Flat singularities of chained systems, illustrated with an aircraft model

Yirmeyahu Kaminski, François Ollivier

FLAT SINGULARITIES OF CHAINED SYSTEMS, ILLUSTRATED WITH AN AIRCRAFT MODEL

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Yirmeyahu J. Kaminski
Holon Institute of Technology
Holon, Israel
kaminsj@hit.ac.il

François Ollivier
LIX, CNRS–École Polytechnique
91128 Palaiseau Cedex, France
ollivier@lix.polytechnique.fr

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ABSTRACT

We consider flat differential control systems for which there exist flat outputs that are part of the state variables and study them using Jacobi bound. We introduce a notion of saddle Jacobi bound for an ordinary differential system for n equations in $n+m$ variables. Systems with saddle Jacobi number generalize various notions of chained and diagonal systems and form the widest class of systems admitting subsets of state variables as flat output, for which flat parametrization may be computed without differentiating the initial equations. We investigate apparent and intrinsic flat singularities of such systems. As an illustration, we consider the case of a simplified aircraft model, providing new flat outputs and showing that it is flat at all points except possibly in stalling conditions. Finally, we present numerical simulations showing that a feedback using those flat outputs is robust to perturbations and can also compensate model errors, when using a more realistic aerodynamic model.

RÉSUMÉ

Nous considérons des systèmes différentiellement plats pour lesquels il existe des sorties plates qui font partie des variables d'état et nous les étudions en utilisant la borne de Jacobi. Nous introduisons une notion de nombre-selle de Jacobi pour un système pour n équations en $n+m$ variables. Les systèmes avec un nombre-selle de Jacobi égal à 0 généralisent diverses notions de systèmes chaînés et diagonaux et forment la classe la plus large de systèmes admettant des sous-ensembles de variables d'état en tant que sortie plate, pour lesquels une paramétrisation plate peut être calculée sans différencier les équations initiales. Nous étudions les singularités plates apparentes et intrinsèques de ces systèmes. À titre d'illustration, nous considérons le cas d'un modèle d'avion simplifié, en fournissant de nouvelles sorties plates et en montrant qu'il est plat en tout point, sauf éventuellement en situation de décrochage. Enfin, nous présentons des simulations numériques montrant qu'un bouclage utilisant ces sorties plates est robuste aux perturbations et peut également compenser les erreurs de modèle, lors de l'utilisation d'un modèle aérodynamique plus réaliste.

AMS classifiaction: 93-10, 93B27, 93D15, 68W30, 12H05, 90C27

Key words: differentially flat systems, flat singularities, flat outputs, aircraft aerodynamics models, gravity-free flight, engine failure, rudder jam, differential thrust, forward sleep landing, Jacobi's bound, Hungarian method

1 Introduction

1.1 Mathematical context

Differentially flat systems, introduced by Fliess, Lévine, Martin and Rouchon [14, 15] are ordinary differential systems $P_i(x_1, \cdots, x_{n+m})$, $1 \leq i \leq n$, the solutions of which can be parametrized in a simple way. Indeed, they admit differential functions $\zeta_j(x)$, $1 \leq i \leq m$, such that for all $1 \leq i \leq n+m$, $x_i = X_i(\zeta)$, where X_i is a differential function, i.e. also depending on the derivatives of ζ up to some finite order. Examples of such systems where considered by Monge [49], and Monge problem, studied by Hilbert [26] and Cartan [10, 11] is precisely to test if a differential

system satisfies this property[1]. Flat systems, for which motion planning and feed-back stabilization are very easy have proven their importance in non-linear control. We use the theoretical framework of *diffiety theory* [36, 71], and extend to it the notion of *defect*, introduced by Fliess *et al.* [14] in the setting of Ritt's *differential algebra* [61]. The defect is a non negative integer that express the distance of a system to flatness. It is 0 iff the system is flat.

Jacobi's bound [29, 30] is an *a priori* bound on the order of a differential system $P_i(x_1, \dots, x_n)$, $1 \leq i \leq n$ that is expressed by the *tropical determinant* of the order matrix $(a_{i,j})$, with $a_{i,j} := \mathrm{ord}_{x_j} P_i$, which is given by the formula $\mathcal{O}_\Sigma := \max_{\sigma \in S_n} \sum_{i=1}^n a_{i,\sigma(i)}$. This bound is conjectural in the general case, but was proved by Kondrateva *et al.* [35] under genericity hypotheses that stand when Jacobi's truncated determinant does not vanish. Then, the bound is precisely the order. Harold Kuhn's *Hungarian method* [37], discovered independently, is very close to Jacobi's polynomial time algorithm for computing the bound and is an important step in the history of combinatorial optimization (see Burkard *et al.* [9]. One may notice that that this result was much probably inspired to Jacobi by *isoperimetrical systems*, satisfied by functions $x_i(t)$ such that $\int_{t_1}^{t_2} L(x)\mathrm{d}t$ is extremal.

1.2 Aims of this paper

We continue the investigation of intrinsic and apparent singularities of flat control system [17, 18, 44, 45], initiated in our previous papers [31, 32], with a study of block triangular systems that generalizes *extended chained form* [23] and an application to aircraft control. We recall that flat systems are systems for which the trajectory can be parametrized using a finite set of state functions, call *flat outputs*, and a finite number of their derivatives. This important notion of non linear control simplifies motion planning, feed-back design and also optimization [51, 62, 19, 13, 4].

The class of block triangular system is important in practice as it includes various notions of "chained systems" or triangular systems [39, 40, 41, 34, 66], containing many classical examples such as robot arms [20, 70], cars with many trailers [64] or discretizations of PDE flat systems [59, 68]. For them, testing flatness reduces to computing the rank of Jacobian matrices and finding the flat outputs to an combinatorial problem.

Our goal is to investigate if it is possible to choose flat outputs among the state functions, and the associated regularity conditions. This is a main difference with preceding papers on chained systems that investigated the existence of change of variables and static feedback allowing to reduce to some more restricted class of chained or triangular form.

We try to enlarge as much as possible the class of systems for which flatness can be tested by polynomial time combinatorics and computations of rank of Jacobian matrices. For such systems, some lazy flat parametrization may be computed without differentiating the initial equations, meaning that we have a block decomposition of variables $\Xi = \bigcup_{i=1}^r \Xi_i$, such that $\Xi_1 = F_1(Z)$, where F_1 is a differential function of the flat outputs Z, then $\Xi_2 = F_2(Z, \Xi_1)$, where F_2 is a differential function of Z and the first block $\Xi_1, \dots, \Xi_{i+1} = F_{i+1}(Z, \Xi_1, \dots, \Xi_i), \dots$ On the other hand, we do not require those systems to be in normal form: they can be implicit systems and, if we impose that some kind of flat parametrization can be computed without further differentiation, we can nevertheless consider from a theoretical standpoint a much larger class of systems than the usual state space representation, that may require to be computed a great number of derivation, again bounded by some Jacobi number.

1.3 Main theoretical results

Considering underdetermined systems $P_i(x_1, \dots, x_{n+m})$, we define the saddle Jacobi bound $\hat{\mathcal{O}}_\Sigma$ as being the minimal Jacobi bound $\mathcal{O}(Y, \Sigma)$, for all $Y \subset \Xi$ with $\sharp Y = n$. If $\hat{\mathcal{O}}_\Sigma = \mathcal{O}(\hat{Y}, \Sigma)$ and the corresponding truncated determinant $\nabla_{\hat{Y},\Sigma}$ does not vanish, then the *defect* of the system, as defined in [14] is at most $\hat{\mathcal{O}}_\Sigma$. This implies that if the saddle Jacobi bound is equal to 0 and the associated truncated determinant does not identically vanish, the system is flat.

We give a sufficient condition of flat singularity for some classes of chained systems, that is enough to prove that the aircraft simplified model admits an intrinsic flat singularity in some stalling condition and some sufficient condition of regularity for block diagonal systems that are enough to show that the simplified aircraft is flat when not in stalling condition.

We show that previously known classes of chained and diagonal systems enter the wider class of oudephippical systems. We further prove that a system Σ, such that a subset $Z \subset \Xi$ of the state variable is a flat output and a lazy flat parametrization can be computed without using any strict derivative of Σ is oudephippical.

[1] Allowing change of independent variable, *i.e.* in control change of time, so that Monge problem is more precisely to test *orbital flatness* [15]

1.4 Flat outputs for the aircraft and regularity conditions

These theoretical results are illustrated with a study of a simplified aircraft model. With 12 states, 4 controls and about 50 parameters, this model is already more complicated than most flat models in the literature, although it is among the first to have been considered.

Martin [46, 47] has shown that a simplified aircraft model where the thrusts related to the actuators and angular velocities are neglected is flat and given the flat output x, y, z, β, where (x, y, z) are the coordinates of the center of gravity and β the sideslip angle. We show that the bank angle μ, the angle of attack α and the engine thrust F can also be used instead of β.

We explicit regularity conditions for those choices of flat outputs and show that the regularity condition for μ is related to some kind of stalling condition. The discovery of 3 new sets of flat outputs just by a systematic application of our theoretical results on chained systems illustrate their usefulness.

We illustrate in two ways the importance of the block decomposition by providing two implementations, using two different kind of feedbacks: a first one in Python relies on the difference of dynamics speeds between the blocks, the second in Maple uses fast computations allowed by the lazy parametrization to work out a feed-back able to reject model errors, keeping the values of the flat outputs close to the planed trajectory, with an acceptable computational complexity.

1.5 Numerical simulations, models and implementations

In our simulations, we used the aircraft model and sets of parameters provided by Grauer and Morelli [22] for various types of aircraft: fighter F16C, STOL utility aircraft DHC-6 Twin Otter and NASA Generic Transport Model (GTM), a subscale airliner model. Such aerodynamics models are not known to be flat, unless one neglects some terms, as Martin did, such as the thrusts created by the control surfaces (ailerons, elevators, rudder) or related to angular speeds.

We investigated first the robustness of the flat control with respect to some failures and some perturbations, for the simplified model, using simulations performed in Python. In a second stage, a Maple implementation was used to test the ability of a suitable feed-back to keep the trajectories of the full model close to the theoretical trajectories computed with the simplified flat one.

We investigate flight situations that are intrinsic singularities for β, such as gravity-free flight, for which we use alternative flat outputs, including bank angle μ. A set of flat outputs including the thrust F may also be used when $\beta \neq 0$ and is suitable to control a slip-forward maneuver for dead-stick emergency landing [5, 6]. See simulations in [52].

1.6 Plan of the paper

We present flat systems in sec. 2, giving first definitions and main properties sec. 2, considering examples sec. 2.2 and providing characterization of flat singularities sec. 2.3 and generalizing the notion of defect 2.4.

We then present Jacobi's bound sec. 3, starting with combinatorial definition sec. 3.1 with some emphasis on Kőnig's theorem sec. 3.2 before coming to evaluation of the order and computations of normal forms sec. 3.3. We then introduce the saddle Jacobi number sec. 3.4.

We can then define $\bar{\text{o}}$-system sec. 4 and provide an algorithmic criterion sec. 4.1 with a sufficient condition of regularity sec. 4.2 followed by an algorithmic criterion for a $\bar{\text{o}}$-system to be regular at a given point sec. 4.3. We review examples of chained or triangular systems sec. 4.4 that enter our category of $\bar{\text{o}}$-systems and conclude with special sufficient conditions of regularity or singularity for block triangular systems sec. 4.5.

We then consider applications to an aircraft model sec. 5, first describing the equations sec. 5.1 and the GNA aerodynamic model sec. 5.2. We show that the model is block triangular under some simplification sec. 5.3. We then investigate the four main choices of flat outputs sec. 5.4 and consider stalling conditions and their relation to flat singularity sec. 5.5.

The last sections are devoted to simulations using first the simplified model sec. 6, then using the full model sec. 7.

1.7 Notations

This paper mixes theoretical results from various fields, with different habits for notations, and an aircraft model coming with classical notations from aircraft engineering. We tried to use uniform notations, as long as it did not become an obstacle to readability or made the access to references too difficult. Regarding diffieties in some abstract setting, it is convenient to denote the derivation operators by different symbols, such as δ, d or even d_t in the case of jet space.

When we start considering control systems, we prefer to use the more comfortable notations x', x'', ..., $x^{(k)}$. We will also use ∂_x for the derivation $\partial/\partial x$.

Considering control systems, it is natural to denote by $\bar{n}$ the number of state variables, which is also the number of state equations and by $\bar{m}$ the number of control. When considering abstract systems, it is easier to denote by $n(=\bar{n}+\bar{m})$ the total number of variables and by $s(=\bar{n})$ the number of equations.

Considering the aircraft equations, we need use then notations that are common to most textbooks and technical papers : so x, y, z are space coordinates, X, Y, Z are the coordinates of the thrust in the wind referential and not sets of state variables, δ_ℓ is not a derivation but the rudder control etc. To avoid conflicts, we basically restrict the notations of previous sections used here to the sets of variables $(\Xi_h)_h$.

2 Flatness

For more details on flat systems, we refer to Fliess *et al.* [17, 18] or Lévine [44, 45]. Roughly speaking, the solutions of flat systems are parametrized by m differentially independent functions, called flat outputs, and a finite number of their derivatives. This property, which characterize them, is specially important for motion planning. We present here flat systems in the framework of diffiety theory [36, 71].

2.1 Definitions and properties

We will be concerned here with systems of the following shape:

$$x_i' = f_i(x, u, t), \text{ for } 1 \leq i \leq \bar{n}, \tag{1}$$

where $x_1, \ldots, x_{\bar{n}}$ are the state variables and $u_1, \ldots, u_{\bar{m}}$ the controls.

In the sequel, we may sometimes denote ∂/∂_x by ∂_x, for short.

Definition 1. *A diffiety is a $\mathcal{C}^\infty$ manifold V of denumerable dimension equipped with a global derivation δ (that is a vector field), the* Cartan derivation *of the diffiety. The ring of functions $\mathcal{O}(V)$ is the ring of $\mathcal{C}^\infty$ function on V depending on a* finite *number of coordinates. The topology on the diffiety is the coarsest topology that makes coordinate functions continuous, i.e. the topology defined by open sets on submanifolds of finite dimensions.*

We can give a few example as an illustration.

Example 2. *The point* 0 *with derivation $\delta := 0$ is considered as a diffiety.*

Example 3. *The* trivial diffiety $\mathbf{T}^m$ *is $\left(\mathbb{R}^{\mathbb{N}}\right)^m$ equipped with the derivation*

$$\delta := \sum_{i=1}^{m} \sum_{k \in \mathbb{N}} z_i^{(k+1)} \partial/\partial_{z_i^{(k+1)}}.$$

Example 4. *The* time diffiety $\mathbb{R}_t$ *is $\mathbb{R}$ equipped with the derivation $\delta_t := \partial/\partial t$.*

Definition 5. *A morphism of diffiety*[2] *$\phi : V_1 \mapsto V_2$ is a smooth map between manifolds such that $\phi^* \circ \delta_2 = \delta_1 \circ \phi^*$, where $\phi^* : \mathcal{O}(V_2) \mapsto \mathcal{O}(V_1)$ is the dual application, defined by $\phi^*(f) = f \circ \phi$ for $f \in \mathcal{O}(V_2)$.*

We may illustrate this definition with the next example.

Example 6. *The product diffiety $\mathbb{R}_t \times \mathbf{T}^m$ is isomorphic to the jet space $\mathbf{J}(\mathbb{R}, \mathbb{R}^m)$. Indeed, points of this jet space can be seen as couples*

$$\left(t, \sum_{i=1}^{m} \sum_{k \in \mathbb{N}} \frac{y_i^{(k)}(t)}{k!} \tau_i^k\right)$$

where $\tau_1, \cdots, \tau_m$ are indeterminates and there is a natural action of the derivation d_t on the ring of function on the jet space $\mathcal{O}(\mathbf{J}(\mathbb{R}, \mathbb{R}^m))$ that defines a diffiety structure on it. Using $(t, y_1, y_1', \ldots, y_m, y_m', \ldots)$, as coordinates, there is a natural bijection ϕ between $\mathbb{R}_t \times \mathbf{T}^m$ and $\mathbf{J}(\mathbb{R}, \mathbb{R}^m)$, given by:

$$\phi(t, u_1, \cdots, u_m, u_1', \cdots, u_m', \cdots) = (t, \sum_{i=1}^{m} \sum_{k \in \mathbb{N}} \frac{u_i^{(k)}(t)}{k!} \tau_i^k)$$

[2] Or Lie-Bäcklund transform.

The derivation d_t *on the jet space is defined by*

$$\mathrm{d}_t := \partial_t + \sum_{i=1}^{m} \sum_{k\in\mathbb{N}} u_i^{(k+1)} \frac{\partial}{\partial u_i^{(k)}},$$

so that ϕ *is compatible with the derivations on both diffiety and is a diffiety morphism.*

We can now define flat diffieties.

Definition 7. *A point* η *of a diffiety* V *is called flat, if it admits a neighborhood* O *that is diffeomorphic by* ϕ *to an open set of* $\mathbb{R}_t \times \mathbf{T}^m$*. Let the generators of* $\mathbf{T}^m$ *be* u_i *their images by the dual automorphism* $z_i := \phi^*(u_i)$ *are called* linearizing outputs *or* flat outputs*. A diffiety* V *is* flat *if there exists a dense open set* $W \subset V$ *of* flat points*.*

A set of such flat outputs defines a Lie-Backlünd atlas, as defined in [31].

For a given set of flat outputs Z*, a point* η *is called* singular *related to this set if it is outside the domain of definition of the local diffiety diffeomorphism that defines it. A point in called an* intrinsic flat singularity *if no flat output is defined in a neighborhood of it. Otherwise it is called an* apparent singularity*.*

2.2 Examples

We illustrate this definition by associating a diffiety to the system (1) considered above.

Example 8. *Any system* (1) *defines a diffiety* $U \times \left(\mathbb{R}^{\mathbb{N}}\right)^{\bar{m}}$*, where* $U \subset \mathbb{R}^{\bar{n}+\bar{m}+1}$ *is the domain of definition of the functions* f_i*, equipped with the Cartan derivation*

$$\mathrm{d}_t := \partial_t + \sum_{i=1}^{\bar{n}} f_i(x,u,t)\partial_{x_i} + \sum_{j=1}^{\bar{m}} \sum_{k\in\mathbb{N}} u_j^{(k+1)} \partial_{u_j^{(k)}} \tag{2}$$

Such a system is a normal form *defining the diffiety.*

Remark 9. *Making the abstract terminology of def. 7 more concrete, flatness means that both the state and input variables* x_i*,* u_i *are functions of the flat outputs* z_i *and a finite number of their derivatives on one hand. On the other hand, this also means that the* z_i *are functions of the state and input variables and a finite number of their derivatives, and that the differential* $\mathrm{d}z_i$ *and all their derivatives are linearly independent and generate the vector space of differentials* $\langle \mathrm{d}x_i,\ 1 \leq i \leq n; \mathrm{d}u_j^{(k)},\ 1 \leq j \leq m,\ k \in \mathbb{N}\rangle$*.*

Flatness may be illustrated by the classical car example.

