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Killing vector fields for some meromorphic affine connections

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Abstract

We give a geometric condition on a meromorphic affine connection for its Killing vector fields to be single valued. More precisely, this condition relies on the pole of the connection and its geodesics, and defines a subcategory. To this end, we use the equivalence between these objects and meromorphic affine Cartan geometries. The proof of the previous result is then a consequence of a more general result linking the distinguished curves of meromorphic Cartan geometries, their poles and their infinitesimal automorphisms. This enables to extend the result of [1] to the subcategory of meromorphic affine connection described before.

Keywords: Meromorphic affine connections, Killing vector fields, Cartan geometries

2010 MSC Classification: 53B21 , 53C56 , 53A55

1 Introduction

1.1 Geometric structures and Cartan geometries

A class of smooth geometric structures on real manifolds, or holomorphic geometric structures on complex manifolds (see [2] for a modern definition) is obtained as infinitesimal versions of a model geometry. As an example in the smooth category, the notion of Riemannian metric is obtained as an infinitesimal version of Euclidean geometry, and affine connections as infinitesimal versions of affine geometry. These two classes of geometric structures were

intensively studied, in particular by Riemann who initiated with Gauss the *Riemannian geometry*.

In the two examples above, we remark that the group of global *automorphisms* of the model geometry, i.e global transformations preserving the caracteristics of this geometry, acts transitively on the base space, namely $\mathbb{R}^n$. We say that the geometric structure corresponding to the model geometry is *homogeneous*. This property was later proposed by Klein to give a definition of a geometry, in his famous program aimed at classifying all the geometries. A *Klein geometry* is a couple (G, P) formed by a Lie group G, seen as the group of global automorphisms of the geometry, and a closed Lie subgroup P seen as the subgroup of isotropy at a fixed point of the space G/P.

Geometric structures underlying a Klein geometry are of diverse kinds. A general fact is that the model space G/P is endowed with a Q-*structure* where Q is a linear subgroup naturally associated with P. The geometric structures obtained in this way are of order one, but some Klein geometries define higher order geometric structures. For example, for the affine Klein geometry, where G is the affine group of $\mathbb{R}^n$ and P the linear subgroup, the group G is exactly the group of global automorphisms for the canonical flat affine connection on $\mathbb{R}^n$. In general, the geometric structure underlying (G, P) is defined using the P-principal bundle $G \longrightarrow G/P$ and the Maurer-Cartan form ω_G of G.

In a series of papers, in particular [3], Cartan described affine connections as infinitesimal versions of the affine Klein geometry, and proposed to generalize this principle to any Klein geometry to obtain a *Cartan geometry*. The formalism used nowadays came, in affine case, from the works of Ehresmann, who gave a purely geometric definition of an affine connection in terms of a principal bundle, a *principal connection* ([4]), and a soldering form to give a geometric meaning to the principal bundle. There is also an equivariant definition which was proposed by Atiyah in [5], and which is useful to extend some results in the meromorphic setting (see subsection 5.2). The definition of a Cartan geometry on M modelled on (G, P) that will be adopted in this article is the following: a couple (E, ω) formed by a P-principal bundle over M and a $\mathfrak{g}$-valued equivariant one form on E mimicking the infinitesimal properties of the Maurer-Cartan form of G. It originates from the paper [6].

In this way, the principle constructing a geometric structure on G/P from a Klein geometry (G, P) can be generalized to Cartan geometries, except that it produces non-homogeneous geometric structures in general: the global *automorphisms* of the Cartan geometry doesn't act transitively on the base manifold.

1.2 Classification of quasi-homogeneous geometric structures on simply connected manifolds

A natural question then, going back to the work of Riemann, Hopf and Killing for Riemannian metrics (see for example [7]), is to classify *locally homogenous* geometric structures, i.e for which the *infinitesimal automorphisms* span the

tangent space of the base manifold at any point. As an example, it is well-known that any locally homogeneous and complete Riemannian metric on a simply connected manifold is homogeneous.

The above question is more relevant in the holomorphic category, for two principal reasons. First, the existence of a holomorphic geometric structure on a complex compact manifold gives some restrictions on its geometry or its topology. Secondly, local homogeneity is sometimes deduced from the complex geometry of the base manifold, at least on an open dense subset. These two reasons are well illustrated by the holomorphic version of Riemannian metrics, i.e holomorphic fields of nondegenerate bilinear forms on the tangent spaces of a complex manifold (see for example [8],[9]). Indeed, on a general complex manifold, such an object gives a trivialisation of some power of the canonical bundle. On a compact complex surface, the curvature of such a object is a constant function, implying local homogeneity.

In dimension two, we can also mention the work by Inoue, Kobayashi and Ochiai in [10]. Using the vanishing of the first two Chern classes of a complex compact surface in presence of a holomorphic affine connection, and the Enriques-Kodaira classification, they gave a complete classification of such objects. In particular, any compact complex surface admitting a holomorphic affine connection admits a flat holomorphic affine connection, which is thus locally homogeneous.

In [11], McKay showed that the existence of an arbitrary holomorphic Cartan geometry on a complex kähler manifold imposes relations on its Chern classes. In a common paper with Dumitrescu ([12]), they proved that a simply connected compact complex manifold, with *algebraic dimension* zero (i.e whose meromorphic functions are the constants), does not bear any holomorphic affine connection.

Dumitrescu gave in [13] a result in arbitrary dimension which implies that on compact complex manifolds with only constant meromorphic functions, any holomorphic Cartan geometry must be *quasi-homogeneous*, i.e locally homogeneous on an open dense subset. This was used in [1] by Biswas and the two previous authors to improve the above result:

Theorem 1 *Compact complex manifolds of algebraic dimension zero bearing a holomorphic Cartan geometry of algebraic type have infinite fundamental group.*

Two important facts are used in the proof. First, it is proved that any germ of infinitesimal automorphism of the Cartan geometry is the germ of a global infinitesimal automorphism: this is a generalization of a result by Nomizu ([14]) for analytic Riemannian metrics, and follows from the fact that the former objects form a local system on M. Hence, using the result by Dumitrescu mentioned above, there are $n = dim(M)$ independent germs of infinitesimal automorphisms at some point of M, extending as a family of n global holomorphic vector fields which are infinitesimal automorphisms. Next, it is proved that there exists such a family made of commuting vector fields:

hence, it integrates to a complex abelian Lie group L with an open dense orbit in M. The conclusion follow from detailed study of the geometry of such manifolds M, which implies that the Cartan geometry is flat.

1.3 The meromorphic setting: results and plan of the paper

In this paper, we consider the meromorphic generalization of the holomorphic geometric structures, in particular the *meromorphic affine connections*. In the meromorphic category, the two above facts no longer stand: infinitesimal automorphisms could be multivalued (see [15], example 3.8). Moreover, meromorphic single valued infinitesimal automorphism may not have a well defined flow at some point of the pole.

We give a sufficient condition on some meromorphic Cartan geometries to recover the first fact. Let's explain briefly the condition. The holomorphic P-principal bundle E of a meromorphic Cartan geometry (E, ω_0) on a pair (M, D), modelled on a complex Klein geometry (G, P), comes equipped with holomorphic foliations (possibly singular) $\mathcal{T}_A$ whose leaves are the A-distinguished curves. Such a leaf $\tilde{\Sigma}$ is a 1-dimensional complex submanifold of E whose tangent directions are seen as of constant direction through ω_0, i.e in the line $\mathbb{C}A$ spanned by a vector $A \in \mathfrak{g}$. In the case of the affine model, it is natural to consider these leaves because their projections Σ on the base manifold M are exactly the spirals of the corresponding meromorphic affine connection (see [16]).

Moreover, there is a well-known result in Riemannian geometry stating that any Killing vector field X for a Riemannian metric g is a Jacobi field, i.e for any geodesic Σ of ∇, the scalar product $g(\gamma'(t), X(\gamma(t)))$ is constant along γ. We can translate the two objects in terms of the Cartan geometry (E, ω) corresponding to the Levi-Civita connection of g. The proof can then be recovered from the fact that (E, ω) is torsionfree (see **??**) and from the structure of the Lie algebra $\mathfrak{g}$ of the complex euclidean group G.

In the meromorphic setting, and for an arbitrary model, we prove the result below, where $\mathcal{V}$ stands for the sheaf of holomorphic functions on E with values in $\mathfrak{g}$ and, for any $A \in \mathfrak{g}$, $\mathcal{V}_A$ the subsheaf of functions with values in $\mathbb{C}A$.

Theorem 1 *Let (G, P) be a complex Klein geometry, and (M, D) be a pair with $dim(M) = dim(G/P)$. Let (E, ω_0) be a meromorphic (G, P)-Cartan geometry on (M, D). Pick a point $x_0 \in D$ belonging to an unique irreducible component D_α, and suppose that there exists $A \in \mathfrak{g} \setminus \mathfrak{p}$ and a the projection $\overline{\Sigma}$ of a A-distinguished curve for (E, ω_0) such that $\overline{\Sigma} \cap D_\alpha = \{x_0\}$. Then there exists a neighborhood U of x_0 with the following properties:*

1. *Let s be a the image of a Killing vector field through ω_0, on an open subset $V \subset p^{-1}(U)$. Then its class in $\mathcal{V}/\mathcal{V}_A$ extends as a single valued section of this sheaf on $p^{-1}(U \setminus D)$.*
2. *The class mentionned in 1. is moreover a section of $\mathcal{V}/\mathcal{V}_A[\star\tilde{D}]$ on $p^{-1}(U)$.*

3. *Suppose moreover that (E, ω_0) is holomorphic branched on (M, D) and let s be as in 2. Then s is in fact a section of $\mathcal{V}/\mathcal{V}_A$ on $p^{-1}(U)$.*

The meromorphic Cartan geometries satisfying this condition of the above theorem for a generic point $x_0 \in D$ on the pole are said to be *totally geodesic* in reference to the affine case. This theorem implies that Killing fields of meromorphic affine connections which are totally geodesic are single valued and meromorphic. If we restrict us to the subcategory of *branched holomorphic affine connections*, i.e those arising from *branched holomorphic Cartan geometries* (see [12]), we obtain that the Killing vector fields may be seen as holomorphic sections of submodule $\mathcal{E} \subset TM[\star D]$ satisfying $TM \subset \mathcal{E}$. Using this fact and some results in complex geomety, we obtained the following partial generalization of Theorem 1:

Theorem 2 *Let M be a compact complex manifold with finite fundamental group, and whose meromorphic functions are constants. Then M doesn't bear any totally geodesic branched holomorphic affine connection.*

The plan of the paper is as follow. In section 2, we recall the dictionnary between locally free modules of finite rank and vector bundles, the corresponding meromorphic sections, and recall the definition of Atiyah's exact sequence associated with a principal bundle. In section 3, we introduce meromorphic Cartan geometries and the holomorphic vector bundles naturally associated to these objects. In section 4, we give sufficient conditions for two classes of regular meromorphic *parabolic* geometries (subsection 4.1) for their infinitesimal automorphisms to be single valued. In section 5, we prove the equivalence between meromorphic affine Cartan connections and meromorphic affine connections, and we introduce the τ-connections which are always induced by a meromorphic affine Cartan geometry satisfying the sufficient condition described before. We then use the previous results to prove Theorem 2. Finally, in the last section, we discuss the genericity of the τ-connections, with some examples on compact complex manifolds.

