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Abstract

Multifractional processes are extensions of Fractional Brownian Motion obtained by replacing its constant Hurst parameter by a deterministic or a random function $H(\cdot)$, called the Hurst function, which allows to prescribe their local sample paths roughness at each point. For that reason statistical estimation of $H(\cdot)$ is an important issue. Many articles have dealt with this issue in the case where $H(\cdot)$ is deterministic. However, statistical estimation of $H(\cdot)$ when it is random remains an open problem. The main goal of our present article is to propose, under a weak local Hölder condition on $H(\cdot)$, a solution for this problem in the framework of Moving Average Multifractional Process with Random Exponent (MAMPRE), denoted by X. From the data consisting in a discrete realization of X on the interval $[0, 1]$, we construct a continuous piecewise linear random function which almost surely converges to $H(\cdot)$ for the uniform norm, when the size of the discretization mesh goes to zero; also we provide an almost sure estimate of the uniform rate of convergence. It is worth noticing that such kind of strong

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consistency result in uniform norm is rather unusual in literature on statistical estimation of functions.

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1 Introduction and background

Throughout the article the underlying probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. It is endowed with a complete filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$, and $\{B(s)\}_{s\in\mathbb{R}}$ is a standard Brownian Motion with respect to this filtration. Fractional Brownian Motion (FBM) of constant Hurst parameter $H \in (0,1)$, denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$, is a very classical centred self-similar Gaussian process with stationary increments. It has continuous paths and it can be defined, for all $t \in \mathbb{R}$, through the non-anticipative moving average Wiener integral:

$$B_H(t) := \int_{-\infty}^{t} \left((t-s)^{H-1/2} - (-s)_+^{H-1/2} \right) dB(s)\,, \tag{1.1}$$

with the convention that, for each $(y,b) \in \mathbb{R}^2$, one has

$$y_+^b := \begin{cases} y^b, \text{if } y > 0, \\ 0, \text{else.} \end{cases} \tag{1.2}$$

One refers to the two well-known books [20, 31] for a detailed presentation of FBM and many other related topics. FBM is a widespread model in signal processing (see e.g. [19]). Unfortunately, in many situations, it does not fit very well to modeling of erratic real-life signals since it lacks of flexibility. An important drawback of FBM model comes from the fact that local roughness of its (sample) paths is not allowed to change from point to point. More precisely, For a generic stochastic process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ with continuous and nowhere differentiable paths, their roughness in a neighborhood of any arbitrary fixed point $\tau \in \mathbb{R}$ is usually measured through $\alpha_Y(\tau)$ and $\widetilde{\alpha}_Y(\tau)$, the pointwise Hölder exponent and local Hölder exponent of Y at τ, defined, for all $\omega \in \Omega$, as:

$$\alpha_Y(\tau,\omega) := \sup\left\{ a \in [0,1] : \limsup_{t\to\tau} \frac{|Y(t,\omega) - Y(\tau,\omega)|}{|t-\tau|^a} < +\infty \right\} \tag{1.3}$$

and

$$\widetilde{\alpha}_Y(\tau,\omega) := \sup\left\{\widetilde{a}\in[0,1]:\limsup_{(t',t'')\to(\tau,\tau)}\frac{|Y(t',\omega)-Y(t'',\omega)|}{|t'-t''|^{\widetilde{a}}}<+\infty\right\}. \tag{1.4}$$

For a given $\omega\in\Omega$, the more close to zero are $\alpha_Y(\tau,\omega)$ and $\widetilde{\alpha}_Y(\tau,\omega)$, the more rough is the path $t\mapsto Y(t,\omega)$ in the vicinity of τ. In the case of FBM $\{B_H(t)\}_{t\in\mathbb{R}}$, path roughness remains everywhere the same because one has (see for instance [32])

$$\mathbb{P}\big(\forall\,\tau\in\mathbb{R}\,,\,\alpha_{B_H}(\tau)=\widetilde{\alpha}_{B_H}(\tau)=H\big)=1\,. \tag{1.5}$$

In order to overcome the latter limitation of FBM, multifractional processes have started to be constructed and studied since the mid 1990s. One refers to the recent book [1] for a detailed presentation of such processes and their connections to wavelet methods. The paradigmatic example of them is the Classical Multifractional Brownian Motion (CMBM) which was introduced independently in the two pioneering articles [12, 29]. CMBM is a Gaussian process with non-stationary increments and continuous paths. It is obtained by replacing the constant Hurst parameter H in a stochastic integral representation of FBM (as for instance the moving average representation (1.1)) by a deterministic continuous function $t\mapsto H(t)$, with values in an arbitrary compact interval $[\underline{H},\overline{H}]\subset(0,1)$, which depends on the time variable t, that is the index of the process. The latter function is called the Hurst function. Under a local Hölder condition on it, the articles [12, 29] have shown that this deterministic function can be used for prescribing local path roughness of CMBM paths which is thus allowed to change from point to point in a deterministic way. Namely, for any point $\tau\in\mathbb{R}$ at which the local Hölder exponent $\widetilde{\alpha}_H(\tau)$ of the function $t\mapsto H(t)$ satisfies the inequality

$$H(\tau)<\widetilde{\alpha}_H(\tau)\,, \tag{1.6}$$

one has, almost surely,

$$\alpha_{\text{CMBM}}(\tau)=\widetilde{\alpha}_{\text{CMBM}}(\tau)=H(\tau)\,, \tag{1.7}$$

where $\alpha_{\text{CMBM}}(\tau)$ and $\widetilde{\alpha}_{\text{CMBM}}(\tau)$ are the pointwise Hölder exponent and the local Hölder exponent of the CMBM at τ. Even if the Gaussian CMBM is a more flexible model than FBM, it still has some limitations, a major one of them is that the two exponents $\alpha_{\text{CMBM}}(\tau)$ and $\widetilde{\alpha}_{\text{CMBM}}(\tau)$ are deterministic quantities since the Hurst function $t\mapsto H(t)$ itself is deterministic. The difficulty for overcoming the latter limitation of the CMBM

comes from the fact that one can not replace the Hurst parameter H in (1.1), or in another stochastic integral representation of FBM, by a random variable $H(t)$ without imposing to it to be (stochastically) independent of the Brownian Motion $\{B(s)\}_{s\in\mathbb{R}}$ generating the stochastic integral. Indeed, when this very restrictive independence condition is dropped, then the stochastic integral, in which H is substituted by $H(t)$, is no longer well-defined. Therefore, the two articles [8, 6] have proposed to use a random wavelet series representation of FBM, instead of a stochastic integral representation of it, in order to be allowed to make this substitution. The multifactional process with random Hurst function obtained in this way is called, in our present article, the Wavelet Multifractional Process with Random Exponent (WMPRE). It is a non-Gaussian process with non-stationary increments and continuous paths. Thanks to wavelet methods, the paper [8] has shown that, under the condition (1.6), the two fundamental equalities (1.7), relating $H(\cdot)$ to local path roughness, can be extended to the WMPRE. Moreover, the latter result has been significantly strengthened in the paper [6] in which it has been established that these two fundamental equalities are even valid on a universal event of probability 1 not depending on τ, and for a much more general class of multifractional processes.

It worth mentioning that several articles (see for instance [13, 14, 15]) have pointed out that multifractional processes with random Hurst function are good candidates for modeling of financial time series. Indeed, they allow to replicate main stylized facts (non-Gaussianity, volatility clustering and so on) of such time series. Moreover, analysis of evolution over time of their random pointwise and local Hölder exponents can provide explanations for trading mechanisms over financial markets. For instance, at a given time one or the other of these two exponents can be viewed as a weight that investors assign to past prices in taking their trading decisions.

As we have already mentioned, there are significant difficulties in construction and study of multifractional processes with random Hurst functions. Even if the WMPRE, constructed a long time ago in [8], is a first breakthrough in this area, it does not at all clear how this process can be represented via Itô integral and how Itô calculus can be applied in its framework. In the last few years, another type of non-Gaussian multifractional process with random Hurst function having a natural representation via Itô intergral was introduced in [3]. It has non-stationary increments and continuous paths. It is called the Moving Average Multifractional Process with Random Exponent

(MAMPRE) in our present article. In contrast with the WMPRE for which the random Hurst function depends on the time variable t, in the case of the MAMPRE this function depends on the integration variable s. Indeed, the MAMPRE, denoted by $\{X(t)\}_{t\in\mathbb{R}}$, is obtained by substituting to the constant Hurst parameter H in (1.1) a stochastic process $\{H(s)\}_{s\in\mathbb{R}}$ with continuous paths, indexed by the integration variable s, which is adapted to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ and satisfies

$$0 < \underline{H} \leq H(s) \leq \overline{H} < 1\,, \quad \text{for all } s \in \mathbb{R}, \tag{1.8}$$

for some deterministic constants $\underline{H}$ and $\overline{H}$ belonging to the open interval $(0,1)$. More formally, the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$ is defined, for all $t \in \mathbb{R}$, as the Itô integral:

$$X(t) := \int_{-\infty}^{t} \left((t-s)^{H(s)-1/2} - (-s)_+^{H(s)-1/2} \right) dB(s)\,. \tag{1.9}$$

Recently, in the article [26], under a very weak global regularity condition on paths of the process $\{H(s)\}_{s\in\mathbb{R}}$, for all $\tau \in \mathbb{R}$, the two fundamental equalities (1.7), relating $H(\cdot)$ to local path roughness, have been extended to MAMPRE. A short time later, the article [2] has shown that they are even valid on a universal event of probability 1 not depending on τ, as soon as paths of $\{H(s)\}_{s\in\mathbb{R}}$ are, on each compact interval, Hölder functions of any arbitrarily small deterministic order $\gamma > 0$. Notice that in contrast with the condition (1.6) there is no need to impose to γ to be greater than $H(\tau)$.

As we have already emphasized, local roughness of paths of a multifractional process is governed by its deterministic or random Hurst function $H(\cdot)$. For that reason statistical estimation of values of $H(\cdot)$ is an important issue both form a practical point of view and from a theoretical one. Many articles have dealt with this issue in the case where $H(\cdot)$ is deterministic (see e.g. [18, 25, 5, 4, 9, 27, 24, 16, 17, 23, 7, 10, 11]). However, statistical estimation of $H(\cdot)$ when it is random remains an open problem. A major difficulty in it is that few information is available on finite-dimensional distributions of multifractional process with random exponent. Another one is that the dependence structure of such a process is very complex. The main goal of our present article is to propose a solution for this problem in the framework of the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$, defined through (1.9), under a weak local Hölder condition on paths of the stochastic process $\{H(s)\}_{s\in\mathbb{R}}$.

