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# Evolution of multimodal final user equilibrium considering public transport network design history

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## 1. Introduction

Analyzing the properties of a network equilibrium (uniqueness and stability) can help to have a better view about network state, robustness, and the effect of any variation in the network. The issue of unicity for User Equilibrium (UE) has long been a subject of concern in the literature on traffic assignment problems (Beckmann *et al.* (1956); Daganzo (1985); Mounce & Smith (2007); Iryo & Smith (2018)). Much research has been performed on the unicity of traffic assignment solutions with several assumptions and limitations on the traffic network model (Beckmann *et al.* (1956); Dafermos (1982); Nagurney (1984); Wie *et al.* (2002); Florian & Morosan (2014); Sun *et al.* (2014); Ameli *et al.* (2018)). In practice, the strong mathematical assumptions for unicity (e.g., FIFO or monotonicity) simply do not hold (Boyles *et al.*, 2013). Therefore, the existence of multiple equilibria can be expected mathematically for real test cases (Levin *et al.*, 2014). There are few analytical studies (Netter, 1972, Wynter, 2001) that showed the existence of multiple equilibria in small networks in the traffic assignment context. Most of the studies focused on check the uniqueness of equilibrium and not use this property to analyze and improve the system.

A key argument (sufficient condition) for unicity is a strictly monotone travel time function with respect to the number of travelers that use a path (Smith (1979); Aashtiani & Magnanti (1981); Florian & Hearn (1995)). This condition is quite restrictive as monotonicity can be guaranteed at the link level for mono-modal flow but hardly at the path level because of the intersection functioning. Traffic assignment models address the network equilibrium problem, including the travel time calculation, mathematically.

In the real multimodal networks, the network design can be changed in the long term, e.g., several new transportation facilities are added to the system. There are few studies on Static Traffic Assignment (STA) and Dynamic Traffic Assignment (DTA) models by considering the evolution of the network design. However, none of them investigate the nonunicity and history of the network together. The scientific question of this study is, does the network converge to different equilibria with different network history and the same final network design? In other words, is the current network situation sufficient to grasp the real user distribution inside the network?

To investigate the multiple equilibria problem with multimodal settings, we consider a day-to-day convergence process as a projected dynamical system. When unicity holds, this process converges to the single equilibrium loading (Zhao *et al.*, 2018). Here, we focus on what happens when multiple solutions can be reached, as we aim to highlight what may drive the system to one equilibrium rather

than another. More specifically, when considering a long-term day-to-day process, the final network may not be built at once but results from the successive opening of additional facilities. In this study, we consider that the road network is stable over the entire time horizon and that the public transport network is subject to the regular new openings (bus or metro line). We define this opening process as the network history and investigate how it would affect the final equilibrium state of the network.

First, we perform an analytical investigation and demonstrate the existence of multiple equilibria with respect to the network history of day-to-day STA. Second, we address the same question through simulation for a more complex test case (large-scale network, dynamic traffic assignment, multiple configurations for the network history). The analytical study highlights the causes for the existence of multiple solutions and the influence of network history. The numerical study aims to provide results considering a realistic urban setting. The influence of network history is investigated with a different angle as we are more interested in the convergence of multiple equilibria, i.e., the description of the differences between final possible states. A specific finding is that several network history configurations lead to shorter total travel times for the system than others. This may be of interest when considering public transport planning.

## 2. Analytical investigation

Let us consider a network with three modes of transportation, which are referred to as car, bus, and train (metro) (figure 1). There are two bus lines between origin and destination. Paths 1 and 2 are shared between car and bus, and path 3 is the train line. Therefore, there are five feasible path choices to reach the destination: car by path 1, car by path 2, bus line on path 1, bus line on path 2, and train on path 3, figure 1.

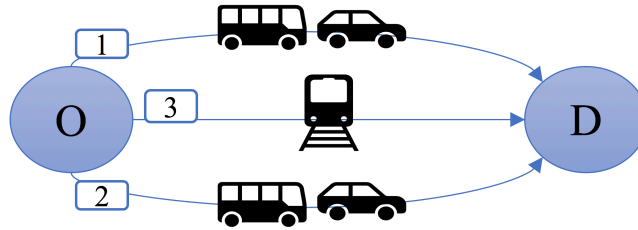


Figure 1: A single origin-destination network of this study

Note that the total demand for public transportation  $D_{PT}$  and cars  $D^C$  are given. The demand for public transportation is disaggregated per mode bus and train. The cost function for motor vehicles (car and bus) depends on not only the path flow for this mode but also the path flow of other modes as all vehicles interact. For example, on paths 1 and 2, bus flow and car flow result from a global congestion level that influences both bus and cars travel costs. The cost of mode  $m \in \{B: \text{Bus}, C: \text{Car}\}$  on path  $a \in 1, 2$  is as follow:

$$c_a^m = g_a^m + \sum_{\mu \in \{B, C\}} h_a^{m, \mu} f_a^\mu, \quad \forall m \in \{B, C\}, a \in 1, 2 \quad (1)$$