2 Preliminaries and notations

This preliminary section is devoted to recall the notion of meromorphic connections on a locally free module, and the meromorphic version of Atiyah's exact sequence associated with a principal bundle.

2.1 Locally free modules and meromorphic connections

Let (M, D) be a *pair*, i.e a complex manifold M equipped with a divisor D, we denote by $\mathcal{O}_M$ the sheaf of holomorphic functions on M and $\mathcal{M}_M$ the sheaf of meromorphic functions on it. In order to write statements about meromorphic

objects with poles at D, we may use the sheaf $\mathcal{O}_M[\star D]$ of meromorphic functions with poles supported on the irreducible components D_α of $D = \sum_\alpha n_\alpha D_\alpha$ (see [17]). Let $\mathcal{L}$ be a coherent $\mathcal{O}_M$-module. Then we can consider the sheaf:

$$\mathcal{L}[\star D] = \mathcal{O}_M[\star D] \underset{\mathcal{O}_M}{\otimes} \mathcal{L} \tag{1}$$

of meromorphic sections of $\mathcal{L}$ with poles supported at the irreducible components of D. The *order* $ord_D^{\mathcal{L}}(s)$ at D of a section s of $\mathcal{L}[\star D]$ defined on an open subset $U \subset M$ is the greatest integer $d \in \mathbb{Z}$ such that s is also a section of $\mathcal{L}(-dD)$ on U.

Definition 1 A *meromorphic connection* on (M, D) is a couple $(\mathcal{V}, \nabla)$ where $\mathcal{V}$ is a locally free $\mathcal{O}_M$-module of finite rank, and ∇ is a morphism of $\mathbb{C}$-sheaves from $\mathcal{V}[\star D]$ to $\Omega_M^1 \otimes \mathcal{V}[\star D]$ satisfying the Leibniz identity $\nabla(fs) = d(f)s + f\nabla(s)$ for any $s \in \mathcal{V}(U)$ and $f \in \mathcal{O}_M[\star D](U)$ (U is an open subset of M).

If $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla)$ are two meromorphic connections such that $\mathcal{L} = \bigoplus_{i=1}^r \mathcal{O}_M s_i$, $\mathcal{L}' = \bigoplus_{i=1}^r \mathcal{O}_M t_i$ and $t_i = \sum_{j=1}^r q_{ji} s_j$ for a meromorphic matrix Q on M, then the matrices A and A' respectively associated to the basis $(s_i)_{i=1,\dots,r}$ and $(t_i)_{i=1,\dots,r}$ are linked by the gauge-transformation formula:

$$A' = Q^{-1} dQ + Q^{-1} A Q \tag{2}$$

where d stands for the de Rham derivative.

A *meromorphic affine connection* on (M, D) is a meromorphic connection ∇ on TM with poles supported at D. The *torsion* of a meromorphic affine connection ∇ on (M, D) is the meromorphic section T_∇ of $\Omega_M^1 \otimes End(TM)$ defined by:

$$T_\nabla(X)(Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y]_{TM} \tag{3}$$

Now, let describe the categories associated with the above objects. Let M, M' be two complex manifolds and $f : M \longrightarrow M'$ be a holomorphic map. We denote by

$$f^{-1}\mathcal{O}_{M'} \tag{4}$$

the pullback in the sheaf theoretic sense. This is a sheaf of algebras over the constant sheaf $\underline{\mathbb{C}}_M$. Then $\mathcal{O}_M$ is naturally a $f^{-1}\mathcal{O}_{M'}$-algebra through the $\underline{\mathbb{C}}_M$-algebras morphism:

$$\begin{array}{rccc} f_\# : & f^{-1}\mathcal{O}_{M'} & \longrightarrow & \mathcal{O}_M \\ & s & \mapsto & s \circ f \end{array} \tag{5}$$

Hence, any $f^{-1}\mathcal{O}_{M'}$-module defines a $\mathcal{O}_M$-module obtained by tensorizing with $\mathcal{O}_M$ in the category of $f^{-1}\mathcal{O}_{M'}$-modules.

Recall that there is a well-known equivalence between the category of holomorphic vector bundles V of rank $r \geq 1$ over complex manifolds and local free $\mathcal{O}_M$-modules of the same rank obtained by considering the sheaf of local holomorphic sections of V. The image of a an isomorphism $\hat{\Psi} : V_1 \longrightarrow V_2$ of vector bundles with associated sheaves of sections $\mathcal{E}_1, \mathcal{E}_2$, covering an isomorphism $f : M \longrightarrow M'$ of complex manifolds, is the isomorphism Φ of $\mathcal{O}_M$-modules:

$$\begin{array}{rcl} \Phi : \mathcal{E}_1 & \longrightarrow & \mathcal{O}_M \underset{f^{-1}\mathcal{O}_{M'}}{\otimes} f^{-1}\mathcal{E}_2 \\ s & \longrightarrow & \hat{\Psi} \circ s \circ \varphi^{-1} \end{array} \tag{6}$$

where U is an open subset of M.

Definition 2 A couple (φ, Φ) as above will be called a *isomorphism of vector bundles* between $\mathcal{E}_1$ and $\mathcal{E}_2$. More generally, we define a *isomorphism of meromorphic bundles* by replacing the sheaves of holomorphic sections by the corresponding of meromorphic sections with poles at a divisor D of M and D' of M'.

The following definition of the pullback of a meromorphic connection is a particular case of the construction of inverse images for D-modules (see [18]). Let ∇' be a meromorphic connection on the $\mathcal{O}_{M'}$-module $\mathcal{E}'$ with poles at D' and $f : M \longrightarrow M'$ be a morphism of complex manifolds. We obtain a holomorphic connection $f^\star\nabla'$ on the module $\mathcal{E} = \mathcal{O}_M \underset{f^{-1}\mathcal{O}_{M'}}{\otimes} f^{-1}(\mathcal{E}'[\star D'])$ defined by the following commutative diagram:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "(f,\phi)^\star\nabla'"] \arrow[d, "\phi"'] & \Omega^1_M \otimes \mathcal{E}[\star D] \\ \mathcal{O}_M \underset{f^{-1}\mathcal{O}_{M'}}{\otimes} f^{-1}(\mathcal{E}'[\star D']) \arrow[r, "\overline{f^{-1}\nabla'}"'] & f^{-1}\Omega^1_{M'} \otimes f^{-1}(\mathcal{E}'[\star D']) \arrow[u, "(df)^*\otimes\phi^{-1}"'] \end{tikzcd} \tag{7}$$

where:

- $\overline{f^{-1}\nabla'}$ is the $\underline{\mathbb{C}}_M$-linear morphism obtained by extending $f^{-1}\nabla'$ by the Leibniz rule for $\mathcal{O}_M$.
- $(df)^* : \mathcal{O}_M \otimes f^{-1}\Omega^1_{M'} \longrightarrow \Omega^1_M$ is the $\mathcal{O}_M$-linear morphism induced by (5) (see [19]).

Suppose now the existence of a divisor D of M and of an isomorphism (f, ϕ) of meromorphic bundles (see above) between $\mathcal{E}[\star D]$ and $\mathcal{O}_M \underset{f^{-1}\mathcal{O}_{M'}}{\otimes} f^{-1}(\mathcal{E}'[\star D'])$. Using this identification, Definition 3 defines a meromorphic connection $(f, \phi)^\star\nabla'$ on $\mathcal{E}$ with poles at D.

Definition 3 Let (f, ϕ) and ∇' be as above. The meromorphic connection $(\mathcal{E}, (f, \phi)^\star\nabla')$ on (M, D) is called the *pullback* of $(\mathcal{E}', \nabla')$ through (f, ϕ).

As an important example, suppose that there is a submersion of complex n-manifolds $f : M \longrightarrow M'$ and denote by D the divisor of the zeroes of $\bigwedge^n (df)^*$ where $(df)^*$ is the morphism defined in (7), seen as an element of $H^0(M, Hom_{\mathcal{O}_M}(TM, \mathcal{O}_M \underset{f^{-1}\mathcal{O}_{M'}}{\otimes} f^{-1}TM'))$. Denote also D' its image through f, as a subvariety of M. Then $(f, (df)^*)$, TM and $\mathcal{O}_M \underset{f^{-1}\mathcal{O}_{M'}}{\otimes} f^{-1}TM'$ are as in Definition 3, and starting with any meromorphic affine connection ∇' on (M', D'), we obtain a meromorphic affine connection on (M, D) denote by:

$$f^\star \nabla' \tag{8}$$

We finish with the Riemann-Hilbert correspondance. A *flat* meromorphic connection ∇ on $\mathcal{E}$ with poles at D is a meromorphic connection such that the subsheaf of *horizontal sections* $ker(\nabla)$ on $M \setminus D$ defined by:

$$\forall U \subset M \setminus D,\ ker(\nabla)(U) = \{s \in \mathcal{E}(U)\, s.t.\, \nabla(s) = 0\} \tag{9}$$

is a *local system* (see [20]).

We recall that there is an equivalence of categories between the category of local systems of rank r on $M \setminus D$ with arrows being the isomorphisms, and the category of representations $\rho : \pi_1(M \setminus D, x) \longrightarrow K$ (for any $x \in M \setminus D$, and K is a $\mathbb{C}$-vector space of dimension r) with arrows being the isomorphisms of representations. Once a point $x \in M \setminus D$ is chosen, this equivalence is obtained by associating to any local system $\mathcal{K}$, the *monodromy map* $Mon^x(\mathcal{K}) : \pi_1(M \setminus D, x) \longrightarrow Aut(\mathcal{K}_x)$ (see [20]).

