L1 solution to scalar BSDEs with logarithmic sub-linear growth generators

Shengjun Fan, Ying Hu, Shanjian Tang

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Abstract

By developing the test function method and combining the localization technique, we prove existence of an L^1 solution to a one-dimensional backward stochastic differential equation (BSDE for short) with L^1 terminal condition when the generator g has a one-sided linear growth in the first unknown variable y and a logarithmic sub-linear growth in the second unknown variable z, which improves some existing results. A new idea to study existence of an adapted solution to a BSDE is given. When the generator g additionally satisfies a one-sided Osgood condition in y and a logarithmic uniform continuity condition in z, we further establish a comparison theorem for the L^1 solutions to the above BSDEs, which yields immediately the uniqueness of the solution.

Keywords: Backward stochastic differential equation, L^1 solution, Existence and uniqueness, Logarithmic sub-linear growth generator, Comparison theorem.

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1. Introduction

Fix a positive integer d and a real number $T > 0$. For $a, b \in \mathbb{R}$, define $a \wedge b := \min\{a, b\}$, $a^+ := \max\{a, 0\}$ and $a^- := -\min\{a, 0\}$, and $\operatorname{sgn}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x\leq 0}$ with $\mathbf{1}_A$ being the indicator function of set A. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with augmented filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by a d-dimensional standard Brownian motion $(B_t)_{t\in[0,T]}$. For each $p > 0$, denote by L^p the collection of all $\mathcal{F}_T$-measurable real-valued random variables ξ satisfying $\mathbb{E}[|\xi|^p] < +\infty$, $\mathcal{L}^p$ the collection of all $(\mathcal{F}_t)$-adapted real-valued processes $(X_t)_{t\in[0,T]}$ satisfying

$$\|X\|_{\mathcal{L}^p} := \left\{\mathbb{E}\left[\left(\int_0^T |X_t|\mathrm{d}t\right)^p\right]\right\}^{\frac{1}{p}\wedge 1} < +\infty,$$

$\mathcal{S}^p$ the collection of $(\mathcal{F}_t)$-adapted continuous real-valued processes $(Y_t)_{t\in[0,T]}$ satisfying

$$\|Y\|_{\mathcal{S}^p} := \left(\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^p]\right)^{\frac{1}{p}\wedge 1} < +\infty,$$

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and $\mathcal{M}^p$ the collection of all $(\mathcal{F}_t)$-adapted $\mathbb{R}^{1\times d}$-valued processes $(Z_t)_{t\in[0,T]}$ satisfying

$$\|Z\|_{\mathcal{M}^p} := \left\{\mathbb{E}\left[\left(\int_0^T |Z_t|^2 \mathrm{d}t\right)^{p/2}\right]\right\}^{\frac{1}{p}\wedge 1} < +\infty.$$

Recall that an $(\mathcal{F}_t)$-adapted real-valued process $(X_t)_{t\in[0,T]}$ belongs to class (D) if the $\{X_\tau : \tau \in \Sigma_T\}$ is a family of uniformly integrable random variables, where Σ_T represents the collection of all $(\mathcal{F}_t)$-stopping times τ taking values in $[0,T]$. The equality and inequality between random elements are stood in $\mathbb{P}-a.s.$.

We study the following scalar backward stochastic differential equation (BSDE for short):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad t \in [0,T], \tag{1.1}$$

where ξ is an $\mathcal{F}_T$-measurable real-valued random variable called the terminal condition, the function (called the generator)

$$g(\omega, t, y, z) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d} \to \mathbb{R}$$

is $(\mathcal{F}_t)$-adapted for each (y,z), and the pair of processes $(Y_t, Z_t)_{t\in[0,T]}$ taking values in $\mathbb{R}\times\mathbb{R}^{1\times d}$ is called the adapted solution of (1.1), which is $(\mathcal{F}_t)$-adapted such that $\mathbb{P}-a.s.$, $t \mapsto Y_t$ is continuous, $t \mapsto |g(t, Y_t, Z_t)| + |Z_t|^2$ is integrable, and verifies (1.1).

Throughout the paper, we always suppose that $\beta \geq 0$ and $\gamma, c > 0$ are three given nonnegative constants, and $(f_t)_{t\in[0,T]}$ is a given $(\mathcal{F}_t)$-adapted nonnegative process. By BSDE(ξ, g), we mean the BSDE with terminal condition ξ and generator g. For narrative convenience, let $\alpha > 0$ and the generator g satisfy the following growth condition:

$$|g(\omega, t, y, z)| \leq f_t(\omega)| + \beta|y| + \gamma|z|^\alpha, \quad (\omega, t, y, z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}. \tag{1.2}$$

In the case of $\alpha \in (0,1)$, $\alpha = 1$, $\alpha \in (1,2)$, $\alpha = 2$ and $\alpha > 2$, it is usually called that the generator g has a linear growth in y, and a sub-linear, linear, sub-quadratic, quadratic and super-quadratic growth in z respectively. The following existence results are well known and classical. For the case of $\alpha \in (0,1)$, it follows from [1, 8] that if $(\xi, f_\cdot) \in L^1 \times \mathcal{L}^1$, then BSDE$(\xi, g)$ admits a solution $(Y_t, Z_t)_{t\in[0,T]}$ such that $Y_\cdot$ belongs to class (D) and $(Y_\cdot, Z_\cdot) \in \mathcal{S}^p \times \mathcal{M}^p$ for each $p \in (0,1)$. For the case of $\alpha = 1$, if $(\xi, f_\cdot) \in L^p \times \mathcal{L}^p$ for some $p > 1$, then BSDE(ξ, g) admits a solution $(Y_\cdot, Z_\cdot) \in \mathcal{S}^p \times \mathcal{M}^p$, see e.g. [18, 7, 16, 1, 13] for more details. For the case of $\alpha = 2$, by [15] we know that if $(\xi, f_\cdot)$ is bounded, then BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t\in[0,T]}$ such that $Y_\cdot$ is a bounded process and $Z_\cdot \in \mathcal{M}^2$. For the case of $\alpha > 2$, it follows from [5] that even if $(\xi, f_\cdot)$ is bounded, BSDE(ξ, g) may not admit a solution.