Let us describe in a more precise way the main contribution of our present article. Similarly to the previous literature on statistical estimation of $H(\cdot)$, we assume that,

on the interval $[0,1]$, the discrete realization: $\big\{X(k/N) : k \in \{0,\ldots,N\}\big\}$ of the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$ is available for all integer N large enough; notice that our main result can be extended without great difficulty to the general case where the interval $[0,1]$ is replaced by any other compact interval with non-empty interior. Also, we suppose that, for some deterministic constants $\gamma \in (0,1)$ and $\rho \in (0,+\infty)$ paths of $\{H(s)\}_{s\in\mathbb{R}}$ satisfy, on the interval $[-1,1]$, the Hölder condition:

$$\big|H(s') - H(s'')\big| \leq \rho\,|s' - s''|^{\gamma}\,, \quad \text{for all } (s',s'') \in [-1,1]^2. \tag{1.10}$$

Then, we construct from generalized quadratic variations associated with $\big\{X(k/N) : k \in \{0,\ldots,N\}\big\}$ and $\big\{X(k/QN) : k \in \{0,\ldots,QN\}\big\}$, the integer $Q \geq 2$ being arbitrary and fixed, a continuous piecewise linear random function on $[0,1]$, denoted by $\widetilde{H}_N(\cdot)$, which provides a uniformly and strongly consistent estimator of the whole random Hurst function $H(\cdot)$ on $[0,1]$. More precisely we show that, when N goes to $+\infty$, the uniform norm $\sup_{s\in[0,1]}\big|H(s) - \widetilde{H}_N(s)\big|$ converges almost surely to zero at the rate $N^{-\beta}$, where the positive exponent β belongs to some known interval. It is worth noticing that such kind of strong consistency result in uniform norm is rather unusual in literature on statistical estimation of functions.

Remark 1.1 *It might seem restrictive to impose to the positive constant ρ, in the Hölder condition (1.10), to be deterministic. In fact, thanks to a localization procedure via stopping times (see for instance Section 4.4.1 in [21]) which is explained in the setting of MAMPRE in Section 3 of [26], our main result (Theorem 2.2) remains valid when ρ is an almost surely finite random variable. Moreover, a careful inspection of the proof of this same theorem shows that it also remains valid when the interval $[-1,1]$ in (1.10) is replaced by any other compact interval of the form $[-b,1]$, where b is a fixed arbitrarily small positive real number.*

The remaining of our article is organized as follows. In section 2, the way of construction via generalized quadratic variations of X of the estimator $\widetilde{H}_N(\cdot)$ is precisely explained, our main result is stated, and $\widetilde{H}_N(\cdot)$ is tested on simulated data. In section 3, basically it is shown that generalized quadratic variations of X can be simplified since some parts of them are negligible for our purpose. The goal of section 4 is to precisely determine their asymptotic behavior when N goes to $+\infty$. At last, section 5 is devoted to complete the proof of our main result.

2 Statement of the main result and simulations

In order to state our main result, first one needs to introduce several notations. From now till the end of our article, the integer $L \geq 2$ is arbitrary and fixed. The coefficients $a_0, a_1, \ldots, a_L$ are defined, for every $l \in \{0, \ldots, L\}$, as:

$$a_l := (-1)^{L-l} \binom{L}{l} := (-1)^{L-l} \frac{L!}{l!\,(L-l)!}\,. \tag{2.1}$$

Observe that one can derive from (2.1) that the finite sequence of real numbers $(a_l)_{0 \leq l \leq L}$ has exactly L vanishing first moments; that is, for all $q \in \{0, \ldots, L-1\}$, one has

$$\sum_{l=0}^{L} l^q a_l = 0 \quad \text{(with the convention } 0^0 = 1\text{), and} \quad \sum_{l=0}^{L} l^L a_l \neq 0. \tag{2.2}$$

For each integer N large enough, the estimator $\{\widetilde{H}_N(s)\}_{s \in [0,1]}$ for paths of the stochastic process $\{H(s)\}_{s \in [0,1]}$ is built from generalized quadratic variations of the MAMPRE X (see (1.9)) associated with its generalized increments $d_{N,k}$, $0 \leq k \leq N-L$, defined, for all $k \in \{0, \ldots, N-L\}$, as:

$$d_{N,k} = \sum_{l=0}^{L} a_l X\big((k+l)/N\big). \tag{2.3}$$

For any compact interval, with non-empty interior, $I \subseteq [0,1]$, the generalized quadratic variation of X on I is denoted by $V_N(I)$ and defined as the empirical mean:

$$V_N(I) := |\nu_N(I)|^{-1} \sum_{k \in \nu_N(I)} |d_{N,k}|^2\,, \tag{2.4}$$

where the finite set of indices

$$\nu_N(I) := \big\{k \in \{0, \ldots, N-L\} : k/N \in I\big\}, \tag{2.5}$$

and $|\nu_N(I)|$ is the cardinality of $\nu_N(I)$. Observe that $|\nu_N(I)|$ does not really depends on the position of I, but mainly on $\lambda(I)$, the Lebesgue measure of this interval. Indeed, it can easily be seen that one has

$$N\lambda(I) - L - 1 < |\nu_N(I)| \leq N\lambda(I) + 1; \tag{2.6}$$

thus, as soon as $N \geq 2(L+1)\lambda(I)^{-1}$, one gets that

$$N\lambda(I)/2 < |\nu_N(I)| \leq 7N\lambda(I)/6. \tag{2.7}$$

Before giving a formal definition of the estimator $\{\widetilde{H}_N(s)\}_{s\in[0,1]}$, let us explain, in a few sentences, its way of construction. Let $(\theta_N)_N$ be an arbitrary sequence of real numbers in the interval $(0, 1/2]$ which converges to zero at a convenient rate (see (2.10) and (2.11)), when N goes to $+\infty$. One splits the interval $[0, 1]$ into a finite sequence $(\mathcal{I}_{N,n})_{0\leq n<\lfloor\theta_N^{-1}\rfloor}$ of compact subintervals with the same length θ_N (except the last one $\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1}$ having a length lying between θ_N and $2\theta_N$), where $\lfloor\theta_N^{-1}\rfloor$ is the integer part of θ_N^{-1}. Then, for any fixed integer $Q \geq 2$, the estimator $\{\widetilde{H}_N(s)\}_{s\in[0,1]} = \{\widetilde{H}^Q_{N,\theta_N}(s))\}_{s\in[0,1]}$ is obtained as the linear interpolation between the $\lfloor\theta_N^{-1}\rfloor + 1$ random points having the coordinates

$$\big(0, \widehat{H}^Q_N\big(\mathcal{I}_{N,0}\big)\big), \ldots, \big((\lfloor\theta_N^{-1}\rfloor - 1)\theta_N, \widehat{H}^Q_N\big(\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1}\big)\big), \big(1, \widehat{H}^Q_N\big(\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1}\big)\big),$$

where, for all $n \in \{0, \ldots, \lfloor\theta_N^{-1}\rfloor - 1\}$,

$$\widehat{H}^Q_N\big(\mathcal{I}_{N,n}\big) := \min\left\{\max\left\{\log_{Q^2}\left(\frac{V_N(\mathcal{I}_{N,n})}{V_{QN}(\mathcal{I}_{N,n})}\right), 0\right\}, 1\right\}. \tag{2.8}$$

Notice that, for every $x \in (0, +\infty)$, $\log_{Q^2}(x) := \log(x)/\log(Q^2)$, with the convention that log is the Napierian logarithm. Also, notice that the ordinate of the last point is assumed to be the same as that of the previous one. This weak assumption comes from the fact that the set of the indices t of MAMPRE has been restricted to the interval $[0, 1]$; it does not significantly alter the results on the estimation of $H(\cdot)$ on this interval. Let us now define the estimator $\widetilde{H}_N(\cdot) = \widetilde{H}^Q_{N,\theta_N}(\cdot)$ in a formal and very precise way.

Definition 2.1 *Assume that the integer $L \geq 2$ is arbitrary and fixed, and that the integer N_0 is defined as*

$$N_0 := \min\left\{N \in \mathbb{N} : 0 < 9(L+1)\, N^{-1}(\log N)^2 \leq 1\right\}. \tag{2.9}$$

Observe that (2.9) implies that $N_0 > 9(L+1) \geq 27$. Let $(\theta_N)_{N\geq N_0}$ be an arbitrary sequence of real numbers belonging to the interval $(0, 1/2]$ and satisfying, for all integer $N \geq N_0$,

$$\theta_N \leq \kappa\, N^{-\mu} \tag{2.10}$$

and

$$\theta_N \geq \kappa' N^{-\mu'} + 4(L+1)\, N^{-1} (\log N)^2 \,, \tag{2.11}$$

where $\kappa > 0$, $\mu \in (0,1)$, $\kappa' > 0$ *and* $\mu' \in [\mu, 1)$ *are four constants not depending on* N. *For each* $n \in \big\{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\big\}$, *we denote by* $\mathcal{I}_{N,n}$ *the compact subinterval of* $[0,1]$, *defined as:*

$$\mathcal{I}_{N,n} := \big[n\theta_N, (n+1)\theta_N\big] \text{ when } n < \lfloor \theta_N^{-1} \rfloor - 1, \text{ and } \mathcal{I}_{N,\lfloor \theta_N^{-1} \rfloor - 1} := \big[(\lfloor \theta_N^{-1} \rfloor - 1)\theta_N, 1\big]. \tag{2.12}$$

Observe that it follows from (2.7), (2.12) and (2.11) that, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ *and* $n \in \{0, \ldots \lfloor \theta_N^{-1} \rfloor - 1\}$, *the cardinality* $|\nu_{QN}(\mathcal{I}_{N,n})|$ *of* $\nu_{QN}(\mathcal{I}_{N,n})$ *satisfies*

$$QN\theta_N/2 < |\nu_{QN}(\mathcal{I}_{N,n})| \leq 7QN\theta_N/3 \,, \tag{2.13}$$

which in particular implies that $\nu_{QN}(\mathcal{I}_{N,n})$ *is non-empty. At last, for all fixed integer* $Q \geq 2$ *and for every integer* $N \geq N_0$, *we denote by* $\{\widetilde{H}^Q_{N,\theta_N}(s)\}_{s\in[0,1]}$ *the stochastic process with continuous piecewise linear paths, defined as:*

$$\widetilde{H}^Q_{N,\theta_N}(s) := \widehat{H}^Q_N\big(\mathcal{I}_{N,\lfloor \theta_N^{-1} \rfloor - 1}\big), \text{ for all } s \in \mathcal{I}_{N,[\theta_N^{-1}]-1}, \tag{2.14}$$

and, for every $n \in \{0, \ldots, [\theta_N^{-1}] - 2\}$ *and* $s \in \mathcal{I}_{N,n}$, *as:*

$$\widetilde{H}^Q_{N,\theta_N}(s) := \big(1 - \theta_N^{-1}(s - n\theta_N)\big)\widehat{H}^Q_N\big(\mathcal{I}_{N,n}\big) + \theta_N^{-1}(s - n\theta_N)\widehat{H}^Q_N\big(\mathcal{I}_{N,n+1}\big) \,, \tag{2.15}$$

where $\widehat{H}^Q_N\big(\mathcal{I}_{N,n}\big)$ *is defined through (2.8) for all* $n \in \{0, \ldots, [\theta_N^{-1}] - 1\}$.

Let us now state the main result of our article.