Here, we assume that  $c_a^m$  is defined as a linear function where  $g_a^m$  is the free flow cost of mode  $m$  on path  $a$  and  $h_a^{m, \mu}$  is the impact factor of the flow of mode  $\mu$  on path  $a$  ( $f_a^\mu$ ) on the cost of mode  $m$ . Using linear functions may look as a strong assumption compared to the more realistic polynomial or exponential shape, e.g., BPR function (Bureau of Public Roads, 1964) or considering path capacity constraints (Beckmann *et al.*, 1956). It is not here as our main purpose is to understand the causes for multiple equilibria to appear. What triggers this phenomenon is the cross-dependency of the cost among modes, which is properly accounted for here.

The cost function for path 3 is even much simpler. First, only a train option is available, as the cost should only depend on the demand for the train. Second, train service is only adjusted to the

demand for the long run (every year or twice a year when a new timetable is proposed). So during the day-to-day STA process, the perceived cost by users is mainly fixed as a combination of the travel time and the cost. In the end:  $c_3^T = \lambda$  for all path flow values ( $f_3^T$ ), where  $\lambda$  is considered as a constant value.

According to the Wardrop equilibrium definition, the network is at the UE flow distribution if and only if for every path  $a$ :

$$f_a^m (c_a^m - c^m) = 0, \quad \forall m \in M, a \in A \quad (2)$$

Therefore, when the equilibrium is reached, the cost of all used paths of mode  $m$  is equal to  $c^m$ . The costs are assumed to be asymmetric: the effect of cars on buses is not the same as the effect of buses on cars. In other words, in the network (figure 1) for one or more modes  $m_1 \neq m_2$ :

$$\frac{\partial c_a^{m_1}}{\partial f_a^{m_2}} \neq \frac{\partial c_a^{m_2}}{\partial f_a^{m_1}} \iff h_a^{m_1, m_2} \neq h_a^{m_2, m_1} \quad \forall a = 1, 2, \forall m_1, m_2 = B, C \quad (3)$$

## 2.1. Network history design

In this study, we are interested in investigating the equilibrium when intermediate changes in the network design occur. In other words, the final network layout is always the same, but it may result from different intermediate steps, see table 1. Scenario 1 has no train line at first place. Thus, an intermediate equilibrium state (UE) is first achieved through the day-to-day learning process before the train line is added. Then, the metro line is added, and the second convergence process proceeds. Scenario 2 assumes that there are only cars and trains, and no buses during the first convergence period. Then buses are added, and a second convergence process is initiated starting from the equilibrium obtained by the first process. Scenario 3 is when all modes are active from the beginning. We will calculate the final network equilibrium for all the scenarios in table 1 and examine why different equilibrium may be raised.

Table 1: The scenarios of network design for the mono-OD test case.

Scenario	Intermediate equilibrium state	Final state
1		
2		
3	—	

## 2.2. Non unicity of equilibrium states in multimodal STA

In order to calculate and visualize the intermediate equilibrium for Scenario 1, we first explore the non-uniqueness of the network without train ( $D^T = 0$ ), i.e., we analyze the equilibrium solution(s) for the initial equilibrium state of scenario 1 in table 1. According to equation 3, on each path, the impacts of two modes on each other are not the same, which represents the real interaction between

modes. In other words, the Jacobian matrix of the car and the bus cost functions is not symmetric positive definite, which means the cost functions for path 1 and 2 are not monotonic (Iryo, 2013).

Let us express costs on paths 1 and 2 as functions of the flows  $f_1^m$ . In order to visualize the equilibrium, we consider  $f_1^m$  as an independent and  $f_2^m$  as a dependent variable. The flows  $f_2^m$  can be considered dependent variables. We draw the curves  $\Delta c^m = c_2^m - c_1^m = 0$  of the two vehicular modes on the flow diagram of path 1 in the  $(f_1^C, f_1^B)$  plane. We perform a complete analysis of different configurations of  $h_a^{m,\mu}$  and highlights in which conditions we will have more than one equilibrium.