Example 10. *A very simplified car model is the following:*

$$\theta' = \frac{cos(\theta)y' - \sin(\theta)x'}{\ell} \tag{3}$$

The state vector is made of the coordinates (x, y) *of a point at distance* ℓ *of the rear axle's center and of the angle* θ *between the car's axis and the* x*-axis. The controls may be taken to be* $u = x'$ *and* $v = y'$*.*

One can define different sets of flat outputs depending on the actual open set, where they are defined, as follows.

1. *Over* $U_1 = \{\zeta_1' \neq 0\}$*, we take* $Z_{(1)} = \{\zeta_1 := x - \ell\cos(\theta), \zeta_2 := y - \ell\sin(\theta)\}$ *and the inverse Lie-Bäcklund transform given by:* $\theta = \tan^{-1}(\zeta_2'/\zeta_1')$*,* $x = \zeta_1 + \ell\cos(\theta)$ *and* $y = \zeta_2 + \ell\sin(\theta)$*.*
2. *Over* $U_2 = \{\zeta_2' \neq 0\}$*, we take again* $Z_{(1)} = \{\zeta_1 := x - \ell\cos(\theta), \zeta_2 := y - \ell\sin(\theta)\}$ *and the inverse Lie-Bäcklund transform given by:* $\theta = \mathrm{cotan}^{-1}(\zeta_1'/\zeta_2')$*,* $x = \zeta_1 + \ell\cos(\theta)$ *and* $y = \zeta_2 + \ell\sin(\theta)$*.*
3. *Over* $U_3 = \{\theta' \neq 0\}$*, we take* $Z_{(3)} = \{\zeta_1 = \theta, \zeta_2 = cos(\theta)y - \sin(\theta)x\}$*. Here, the inverse Lie-Bäcklund transform is given by:* $x = -\sin(\zeta_1)\zeta_2 - \cos(\zeta_1)(\zeta_2' - \ell\zeta_1')/\zeta_1'$*,* $y = \cos(\zeta_1)\zeta_2 - \sin(\zeta_1)(\zeta_2' - \ell\zeta_1')/\zeta_1'$ *and* $\theta = \zeta_1$*.*

See [31] for more details using a more realistic model.

2.3 Characterization of intrinsic singularities

The linearized tangent system of (1) at a point (x_0, u_0) is classically defined as:

$$\dot{\delta x} = \frac{\partial f}{\partial x}(x_0, u_0)\delta x + \frac{\partial f}{\partial u}(x_0, u_0)\delta u. \tag{4}$$

We use here a slightly different definition, more suited to a local study.

Definition 11. *Let η be a point of a diffiety V. For any function $g \in \mathcal{O}(V)$, we denote by $\mathrm{j}_\eta(g)$ the power series $\sum_{k\in\mathbb{N}} g^{(k)}(\eta)\tau^k \in \mathbb{R}[\![\tau]\!]$.*

For any system Σ defined by (1)*, we define the* linearized system at the point η*, denoted by $\mathrm{d}_\eta\Sigma$, to be*

$$\mathrm{d}\dot{x}_i = \sum_{i=1}^{n} \mathrm{j}_\eta\left(\frac{\partial f_i(x,u)}{\partial x_i}\right)\mathrm{d}x_i + \sum_{j=1}^{m} \mathrm{j}_\eta\left(\frac{\partial f_i(x,u)}{\partial u_j}\right)\mathrm{d}u_j. \tag{5}$$

There exists a whole algebraic approach to flat systems and their linear tangent systems. For details, we refer to [16]. We will limit ourselves here to mention that for a flat system, the module generated by the differentials of the flat outputs is free, as stated by the following lemma, which provides a necessary condition for local flatness.

Theorem 12. *At any flat regular point η, the linearized system defines a free module.*

Proof. For a linear system, controllable means flat: from the algebraic standpoint, the associated *module* (that is the module generated by the differentials of the flat outputs) is *free*, which just means that it is generated by a *basis*. If z is a flat output, then at any flat point η, $\mathrm{d}_\eta z$ is a basis of the module defined by the linearized system. Indeed, for any function $H(z)$ depending on z and its derivatives up to order r, we have

$$\mathrm{d}H(z) = \sum_{i=1}^{m}\sum_{k=0}^{r} \frac{\partial H}{\partial z_i^{(k)}}\mathrm{d}z_i^{(k)},$$

so that it is a linear combination of derivatives of the $\mathrm{d}z_i$. □

This criterion may be illustrated by the car example.

Example 13. *The system defined at ex. 10 is non flat at all trajectories such that $x' = y' = 0$. This result is stated in [40] without complete proof in the general case, but the authors show that no flat output depending only on x, y, z and not on their derivatives can be regular on such points.*

The linearized system is then

$$\begin{cases} \mathrm{d}x' &= \cos\theta \mathrm{d}u \\ \mathrm{d}y' &= \sin\theta \mathrm{d}u \\ \mathrm{d}\theta' &= \frac{\tan\phi}{\ell}\mathrm{d}u, \end{cases} \tag{6}$$

where u is the speed of the car.

This implies that $(\sin\theta x - \cos\theta y)' = 0$, so the associated module contains a torsion element and is not flat, according to th. 12. (See also [31].)

A general criterion of freeness for modules other power-series would allow a wider use of the theorem.

2.4 Defect

We propose the following definition to extend the notion of defect [14] of a diffiety to the framework of diffiety theory.

Definition 14. *Any diffiety V defined by a finite set of equations, as a subdiffiety of a trivial diffiety $\mathbf{T}^r$, may locally be described, in the neighborhood of some point η, as an open subset of $\mathbb{R}^n \times \left(\mathbb{R}^{\mathbb{N}}\right)^m$, with coordinate functions x_i, $1 \leq i \leq n$, for $\mathbb{R}^n$ and $u_{i,k}$, $(i,k) \in \mathbb{N} \times [1,m]$, for $\left(\mathbb{R}^{\mathbb{N}}\right)^m$. This is called a* representation of V at η*, and n is the* order of the representation*. (See [58] for more details.)*

The defect of V at η *is the smallest integer n such that V admits a representation of order n in a neighborhood of η.*

Remark 15. *If the diffiety is defined by an explicit normal form*

$$x_i^{(r_i)} = f_i(x), \quad 1 \leq i \leq s, \tag{7}$$

where the f_i do no depend of derivatives of the x_i, $1 \leq i \leq s$ with order greater or equal to r_i, then, reducing to an order 1 system by adding new variables $x_{i,k}$ standing to $x_i^{(k)}$, for $0 \leq k < r_i$, and new equations $x'_{i,k} = x_{i,k+1}$, for $0 \leq k < r_i - 1$, we see that the order of the representation is $\sum_{i=1}^{s} r_i$. See [53, 4.2] for more details.

It is obvious that if the defect of V at η is 0, then η is a flat point of V.

3 Jacobi's bound

Jacobi's bound was introduced by Jacobi in posthumous manuscripts [56, 57]. It is a bound on the order of a differential system, that is still conjectural in the general case, but was proved by Kondratieva *et al.* [35] under regularity hypotheses in the framework of differential algebra. A proof in the framework of diffiety theory is available in [58] and one may find complete proofs of all main results of Jacobi in [52], in the setting of differential algebra. We will refer to this paper for combinatorial aspects which are the same for differential algebra and diffiety theory.

If P_i is a function of the x_j and their derivatives, we denote by $\mathrm{ord}_{x_{j_0}} P_i$ the order of P_i considered as a function of x_{j_0} and its derivatives, which is the maximal order of derivation at which x_{j_0} appears in P_i.

Warning. From now on, all functions will be assumed to be analytic on their definition domain, so that the order does not depend on the considered point.

3.1 Combinatorial definitions

We recall briefly a few basic definitions and properties.

Definition 16. *We denote by $S_{s,n}$ the set of injections from $\{1, \ldots, s\}$ to $\{1, \ldots, n\}$.*

Let P_i, $1 \leq i \leq s$ be a differential system in $n \geq s$ variables x_j. By convention, if P_i is free from x_j and its derivatives, i.e. if P_i does not depend on x_j and its derivatives, we define $\mathrm{ord}_{x_j} P_i = -\infty$[3].

With this convention we define the order matrix *of Σ, denoted $A_\Sigma := (a_{i,j})$, where $a_{i,j} := \mathrm{ord}_{x_j} P_i$. The* Jacobi number *of the system Σ is the* tropical determinant *of A_Σ:*

$$\mathcal{O}_\Sigma := \mathcal{O}_{A_\Sigma} := \max_{\sigma \in S_{s,n}} \sum_{i=1}^{s} a_{i,\sigma(i)}.$$

Let $Y \subset X := \{x_1, \ldots, x_n\}$ be a subset of s variables, then $\mathcal{O}_{Y,\Sigma}$ denotes the Jacobi number of Σ, considered as a system in the variables of Y alone.

The tropical determinant may be computed in polynomial time using Jacobi's algorithm [53, 2.2] that relies on the notion of *canon*, and is equivalent to Kuhn's Hungarian method [37] that relies on the notion of *minimal cover*. We shall now detail these notions.

Definition 17. *For a $s \times n$ matrix of integers A, denoting by $S_{s,n}$ the set of injections of integer sets $[1, s] \mapsto [1, n]$, a* canon *is a vector of integers $(\ell_1, \ldots, \ell_s)$, such that there exists $\sigma_0 \in S_{s,n}$ that satisfies, for all $j \in im\sigma_0$, $a_{\sigma_0^{-1}(j),j} + \ell_{\sigma_0^{-1}(j)} = \max_{i=1}^{n}(a_{i,j} + \ell_i)$. The $a_{i,\sigma(i)}$, for $1 \leq i \leq s$, are a* maximal family of transversal maxima.

The following proposition is easy and yet important.

Proposition 18. *If ℓ is a canon with a maximal family of transversal maxima described by the permutation σ_0, then the tropical determinant of A is $\sum_{i=1}^{n} a_{i,\sigma_0(i)}$.*

Remark 19. *This proposition is the first of the two main reasons to introduce the concept of canon. Indeed computing such a canon can be performed in polynomial time, while computing directly the Jacobi number has exponential complexity. The other main justification for using canons will appear below in the context of application to flatness in proposition 38.*

[3] This convention, introduced by Ritt[4], is known as the *strong bound*. The convention $\mathrm{ord}_{x_j} P_i = 0$ is the *weak bound*.

Definition 20. *Assuming $s = n$, a* cover *is a couple of integer vectors μ, ν, such that $a_{i,j} \leq \mu_i + \nu_j$. A* minimal cover *is a cover such that the tropical determinant of A satisfies: $\mathcal{O}_A = \sum_{i=1}^{n}(\mu_i + \nu_i)$.*

The next proposition describes the equivalence between minimal covers and canons.

Proposition 21. *Assuming $s = n$, to any canon ℓ, one may associate a minimal cover $\mu_i := \max_{\kappa=1}^{n} \ell_\kappa - \ell_i$ and $\nu_j := \max_{i=1}^{n}(a_{i,j} - \mu_i)$.*

Reciprocally, to any minimal cover μ, ν, one may associate a canon $\ell_i = \max_{h=1}^{n} \mu_h - \mu_i$.

Proof. See [53, prop. 20]. □

The following theorem shows the existence of a unique minimal canon, that is computed in polynomial time by Jacobi's algorithm [53, alg. 9].

Theorem 22. *Using the partial order defined by $\ell \leq \ell'$ if $\ell_i \leq \ell'_i$ for all $1 \leq i \leq s$, there exists a unique minimal canon λ, that satisfies $\lambda \leq \ell$ for any canon ℓ.*

Proof. See [53, th. 13] for more details. □

The minimal cover associated to the minimal canon will be used for prop. 27.

Definition 23. *Assuming $s = n$, to this minimal canon, we associate the minimal cover α, β that we call* Jacobi's cover.

3.2 Kőnig's theorem

Matrices of 0 and 1 are a case of special interest that has been considered by Frobenius [21] and Kőnig [38, 69], to whom is due the following theorem.

Theorem 24. *Let $A = (a_{i,j})$ be a $s \times n$ matrix such that $a_{i,j} \in \{0, 1\}$, $(i, j) \in [1, s] \times [1, n]$, then $\mathcal{O}_A$ is the smallest integer r such that all non 0 elements in A belong to the union of p rows and $r - p$ columns.*

Proof. See [52, th. 17]. □

Example 25. *For the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, we have $p = 1$ and $r = 2$, since all non zero entries appear in the union of the first row and the first column. It is also cleat that the tropical determinant of A is $\mathcal{O}_A = 2 = a_{1,2} + a_{2,1} + a_{3,3}$.*

In sec. 4.1, we will be concerned with order matrices A containing 0 and $-\infty$ entries. According to Kőnig theorem 24 and changing 0 to 1 and $-\infty$ to 0, if one may find at most r entries equal 0 located in all different rows and columns of A, then one may find p rows R and $r - p$ columns C such that all entries 0 belong to a row in R or a column in C.

Definition 26. *We call such 0 or any family of elements placed in all mutually different rows and columns* transversal elements *and r the* maximum number of transversal 0.

We can even be more precise with the following result.

Proposition 27. *Considering a matrix A the entries of which are either 0 or $-\infty$, there exists a unique set R_0, maximal for inclusion and a unique set C_0, minimal for inclusion, such that $\sharp R_0 + \sharp C_0 = r$ and all entries 0 belong to rows in R_0 or columns in C_0.*

Proof. The basic idea is to transform the matrix of 0 and $-\infty$ to a matrix of 1 and 0 entries. SO r is now the number of non zero entries.

We refer to [53, prop. 58] for details. First, if a matrix A is a matrix of 0 and 1, we may restricts to covers μ, ν that are vectors of 0 and 1, as well as the associated canon ℓ (see [53, prop. 26].

Then, we can make the matrix A a $n \times n$ square matrix by adding rows or columns of 0, which does not change the value of r. Then, let λ be the minimal canon of A. Provided that some λ_{i_0} is equal to 1, the rows in R_0 are the rows of index i, with $\lambda_i = 0$, which is equivalent to $\alpha_i = 1$ for the Jacobi cover (def. 23). Without loss of generality, as there exist r transversal 1, we may assume that $a_{i,i} = 1$, for $1 \leq i \leq r$. The column j belongs to C iff $j \notin R$, so the minimality of λ imply that R_0 is maximal for inclusion and implies the minimality of C_0.

The result is straightforward when $r = s$. When $r < s$ and all λ_i are 0, we only have to add a row of n 1 and a column of $n + 1$ 0 to reduce to the previous case. □

We will also need to use the *path relation* associated to the minimal canon, which is the key ingredient of Jacobi's algorithm to compute a minimal canon and $\mathcal{O}_A$ in polynomial time [29].

Definition 28. *Let ℓ be a canon associated to a square $s \times s$ matrix A with $\mathcal{O}_A \neq -\infty$. Let $\sigma : [1, s] \mapsto [1, s]$ be permutation such that $\mathcal{O}_A = \sum_{i=1}^{s} a_{i,\sigma(i)}$. We say that* there is an elementary path from row i_1 to row i_2 *if $a_{i_1,\sigma(i_1)} = a_{i_2,\sigma(i_1)}$ (which just expresses the fact the maximal element $a_{i_1,\sigma(i_1)}$ in column $\sigma(i_1)$ also appears in row i_2). We define the* path relation defined by ℓ *as being the reflexive and transitive closure of the elementary path relation.*

The path relation does not depend on the choice of a permutation σ, provided that $\mathcal{O}_A = \sum_{i=1}^{s} a_{i,\sigma(i)}$ [53, prop. 54]. We have the following characterization of the minimal canon [53, lem. 51 i)].

Lemma 29. *A canon ℓ is the minimal canon iff for any row i_1, there is a path from it to a row i_2 with $\ell_{i_2} = 0$.*

We can now conclude these combinatorial preliminaries by state the following algorithmic result.

Theorem 30. *Let A be a $s \times n$ matrix of 0 and 1, with $\mathcal{O}_A = r$, there exists an algorithm to construct the sets of rows R_0 and columns C_0 of prop. 27 in $O(r^{1/2}sn)$ elementary operations.*

Proof. See [53, algo. 60].

We sketch here the idea of the proof. Using Hopcroft and Karp [27] algorithm, we may build a maximal set of transversal 1 in $O(r^{1/2}sn)$ operations (see also [53, § 3]). This algorithm does not compute the minimal canon. Using Jacobi's algorithm [53, § 2.2], we need to compute third class rows [53, § 2.2], and increase them by 1, which is to be done only once, as the minimal canon only contains 0 and 1 entries, as already stated in the proof of prop. 27. The computation of third class row can be done in $O(sn)$ operations. See [53, algo. 9 e)] for more details. □

We assume that this process is implemented in the procedure `HK`.

3.3 Order and normal forms

Definition 31. *Let $\Sigma = \{P_i, 1 \leq i \leq n\}$ be a square system in n differential indeterminates $x_1, \ldots, x_n$. The* system determinant *or* truncated determinant[5] *is*

$$\nabla_\Sigma := \left| \frac{\partial P_i}{\partial x_j^{\alpha_i+\beta_j}} \right|,$$

where $(\alpha_i)_i$ and $(\beta_j)_j$ define the Jacobi cover.

With this definition, we may state the following result, due to Jacobi.

Theorem 32. *Let P_i, $1 \leq i \leq n$ be a system in n differential indeterminates $x_1, \ldots, x_n$ that defines a diffiety V in a neighborhood of a point $\eta \in \mathbf{J}(\mathbb{R}, \mathbb{R}^m)$.*

If ∇_P does not vanish at η, there exists $\sigma \in S_n$ and an open set $W \ni \eta$ such that the diffiety admits in W a normal form

$$x_j^{(\alpha_{\sigma^{-1}(j)}+\beta_j)} = f_j(x),$$

so that the order of the diffiety is $\mathcal{O}_\Sigma$.

Proof. We give a sketch of the proof. See [58, th. 0.3 (ii)] for more details.

First, let $\{P_1, \ldots, P_n\} = \bigcup_{k=1}^{q} \Sigma_k$ be a partition of the set P of equations such that $\lambda_{i_1} = \lambda_{i_2}$, where λ is the minimal canon iff P_{i_1} and P_{i_2} belong to the same subset Σ_k of the partition. One may find a permutation $\sigma \in S_n$ such that, for all $1 \leq h \leq q$,

$$D_h := \left| \frac{\partial P}{\partial x}; (P, x) \in (\bigcup_{k=1}^{h} \Sigma_k, \sigma(\bigcup_{k=1}^{h} \Sigma_k) \right|$$

does not vanish at point η.

[5] Jacobi named it *determinans mancum sive determinans mutilatum* because only the terms $\partial P_i / \partial x_j^{a_{i,j}}$ such that $a_{i,j} = \alpha_i + \beta_j$ appear in it.