2.2 Atiyah sequence of the frame bundle

The *frame bundle* of a locally free $\mathcal{O}_M$-module $\mathcal{E}$ of rank r is the holomorphic $GL_r(\mathbb{C})$-principal bundle $E \xrightarrow{p} M$ whose fiber at $x \in M$ is the set of isomorphisms $\mathbb{C}^r \simeq \mathcal{E}(x)$. Here $\mathcal{E}(x) = \mathcal{E}_x/\mathfrak{m}_x$ stands for the fiber of $\mathcal{E}$ at x.

We recall that for any complex Lie group P and a holomorphic P-principal bundle $E \xrightarrow{p} M$, there is a notion of *P-linearization* for a $\mathcal{O}_E$-module $\mathcal{V}$: this is a family $(\phi_b)_{b\in P}$ of isomorphisms $\phi_b : \mathcal{V} \simeq r_b^*\mathcal{V}$ (where r_b is the right action of P) with nice properties (see [21]). A $\mathcal{O}_E$-module equipped with a P-linearization is said to be *P-equivariant*. In this context, there is an equivalence between the P-equivariant locally free $\mathcal{O}_E$-modules and the locally free $\mathcal{O}_M$-modules, and between the P-equivariant morphisms and the morphisms between the corresponding $\mathcal{O}_M$-modules (see [22]) . For any representation $\rho : P \longrightarrow GL(\mathbb{V})$, and any holomorphic P-principal bundle $E \xrightarrow{p} M$, we denote by $E(\mathbb{V})$ the $\mathcal{O}_M$-module associated with the $\mathcal{O}_E$-module $\mathcal{O}_E \otimes \mathbb{V}$, where the P-linearization $(\phi_b)_{b\in P}$ is given by $\phi_b = r_b^* \otimes \rho(b^{-1})$. We call it the *representation module* associated with E and $\mathbb{V}$. For any isomorphism $\Psi : E \longrightarrow E'$ of holomorphic P-principal bundles covering $\varphi : M \longrightarrow M'$, the *representation isomorphism of associated vector bundles* corresponding to Ψ

is the isomorphism

$$\Psi(\mathbb{V}) : E(\mathbb{V}) \longrightarrow \varphi^* E'(\mathbb{V}) \tag{10}$$

associated to the P-equivariant isomorphism $\Psi^* \otimes Id_{\mathbb{V}}$ of trivial $\mathcal{O}_E$-modules.

Definition 4 Let $\mathbb{V}$ be a representation of a complex Lie group P. Let $E \xrightarrow{p_1} M$ and $E' \xrightarrow{p_1} M'$ be two holomorphic P-principal bundles, and D, D' be respectively two divisors of M and M'.

1. An isomorphism $\Psi : E|_{M\setminus D} \longrightarrow E'|_{M'\setminus D'}$ of holomorphic P-principal bundles is $\mathbb{V}$-*meromorphic* between (M, D) and (M', D') iff the representation isomorphism $\Phi = \Psi(\mathbb{V})$ restricts to an isomorphism $\Phi : E(\mathbb{V})[\star D] \longrightarrow \varphi^* E'(\mathbb{V})[\star D']$ (see 10).
2. A $\mathbb{V}$-*meromorphic section* of a holomorphic P-principal bundle $E \xrightarrow{p} M$ on U with pole at D is a holomorphic section $\sigma : U \setminus D \longrightarrow E$ such that the corresponding trivialisation ψ_σ of $E(\mathbb{V})|_{U\setminus D}$ induces an isomorphism of meromorphic bundles between $E(\mathbb{V})$ and $\mathcal{O}_U \otimes \mathbb{V}$.

In particular, mapping holomorphic $GL_r(\mathbb{C})$-principal bundles E over M to the associated representation modules $E(\mathbb{C}^r)$ gives an equivalence of categories. A pseudo-inverse is given by mapping a locally free $\mathcal{O}_M$-module $\mathcal{E}$ of rank r to its frame bundle E.

Consider $\mathfrak{p} = Lie(P)$ which is the adjoint representation of P. Let $At(E)$ be the $\mathcal{O}_M$-module associated with the P-equivariant locally free $\mathcal{O}_E$-module TE equipped with the P-linearization induced by the infinitesimal action of P on E: it is called the *Atiyah bundle* of E, and fits into the short exact sequence:

$$0 \longrightarrow E(\mathfrak{p}) \xrightarrow{\iota} At(E) \xrightarrow{q} TM \longrightarrow 0 \tag{11}$$

where ι is the morphism associated with the P-equivariant morphism which to any $A \in \mathcal{O}_E \otimes \mathfrak{p}$ associates the corresponding fundamental vector field on E, and q is the one associated with the P-equivariant morphism $dp : TE \longrightarrow p^*TM$.

The previous equivalence implies that P-equivariant meromorphic one forms on E, with poles at $\tilde{D} = p^{-1}(D)$, and values in $\mathbb{V}$ are in bijection with morphisms $\beta : At(E)[\star D] \longrightarrow E(\mathbb{V})[\star D]$, or equivalently with sections of $At(E) \otimes E(\mathbb{V})[\star D]$. This correspondance restricts to a bijective correspondance between:

- The set of morphisms β as above vanishing on the image of ι in (11), equivalently sections of $\Omega^1_M[\star D] \otimes E(\mathbb{V})$
- The set of meromorphic one forms $\tilde{\omega}$ on $(E, \tilde{D})$ with values in $\mathbb{V}$ vanishing on $ker(dp)$

3 Holomorphic branched Cartan geometries and the Killing connection

In this section, we fix a pair (M, D) where M is of complex dimension n. We define meromorphic Cartan geometries, and the subcategory of branched holomorphic Cartan geometries. We describe their infinitesimal automorphisms as sections for a meromorphic connection either on a trivial module over the principal bundle of the geometry, or on the corresponding module over the base manifold. We introduce the subcategory of *totally geodesic* meromorphic Cartan geometries: in the next section, we will see that their infinitesimal automorphisms are single valued, in a sense that will be defined.

3.1 Meromorphic and holomorphic branched Cartan geometries

First, we have to define the models for Cartan geometries:

Definition 5 A *complex Klein geometry* of dimension $n \geq 1$ is a couple (G, P) where G is a complex Lie group, and P is a complex Lie subgroup with $dim(G) - dim(P) = n$.

Let (G, P) be as in Definition 5 and let $P' = ker(\underline{ad})$ where $\underline{ad} : P \longrightarrow GL(\mathfrak{g}/\mathfrak{p})$ is the representation induced by the adjoint representation. Then any choice of a basis for $\mathfrak{g}/\mathfrak{p}$ identifies $Q = P/P'$ with a linear complex subgroup, and TG/P with the module $G(\mathfrak{g}/\mathfrak{p})$ associated to the P-principal bundle E and the representation $\mathfrak{g}/\mathfrak{p}$. Thus, the complex manifold G/P comes equiped with a holomorphic reduction $G \underset{P}{\times} Q$ of its holomorphic frame bundle $R^1(G/P)$, i.e a holomorphic Q-structure: namely G/P'.

This is in fact only due to the presence of a $\mathfrak{g}$-valued holomorphic 1-form with special properties on the total space of the holomorphic P-principal bundle $G \longrightarrow G/P$, namely the *Maurer-Cartan form* ω_G of G. We can consider curved versions of theses objects for which the above fact is still true replacing G by a suitable holomorphic P-principal bundle (see next subsection). Authorizing the one form to have poles on the P-principal bundle, we obtain their meromorphic analogues:

Definition 6 Let (G, P) be a complex Klein geometry with $dim(G/P) = n$ and (M, D) be a pair. A *meromorphic* (G, P)*-Cartan geometry* is a couple (E, ω_0) where $E \xrightarrow{p} M$ is a holomorphic P-principal bundle, and ω_0 is a $\mathfrak{g}$-valued meromorphic one form on E, with poles on $\tilde{D} = p^{-1}(D)$, such that:

(i) For any $x \in M \setminus D$, $\iota_x^\star \omega_0$ coïncides with the Maurer-Cartan form $\omega_{E,x}$ (see above).

(ii) ω_0 is P-equivariant.

(iii) For any $e \in E \setminus \tilde{D}$, $\omega_0(e)$ is an isomorphism between T_eE and $\mathfrak{g}$.

These objects form a category:

Definition 7 Let (G, P) be a complex Klein geometry and (M, D),(M', D') be two pairs with $dim(M) = dim(M') = dim(G/P)$. Let (E, ω_0) and (E', ω_0') be respectively two meromorphic (G, P)-Cartan geometries on (M, D) and (M', D'). An *isomorphism* between (E, ω_0) and (E', ω_0') is an isomorphism of $\mathfrak{g}$-meromorphic P-principal fiber bundles $\Psi : E \backslash \tilde{D} \overset{\sim}{\to} E' \backslash \tilde{D}'$ (see Definition 4) such that $\Psi^\star \omega_0' = \omega_0$.

The following object is central in the study of Cartan geometries:

Definition 8 Let (E, ω_0) be a meromorphic (G, P)-Cartan geometry on (M, D). Its *curvature function* (or *curvature*) is the meromorphic function k_{ω_0} on E with values in $\mathbb{W} = \mathfrak{g}^* \wedge \mathfrak{g}^* \otimes \mathfrak{g}$ and defined by:

$$k_{\omega_0} = d\omega_0 \circ (\omega_0^{-1} \wedge \omega_0^{-1}) + [,]_{\mathfrak{g}} \tag{12}$$

where $[,]_{\mathfrak{g}}$ is the Lie-bracket of $\mathfrak{g}$ identified with an element of $\mathbb{W}$.

Fix a Klein geometry (G, P) and choose a basis $(\mathfrak{e}_i)_{i=1,\dots,N}$ of $\mathfrak{g}$, with $(\mathfrak{e}_i)_{i=1,\dots,n}$ spanning a subspace $\mathfrak{g}_-$ complementary to $\mathfrak{p}$. Denote by $(\mathfrak{e}_i^*)_{i=1,\dots,N}$ the dual basis of $\mathfrak{g}^*$.

Definition 9 Let (E, ω_0) be a meromorphic (G, P)-Cartan geometry on (M, D). The meromorphic functions:

$$\gamma_{i,j}^k = \mathfrak{e}_k^* \circ k_{\omega_0}(\mathfrak{e}_i, \mathfrak{e}_j) \tag{13}$$

are called the *structure coefficients* (or *structure functions*) of (E, ω_0).