When the generator g satisfies (1.2) with $\alpha = 2$, [2] gave a general existence result on an adapted solution to BSDE(ξ, g), which weakens the boundedness condition on $(\xi, f_\cdot)$ in [15] to a certain exponential moment condition. To the best of our knowledge, this is the first work investigating the weakest possible integrability condition on $(\xi, f_\cdot)$ for existence of an adapted solution to BSDE(ξ, g) when the generator g has a certain growth condition on both unknown variables. In this spirit, [14] and [9] studied the same problem in the case that the generator g satisfies (1.2) with $\alpha = 1$ by applying the dual representation of solution to BSDE with convex generator used in [7, 19] and utilizing the test function method respectively.

Very recently, the case that the generator g satisfies (1.2) with $\alpha \in (1,2)$ was tracked in [10], and the case that the generator g satisfies (1.2) with $|z||\ln|z||^{\alpha}$ instead of $|z|^{\alpha}$ was dealt with in [12], where g has a super-linear growth in z. We would like to especially mention that in some of known results mentioned above, the generator g can be allowed to have a one-sided linear growth in y.

Generally speaking, more assumptions are required in order to obtain the uniqueness and the comparison theorem of solutions of scalar BSDEs. In particular, when the generator g is of a linear growth in z, it is additionally supposed that g satisfies a one-sided Osgood condition in y and a uniform continuity condition in z to obtain the desired results (see [7, 8, 4, 14, 17] for details, where Gronwall's inequality, Bihari's inequality and Girsanov's transform are usually used as important tools). And, when the generator g has a super-linear, sub-quadratic or quadratic growth in z, some extended convexity (concavity) assumptions of the generator g in (y,z) are added to get the uniqueness or the comparison theorem (see [3, 6, 8, 11, 10, 12] for more details, where the θ difference method and the a priori estimate technique usually play the key roles in the proof). In addition, we would like to mention that under the condition that the generator g satisfies (1.2) with $\alpha \in (0,1)$, if g further satisfies a one-sided Osgood condition in y and a uniform continuity condition with a sub-linear growth condition in z, then the uniqueness and the comparison theorem of solutions of scalar BSDEs hold, see [1] and [8] for details.

In this paper, we put forward a new idea to study existence of an adapted solution to BSDE(ξ, g). More specially, the following question is the starting point of this paper. That is, if only $(\xi, f_\cdot) \in L^1 \times \mathcal{L}^1$ is satisfied, then what growth conditions (as weakest as possible) of the generator g in (y,z) can ensure existence of the adapted solution to BSDE(ξ, g)? As mentioned before, [1] gave an answer of this problem where the generator g has a linear growth in y and a sub-linear growth in z, i.e., g satisfies (1.2) with $\alpha \in (0,1)$. Then, a question naturally arises: can we find some weaker conditions? This paper gives an affirmative answer of this question. Roughly speaking, by utilizing the localization technique put forward initially in [2], we prove that BSDE(ξ, g) admits a solution $(Y_t, Z_t)_{t\in[0,T]}$ such that $Y_\cdot$ belongs to class (D) and $(Y_\cdot, Z_\cdot) \in \mathcal{S}^p \times \mathcal{M}^p$ for each $p \in (0,1)$ when $(\xi, f_\cdot) \in L^1 \times \mathcal{L}^1$ and the generator g has a one-sided linear growth in y and a logarithmic sub-linear growth in z, see Theorem 2.1 in Section 2. It is a key step in the proof to search for a suitable test function (see Proposition 3.3 in Section 3) to apply Itô's formula and then obtain the desired a priori bound on the first parts of adapted solutions to the approached BSDEs. Before that, we need to prove an important technical inequality, see Proposition 3.2 in Section 3. Furthermore, by using the same a priori estimate technique as above and combining Theorem 2.1 in [8], we establish a comparison theorem for the adapted L^1 solutions to the previous BSDEs when the generator g additionally satisfies a one-sided Osgood condition in y and a logarithmic uniform continuity condition in z, which naturally yields the uniqueness of the L^1 solution, see Proposition 2.5 in Section 2.

The rest of this paper is organized as follows. In next section we state the main result, and in section 3 we prove it.

2. Statement of main result

Let us first introduce the following assumptions on the generator g.

(H1) $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, $g(\omega,t,\cdot,\cdot)$ is continuous.

(H2) g has a one-sided linear growth in y and a logarithmic sub-linear growth in z, i.e., there exists a constant $\lambda\in(1/2,1]$ that $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^{1\times d}$,

$$\mathrm{sgn}(y)g(\omega,t,y,z)\leq f_t(\omega)+\beta|y|+\frac{\gamma|z|}{(\ln(e+|z|))^{\lambda}}.$$

(H3) g has a general growth in (y,z), i.e., there exists a continuous nondecreasing function $h(\cdot):[0,+\infty)\to[0,+\infty)$ with $h(0)=0$ such that $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^{1\times d}$,

$$|g(\omega,t,y,z)|\leq f_t(\omega)+h(|y|)+c|z|^2.$$

(H4) g satisfies a one-sided Osgood condition in y, i.e., there exists a continuous, nondecreasing and concave function $\rho(\cdot):[0,+\infty)\to[0,+\infty)$ with $\rho(0)=0$, $\rho(u)>0$ for $u>0$, and $\int_{0+}\frac{\mathrm{d}u}{\rho(u)}=+\infty$ such that $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y_1,y_2,z)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{1\times d}$,

$$\mathrm{sgn}(y_1-y_2)(g(\omega,t,y_1,z)-g(\omega,t,y_1,z))\leq\rho(|y_1-y_2|).$$

(H5) g satisfies a logarithmic uniformly continuity condition in z, i.e., there exists a nondecreasing continuous function $\kappa(\cdot):[0,+\infty)\to[0,+\infty)$ with linear growth and $\kappa(0)=0$ such that $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y,z_1,z_2)\in\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{R}^{1\times d}$,

$$|g(\omega,t,y,z_1)-g(\omega,t,y,z_2)|\leq\kappa\left(\frac{|z_1-z_2|}{(\ln(e+|z_1-z_2|))^{\lambda}}\right),$$

where the constant $\lambda\in(1/2,1]$ is the same as in (H2).