### 2.3. The projected dynamical system of traffic assignment

In order to address the day-to-day process in the static case, We define the network equilibrium problem as a projected dynamical system. It has been proved by several studies (see e.g., Nagurney & Zhang (2012), Smith (1993), Jin (2007), Lebacque *et al.* (2009)) that projected dynamical systems find the equilibrium point(s) by producing the solution trajectory (mapping function) based on a fixed point theory. We define 2D vector fields for the intermediate equilibrium of Scenario 1 and 3D vector fields for the final state of all scenarios.

We also analyze the stability of the solution based on the projected dynamical system. In summary, the UE solution  $E$  is stable only if the day-to-day process brings the solution back to  $E$  when a small perturbation  $\epsilon$  moves it from  $E$  in any direction of the feasible region (converging arrows to  $E$ ).

Figure 2 presents an example of a test case considered wherein we have multiple equilibria. It shows the trajectory of the different starting points in the solution space. Almost all of the initial path flow distribution converges to the two stable equilibria on the corners while there is a rare situation wherein the starting point is located on a line whereon the field lines direction is thorough to the unstable equilibrium (the intersection of  $\Delta c^C$  and  $\Delta c^B$ ).

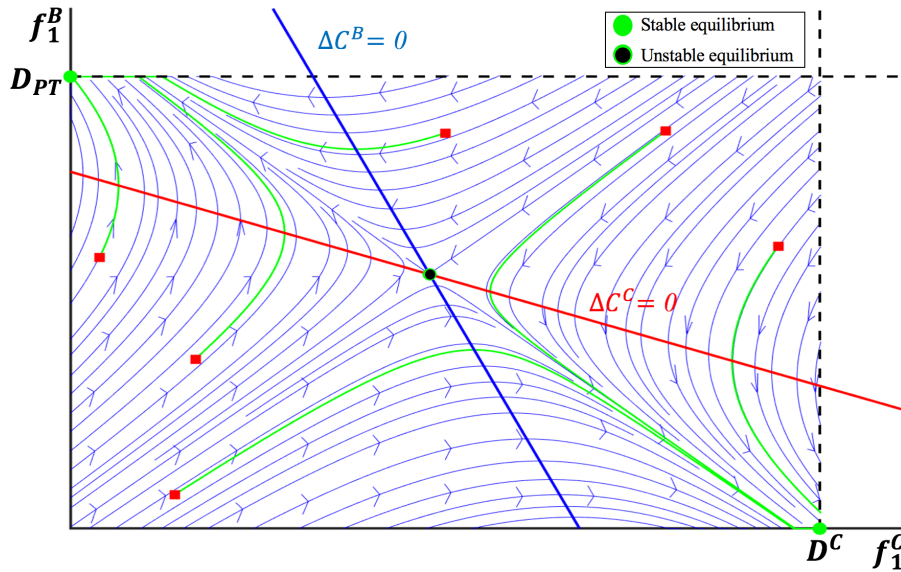


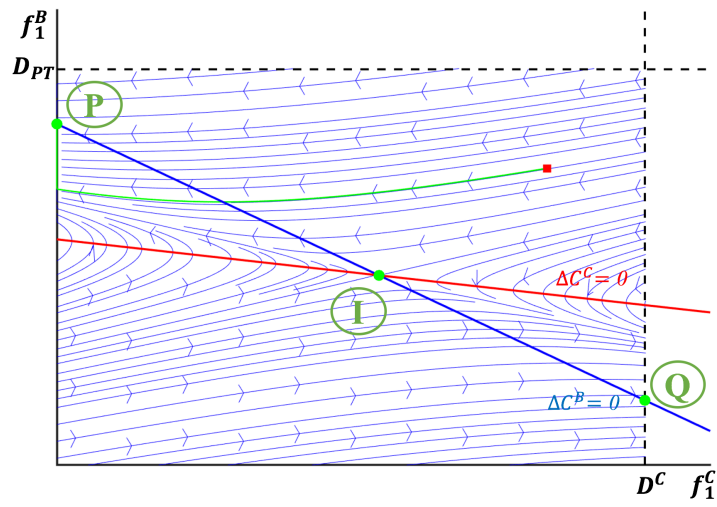
Figure 2: The trajectory of different initial solutions (red squares) on flow diagram of path 1