A natural subcategory of the meromorphic (G, P)-Cartan geometries on pairs is the following:

Definition 10 A *branched holomorphic* (G, P)*-Cartan geometry* on a pair (M, D) is a meromorphic (G, P)-Cartan geometry (E, ω_0) on (M, D) such that ω_0 extends as a holomorphic one form on E.

An important feature of these objects for the classification is the existence of a holomorphic connection on the adjoint vector bundle. Indeed, let (E, ω_0) be a branched holomorphic (G, P)-Cartan geometry on (M, D), and $E_G = E \underset{P}{\times} G$ the extension of the holomorphic P-principal bundle E to the group G. By definition, E_G is the quotient of the product $E \times G$ by the action of G given by $(e, g) \cdot h = (e \cdot h, h^{-1} g)$. Consider the G-equivariant holomorphic one form $\overline{\omega}$ on $E \times G$ with values in $\mathfrak{g}$ given by:

$$\overline{\omega} = ad(\pi_2) \circ \pi_1^\star \omega_0 + \pi_2^\star \omega_G \tag{14}$$

where π_1, π_2 are the projections on each factor and ω_G is the Maurer-Cartan form of G. It is straightforward to verify that for any $A \in \mathfrak{g}_0$, $\overline{\omega}(\frac{d}{dt}|_{t=0}(e,h) \cdot exp_G(tA)) = 0$, i.e the vectors tangent to the fibers of

$$\pi_G : E \times G \longrightarrow E_G \tag{15}$$

are in the kernel of $\overline{\omega}$. Moreover, $\overline{\omega}$ is invariant under the action of G on $E \times G$. Thus, $\overline{\omega}$ induces a holomorphic one form on E_G.

Definition 11 The holomorphic G-principal connection $\tilde{\omega}$ on E_G induced by $\overline{\omega}$ is the *tractor-connection* of (E, ω_0). We denote by ∇^{ω_0} the corresponding holomorphic connection on $E_G(\mathfrak{g}) = E(\mathfrak{g})$ (see Theorem 14).

The pullback $p^*E(\mathfrak{g})$ is the trivial module $\mathcal{V} = \mathcal{O}_E \otimes \mathfrak{g}$.

Lemma 1 *The pullback $p^\star\nabla^{\omega_0}$ is $d - Ad(\omega_0)$ where d is the de Rham diffential on the trivial module $\mathcal{V}$, $Ad(\omega_0)$ is the section of $\Omega^1_E \otimes End(\mathfrak{g}) = \Omega^1_E \otimes End(\mathcal{V})$ defined by:*

$$X \lrcorner Ad(\omega_0)(s) = [\omega_0(X), s]_{\mathfrak{g}}$$

for any holomorphic vector field X of E and section s of $\mathcal{V}$. In particular, its curvature is $R_{p^\star\nabla^{\omega_0}} = Ad(d\omega_0) + Ad(\omega_0 \wedge \omega_0)$.

Proof Since the ω_0-constant vector fields on E span T_eE at any $e \in E \setminus \tilde{D}$, we can choose $\tilde{A} = \omega_0^{-1}(A)$ for $A \in \mathfrak{g}$ as a holomorphic vector field on $E \setminus \tilde{D}$. Let s be any section of $\mathcal{V}(U)$, $U \subset E \setminus \tilde{D}$ an open subset. By definition of ∇^{ω_0} and the remarks preceding Definition 11, we have:

$$\tilde{A} \lrcorner p^\star\nabla^{\omega_0}(s) = (\overline{A} - \hat{A}) \lrcorner d(\tilde{s})$$

where $\tilde{s}$ is the unique G-equivariant section of $\mathcal{O}_{E\times G} \otimes \mathfrak{g}$ which coïncides with s in restriction to $E \subset E \times G$, $\overline{A}$ is the unique G-invariant meromorphic vector field whose restriction to E coïncides with $\tilde{A}$, and $\hat{A}$ is the holomorphic vector field tangent to the fibers of $E \times G \xrightarrow{\pi_1} E$ such that $\pi_2^\star\omega_G(\hat{A}) = A$. Indeed, $\overline{A} - \hat{A}$ is the unique vector field which belongs to $ker(\overline{\omega})$ and projects to $\tilde{A}$ via $\pi_1 : E \times G \longrightarrow E$. Now, $\overline{A} \lrcorner d(\tilde{s})$ coïncides with $\tilde{A} \lrcorner d(s)$ in restriction to E, while $\hat{A} \lrcorner d(\tilde{s}) = [A, s]_{\mathfrak{g}}$ because $\tilde{s}$ is G-equivariant. The first formula follows. For the curvature, it corresponds to the classical computation of the curvature in a trivialisation of a vector bundle. □

3.2 Meromorphic extension of the tangent sheaf

We now describe an object induced by any meromorphic Cartan geometry, which plays the same role as the tangent bundle of the base manifold in the regular case. It is a particular case of the following objects:

Definition 12 Let (M, D) be a pair.

1. A meromorphic extension of (M, D) is a couple $(\phi_0, \mathcal{E})$ where $\mathcal{E}$ is a locally free $\mathcal{O}_M$-module and $\phi_0 : TM[\star D] \longrightarrow \mathcal{E}[\star D]$ is an isomorphism of $\mathcal{O}_M$-modules.
2. A holomorphic extension of (M, D) is a meromorphic extension $(\phi_0, \mathcal{E})$ such that $\phi_0(TM) \subset \mathcal{E}$.
3. The category $\mathcal{F}$ (resp. $\mathcal{F}^0$) of meromorphic extensions (resp. holomorphic extensions) over pairs is defined as follow. An arrow between two meromorphic extensions $(\phi_0, \mathcal{E})$ and $(\phi_0', \mathcal{E}')$ over (M, D) and (M', D') is a an isomorphism (φ, Φ) of meromorphic bundles (resp. of vector bundles, see Definition 2) between $\mathcal{E}$ and $\mathcal{E}'$ such that the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "\phi_0"] \arrow[d, "d\varphi"'] & \mathcal{E}[\star D] \arrow[d, "\Phi"] \\ \varphi^* TM' \arrow[r, "\varphi^*\phi_0'"'] & \varphi^*\mathcal{E}'[\star D] \end{tikzcd} \tag{16}$$

4. The category obtained by restricting to meromorphic extensions of (M, D) and to isomorphisms of meromorphic bundles of the form (Id_M, Φ) is denoted by $\mathcal{F}_{M,D}$ (resp. $\mathcal{F}^0_{M,D}$).

Meromorphic extensions on (M, D) are thus canonically isomorphic to submodules of maximal rank of the sheaf of tangent vector fields with poles at D. The restriction of the corresponding frame bundle to $M \setminus D$ can thus be canonically identified with the frame bundle of $M \setminus D$. This gives the following alternative description:

Definition 13 Let (M, D) be a pair.

1. Let $E \xrightarrow{p} M$ be a holomorphic P-principal bundle and $\tilde{D} = p^{-1}(D)$. A *meromorphic solderform* on $(E, \tilde{D})$ is a P-equivariant $\mathbb{C}^n$-valued meromorphic 1-form θ_0 on E, with poles supported at $\tilde{D}$, vanishing on $ker(dp)$, and such that $\theta_0(e)$ is surjective for any $e \in E \setminus \tilde{D}$. A couple (E, θ_0) is called a meromorphic solder form over (M, D)
2. An arrow between two meromorphic solderforms (E, θ_0) and (E', θ_0') over (M, D) and (M', D') is an isomorphism of holomorphic P-principal bundles $\tilde{\Psi} : E \longrightarrow E'$ such that $\theta_0 = \tilde{\Psi}^\star \theta_0'$. This defines the category $\mathcal{D}$ of meromorphic solderforms over pairs.

Proposition 2 *The map which to any meromorphic solder form (E, θ_0) over (M, D) (Definition 13) associates the meromorphic extension $(\phi_0, \mathcal{E})$ where $\phi_0 : TM[\star D] \xrightarrow{\sim} \mathcal{E}[\star D]$ is the isomorphism which corresponds to θ_0 (see remarks above), extends to an equivalence of categories $m : \mathcal{D} \longrightarrow \mathcal{E}$.*

Proof If $\tilde{\Psi} : E \longrightarrow E'$ is an arrow between two objects (E, θ_0) and (E', θ_0') of the category of solderforms over (M, D), we define $m(\tilde{\Psi}) = \Phi$ as the image of $\tilde{\Psi}$ through the equivalence of categories described in subsection 2.2. Consider the images $(\phi_0, \mathcal{E})$

and $(\phi_0', \mathcal{E})$ of (E, θ_0) and (E', θ_0'). Since $\theta_0' = \tilde{\Psi}^\star \theta_0$, by definition, $\Phi \circ \phi_0 = \phi_0'$ so m is an essentially surjective functor. Since it is the restriction of the equivalence of categories described in subsection 2.2, it is an equivalence of categories. □

Now let (E, ω_0) be any meromorphic (G, P)-Cartan geometry on (M, D). Then the meromorphic one form $\pi_{\mathfrak{g}/\mathfrak{p}} \circ \omega_0$ obtained by projecting ω_0 on $\mathfrak{g}/\mathfrak{p}$ is P-equivariant for the quotient adjoint action on $\mathfrak{g}/\mathfrak{p}$, and pointwise surjective on $E \setminus p^{-1}(D)$. Moreover, its kernel contains $ker(dp)$. By the subsection 2.2, it thus corresponds to a morphism of $\mathcal{O}_M$-modules:

$$\phi_0 : TM[\star D] \longrightarrow \mathcal{E}[\star D] \tag{17}$$

where we set $\mathcal{E} = E(\mathfrak{g}/\mathfrak{p})$. By construction, ϕ_0 is an isomorphism of meromorphic bundles and $(\mathcal{E}, \phi_0)$ is thus a meromorphic extension on (M, D).

Definition 14 The meromorphic extension $(\mathcal{E}, \phi_0)$ obtained as above is the *meromorphic extension induced by* (E, ω_0). We denote by f the map from the set of meromorphic (G, P)-Cartan geometries on pairs to the set of meromorphic extensions which maps (E, ω_0) to its induced meromorphic extension $(\mathcal{E}, \phi_0)$. This extends as a functor f between the corresponding categories.