The following existence and uniqueness theorem is the main result of this paper.

Theorem 2.1. *Let the terminal condition $\xi\in L^1$ and the generator g satisfy (H1)-(H3) with $f_\cdot\in\mathcal{L}^1$. Then BSDE(ξ,g) admits a solution $(Y_t,Z_t)_{t\in[0,T]}$ such that $Y_\cdot$ belongs to class (D) and $(Y_\cdot,Z_\cdot)\in\mathcal{S}^p\times\mathrm{M}^p$ for each $p\in(0,1)$. And, there exists a constant $C>0$ depending only on (β,γ,λ,T) such that*

$$|Y_t|\leq|Y_t|+\int_0^t f_s\mathrm{d}s\leq C\mathbb{E}\left[|\xi|+\int_0^T f_t\mathrm{d}t\middle|\mathcal{F}_t\right]+C,\quad t\in[0,T].\tag{2.3}$$

Moreover, if g also satisfies assumptions (H4) and (H5), then the solution $(Y_\cdot,Z_\cdot)$ such that $Y_\cdot$ belongs to class (D) is unique.

Remark 2.2. *With respect to Theorem 2.1, we make the following several remarks.*

(i) It is not hard to verify that for each $\alpha\in(0,1)$, there exists a constant $k_\alpha>0$ depending only on α such that for each $z\in\mathbb{R}^{1\times d}$,

$$|z|^{\alpha}\leq k_\alpha+\frac{|z|}{\ln(e+|z|)}.$$

This means that for the one-dimensional case, (H2) is weaker than the sub-linear growth assumption (H6) used in [1], and then Theorem 2.1 improves the corresponding existence result of [1].

(ii) Note that for each $\lambda > 1$ and $z \in \mathbb{R}^{1\times d}$,

$$\frac{|z|}{(\ln(e+|z|))^{\lambda}} \leq \frac{|z|}{\ln(e+|z|)}.$$

Therefore, the constant λ appearing in assumption (H2) can take values in $(1/2, +\infty)$. However, it is still open whether Theorem 2.1 holds still when $\alpha \in (0, 1/2]$.

(iii) It is not hard to verify that the constant e appearing in (H2) and (H5) can be replaced with any constant bigger than one. And, it is clear that if g satisfies assumption (H5), then it is uniformly continuous in z. In addition, it can be proved that assumptions (H3)-(H5) imply (H2).

In the sequel, we give two examples to which Theorem 2.1 applies, but no existing results could.

Example 2.3. *For each $(\omega, t, y, z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$, define*

$$g(\omega, t, y, z) := e^{B_t(\omega)} + e^{-y}\cos^2|z| + \frac{|z|\sin|z|}{(\ln(e+|z|))^{2/3}} - |z|^2\mathrm{sgn}(y).$$

It is easy to check that this g satisfies (H1)-(H3) with

$$f_\cdot = e^{B_\cdot} + 1,\ \beta = 0,\ \gamma = 1,\ \lambda = 2/3,\ c = 1 \quad \text{and} \quad h(u) = e^u.$$

It then follows from Theorem 2.1 that for each $\xi \in L^1$, BSDE(ξ, g) admits an adapted solution $(Y_t, Z_t)_{t\in[0,T]}$ such that $Y_\cdot$ belongs to class (D) and $(Y_\cdot, Z_\cdot) \in \mathcal{S}^p \times \mathrm{M}^p$ for each $p \in (0,1)$.

Example 2.4. *For each $(\omega, t, y, z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$, define*

$$\bar{g}(\omega, t, y, z) := B_t(\omega) + y^2\mathbf{1}_{y\leq 0} + l(|y|) + l\left(\frac{|z|}{\ln(e+|z|)}\right) + \sqrt{|z|},$$

where

$$l(u) := u|\ln u|\mathbf{1}_{0\leq u\leq \varepsilon} + l'_-(\varepsilon)(u - l(\varepsilon)), \quad u \in (0, +\infty)$$

with $\varepsilon > 0$ being sufficiently small. Note that if $F(\cdot) : [0,+\infty) \to [0,+\infty)$ is a nondecreasing concave function with $F(0) = 0$, then for each $0 < x < y$, by letting $k := x/y \in (0,1)$ we have

$$\begin{aligned} F(y) - F(x) &= F(y) - F(ky + (1-k)\cdot 0) \\ &\leq F(y) - (kF(y) + (1-k)F(0)) \\ &= (1-k)F(y) + kF(0) \\ &\leq F((1-k)y + k\cdot 0) = F(y-x), \end{aligned}$$

and then for each $x_1, x_2 \geq 0$,

$$|F(x_1) - F(x_2)| \leq F(|x_1 - x_2|).$$

With the above observation, it is not very hard to verify that for each $x_1, x_2 \geq 0$,

$$|l(x_1) - l(x_2)| \leq l(|x_1 - x_2|) \quad \text{and} \quad \int_{0+} \frac{\mathrm{d}u}{l(u)} = +\infty,$$

$$\left|\frac{x_1}{\ln(e+x_1)} - \frac{x_2}{\ln(e+x_2)}\right| \leq \frac{|x_1 - x_2|}{\ln(e+|x_1-x_2|)}$$

and

$$|\sqrt{x_1}-\sqrt{x_2}| \le \sqrt{|x_1-x_2|} \le 2\left(\sqrt{\frac{|x_1-x_2|}{\ln(e+|x_1-x_2|)}}+\frac{|x_1-x_2|}{\ln(e+|x_1-x_2|)}\right).$$

Thus, it can be checked that this $\bar{g}$ satisfies assumptions (H1)-(H5) with

$$f_\cdot = B_\cdot + 2,\ \beta = 1,\ \gamma = 2,\ \lambda = 1,\ c = 2,\ h(u) = u^2 + u,\ \rho(u) = l(u) \text{ and } \kappa(u) = l(u) + u + 2\sqrt{u}.$$

It then follows from Theorem 2.1 that for each $\xi \in L^1$, BSDE(ξ, g) admits a unique adapted solution $(Y_t, Z_t)_{t\in[0,T]}$ such that $Y_\cdot$ belongs to class (D) and $(Y_\cdot, Z_\cdot) \in \mathcal{S}^p \times \mathrm{M}^p$ for each $p \in (0,1)$.