Theorem 2.2 *Assume that the conditions (1.8), (1.10), (2.10) and (2.11) hold. Let* β *be an arbitrary real number satisfying*

$$0 < \beta < \min\Big\{\gamma\big((1-\delta)\wedge\mu\big),\, \delta(L - \overline{H}) + \underline{H} - \overline{H},\, 2^{-1}(1-\mu')\Big\} \,,$$

where $L \geq 2$ *is as in (2.2), and where* δ *is an arbitrary fixed real number such that*

$$\frac{\overline{H} - \underline{H}}{L - \overline{H}} < \delta < 1 \,.$$

Then, one has almost surely, for all $Q \in \mathbb{N}$,

$$\lim_{N\to+\infty} \Big\{ N^\beta \sup_{t\in[0,1]} \big|H(t) - \widetilde{H}^Q_{N,\theta_N}(t)\big| \Big\} = 0 \,. \tag{2.16}$$

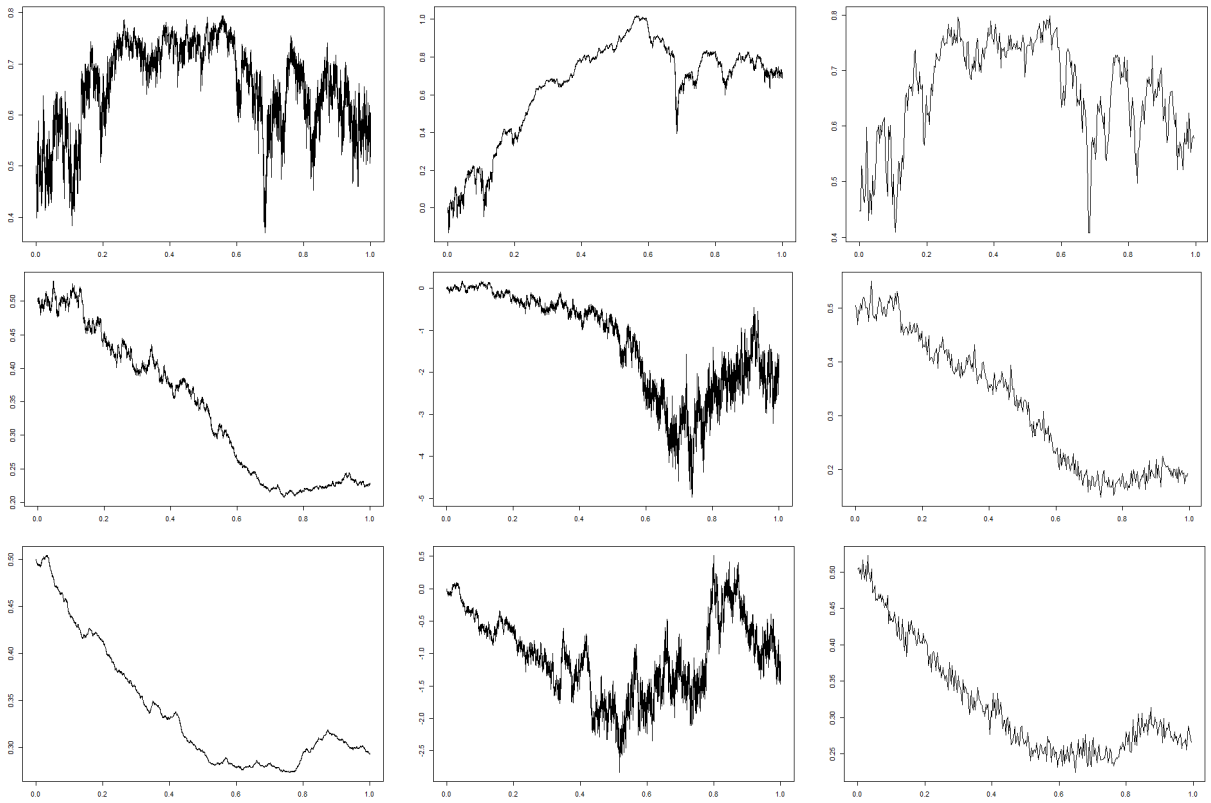
We mention in passing that $x \wedge y := \min\{x, y\}$, for all $(x, y) \in \mathbb{R}^2$. It is noteworthy that a major ingredient of the proof of Theorem 2.2 is the important Burkholder-Davis-Gundy inequality (see for instance [28, 30]) as formulated in the following proposition:

Proposition 2.3 *Let $p \in [1, +\infty[$ be arbitrary and fixed. There is a universal deterministic finite constant $a(p)$ for which the following result holds: for any $(\mathcal{F}_s)_{s\in\mathbb{R}}$-adapted stochastic process $f = \{f(s)\}_{s\in\mathbb{R}}$ satisfying almost surely $\int_{-\infty}^{+\infty} |f(s)|^2\, ds < +\infty$, one has*

$$\mathbb{E}\left(\left| \int_{-\infty}^{+\infty} f(s)\, dB(s) \right|^p \right) \leq a(p)\mathbb{E}\left(\left(\int_{-\infty}^{+\infty} |f(s)|^2\, ds \right)^{p/2} \right), \tag{2.17}$$

where $\int_{-\infty}^{+\infty} f(s)\, dB(s)$ denotes the Itô integral of f on $\mathbb{R}$.

The statistical estimator of random Hurst functions, introduced in Definition 2.1, has been tested in the following simulations:



The three random Hurst functions $H^1(s) := \psi(B_{0.3}(s))$, $H^2(s) := \psi(B_{0.55}(s))$ and $H^3(s) := \psi(B_{0.75}(s))$ have been successively simulated on the interval $[0, 1]$ in the first

column. The corresponding MAMPREs have been simulated on the same interval in the second column, by using a simulation method, relying on the Haar wavelet basis, which is rather similar to that introduced in [3]. The estimated versions of these three random Hurst functions, via the statistical estimator introduced in Definition 2.1 with $L = 2$, $Q = 2$, $\theta_N = N^{-0.6}$ and $N = 2^{14}$ have been simulated on the interval $[0, 1]$ in the third column. Notice that ψ is the deterministic function from $\mathbb{R}$ into the interval $[0.1, 0.9] \subset (0, 1)$, defined, for all $x \in \mathbb{R}$, as $\psi(x) := 0.8(\pi^{-1} \arctan(x)) + 0.5) + 0.1$. Also, notice that $\{B_{0.3}(s)\}_s$, $\{B_{0.5}(s)\}_s$ and $\{B_{0.75}(s)\}_s$ are FBMs (see (1.1)) which are adapted to the filtration $(\mathcal{F}_s)_s$ and whose Hurst parameters are respectively equal to 0.3, 0.55 and 0.75.

In view of the simulations the statistical estimator of random Hurst functions, introduced in Definition 2.1, seems to work fairly well. Indeed, the simulations show that it allows to reconstruct random Hurst functions in a rather precise way, even when they are very erratic, as for instance the random Hurst function $H^1(s)$.

Before ending the present section let us point out that:

Remark 2.4 *From now till the end of the article we always assume that the four conditions (1.8), (1.10), (2.10) and (2.11) hold, without mentioning it explicitly in the statements of the intermediate results, obtained in the remaining sections, which will allow us to prove Theorem 2.2.*

3 Negligible parts of generalized quadratic variations of X

In view of (2.8), (2.14) and (2.15), for proving Theorem 2.2 it is useful to study, for any fixed positive integer Q, asymptotic behavior of the generalized quadratic variations $V_{QN}(\mathcal{I}_{N,n})$, $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$, when N goes to $+\infty$. A first difficulty in this matter is that the domains of integration of the Itô integrals representing the generalized increments $d_{N,k}$ are unbounded intervals. Indeed, one can derive from (1.9), (2.3), (2.2), (1.2) and easy computations, that, for all $k \in \{0, \ldots, N - L\}$,

$$d_{N,k} = \int_{-\infty}^{N^{-1}(k+L)} N^{-(H(s)-1/2)} \, \Phi\big(Ns - k, H(s)\big) \, dB(s) \,, \tag{3.1}$$

where Φ is the real-valued deterministic function defined, for all $(u,v) \in \mathbb{R} \times (0,1)$, as:

$$\Phi(u,v) := \sum_{l=0}^{L} a_l (l-u)_+^{v-1/2}. \tag{3.2}$$

Roughly speaking, our first goal will be to show that $d_{N,k}$ can be expressed as the sum of an Itô integral over a well-chosen bounded interval and another term which is negligible in some sense. From now till the end of our article, we assume that $\delta \in (0,1)$ is arbitrary and fixed, and that, for every integer $N \geq N_0$, $e_N = e_N(\delta)$ is the positive integer defined as:

$$e_N := \lfloor N^\delta \rfloor. \tag{3.3}$$

Then we can derive from (3.1), that, for all $k \in \{0, \ldots, N-L\}$,

$$d_{N,k} = \widetilde{d}^{\,\delta}_{N,k} + \breve{d}^{\,\delta}_{N,k}\,, \tag{3.4}$$

where

$$\widetilde{d}^{\,\delta}_{N,k} := \int_{N^{-1}(k-e_N+L)}^{N^{-1}(k+L)} N^{-(H(s)-1/2)}\, \Phi\big(Ns-k, H(s)\big)\, dB(s) \tag{3.5}$$

and

$$\breve{d}^{\,\delta}_{N,k} := \int_{-\infty}^{N^{-1}(k-e_N+L)} N^{-(H(s)-1/2)}\, \Phi\big(Ns-k, H(s)\big)\, dB(s)\,. \tag{3.6}$$

Definition 3.1 *For any $\delta \in (0,1)$ and integers $Q \geq 1$, $N \geq N_0$ and $n \in \big\{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\big\}$, the generalized quadratic variations $\widetilde{V}^{\,\delta}_{QN}(\mathcal{I}_{N,n})$ and $\breve{V}^{\,\delta}_{QN}(\mathcal{I}_{N,n})$ are defined as:*

$$\widetilde{V}^{\,\delta}_{QN}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\widetilde{d}^{\,\delta}_{QN,k}|^2 \tag{3.7}$$

and

$$\breve{V}^{\,\delta}_{QN}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\breve{d}^{\,\delta}_{QN,k}|^2. \tag{3.8}$$

Basically, the following lemma shows that the generalized quadratic variations $\breve{V}^{\,\delta}_{QN}(\mathcal{I}_{N,n})$, $n \in \big\{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\big\}$, are negligible when N goes to $+\infty$. In other words, when N goes to $+\infty$, the asymptotic behavior of $V_{QN}(\mathcal{I}_{N,n})$ is similar to that of the "less complicated" generalized quadratic variations $\widetilde{V}^{\,\delta}_{QN}(\mathcal{I}_{N,n})$.

Lemma 3.2 *Let $\underline{H}$, $\overline{H}$ and L be as in (1.8) and (2.2). Let $\delta \in (0,1)$ be arbitrary and fixed. One has almost surely*

$$\limsup_{N\to+\infty}\left\{N^{\beta}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\right\}=0\,,\quad \text{for all } Q\in\mathbb{N} \text{ and } \beta<2\delta(L-\overline{H})+2\underline{H}. \tag{3.9}$$

For proving Lemma 3.2, one needs the following lemma whose proof will be given in the sequel.

Lemma 3.3 *Let c be the same constant as in (3.15). For all real number $p\in[1,+\infty)$ and integers $Q\ge 1$ and $N\ge(3L)^{1/\delta}+N_0$, the following inequality is satisfied:*

$$\max_{0\le k\le N-L}\mathbb{E}\big(|\breve{d}_{QN,k}^{\delta}|^{2p}\big)\le c^{2p}\,a(2p)N^{-2p(\delta(L-\overline{H})+\underline{H})}\,, \tag{3.10}$$

where $a(2p)$ is the same constant only depending on p as in Proposition 2.3.