3.3 Infinitesimal automorphisms as horizontal sections

Important objects in the study of meromorphic Cartan geometries are the following:

Definition 15 Let (E, ω_0) be a meromorphic (G, P)-Cartan geometry on (M, D). An *infinitesimal automorphism* of (E, ω_0) is a holomorphic vector field $\overline{X}$ on an open subset $U \subset M \setminus D$, lifting to a vector field X on $p^{-1}(U)$ such that $\phi_X^{\iota\star}\omega_0 = \omega_0$. We write $\mathfrak{kill}^{loc}_{M,\omega_0}$ for the subsheaf of $TM \setminus D$ whose sections are the local infinitesimal automorphisms, and $\mathfrak{kill}^{loc}_{E,\omega_0}$ for the subsheaf of $TE \setminus \tilde{D}$ whose sections are the lifts of sections of $\mathfrak{kill}^{loc}_{M,\omega_0}$.

In order to study the sections of $\mathfrak{kill}^{loc}_{M,\omega_0}$, it is convenient to identify them with horizontal sections for a meromorphic connection on a trivial module over E. This is also a classical approach for general meromorphic parallelisms (see for example [15]). Indeed, let's denote by T the torsion of the flat meromorphic connection ∇^0 whose horizontal sections are the ω_0-constant vector fields on E. Then:

Proposition 3 *Let (E, ω_0) be a meromorphic (G, P)-Cartan geometry on (M, D).*

1. *The sheaf $\mathfrak{kill}^{loc}_{E,\omega_0}$ coïncides with the sheaf $ker(\nabla^{rec}_{\omega_0})$ of horizontal sections for the reciprocal connection ∇^{rec} defined by:*
$$\nabla^{rec}_X = \nabla^0_X + T(X, \cdot) \tag{18}$$
for any local vector field X.

2. *The connection $\nabla^{rec}_{\omega_0}$ is invariant by the P-linearization $(dr_b)_{b\in P}$ corresponding to the action of principal P-bundle.*

Proof 1. See Lemma 3.2 in [15].

2. This straightforwardly follows from the fact that the torsion of ∇_0 is P-invariant by definition.

□

Definition 16 The *Killing connection* of a meromorphic (G, P)-Cartan geometry (E, ω_0) on (M, D) is the meromorphic connection $(\mathcal{V}, \nabla^{\omega_0})$ where $\mathcal{V} = \mathcal{O}_E \otimes \mathfrak{g}$ and

$$\nabla^{\omega_0} = \Phi^{-1}_{\omega_0} \nabla^{rec}$$

where Φ_{ω_0} is the isomorphism of $\mathcal{O}_E[\star\tilde{D}]$-modules between $TE[\star\tilde{D}]$ and $\mathcal{V}[\star\tilde{D}]$.

The sheaves $\mathfrak{kill}_{E,\omega_0}$ and $\mathfrak{kill}_{M,\omega_0}$ are respectively local systems on $E \setminus \tilde{D}$ and $M \setminus D$.

As explained in the introduction, our goal is to classify quasi-homogeneous meromorphic Cartan geometries (E, ω_0). This hypothesis is satisfied whenever the base manifold M has only constant meromorphic functions (see [13]). In this case, there exists a point $x_0 \in M$ and n independent germs of Killing vector fields for (E, ω_0) at x_0. We seek for a sufficient condition for these germs to come from global Killing vector fields, i.e for the following property to be satisfied:

Definition 17 Let (E, ω_0) be a meromorphic (G, P)-Cartan geometry on a pair (M, D). It satisfies the *extension property of infinitesimal automorphisms* if and only the local system $\mathfrak{kill}_{M,\omega_0}$ on $M \setminus D$ extends as a local system $\mathfrak{k} \subset TM$ on M.

3.4 Distinguished foliations and totally spiral meromorphic Cartan geometries

We will restrict our attention on the following subcategory of meromorphic Cartan geometries. Let (E, ω_0) be a meromorphic (G, P)-Cartan geometry on (M, D), and $A \in \mathfrak{g} \setminus \{0\}$. Since ω_0 induces an isomorphism of meromorphic bundles between $TE[\star\tilde{D}]$ and $\mathcal{O}_E[\star\tilde{D}] \otimes \mathfrak{g}$, there exists a unique distribution of rank one (thus integrable) $\mathcal{T}_A \subset TE$ with the following property:

$$\omega_0(\mathcal{T}_A) \subset \mathcal{O}_E[\star\tilde{D}]A \tag{19}$$

We will call it the *A-distinguished foliation* of (E, ω_0), and a leaf $\tilde{\Sigma}$ will be called a *A-distinguished curve* for (E, ω_0).

Let $A \in \mathfrak{g} \setminus \{0\}$ and $\tilde{\Sigma}$ a A-distinguished curve for (E, ω_0). If $A \in \mathfrak{p}$, then $\tilde{\Sigma}$ is tangent to the kernel $ker(dp)$ of the differential of the bundle map. If $A \notin \mathfrak{p}$, $\tilde{\Sigma} \cap (E \setminus \tilde{D})$ is transverse to this distribution. Hence, the restriction of p to $\tilde{\Sigma}$ is a cover map from $\tilde{\Sigma}$ to its image $\Sigma \subset M$. In particular, if Σ is simply connected, then it is a biholomorphism.

Definition 18 Let $A \in \mathfrak{g}$ and (E, ω_0) a meromorphic (G, P)-Cartan geometry.

1. A *A-spiral* for (E, ω_0) at $x_0 \in M$ is a complex *smooth* curve Σ embedded in M, containing x_0 and such that $\Sigma \setminus D$ lifts to a A-distinguished curve in E.
2. A *holomorphic A-spiral* is a A-spiral Σ such that the lift $\tilde{\Sigma}$ as in 1. extends to a curve $\tilde{\Sigma}$ projecting onto Σ.

Definition 19 A meromorphic (G, P)-Cartan geometry (E, ω_0) on a pair (M, D) satisfies the *τ-condition* if the following is true. For any irreducible component D_α of D, there exists $x_0 \in D_\alpha$ and a spiral Σ for (E, ω_0) with $\Sigma \cap D = \{x_0\}$

Recall that for any holomorphic foliation $\mathcal{T}_A$ on a complex manifold E admits an analytic subset $Sing(\mathcal{T}_A)$ of codimension at least 2 s.t. for any $e_0 \in E \setminus Sing(\mathcal{T}_A)$, there is a neighborhood U of e_0 and a nonvanishing holomorphic vector field $Z \in TE(U)$ with:

$$\mathcal{T}_A|_U = \mathcal{O}_U Z \tag{20}$$

In this case, we say that Z *defines* $\mathcal{T}_A$ over U, and the leaves of $\mathcal{T}_A$ in U are exactly the orbits of the local flow for Z. From this remark, we can easily deduce the first part of the following:

Lemma 4 *Let E and $\mathcal{T}_A$ be as above. Let $\tilde{D}$ be any submanifold E.*

1. *The following assertions are equivalent:*
   (i) $\tilde{D}$ is a union of leaves for $\mathcal{T}_A$.
   (ii) Any local vector field Z defining $\mathcal{T}_A$ is everywhere tangent to $\tilde{D}$.
   (iii) For any local equation z_1 of $U \cap \tilde{D}$, and local vector field Z defining $\mathcal{T}_A$ over U, the dimension :
   $$dim_{\mathbb{C}}\, \mathcal{O}_{E,e_0}/\langle \mathcal{L}_Z(z_1), z_1 \rangle_{e_0} \tag{21}$$
   is never finite for $e_0 \in \tilde{D} \cap U$.
   We say that a submanifold $\tilde{D} \subset E$ is invariant by $\mathcal{T}_A$.
2. *If $\tilde{D}$ is not invariant by $\mathcal{T}_A$, then there exists an Zariksi-dense subset $\tilde{W} \subset \tilde{D} \setminus Sing(\mathcal{T}_A)$ with the following property: for any $e_0 \in \tilde{W}$, there exists a leaf $\tilde{\Sigma}$ of $\mathcal{T}_A$ through e_0 satisfying $\tilde{\Sigma} \cap \tilde{D} = \{e_0\}$.*

Proof 1. The equivalence between (i) and (ii) is clear from the above remark. The number (21) is the order of tangency of $\mathcal{T}_A$ to $\tilde{D}$ at e_0 (see [23]).

2. Let z_1 be a local equation for $\tilde{D}$, defined on an open subset $U \subset E$ where we can find a holomorphic vector field Z defining $\mathcal{T}_A$. Then the dimension (21) is zero except for a finite number of points in $\tilde{D} \cap U$ (see [23]). Complete z_1 into local coordinates $(z_1, \ldots, z_n)$ on U, and decompose $Z = h\frac{\partial}{\partial z_1} + Z'$ where h is a holomorphic function on U and Z' is a holomorphic vector field on U which belongs to the submodule spanned by $\frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}$. Since $\mathcal{L}_{Z'}(z_1) = 0$, the previous fact implies that h is not a multiple of z_1. This means that Z is generically transverse to $\tilde{D} \cap U$, and the leaves of $\mathcal{T}_A|_U$ are so, completing the proof of 2.

□

Definition 20 Let (E, ω_0) be a meromorphic (G, P)-Cartan geometry on a pair (M, D), and let $(\tilde{D}_\alpha)_{\alpha \in I}$ be the irreducible components of the divisor $\tilde{D} = p^{-1}(D)$. We say that (E, ω_0) satisfies the *strong τ-condition* if for any $\alpha \in I$, there exists $A \in \mathfrak{g}$ such that $\tilde{D}_\alpha$ is not invariant by $\mathcal{T}_A$.

In what follows, we prove that in the case of a branched holomorphic Cartan geometry (Definition 10), the strong τ-condition can be detected in terms of the geometric structures induced by this geometry:

Lemma 5 *Let (E, ω_0) be a branched holomorphic (G, P)-Cartan geometry on (M, D). Let $A \in \mathfrak{g} \setminus \mathfrak{p}$. Then the foliation $\mathcal{T}_A$ is transverse to $ker(dp)$.*

Proof Let $e_0 \in E \setminus Sing(\mathcal{T}_A)$, and U be an open neighborhood of e_0 in $E \setminus Sing(\mathcal{T}_A)$ equiped with coordinates $(z_1, \ldots, z_N)$, with the property that $\frac{\partial}{\partial z_{n+1}}, \ldots, \frac{\partial}{\partial z_N}$ are sections of $ker(dp)$ (i.e vertical vector fields). Fix a basis $(\mathfrak{e}_1, \ldots, \mathfrak{e}_N)$ of $\mathfrak{g}$ obtained by completing a basis $(\mathfrak{e}_{n+1}, \ldots, \mathfrak{e}_N)$ of $\mathfrak{p}$.

Since (E, ω_0) is a branched holomorphic Cartan geometry, the matrix $Q = (q_{ij})_{i,j=1,\ldots,N}$ of ω_0 in the previous basis takes the form:

$$Q = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \tag{22}$$

where:

- A, B, C are holomorphic matrices on U.
- $C' = C^{-1}$ is a holomorphic matrix U.