The following proposition establishes a comparison result for the L^1 solutions of two BSDEs, which naturally yields the uniqueness part of Theorem 2.1.

Proposition 2.5. *Let ξ and ξ' be two terminal conditions, g and g' be two generators, and $(Y_t, Z_t)_{t\in[0,T]}$ and $(Y'_t, Z'_t)_{t\in[0,T]}$ be respectively an adapted solution to BSDE(ξ, g) and BSDE(ξ', g'). Assume that g (resp. g') satisfies assumptions (H4)-(H5), and $(Y_\cdot - Y'_\cdot)^+$ belongs to class (D). If $\xi \le \xi'$ and*

$$\mathbf{1}_{Y_t>Y'_t}\left(g(t,Y'_t,Z'_t) - g'(t,Y'_t,Z'_t)\right) \le 0 \quad (\text{resp. } \mathbf{1}_{Y_t>Y'_t}\left(g(t,Y_t,Z_t) - g'(t,Y_t,Z_t)\right) \le 0\ ), \tag{2.4}$$

then for each $t \in [0,T]$, we have $Y_t \le Y'_t$.

3. Proof of main result

We first introduce the following technical lemma.

Lemma 3.1. *For each $x \ge 0$, we have*

$$(e+x)\ln(e+x) > 2x \tag{3.5}$$

and

$$(e+x)^2 - 2x\ln(e+x) > e. \tag{3.6}$$

Proof. Let $f(x) := (e+x)\ln(e+x) - 2x,\ x \in [0,+\infty)$. A simple calculate gives

$$f'(x) = \ln(e+x) - 1 \ge 0, \quad x \ge 0.$$

Then, (3.5) follows since $f(x) \ge f(0) = e > 0$ for each $x \ge 0$.

In order to prove (3.6), we let $h(x) := (e+x)^2 - 2x\ln(e+x) - e,\ x \in [0,+\infty)$. A simple calculate gives that for each $x \ge 0$,

$$h'(x) = 2(e+x) - 2\ln(e+x) - \frac{2x}{e+x}$$

and

$$h''(x) = 2 - \frac{2}{e+x} - \frac{2e}{(e+x)^2} \ge 2 - \frac{4}{e} > 0.$$

Hence, for each $x \ge 0$ we have

$$h'(x) \ge h'(0) = 2e - 2 > 0$$

and then

$$h(x) \geq h(0) = e^2 - e > 0,$$

which yields the desired inequality (3.6). □

By virtue of the above lemma we can prove the following key inequality.

Proposition 3.2. *For each $x, y \geq 0$ and $\lambda \in [0,1]$, it holds that*

$$\frac{xy}{(\ln(e+y))^\lambda} \leq \frac{2x^2}{(\ln(e+x))^{2\lambda}} + \frac{1}{2}y^2. \tag{3.7}$$

Proof. The conclusion is obvious in the case of $\lambda = 0$. Let us prove the case of $\lambda \in (0,1]$. For $(x,y) \in [0,+\infty) \times [0,+\infty)$, define the function

$$\begin{aligned} f(x,y) &:= y^2 - \frac{2xy}{(\ln(e+y))^\lambda} + \frac{4x^2}{(\ln(e+x))^{2\lambda}} \\ &= \left(y - \frac{x}{(\ln(e+y))^\lambda}\right)^2 + \frac{4x^2}{(\ln(e+x))^{2\lambda}} - \frac{x^2}{(\ln(e+y))^{2\lambda}}. \end{aligned} \tag{3.8}$$

Clearly, in order to prove (3.7), it suffices to prove that $f(x,y) \geq 0$ for each $x, y \geq 0$. By (3.8) it is obvious when $y \geq x$, hence we only need to prove that $f(x,y) \geq 0$ for each $x \in (0,+\infty)$ and $y \in [0,x]$.

Now, fix arbitrarily $x \in (0,+\infty)$ and let $\bar{f}(y) := f(x,y),\ y \in [0,x]$. Then we have

$$\bar{f}'(y) = 2y - \frac{2x}{(\ln(e+y))^\lambda} + \frac{2\lambda xy}{(e+y)(\ln(e+y))^{\lambda+1}}, \quad y \in [0,x] \tag{3.9}$$

and in view of (3.5),

$$\begin{aligned} \bar{f}''(y) &= 2 + \frac{2\lambda x}{(e+y)(\ln(e+y))^{\lambda+1}} + \frac{2\lambda x\,(e\ln(e+y) - (\lambda+1)y)}{(e+y)^2(\ln(e+y))^{\lambda+2}} \\ &= 2 + \frac{2\lambda x\,[(e+y)\ln(e+y) + e\ln(e+y) - (\lambda+1)y]}{(e+y)^2(\ln(e+y))^{\lambda+2}} \\ &> 2 + \frac{2\lambda x\,[(e+y)\ln(e+y) - 2y]}{(e+y)^2(\ln(e+y))^{\lambda+2}} > 0, \quad y \in [0,x]. \end{aligned} \tag{3.10}$$

The previous inequality (3.10) means that $\bar{f}$ is a strictly convex function on $[0,x]$. Furthermore, noticing by (3.9) that $\bar{f}'(0) = -2x < 0$ and

$$\bar{f}'(x) = 2x - \frac{2x}{(\ln(e+x))^\lambda} + \frac{2\lambda x^2}{(e+x)(\ln(e+x))^{\lambda+1}} > 0,$$

we deduce that there exists a unique $y_0 \in (0,x)$ such that $\bar{f}'(y_0) = 0$ and

$$f(x,y) = \bar{f}(y) \geq \bar{f}(y_0) = f(x,y_0), \quad y \in [0,x]. \tag{3.11}$$