We recall that $\alpha_i = (\max_{k=1}^n \lambda_i) - \lambda_i$ and that $\lambda_i = (\max_{k=1}^n \alpha_i) - \alpha_i$ by def. 23. Let $\Lambda := \max_{i=1}^n \lambda_i = \max_{i=1}^n \alpha_i$. We may then consider the set of equations $E := \{P_i^{(k)} | 1 \leq i \leq n,\ 0 \leq k \leq \Lambda - \lambda_i\}$ and the set of derivatives $U := \{x_j^{k+\beta_j} | \alpha_{\sigma^{-1}(j)} \leq k \leq \Lambda\}$. Easy computations show that

$$\Delta := \left| \frac{\partial Q}{\partial v}; (Q, v) \in E \times U \right| = \prod_{h=1}^{q} D_h^{\Lambda - \hat{\lambda}_h},$$

where $\hat{\lambda}_h = \lambda_i$ for $P_i \in \Sigma_i$. So Δ does not vanish in a neighborhood W of η, where we may define a local parametrization of the variety defined by the system E, using the implicit function theorem: $x_j^{k+\beta_j} = f_{k,j}(x)$, for $x_j^{k+\beta_j} \in U$, where $f_{k,j}$ only depends on the derivatives $x_{j'}^{k'+\beta_{j'}}$, for $1 \leq j \leq n$, with $k' \leq \min(k, \alpha_{\sigma^{-1}(j)} - 1)$.

It is then easily seen that the equations $x_j^{alpha_{\sigma^{-1}(j)}+\beta_j} - f_{k,j}(x) = 0$, for $1 \leq j \leq n$, and and their derivatives locally define $W \cap V$, so that $x_j^{alpha_{\sigma^{-1}(j)}+\beta_j} = f_{k,j}(x)$ is a normal form of the diffiety $W \cap V$.

We may then check that the order of the diffiety is the sum $\sum_{j=1}^n (\alpha_{\sigma^{-1}(j)} + \beta_j) = \sum_{i=1}^n \alpha_i + \beta_i = \mathcal{O}_P$. □

When the system determinant vanishes, Jacobi's number provides a majoration of the order, under genericity hypotheses.

Considering algebraic systems, one may also refer to [53, 6 and 7.1]. The basic idea is to use a new ordering on derivatives, compatible with Jacobi's cover: $\mathrm{ord}_{x_j}^{\mathrm{J}} P := \mathrm{ord}_{x_j} P - \beta_j$. The non vanishing of the system determinant is then precisely the condition required to get a normal formal by applying the implicit function theorem. We illustrate the result with a linear example to make computations easier.

Example 33. *Assume that one wants to minimize or maximize the integral*

$$\int_a^b U(x_1(t), \ldots, x_n(t)) \mathrm{d}t. \tag{8}$$

Here we have used a shortened notation, the function U actually also depends on the derivatives of the funcitons $x_1, \cdots x_n$ up to certains orders.

The functions x_j such that this integral is extremal are solutions of the isoperimetrical system Σ defined by the equations

$$P_i(x) := \sum_{k=0}^{e_i} (-1)^k \frac{\mathrm{d}^k \frac{\partial U}{\partial x_i^{(k)}}}{\mathrm{d}t^k} = 0, \tag{9}$$

with $e_i := \mathrm{ord}_{x_i} U$. We have $a_{i,j} := \mathrm{ord}_{x_j} P_i = e_i + e_j$, so that the minimal canon is $\lambda_i = \max_{k=1}^n e_k - e_i$ and the Jacobi cover is $\alpha_i = e_i - \min_{k=1}^n e_k$, $\beta_j = e_j + \min_{k=1}^n e_k$. The order is equal to Jacobi's bound $\mathcal{O}_\Sigma = 2\sum_{i=1}^n e_i$ when the system determinant ∇_Σ, which is equal to the Hessian determinant $|\partial^2 U / \partial x_i \partial x_j|$ does not vanish. See [53, § 1.2] for details.

Example 34. *Consider the system $P_1 := x_1 + x_2'$, $P_2 := x_1' - x_2'' + x_3$, $P_3 := x_2''' + x_3'$. We have $\mathcal{O}_P = 3$, $\alpha = (0, 1, 2)$ and $\beta = (0, 1, -1)$. The normal forms compatible with Jacobi's ordering are $x_1 = -x_2'$, $x_2'' = x_3/2$, $x_3' = 0$; $x_1 = -x_2'$, $x_3 = 2x_2$, $x_2''' = 0$; $x_2' = x_1$, $x_1' = -x_3/2$, $x_3' = 0$ and $x_2' = x_1$, $x_3 = 2x_1'$, $x_1'' = 0$.*

One may use th. 32 for systems Σ of s equations in $n > s$ variables, by choosing, when it is possible, a subset $Y \subset X$, with $\nabla_{Y,\Sigma} \neq 0$.

3.4 The saddle Jacobi number

For systems with less equations than variables, one may define the *saddle Jacobi number*.

Definition 35. *Using the same notations as in def. 16, we define the* saddle Jacobi number *of the system Σ as being*

$$\hat{\mathcal{O}}_\Sigma := \min_{Y \subset X, \sharp Y = s, \mathcal{O}(Y,\Sigma) \neq -\infty} \mathcal{O}_{Y,\Sigma}.$$

Recall that $\mathcal{O}_{Y,\Sigma}$ is the tropical determinant of the square system obtained by restricting our attention to the variables in Y.

By convention if $\mathcal{O}_{Y,\Sigma} = -\infty$ for all Y, or if $s > n$, we set $\hat{\mathcal{O}}_\Sigma = -\infty$. If A is a matrix with entries in $\mathbb{N} \cup \{-\infty\}$, we define $\hat{\mathcal{O}}_A$ accordingly.

Systems Σ such that$\hat{\mathcal{O}}_\Sigma = 0$ are called oudephippical systems *or* ō-systems. *A ō-system is called* regular *if there exists $Y \subset X$ such that $\hat{\mathcal{O}}_\Sigma = \mathcal{O}_{Y,\Sigma}$ and $\nabla_{Y,\Sigma}$ does not identically vanish. It is said to be* regular at point η *if there exists $Y \subset X$ such that $\hat{\mathcal{O}}_\Sigma = \mathcal{O}_{Y,\Sigma}$ and $\nabla_{Y,\Sigma}$ does not vanish at η.*

We do not know an algorithm to compute the saddle Jacobi number faster than by testing all possible subsets $Y \subset X$, but we will see that it is possible to test in polynomial time if it is 0.

Definition 36. *We say that a system $\Sigma \subset O(\mathbf{J}(\mathbb{R}, \mathbb{R}^n))$ of s differential equations in n variables $x_1, \ldots, x_n$ admits a lazy flat parametrization at $\eta \in \mathbf{J}(\mathbb{R}, \mathbb{R}^n)$ with flat output Z if there exists a partition $X = \{x_1, \ldots, x_n\} = \bigcup_{h=0}^{r} \Xi_h$, with $\Xi_0 = Z$, and an open neighborhood V of η, such that for all $0 < h \leq r$ and all $x_{i_0} \in \Xi_h$, there exists an equation $x_{i_0} - H_{i_0}(\Xi_0, \ldots, \Xi_{r-1})$, where H_{i_0} is a differential function defined on V that belongs to the algebraic ideal[6] generated by Σ in $O(V)$.*

Remark 37. *It is easilly checked that a system Σ admitting a lazy flat parametrization with flat output $Z = \Xi_0$ is flat.*

We may indeed rewrite the parametrization $\Xi_h = \tilde{H}_h(\Xi_0, \ldots, \Xi_{h-1})$, for $1 \leq h \leq r$. So, for $h = 1$ we have an expression $\Xi_1 = \hat{H}_1(Z) := \tilde{H}_1(Z)$. We may then recursively define $\hat{H}_h$, for $2 \leq h \leq r$ by setting $\Xi_h = \hat{H}_h(Z) := \tilde{H}_h(Z, \hat{H}_1(Z), \ldots, \hat{H}_{h-1}(Z))$.

The following proposition relies on a theorem of Jacobi [30, § 1 and § 3]. See also [53, 7.3 or 9.2] in the algebraic case or [58, proof of th. 4.3 ii)] in the case of diffieties.

Proposition 38. *In order to compute explicitly the full flat parametrization, we need to differentiate equation P_i at most λ_i times, if λ is the minimal canon of the order matrix $A_{\bigcup_{h=1}^{r} \Xi_h, \Sigma}$. Then for a flat output $\zeta \in Z = \Xi_0$, assuming that $\mathrm{ord}_\zeta H_i = e_i$, the maximal order of ζ in the full flat parametrization is at most $\max_{\Sigma_i \in \bigcup_{h=1}^{r} \Sigma_h} \lambda_i + e_i$.*

Proof. This is a straightforward consequence of th. 32. □

Remark 39. *This result exhibits the second main reason for which the use of canon in our context has a very important impact.*

The next example will help to understand the situation.

Example 40. *Consider the system $P_1 := x_5 - (x_4' + x_3'') = 0$, $P_2 := x_6 - (x_4'' + x_3' + x_1^{(4)}) = 0$, $P_3 := x_3 - (x_1' + x_2) = 0$, $P_4 := x_4 - (x_2' + x_1') = 0$. We have then a full flat parametrization, with flat outputs $Z = \{x_1, x_2\}$, $x_3 = x_1' + x_2$, $x_4 = x_2' + x_1'$, $x_5 = x_1''' + x_2'' + x_1'' + 2x_2''$ and $x_6 = x_1^{(4)} + x_1''' + x_2''' + x_1'' + x_2^{(3)} + x_2'$, that may be computed using derivatives of P_3 up to order 2 and P_4 up to order 2. The vector $(0, 0, 2, 2)$ is indeed the minimal canon of the order matrix*

$$A_{\{x_3, x_4, x_5, x_6\}, \Sigma} = \begin{pmatrix} 2 & 1 & 0 & -\infty \\ 1 & 2 & -\infty & 0 \\ 0 & -\infty & -\infty & -\infty \\ -\infty & 0 & -\infty & -\infty \end{pmatrix}$$

One may remark that such expressions may be much bigger and much harder to compute, as shown by the next example, so that we have advantage to achieve numerical computations with lazy parametrizations.

Example 41. *Consider the system $x_3 = x_1 x_2$, $x_4 = \left(x_3^{(k)}\right)^d$. It is a lazy flat parametrization with flat outputs x_1 and x_2. If we develop $\left((x_1 x_2)^{(k)}\right)^d$, we get a expression with $\binom{d+k}{k}$ monomials.*

In instead of computing the flat parametrization itself, one can first to choose for all flat outputs z_j a function of the time $\zeta_j(t)$. If the function $\hat{H}_r$ is of order e_j in z_j, then, the best is to substitute to z_i in Z the sum $\mathrm{j}\zeta_i(t) = \sum_{k=0}^{e_i} \zeta_i^{(k)}(t)\tau^k/k!$ before achieving the substitutions of rem. 37.

So, the size of intermediate results is only proportional to $\max \mathrm{e}_i$, allowing much faster computations. In fact, as we will see is sec. , it is enough to have a non vanishing system determinant to work with series, e.g. by using Newton's method, without actually computing the lazy flat parametrization. See sec. 7.1.

We can now conclude this section with the following theorem that characterizes flat $\bar{o}$-systems. As a lazy flat parametrization is a special kind of regular ō-system, systems that admit a lazy flat parametrization are equivalent to a regular ō-system using simple elimination tools, such as Gröbner bases or characteristic set computations, without differentiation and without solving PDE systems.

[6] The algebraic ideal is a proper subset of the differential ideal.

Theorem 42. *With the notations of def. 36, we have the following propositions.*

i) A $\bar{o}$-system Σ, which is regular at point η, admits a lazy flat parametrization *at point η.*

ii) A system Σ that admits a lazy flat parametrization *at point η with flat output Z and such that $\nabla_{X\setminus Z,\Sigma}(\eta) \neq 0$ is a regular $\bar{o}$-system at point η.*

iii) If the system Σ is a $\bar{o}$-system, regular at point η it is flat at η.

Proof. i) Let Y be such that $\mathcal{O}_{Y,\Sigma} = 0$ and $\nabla_{Y,\Sigma}(\eta) \neq 0$ and λ be the minimal canon of the order matrix A_Σ restricted to the columns of Y. Then, there is a partition $\Sigma = \bigcup_{h=1}^{r} \Sigma_h$, such that $\lambda_i = \lambda_{i'}$ iff P_i and $P_{i'}$ belong to the same subset Σ_h. We further assume that the sets Σ_h are indexed so that the corresponding λ_i for $P_i \in \Xi_h$ are decreasing. Let $\sigma \in S_{s,n}$ be such that its values are the columns defined by Y and such that $\sum_{i=1}^{s} a_{i,\sigma(i)} = 0$. Then we define Ξ_0 to be $X \setminus Y$ and Ξ_h to be the variables with indexes $\sigma(i)$, where i runs over the indexes of the equations in Σ_h.

As $a_{i,\sigma(i)} + \lambda_i \geq a_{i',\sigma(i)} + \lambda_{i'}$ by the definition of a canon (def. 17), the equations of Σ_h do not depend on the variables in $\Xi_{h'}$ if $h < h'$, so that the system is block triangular and $\nabla_{Y,\Sigma}(\eta) = \prod_{h=1}^{r} D_h(\eta)$, where D_h is the Jacobian determinant of Σ_h with respect to variables in Ξ_h, so that for all $1 \leq h \leq r$ $D_h(\eta) \neq 0$. We only have to use the implicit function theorem to get the requested lazy flat parametrization.

ii) As $\nabla_{X\setminus Z,\Sigma}(\eta) \neq 0$, the order of Σ considered as a system in the subset of variables $X \setminus Z$ is equal to $\mathcal{O}_{X\setminus Z,\Sigma}$ by th. 32. If a lazy flat parametrization exists with flat output Z, then this order must be 0.

iii) This is a consequence of i) and also a special case of th. 32. □

This is particularly important for the complexity as the size of a non linear expression grows exponentially with the order of derivation, as shown by ex. 41. This result provides a fast flat parametrization.

Remark 43. *Any flat parametrization is a $\bar{o}$-system, which shows that flat parametrization can be very far from the usual state space representation of control theory. It is known that a flat parametrization may be sometimes easier to compute from the physical equations.*

E.g. the system $x_1 = x_4$, $x_2 = x_5'/x_4'$, $x_3 = x_4'x_5''/x_4'' - x_5$ is a flat parametrization, and so a $\bar{o}$-system, that corresponds to the system of Rouchon $x_3' = x_1'x_2'$ with flat outputs $x_5 = x_1$ and $x_6 = x_1'x_2 - x_3$. See [54, 55] for more details on this classical example.

4 $\bar{O}$-systems and flatness

In this section, we give an efficient criteria to test if a system is a ō-system or a regular ō-system.

Remark 44. *In this section, we consider a submatrix B of a matrix $A = (a_{i,j})$ to be defined by a set R of rows and a set C of columns, together with the values $a_{i,j}$, for $(i,j) \in R \times C$. The empty submatrix corresponds to $R = C = \emptyset$.*

By abuse of notation, we identify subsets of equations P_i or of variables x_j with the corresponding subsets of indices.

4.1 An algorithmic criterion for ō-systems

In this section, we consider a matrix A of positive integer and $-\infty$ elements and provide an algorithm to test if $\hat{\mathcal{O}} = 0$.

We may first make some obvious simplification to spare useless computations.

Remark 45. *If $s > n$ or if A contains a row of $-\infty$ elements, then $\hat{\mathcal{O}}_A = -\infty$. One may remove from A all columns that contain only $-\infty$ elements.*

The basic idea of the algorith relies then on the following lemma.

Lemma 46. *Assume that A is a $s \times n$ matrix with $s \leq n$ such that $\hat{\mathcal{O}}_A = 0$. Let B denote the submatrix formed of the rows R of A that contain only 0 or $-\infty$ entries and columns C that contain at least one element equal to 0, R_0 its maximal set of rows and C_0 its corresponding minimal set of columns, according to prop. 27. With these hypotheses, we have the following propositions.*

i) For any subset of columns Y such that $\mathcal{O}_{Y,A} = 0$, let λ be the minimal canon of A restricted to the columns of Y. The set R_0 contains the rows i with $\lambda_i = 0$, so that it is non empty.

ii) the matrix A_2 formed of rows R_2 not in R_0 and columns C_2 not in $C \setminus C_0$ is such that there exists a set of columns Y_2 that satisfies $\hat{\mathcal{O}}_{A_2} = \mathcal{O}_{Y_2,A_2} = 0$.

iii) The matrix B' formed of rows of B in R_0 and columns not in C_0 is such that there exists Y_1 with $\mathcal{O}_{Y_1,B'} = 0$, which implies $\mathcal{O}_{Y_2 \cup Y_1, A} = 0$.

Proof. i) Let σ be an injection $[1, s] \mapsto Y$ such that $a_{i,\sigma(i)} = 0$, for $1 \leq i \leq s$. If $\lambda_i = 0$, as $a_{i,\sigma(i)} + \lambda_i \geq a_{i',\sigma(i)} + \lambda_{i'}$ according to the canon definition, we need have

$$a_{i',\sigma(i)} = -\infty \text{ if } \lambda_{i'} > 0 \text{ and } a_{i',\sigma(i)} = 0 \text{ if } \lambda_{i'} = 0. \tag{10}$$

So the column $\sigma(i)$ belongs to the columns of B.

Let $R_0' := \{i | \lambda_i = 0 \text{ and } i \notin R_0\}$. Then, by (10), the columns of $\sigma(R_0')$ cannot contain elements equal to 0 in rows i with $\lambda_i > 0$, so not in R. This means that all 0 elements are located in rows $R_0 \cup R_0'$ and columns $C_0 \setminus \sigma(R_0)$, which contradicts the maximality of R_0 unless $R_0' = \emptyset$, so that all rows with $\lambda_i = 0$ belong to R_0.

ii) With the same notations as in the proof of i), as the columns of $C \setminus C_0$ contain no element equal to 0 outside the rows of R_0, $Y_2 := \sigma(R_2) \subset C_2$ and $\hat{\mathcal{O}}_{A_2} = \mathcal{O}_{Y_2,A_2} = 0$.

iii) By the definition of the sets R_0 and C_0, one may find a family of r transversal elements of B $a_{i_k,j_k} = 0$, for $1 \leq k \leq r$. Let $Y_1 := \{j_k | i_k \notin C_0\}$, we have $\sharp Y_i = \sharp R_0$, so that $\mathcal{O}_{Y_1,B'} = 0$. □

This provides the following recursive algorithm, denoted `ō-test`. We use here the subroutine `HK` that implements the algorithm described in th. 30. We assume moreover that it also returns the set Y_1, with the notations of the above lemma.

If B contains only $-\infty$ entries, R_0 is $\emptyset$ and Y_1 too.