Hence, the matrix Q^{-1} of the ω_0-constant vector fields associated with $(\mathfrak{e}_j)_{j=1,\ldots,N}$ in $(\frac{\partial}{\partial z_i})_{i=1,\ldots,N}$ is:

$$Q^{-1} = \begin{pmatrix} A_1 & 0 \\ B_1 & C_1 \end{pmatrix} \tag{23}$$

with:

$$CB_1 + BA_1 = 0$$

Consider any irreducible component $\tilde{D}_\alpha$ of $\tilde{D} = p^{-1}(D)$, and any $j \in \{n+1, \dots, N\}$. Since C^{-1} is holomorphic on U, the above equation implies that for any vector in $\mathbb{C}^n$:

$$ord_{\tilde{D}_\alpha \cap U}(B_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix})_j \geq \min_{i=1,\dots,n} ord_{\tilde{D}_\alpha \cap U}(A_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix})_i \tag{24}$$

where the subscript stands for the i-th component.

We now interpret the inequality (24) geometrically. Let Z be a holomorphic vector field defining $\mathcal{T}_A$ on U. Define $\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$ to be the coordinates of A in the basis $(\mathfrak{e}_i)_{i=1,\dots,N}$. Thus:

$$Z = h\tilde{A} \tag{25}$$

with h a meromorphic function on U satisfying:

$$ord_{\tilde{D}_\alpha \cap U}(h) = -\min_{j=1,\dots,N} ord_{\tilde{D}_\alpha \cap U}(\sum_{i=1}^{N} q_{ji}^{-1} a_i)$$

$$= -min(\min_{i'=1,\dots,n} ord_{\tilde{D}_\alpha \cap U}(A_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix})_{i'}, \min_{j'=1,\dots,n} ord_{\tilde{D}_\alpha \cap U}(B_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix})_{j'})$$

Using (24), we obtain that $ord_{\tilde{D}_\alpha \cap U}(h) = -\min_{i'=1,\dots,n} ord_{\tilde{D}_\alpha \cap U}(A_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix})_{i'}$.

Decompose $Z = Z' + Z''$ with Z' in the subsheaf of holomorphic vector fields spanned by $(\frac{\partial}{\partial z_i})_{i=1,\dots,n}$ and Z'' in the subsheaf $ker(dp)$. Then the coordinates of Z' (resp. Z'') in $(\frac{\partial}{\partial z_i})_{i=1,\dots,n}$ (resp. $(\frac{\partial}{\partial z_i})_{i=n+1,\dots,N}$) are

$$hA_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

respectively:

$$hB_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + hC_1 \begin{pmatrix} a_{n+1} \\ \vdots \\ a_N \end{pmatrix}$$

Using the above remark, and (24) again, we conclude that Z' has order zero along $\tilde{D}_\alpha \cap U$ since $a_{i_0} \neq 0$ for some $i_0 \in \{1, \dots, n\}$, while Z'' is holomorphic. Since Z' never vanishes on the regular part of (E, ω_0) (up to restriction of U), we conclude that Z is nowhere tangent to $ker(dp)$, concluding the proof. □

Proposition 6 *Let (E, ω_0) be a branched holomorphic (G, P)-Cartan geometry on (M, D). Suppose that no irreducible component D_α of D is invariant by the spirals of (E, ω_0). Then (E, ω_0) satisfies the strong τ-condition.*

Proof Pick an irreducible component $\tilde{D}_\alpha$ of $\tilde{D} = p^{-1}(D)$. We have to prove that there exists $A \in \mathfrak{g}$ and a A-distinguished curve $\tilde{\Sigma}$ transverse to $\tilde{D}_\alpha$. But by the hypothesis, there exists a spiral Σ of (E, ω_0) which is transverse to $D_\alpha = p(\tilde{D}_\alpha)$. Thus, there exists $A \in \mathfrak{g} \setminus \mathfrak{p}$ and a A-distinguished curve $\tilde{\Sigma}$ with $p(\tilde{\Sigma}) = \Sigma \setminus D_\alpha$.

Remark that $\mathcal{T}_A$ is the kernel of an unique holomorphic P-principal connection on the restricted bundle $E|_{\Sigma \setminus D_\alpha}$, and $\tilde{\Sigma}$ is a horizontal section of this connection. By Theorem 5, this holomorphic principal connection extends as a holomorphic principal connection on $E|_\Sigma$. Hence, $\tilde{\Sigma}$ extends as a horizontal section of $E|_\Sigma$, i.e $p(\tilde{\Sigma}) = \Sigma$. Up to restriction of Σ, we can assume that $\Sigma \cap D_\alpha = \{x_0\}$. Thus $\tilde{\Sigma}$ intersects $\tilde{D}_\alpha$ in some point in the fiber of x_0, concluding the proof.

□

4 Infinitesimal automorphisms of meromorphic parabolic geometries

A classical result in Riemannian geometry states that any Killing vector field X for a Riemannian metric g is a Jacobi field: for any geodesic γ, its scalar product $g(X(\gamma(t)), \gamma'(t)))$ with the velocity of γ is constant. There is a natural generalization of Riemannian metrics to the holomorphic category, and the corresponding objects are equivalent to torsionfree holomorphic affine connections preserving a holomorphic reduction to the orthogonal group. The holomorphic version of the previous result can be seen as a result on some torsionfree holomorphic affine Cartan geometries (see Theorem 17). In this section, we will see a general result for meromorphic Cartan geometries. In particular, this will imply that the local system of infinitesimal automorphisms for any totally geodesic *regular meromorphic parabolic geometry* extends as a local system on the whole base manifold.

4.1 Regular meromorphic parabolic geometries

A *complex parabolic Klein geometry* is a complex Klein geometry (G, P) where G is a complex semi-simple Lie group, and P a parabolic subgroup. A *meromorphic parabolic geometry* is a meromorphic (G, P)-Cartan geometry for some complex parabolic Klein geometry. We refer the reader to [24] for a detailed introduction.

With the subgroup P is associated a grading $(\mathfrak{g}_i)_{i \in \mathbb{Z}}$ of the Lie algebra $\mathfrak{g} = Lie(G)$, meaning $[\mathfrak{g}_{i_1}, \mathfrak{g}_{i_2}]_\mathfrak{g} \subset \mathfrak{g}_{i+j}$ for any indices $i_1, i_2 \in \mathbb{Z}$. We call it the *parabolic filtration* associated with P. It induces a grading of any representation of G, in particular $\mathbb{W} = (\bigwedge^2 \mathfrak{g}_-^*) \otimes \mathfrak{g}$ is graded by *homogeneous degrees* $\mathbb{W}_l$, and we denote by π_l the corresponding projections.

The *parabolic degree* of (G, P) is the smallest positive integer $k \geq 1$ such that $\mathfrak{g}_i = \{0\}$ for any $|i| > k$. The subspaces $\mathfrak{p} = Lie(P)$ and the subspace

$$\mathfrak{g}_- = \bigoplus_{i=-k}^{-1} \mathfrak{g}_i \tag{26}$$

are clearly subalgebras of $\mathfrak{g}$. For any $i \in \{-k, \dots, k\}$, we will denote $\mathfrak{g}^i = \bigoplus_{i' \geq i} \mathfrak{g}_{i'}$, inducing a filtration $(\mathfrak{g}^i)_{i=-k,\dots,k}$ of $\mathfrak{g}$.

By a result of C. Chevalley, we can always pick a basis $(\mathfrak{e}^i_j)_{\substack{i=-k,\dots,k \\ j=1,\dots,n_i}}$ of $\mathfrak{g}$, such that $(\mathfrak{e}^i_j)_{j=1,\dots,n_i}$ is a basis for $\mathfrak{g}_i$ for any $i \in \{-k, \dots, k\}$, and $[\mathfrak{e}^{i_1}_{j_1}, \mathfrak{e}^{i_2}_{j_2}]_{\mathfrak{g}}$ is a vector in $\mathbb{Z}\mathfrak{e}^{i_1+i_2}_j$ for some $j \in \{1, \dots, n_{i_1+i_2}\}$. We will refer to it as a *graded basis* of $\mathfrak{g}$ for (G, P).

The homogeneous space G/P associated with a complex parabolic Klein geometry (G, P) bears the following holomorphic geometric structure. Its tangent bundle is filtered by subbundles $(T^{-i}G/P)_{i=1,\dots,k}$ where $T^{-i}G/P$ is the projection of $\omega_G^{-1}(\mathfrak{g}^i)$ through the tangent map $Tp_{G/P}$ of the projection $p_{G/P} : G \longrightarrow G/P$. The Lie bracket of holomorphic vector fields on G/P induces a Lie bracket of holomorphic vector bundle on the corresponding graded bundle $gr(TG/P)$. The Lie algebra bundle thus obtained is locally isomorphic to $(U \times \mathfrak{g}_-, [,]_{\mathfrak{g}_-})$.

The *regular meromorphic parabolic geometries* are the infinitesimal versions of this model. More precisely, these are meromorphic (G, P)-Cartan geometries (E, ω_0) on (M, D) for which the homogeneous component $\pi_l(k_{\omega_0})$ of degree l of the Cartan curvature vanishes identically whenever $l \leq 0$ (see above). This amounts to the following property. Let $T^{-i}M[\star D]$ be the image of $\omega_0^{-1}(\mathfrak{g}^{-i})[\star\tilde{D}]$ through Tp. This gives a filtration of $TM[\star\tilde{D}]$, and (E, ω_0) is regular if and only if the Lie bracket of vector fields on M induces a structure of Lie algebras bundle on the graded $gr(TM \setminus D)$, locally isomorphic to $(U \times \mathfrak{g}_-, [,]_{\mathfrak{g}_-})$.