In the sequel, let

$$y_1 = \frac{x}{2(\ln(e+y_1))^\lambda}. \tag{3.12}$$

Then, $y_1 \in (0, x/2)$ and it follows from (3.9), (3.12) and (3.5) that

$$\begin{aligned} \bar{f}'(y_1) &= \frac{x}{(\ln(e+y_1))^\lambda} - \frac{2x}{(\ln(e+y_1))^\lambda} + \frac{2\lambda x y_1}{(e+y_1)(\ln(e+y_1))^{\lambda+1}} \\ &= \frac{x\,[2\lambda y_1 - (e+y_1)\ln(e+y_1)]}{(e+y_1)\ln(e+y_1))^{\lambda+1}} \\ &\leq \frac{x\,[2y_1 - (e+y_1)\ln(e+y_1)]}{(e+y_1)(\ln(e+y_1))^{\lambda+1}} < 0. \end{aligned}$$

Therefore, $y_1 < y_0$ and then it follows from (3.11) and (3.8) that

$$\begin{aligned} f(x,y) &\geq f(x,y_0) \\ &= \left(y_0 - \frac{x}{(\ln(e+y_0))^\lambda}\right)^2 + \frac{4x^2}{(\ln(e+x))^{2\lambda}} - \frac{x^2}{(\ln(e+y_0))^{2\lambda}} \\ &\geq \frac{4x^2}{(\ln(e+x))^{2\lambda}} - \frac{x^2}{(\ln(e+y_1))^{2\lambda}}, \quad y \in [0,x]. \end{aligned} \tag{3.13}$$

Finally, by (3.6) we can deduce that

$$(e+y_1)^{2^{\frac{1}{\lambda}}} - 2y_1(\ln(e+y_1))^\lambda \geq (e+y_1)^2 - 2y_1\ln(e+y_1) > e.$$

It follows from the last inequality and (3.12) that

$$2^{\frac{1}{\lambda}}\ln(e+y_1) \geq \ln(e+2y_1(\ln(e+y_1))^\lambda) = \ln(e+x),$$

which means that

$$4(\ln(e+y_1))^{2\lambda} \geq (\ln(e+x))^{2\lambda}. \tag{3.14}$$

Combining (3.13) and (3.14) yields that $f(x,y) \geq 0$ for each $x \in (0,+\infty)$ and $y \in [0,x]$, which is the desired assertion. □

Now, we fix arbitrarily $\lambda \in (1/2, 1]$ and search for a positive, continuous, strictly increasing and strictly convex function $\phi(s,x) : [0,T]\times[0,+\infty) \to (0,+\infty)$ satisfying

$$\begin{gathered} -\beta\phi_x(s,x)x - \phi_x(s,x)\frac{\gamma|z|}{(\ln(e+|z|))^\lambda} + \frac{1}{2}\phi_{xx}(s,x)|z|^2 + \phi_s(s,x) \geq 0, \\ (s,x,z) \in [0,T]\times[0,+\infty)\times\mathbb{R}^{1\times d}, \end{gathered} \tag{3.15}$$

where and hereafter, $\phi_s(\cdot,\cdot)$ denotes the first-order partial derivative of $\phi(\cdot,\cdot)$ with respect to the first variable, and $\phi_x(\cdot,\cdot)$ and $\phi_{xx}(\cdot,\cdot)$ respectively the first-order and second order partial derivative of $\phi(\cdot,\cdot)$ with respect to the second variable. Observe from (3.7) that

$$\begin{aligned} &-\phi_x(s,x)\frac{\gamma|z|}{(\ln(e+|z|))^\lambda} + \frac{1}{2}\phi_{xx}(s,x)|z|^2 \\ &= \phi_{xx}(s,x)\left(-\frac{\gamma\phi_x(s,x)}{\phi_{xx}(s,x)}\frac{|z|}{(\ln(e+|z|))^\lambda} + \frac{1}{2}|z|^2\right) \\ &\geq -2\gamma^2\frac{\phi_x^2(s,x)}{\phi_{xx}(s,x)}\left(\ln\left(e+\frac{\gamma\phi_x(s,x)}{\phi_{xx}(s,x)}\right)\right)^{-2\lambda}. \end{aligned}$$

Hence, it is sufficient for the function $\phi(\cdot,\cdot)$ to satisfy that for each $(s,x) \in [0,T]\times[0,+\infty)$,

$$-\beta\phi_x(s,x)x - 2\gamma^2\frac{\phi_x^2(s,x)}{\phi_{xx}(s,x)}\left(\ln\left(e+\frac{\gamma\phi_x(s,x)}{\phi_{xx}(s,x)}\right)\right)^{-2\lambda} + \phi_s(s,x) \geq 0. \tag{3.16}$$

In the sequel, we choose the following test function

$$\phi(s,x) := (k+x)\left(1 - \frac{1}{(\ln(k+x))^\delta}\right)\mu_s, \quad (s,x) \in [0,T]\times[0,+\infty)$$

to explicitly solve the inequality (3.16), where δ is a positive constant to be assigned, $\mu_s : [0,T] \to (0,+\infty)$ is a nondecreasing and continuous differentiable function to be assigned and

$$k := \exp\left(\frac{2}{\delta} + 2(\delta+1) + \frac{2\delta}{(\delta+1)\gamma}\right). \tag{3.17}$$

First of all, a simple computation gives that for each $(s,x) \in [0,T] \times [0,+\infty)$,

$$\phi_x(s,x) = \left[1 - \frac{1}{(\ln(k+x))^\delta}\left(1 - \frac{\delta}{\ln(k+x)}\right)\right]\mu_s > 0,$$

$$\phi_{xx}(s,x) = \frac{\delta}{(k+x)(\ln(k+x))^{\delta+1}}\left(1 - \frac{\delta+1}{\ln(k+x)}\right)\mu_s > 0,$$

and

$$\phi_s(s,x) = (k+x)\left(1 - \frac{1}{(\ln(k+x))^\delta}\right)\mu_s' > 0.$$

Combining (3.17) and the last three inequalities yields that for $(s,x) \in [0,T] \times [0,+\infty)$,

$$\frac{1}{2}\mu_s \le \phi_x(s,x) \le \mu_s, \tag{3.18}$$

$$\frac{\delta\mu_s}{2(k+x)(\ln(k+x))^{\delta+1}} \le \phi_{xx}(s,x) \le \frac{\delta\mu_s}{(k+x)(\ln(k+x))^{\delta+1}} \tag{3.19}$$

and

$$\phi_s(s,x) \ge \frac{1}{2}(k+x)\mu_s'. \tag{3.20}$$

It follows from (3.18), (3.19) and (3.17) that

$$\frac{\gamma\phi_x(s,x)}{\phi_{xx}(s,x)} \ge \frac{\gamma}{2\delta}(k+x)(\ln(k+x))^{\delta+1} \ge k+x. \tag{3.21}$$