Algorithm 47. ***Imput:*** *A $s \times n$ matrix A with entries $a_{i,j} \in \mathbb{N} \cup \{-\infty\}$.*

Output: *"failed" or a set Y of rows such that $\mathcal{O}_{Y,A} = 0$.*

if *$s > n$ or A contains a full row of $-\infty$ then* ***return*** *"failed";*

Suppress from A all columns of $-\infty$;

Build B the submatrix of A of columns containing only $-\infty$ or 0 elements.

Compute $(R_0, C_0, Y_1) :=$`HK`(B);

if *$R_0 = \emptyset$* ***then return*** *"failed";*

$Y_2 :=$`ō-test`$(A \setminus R_0)$[7];

if *$Y_2 =$ "failed"* ***then return*** *"failed";*

return *$Y_2 \cup Y_1$;*

In the previous algorithm, finding the columns of B can be made faster using balanced trees.

Remark 48. *We may sort the elements in the columns of a $s \times n$ matrix A and store the results in* balanced (or AVL) trees *(Adel'son-Vel'skii and Landis [1] or Knuth [33, sec. 6.2.3], with complexity $ns \ln s$, which allows to delete an element in a column with cost $\ln s$, preserving the order, and to get the greatest element or a column with cost $\ln s$.*

The following theorem provides an evaluation of the complexity.

Theorem 49. *i) This algorithm tests if $\hat{\mathcal{O}}_A = 0$ and if yes returns Y such that $\hat{\mathcal{O}}_A = \mathcal{O}_{Y,A}$.*

ii) It works in $O(d^{1/2}psn)$ elementary operations, where d is the maximal number of transversal 0 in the sets B built at each call of the algorithm `HK` and p the number of recursive call of the main algorithm `ō-test`.

iii) If at each step $C_0 = \emptyset$, then the algorithm works in $O((d^{1/2} + \ln s)sn)$ operations, using balanced trees as in rem. 48.

Proof. i) The algorithms produces the correct result as a consequence of lem. 29. Indeed, by lem. 29 i) if `HK`(B) returns "failed", we know that $\hat{\mathcal{O}}_A \neq 0$. In the same way if `ō-test`$(A \setminus R_0)$ returns "failed", $\hat{\mathcal{O}}_A$ is not 0 by lem. 29 ii). Furthermore, except in those two cases, we know that $\mathcal{O}_{Y_2 \cup Y_1', A}$ is 0 by lem. 29 iii).

ii) For the complexity, at each call the computation of the set B requires $O(sn)$, which is proportional to size of the matrix. Then, computing a maximal set of diagonal 0 requires at most $O(d^{1/2}sn)$ operations, using Hopcroft and Karp [27] algorithm, by th. 30. So the total cost is $O(d^{1/2}psn)$, where p is the number of recursive call to `ō-test`.

[7] To simplify the presentation, we denote in this way the matrix A where the rows in R_0 have been suppressed.

iii) Using a balanced tree, the cost of the first inspection of the matrix becomes $O(\ln ssn)$. Let b_i and c_i respectively denote the cardinal of R_0 and the number of columns in B at step i, c_i, and let p be the total number of steps. At step i, we just have to remove at most $b_i n$ element from the avl trees with cost $O(\ln sb_i n)$ and from the matrix with cost $O(b_i n)$. The detection of rows in B at each step can be made with cost $O(\ln sn)$. This provides a total cost $O(\ln s^2 n)$.

The remaining costs lie in the HK routine, for which the cost is $O(d^{1/2}b_i c_i)$ at step i by th. 30. So, the remaining total cost is $O(\sum_{i=1}^{p} d_i^{1/2} d_i c_i) \leq O(d^{1/2} sn)$, which concludes the proof. □

Examples will be given in the section 4.4.

4.2 Sufficient condition for regularity

The next theorem is an easy sufficient criterion for the regularity of ō-systems.

Theorem 50. *Let Ξ be a ō-system defining a diffiety V in some neighborhood of a point η. Using algo. 47, one may assume that we have a partition of the equations $\Sigma = \bigcup_{h=1}^{p} \Sigma_h$, where Σ_{p-h} corresponds to rows in R_0 at step h of the algorithm, and a partition of the variables $X = \bigcup_{h=1}^{p} \Xi_h$, where Ξ_h corresponds to columns in B and not in C_0 at step h.*

With these notations, if for all $1 \leq h \leq p$,

$$\text{rank } \left(\frac{\partial P_i}{\partial x}(\eta) \mid P_i \in \Sigma_h,\ x \in \Xi_h \right) = \sharp \Sigma_i, \tag{11}$$

then Ξ is ō-regular at point η.

Proof. Eq. 11 implies that for all $1 \leq i \leq p$ there exists a subset $Y_h \subset \Xi_h$ such that $\nabla_{Y_h,\Sigma_h}(\eta) \neq 0$. By construction, setting $Y := \bigcup_{h=1}^{p} Y_h$, we have then

$$\mathcal{O}_{Y_h,\Sigma_h} = 0,$$

so that

$$\mathcal{O}_{Y,\Sigma}(\eta) = \sum_{h=1}^{p} \mathcal{O}_{Y_h,\Sigma_h} = 0,$$

as the equations in Σ_h do not depend on the variables in Ξ_ℓ for $\ell > h$. Furthermore, we have

$$\nabla_{Y,\Sigma}(\eta) = \prod_{h=1}^{p} \nabla_{Y_h,\Sigma_h}(\eta) \neq 0,$$

so that Σ is ō-regular at point η. □

This condition is not necessary, as shown by the next example.

Example 51. *Let be the system $\Sigma := \{P_1, \ldots, P_4\}$ with $P_1 := x_1 + x_3' + x_6$, $P_2 = x_1 + x_4' + x_5^3$, $P_3 := x_1' + x_3$, $P_4 := x_2' + x_4$. We have $\Xi_1 = \{x_5, x_6\}$, $\Xi_2 = \{x_3, x_4\}$ and $\Sigma_1 = \{P_1, P_2\}$, $\Sigma_2 = \{P_3, P_4\}$. The only set Y_i that may be extracted from the Ξ_i are the Ξ_i themselves, as $\sharp \Xi_i = \sharp \Sigma_i$, for $i = 1, 2$. The corresponding flat outputs are then x_1 and x_2. With this choice, $\nabla_{Y,P} = \nabla_{\Xi_1,\Sigma_1} \nabla_{\Xi_2,\Sigma_2} = \nabla_{\Xi_1,\Sigma_1} = 3x_5^2$, which vanishes when $x_5 = 0$.*

But at such a point, we can use the alternative values $Y = \{x_1, x_3, x_4, x_6\}$, with $\nabla_{Y,\Sigma} = 1$.

So we need a more precise criterion of ō-regularity, that will be described bellow.

4.3 Characterization of regular ō-systems

We work at a point η of the diffiety. We assume that the coordinates of this point belong to some effective subfield $\mathbb{K}$ of $\mathbb{R}$. If the equations of Σ are algebraic, $\mathbb{K}$ can be $\mathbb{Q}$ or an algebraic extension of $\mathbb{Q}$. We do not consider here the size of the elements of $\mathbb{K}$ and bit complexity and only evaluate the number of elementary operations in $\mathbb{K}$, counting in them operations in $\mathbb{N}$ and all other faster elementary operations.

The basic principle looks much like that of sec. 4.1. We will need the following easy technical lemma.

Lemma 52. *Let $v_i \in \mathbb{R}^n$ with $v_i = (c_{i,1}, \ldots, c_{i,n})$, for $1 \leq i \leq s$. Let K be the vector space generated by the linear relations between the vectors v_i (K is the a subspace of the dual of $(\mathbb{R}^n)^s$.). Let $e_k = \sum_{i=1}^{s} b_{k,i} v_i$, for $1 \leq k \leq r$, be a basis of K and $R_{[\mathrm{K}]} := \{i \mid \exists k, 1 \leq k \leq r\ , b_{k,i} \neq 0\}$.*

The set $R_{[\mathrm{K}]}$ does not depend on the choice of the basis e.

Proof. Let $\bar{e}$, with $\bar{e}_k = \sum_{\ell=1}^{r} \gamma_{k,\ell} e_\ell$ be another basis, defining another set of rows $\bar{R}_{[\mathrm{K}]}$. If $\bar{b}_{k,i} = \sum_{\ell=1}^{r} \gamma_{k,\ell} b_{\ell,i}$ is non zero, then $b_{\ell,i}$ is non zero. So $\bar{R}_{[\mathrm{K}]} \subset R_{[\mathrm{K}]}$. We prove in the same way the reciprocal inclusion. □

The following lemma will allow us to design a recursive process.

Lemma 53. *Assume that a system Σ of s equations in the variables $X := \{x_1, \ldots, x_n\}$ is $\bar{o}$-regular at point η, with $\nabla_{Y,\Sigma}(\eta) \neq 0$. We use the same notations and hypotheses as in lemma 46, with $A := A_\Sigma$, the order matrix of Σ.*

We consider the minimal canon λ of the order matrix $A_{Y,\Sigma}$ restricted to the columns in Y and a bijection $\sigma : [1, s] \mapsto Y$ such that $\sum_{i=1}^{s} a_{i,\sigma(i)} = 0$.

Let B be defined as in lem. 46 and B' be a submatrix of B restricted to a set of columns C that contains $\sigma(\bar{R})$, with $\bar{R} := \{i \mid \lambda_i = 0\}$.

i)Let R_0 and C_0 be respectively sets of rows and columns that contain all entries of B' equal to 0, with $\sharp R_0 + \sharp C_0 = \mathcal{O}_{B'}$ and R_0 maximal (as in th. 24), then $\bar{R} \subset R_0$ and $\sigma(R_0) \subset C \setminus C_0$.

ii) Let J be the Jacobian matrix $(\partial P_i/\partial x_j \mid a_{i,j} \in B')$ and $R_{[k]}$ be the set of rows associated to a basis of linear relations between the rows of J as in lem. 52. We also define the set of rows $C_{[\mathrm{k}]} := \{j \mid \exists i \in R_{[\mathrm{k}]}\ a_{i,j} = 0$. Then, $\bar{R} \subset R \setminus R_{[\mathrm{k}]}$ and $\sigma(\bar{R}) \subset C \setminus C_{[\mathrm{k}]}$.

Proof. i) We proceed as in the proof of lem. 46 i). Let $R'_0 := \{i | \lambda_i = 0$ and $i \notin R_0\}$. Then, by (10), the columns of $\sigma(R'_0)$ cannot contain elements equal to 0 in rows i with $\lambda_i > 0$, which means that all 0 elements are located in rows $R_0 \cup R'_0$ and columns $C_0 \setminus \sigma(R_0)$, which contradicts the maximality of R_0 unless $R'_0 = \emptyset$, so that all rows with $\lambda_i = 0$ belong to R_0.

ii) Assume that $\bar{R} \cap R_{[\mathrm{K}]}$ is non empty and contains row i. Then, by (10), the columns of $\sigma(\bar{R})$ cannot contain entries equal to 0 and not located in the rows of $\bar{R}$. So, the rows of $\bar{R} \cap R_{[\mathrm{K}]}$ in J are linearly dependent, so that $\nabla_{\sigma(\bar{R}),\bar{\Sigma}}$, where $\bar{\Sigma}$ is the subset of equations that corresponds to the rows of $\bar{R}$, is a factor of $\nabla_{Y,\Sigma}$ and must vanish at η which contradicts $\nabla_{Y,\Sigma}(\eta) \neq 0$.

This implies that the columns of $\sigma(\bar{R})$ cannot contain entries equal to 0 in the rows of $R_{[\mathrm{k}]}$, so that $\sigma(\bar{R}) \cap C_{[\mathrm{k}]} = \emptyset$. □

Definition 54. *With the notations of lem. 53, we define the* row kernel support *of B' at point η to be $R_{[\mathrm{K}]}$ and* column kernel support *of B' at point η to be $C_{[\mathrm{K}]}$.* Non trivial rows *of B' are those that contain entries equal to* 0.

The following algorithm `KS` computes the row and column kernel support of the Jacobian matrix J at point η and returns B' retricted to columns not in $C_{[\mathrm{K}]}$.

Algorithm 55. ***Imput:*** *A matrix $B' \subset B$ of integers and $-\infty$ elements and a matrix J' of real elements with the same number of rows and columns.*

Output: *The submatrix B'' of B' where columns in $C_{[\mathrm{K}]}$ have been suppressed, with the notations of lem. 53.*

Compute the kernel K of the linear mapping L defined by the rows of J'.

Compute the column support $C_{[\mathrm{K}]}$ of K.

return $(B'' := B' \setminus C_{[\mathrm{K}]}, J'' := J' \setminus C_{[\mathrm{K}]})$

Remark 56. *It is easilly seen that the complexity of this algorithm is $O(s^2 n)$ elementary operations in $\mathbb{K}$, using Gaussian elimination. In fact we may only consider non trivial rows of B' and the corresponding rows of J'. If their number is q, the complexity is $O(q^2 n + sn)$.*

We may now iterate `HK` and `KS` until the returned matrix B'' is equal to B'. We call the process `Seq`. We do not need any more to assume that `HK` returns a set Y_1 of columns, but only the sets of rows and columns R_0 and C_0.

Algorithm 57. ***Imput:*** *A matrix $B' \subset B$ of integers and $-\infty$ elements and a matrix J' of real elements with the same number of rows and columns.*

Output: *A submatrix B'' of B' such that $\mathtt{HK}(B) = (R_0, C_0)$ with $C_0 = \emptyset$ and $\mathtt{KS}(B'') = B''$.*

$(R_0, C_0) :=\mathtt{HK}(B')$;

$B'' := B' \setminus C_0$; $J'' := J \setminus C_0$;

$(B'', Y'') :=\mathtt{KS}(B', J')$

if $B'' = B'$ ***then return*** (B'', J'').

else return $\mathtt{Seq}(B'', J'')$ ***fi***

To alleviate the presentation, we avoid going too deeply in computational details. The following remark should be enough for our purpose.

Remark 58. *As the number of rows of B' decreases at each recursive call the number of iterations is bounded by s, so that the complexity is $O(s^3 n)$. We can obviously neglect the trivial rows of B' and the corresponding rows of J' that are rows of 0 elements. If d is the number of non trivial rows in B' and q the number of iterations, then the computations only require $O(qd^2 n')$, where n' is the number of columns of B. operations.*

We have then the following lemma, where ii) and iii) are analogs of lem. 46 ii) and iii).

Lemma 59. *i) With the hypotheses and notations of lem. 53, the process $\mathtt{Seq}$ returns a matrix B'', which is a submatrix of B' containing at least one entry equal to 0 and such that $\mathtt{HK}(B) = (R_0, C_0)$ with $C_0 = \emptyset$ and $\mathtt{KS}(B'', J'') = (B'', J'')$. Let C_1 denote the rows of B'' and R_1 denote the non trivial rows i of B'' that contain at least one entry $a_{i,j} = 0$, where column j belong to the columns of B''.*

ii) the matrix A_2 formed of rows R_2 not in R_1 and columns C_2 not in $C \setminus C_0$ is such that there exists a set of columns Y_2 that satisfies $\hat{\mathcal{O}}_{A_2} = \mathcal{O}_{Y_2,A_2} = 0$ and $\nabla_{Y_2,\Sigma_2}(\eta) \neq 0$, where Σ_2 is the subset of Σ $\Sigma_2 := \{P_i \mid i \in R_2\}$.

iii) The matrix B'' is such that there exists Y_1 with $\mathcal{O}_{Y_1,B''} = 0$ and $\nabla_{Y_1,B''}(\eta) \neq 0$, which implies $\mathcal{O}_{Y_2 \cup Y_1,A} = 0$ and $\nabla_{Y_2 \cup Y_1,A}(\eta) \neq 0$.

Proof. i) By lem. 53 i) and ii), the columns of $\sigma(\bar{R})$ belong to B'', so that B'' contains $a_{i,\sigma(i)} = 0$ for $i \in \bar{R}$. It is then straightforward as when the process Seq stops, it must return a B'' such that $\mathtt{HK}(B) = (R_0, C_0)$ with $C_0 = \emptyset$ and $\mathtt{KS}(B'', J'') = (B'', J'', Y_1)$.

ii) We know that there exists Y such that $\mathcal{O}_{Y,\Sigma} = 0$ and $\nabla_{Y,\Sigma}(\eta) \neq 0$. We have

$$\mathcal{O}_{Y,\Sigma} = \max_{Y_2 \subset Y, \sharp Y_2 = \sharp \Sigma_2} \mathcal{O}_{Y_2,\Sigma_2} + \mathcal{O}_{Y \setminus Y_2, \Sigma \setminus \Sigma_2} = 0 \text{ and } \nabla_{Y,\Sigma}(\eta) = \sum_{Y_2 \subset Y, \sharp Y_2 = \sharp \Sigma_2} \nabla_{Y_2,\Sigma_2}(\eta) + \nabla_{Y \setminus Y_2, \Sigma \setminus \Sigma_2}(\eta) \neq 0,$$

so that there exists so $Y_2 \subset Y$ with $\nabla_{Y_2,\Sigma_2}(\eta) \neq 0$, which implies $\nabla_{Y_2,\Sigma_2} = 0$.

iii) As $\mathtt{KS}(B'', J'') = (B'', J'')$, the matrix J'' restricted to the non trivial rows R_1 has full rank. So there exists Y_1 such that $\mathcal{O}_{Y_1,B''} = 0$ and $\nabla_{Y_1,\Sigma \setminus \Sigma_2}(\eta) \neq 0$, which implies $\mathcal{O}_{Y_2 \cup Y_1,A} = 0$ and $\nabla_{Y_2 \cup Y_1,\Sigma_2}(\eta) \neq 0$, as equations of Σ_2 do not depend on variables x_j, with $j \in C_1$. □

It is then easy to design an algorithm $\bar{\mathtt{o}}$-reg to test $\bar{\mathtt{o}}$-regularity at point η. The routine Seq2 is assumed to be a variant that returns the set of columns Y_1 and the set of equations Σ_2 of lem. 59 iii) or "failed" if B'' is empty or contains no entry equal to 0.

Algorithm 60. ***Imput:*** *A differential system Σ of s equations in n variables $x_1, \ldots, x_n$, defining a diffiety in the neighborhood of $\eta \in \mathbf{J}^n$.*

Output: *"failed" or a set Y of rows such that $\mathcal{O}_{Y,\Sigma} = 0$ and $\nabla_{Y,\Sigma}(\eta) \neq 0$.*

$A := A_\Sigma$;

if *$s > n$ or A contains a full row of $-\infty$ then* ***return*** *"failed";*

Suppress from A all columns of $-\infty$;

Build B the submatrix of A of columns C containing only $-\infty$ or 0 elements and $J := (\partial P_i / \partial x_j \mid 1 \leq i \leq s,\ j \in C)$

if $\mathtt{Seq2}(B, J) =$ *"failed"* ***return*** *"failed"*

else $(Y, \Sigma_2) :=$$\mathtt{Seq2}(B, J)$ ***fi***

if $\bar{\mathtt{o}}\mathtt{reg}(\Sigma_2, \eta) =$ *"failed"* ***return*** *"failed"*

else $Y_2 :=$$\mathtt{Seq2}(B, J)$ ***fi***

return $Y_1 \cup Y_2$

The following theorem provides an evaluation of the complexity.