Proof of Lemma 3.2 Let $Q\in\mathbb{N}$ and b a fixed real number such that

$$\beta<b<2\delta(L-\overline{H})+2\underline{H}\,, \tag{3.11}$$

where β is as in (3.9). Let $p\in[1,+\infty)$ be fixed and such that

$$p(2\delta(L-\overline{H})+2\underline{H}-b)>2\,. \tag{3.12}$$

Using (3.8), Markov inequality, the fact that $z\mapsto|z|^p$ is a convex function on $\mathbb{R}$, (3.10) and (2.11), one gets that

$$\begin{aligned}
&\mathbb{P}\Big(N^b\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})>1\Big)\le\sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1}\mathbb{P}\Big(N^b\,\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})>1\Big)\\
&\le N^{pb}\sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1}\mathbb{E}\Big(\big|\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|^p\Big)\le N^{pb}\sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1}|\nu_{QN}(\mathcal{I}_{N,n})|^{-1}\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}\mathbb{E}\big(|\breve{d}_{QN,k}^{\delta}|^{2p}\big)\\
&\le c_1\lfloor\theta_N^{-1}\rfloor N^{-p(2\delta(L-\overline{H})+2\underline{H}-b)}\le c_1N^{1-p(2\delta(L-\overline{H})+2\underline{H}-b)}\,,
\end{aligned} \tag{3.13}$$

where $c_1>0$ is a constant not depending on N and Q. Next, combining (3.12) and (3.13), one obtains that

$$\sum_{N=N_0}^{+\infty}\mathbb{P}\Big(N^b\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})>1\Big)<+\infty\,.$$

Thus, it results from the Borel-Cantelli Lemma that one has, almost surely,

$$\sup_{N\geq N_0}\left\{N^b \max_{0\leq n<\lfloor\theta_N^{-1}\rfloor} \breve{V}_{Q_N}^{\delta}(\mathcal{I}_{N,n})\right\} < +\infty\,. \tag{3.14}$$

Finally, (3.11) and (3.14) imply that (3.9) holds. ■

Let us now focus on the proof of Lemma 3.3. It mainly relies on Proposition 2.3 and the following proposition.

Proposition 3.4 *One has*

$$c := \sup\left\{\big(1+L+|u|\big)^{L+1/2-\overline{H}}\,\big|\Phi(u,v)\big| \,:\, (u,v)\in(-\infty,-2L]\times[\underline{H},\overline{H}]\right\} < +\infty \tag{3.15}$$

and

$$c' := \sup\left\{\frac{\big(1+L+|u|\big)^{L+1/2-\overline{H}}}{\log\big(1+L+|u|\big)}\,\big|(\partial_v\Phi)(u,v)\big| \,:\, (u,v)\in(-\infty,-2L]\times[\underline{H},\overline{H}]\right\} < +\infty. \tag{3.16}$$

Proof of Proposition 3.4 Combining (3.2) and (1.2) one gets, for all $(u,v)\in(-\infty,-L)\times(0,1)$, that

$$\Phi(u,v) = |u|^{v-1/2}\sum_{l=0}^{L} a_l f(lu^{-1},v) \tag{3.17}$$

and

$$(\partial_v\Phi)(u,v) = |u|^{v-1/2}\log(|u|)\sum_{l=0}^{L} a_l f(lu^{-1},v) + |u|^{v-1/2}\sum_{l=0}^{L} a_l(\partial_v f)(lu^{-1},v), \tag{3.18}$$

where f is the C^∞ function on $(-1,1)\times(-2,2)$ defined, for all $(y,v)\in(-1,1)\times(-2,2)$, as $f(y,v) = (1-y)^{v-1/2}$. Then noticing that when u belongs to $(-\infty,-2L]$ one equivalently has that $z=u^{-1}$ belongs to $\big[-2^{-1}L^{-1},0\big)\subset\big(-L^{-1},L^{-1}\big)$, one can easily derive from (3.17), (3.18) and Lemma 3.5 below that (3.15) and (3.16) are satisfied. ■

Lemma 3.5 *Assume that y_0 and v_0 are two arbitrary and fixed positive real numbers. Let φ be an arbitrary real-valued C^∞ function on $(-y_0,y_0)\times(-v_0,v_0)$ and let g be the*

C^∞ function on $\big(-L^{-1}y_0, L^{-1}y_0\big)\times(-v_0, v_0)$ defined, for all $(z,v)\in\big(-L^{-1}y_0, L^{-1}y_0\big)\times(-v_0, v_0)$, as:

$$g(z,v) := \sum_{l=0}^{L} a_l \varphi(lz, v).$$

Then, one has, for every $(z,v)\in\big[-2^{-1}L^{-1}y_0, 2^{-1}L^{-1}y_0\big]\times\big[-2^{-1}v_0, 2^{-1}v_0\big]$,

$$\big|g(z,v)\big| \le c|z|^L,$$

where c is the finite constant defined as

$$c := (L!)^{-1}\sup\big\{|(\partial_z^L g)(z,v)| : (z,v)\in\big[-2^{-1}L^{-1}y_0, 2^{-1}L^{-1}y_0\big]\times\big[-2^{-1}v_0, 2^{-1}v_0\big]\big\}.$$

Proof of Lemma 3.5 Assume that $v\in(-v_0, v_0)$ is arbitrary and fixed. Applying Taylor formula to the function $z\mapsto g(z,v)$ it follows, for all $z\in\big[-2^{-1}L^{-1}y_0, 2^{-1}L^{-1}y_0\big]$, that

$$g(z,v) = \Big(\sum_{q=0}^{L-1}\frac{(\partial_z^q g)(0,v)}{q!}z^q\Big) + \frac{(\partial_z^L g)(\theta,v)}{L!}z^L\,, \tag{3.19}$$

where $\theta\in\big(-2^{-1}L^{-1}y_0, 2^{-1}L^{-1}y_0\big)$. Next, observe that, for each $z\in\big(-L^{-1}y_0, L^{-1}y_0\big)$ and $q\in\mathbb{N}$, one has

$$(\partial_z^q g)(z,v) = \sum_{l=0}^{L} l^q a_l (\partial_y^q\varphi)(lz, v)\,.$$

Therefore, one gets that

$$(\partial_z^q g)(0,v) = (\partial_y^q\varphi)(0,v)\Big(\sum_{l=0}^{L} l^q a_l\Big).$$

Then, in view of (2.2), it turns out that $(\partial_z^q g)(0,v) = 0$, for all $q\in\{0,\dots,L-1\}$. Finally, combining the latter equality with (3.19), one obtains the lemma. ■

We are now ready to prove Lemma 3.3.

Proof of Lemma 3.3 Using (3.6), (2.17), (1.8), (3.15) and (3.3) one gets, for all integers $Q\ge 1$, $N\ge(3L)^{1/\delta}+N_0$ and $k\in\{0,\dots,N-L\}$, that

$$\begin{aligned}
&\mathbb{E}\big(|\breve{d}^{\,\delta}_{QN,k}|^{2p}\big) \le a(2p)\mathbb{E}\bigg(\Big(\int_{-\infty}^{(QN)^{-1}(k-e_{QN}+L)}(QN)^{-2H(s)+1}\,\big|\Phi\big(QNs-k, H(s)\big)\big|^2\,ds\Big)^p\bigg)\\
&\le c^{2p}\,a(2p)N^{-2p\underline{H}}\Big(QN\int_{-\infty}^{(QN)^{-1}(k-e_{QN}+L)}\big(1+L+k-QNs\big)^{2\overline{H}-2L-1}\,ds\Big)^p\\
&\le c^{2p}\,a(2p)N^{-2p\underline{H}}\,(1+e_{QN})^{2p(\overline{H}-L)} \le c^{2p}\,a(2p)N^{2p(\delta(\overline{H}-L)-\underline{H})}\,,
\end{aligned}$$

which proves that (3.10) holds. ■

Roughly speaking, so far we have shown that, when N goes to $+\infty$, the asymptotic behavior of $V_{QN}(\mathcal{I}_{N,n})$ is similar to that of $\widetilde{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$ defined in (3.7). There is still a difficulty in the study of the latter behavior. Basically, it comes from the $H(s)$ which figures in (3.5). It is convenient to replace $H(s)$ by a well-chosen random variable not depending on s. This is the main motivation behind the following definition.

Definition 3.6 *For every $\delta \in (0,1)$ and integers $N \geq N_0$ and $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$, one sets*

$$\zeta_{N,n} := n\theta_N - N^{-(1-\delta)}\,; \tag{3.20}$$

moreover, for each $Q \in \mathbb{N}$, the generalized quadratic variations $\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$ and $\overline{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$ are defined as:

$$\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\widehat{d}^{\delta,n}_{QN,k}|^2 \tag{3.21}$$

and

$$\overline{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\overline{d}^{\delta,n}_{QN,k}|^2\,, \tag{3.22}$$

where, for all $k \in \nu_{QN}(\mathcal{I}_{N,n})$,

$$\widehat{d}^{\delta,n}_{QN,k} := \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} (QN)^{-(H(\zeta_{N,n})-1/2)}\, \Phi\big(QNs - k, H(\zeta_{N,n})\big)\, dB(s) \tag{3.23}$$

and

$$\begin{aligned} \overline{d}^{\delta,n}_{QN,k} &:= \widetilde{d}^{\delta}_{QN,k} - \widehat{d}^{\delta,n}_{QN,k} \\ &= \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} \Big((QN)^{-(H(s)-1/2)}\, \Phi\big(QNs - k, H(s)\big) \\ &\qquad\qquad -(QN)^{-(H(\zeta_{N,n})-1/2)}\, \Phi\big(QNs - k, H(\zeta_{N,n})\big) \Big)\, dB(s)\,. \end{aligned} \tag{3.24}$$

Basically, the following lemma shows that the generalized quadratic variations $\overline{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$, $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$, are negligible when N goes to $+\infty$. In other words, when N goes to $+\infty$, the asymptotic behavior of $\widetilde{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$ (and consequently that of $V_{QN}(\mathcal{I}_{N,n})$) is similar to that of the "less complicated" generalized quadratic variation $\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$.

Lemma 3.7 *Let γ and μ be as in (1.10) and (2.10). Let $\delta \in (0,1)$ be arbitrary and fixed. One has almost surely*

$$\limsup_{N\to+\infty} \left\{ N^{\beta} \max_{0\le n<\lfloor \theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\, \overline{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) \right\} = 0\,, \quad \text{for all } Q\in\mathbb{N} \text{ and } \beta < 2\gamma((1-\delta)\wedge\mu). \tag{3.25}$$

In order to show that Lemma 3.7 holds, one needs the following lemma.