Substituting (3.18)-(3.21) into the left hand side of (3.16) we can deduce that

$$\begin{aligned}
&-\beta\phi_x(s,x)x - 2\gamma^2\frac{\phi_x^2(s,x)}{\phi_{xx}(s,x)}\left(\ln\left(e + \frac{\gamma\phi_x(s,x)}{\phi_{xx}(s,x)}\right)\right)^{-2\lambda} + \phi_s(s,x)\\
&\ge -\beta(k+x)\mu_s - \frac{2\gamma^2\mu_s^2}{\frac{\delta\mu_s}{2(k+x)(\ln(k+x))^{\delta+1}}(\ln(e+k+x))^{2\lambda}} + \frac{1}{2}(k+x)\mu_s'\\
&\ge (k+x)\left[-\left(\beta + \frac{4\gamma^2}{\delta}(\ln(k+x))^{\delta-(2\lambda-1)}\right)\mu_s + \frac{1}{2}\mu_s'\right], \quad (s,x) \in [0,T] \times [0,+\infty).
\end{aligned}$$

Thus, if we pick $\delta = 2\lambda - 1$ and

$$\mu_s := \exp\left[2\left(\beta + \frac{4\gamma^2}{\delta}\right)s\right] = \exp\left[2\left(\beta + \frac{4\gamma^2}{2\lambda-1}\right)s\right], \quad s \in [0,T],$$

then (3.16) and then (3.15) holds.

In conclusion, we have the following proposition on the test function.

Proposition 3.3. *Let $\lambda \in (1/2, 1]$ and*

$$k_0 := \exp\left(\frac{2}{2\lambda-1} + 4\lambda + \frac{2\lambda-1}{\lambda\gamma}\right). \tag{3.22}$$

For $(s,x) \in [0,T] \times [0,+\infty)$, define

$$\varphi(s,x) := (k_0+x)\left(1-\frac{1}{(\ln(k_0+x))^{2\lambda-1}}\right)\exp\left[2\left(\beta+\frac{4\gamma^2}{2\lambda-1}\right)s\right]. \tag{3.23}$$

Then, the test function $\varphi(\cdot,\cdot)$ satisfies that for each $(s,x,z)\in[0,T]\times[0,+\infty)\times\mathbb{R}^{1\times d}$,

$$-\beta\varphi_x(s,x)x-\varphi_x(s,x)\frac{\gamma|z|}{(\ln(e+|z|))^{\lambda}}+\frac{1}{2}\varphi_{xx}(s,x)|z|^2+\varphi_s(s,x)\geq 0. \tag{3.24}$$

The following Proposition 3.4 establishes an important a priori estimate for the solution to a BSDE.

Proposition 3.4. *Assume that ξ is a terminal condition, g is a generator satisfying assumption (H2), and $(Y_t,Z_t)_{t\in[0,T]}$ is a solution of BSDE(ξ,g). If the process $|Y_\cdot|+\int_0^\cdot f_s\mathrm{d}s$ belongs to class (D), then there exists a constant $C>0$ depending only on (β,γ,λ,T) such that for each $t\in[0,T]$,*

$$|Y_t|\leq|Y_t|+\int_0^t f_s\mathrm{d}s\leq C\mathbb{E}\left[|\xi|+\int_0^T f_t\mathrm{d}t\middle|\mathcal{F}_t\right]+C. \tag{3.25}$$

Proof. Define

$$\bar{Y}_t:=|Y_t|+\int_0^t f_s\mathrm{d}s\quad\text{ and }\quad\bar{Z}_t:=Z_t\mathrm{sgn}(Y_t),\quad t\in[0,T].$$

Itô-Tanaka's formula yields

$$\bar{Y}_t=\bar{Y}_T+\int_t^T\left(\mathrm{sgn}(Y_s)g(s,Y_s,Z_s)-f_s\right)\mathrm{d}s-\int_t^T\bar{Z}_s\mathrm{d}B_s-\int_t^T\mathrm{d}L_s,\quad t\in[0,T],$$

where $L_\cdot$ is the local time of $Y_\cdot$ at 0. Now, let the function $\varphi(\cdot,\cdot)$ be defined in (3.23) and apply Itô-Tanaka's formula to the process $\varphi(s,\bar{Y}_s)$ to deduce, in view of (H2),

$$\begin{aligned}\mathrm{d}\varphi(s,\bar{Y}_s) &= \varphi_x(s,\bar{Y}_s)\left(-\mathrm{sgn}(Y_s)g(s,Y_s,Z_s)+f_s\right)\mathrm{d}s+\varphi_x(s,\bar{Y}_s)\bar{Z}_s\mathrm{d}B_s\\ &\quad+\varphi_x(s,\bar{Y}_s)\mathrm{d}L_s+\frac{1}{2}\varphi_{xx}(s,\bar{Y}_s)|\bar{Z}_s|^2\mathrm{d}s+\varphi_s(s,\bar{Y}_s)\mathrm{d}s\\ &\geq \left[-\varphi_x(s,\bar{Y}_s)\left(\beta|Y_s|+\frac{\gamma|Z_s|}{(\ln(e+|Z_s|))^{\lambda}}\right)+\frac{1}{2}\varphi_{xx}(s,\bar{Y}_s)|Z_s|^2+\varphi_s(s,\bar{Y}_s)\right]\mathrm{d}s\\ &\quad+\varphi_x(s,\bar{Y}_s)\bar{Z}_s\mathrm{d}B_s,\quad s\in[0,T].\end{aligned}$$

Then, in view of $|Y_s|\leq\bar{Y}_s$, by Proposition 3.3 we have

$$\mathrm{d}\varphi(s,\bar{Y}_s)\geq\varphi_x(s,\bar{Y}_s)\bar{Z}_s\mathrm{d}B_s,\quad s\in[0,T], \tag{3.26}$$

which means that $\varphi(s,\bar{Y}_s)$ is a local submartingale.