Theorem 61. *i) The algorithm* `ō-reg` *60 tests if Σ is $\bar{o}$-regular at point η and then returns Y such that $\mathcal{O}_{Y,\Sigma} = 0$ and $\nabla_{Y,\Sigma}(\eta) \neq 0$.*

ii) It works in $O(p(qd^2 + s)n)$ elementary operations in $\mathbb{K}$, where

— p is the number of recursive calls of `ō-reg`*;*

— q and d are respectively the maximal number of iterations of `Seq` *and the maximal number of non trivial rows in B at each recursive call of* `ō-reg`*.*

iii) If q is equal to 1 and $C_0 = \emptyset$ at each recursive call of `ō-reg`*, then using balanced trees (rem. 48) as in algo. 47, we can achieve a complexity of $O((d^2 + \ln(s)s)n$.*

Proof. i) Proceeding as in the proof of th. 49, it is a straightforward consequence of lem. 59.

ii) The construction of B at each recursive call requires $O(sn)$ operations and `Seq` $O(qd^2n)$ operations, using rem. 58.

iii) Using balanced trees, the first construction of B requires $O(\ln(s)sn)$ operations and further constructions only $O(\ln(s)n)$. Let n_h be the number of columns in B at recursive call h of `ō-reg`. By rem. 58, `Seq` requires then $O(d^2 n_h)$ operations, so that the total cost is $O(\ln(s)sn) + \sum_{h=1}^{p} O(d^2 n_h) = O((\ln(s)s + d^2)n)$ operations. □

The following example show that those evaluation of complexity are sharp.

Example 62. *We consider the system $P_i := x_i^2 + x_{i+1} + x_{2s-i+1} + x'_{2s-i+2}$, for $1 \leq i \leq s$ in $2s+1$ variables x_j. We have then $p = s$ and at iteration h of* `ō-reg`*, we have $q = s - h + 1$ iterations in* `Seq` *with columns of B' reducing from $\{1, \ldots, h, 2s - h + 1\}$ to $\{h, 2s - h + 1\}$.*

4.4 Examples

We will consider here some examples of classes of "chained" or "triangular" flat systems in the literature. We stress of the fact that in the papers quoted here, the main issue is to test the existence of a change of variables that may reduce a given system to such a form, where as our problem here is to test if a system is already in a such a form, up to a permutation of indices.

4.4.1 Goursat normal form

It is known that all driftless systems with two controls of the general form

$$x'_i = f_i(x)u + g_i(x)v, \text{ for } 1 \leq i \leq n \tag{12}$$

can be reduced to the Goursat normal form

$$\begin{array}{lcl} z'_0 & = & v_0; \\ z'_i & = & z_{i+1}v_0 \text{ for } 1 \leq i \leq n-2; \\ z'_{n-1} & = & v_1, \end{array} \tag{13}$$

iff it is flat and we may use the following flatness criterion.

Theorem 63. *A driftless systems with two control* (12) *is flat iff the vector spaces E_i, $0 \leq i \leq n-2$ defined by $E_0 := \{f, g\}$ and $E_{i+1} = E_i + \{[E_i, E_i]\}$ satisfy $\dim E_i = i + 2$ for all $0 \leq i \leq n - 2$.*

This result goes back to the work of Cartan [10, 11] and has been adapted to control by Martin and Rouchon [65]. See also Li *et al.* [40].

The system (13) is obviously a ō-system, with a single set of possible flat outputs; $\{z_0, z_1\}$, which is regular iff $v_0 \neq 0$.

4.4.2 Complexity issues

Without going into useless details, for which we refer to the references quoted above, we need to give some idea of the complexity of computations involved to work out a flat parametrization after having proved the existence of a suitable change of variables. We assume here for simplicity that the fields f, g, ... are defined by rational functions of the state variables and that vector spaces are $\mathbb{R}(x)$-vector spaces. For simplicity, we identify the fields f, g, ... with the associated derivations.

The next theorem is the basis of a step by step reduction of a two inputs driftless system in Goursat normal form.

Proposition 64. *Assume that a two input driftless system, with $n > 3$, states admits a change of variable $y_i = Y_i(x)$ such that the system becomes*

$$\begin{array}{rcl} y_i' &=& (\bar{f}_i(y_1, \dots, y_{n-1}) + \bar{g}(y_1, \dots, y_{n-1})y_n)\bar{u}; \text{ for } 1 \leq i \leq n-1; \\ y_n' &=& \bar{w}, \end{array} \tag{14}$$

with $[\bar{f}, \bar{g}] \notin \langle \bar{f}, \bar{g} \rangle$. Then $\dim E_1 = 3$ and $\dim E_2 = 4$.

Proof. Using the new coordinates, we have have $E_3 = \langle \partial_{y_n}, \bar{f}, \bar{g} \rangle$ and $E_4 = \langle \partial_{y_n}, \bar{f}, \bar{g}, [\bar{f}, \bar{g}] \rangle$, that must have respective dimensions 3 and 4, according to our independence hypothesis. □

Easy computations imply the following corollary.

Corollary 65. *Under the hypotheses of the proposition, there exists $(a, b) \neq (0, 0)$ such that $a[f, [f, g]] + b[g, [f, g]] = 0$ modulo E_3. Then, $af + bg = c\partial_{y_n}$ and the y_i, $1 \leq i \leq n-1$ are functionally independent first integrals common to the field $af + bg$.*

Using this lemma, we are reduced to a new two input system

$$\begin{array}{rcl} y_i' &=& (\bar{f}_i(y_1, \dots, y_{n-1}) + \bar{g}(y_1, \dots, y_{n-1})y_n)\bar{u}; \text{ for } 1 \leq i \leq n-1; \\ &=& \bar{f}_i(y_1, \dots, y_{n-1})\bar{u} + \bar{g}(y_1, \dots, y_{n-1})\bar{v}, \end{array} \tag{15}$$

with $\bar{v} = z_n\bar{u}$.

Successive applications of this process reduce the state dimension and produce a Goursat normal form and a flat parametrization. One may notice that for $n = 3$, all combinations $a(x)f + b(x)g$ work.

The main issue then is to look for first integrals. There can exist no rational solutions, there is no general method to test if a rational solution exists, even when looking for first integral of a field in the affine plane, and already looking for the existence of such an integral up to to a given degree is computationally difficult. See *e.g.* Chèze and Combot [12] and the references therein for more details.

One may also look for closed forms solutions as did Rouchon for the car with one trailer in the general case [63], but again there might not exist any. This does not mean that the task is hopeless, but justifies some special interest to situations where the computations are much easier, although not completely trivial, mostly when the size of initial equations is already appreciable. Such situations are obviously non generic, but may be encountered in practice with the help of some simplifications, which is not uncommon with flat systems, that are themselves non generic but quite ubiquitous in engineering practice.

Example 66. *An affine generalization with two inputs has been considered by Silveira [67] and Silveira et al. [66]:*

$$\begin{array}{rcl} z_0' &=& v_0; \\ z_i' &=& f_i(z_0, z_1, \dots, z_{i+1}) + z_{i+1}v_0 \text{ for } 1 \leq i \leq n-2; \\ z_{n-1}' &=& v_1. \end{array} \tag{16}$$

They provide necessary and sufficient conditions to reduce a system of the form $x' = f(x) + g_1(x)u_1 + g_2(x)u_2$ to the form (16)*.*

Such a system is a oudephippical. Using algo. 47, one can conclude that all matrices B are such that $C_0 = \emptyset$. The sets Ξ_h of th. 50 are $\Xi_h = \{v_0, v_1\}$ if $h = n-1$ and $\Xi_h = \{z_h\}$ if $1 < h < n-1$. For best efficiency, a sparse version of the algorithm should be designed for such sparse systems. The only possible flat outputs set in the setting of th. 50 is $\{z_0, z_1\}$ and the regularity condition is $v_0 + \partial f_i/\partial z_{i+1} \neq 0$, for all $1 \leq i \leq n-2$.

4.4.3 Multi input chained forms

Some notions of chained forms may be found in the literature for systems with many inputs.

Example 67. *A multi-input generalization, the "m-chained form", has been proposed by Li et al. [41]:*

$$\begin{array}{rcl} z_0' &=& v_0; \\ z_{i,\ell}' &=& f_{i,\ell}(z_0, \bar{z}_\ell) + z_{i,\ell+1,}v_0 \text{ for } 1 \leq i \leq m \text{ and } 1 \leq \ell < k; \\ z_{i,k}' &=& v_i, \end{array} \tag{17}$$

where $\bar{z}_\ell := (z_{1,1}, \dots, z_{1,\ell}, \dots, z_{m,1}, \dots, z_{m,\ell})$.

This system is also oudephippical. All matrices B of algo. 47 are again such that $C_0 = \emptyset$. The sets Ξ_h of th. 50 are $\Xi_h = \{v_0, \dots, v_m\}$ if $h = k-1$ and $\Xi_h = \{z_{1,h+1}, \dots, z_{m,h+1}\}$ if $1 \leq h \leq k-1$. The only possible flat outputs set in the setting of th. 50 is $\{z_0, z_{1,1}, \dots, z_{m,1}\}$ and the regularity condition is $|v_0\delta(i,j) + \partial f_i/\partial z_{j,\ell+1}| \neq 0$, for all $1 \leq i \leq k-1$.

The authors consider the case of a rolling coin on a moving table.

$$\begin{pmatrix} x' \\ y' \\ \theta' \\ \phi' \end{pmatrix} = \begin{pmatrix} \cos\theta(\alpha\cos\theta + \beta\sin\theta) \\ \sin\theta(\alpha\cos\theta + \beta\sin\theta) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix} u_2$$

where α and β are known functions of the time, that describe the motion of the table and R is a constant. They show that it can be reduced to the form (67) only if the case of a constant speed rotation of the table. Nevertheless, one sees that one may choose θ as a flat output. The system becomes then linear in the remaining variables and is flat is $\theta' \neq 0$ and $\theta'' \neq 0$.

When the table does not move, one recovers the example $x' = \cos(\phi)\theta'$, $y' = \sin(\phi)\theta'$, which may be traced back to Monge [49] under the form $\mathrm{d}\theta^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ and which has been rediscovered independently by Petitot [60].

We see that it seems difficult to find applications where the changes of coordinates described by EDP remain tractable. Working with such functions, when they cannot be expressed by closed form formulas remains a challenge.

All these examples enter in more general notions of block diagonal or chained systems.

4.4.4 Block diagonal and chained systems

"Almost chained systems", a described in [15, 52]

$$(Z_h', X_h') = G_h(Z_1, \dots Z_{h+1}, X_1, \dots, X_{h+1}) + H_h(X_{h+2}, \dots, X_{h+\ell_h}), 1 \leq h \leq r, \tag{18}$$

are chained when the extra functions H_h are 0. We propose here some more precise definition.

Definition 68. *An* order 1 block triangular systems *is a system Σ in the variables Ξ such that there exist partitions $\Sigma = \bigcup_{h=1}^{p} \Sigma_h$ and $\Xi = \bigcup_{h=0}^{p} \Xi_h$ such that all equations in Σ_i depend only in variables in $\bigcup_{k=0}^{i} \Xi_k$ and are of order 1 in variables of Ξ_{i-1} and 0 at most in the other variables, with*

i) $\sharp\Xi_{h-1} = \sharp\Sigma_h$;

ii) $\mathcal{O}_{X_{i_{h-1}},\Sigma_h} = \sharp\Xi_{h-1}$;

iii) $\mathcal{O}_{X_{i_h},\Sigma_h} = 0$.

It is said to be dense *if moreover we have*

iv) $a_{i,j} = 0$ in A_Σ for all $1 \leq h \leq p$ and all $(i,j) \in \Sigma_h \times \Xi_h$.

An order 1 block triangular system is said to be chained *at level $h > 0$ if all equations in Σ_h depend only in variables in $\Xi_h \cap \Xi_{h-1}$. It is said to be* stricly chained *if is chained at level h and the equation in Σ_h depend only in derivatives of order 1 of the variables in Ξ_{h-1} and not of those variables themselves.*

Remark 69. *Condition iii) means that there is an injection $\sigma : \Sigma_h \mapsto \Xi_h$ with $\sum_{P\in\Sigma_h} \mathrm{ord}_{\sigma(P)} P = 0$. Then, considered as a system in the variables of $\bigcup_{h=1}^{p} \sigma(\Sigma_h)$, or in the variables of $\bigcup_{h=1}^{p} \Sigma_{h-1}$, the system Σ is block triangular according to the definition of [52, 5.2].*

We have the following proposition, of which the easy proof is left to the reader.

Proposition 70. *Order 1 block triangular systems are $\bar{o}$-systems such that $C_0 = \emptyset$ at each step of algo. 47.*

With the notations of this algorithm and of the previous definition, the set Ξ_{p-h+1} (resp. Σ_{p-h+1}), for $1 \leq h \leq p$ corresponds to the columns (resp. the rows) of B at step h of the algorithm.

The next subsection will provide some sufficient conditions of regularity and singularity in the block diagonal case.

4.5 Some special results for block triangular systems

4.5.1 A sufficient condition for $\bar{o}$-regularity

We have seen with th. 50 a sufficient condition for regularity. Example 51 shows that it is not a necessary solution for $\bar{o}$-regularity for all $\bar{o}$-systems. The following corollary shows that the condition of th. 50 is indeed an necessary and sufficient condition of $\bar{o}$-regularity in the case of block diagonal systems.

Corollary 71. *With the hypotheses of th. 50, assume that*

a) Ξ is a dense order 1 block triangular system or that

b) for all $1 \leq h \leq p$, the Jacobian matrix $(\partial P/\partial x \mid (P,x) \in \Sigma_h \times \Xi_h)$ has rank at least $\sharp\Sigma_h - 1$.

Then, it is $\bar{o}$-regular at point η iff for all $1 \leq h \leq p$, we have eq. 11, i.e.

$$\text{rank } \left(\frac{\partial P}{\partial x}(\eta) \mid P \in \Sigma_h,\ x \in \Xi_h \right) = \sharp\Sigma_h. \tag{19}$$

Proof. The sufficient part is th. 50.

The necessary part is a consequence of the correctness of algo. 60 th. 61 i). If a), then if the rank of (19) is not $\sharp\Sigma_h$ at level h, then at stage $p-h$ of the algorithm, all columns of B are suppressed, so that the system is not regular. The same applies using hypothesis b), as then relations between the rows of the Jacobian matrix must imply all $\sharp\Sigma_h$ rows, so that again we need to suppress all columns of B. □

4.5.2 A sufficient condition for singularity

We will need some technical folkloric lemma about linear systems.

Lemma 72. *If a block diagonal linear system Σ, chained at level i_0 is flat, then, using the notations of def. 68, for all $1 \leq i_0 < i_1 \leq p$, the block diagonal system $\Sigma_2 := \bigcup_{i=i_0}^{i_1} \Sigma_i$ in the variables $\Xi_2 \bigcup_{i=i_0-1}^{i_1} \Xi_i$ is flat.*

Proof. By the structure theorem, the linear system Σ is flat iff the module

$$\left[\sum_{x \in \Xi} \mathbb{R}(t)[\mathrm{d}/\mathrm{d}t]x \right] / \mathbb{R}(t)[\mathrm{d}/\mathrm{d}t]\Sigma$$

contains no torsion element, which implies that the module $\left[\sum_{x\in\Xi_2} \mathbb{R}(t)[\mathrm{d}/\mathrm{d}t]x\right] / \mathbb{R}(t)[\mathrm{d}/\mathrm{d}t]\Sigma_2$ contains no torsion element. □

We can now state the following sufficient condition of singularity.

Theorem 73. *Let $\Sigma = \bigcup_{i=1}^{p} \Sigma_h$ be a block diagonal system in the variables $\bigcup_{h=0}^{p} \Xi_h$, chained at level h_0 and strictly chained at level $h_0 - 1$, and such that all variables in $\Xi_{h_0-1} \cup \Xi_{h_0}$ are constants along a given trajectory. Let η denote a point of this trajectory.*

Assume moreover that the Jacobian matrix

$$\left(\frac{\partial P}{\partial x} \mid (P,x) \in \Sigma_{h_0} \times \Xi_{h_0} \right) \tag{20}$$

has rank $m_0 < n_0 := \sharp\Xi_{h_0-1}$, that we have $\sharp\Xi_{h_0-1} = n_0$ and that the Jacobian determinants

$$\left| \frac{\partial P}{\partial x} \mid (P,x) \in \Sigma_{h_0} \times \Xi_{h_0-1} \right| \tag{21}$$

do not vanish at η for all $1 \leq h \leq p$.

With these hypotheses, Σ is not flat at η.

Proof. The non vanishing of the determinants (21) implies the existence of explicit differential equations

$$x' = f_x(\Xi_0, \ldots, \Xi_{h-1}),\ x \in \Xi_h \text{ for } 0 \leq h < p. \tag{22}$$

We will consider the linearized system

$$\mathrm{d}x' = \mathrm{d}f_x(\mathrm{d}\Xi_0, \ldots, \mathrm{d}\Xi_{h-1}),\ x \in \Xi_h \text{ for } 0 \leq h < p. \tag{23}$$

As the system is chained at level h_0, for $x \in \Xi_{h_0-1}$, f_x and $\mathrm{d}f_x$ only depend on $\Xi_h \cup \Xi_{h_0-1}$ (and $\mathrm{d}\Xi_h \cup \Xi_{h_0-1}$). In the same way, as the system is strictly chained at level $h_0 - 1$, for $x \in \Xi_{h_0-2}$, f_x and $\mathrm{d}f_x$ only depend on Ξ_{h-1} (and $\mathrm{d}\Xi_h \cup \Xi_{h_0-1}$). Moreover, as the $x \in \Xi_{h_0-1} \cup \Xi_{h_0}$ are constants, the coefficients in the f_x, for $x \in \Xi_{h_0-1} \cup \Xi_{h_0}$, are constants too.

We will show that the linearized system, truncated at level h_0, admits torsion elements, so that the system Ξ is not flat by lem. 72. Such torsion elements are first integrals of the Lie algebra generated by ∂_x, for $x \in \Xi_h$ and

$$\tau := \partial_t + \sum_{x \in \bigcup_{h=0}^{h_0-1} \Xi_h} f_x \partial_x,$$

where ∂_x denotes $\partial/\partial x$. See [32, 4.3] for more details on such constructions.

