}^{-1} = \frac{1}{s-r} \begin{pmatrix} s & -1 \\ -r & 1 \end{pmatrix} \quad (6)$$

6) We have

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ r & s \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \quad (7)$$

Hence (4).

7) Again, from equation (4), one has in  $\mathcal{M}(2, R)^\times$  and  $R^\times$

$$\text{ord}\left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right) = \text{lcm}(\text{ord}(r), \text{ord}(s)) \quad (8)$$

□

We are now in the position to state

**Proposition 2.2.** *We suppose the hypotheses of Lemma 2.1. Then*

1. *The roots,  $r, s$  of (E) are both of finite or infinite order.*
2. *In the case when both roots are of finite order*
  - (a) *Either  $2.1_R = 0_R$  and  $\text{ord}(r) = \text{ord}(s) = 3$ .*
  - (b) *Or ( $2 \neq 0$  and) one of the orders of  $r, s$  is odd (say  $\text{ord}(r) \equiv 1 \pmod{2}$ ) and the other one is the double of it ( $\text{ord}(s) = 2 \text{ord}(r)$ ). In this case, the Pisano period is  $\text{ord}(s) = 2 \text{ord}(r)$*
  - (c) *Or ( $2 \neq 0$  and) their orders are all even. In this case  $\text{ord}(s) = \text{ord}(r)$  and the Pisano period is  $\text{ord}(s) = \text{ord}(r)$ .*

*Proof.* 1) Due to the fact that

$$s = (-1/r) \text{ and } r = (-1/s) \quad (9)$$

2a) For  $g \in \{r, s\}$ , the group generated by  $g$  is  $\{1, g, g+1\}$  hence  $g$  is of order 3. From now on we assume that  $2 \neq 0$ .

2b) Let us now suppose that  $\text{ord}(r) \equiv 1 \pmod{2}$ . Then, due to (9), we have

$$1 = \left(s^{\text{ord}(s)}\right)^{\text{ord}(r)} = \left((-1/r)^{\text{ord}(r)}\right)^{\text{ord}(s)} = (-1)^{\text{ord}(s)} \quad (10)$$

which, due to the fact that  $2.1_R \neq 0_R$ , entails that  $s$  is of even order.

But, in the same way, equation (9) shows that  $s^{2 \cdot \text{ord}(r)} = (-1)^{2 \cdot \text{ord}(r)} = 1$  and then  $\text{ord}(s) | 2 \cdot \text{ord}(r)$ .

On the other hand

$$r^{\text{ord}(s)} = (-1/s)^{\text{ord}(s)} = (-1)^{\text{ord}(s)} = 1$$

this proves that  $\text{ord}(r) | \text{ord}(s)$ . By Gauss lemma and taking into account that  $\text{ord}(r)$  is odd and  $\text{ord}(s)$  is even, we also have  $2 \cdot \text{ord}(r) | \text{ord}(s)$ .

Finally  $\text{ord}(s) | 2 \cdot \text{ord}(r) | \text{ord}(s)$  therefore  $\text{ord}(s) = 2 \cdot \text{ord}(r)$ .

2c) Similar reasoning using (9). □

**Second case:**  $x^2 - x - 1 = 0$  has no roots in  $R$ .

We consider  $R_1 := R[x] / (x^2 - x - 1)$  and, with  $r = \bar{x}$  (considering that 5 is still a unit in  $R_1$ ), we are brought back to the preceding case. In this case, however, we remark that we have the automorphism  $\sigma$  sending  $x$  to  $(1 - x)$  which preserves the ideal  $(x^2 - x - 1)$  and then passes to quotients (see Figure 1). this proves that  $r$  and  $\bar{\sigma}(r) = 1 - r$  are of the same order<sup>2</sup>.

$$\begin{array}{ccc}
 R[x] & \xrightarrow{\sigma} & R[x] \\
 \downarrow p & & \downarrow p \\
 R[x] / (x^2 - x - 1) & \xrightarrow{\bar{\sigma}} & R[x] / (x^2 - x - 1)
 \end{array}$$

Figure 1: Involutive automorphism of  $R[x] / (x^2 - x - 1)$  without Galois theory.

## References

- [1] Library of Alexandria  
[https://en.wikipedia.org/wiki/Library\\_of\\_Alexandria](https://en.wikipedia.org/wiki/Library_of_Alexandria)
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[https://en.wikipedia.org/wiki/Pisano\\_period](https://en.wikipedia.org/wiki/Pisano_period)
- [3] Complexity of a Fibonacci Numbers discrete log variation  
<https://mathoverflow.net/questions/287262>

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<sup>2</sup>This common order is necessarily even by the examination of the first case.