Lemma 3.8 *For any fixed $Q\in\mathbb{N}$ and $p\in[1,+\infty)$, there exists a finite constant $\overline{c}(Q,p)$ such that, for all integer $N\ge N_0$, one has*

$$\max_{0\le n<\lfloor \theta_N^{-1}\rfloor} \max_{k\in\nu_{QN}(\mathcal{I}_{N,n})} \mathbb{E}\Big(\big| N^{H(\zeta_{N,n})}\, \overline{d}^{\delta,n}_{QN,k} \big|^{2p} \Big) \le \overline{c}(Q,p) (\log N)^{2p} N^{-2p\gamma((1-\delta)\wedge\mu)}\,. \tag{3.26}$$

Proof of Lemma 3.8 The integers $Q\in\mathbb{N}$, $N\ge N_0$, $n\in\big\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\big\}$ and $k\in\nu_{QN}(\mathcal{I}_{N,n})$ are arbitrary and fixed. It follows from (3.24), (2.17), the inequality $Q^{-2H(\zeta_{N,n})}\le 1$, the inequality

$$|x+y|^2 \le 2\big(|x|^2+|y|^2\big), \quad \text{for all } (x,y)\in\mathbb{R}^2, \tag{3.27}$$

and the convexity on $\mathbb{R}_+$ of the function $z\mapsto z^p$ that

$$\begin{aligned}
&\mathbb{E}\Big(\big| N^{H(\zeta_{N,n})}\, \overline{d}^{\delta,n}_{QN,k} \big|^{2p} \Big) \\
&\le a(2p)\mathbb{E}\Bigg(\bigg(QN \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} \Big| (QN)^{(H(\zeta_{N,n})-H(s))}\, \Phi\big(QNs-k,H(s)\big) \\
&\qquad\qquad -\Phi\big(QNs-k,H(\zeta_{N,n})\big)\Big|^2 ds \bigg)^p \Bigg) \\
&\le 2^{2p-1} a(2p) \Big(\mathbb{E}\big((A^{\delta,n}_{QN,k})^p\big) + \mathbb{E}\big((B^{\delta,n}_{QN,k})^p\big) \Big)\,,
\end{aligned} \tag{3.28}$$

where

$$A^{\delta,n}_{QN,k} := QN \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} \Big| (QN)^{(H(\zeta_{N,n})-H(s))} - 1 \Big|^2 \Big| \Phi\big(QNs-k,H(s)\big)\Big|^2 ds \tag{3.29}$$

and

$$B^{\delta,n}_{QN,k} := QN \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} \Big| \Phi\big(QNs-k,H(s)\big) - \Phi\big(QNs-k,H(\zeta_{N,n})\big)\Big|^2 ds\,. \tag{3.30}$$

Next, using the mean value Theorem, (1.10), (3.20), the fact that $(QN)^{-1}k \in \mathcal{I}_{N,n}$ (see (2.5) and (2.12)), (3.3) and (2.10), for all $s \in \big[(QN)^{-1}(k - e_{QN} + L), (QN)^{-1}(k + L)\big]$, one gets that

$$\begin{aligned}\left|(QN)^{(H(\zeta_{N,n})-H(s))} - 1\right| &\leq c_1\rho \exp\big(c_1\rho(\log QN)N^{-\gamma((1-\delta)\wedge\mu)}\big)(\log QN)N^{-\gamma((1-\delta)\wedge\mu)} \\ &\leq \exp\big(c_2(Q)\rho\big)(\log N)N^{-\gamma((1-\delta)\wedge\mu)}\,, \end{aligned} \tag{3.31}$$

where the deterministic finite constants c_1 and $c_2(Q)$ are defined as: $c_1 := (2\kappa + L + 1)^\gamma$ and $c_2(Q) := c_1\big(2\log(3+Q) + \sup_{N\geq N_0}(\log QN)N^{-\gamma((1-\delta)\wedge\mu)}\big)$. Next, putting together (3.29), (3.31), the change of variable $u = QNs - k$, (3.15) and (3.2), one obtains that

$$\begin{aligned} A_{N,k}^{\delta,n} &\leq \exp\big(2c_2(Q)\rho\big)(\log N)^2 N^{-2\gamma((1-\delta)\wedge\mu)} \int_{-\infty}^{L} \left|\Phi\big(u, H((QN)^{-1}(u+k))\big)\right|^2 du \\ &\leq c_3 \exp\big(2c_2(Q)\rho\big)(\log N)^2 N^{-2\gamma((1-\delta)\wedge\mu)}, \end{aligned} \tag{3.32}$$

where the deterministic finite constant

$$c_3 := c_4^2 \int_{-\infty}^{-2L} \big(1 + L + |u|\big)^{2\overline{H}-2L-1}\, du + \int_{-2L}^{L} \Big(\sum_{l=0}^{L} |a_l|\big((l-u)_+^{\underline{H}-1/2} + (l-u)_+^{\overline{H}-1/2}\big)\Big)^2 du, \tag{3.33}$$

c_4 being the constant c in (3.15). Next, notice that in view of (3.3) one can assume without any restriction that N is big enough so that $L - e_{QN} \leq L - e_N < -2L$. Then, it follows from (3.30), the change of variable $u = QNs - k$ and (3.2) that

$$\begin{aligned} B_{QN,k}^{\delta,n} &= \int_{L-e_{QN}}^{L} \left|\Phi\big(u, H((QN)^{-1}(u+k))\big) - \Phi\big(u, H(\zeta_{N,n})\big)\right|^2 du \\ &= \int_{L-e_{QN}}^{-2L} \left|\Phi\big(u, H((QN)^{-1}(u+k))\big) - \Phi\big(u, H(\zeta_{N,n})\big)\right|^2 du \\ &\quad + \int_{-2L}^{0} \Big(\sum_{l=0}^{L} a_l(l-u)^{H((QN)^{-1}(u+k))-1/2} - \sum_{l=0}^{L} a_l(l-u)^{H(\zeta_{N,n})-1/2}\Big)^2 du \\ &\quad + \sum_{p=0}^{L-1} \int_{p}^{p+1} \Big(\sum_{l=p+1}^{L} a_l(l-u)^{H((QN)^{-1}(u+k))-1/2} - \sum_{l=p+1}^{L} a_l(l-u)^{H(\zeta_{N,n})-1/2}\Big)^2 du\,. \end{aligned}$$

Thus, one can derive from the mean value Theorem, (1.10), (3.20), the fact that $(QN)^{-1}k \in \mathcal{I}_{N,n}$, (3.3), (2.10), (3.16) and (1.8) that

$$B_{QN,k}^{\delta,n} \leq c_5\rho^2 N^{-2\gamma((1-\delta)\wedge\mu)}\,, \tag{3.34}$$

where the deterministic finite constant

$$
\begin{aligned}
c_5 \;:=\; & c_7^2\, c_6^2 \int_{-\infty}^{-2L} \big(1+L+|u|\big)^{2\overline{H}-2L-1} \log^2\big(1+L+|u|\big)\, du \\
& + c_6^2 \int_{-2L}^{0} \Big(\sum_{l=0}^{L} |a_l| \big((l-u)^{\underline{H}-1/2} + (l-u)^{\overline{H}-1/2} \big) \big| \log(l-u) \big| \Big)^2 du \\
& + c_6^2 \sum_{p=0}^{L-1} \int_{p}^{p+1} \Big(\sum_{l=p+1}^{L} |a_l| \big((l-u)^{\underline{H}-1/2} + (l-u)^{\overline{H}-1/2} \big) \big| \log(l-u) \big| \Big)^2 du \,,
\end{aligned}
$$

$c_6 := (2\kappa+2)^\gamma$ and c_7 being the constant c' in (3.16). Finally, putting together (3.28), (3.32) and (3.34), one obtains (3.26). ■

We are now ready to prove Lemma 3.7.

Proof of Lemma 3.7 Let $Q \in \mathbb{N}$ and b be a fixed real number such that

$$
\beta < b < 2\gamma((1-\delta)\wedge\mu)\,, \tag{3.35}
$$

where β is as in (3.25). Let $p \in [1,+\infty)$ be fixed and such that

$$
p(2\gamma((1-\delta)\wedge\mu) - b) > 2\,. \tag{3.36}
$$

Using (3.22), Markov inequality, the fact that $z \mapsto |z|^p$ is a convex function on $\mathbb{R}$, (3.26) and (2.11), one gets that

$$
\begin{aligned}
& \mathbb{P}\Big(N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\, \overline{V}^{\,\delta}_{QN}(\mathcal{I}_{N,n}) > 1 \Big) \le \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} \mathbb{P}\Big(N^{b+2H(\zeta_{N,n})}\, \overline{V}^{\,\delta}_{QN}(\mathcal{I}_{N,n}) > 1 \Big) \\
& \le N^{pb} \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} \mathbb{E}\Big(\big| N^{2H(\zeta_{N,n})}\, \overline{V}^{\,\delta}_{QN}(\mathcal{I}_{N,n}) \big|^p \Big) \\
& \le N^{pb} \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})} \mathbb{E}\Big(\big| N^{H(\zeta_{N,n})}\, \overline{d}^{\,\delta,n}_{QN,k} \big|^{2p} \Big) \\
& \le c_1 \lfloor\theta_N^{-1}\rfloor (\log N)^{2p} N^{-p(2\gamma((1-\delta)\wedge\mu)-b)} \le c_1 (\log N)^{2p} N^{1-p(2\gamma((1-\delta)\wedge\mu)-b)}\,,
\end{aligned} \tag{3.37}
$$

where $c_1 > 0$ is a constant not depending on N. Next, combining (3.36) and (3.37), one obtains that

$$
\sum_{N=N_0}^{+\infty} \mathbb{P}\Big(N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\, \overline{V}^{\,\delta}_{N}(\mathcal{I}_{N,n}) > 1 \Big) < +\infty\,.
$$

Thus, it results from the Borel-Cantelli Lemma that one has, almost surely,

$$\sup_{N\geq N_0}\left\{N^b \max_{0\leq n<\lfloor\theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\,\overline{V}_N^{\,\delta}(\mathcal{I}_{N,n})\right\}<+\infty\,. \tag{3.38}$$

Finally, (3.35) and (3.38) imply that (3.25) holds. ■

4 Asymptotic behavior of generalized quadratic variation of X

The main goal of the present section is to prove the following lemma.

Lemma 4.1 *Let $\underline{H}$, $\overline{H}$, γ, L, μ and μ' are as in (1.8), (1.10), (2.2), (2.10) and (2.11). Let β be an arbitrary real number satisfying*

$$0<\beta<\min\left\{\gamma\big((1-\delta)\wedge\mu\big),\,\delta(L-\overline{H})+\underline{H}-\overline{H},\,2^{-1}(1-\mu')\right\}, \tag{4.1}$$

where δ is an arbitrary fixed real number such that

$$\frac{\overline{H}-\underline{H}}{L-\overline{H}}<\delta<1\,. \tag{4.2}$$

Then, one has almost surely, for all $Q\in\mathbb{N}$,

$$\limsup_{N\to+\infty}\left\{N^\beta \max_{0\leq n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|\right\}=0\,, \tag{4.3}$$

where $V_{QN}(\mathcal{I}_{N,n})$, $\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})$ and $\zeta_{N,n}$ are defined through (2.4), (3.21) and (3.20). Notice that $\mathbb{E}\big(\cdot\,|\mathcal{F}_{\zeta_{N,n}}\big)$ is the conditional expectation operator with respect to the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$.

The proof of Lemma 4.1, which will be given at the end of the present section, relies on Lemma 3.2, Lemma 3.7, and the following crucial lemma.