As the rank of the Jacobian matrix (20) is $n_0 < m$,

$$\text{Lie}(\tau; \partial_x \mid x \in \Xi_{h_0}) = \langle \partial_x \mid x \in \Xi_{h_0} \rangle + \text{Lie}(\tau; [\tau, \partial_x] \mid x \in W \subset \Xi_{h_0}),$$

for some subset of variables W, such that $\sharp W = n_0$. We denote $[\tau, \partial x]$ by $\hat{\tau}\partial_x$ and $\hat{\tau}^{k+1}\partial x := [\tau, \hat{\tau}^k \partial x]$. One may find integers e_x, $x \in \Xi_h$ such that the $\pi_{h_0-1}\hat{\tau}^k\partial_x$, for $x \in \Xi_h$ and $1 \leq k \leq e_x$ are linearly independent, where π_{h_0-1} denotes the projection on the vector space $\langle \partial_x \mid x \in \Xi_{h_0-1} \rangle$. Those integers may be chosen so that $\sum_{x \in \Xi_h} e_x$ is maximal.

Then, we have

$$\begin{aligned} \text{Lie}(\tau; \partial_x \mid x \in \Xi_{h_0}) = \quad & \langle \partial_x \mid x \in \Xi_{h_0} \rangle + \langle \hat{\tau}^k \delta_x \mid x \in W,\ 1 \leq k \leq e_x \rangle \\ & + \text{Lie}(\tau; \hat{\tau}^{e_x+1}\delta_x - \pi_{h_0-1}\hat{\tau}^{e_x+1}\delta_x \mid x \in W). \end{aligned}$$

The brackets $[\tau, \hat{\tau}^{e_x+1}\delta_x - \pi_{h_0-1}\hat{\tau}^{e_x+1}\delta_x]$ do not involve derivations ∂_x, for $x \in \Xi_{h_0-1} \cup \Xi_{h_0-2}$, as the system is strictly chained at level $h_0 - 1$. So the intersection of $\text{Lie}(\tau, \partial_x \mid x \in \Xi_{h_0})$ with $\langle \partial_x \mid x \in \Xi_{h_0-2} \rangle$ has dimension at most $\sharp W = n_0 < \sharp \Xi_{h_0-2}$, so that the dimension of $\text{Lie}(\tau, \partial_x \mid x \in \Xi_{h_0})$ is not the maximal dimension $\sum_{h=0}^{h_0} \sharp \Xi_h$ and non trivial torsion elements exist, which concludes the proof. □

5 The simplified aircraft. A block diagonal system

5.1 The aircraft model

5.1.1 Nomenclature

Roman

a: wing span

C_x, C_y, C_z: aerodynamic force coefficients, wind frame

C_D, C_L: lift and drag coefficient in aircraft frame

C_l, C_m, C_n: aerodynamic moment coefficients

c: mean aerodynamic chord

F: thrust

L, M, N: aerodynamic moments

m: mass

p, q, r: roll, pitch and yaw rates

S: wing area

V: airspeed

X, Y, Z: aerodynamic forces

y_p: distance of the engines to the plane of symmetry

Greek

α angle of attack

β sideslip angle

γ: flight path angle

η: differential thrust ratio

ϑ: pitch angle

θ: parameters

μ: bank angle

ϕ: roll angle

χ: aerodynamic azimuth or heading angle

ψ: yaw angle

δ_l, δ_m, δ_n: aileron, elevator, rudder deflexion

θ: model parameters

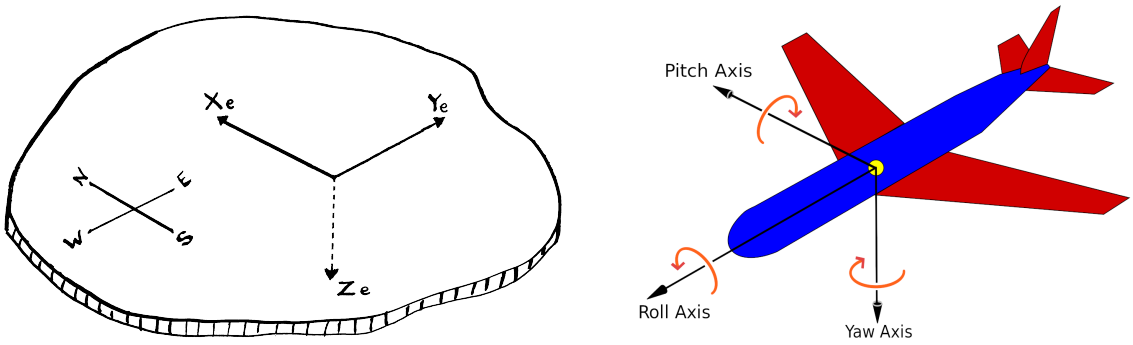
The model presented here relies on Martin [46, 47]. On may also refer to Asselin [2], Gudmundsson [24] or McLean [48] for more details.

5.1.2 Earth frame, wind frame and body frame

We use an earth frame with origin at ground level, with z-axis pointing downward, as in the figure 1 (a). The coordinates of the gravity center of the aircraft are given in this referential.

The body frame or aircraft referential is defined as in figure 1 (b), where x_b corresponds to the roll axis, y_b to the pitch axes and z_b to the yaw axes, oriented downward. The angular velocity vector (p, q, r) is given in this referential, or to be more precise, at each time, in the Galilean referential that is tangent to this referential.

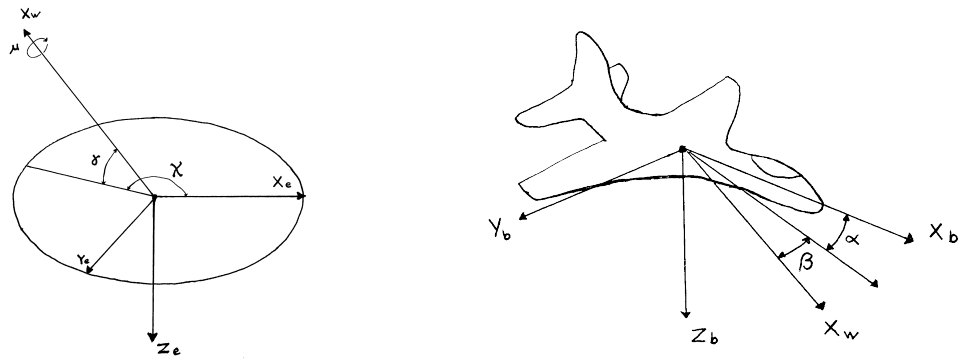
The wind frame, with origin the center of gravity of the aircraft has an axis x_w, in the direction of the velocity of the aircraft, the axis z_w being in the plane of symmetry of the aircraft. The Euler angles that define the orientation of the wind frame in the earth frame are denoted $\chi(t), \gamma(t), \mu(t)$, and are respectively the *aerodynamic azimuth* or *heading*



a) b)

Figure 1: a) Earth frame and b) body frame.

angle, the *flight path angle* and the *aerodynamic bank angle*, positive if the port side of the aircraft is higher than the starboard side. See figure 2 (a). We go from earth referential to wind referential using first a rotation with respect to z axis by the *heading angle* χ, then a rotation with respect to y axis by the *flight path angle* γ, and last a rotation with respect to x axis by the *bank angle* μ.



a) b)

Figure 2: a) Wind frame and b) From wind to body frame.

The orientation of the wind frame with respect to the body frame is defined by two angles: the *angle of attack* $\alpha(t)$ and the *sideslip angle* $\beta(t)$, which is positive when the wind is on the starboard side of the aircraft, as in figure 2 (b). We go from the wind referential to the body referential using first a rotation with respect to z axis by the side slip angle β and then a rotation with respect to y axis by the angle of attack α.

5.1.3 Dynamics

In the sequel, we shall write $p(t), q(t), r(t)$ the coordinates of the rotation vector of the body frame with respect to the the earth frame expressed in the Galilean referential that coincide at time t with the body frame, and $L(t)$, $M(t)$, $N(t)$ the corresponding torques. In the same way $X(t)$, $Y(t)$ and $Z(t)$ denote the forces applied on the aircraft, expressed in the Galilean referential that coincide at time t with the the wind referential.

5.1.4 Aircraft geometry

The mass of the aircraft is denoted by m, S is the surface of the wings. In the body frame, we assume that the aircraft is symmetrical with respect to the xz-plane, so that the tensor of inertia has the following form:

$$J := \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix}. \tag{24}$$

In the standard equations (28), we also need a, that stands for the *wing span* and b for the *mean aerodynamic chord*.

5.1.5 Forces and torques

The force (X, Y, Z) in the wind frame and the torque (L, M, N) are expressed by the next formulas:

$$X = F(t)\cos(\alpha+\epsilon)\cos(\beta(t)) - \frac{\rho}{2}SV(t)^2C_x - gm\sin(\gamma(t)); \tag{25a}$$

$$Y = F(t)\cos(\alpha+\epsilon)\sin(\beta(t)) + \frac{\rho}{2}SV(t)^2C_y + gm\cos(\gamma(t))\sin(\mu(t)); \tag{25b}$$

$$Z = -F\sin(\alpha+\epsilon) - \frac{\rho}{2}SV(t)^2C_z + gm\cos(\gamma(t))\cos(\mu(t)); \tag{25c}$$

$$L = -y_p\sin(\epsilon)(F_1(t) - F_2(t)) + \frac{\rho}{2}SV(t)^2aC_l; \tag{25d}$$

$$M = \frac{\rho}{2}SV(t)^2bC_m; \tag{25e}$$

$$N = y_p\cos(\epsilon)(F_1(t) - F_2(t)) + \frac{\rho}{2}SV(t)^2aC_n. \tag{25f}$$

The angle ϵ is related to the lack of parallelism of the reactors with respect to the $xy-$plane in the body frame and is small.

The aerodynamic coefficients $C_x, C_y, C_z, C_l, C_m, C_n$ depend on α and β and also on the angular speeds p, q, r and the controls are virtual angles δ_l, δ_m and δ_n, that respectively express the positions of the ailerons, elevator and rudder.

Remark 74. *One may use $\eta = (F_1 - F_2)/(F_1 + F_2)$ as an alternative control instead of δ_n in case of rudder jam.*

5.1.6 Equations

Following Martin [46, 47], the dynamics of the system is modeled by the following set of explicit differential equations (26a–26i,27):

$$\frac{d}{dt}x(t) = V(t)\cos(\chi(t))\cos(\gamma(t)); \tag{26a}$$

$$\frac{d}{dt}y(t) = V(t)\sin(\chi(t))\cos(\gamma(t)); \tag{26b}$$

$$\frac{d}{dt}z(t) = -V(t)\sin(\gamma(t)); \tag{26c}$$

$$\frac{d}{dt}V(t) = \frac{X}{m}; \tag{26d}$$

$$\frac{d}{dt}\gamma(t) = -\frac{Y\sin(\mu(t)) + Z\cos(\mu(t))}{mV(t)}; \tag{26e}$$

$$\frac{d}{dt}\chi(t) = \frac{Y\cos(\mu(t)) - Z\sin(\mu(t))}{\cos(\gamma(t))mV(t)}; \tag{26f}$$

$$\frac{d}{dt}\alpha(t) = \frac{1}{\cos(\beta(t))}(-p\cos(\alpha(t))sin(\beta(t)) + q\cos(\beta(t)) - r\sin(\alpha(t))\sin(\beta(t)) + \frac{Z}{mV(t)}); \tag{26g}$$

$$\frac{d}{dt}\beta(t) = +p\sin(\alpha(t)) - r\cos(\alpha(t)) + \frac{Y}{mV(t)}; \tag{26h}$$

$$\begin{aligned}\frac{d}{dt}\mu(t) = \ & \tfrac{1}{\cos(\beta(t))}(p\cos(\alpha(t)) + r\sin(\alpha(t)) \\ & + \tfrac{1}{mV(t)}(Y\cos(\mu(t))\tan(\gamma(t))\cos(\beta(t)) - Z(\sin(\mu(t))\tan(\gamma(t))\cos(\beta(t)) + sin(\beta(t))));\end{aligned} \tag{26i}$$

$-4^\circ \leq \alpha \leq 30^\circ$	$-20^\circ \leq \beta \leq 20^\circ$,	
$-100^\circ/\text{s} \leq p \leq 100^\circ/\text{s}$	$-50^\circ/\text{s} \leq q \leq 50^\circ/\text{s}$	$-50^\circ/\text{s} \leq r \leq 50^\circ/\text{s}$
$-10^\circ \leq \delta_l \leq 10^\circ$	$-20^\circ \leq \delta_m \leq 10^\circ$	$-30^\circ \leq \delta_n \leq 30^\circ$

Table 1: Range of values for the GNA model

$$\begin{pmatrix} \frac{d}{dt}p(t) \\ \frac{d}{dt}q(t) \\ \frac{d}{dt}r(t) \end{pmatrix} = J^{-1} \begin{pmatrix} (I_{yy} - I_{zz})qr + I_{xz}pq + L \\ (I_{zz} - I_{xx})pr + I_{xz}(r^2 - p^2) + M \\ (I_{xx} - I_{yy})pq - I_{xz}rq + N \end{pmatrix}. \tag{27}$$

In the last expressions, the terms depending on gravity have been incorporated to the expressions X, Y and Z, as in [47].

We notice with Martin that this set of equations imply $\cos(\beta)\cos(\gamma)V \neq 0$. The non vanishing of V and $\cos(\beta)$ seems granted in most situations; the vanishing of V may occur with aircrafts equipped with vectorial thrust, which means a larger set of controls, that we won't consider here. The vanishing of $\cos(\gamma)$ can occur with loopings etc. and would require the choice of a second chart with other sets of Euler angles.

5.2 The GNA model

In the last equations, ρ can depend on z, as the air density vary with altitude. The expression of C_x and C_z could also depend on z to take in account ground effect. These expressions that depend on α, β, p, q, r, δ_l, δ_m and δ_n, should also depend on the Mach number, but most available formulas are given for a limited speed range and the dependency on V is limited to the V^2 term in factor. In the literature, the available expressions are often partial or limited to linear approximations. McLean [48] provides such data for various types of aircrafts; for different speed and flight conditions, including landing conditions with gears and flaps extended.

We have chosen here to use the Generic Nonlinear Aerodynamic (GNA) subsonic models, given by Grauer and Morelli [22] that cover a wider range of values, given in the following table. Among the 8 aircrafts in their database, we have made simulations with 4: fighters F-4 and F-16C, STOL utility aircraft DHC-6 Twin Otter and the sub-scale model of a transport aircraft GTM (see [28]).

The GNA model depends on 45 coefficients:

$$\begin{array}{rcl} C_D &=& \theta_1 + \theta_2\alpha + \theta_3\alpha\tilde{q} + \theta_4\alpha\delta_m + \theta_5\alpha^2 + \theta_6\alpha^2\tilde{q} + \theta_7\delta_m + \theta_8\alpha^3 + \theta_9\alpha^3\tilde{q} + \theta_{10}\alpha^4, \\ C_y &=& \theta_{11}\beta + \theta_{12}\tilde{p} + \theta_{13}\tilde{r} + \theta_{14}\delta_l + \theta_{15}\delta_n, \\ C_L &=& \theta_{16} + \theta_{17}\alpha + \theta_{18}\tilde{q} + \theta_{19}\delta_n + \theta_{20}\alpha\tilde{q} + \theta_{21}\alpha^2 + \theta_{22}\alpha^3 + \theta_{23}\alpha^4, \\ C_l &=& \theta_{24}\beta + \theta_{25}\tilde{p} + \theta_{26}\tilde{r} + \theta_{27}\delta_l + \theta_{28}\delta_n \\ C_m &=& \theta_{29} + \theta_{30}\alpha + \theta_{31}\tilde{q} + \theta_{32}\delta_e + \theta_{33}\alpha\tilde{q} + \theta_{34}\alpha^2\tilde{q} + \theta_{35}\alpha^2\delta_e + \theta_{36}\alpha^3\tilde{q} + \theta_{37}\alpha^3\delta_e + \theta_{38}\alpha^4, \\ C_n &=& \theta_{39}\beta + \theta_{40}\tilde{p} + \theta_{41}\tilde{r} + \theta_{42}\delta_l + \theta_{43}\delta_n + \theta_{44}\beta^2 + \theta_{45}\beta^3, \end{array} \tag{28}$$

where $\tilde{p} = ap$, $\tilde{r} = ar$, $\tilde{q} = bq$ (see 5.1.4 for the meaning of a and b), C_D and C_L correspond to the lift and drag coefficients in the aircraft frame. The coefficients C_x and C_z in the wind frame are then obtained by the formulas:

$$\begin{array}{rcl} C_x &=& \cos(\alpha)C_D + \sin(\alpha)C_L, \\ C_z &=& \cos(\alpha)C_L - \sin(\alpha)C_D. \end{array} \tag{29}$$

Definition 75. *The simplified model is obtained by replacing p, q, r, δ_ℓ, δ_m an δ_n by 0 (or some known constants) in the expressions of C_x, C_y and C_z.*

5.3 Block triangular structure of the simplified model

Using the simplified model, with $\Xi_0 := (x, y, z)$, $\Xi_1 := (V, \gamma, \chi)$, $\Xi_2 := (\alpha, \beta, \mu, F)$, $\Xi_3 := (p, q, r)$ and $\Xi_4 := (\delta_\ell, \delta_m, \delta_n)$ or $\Xi_4 := (\delta_\ell, \delta_m, \eta)$ when differential thrust is used (see rem. 74), we see that, for $1 \leq k \leq 3$, $\dot{\Xi}_i$ only depends on $\bigcup_{\kappa=0}^{k+1} \Xi_\kappa$, so that the model is block diagonal with Σ_1 corresponding to (26a–26c), Σ_2 corresponding to (26d–26f), Σ_3 corresponding to (26g–26i) and Σ_4 to (27).

Simple computations show that

$$\left|\frac{\partial P}{\partial \xi}|P \in \Sigma_1;\ \xi \in \Xi_1\right| = -V^2\cos(\gamma); \quad \left|\frac{\partial P}{\partial \xi}|P \in \Sigma_3;\ \xi \in \Xi_3\right| = \frac{1}{\cos}(\beta), \tag{30}$$

That do not vanish. In the same way,

$$\left|\frac{\partial P}{\partial \xi}|P \in \Sigma_4;\ \xi \in \Xi_3\right| = \frac{\rho^3}{8} S^3 V^6 a^2 b |J|^{-1} \left|\frac{\partial C_i}{\partial \delta_j}|(i,j) \in \{\ell, m, n\}^2\right|, \tag{31}$$

does not vanish as the diagonal terms $\partial C_i/\partial \delta_i$, $i = \ell, m, n$, are much bigger than the others.

So, to apply th. 50, we only need to consider the rank of the Jacobian matrix

$$\left(\frac{\partial P}{\partial \xi}|P \in \Sigma_2;\ \xi \in \Xi_2\right),$$

which is equal to the rank of the Jacobian matrix

$$\Delta := \left(\frac{\partial Q}{\partial \xi}|Q \in \{X, \sin(\mu)Y + \cos(\mu)Z, -\cos(\mu)Y + \sin(\mu)Z\};\ \xi \in \Xi_2\right). \tag{32}$$

Proposition 76. *The simplified aircraft is flat when the matrix Δ has full rank.*

Let Δ_ξ be the matrix Δ where the column corresponding to $\xi \in \Xi_2$ has been suppressed. If $|\Delta_\xi| \neq 0$ at some point, then x, y, z, ξ is a regular flat output at that point.