For each $n\geq1$ and $t\in[0,T]$, define the following stopping time

$$\tau_n^t:=\inf\left\{s\in[t,T]:\varphi(s,\bar{Y}_s)+\int_t^s\left[\varphi_x(r,\bar{Y}_r)\right]^2|\bar{Z}_r|^2\mathrm{d}r\geq n\right\}\wedge T$$

with the convention that $\inf\emptyset=+\infty$. It follows from the definition of τ_n^t and (3.26) that for each $n\geq1$ and $t\in[0,T]$,

$$\varphi(t,\bar{Y}_t)\leq\mathbb{E}\left[\varphi(\tau_n^t,\bar{Y}_{\tau_n^t})\middle|\mathcal{F}_t\right]. \tag{3.27}$$

On the other hand, observe from (3.23) and (3.22) that

$$\frac{1}{2}(k_0+x)\leq\varphi(s,x)\leq k_1(k_0+x),\quad(s,x)\in[0,T]\times[0,+\infty), \tag{3.28}$$

where k_0 is defined in (3.22) and

$$k_1 := \exp\left[2\left(\beta + \frac{4\gamma^2}{2\lambda - 1}\right)T\right]. \tag{3.29}$$

Combining (3.27) and (3.28) yields that for each $n \geq 1$ and $t \in [0,T]$,

$$\frac{1}{2}(k_0 + \bar{Y}_t) \leq \varphi(t, \bar{Y}_t) \leq \mathbb{E}\left[\varphi(\tau_n^t, \bar{Y}_{\tau_n^t})\middle|\mathcal{F}_t\right] \leq k_1 \mathbb{E}\left[(k_0 + \bar{Y}_{\tau_n^t})\middle|\mathcal{F}_t\right].$$

Since $\bar{Y}_\cdot$ belongs to class (D) and $\tau_n^t \to T$ as $n \to \infty$ for each $t \in [0,T]$, by sending n to infinity in the last inequality we deduce that

$$\bar{Y}_t \leq 2k_1 k_0 + 2k_1 \mathbb{E}[\bar{Y}_T|\mathcal{F}_t] - k_0, \quad t \in [0,T],$$

which is the desired assertion. □

Before proving Theorem 2.1, we introduce the following proposition, which is a direct corollary of Proposition 3.8 in [12].

Proposition 3.5. *Assume that the generator g satisfies assumption (H2) and $(Y_t, Z_t)_{t\in[0,T]}$ is a solution of BSDE(ξ, g). If $|Y_\cdot| + \int_0^\cdot f_s \mathrm{d}s \in \mathcal{S}^p$ for $p > 0$, then $Z_\cdot \in \mathcal{M}^q$ for each $q \in (0,p)$.*

Proof of Theorem 2.1. For positive integers $n, p \geq 1$, let

$$\xi^{n,p} := \xi^+ \wedge n - \xi^- \wedge p \quad \text{and} \quad g^{n,p}(\omega,t,y,z) := g^+(\omega,t,y,z) \wedge n - g^-(\omega,t,y,z) \wedge p.$$

As the terminal condition $\xi^{n,p}$ is bounded and $g^{n,p}$ is a bounded generator, it follows from [15] that BSDE$(\xi^{n,p}, g^{n,p})$ admits a maximal bounded solution $(Y_\cdot^{n,p}, Z_\cdot^{n,p})$ such that $Y_\cdot^{n,p}$ is bounded and $Z_\cdot^{n,p} \in \mathcal{M}^2$. It is not hard to verify that $g^{n,p}$ satisfies assumption (H2) with $f_\cdot \wedge (n \vee p)$ instead of $f_\cdot$. It then follows from Proposition 3.4 that there exists a constant $C > 0$ depending only on $(\beta, \gamma, \lambda, T)$ such that for each $t \in [0,T]$ and $n, p \geq 1$,

$$\begin{aligned} |Y_t^{n,p}| &\leq |Y_t^{n,p}| + \int_0^t [f_s \wedge (n \vee p)]\mathrm{d}t \\ &\leq C\mathbb{E}\left[|\xi^{n,p}| + \int_0^T [f_t \wedge (n \vee p)]\mathrm{d}t \middle| \mathcal{F}_t\right] + C \\ &\leq C\mathbb{E}\left[|\xi| + \int_0^T f_t \mathrm{d}t \middle| \mathcal{F}_t\right] + C. \end{aligned} \tag{3.30}$$

It follows from Theorem 2.3 in [8] that $Y_\cdot^{n,p}$ is nondecreasing in n and non-increasing in p. Then, in view of (3.30) and assumptions (H1) and (H3), by using the localization technique put forward initially in [2] we deduce that there exists an $(\mathcal{F}_t)$-adapted process $(Z_t)_{t\in[0,T]}$ such that $(Y_\cdot := \inf_p \sup_n Y_\cdot^{n,p},\ Z_\cdot)$ is a solution to BSDE(ξ, g). Furthermore, by letting n and p to infinity in (3.30) yields the desired inequality (2.3), and then the process $Y_\cdot$ belongs to class (D).

Let us further show that $(Y_\cdot, Z_\cdot) \in \mathcal{S}^p \times \mathcal{M}^p$ for each $p \in (0,1)$. Indeed, in view of (2.3), it follows from Lemma 6.1 in [1] that for each $p \in (0,1)$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(|Y_t| + \int_0^t f_s \mathrm{d}s\right)^p\right] \leq (2C)^p \left[\frac{1}{1-p}\left(\mathbb{E}\left[|\xi| + \int_0^T f_s \mathrm{d}s\right]\right)^p + 1\right] < +\infty,$$

which means that $Y_\cdot \in \mathcal{S}^p$ for each $p \in (0,1)$, and then by Proposition 3.5, $Z_\cdot \in \mathcal{M}^p$ for each $p \in (0,1)$. Thus, the existence part of Theorem 2.1 is proved.