### 2.4. Primary findings

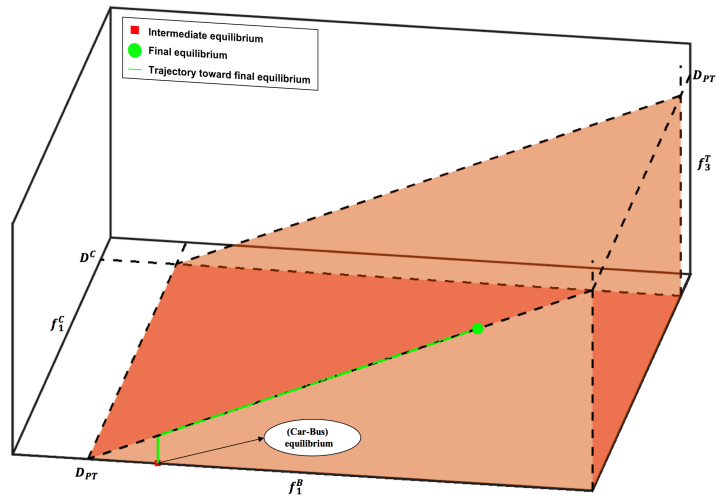
Figure 3 presents an example wherein with different network design history, the system converges to different equilibrium. In this figure, the unstable equilibrium is the intersection of two lines and presented by a point  $(I)$ , and the two stable equilibria are points  $(P)$  and  $(Q)$ .

Figures 3a and 3b present Scenario 1. We choose the uniform flow propagation for the initial solution. The intermediate equilibrium in the car-bus network is obtained based on the projected dynamical system (figure 3a). Then we add the train line to the system and calculate the final equilibrium (figure 3b). Figure 3c presents Scenario 2, wherein the intermediate equilibrium is the unique

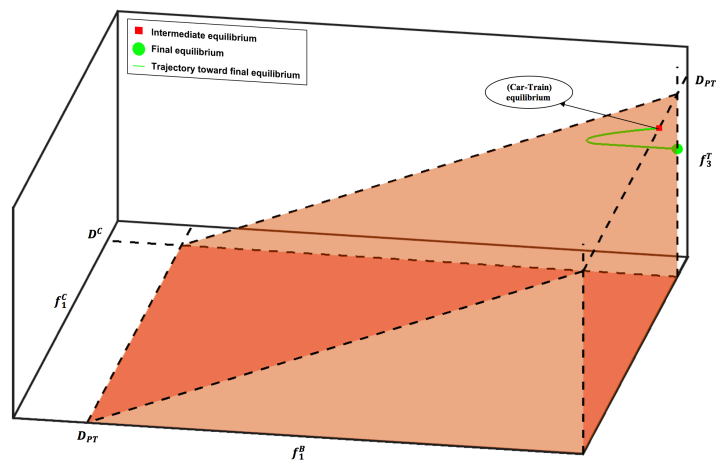
one from the car-train network. The results show that with a different order of mode activation, the system reaches to different equilibrium.



(a) Intermediate equilibrium convergence of Scenario 1 with the initial solution:  $f_1^m = (\frac{3}{4}D^C, \frac{3}{4}D_{PT})$ .



(b) Final equilibrium convergence of Scenario 1. The intermediate equilibrium comes from Figure 3a



(c) Equilibrium convergence of Scenario 2.

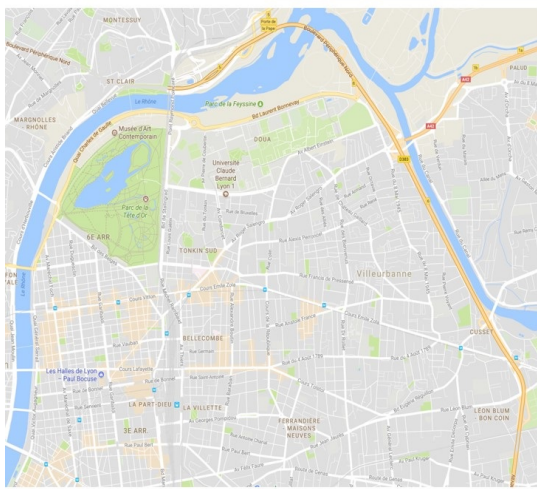
Figure 3: The impact of network design history on the final equilibrium.

### 3. Multi-modal simulation-based day-to-day DTA

In this section, we address the question of network history and multiple multimodal user equilibria in a more realistic framework. Now, we resort to a dynamic traffic simulator for the network loading and focus on a real network. Although the setting is more complex, mechanisms similar to those described in the previous section apply and induce non unicity of equilibria, and dependence of equilibria to the order of activation of facilities.