Proof. This is a direct consequence of th. 50 ii). □

Example 77. *We can associate a diffiety to our aircraft model, which is defined by $\mathbb{R}^{12} \times \left(\mathbb{R}^{\mathbb{N}}\right)^4$, and a derivation $\eth$ defined by $\eth := \eth_0 + \eth_1$, where $\eth_1$ is the trivial derivation on $\left(\mathbb{R}^{\mathbb{N}}\right)^4$: $\eth_1 := \sum_{z\in\{F,\delta_l,\delta_m,\delta_n\}}^{m} \sum_{k\in\mathbb{N}} z_i^{(k+1)} \partial/\partial z_i^{(k+1)}$ and $\eth_0$ is defined on $\mathbb{R}^{12}$ by the differential equations (26a–26i,27):*

$$\delta_0 := V(t)\cos(\chi(t))\cos(\gamma(t))\frac{\partial}{\partial x} + V(t)\sin(\chi(t))\cos(\gamma(t))\frac{\partial}{\partial y} + \cdots$$

In practice, the diffiety is defined a smaller open set, because of the bounds that exist on the variables. The values of the controls are bounded and one wishes to restrict the values of attack angle α or side-slip angle β for safety reasons. The maximal values for the GNA model are given below in table 1. Other limitations must be included, such as the maximal value of the thrust. The speed V should also be greater than the stalling speed (see 5.5).

We will now consider more closely the possible choices of flat outputs.

5.4 Choices of flat outputs

5.4.1 The side-slip angle choice

Martin [46, 47] has used the set of flat outputs: x, y, z, β. We need to explicit under which condition such a flat output may be chosen, *i.e.* when the Jacobian determinant

$$\Delta_\beta = \begin{vmatrix} \frac{\partial X}{\partial \alpha} & \frac{\partial X}{\partial \mu} & \frac{\partial X}{\partial F} \\ \frac{\partial(\sin\mu\, Y + \cos\mu\, Z)}{\partial \alpha} & \frac{\partial(\sin\mu\, Y + \cos\mu\, Z)}{\partial \mu} & \frac{\partial(\sin\mu\, Y + \cos\mu\, Z)}{\partial F} \\ \frac{\partial(\cos\mu\, Y - \sin\mu\, Z)}{\partial \alpha} & \frac{\partial(\cos\mu\, Y - \sin\mu\, Z)}{\partial \mu} & \frac{\partial(\cos\mu\, Y - \sin\mu\, Z)}{\partial F} \end{vmatrix}$$

does not vanish, according to prop. 76. First, we remark, following Martin [46, p. 80] that when $Y = gm\cos(\gamma)\sin(\mu)$ and $Z = gm\sin(\gamma)\sin(\mu)$, *i.e.* when the lift is zero, $\sin\mu\, Y + \cos\mu\, Z = gm\cos(\gamma)$ and $\cos\mu\, Y - \sin\mu\, Z = 0$, so that the last row of D_1 is zero and the determinant vanish. This mean that 0-g flight trajectories are singular for this flat output. On the other hand, when β and α are close to 0, which is the case in straight and level flight, easy computations using eq. (25a–25c) allow Martin to conclude that

$$|\Delta_\beta| \approx -Z\left(\frac{\rho}{2} S V^2 \frac{\partial C_z}{\partial \alpha} + F\right) \gg 0.$$

To go further, one may use the expression of X eq (25a) and deduce from it

$$F = \frac{X + \frac{\rho}{2} S V^2 C_x + gm\sin(\gamma)}{\cos(\alpha + \epsilon)\cos(\beta)}, \tag{33}$$

assuming $\cos(\alpha+\epsilon)\cos(\beta)$. Substituting this expression in Y and Z, we define $\tilde{Y}$ and $\tilde{Z}$ and further define $\hat{Y} := \cos(\mu)\tilde{Y} - \sin(\mu)\tilde{Z}$ and $\hat{Z} := \sin(\mu)\tilde{Y} + \cos(\mu)\tilde{Z}$. Then, $|\Delta_\beta| \neq 0$ when

$$\begin{vmatrix} \frac{\partial \hat{Y}}{\partial \alpha} & \frac{\partial \hat{Y}}{\partial \mu} \\ \frac{\partial \hat{Z}}{\partial \alpha} & \frac{\partial \hat{Z}}{\partial \mu} \end{vmatrix} \neq 0. \tag{34}$$

The main interest of this choice is to be able to impose easily $\beta = 0$, which is almost always required.

5.4.2 The bank angle choice

As the angle μ is known, we may compute from Ξ_1' and Ξ_1'' the values X, Y and Z. So, singularities for this flat outputs are such that

$$|\Delta_\mu| = \begin{vmatrix} \frac{\partial X}{\partial \alpha} & \frac{\partial X}{\partial \beta} & \frac{\partial X}{\partial F} \\ \frac{\partial Y}{\partial \alpha} & \frac{\partial Y}{\partial \beta} & \frac{\partial Y}{\partial F} \\ \frac{\partial Z}{\partial \alpha} & \frac{\partial Z}{\partial \beta} & \frac{\partial Z}{\partial F} \end{vmatrix} \neq 0 \tag{35}$$

Using $\tilde{Y}$ and $\tilde{Z}$, as defined in subsec. 5.4.1, we see that this is equivalent to

$$\begin{vmatrix} \frac{\partial \hat{Y}}{\partial \alpha} & \frac{\partial \hat{Y}}{\partial \beta} \\ \frac{\partial \hat{Z}}{\partial \alpha} & \frac{\partial \hat{Z}}{\partial \beta} \end{vmatrix} \neq 0. \tag{36}$$

When β is 0, $\partial\tilde{Z}/\partial\beta$ is also 0, due to the aircraft symmetry. Using the GNA model (see 5.2), we have $\partial C_x/\beta = 0$ and $\partial C_z/\partial\beta = 0$, so that $\partial\tilde{Z}/\partial\beta = 0$. The value of the determinant (36) is then

$$-\frac{\partial \hat{Z}}{\partial \alpha}\frac{\partial \hat{Y}}{\partial \beta}. \tag{37}$$

For most aircrafts, $\partial C_y/\partial\beta$ is negative at $\beta = 0$, with values in the range $[-1., -0.5]$. Delta wing aircrafts seem to be a exception, with smaller absolute values (-0.014 for the X-31) or even negative ones ($+0.099$ for the F-16XL). It seems granted that for regular transport planes, $\partial C_y/\partial\beta$ is negative, so that the determinant vanishes only when $\partial\tilde{Z}/\partial\alpha$ is 0. We will see in 5.5 that this may be interpreted as stalling condition and that the vanishing of (37) on a trajectory with constant controls means that the points of this trajectory are flat singularities, so that no other flat outputs could work.

This choice is the best to impose $\mu = 0$ and is natural for decrabe maneuver, that is when landing with a lateral wind, which implies $\beta \neq 0$. We then need to maintain μ close to 0 to avoid the wings hitting the runway.

It is also a good choice when $Y = Z = 0$, a situation that may be encountered in aerobatics or when training for space condition with 0-g flights (see subsec 7.2). The choices β and μ are compared in [52, 7.1] with the simulation of a twin otter flying with one engine.

5.4.3 The thrust choice

The choice of thrust F has one main interest: to set $F = 0$ and consider the case of a aircraft having lost all its engines. The aircraft must land by gliding when all engines are lost. This is a rare situation, but many successful examples are known, including the famous US Airways Flight 1549 [6]. The singularities of this flat output are such that

$$|\Delta_F| = \begin{vmatrix} \frac{\partial X}{\partial \alpha} & \frac{\partial X}{\partial \mu} & \frac{\partial X}{\partial \beta} \\ \frac{\partial(\sin\mu\, Y + \cos\mu\, Z)}{\partial \alpha} & \frac{\partial(\sin\mu\, Y + \cos\mu\, Z)}{\partial \mu} & \frac{\partial(\sin\mu\, Y + \cos\mu\, Z)}{\partial \beta} \\ \frac{\partial(\cos\mu\, Y - \sin\mu\, Z)}{\partial \alpha} & \frac{\partial(\cos\mu\, Y - \sin\mu\, Z)}{\partial \mu} & \frac{\partial(\cos\mu\, Y - \sin\mu\, Z)}{\partial \beta} \end{vmatrix}$$

vanishes. When $F = 0$, by eq. (25a–25c), the vanishing of D_3 is equivalent to

$$\begin{vmatrix} \frac{\partial C_x}{\partial \alpha} & 0 & \frac{\partial C_x}{\partial \beta} \\ \frac{\partial(\cos\mu\, C_z)}{\partial \alpha} & \frac{\partial(\sin\mu\, C_y + \cos\mu\, C_z)}{\partial \mu} & \frac{\partial(\sin\mu\, C_y)}{\partial \beta} \\ \frac{\partial(-\sin\mu\, C_z)}{\partial \alpha} & \frac{\partial(\cos\mu\, C_y - \sin\mu\, C_z)}{\partial \mu} & \frac{\partial(\cos\mu\, C_y)}{\partial \beta} \end{vmatrix} = \frac{\partial C_x}{\partial \alpha}\left(C_y\frac{\partial C_y}{\partial \beta} - C_z\frac{\partial C_x}{\partial \beta}\right).$$

When β vanishes, C_y and $\partial C_z/\partial\beta$ also vanish, due to the aircraft symmetry with respect to the xz-plane. So, we need have $\beta \neq 0$ to use those flat outputs. Using the GNA model, C_x and C_z depend only on α and C_y depends linearly on β.

In the case of a gliding aircraft, situations with $\beta \neq 0$ could precisely be useful to achieve the forward slip maneuver. When the aircraft is too high, combining non zero β and μ precisely allows a fast descent while keeping a moderate speed. This is very useful when gliding, as there is no option for a go around when approaching the landing strip too high or too fast. This maneuver was performed with success by the "Gimli Glider" [42], Air Canada Flight 143, that ran out of fuel on July 23, 1983, which could land safely in Gimli former Air Force base [42]. A simulation of the forward slip may be found in [52, 7.2].

5.4.4 Other sets of flat outputs

Among the other possible choices for completing the set Ξ_1 in order to get flat outputs, the interest of α is mostly academic. Indeed,

$$\Delta_\alpha = \begin{vmatrix} \frac{\partial X}{\partial \beta} & \frac{\partial X}{\partial \mu} & \frac{\partial X}{\partial F} \\ \frac{\partial(\sin\mu\, Y + \cos\mu\, Z)}{\partial \beta} & \frac{\partial(\sin\mu\, Y + \cos\mu\, Z)}{\partial \mu} & \frac{\partial(\sin\mu\, Y + \cos\mu\, Z)}{\partial F} \\ \frac{\partial(\cos\mu\, Y - \sin\mu\, Z)}{\partial \beta} & \frac{\partial(\cos\mu\, Y - \sin\mu\, Z)}{\partial \mu} & \frac{\partial(\cos\mu\, Y - \sin\mu\, Z)}{\partial F} \end{vmatrix}$$

Then, $|\Delta_\alpha| = 0$ when

$$\begin{vmatrix} \frac{\partial \hat{Y}}{\partial \beta} & \frac{\partial \hat{Y}}{\partial \mu} \\ \frac{\partial \hat{Z}}{\partial \beta} & \frac{\partial \hat{Z}}{\partial \mu} \end{vmatrix} = 0. \tag{38}$$

Easy computations show that it is the case when $\mu = \beta = 0$, so that α is not a suitable alternative input near stalling conditions, except possibly when $\mu \neq 0$. But mostly, our aerodynamic model cannot fully reflect the real behavior near stalling so that it seems safe to exclude stalling a.o.a from the domain of definition.

One may also consider time varying expressions, e.g. linear combinations of β and μ, to smoothly go from one choice to another.

5.5 Stalling conditions

It is known that the lift of a wing reaches a maximum at a critical angle of attack, due to flow separation. This phenomenon can be hardly reversible and create a sudden drop of the lift force Z from its pick value. Our mathematical model is too poor to fully reflect such behavior, but a maximum for the lift can still be computed.

We need to take also in account the contribution of the thrust in the expression of Z, and simple computations show that the critical angle of attack corresponds in our setting to a maximum of $\tilde{Z}$, that is $\partial\tilde{Z}/\partial\alpha = 0$, which corresponds to the singularity for flat output μ already observed above (37).

Three cases may appear with stalling:

1) to reach an extremal value of $\tilde{Z}$, meaning that $\partial\tilde{Z}/\partial\alpha$ vanishes;

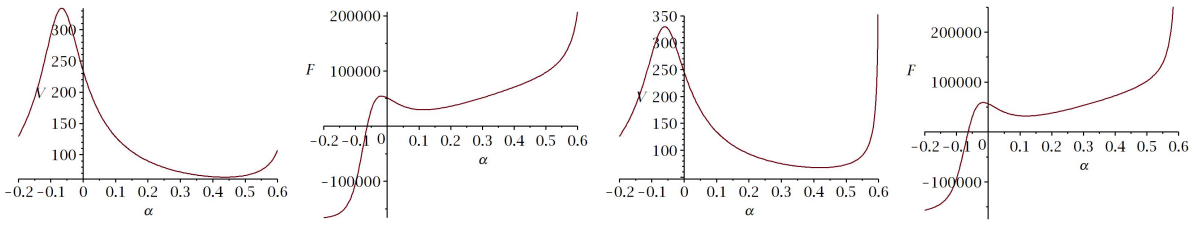
2) to reach the maximum thrust $F_{\max}$ before reaching a maximum of $\tilde{Z}$;

3) reaching no maximum of $\tilde{Z}$ with a aircraft with trust/weight ratio greater than 1: in such a case, there is no stalling.

For horizontal straight line trajectories, we may compute the speed V and the thrust F, depending on α, for $\beta = \mu = 0$, using the simplified model. We may also use the full model. As α, β and μ are constants, $p = q = r = 0$, which further allows to express δ_ℓ, δ_m and δ_n depending on α, so that $C_\ell = C_m = C_n = 0$, which is easy with the GNA model that is linear in those quantities.

E.g., For the F4, setting the weight to 38924lb [22], the evaluated stall speed, angle of attack and thrusts are respectively 67.6789m/s (131.56kn), 0.4200rad (24.07°) and 77.0436 for the full model and 64.0904m/s, 0.4366rad and 78.8806 for the simplified one, without thrust limitations. These thrust values are bellow the thrust of J79-GE-17A engines of later versions (79.38kN with afterburning). Assuming a maximal thrust of 2×71.8kN, that corresponds to the J79-GE-2 engines of the first production aircrafts, the stall speed and angle of attack are 67.9835m/s and 0.3969rad with the full model, 64.5515m/s and 0.4057rad with the simplified one. The full model stall speed of about 132kn agrees more or so with the stall speed values of 146KCAS or 148KCAS, according to the models, computed with the NATOPS manual [50, fig. 4.1 and 4.2] at 10000 ft and below and the computed a.o.a of 23° with the indicated stall a.o.a of 27 to 28 "units", keeping in mind that those units are not exactly degrees and that our mathematical definition of stalling cannot fully match the actual behavior. See fig. 3.

We have the following theorem, that shows that stalling condition means that the system is not flat for some trajectories with $\beta = \mu = 0$.



a) b) c) d)

Figure 3: F-4: values of V and F depending on α. a) & b) real model; c) & d) simplified model

Theorem 78. *Let a trajectory be such that α, β, μ, F, γ, χ and V are constants, with moreover $\beta = \mu = 0$, α and V respectively equal to the stall a.o.a. A point eta of this trajectory is not flat.*

Proof. Simple computations show that when $\beta = \mu = 0$ and $alpha$ is equal to stalling a.o.a, the rank of the Jacobian matrix $(\partial P/\partial x \mid P \in \Sigma_2,\ x \in \Xi_3)$. This is then a straightforward consequence of th. 73. □

6 Simulations using the simplified flat model

In this section, we show simulations done with the flat approximation of the model. These simulations are conducted with the classical set of flat outputs described in section 5.4.1, that is x, y, z, β.

As mentioned above, the flat approximation consists in neglecting the dependency of C_x, C_y, C_z on $p, q, r, \delta_l, \delta_m, \delta_n$. While this approximation, at first sight, may result in some noticeable divergence from the real aircraft, we show here that the model remains robust to various perturbations in the expression of the forces.

This tends to show that in many contexts the flat approximation is quite sufficient.

Moreover, we show that the flat model allows a high flexibility in trajectory planning and tracking.

All these suitable properties remain when one reactor is out of order.

6.1 Theoretical setting for feed-back design

The great advantage of flatness is that the flat motion planning makes an open loop control immediately available. When a closed loop is required, the feedback is designed from the difference between the actual values of the flat outputs and their reference values, so that this difference, being the solution of some differential equation, tends to zero.

In the framework of the flat aircraft model, the feedback is done is two stages. Indeed the dependency of the system variables on F, p, q, r has a slow dynamics in comparison to the rapidity of the dynamics that controls p, q, r from $\delta_l, \delta_m, \delta_n$. This allows to construct a cascade feedback, as done in [47]. More precisely, one can build a dynamic linearizing feedback that allows controlling the partial state vector $\Xi = (x, y, z, V, \alpha, \beta, \gamma, \chi, \mu, F)$ using the command $\dot{F}, p, q, r$, which allows following the reference trajectories of the flat outputs x, y, z, β, using static linearizing feedback. More precisely, one can compute a vector valued function Δ_0 and a matrix valued function Δ_1, both depending on $x, y, z, V, \alpha, \beta, \gamma, \chi, \mu, F$ such that:

$$\begin{pmatrix} x^{(3)} \\ y^{(3)} \\ z^{(3)} \\ \dot{\beta} \end{pmatrix} = \Delta_0 + \Delta_1 \begin{pmatrix} p \\ q \\ r \\ \dot{F} \end{pmatrix}$$

At this stage, the variables $p, q, r, \dot{F}$ are seen as commands. In order to make the system linear, one introduces a new vector valued command v, such that:

$$\begin{pmatrix} p \\ q \\ r \\ \dot{F} \end{pmatrix} = \Delta_1^{-1}(v - \Delta_0)$$

Eventually the command v is chosen of the form:

$$v(t) = \begin{pmatrix} P_0(x_{ref}(t) - x(t)) + P_1(\dot{x}_{ref}(t) - \dot{x}(t)) + P_2(\ddot{x}_{ref}(t) - \ddot{x}(t)) + P_3 x^{(3)}_{ref}(t) \\ P_0(y_{ref}(t) - y(t)) + P_1(\dot{y}_{ref}(t) - \dot{y}(t)) + P_2(\ddot{y}_{ref}(t) - \ddot{y}(t)) + P_3 y^{(3)}_{ref}(t) \\ P_0(z_{ref}(t) - z(t)) + P_1(\dot{z}_{ref}(t) - \dot{z}(t)) + P_2(\ddot{z}_{ref}(t) - \ddot{z}(t)) + P_3 z^{(3)}_{ref}(t) \\ -k_1(\beta_{ref}(t) - \beta(t)) + \dot{\beta}_{ref}(t), \end{pmatrix}$$

where $P(X) = P_0 + P_1 X + P_2 X^2 + P_X^3$ is actually the following polynomial $P(X) = (X - k_1)^3$. Therefore the error function $e_s(t) = s_{ref}(t) - s(t)$ satisfies the following differential equations $P(e_s(t)) = 0$. In our experiments, $k_1 = -5$, so that $e_s(t) \underset{t \to +\infty}{\longrightarrow} 0$, for each value of s in x, y, z, β.