The uniqueness part is a direct corollary of Proposition 2.5, which will be proved below. The proof of Theorem 2.1 is then complete. □

Proof of Proposition 2.5. We only prove the case that the generator g satisfies assumptions (H4)-(H5), and $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$\mathbf{1}_{Y_t > Y'_t} \left(g(t, Y'_t, Z'_t) - g'(t, Y'_t, Z'_t)\right) \leq 0. \tag{3.31}$$

The other case can be proved in the same way. Without loss of generality, we always assume that the function $\rho(\cdot)$ in assumption (H4) and the function $\kappa(\cdot)$ in assumption (H5) verify that for each $x \geq 0$,

$$\rho(x) \leq Ax + A \quad \text{and} \quad \kappa(x) \leq Ax + A, \tag{3.32}$$

where $A > 0$ is a universal constant.

Define $\hat{Y}_\cdot := Y_\cdot - Y'_\cdot$ and $\hat{Z}_\cdot := Z_\cdot - Z'_\cdot$. Then, $(\hat{Y}_\cdot, \hat{Z}_\cdot)$ verifies

$$\hat{Y}_t = \xi - \xi' + \int_t^T \left(g(s, Y_s, Z_s) - g'(s, Y'_s, Z'_s)\right) \mathrm{d}s - \int_t^T \hat{Z}_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{3.33}$$

Let

$$\tilde{Y}_t := \hat{Y}_t^+ + 2At \quad \text{and} \quad \tilde{Z}_t := \mathbf{1}_{\hat{Y}_t > 0} \hat{Z}_t, \quad t \in [0,T].$$

Itô-Tanaka's formula yields

$$\tilde{Y}_t = \tilde{Y}_T + \int_t^T \left(\mathbf{1}_{\hat{Y}_s > 0} \left(g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)\right) - 2A\right) \mathrm{d}s - \int_t^T \tilde{Z}_s \mathrm{d}B_s - \int_t^T \mathrm{d}L_s, \quad t \in [0,T],$$

where $L_\cdot$ is the local time of $\hat{Y}_\cdot$ at 0.

Furthermore, in view of assumptions (H4) and (H5) of the generator g together with (3.31) and (3.32), we have $\mathrm{d}\mathbb{P} \times \mathrm{d}s - a.e.$,

$$\begin{aligned}
&\mathbf{1}_{\hat{Y}_s > 0} \left(g(s, Y_s, Z_s) - g'(s, Y'_s, Z'_s)\right) - 2A \\
&\leq \rho(\hat{Y}_s^+) + \mathbf{1}_{\hat{Y}_s > 0} \kappa\left(\frac{|\hat{Z}_s|}{(\ln(e + |\hat{Z}_s|))^\lambda}\right) - 2A \\
&\leq A\hat{Y}_s^+ + A\frac{\mathbf{1}_{\hat{Y}_s > 0}|\hat{Z}_s|}{(\ln(e + |\hat{Z}_s|))^\lambda} \leq A\tilde{Y}_s + \frac{A|\tilde{Z}_s|}{(\ln(e + |\tilde{Z}_s|))^\lambda}.
\end{aligned} \tag{3.34}$$

Let the function $\varphi(\cdot,\cdot)$ be defined in Proposition 3.3 with A instead of β and γ in (3.22) and (3.23). In view of (3.34), applying Itô-Tanaka's formula to the process $\varphi(s, \tilde{Y}_s)$ yields that

$$\begin{aligned}
\mathrm{d}\varphi(s, \tilde{Y}_s) &= -\varphi_x(s, \tilde{Y}_s) \left(\mathbf{1}_{\hat{Y}_s > 0} \left(g(s, Y_s, Z_s) - g(s, Y'_s, Z'_s)\right) - 2A\right) \mathrm{d}s + \varphi_x(s, \tilde{Y}_s) \tilde{Z}_s \mathrm{d}B_s \\
&\quad + \varphi_x(s, \tilde{Y}_s) \mathrm{d}L_s + \frac{1}{2}\varphi_{xx}(s, \tilde{Y}_s)|\tilde{Z}_s|^2 \mathrm{d}s + \varphi_s(s, \tilde{Y}_s) \mathrm{d}s \\
&\geq \left[-\varphi_x(s, \tilde{Y}_s)\left(A\tilde{Y}_s + \frac{A|\tilde{Z}_s|}{(\ln(e + |\tilde{Z}_s|))^\lambda}\right) + \frac{1}{2}\varphi_{xx}(s, \tilde{Y}_s)|\tilde{Z}_s|^2 + \varphi_s(s, \tilde{Y}_s)\right] \mathrm{d}s \\
&\quad + \varphi_x(s, \tilde{Y}_s) \tilde{Z}_s \mathrm{d}B_s, \quad s \in [0,T].
\end{aligned}$$

Then, by (3.24) in Proposition 3.3 with A instead of β and γ we have

$$\mathrm{d}\varphi(s,\tilde{Y}_s) \geq \varphi_x(s,\tilde{Y}_s)\tilde{Z}_s \mathrm{d}B_s, \quad s \in [0,T].$$

With the last inequality in hand, noticing that $\tilde{Y}_\cdot$ belongs to class (D) and using a same argument as that in the proof of Proposition 3.4 we deduce that for each $t \in [0,T]$,

$$\tilde{Y}_t \leq 2k_1k_0 + 2k_1\mathbb{E}[\tilde{Y}_T|\mathcal{F}_t] - k_0,$$

where the constants k_0 and k_1 are defined respectively in (3.22) and (3.29) with A instead of β and γ. Then, we have for each $t \in [0,T]$,

$$(Y_t - Y'_t)^+ + 2At \leq 2k_1k_0 + 2k_1\left(\mathbb{E}[(\xi-\xi')^+|\mathcal{F}_t] + 2AT\right) - k_0 \leq 2k_1k_0 + 4Ak_1T,$$

which means that $(Y_\cdot - Y'_\cdot)^+$ is a bounded process.

Finally, in view of assumptions (H4)-(H5) of g, $\xi \leq \xi'$ and (3.31) together with the fact that $(Y_\cdot - Y'_\cdot)^+$ is a bounded process, we can apply Theorem 2.1 in [8] to obtain the desired assertion. □

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