Lemma 4.2 *Let $\mu'\in[\mu,1)\subset(0,1)$ be as in (2.11). Let $\delta\in(0,1)$ be arbitrary and fixed. One has almost surely*

$$\limsup_{N\to+\infty}\left\{N^{\beta(1-\mu')} \max_{0\leq n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|\right\}=0\,,\quad \text{for all } Q\in\mathbb{N} \text{ and } \beta<1/2. \tag{4.4}$$

In order to show that Lemma 4.2 holds, one needs some preliminary results.

Lemma 4.3 *For all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, and for each finite sequence $(z_k)_{k \in \nu_{QN}(\mathcal{I}_{N,n})}$ of real numbers, one has, almost surely,*

$$\begin{aligned} &\mathbb{E}\Big(\exp\Big(i \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} z_k \, \widehat{d}^{\,\delta,n}_{QN,k} \Big) \Big| \mathcal{F}_{\zeta_{N,n}} \Big) \\ &= \exp\Bigg(\\ &\quad -2^{-1} \int_{\zeta_{N,n}}^1 \Big| \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} z_k \mathbb{1}_{\mathcal{D}_{QN,k}}(s) (QN)^{-(H(\zeta_{N,n})-1/2)} \, \Phi\big(QNs - k, H(\zeta_{N,n})\big) \Big|^2 ds \Bigg) , \end{aligned} \tag{4.5}$$

where $\zeta_{N,n}$ is an in (3.20), and

$$\mathcal{D}_{QN,k} := \big[(QN)^{-1}(k - e_{QN} + L), (QN)^{-1}(k + L) \big] . \tag{4.6}$$

Notice that (4.5) means that, for each $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, conditionally to the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$, the random vector $(\widehat{d}^{\,\delta,n}_{QN,k})_{k \in \nu_{QN}(\mathcal{I}_{N,n})}$ has a centred Gaussian distribution with covariance matrix $\big(\mathbb{E}(\widehat{d}^{\,\delta,n}_{QN,k'} \, \widehat{d}^{\,\delta,n}_{QN,k''} | \mathcal{F}_{\zeta_{N,n}}) \big)_{k',k'' \in \nu_{QN}(\mathcal{I}_{N,n})}$ such that, for every $k', k'' \in \nu_{QN}(\mathcal{I}_{N,n})$,

$$\begin{aligned} &\mathbb{E}\big(\widehat{d}^{\,\delta,n}_{QN,k'} \, \widehat{d}^{\,\delta,n}_{QN,k''} \big| \mathcal{F}_{\zeta_{N,n}} \big) \\ &= (QN)^{1-2H(\zeta_{N,n})} \\ &\quad \times \int_{\zeta_{N,n}}^1 \mathbb{1}_{\mathcal{D}_{QN,k'}}(s) \mathbb{1}_{\mathcal{D}_{QN,k''}}(s) \Phi\big(QNs - k', H(\zeta_{N,n})\big) \Phi\big(QNs - k'', H(\zeta_{N,n})\big) \, ds . \end{aligned} \tag{4.7}$$

Proof of Lemma 4.3 First observe that one can derive from (3.23), (2.5), (2.12), (3.20) and (4.6) that, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, and for each finite sequence $(z_k)_{k \in \nu_{QN}(\mathcal{I}_{N,n})}$ of real numbers, one has

$$\begin{aligned} &\sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} z_k \, \widehat{d}^{\,\delta,n}_{QN,k} \\ &= \int_{\zeta_{N,n}}^1 \Bigg(\sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} z_k \mathbb{1}_{\mathcal{D}_{QN,k}}(s) (QN)^{-(H(\zeta_{N,n})-1/2)} \, \Phi\big(QNs - k, H(\zeta_{N,n})\big) \Bigg) \, dB(s) . \end{aligned} \tag{4.8}$$

The main idea of the proof of this lemma consists in the observation that the Brownian motion B in (4.8) can be replaced by the Brownian motion $W_{N,n} = \{W_{N,n}(x)\}_{x \in \mathbb{R}_+} :=$

$\{B(x+\zeta_{N,n})-B(\zeta_{N,n})\}_{x\in\mathbb{R}_+}$ which is independent of the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$. Therefore $W_{N,n}$ is independent of the integrand in (4.8), denoted by $K_{N,n}$, which is $\mathcal{F}_{\zeta_{N,n}}$-measurable. Having made this observation the proof becomes classical: it can be done in a standard way by approximating the integrand $K_{N,n}=\{K_{N,n}(s)\}_{s\in[\zeta_{N,n},1]}$ by a sequence $\big(K^j_{N,n}\big)_{j\in\mathbb{N}}=\big(\{K^j_{N,n}(s)\}_{s\in[\zeta_{N,n},1]}\big)_{j\in\mathbb{N}}$ of elementary processes of the form:

$$K^j_{N,n}(s)=\sum_{p=0}^{q-1}A_p\mathbb{1}_{[t_p,t_{p+1})}(s)\,,$$

where the random variables A_p, $0\le p<q$, are $\mathcal{F}_{\zeta_{N,n}}$-measurable, and the finite sequence $(t_p)_{0\le p\le q}$ is a subdivision of the interval $[\zeta_{N,n},1]$. ■

Roughly speaking, the following lemma shows that $\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)$ behaves in the same way as $(QN)^{-2H(\zeta_{N,n})}$

Lemma 4.4 *For every $\delta\in(0,1)$ one has almost surely, for all integers $Q\in\mathbb{N}$, $N\ge N_0$ and $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$, that*

$$\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)=(QN)^{-2H(\zeta_{N,n})}\int_{L-e_{QN}}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du \tag{4.9}$$

and consequently that

$$c'(QN)^{-2H(\zeta_{N,n})}\le\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)\le c''(QN)^{-2H(\zeta_{N,n})}\,, \tag{4.10}$$

where c' and c'' are two finite, deterministic and strictly positive constants not depending on δ, Q, N and n.

Proof of Lemma 4.4 One can derive from (4.7), (4.6) and the change of variable $u=(QN)s-k$ that one has almost surely, for integers $Q\in\mathbb{N}$, $N\ge N_0$, $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$ and $k\in\nu_{QN}(\mathcal{I}_{N,n})$, that

$$\mathbb{E}\big(|\widehat{d}^{\delta,n}_{QN,k}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big)=(QN)^{-2H(\zeta_{N,n})}\int_{L-e_{QN}}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du\,. \tag{4.11}$$

Thus combining (3.21) and (4.11) one obtains (4.9). Then notice that (3.2), (1.2), (2.1), (3.3), (3.15) and (1.8) entail that

$$\int_{L-e_{QN}}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du\ge\int_{L-1}^{L}(L-u)^{2H(\zeta_{N,n})-1}\,du\ge c' \tag{4.12}$$

and

$$\int_{L-e_{QN}}^{L} \Big|\Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 du \leq \int_{-\infty}^{L} \Big|\Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 du \leq c'', \tag{4.13}$$

where the strictly positive constant $c' := (2\overline{H})^{-1}$ and the constant c'' is equal to the constant c_3 defined in (3.33). Finally, putting together (4.9), (4.12) and (4.13) one gets (4.10). ■

Remark 4.5 *The integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$ are arbitrary and fixed. One denotes by $\mathcal{G}$ a Gaussian Hilbert space on $\mathbb{R}$ spanned by a centred real-valued Gaussian vector $(G_k)_{k \in \nu_{QN}(\mathcal{I}_{N,n})}$ whose distribution is equal to the conditional distribution of the random vector $(\widehat{d}^{\,\delta,n}_{QN,k})_{k \in \nu_{QN}(\mathcal{I}_{N,n})}$ with respect to the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$ (see Lemma 4.3) for some given arbitrary value of the random variable $H(\zeta_{N,n})$. Then, for the same given value of $H(\zeta_{N,n})$, the conditional distribution with respect to $\mathcal{F}_{\zeta_{N,n}}$ of the random variable $\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)$ (see (3.21)) is equal to the distribution of the random variable $|\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} \big(|G_k|^2 - \mathbb{E}(|G_k|^2)\big)$. Since the latter random variable belongs to $\overline{\mathcal{P}}_2(\mathcal{G})$ the second order chaos associated to $\mathcal{G}$ (see Definition 2.1 on page 17 in [22]), one knows from Theorem 5.10 on page 62 in [22] that, for any fixed $q \in \mathbb{N}$, there exists a universal deterministic finite constant $\widehat{c}(q)$, only depending on q, such that*

$$\mathbb{E}\left(\Big||\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} \big(|G_k|^2 - \mathbb{E}(|G_k|^2)\big)\Big|^{2q}\right)$$
$$\leq \widehat{c}(q)\left(\mathbb{E}\left(\Big||\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} \big(|G_k|^2 - \mathbb{E}(|G_k|^2)\big)\Big|^{2}\right)\right)^{q}.$$

Therefore, one has

$$\mathbb{E}\left(\Big|\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big|^{2q}\Big|\mathcal{F}_{\zeta_{N,n}}\right) \tag{4.14}$$
$$\leq \widehat{c}(q)\left(\mathbb{E}\left(\Big|\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big|^{2}\Big|\mathcal{F}_{\zeta_{N,n}}\right)\right)^{q}.$$

Also, notice that one can derive from Theorem 3.9 on page 26 in [22] that

$$\mathbb{E}\Big(\big(|G_{k'}|^2 - \mathbb{E}(|G_{k'}|^2)\big)\big(|G_{k''}|^2 - \mathbb{E}(|G_{k''}|^2)\big)\Big) = 2\big(\mathbb{E}(G_{k'}G_{k''})\big)^2, \quad \text{for all } k', k'' \in \nu_{QN}(\mathcal{I}_{N,n}),$$

which implies that

$$\mathbb{E}\Big(\Big(|\widehat{d}^{\delta,n}_{QN,k'}|^2-\mathbb{E}\big(|\widehat{d}^{\delta,n}_{QN,k'}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big)\Big(|\widehat{d}^{\delta,n}_{QN,k''}|^2-\mathbb{E}\big(|\widehat{d}^{\delta,n}_{QN,k''}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big)\Big|\mathcal{F}_{\zeta_{N,n}}\Big)$$
$$=2\Big(\mathbb{E}\big(\widehat{d}^{\delta,n}_{QN,k'}\,\widehat{d}^{\delta,n}_{QN,k''}|\mathcal{F}_{\zeta_{N,n}}\big)\Big)^2,\quad \textit{for all } k',k''\in\nu_{QN}(\mathcal{I}_{N,n}). \tag{4.15}$$

Lemma 4.6 *There exists a finite deterministic constant c such that, for all integers $Q\in\mathbb{N}$, $N\geq N_0$, $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$ and $k',k''\in\nu_{QN}(\mathcal{I}_{N,n})$, one has*

$$\big|\mathbb{E}(\widehat{d}^{\delta,n}_{QN,k'}\,\widehat{d}^{\delta,n}_{QN,k''}|\mathcal{F}_{\zeta_{N,n}})\big|\leq c\,(QN)^{-2H(\zeta_{N,n})}\big(1+|k'-k''|\big)^{-(L-\overline{H})}. \tag{4.16}$$