In a second stage the variables p, q, r are controlled through a static linearizing feedback based on $\delta_l, \delta_m, \delta_n$. This part of the system, as mentioned above, is fast in comparison to the first part. More precisely, one can compute the vector valued function Λ_0 and a matrix valued function Λ_1, both depending on $V, \alpha, \beta, p, q, r$ such that:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \Lambda_0 + \Lambda_1 \begin{pmatrix} \delta_l \\ \delta_m \\ \delta_n \end{pmatrix}$$

Then as previously, one introduced a new command w such that

$$\begin{pmatrix} \delta_l \\ \delta_m \\ \delta_n \end{pmatrix} = \Lambda_1^{-1}(w - \Lambda_0)$$

and

$$w = \begin{pmatrix} -k_2(p_{ref}(t) - p(t)) + \dot{p}_{ref}(t) \\ -k_2(q_{ref}(t) - q(t)) + \dot{q}_{ref}(t) \\ -k_2(r_{ref}(t) - r(t)) + \dot{r}_{ref}(t) \end{pmatrix},$$

where $k_2 = -15$ in our experiments. Therefore $s(t) - s_{ref}(t) \underset{t \to +\infty}{\longrightarrow} 0$, for $s \in \{p, q, r\}$.

The rational behind this cascade feedback is the following. The variables $(x, y, z, V, \alpha, \beta, \gamma, \chi, \mu, F)$ are slowly controlled through $\dot{F}, p, q, r$. Once the required values of p, q, r are known, they are quickly reached through the control performed with $\delta_l, \delta_m, \delta_n$. The respective values of k_1 and k_2 reflect the disparity of speed between the two dynamics.

6.2 Conventions used in our simulations

We now show a series of experiments that illustrate the strength of the flat approximation to control the aircraft in various situations. Those experiments were all performed with GTM extracted from [28]. For the sake of simplicity, we have left the model is imperial units. The implementation is made in Python, relying on the symbolic library `sympy`, the numerical array library `numpy` and the numerical integration of ODE systems from the library `scipy`.

The experiments are all about following a reference trajectory defined by the following expressions:

$$\begin{cases} x_{ref}(t) &= V_1 \cos(\pi(t - Tinitial)/(Tfinal - Tinitial)) \\ y_{ref}(t) &= V_1 \sin(\pi(t - Tinitial)/(Tfinal - Tinitial)) \\ z_{ref}(t) &= -V_2 t - 1000 \\ \beta_{ref}(t) &= 0 \end{cases},$$

where $Tinitial = 0, Tfinal = 30, V_1 = 30, V_2 = 5$. This reference trajectory is an upward helix.

6.3 Initial perturbation

We carried out experiments where the aircraft started away from the reference trajectory and then joined it after a few seconds. If the initial perturbation is not too big, the feedback alone is capable to attract the aircraft to the reference trajectory. If the initial starting point is really far away from the reference trajectory, the flexibility of the flatness based approach allows designing very easily transition trajectory which can be followed with the feedback and that brings to aircraft to the reference upward helix trajectory.

The experiment is moreover performed when a one reactor is broken. We observe that the actual trajectory of the aircraft merges with the reference one, as shown in figure 4. The reference and the actual trajectories merge perfectly, even when starting from a point off the trajectory.

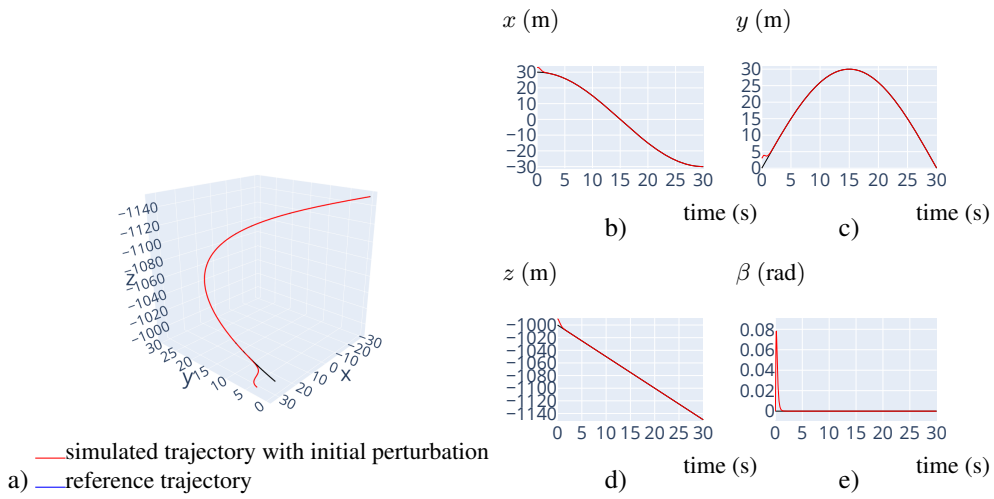


Figure 4: The values of the GTM trajectory, one engine: the aircraft converges toward the reference trajectory. a) Trajectory, 3D view; b)–e) Histories of x, y, z, β

6.4 Variable wind

In this section, we address the most critical problem about the flat approximation. Since the dependency of the aerodynamic coefficient on $p, q, r, \delta_l, \delta_m, \delta_n$ is discarded, one can wonder if the model is robust enough to significant perturbations in the values of the thrust. It turns out that under mild external forces, the model remains reliable.

In the last experiment, the motion of the aircraft is perturbed by a variable wind. This perturbation force is a sinusoidal function which amplitude is 50lbf and frequency is 0.1Hz. This setting is applied to the GTM with the piece-wise trajectory defined above. We observe a very robust behavior of the model, as rendered in figures 5 and 6. The reference and the actual trajectories still merge perfectly with a variable wind. For the variables F, α, p, q, r variations due to the variable wind are noticeable in the graphs.

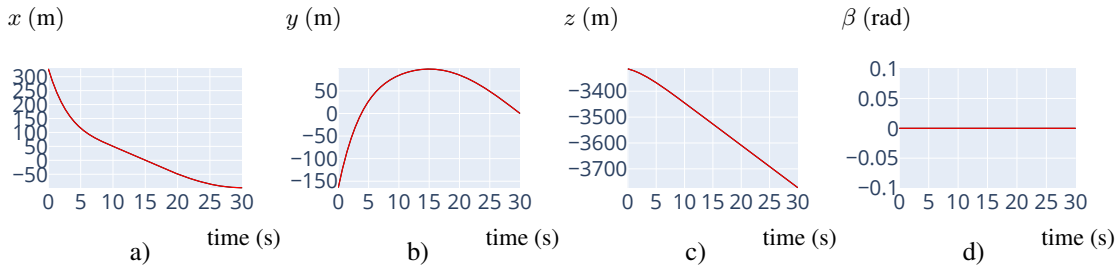


Figure 5: Variable wind. a)–d) Histories of x, y, z, β

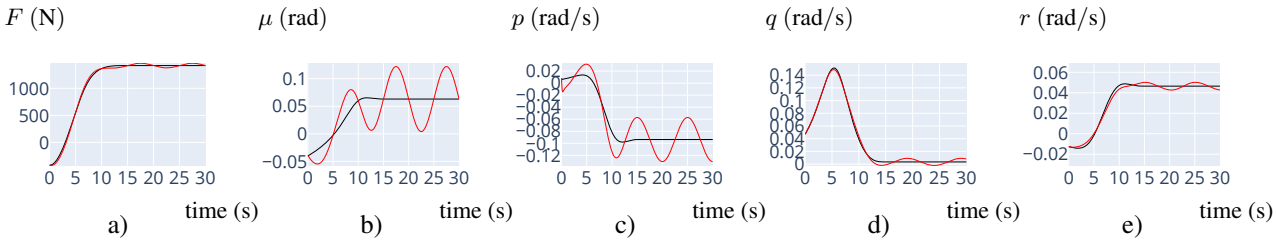


Figure 6: Variable wind. a)–e) Histories of F, μ, p, q, r

7 Simulations using the full model

7.1 Implementation in Maple

Our implementation computes power series approximations of all state variables and control at regular time interval. We proceed in the same way for the feed-back, which is also approximated with power series at the same time interval for better efficiency during numerical integration.

The structure of the equations implies the existence of a lazy flat parametrization, as shown by th. 42. Moreover, it means that the requested power series can be computed in a fast way using Newton's method, when initialized with suitable values. Most of the time, values for state variables such as β or α are close to 0. If not, calibration functions can give *e.g.* the value of α, assuming that V, γ', χ, β and μ are constants. See [52, § 4] for more details.

We denote by $\hat{\xi}(t)$ the planed function for any state variable ξ, according to the motion planning using the simplified flat system and the choice of $\hat{x}$, $\hat{y}$, $\hat{z}$ and $\hat{\beta}$ (or $\hat{\mu}$). We also denote by $\delta\xi$ the difference $\xi - \hat{\xi}$ between the planed trajectory and the trajectory computed with the full model. We did not manage to get a working feed-back without using integrals I_1, I_2, I_3, I_4 of $\cos(\chi)\delta x + \sin(\chi)\delta y$, $-\sin(\chi)\delta x + \cos(\chi)\delta y$, δz and $\delta\beta$ (or $\delta\mu$) respectively. When using the flat outputs x, y, z, F, I_4 is no longer needed.

For the feed-back we choose positive real numbers $\lambda_{i,j}$, with $1 \leq i \leq 4$, and $1 \leq j \leq 5$ for $i = 2$ or $i = 3$ and $1 \leq j \leq 3$ for $i = 1$ or $i = 4$. The value of δF, $\delta\delta_l$, $\delta\delta_m$, $\delta\delta_n$, are computed, so that $\prod_{i=1}^{3}(d/dt + \lambda_{1,i})I_1 = 0$, $\prod_{i=1}^{3}(d/dt + \lambda_{4,i})I_4 = 0$, $\prod_{i=1}^{5}(d/dt + \lambda_{2,i})I_2 = 0$, $\prod_{i=1}^{5}(d/dt + \lambda_{3,i})I_3 = 0$, using the derivation d/dt of the linearized simplified system around the planed trajectory. Then, we use the controls $\hat{\delta}_l + \delta\delta_l$, $\hat{\delta}_m + \delta\delta_m$, $\hat{\delta}_n + \delta\delta_n$, $\hat{F} + \delta F$ in the numerical integration. If the $\delta\xi$ are small enough to behave like the $\mathrm{d}\xi$ of the linearized system, and the solution of the full model not too far from the planed solution of the simplified model, the convergence is granted.

In practice, the choice of suitable $\lambda_{i,j}$ is difficult and empirical: two small, the trajectory is lost, two high, increasing oscillations may appear. We neglect here the dynamics of the actuator, our goal being to show that the feed-back is able to provide a solution for the full model, using the trajectory planed with the simplified one, the linearizing outputs remaining close to their original values.

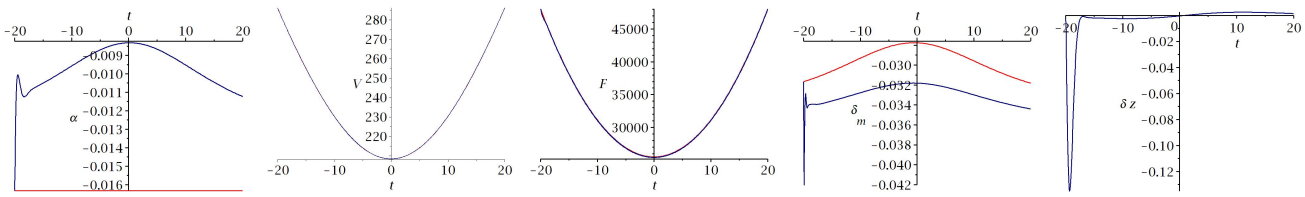
Unless otherwise stated, angles are expressed in radians, lengths in meters, times in seconds, thrusts in Newtons, masses in kg. Curves in red correspond to the planed trajectory $\hat{\xi}$, while curves in blue correspond to the integration of the full model. For more clarity z has been replaced by $-z$ to get positive values when drawing curves.

Computation times are given using Maple 19 with an Intel processor Core i5 2.5GHz. These are just indications that can vary from a session to another.

7.2 Using flat outputs with μ. Gravity-free flight with the F16

We experiment here a 0-g flight with a parabolic trajectory, using the F-16 model for which C_z vanishes for a value of α close to -0.016rad. See fig. 7. For this simulation, we used an expression of air density $\rho(z)$ varying with altitude z, following Martin [46, A.16 p. 97]. The total computation time of the simulation is 371s.

$$\begin{array}{lll} x & = 750t\mathrm{km/h}; \quad y = 0; & \quad z = g\frac{t^2}{2} - 2000; \quad \mu = 0; \\ \lambda_{i,j} & = 5.; & \\ \rho(z) & = 1.225(1 + 0.0065z/288.15)^{(9.80665/287.053\times 0.0065 - 1)} & \end{array} \tag{39}$$



a) b) c) d) e)

Figure 7: F-16 0-g flight. a) α (rad); b) V (m/s); c) F (N); d) δ_m (rad) and e) δz (m).

7.3 Using flat outputs with β. Rudder failure with the Twin Otter

Rudder failure is a classical situation where differential thrust is known to have been applied with success, which requires that the engines are not used at maximal power during take-off and/or themselves lost. Classical examples include Americal Airlines Flight 96 [5] or 2003 Baghdad DHL attempted shootdown incident [3]. One must notice that in those two cases, the elevators control was lost too, making the attitude control more difficult, an issue that is not addressed here. See also on this topic [7, 8, 25, 43].

We consider here differential thrust control of a T-O with rudder failure. We use flat outputs with β, and η is the control that replaces δ_n, set to 20°. This constant value is taken in account in the simplified model. We use a descending near circular trajectory with speed close to $80\text{kn} = 41,1556\text{m/s}$ and vertical speed close to $5.5\text{ft} = 1.6764\text{m/s}$.

$$\begin{array}{ll} x & = 1100\cos(\frac{41,1556}{1100}t); \quad y = 1200\sin(\frac{41,1556}{1100}t); \quad z = 1.6764t + 0,00762t^2 - 500; \quad \beta = 0; \\ \delta_n & = 20^\circ \\ \lambda_{i,j} & = 5.5 \end{array} \tag{40}$$

The value of δ_n is set to 20° and η is used as a new control in order to simulate differential thrust control of the aircraft. The constant value of δ_n is used in the expression of C_x, C_y and C_z for the simplified model, only δ_l and δ_n being set to 0. The strong action of the controls during this circular trajectory implies a rather high value for all the $\lambda_{i,j}$. Precision is good, less to 2.2cm for δz and much less after oscillations during the first 10s of simulation. Nevertheless, small oscillations never stop for q and some other state functions.

The T-O is the model for which designing a suitable feed-back is the most difficult. The maximal power for a single engine during the simulation is 537.4hp, bellow the maximal power 550hp (410 kW) of the Pratt & Whitney PT6A-20 engines of the T-O series 1, 100 and 200.

The total computation time is 3500s. See fig. 8. One may notice the variations of μ and the control δ_m, due to the thrusts neglected in the simplified model.

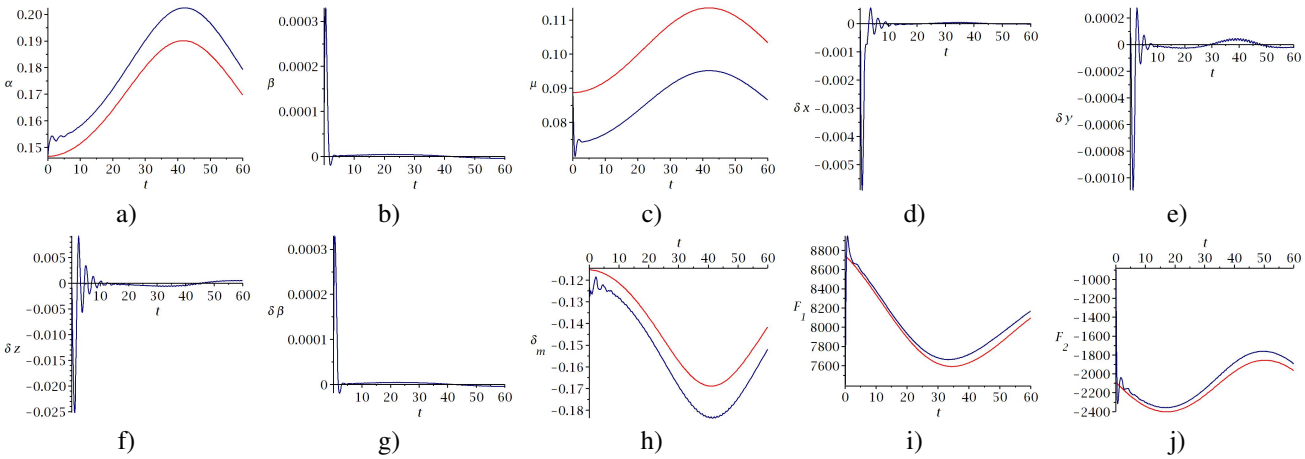


Figure 8: T-O rudder failure flight. a) α (rad); b) β (rad); c) μ (rad); d) δx (m); e) δy (m); f) δz (m); g) $\delta\beta$ (rad); h) δ_m (rad); i) F_1 (N) and j) F_2 (N).

8 Conclusion

We have introduced a notion of oudephippical systems, or ō-systems, that generalizes many special cases of chained or triangular systems previously know in the control literature. We further gave algorithms to test in polynomial time if a given system belongs to this category and to test regularity conditions that are sufficient to imply that the system is flat at a some point.

Then, as the flat outputs are state variables, the computation of the flat parametrization and the design of a feedback to stabilize the system around the planned trajectory are computationally easy, even for non trivial systems, such as the aircraft model used as an illustration. This model is flat if one neglects so terms and our simulations show that our feedback can compensate model errors due to this simplification.

The systematic study of all possible flat outputs in our setting made us discover new flat outputs for the aircraft, some of them of practical interest. They provide a set of charts with flat parametrization, that cover all common flight situations, flat singularities being close to stalling conditions.

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