#### 3.1. Dynamic test case

Using a dynamic simulator permits us to consider the large-scale network of Lyon 6e + Villeurbanne (figure 4a) with 1,883 Nodes, 3,383 Links, 94 Origins, 227 Destinations and 54,190 trips. Walking, buses and private cars are initially available transportation modes in the network. Figure 4b presents 31 bus lines in the Lyon 6e + Villeurbanne network includes 176 bus station (figure 4c). There are three metro lines (A, B and C) and 25 metro stations in the network (figure 4d).



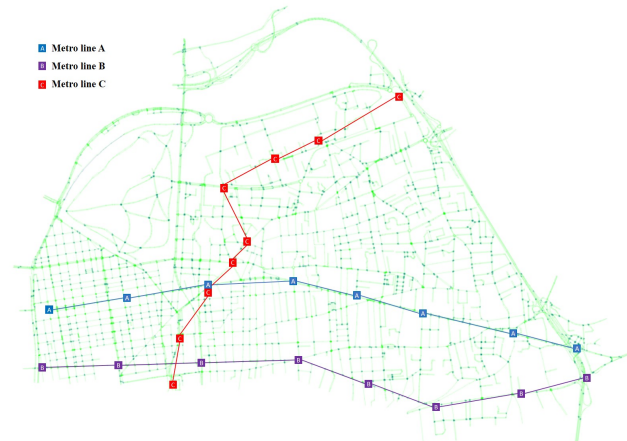
(a) Lyon 6e + Villeurbanne: Mapping data ©Google 2019



(b) 31 bus lines



(c) 176 bus station



(d) 3 metro lines

Figure 4: Multimodal traffic network of Lyon 6e + Villeurbanne

The network is loaded with travelers of all ODs with a given departure time in order to represent 1.5 hours of the network with the demand level based on the study of (Krug *et al.*, 2019). The goal is to analyze the final equilibrium solution obtained by a day-to-day DTA model with different settings corresponding to different successive introductions of the metro lines.

### 3.2. Experiment scenarios

For each scenario of opening metro lines, we run the day-to-day DTA for 300 days. A quarter of users only have access to the public transportation system (bus and metro) and the other three quarters have access to all transportation facilities (private car, bus, and metro) in the network. We can open three metro lines at the same time (called A&B&C scenario) and calculate the equilibrium or successively open one metro line every 100 days (e.g., ACB denotes a scenario wherein we first open metro line A, then after 100 days we open metro line C, and after on day 200, we open metro line B) and look for the final network state after 300 days. There are seven possible orders to activate the metro lines. All the scenarios are started by the final equilibrium solution of the network without metro lines. The initial assignment pattern of each step is the final equilibrium flow distribution of the previous step. For Scenario 1 (A&B&C), all three metro lines are activated at the same time and once the day-to-day process is executed in order to equilibrate the system.

### 3.3. Numerical results

The full day-to-day process is conducted for all the scenarios, and we verify that all the simulations converge to a satisfactory UE solution, i.e., the average delay of all scenarios is less than 12 seconds, which shows good quality for the equilibrium in the large-scale network and given the demand level (Ameli *et al.*, 2020). Next, we evaluate the final solution in order to investigate the unicity of the solution.

Table 2 presents the usage of metro lines at the equilibrium state for all scenarios. The number of users who take metro line A is between 2089 and 2983. This means if we open the metro line in the order ACB, we will have 42% more users that take metro line A than in the CAB scenario. This width of interval for metro lines B and C is 1250 and 640.

Table 2: Primary results

Scenario	Sequence	Number of users used Metro			Total travel time (hours)
		A	B	C	
1	A&B&C	2236	3057	3708	19454.22
2	ABC	2826	2771	3316	18559.33
3	ACB	2983	2636	3678	19070.50
4	BAC	2419	3077	3138	18967.44
5	BCA	2608	3117	3481	19199.75
6	CAB	2089	3886	3564	19701.43
7	CBA	2313	2977	3778	18644.19

The total travel time is standard criteria for evaluating traffic network performance. According to the results in table 2, by opening the metro lines in the order ABC, we can save 600 hours (3%) on average compared to the other scenarios. The total travel time values of the scenarios are in the range of [18559.33, 19701.43], which is the range of the potential equilibrium space. This shows that the intermediate state of the network has a significant impact on the final UE. We plan to measure public transportation systems' reliability and multimodal equilibrium stability to investigate which history of the network design can provide a more reliable transportation service.

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