Proof of Lemma 4.6 The integers $Q\in\mathbb{N}$, $N\geq N_0$, $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$ and $k',k''\in\nu_N(\mathcal{I}_{N,n})$ are arbitrary; moreover one can assume without any restriction that $k''\geq k'$. One can derive from (4.7) and the change of variable $QNs-k'$ that

$$\begin{aligned}&\big|\mathbb{E}(\widehat{d}^{\delta,n}_{QN,k'}\,\widehat{d}^{\delta,n}_{QN,k''}|\mathcal{F}_{\zeta_{N,n}})\big| \\ &\leq (QN)^{-2H(\zeta_{N,n})}\int_{\mathbb{R}}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|\,du\,.\end{aligned} \tag{4.17}$$

One denotes by c_1 the finite deterministic constant c_3 defined in (3.33) which does not depend on Q, N, n, $H(\zeta_{N,n})$, k' and k''. Using (4.17), the Cauchy-Scharwz inequality, (3.2) and (1.2), one gets that

$$\big|\mathbb{E}(\widehat{d}^{\delta,n}_{N,k'}\,\widehat{d}^{\delta,n}_{N,k''}|\mathcal{F}_{\zeta_{N,n}})\big|\leq(QN)^{-2H(\zeta_{N,n})}\int_{-\infty}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du\leq c_1(QN)^{-2H(\zeta_{N,n})} \tag{4.18}$$

and

$$\begin{aligned}&\big|\mathbb{E}(\widehat{d}^{\delta,n}_{QN,k'}\,\widehat{d}^{\delta,n}_{QN,k''}|\mathcal{F}_{\zeta_{N,n}})\big| \\ &\leq (QN)^{-2H(\zeta_{N,n})}\int_{-\infty}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|\,du \\ &= (QN)^{-2H(\zeta_{N,n})}\Big(\int_{2^{-1}(k'-k'')}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|\,du \\ &\qquad\qquad+\int_{-\infty}^{2^{-1}(k'-k'')}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|\,du\Big) \\ &\leq \sqrt{c_1}(QN)^{-2H(\zeta_{N,n})}\Big(\sqrt{\int_{2^{-1}(k'-k'')}^{L}\Big|\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|^2\,du} \\ &\qquad\qquad+\sqrt{\int_{-\infty}^{2^{-1}(k'-k'')}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du}\Big)\,.\end{aligned} \tag{4.19}$$

Next observe that, under the condition that

$$k' - k'' \leq -4L\,, \tag{4.20}$$

one clearly has $2^{-1}(k'-k'') \leq -2L$, and thus one can derive from (3.15) that

$$\begin{aligned} &\int_{-\infty}^{2^{-1}(k'-k'')} \Big|\Phi\big(u, H(\zeta_{N,n})\big)\Big|^2 \, du \\ &\leq c_2^2 \int_{-\infty}^{2^{-1}(k'-k'')} \big(1+L-u\big)^{2\overline{H}-2L-1} \, du \leq c_3 \big(1+L+k''-k'\big)^{-2(L-\overline{H})}\,, \end{aligned} \tag{4.21}$$

where c_2 is the finite deterministic constant c in (3.15) and $c_3 := 2^{2(L-\overline{H})-1}(L-\overline{H})^{-1}c_2^2$. Also, observe that under the condition (4.20), for all $u \in \big[2^{-1}(k'-k''), L\big]$, one has $u + k' - k'' \leq -3L$, and thus one can derive from (3.15) that

$$\begin{aligned} &\int_{2^{-1}(k'-k'')}^{L} \Big|\Phi\big(u+k'-k'', H(\zeta_{N,n})\big)\Big|^2 \, du \leq c_2^2 \int_{-\infty}^{L} \big(1+L-u+k''-k'\big)^{2\overline{H}-2L-1} \, du \\ &\leq \frac{c_2^2}{2(L-\overline{H})} \big(1+k''-k'\big)^{-2(L-\overline{H})} \leq c_3 \big(1+k''-k'\big)^{-2(L-\overline{H})}\,. \end{aligned} \tag{4.22}$$

Finally setting $c := c_1(4L)^{L-\overline{H}} + 2\sqrt{c_1 c_3}$, it follows from (4.18), (4.19), (4.21) and (4.22) that (4.16) is satisfied. ■

We are now ready to prove Lemma 4.2.

Proof of Lemma 4.2 Let b be a fixed real number such that

$$\beta < b < 1/2\,, \tag{4.23}$$

where β is as in (4.4). Let $q \in \mathbb{N}$ be fixed and big enough so that

$$q(1-\mu')(1-2b) > \mu' + 1\,, \tag{4.24}$$

where $\mu' \in (0,1)$ is as in (2.11). Using Markov inequality one obtains, for all integers

$Q \in \mathbb{N}$ and $N \geq N_0$, that

$$\begin{aligned}
&\mathbb{P}\left(N^{b(1-\mu')} \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor} \left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \right| > 1 \right) \\
&\leq \sum_{n=0}^{\lfloor \theta_N^{-1}\rfloor -1} \mathbb{P}\left(N^{b(1-\mu')} \left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \right| > 1 \right) \\
&\leq N^{2qb(1-\mu')} \sum_{n=0}^{\lfloor \theta_N^{-1}\rfloor -1} \mathbb{E}\left(\left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \right|^{2q} \right). \qquad (4.25)
\end{aligned}$$

Moreover, the expectations in (4.25) can be expressed, for all $n \in \{0, \ldots, \lfloor \theta_N^{-1}\rfloor - 1\}$, as:

$$\begin{aligned}
&\mathbb{E}\left(\left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \right|^{2q} \right) \\
&= \mathbb{E}\left(\left(\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big) \right)^{-2q} \left| \widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) \right|^{2q} \right) \qquad (4.26) \\
&= \mathbb{E}\left(\left(\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big) \right)^{-2q} \mathbb{E}\left(\left| \widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) \right|^{2q} \Big| \mathcal{F}_{\zeta_{N,n}} \right) \right).
\end{aligned}$$

On the other hand, it follows from (3.21), (4.15), (4.16) and the inequality $L - \overline{H} > 1$ that, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \ldots, \lfloor \theta_N^{-1}\rfloor - 1\}$, one has

$$\begin{aligned}
&\mathbb{E}\left(\left| \widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) \right|^2 \Big| \mathcal{F}_{\zeta_{N,n}} \right) \\
&= \big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-2} \sum_{k',k''\in\nu_{QN}(\mathcal{I}_{N,n})} \mathbb{E}\Big(\Big(|\widehat{d}_{QN,k'}^{\delta,n}|^2 - \mathbb{E}\big(|\widehat{d}_{QN,k'}^{\delta,n}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big) \Big) \\
&\qquad\qquad \times \Big(|\widehat{d}_{QN,k''}^{\delta,n}|^2 - \mathbb{E}\big(|\widehat{d}_{QN,k''}^{\delta,n}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big) \Big) \Big| \mathcal{F}_{\zeta_{N,n}} \Big) \\
&= 2\big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-2} \sum_{k',k''\in\nu_{QN}(\mathcal{I}_{N,n})} \Big(\mathbb{E}\big(\widehat{d}_{QN,k'}^{\delta,n}\, \widehat{d}_{QN,k''}^{\delta,n}\big|\mathcal{F}_{\zeta_{N,n}}\big) \Big)^2 \\
&\leq 2c_1^2\big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-2} (QN)^{-4H(\zeta_{N,n})} \sum_{k',k''\in\nu_{QN}(\mathcal{I}_{N,n})} \big(1 + |k' - k''|\big)^{-2(L-\overline{H})} \\
&\leq c_2\big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-1} (QN)^{-4H(\zeta_{N,n})}, \qquad (4.27)
\end{aligned}$$

where c_1 denotes the constant c in (4.16) and $c_2 := 4c_1^2 \sum_{j=1}^{+\infty} j^{-2(L-\overline{H})} < +\infty$. Next, putting together (4.26), the first inequality in (4.10), (4.14), (4.27), the first inequality in (2.13) and (2.11), one gets, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$, that

$$\mathbb{E}\left(\left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \middle| \mathcal{F}_{\zeta_{N,n}} \right)} - 1 \right|^{2q} \right) \leq c_3 \left| \nu_{QN}(\mathcal{I}_{N,n}) \right|^{-q} \leq c_4 N^{-q(1-\mu')}, \tag{4.28}$$

where c_3 and c_4 are two deterministic finite constants not depending on Q, N and n. Then, one can derive from (4.25), (4.28) and (2.11) that, for all integers $Q \in \mathbb{N}$ and $N \geq N_0$,

$$\begin{aligned} &\mathbb{P}\left(N^{b(1-\mu')} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \middle| \mathcal{F}_{\zeta_{N,n}} \right)} - 1 \right| > 1 \right) \\ &\leq c_4 \, N^{2qb(1-\mu')} \theta_N^{-1} N^{-q(1-\mu')} \leq \frac{c_4}{\kappa'} \, N^{\mu' - q(1-\mu')(1-2b)}. \end{aligned} \tag{4.29}$$

Thus, it follows from (4.24) and (4.29) that

$$\sum_{N=N_0}^{+\infty} \mathbb{P}\left(N^{b(1-\mu')} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \middle| \mathcal{F}_{\zeta_{N,n}} \right)} - 1 \right| > 1 \right) < +\infty .$$

Then the Borel-Cantelli Lemma entails that one has almost surely

$$\sup_{N \geq N_0} \left\{ N^{b(1-\mu')} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \middle| \mathcal{F}_{\zeta_{N,n}} \right)} - 1 \right| \right\} < +\infty . \tag{4.30}$$

Finally, combining (4.23) and (4.30) one gets (4.4). ■

We are now ready to prove Lemma 4.1.