4.2 Bott connections and infinitesimal automorphisms of Cartan geometries

Now, we come back to a general complex Klein geometry (G, P). Let (M, D) be a complex pair of dimension $n \geq 1$, and (E, ω_0) be a meromorphic (G, P)-Cartan geometry on it. Fix $A \in \mathfrak{g} \setminus \{0\}$ and consider the holomorphic foliation $\mathcal{T}_A$ from (19). To any such holomorphic foliation is associated a $\mathcal{T}_A$-partial holomorphic connection $\nabla^{\mathcal{T}_A}$ on $TE/\mathcal{T}_A$, the *Bott-connection* of $\mathcal{T}_A$, defined as follow. Let X be a holomorphic vector field on $U \subset E$, $[X]$ its class in $TE/\mathcal{T}_A(U)$, and $Z \in \mathcal{T}_A(U)$. Then:

$$Z \lrcorner \nabla^{\mathcal{T}_A}([X]) = [[Z, X]_{TE}] \tag{27}$$

Let $t \in \mathbb{C}$ and $V \subset U$ such that the flow $\phi = \phi^t_Z$ is well defined on V. Then clearly $d\phi(\mathcal{T}_A) \subset \phi^*\mathcal{T}_A$, so ϕ induces a morphism $[d\phi]$ of $\mathcal{O}_V$-modules defined

by the commutative diagram:

$$\begin{array}{ccc} TV & \xrightarrow{d\phi} & \phi^* T\phi(V) \\ \downarrow{\scriptstyle []} & & \downarrow{\scriptstyle \phi^*[]} \\ TV/\mathcal{T}_A & \xrightarrow[{[d\phi]}]{} & \phi^* T\phi(V)/\mathcal{T}_A \end{array} \tag{28}$$

By the formula (27), the horizontal sections for $\nabla^{\mathcal{T}_A}$ are the $[X]$ which are invariant by the isomorphisms of holomorphic vector bundle $(\phi, [d\phi])$ defined as before.

It will be more convenient to work with the images of meromorphic vector fields on E through the isomorphism Φ_{ω_0} between $TE[\star\tilde{D}]$ and $\mathcal{V}[\star\tilde{D}]$, where $\mathcal{V} = \mathcal{O}_E \otimes \mathfrak{g}$. We will write:

$$\mathcal{K} = \Phi_{\omega_0}(\mathfrak{kill}_{E,\omega_0}) \tag{29}$$

for the corresponding local system on $E \setminus \tilde{D}$. Clearly, the image of $\mathcal{T}_A[\star\tilde{D}]$ is $\mathcal{V}_A = \mathcal{O}_E[\star\tilde{D}]A$. The class of a section s of $\mathcal{V}[\star\tilde{D}](U)$ (where $U \subset E$ is an open subset) in $\mathcal{V}/\mathcal{V}_A[\star\tilde{D}]$ will be denoted by $[s]_{\mathcal{V}/\mathcal{V}_A}$. Since Φ_{ω_0} induces an isomorphism of $\mathcal{O}_E$-modules between $TE/\mathcal{T}_A[\star\tilde{D}]$ and $\mathcal{V}/\mathcal{V}_A[\star\tilde{D}]$, for any $Z \in \mathcal{T}_A(U)$, the morphism $[d\phi]$ defined by (28) corresponds to an isomorphism

$$\overline{d\phi} : \mathcal{V}/\mathcal{V}_A[\star\tilde{D}]|_V \longrightarrow \phi^*\mathcal{V}/\mathcal{V}_A[\star\tilde{D}]|_{\phi(V)} \tag{30}$$

and thus an isomorphism $(\phi, \overline{d\phi})$ of meromorphic bundles.

The isomorphism of meromorphic bundles Φ_{ω_0} (see above) maps $\mathcal{T}_A[\star\tilde{D}]$ to $\mathcal{V}_A[\star\tilde{D}]$, and we denote by $\overline{\Phi}_{\omega_0} : TE/\mathcal{T}_A[\star\tilde{D}] \longrightarrow \mathcal{V}/\mathcal{V}_A[\star\tilde{D}]$ the isomorphism induced by Φ_{ω_0}. Then:

Lemma 7 *Let s be a section of $\mathcal{K}$ on an open subset $U \subset E \setminus \tilde{D}$. Then its class $[s]_{\mathcal{V}/\mathcal{V}_A}$ is invariant by any isomorphism of meromorphic bundles $(\phi, \overline{d\phi})$ constructed as above.*

Proof Let X be any holomorphic vector field on $U \subset E$, and $[X]$ its class in $TE/\mathcal{T}_A$. By definition, for any $Z_A = h\tilde{A}$ (where h is a meromorphic function on U and $\tilde{A} = \omega_0^{-1}(A)$) we have:

$$\begin{aligned} 0 = & \quad [\tilde{A}, X]_{TE} \\ = & \ \tfrac{1}{h}[Z_A, X]_{TE} \quad \mod \ \mathcal{T}_A[\star\tilde{D}](U) \end{aligned} \tag{31}$$

In other words, the classes of $d\phi(X)$ and ϕ^*X in $TE/\mathcal{T}_A[\star\tilde{D}]$, well defined on $U \cap \phi(U)$, coïncides i.e s is invariant by $(\phi, \overline{d\phi})$. □

Now, we suppose M to be simply connected. We wish to prove the extension property for (E, ω_0) (Definition 17). We will use the following general fact on meromorphic Cartan geometries:

Theorem 8 *Let (G, P) be a complex Klein geometry, and (M, D) be a pair with $dim(M) = dim(G/P)$. Let (E, ω_0) be a meromorphic (G, P)-Cartan geometry on (M, D). Let $x_0 \in D$ belonging to the smooth part of an unique irreducible component D_α, and $\tilde{D}_\alpha = p^{-1}(D_\alpha)$. Suppose that there exists $A \in \mathfrak{g} \setminus \mathfrak{p}$ and a A-distinguished curve $\tilde{\Sigma}$ for (E, ω_0) such that $\tilde{\Sigma} \cap \tilde{D}_\alpha = \{e_0\}$ for some point e_0 in the fiber of x_0. Then there exists a neighborhood U of x_0 with the following properties:*

1. *Let s be a section of $\mathcal{K}$ on $V \subset p^{-1}(U \setminus D)$. Then the class $[s]_{\mathcal{V}/\mathcal{V}_A}$ of s in $TE/\mathcal{T}_A$ extends as a (univaluated) section of $TE/\mathcal{T}_A$ on $p^{-1}(U \setminus D)$.*
2. *The section of $\mathcal{V}/\mathcal{V}_A$ obtained as above is the restriction of a section of $\mathcal{V}/\mathcal{V}_A[\star D]$ over $p^{-1}(U)$.*
3. *Suppose moreover that (E, ω_0) is holomorphic branched on (M, D). Then the above section lies in $\mathcal{V}/\mathcal{V}_A(p^{-1}(U))$.*

Proof 1. It is a classical result that the image $\mathcal{K}_A = \pi_{\mathcal{V}/\mathcal{V}_A}(\mathcal{K})$ of the local system $\mathcal{K}$ on $E \setminus \tilde{D}$ is a local system on $E \setminus \tilde{D}$. Moreover, $\mathcal{K}$ is P-equivariant, and the same remains valid for $\mathcal{K}_A$. We thus have to prove that $\mathcal{K}_A$ is a constant sheaf on an open subset $\tilde{U} \setminus \tilde{D}$ where $\tilde{U} \subset E$ containing some $e_0 \in p^{-1}(x_0)$. It suffices to prove that there exists a simply connected $\tilde{U}$, with $\tilde{U} \cap \tilde{D}$ simply connected, and $e_0' \in \tilde{U} \cap \tilde{D}$ such that $\mathcal{K}_A$ is a constant sheaf in a neighborhood of e_0'.

The hypothesis says that $\tilde{D}_\alpha$ is not invariant by $\mathcal{T}_A$. By Theorem 4 there exists an open neighborhood $\tilde{U}_0$ of some point $e_0 \in p^{-1}(x_0)$ such that a generic leaf of $\mathcal{T}_A|_{U_0}$ intersects $\tilde{D} \cap U_0$ in exactly one point. Pick e_0' such that there exists a leaf $\tilde{\Sigma}$ of $\mathcal{T}_A$ with $\tilde{\Sigma} \cap \tilde{D} = \{e_0'\}$. Pick $e \in \tilde{\Sigma} \setminus \tilde{D}$, and a simply connected open neighborhood V of e equipped with a basis $(s_1, \ldots, s_r)$ of $\mathcal{K}$.

By the Theorem 7, the family $\overline{d\phi}([s_1]_{\mathcal{V}/\mathcal{V}_A}), \ldots, \overline{d\phi}([s_r]_{\mathcal{V}/\mathcal{V}_A})$, where $\overline{d\phi}$ is the morphism (30), is a basis of $\phi^*\mathcal{K}_A(\phi(V) \setminus \tilde{D})$. Then $\mathcal{K}_A$ is a constant sheaf when restricted to $\tilde{U} \setminus \tilde{D}$, where $\tilde{U} = \phi(V)$ is a neighborhood of e_0'. By the above remarks, this concludes the proof.

2. Since(30) is an isomorphism of meromorphic bundles, we have proved in 1. that the local system $\mathcal{K}_A = \pi_{\mathcal{V}/\mathcal{V}_A}(\mathcal{K})$ extends as a constant sheaf, included in $\mathcal{V}/\mathcal{V}_A[\star\tilde{D}]|_{p^{-1}(U)}$ since $(\phi, \overline{d\phi})$ is an automorphism of meromorphic bundles for $\mathcal{V}/\mathcal{V}_A$.
3. The meromorphic Cartan geometry (E, ω_0) is holomorphic branched on (M, D) if and only if $\Phi_{\omega_0}(TE) \subset \mathcal{V}$. Suppose this is the case. Since the automorphism of meromorphic bundles $(\phi, [d\phi])$ of $TE/\mathcal{T}_A$ defined before (28) is an automorphism of holomorphic vector bundles. Since $\Phi_{\omega_0}(TE)$ and $\mathcal{V}$ coïncides when restricted to $p^{-1}(U \setminus D)$, we obtain that the image of $\mathcal{V}/\mathcal{V}_A|_{p^{-1}(U \setminus D)}$ through the $\overline{d\phi}$ lies in $\mathcal{V}/\mathcal{V}_A|_{\phi(p^{-1}(U \setminus D))}$, where $\phi(p^{-1}(U \setminus D))$ is a neighborhood of e_0' by construction. This proves the assertion.

□

4.3 Affine and degree one parabolic models

We now apply Theorem 8 to prove the extension property for infinitesimal automorphisms of some holomorphic branched (G, P)-Cartan geometries which satisfies the τ-condition (Definition 19). More precisely, we let (G, P) be:

- A complex parabolic Klein geometry of dimension $n \geq 2$ and degree $k = 1$ (see subsection 4.1), with G a complex simply connected simple Lie group,
- Or the complex affine Klein geometry of dimension $n \geq 2$.

For the first model, we denote by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the parabolic filtration associated with P (see subsection 4.1). For the second one, $\mathfrak{g}_{-1}$ will stand for the abelian subalgebra of infinitesimal generators for the translations in $\mathbb{C}^n$.