Proof of Lemma 4.1 First observe that, for all integers $Q \in \mathbb{N}$ and $N \geq N_0$, one

has

$$\begin{aligned}
&\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right| \\
&=\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right|\left|\sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}+1\right| \\
&\le\left(\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right|\right)^2 \\
&\qquad\qquad+2\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right|,
\end{aligned}$$

where δ is an arbitrary real number satisfying (4.2). Thus, in order to prove that (4.3) holds, it is enough to show almost surely that

$$\limsup_{N\to+\infty}\left\{N^{\beta}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right|\right\}=0\,. \tag{4.31}$$

Let us point out that throughout this proof β denotes an arbitrary fixed positive real number satisfying (4.1). Next observe that (2.4), (3.4), (3.24), (3.8), (3.21), (3.22) and the triangle inequality imply, for all integers $Q\in\mathbb{N}$, $N\ge N_0$ and $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$, that

$$\begin{aligned}
&\sqrt{\frac{\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-\sqrt{\frac{\breve{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-\sqrt{\frac{\overline{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} \\
&\le\sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} \\
&\le\sqrt{\frac{\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}+\sqrt{\frac{\breve{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}+\sqrt{\frac{\overline{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}
\end{aligned}$$

and consequently that

$$\begin{aligned}
&\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right| \\
&\le \max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right|+\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{\frac{\breve{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} \\
&\qquad\qquad+\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{\frac{\overline{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} \\
&\le \max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right|+c_1N^{\overline{H}}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{\breve{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})} \\
&\qquad\qquad+c_1\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{N^{2H(\zeta_{N,n})}\,\overline{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}\,, \qquad (4.32)
\end{aligned}$$

where c_1 is a deterministic finite constant not depending on N. Notice that the last inequality in (4.32) results from (4.10) and (1.8). It clearly follows from (4.1), (4.2), Lemma 3.2 and Lemma 3.7 that one has almost surely

$$\limsup_{N\to+\infty}\left\{N^{\beta+\overline{H}}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{\breve{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}\right\}=0$$

and

$$\limsup_{N\to+\infty}\left\{N^{\beta}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\sqrt{N^{2H(\zeta_{N,n})}\,\overline{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}\right\}=0\,.$$

Thus, in view of (4.32), in order to show that (4.31) holds, it is enough to prove that

$$\limsup_{N\to+\infty}\left\{N^{\beta}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\sqrt{\frac{\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\,\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}-1\right|\right\}=0\,. \qquad (4.33)$$

Combining (4.1) with Lemma 4.2 and the inequality $|\sqrt{z}-1|\le|z-1|$, for every $z\in\mathbb{R}_+$, one gets (4.33). ■

5 Final steps of the proof of Theorem 2.2

Lemma 5.1 *Let $\overline{H}$, γ, L and μ be as in (1.8), (1.10), (2.2) and (2.10). Let β be an arbitrary real number satisfying*

$$0 < \beta < \min\left\{\gamma\big((1-\delta)\wedge\mu\big),\, 2\delta(L-\overline{H})\right\}, \tag{5.1}$$

where $\delta \in (0,1)$ is arbitrary and fixed. Then, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,

$$\limsup_{N\to+\infty}\left\{N^{\beta}\sup_{s\in[0,1]}\left|\frac{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}-Q^{2H(s)}\right|\right\}=0\,, \tag{5.2}$$

where, for every $s \in [0,1]$,

$$n_0(N,s) := \begin{cases} \lfloor \theta_N^{-1} s\rfloor & \text{if } s \in \big[0,(\lfloor\theta_N^{-1}\rfloor-1)\theta_N\big), \\ \\ \lfloor\theta_N^{-1}\rfloor - 1 & \text{if } s \in \big[(\lfloor\theta_N^{-1}\rfloor-1)\theta_N, 1\big], \end{cases} \tag{5.3}$$

and

$$n_1(N,s) := \begin{cases} \lfloor \theta_N^{-1} s\rfloor + 1 & \text{if } s \in \big[0,(\lfloor\theta_N^{-1}\rfloor-1)\theta_N\big), \\ \\ \lfloor\theta_N^{-1}\rfloor - 1 & \text{if } s \in \big[(\lfloor\theta_N^{-1}\rfloor-1)\theta_N, 1\big]. \end{cases} \tag{5.4}$$

Proof of Lemma 5.1 One can derive from (4.9) and (1.8), that one has almost surely, for each real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$, that

$$\begin{aligned} &\left|\frac{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}-Q^{2H(s)}\right| \\ &= \left|Q^{2H(\zeta_{N,n_i(N,s)})}\frac{\int_{L-e_N}^{L}\big|\Phi\big(u,H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du}{\int_{L-e_{QN}}^{L}\big|\Phi\big(u,H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du}-Q^{2H(s)}\right| \\ &\leq U_{i,N}(s)+V_{i,N}(s)\,, \end{aligned} \tag{5.5}$$

where

$$U_{i,N}(s) := Q^{2\overline{H}}\left|Q^{2(H(\zeta_{N,n_i(N,s)})-H(s))}-1\right|\frac{\int_{L-e_N}^{L}\big|\Phi\big(u,H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du}{\int_{L-e_{QN}}^{L}\big|\Phi\big(u,H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du} \tag{5.6}$$

and

$$V_{i,N}(s) := Q^{2\overline{H}} \left| \frac{\int_{L-e_N}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du}{\int_{L-e_{QN}}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du} - 1 \right|$$
$$= Q^{2\overline{H}} \frac{\int_{L-e_{QN}}^{L-e_N} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du}{\int_{L-e_{QN}}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du}. \tag{5.7}$$

Next observe that, one can derive from the mean value Theorem, (1.8), (1.10), (3.20), (5.3), (5.4), (2.10) and (3.3) that

$$\begin{aligned} &\left|Q^{2(H(\zeta_{N,n_i(N,s)})-H(s))} - 1\right| \\ &\leq 2(\log Q) \exp\Big(2(\log Q)\overline{H}\Big) \left|H(\zeta_{N,n_i(N,s)}) - H(s)\right| \\ &\leq \rho(2\kappa+1)^{\gamma} \log(Q^2) Q^{2\overline{H}} N^{-\gamma((1-\delta)\wedge\mu)}. \end{aligned} \tag{5.8}$$

Moreover, it easily follows (3.3) that

$$\frac{\int_{L-e_N}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du}{\int_{L-e_{QN}}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du} \leq 1. \tag{5.9}$$

Thus, combining (5.6), (5.8) and (5.9), one gets, that, for all $N \geq N_0$,

$$\sup_{s\in[0,1]} U_{i,N}(s) \leq \rho(2\kappa+1)^{\gamma} \log(Q^2) Q^{4\overline{H}} N^{-\gamma((1-\delta)\wedge\mu)}. \tag{5.10}$$

Next observe that similarly to (4.12) it can be shown that, for all real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$, one has

$$\int_{L-e_{QN}}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du \geq (2\overline{H})^{-1}.$$

Thus, one can derive from (5.7) that, for all real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$,

$$V_{i,N}(s) \leq 2\overline{H} Q^{2\overline{H}} \int_{-\infty}^{L-e_N} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du. \tag{5.11}$$

Notice that there is no restriction to assume that $N \geq (3L)^{1/\delta} + N_0$ which implies that $L - e_N < -2L$. Then, using (3.15), one gets that

$$\begin{aligned} &\int_{-\infty}^{L-e_N} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du \\ &\leq c_1^2 \int_{-\infty}^{L-e_N} \big(1 + L - u\big)^{2\overline{H}-2L-1} du \leq \frac{c_1^2}{2(L-\overline{H})} N^{-2\delta(L-\overline{H})}, \end{aligned} \tag{5.12}$$

where c_1 denotes the finite and deterministic constant c in (3.15) which does not depend on N and $\zeta_{N,n_i(N,s)}$. Then combining (5.11) and (5.12), one obtains, for all integers $i \in \{0,1\}$ and $N \geq (3L)^{1/\delta} + N_0$, that

$$\sup_{s\in[0,1]} V_{i,N}(s) \leq 2\overline{H}Q^{2\overline{H}} \frac{c_1^2}{2(L-\overline{H})} N^{-2\delta(L-\overline{H})} . \tag{5.13}$$

Finally, putting together (5.1), (5.5), (5.10) and (5.13) it follows that (5.2) holds. ■

Lemma 5.2 *Let β be an arbitrary real number satisfying the condition (4.1), where δ is an arbitrary fixed real number satisfying the condition (4.2). Then, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,*

$$\limsup_{N\to+\infty} \left\{ N^{\beta} \sup_{s\in[0,1]} \left| \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} - Q^{2H(s)} \right| \right\} = 0 , \tag{5.14}$$

where $n_0(N,s)$ and $n_1(N,s)$ are as in (5.3) and (5.4).

Proof of Lemma 5.2 First observe that, for each real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$, one has

$$\begin{aligned}
&\left| \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} - Q^{2H(s)} \right| \\
&= \left| R_N^i(s) S_{Q,N}^i(s) \big(Z_{Q,N}^i(s)\big)^{-1} - Q^{2H(s)} \right| \\
&\leq Q^{2H(s)} \left| R_N^i(s) \big(Z_{Q,N}^i(s)\big)^{-1} - 1 \right| + \frac{R_N^i(s)}{Z_{Q,N}^i(s)} \left| S_{Q,N}^i(s) - Q^{2H(s)} \right| \\
&\leq Q^{2\overline{H}} \left| R_N^i(s) - 1 \right| + Q^{2\overline{H}} \frac{R_N^i(s)}{Z_{Q,N}^i(t)} \left| Z_{Q,N}^i(s) - 1 \right| + \frac{R_N^i(s)}{Z_{Q,N}^i(s)} \left| S_{Q,N}^i(s) - Q^{2H(s)} \right| ,
\end{aligned} \tag{5.15}$$

where

$$R_N^i(s) := \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)}) \big| \mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)} , \tag{5.16}$$

$$S_{Q,N}^i(s) := \frac{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)}) \big| \mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)}) \big| \mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)} \tag{5.17}$$

and

$$Z^i_{Q,N}(s) := \frac{V_{QN}(\mathcal{I}_{N,n_i(N,s)})}{\mathbb{E}\Big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}\,. \tag{5.18}$$

Recall that δ is an arbitrary fixed real number satisfying (4.2). Next notice that one knows from (4.1), (5.16), (5.18) and Lemma 4.1 that, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,

$$\sup_{s\in[0,1]} \Big|R^i_N(s) - 1\Big| = o\big(N^{-\beta}\big) \tag{5.19}$$

and

$$\sup_{s\in[0,1]} \Big|Z^i_N(s) - 1\Big| = o\big(N^{-\beta}\big)\,. \tag{5.20}$$

Moreover, it results from (5.19) that, almost surely,

$$\sup_{N\geq N_0}\sup_{s\in[0,1]} R^i_N(s) < +\infty\,, \tag{5.21}$$

and it follows from (5.20) that, almost surely,

$$\inf_{s\in[0,1]} Z^i_N(s) \geq 1/2\,, \quad \text{for all } N \text{ big enough.} \tag{5.22}$$

On the other hand, one knows from (4.1), (5.17) and Lemma 5.1 that, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,

$$\sup_{s\in[0,1]} \Big|S^i_N(s) - Q^{2H(s)}\Big| = o\big(N^{-\beta}\big)\,. \tag{5.23}$$

Finally, putting together (5.15) and (5.19) to (5.23) one obtains (5.14). ■

We are now in position to complete the proof of Theorem 2.2.

End of the proof of Theorem 2.2 One can derive from (2.14), (2.15), (2.8), (5.3), (5.4) and (1.8) that, for all integer $Q \geq 2$ and $N \geq N_0$, one has

$$\begin{aligned}&\sup_{s\in[0,1]} \big|H(s) - \widetilde{H}^Q_{N,\theta_N}(s)\big| \\ &\leq \sum_{i=0}^{1} \sup_{s\in[0,1]} \left|\log_{Q^2}\left(Q^{2H(s)}\right) - \log_{Q^2}\left(\frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})}\right)\right|\,.\end{aligned} \tag{5.24}$$

Next observe that one knows from (5.14) and (1.8) that, one has almost surely, for all N large enough,

$$\inf_{s\in[0,1]} \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} \geq 2^{-1}Q^{2\underline{H}}\,.$$

Thus, one can derive (5.24) and the mean value Theorem that one has almost surely, for all N large enough,

$$\sup_{s\in[0,1]} \left|H(s) - \widetilde{H}^{Q}_{N,\theta_N}(s)\right| \leq \frac{2Q^{-2\underline{H}}}{\log(Q^2)} \sum_{i=0}^{1} \sup_{s\in[0,1]} \left| Q^{2H(s)} - \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} \right| . \tag{5.25}$$

Then, (5.25) and Lemma 5.2 imply that (2.16) holds. ■

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