These two kind of models are of special interest because the Levi subgroup $G_0 = exp_G(\mathfrak{g}_0) \subset P$ acts transitively on the lines in $\mathfrak{g}_{-1}$. This is clear for the second one, since the corresponding action is the standard representation of $GL_n(\mathbb{C})$. For the first one, remark that $P = G_0 \ltimes exp_G(\mathfrak{g}_1)$, and $[\mathfrak{g}_1, \mathfrak{g}_{-1}]_{\mathfrak{g}} \subset \mathfrak{g}_0$. Hence, $[\mathfrak{g}_0, \mathfrak{g}_{-1}]_{\mathfrak{g}}$ must be $\mathfrak{g}_{-1}$. This implies that the action of G_0 on $\mathfrak{g}_{-1}$ is open.

By the above remarks, for any $A \in \mathfrak{g}_{-1} \setminus \{0\}$, there exists a basis $(\mathfrak{e}_i)_{i=1,\ldots,n}$ of the abelian subalgebra $\mathfrak{g}_{-1} \subset \mathfrak{g} = Lie(G)$, and n elements $b_1, \ldots, b_n \in P$ such that

$$ad(b_i^{-1})[A] \in \mathbb{C}\mathfrak{e}_i \tag{32}$$

for $1 \leq i \leq n$.

Lemma 9 *Let (E, ω_0) be a holomorphic branched (G, P)-Cartan geometry on (M, D), and let $x_0 \in D$ belonging to the smooth part of an unique irreducible component D_α. Suppose there exists a spiral $\overline{\Sigma}$ for (E, ω_0) with $\overline{\Sigma} \cap D = \{x_0\}$. Then, for any $1 \leq i \leq n$, there exists a holomorphic $\mathfrak{e}_i$-spiral $\overline{\Sigma}_i$ for (E, ω_0) with $\overline{\Sigma}_i \cap D = \{x_0\}$.*

Proof By Theorem 6, $\overline{\Sigma}$ is a A-holomorphic geodesic for $A \in \mathfrak{g} \setminus \mathfrak{p}$. Thus, there exists $e_0 \in p^{-1}(x_0)$ and a A-distinguished smooth complex curve $\tilde{\Sigma}$ with $p(\tilde{\Sigma}) = \overline{\Sigma}$ and $\tilde{\Sigma} \cap \tilde{D}_\alpha = \{e_0\}$. Let $b_1, \ldots, b_n \in P$ as in (32). By equivariance of ω_0, for any $1 \leq i \leq n$, the b_i-translated $\tilde{\Sigma}_i$ of $\tilde{\Sigma}$ is a $\mathfrak{e}_i$-distinguished smooth complex curve, with $\tilde{\Sigma}_i \cap \tilde{D}_\alpha = \{e_0 \cdot b_i\}$. This means that for any $i \in \{1, \ldots, n\}$, $\tilde{D}_\alpha$ is not invariant by $\mathcal{T}_{\mathfrak{e}_i}$. By Theorem 4, this implies that for any such i, there exists a Zariski open-dense subset W_i of $\tilde{D}_\alpha$ such that, for any $e_0' \in W_i$, there exists a leaf $\tilde{\Sigma}_i$ of $\mathcal{T}_{\mathfrak{e}_i}$ transverse to $\tilde{D}_\alpha$ at e_0'.

Consider the open subset V_{x_0} of the fiber of x_0 in E defined by:

$$\tilde{V}_{x_0} = \bigcap_{i=1}^{n} W_i \cap p^{-1}(x_0) \tag{33}$$

Since each W_i contains a point in the fiber of x_0, $\tilde{V}_{x_0}$ is a non-empty Zariski-dense open subset of this fiber. Moreover by construction, for any $e_0' \in \tilde{V}_{x_0}$, and any $1 \leq i \leq n$ there is a $\mathfrak{e}_i$-distinguished smooth curve $\tilde{\Sigma}_i$ with $\tilde{\Sigma}_i \cap \tilde{D} = \{e_0'\}$. The corresponding projections Σ_i throuhg p are holomorphic $\mathfrak{e}_i$-spirals for (E, ω_0), and the proof is thus achieved. $\square$

Corollary 10 *Let (E, ω_0) be a holomorphic branched (G, P)-Cartan geometry on (M, D), with M simply connected. Suppose it satisfies the τ-condition (Definition 19). Then:*

1. *(E, ω_0) satisfies the extension property for the infinitesimal automorphisms.*
2. *Any section s of $ker(\nabla^{\kappa}_{\omega_0})(U)$ (where $\nabla^{\kappa}_{\omega_0}$ is the Killing connection, see Definition 16, and $U \subset E$ is an open subset of M) is a section of $\mathcal{V}(U)$.*

Proof 1. Since the complement of a codimension 2 subset of M has the same fundamental group as M, and in virtue of the equivalence between local systems and representations of the fundamental group, it suffices to find a codimension 1 subset W of D, pick a point x_0 on an unique irreducible component D_α of $D \cap W$, and show the existence of a neighborhood U of x_0 in M such that the restriction of $ker(\nabla^{\kappa}_{\omega_0})$ to $p^{-1}(U \setminus D)$ extends as a local system on $p^{-1}(U)$, included in $TE[\star D]|_U$.

Pick any irreducible component $\tilde{D}_\alpha$ of $\tilde{D}$. By Theorem 4, there exists $e_0 \in \tilde{D}_\alpha$ and a leaf $\tilde{\Sigma}$ of $\mathcal{T}_A$ with $\tilde{\Sigma} \cap \tilde{D}_\alpha = \{e_0\}$. Thus, there exists a A-spiral $\overline{\Sigma}$ of (E, ω_0) with $\Sigma \cap D_\alpha = \{x_0\}$. By Theorem 6, we can moreover suppose that Σ is a holomorphic A-spiral.

We now apply Theorem 9 to obtain, for any $1 \leq i \leq n$, a holomorphic $\mathfrak{e}_i$-spiral Σ_i with $\Sigma_i \cap D = \{x_0\}$. More precisely, the proof of the lemma implies the existence, for $i \in \{1, \ldots, n\}$ fixed, of $e_0 \in p^{-1}(x_0)$ such that the $\mathfrak{e}_i$-distinguished curve $\tilde{\Sigma}_i$ projecting onto $\overline{\Sigma}_i$ satisfies $\tilde{\Sigma}_i \cap \tilde{D} = \{e_0\}$. Using the Theorem 8 for each geodesic, we obtain neighborhoods U_i of x_0 such that the restriction of the local system $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_i}}(ker(\nabla^{\kappa}_{\omega_0}))$ to $p^{-1}(U_i)$ is a constant sheaf.

Let $U = \bigcap_{i=1}^{n} U_i$. Since $\mathfrak{e}_1, \mathfrak{e}_2$ are independent vectors of $\mathfrak{g}$, the morphism of $\mathcal{O}_E$-modules:

$$\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}} \oplus \pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}} : \mathcal{V}[\star\tilde{D}] \longrightarrow \mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}[\star\tilde{D}] \oplus \mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}[\star\tilde{D}] \tag{34}$$

is an isomorphism onto its image. Thus, it restricts to $ker(\nabla^{\kappa}_{\omega_0})$ as an isomorphism of $\mathbb{C}$-sheaves onto its image, a subsheaf of the local system $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}}(ker(\nabla^{\kappa}_{\omega_0})) \oplus \pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}}(ker(\nabla^{\kappa}_{\omega_0}))$. By the above remark, this local system is a constant sheaf when restricted to $p^{-1}(U)$. Thus, the same is true for $ker(\nabla^{\kappa}_{\omega_0})$, i.e (E, ω_0) satisfies the extension property for the infinitesimal automorphisms.

2. Since (E, ω_0) is a branched holomorphic Cartan geometry, we can apply the point 3. of Theorem 8 to $A = \mathfrak{e}_1$ and $A = \mathfrak{e}_2$. We obtain that the image of $ker(\nabla^{\kappa}_{\omega_0})$ through $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}}$ and $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}}$ respectively extends as subsheaves of $\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}$ and $\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}$ on E. Since the morphism (34) clearly restricts to a morphism between $\mathcal{V}$ and $\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1} \oplus \mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}$, this proves the assertion.

□

Remark 1 The conclusion of point 1. in Theorem 10 remains valid if we consider a meromorphic (G, P)-Cartan geometry (E, ω_0) on (M, D), but replacing the τ-condition by the strong τ-condition (Definition 20). The conclusion of point 2. remains true if $\mathcal{V}$ is replaced by $\mathcal{V}[\star D]$.

4.4 Parabolic geometries of higher degree

Now, we let (G, P) be a complex parabolic Klein geometry of degree $k > 1$, and denote by $\mathfrak{g}_{-k} \oplus \ldots \mathfrak{g}_0 \oplus \ldots \mathfrak{g}_k$ the parabolic filtration. We refer the reader to [24] for the definitions and a complete introduction on this subject.

The group P no longer acts transitively on $\mathbb{P}(\mathfrak{g}/\mathfrak{p})$. Instead, we use a result of [25] which implies the following:

Lemma 11 *Let (E, ω_0) be a regular meromorphic (G, P)-Cartan geometry on a pair (M, D). Then there exists a morphism of $\mathbb{C}$-sheaves:*

$$\mathcal{L} : \mathcal{V}_{\mathfrak{g}_{-k}}[\star\tilde{D}] \longrightarrow \mathcal{V}[\star\tilde{D}] \tag{35}$$

with the following properties:

(i) Let $\pi_{-k} : \mathcal{V}[\star\tilde{D}] \longrightarrow \mathcal{V}_{\mathfrak{g}_{-k}}[\star\tilde{D}]$ be the projection on $\mathcal{V}_{\mathfrak{g}_{-k}}$ with respect to $\mathcal{V}_{\mathfrak{g}^{-k+1}}[\star\tilde{D}]$. Then $\pi_{-k} \circ \mathcal{L} = Id_{\mathcal{V}_{\mathfrak{g}_{-k}}}$.

(ii) The restriction of $\mathcal{L} \circ \pi_{-k}$ to $ker(\nabla^{\kappa}_{\omega_0})$ is the identity on $ker(\nabla^{\kappa}_{\omega_0})$.

Proof The Theorem 4 in [25] is exactly the non-singular version of this lemma, i.e when D is empty. Its proof uses only differential operators constructed with the de Rham differential on trivial modules, and morphisms of modules obtained by tensorizing linear map of complex vector spaces with the identity on holomorphic functions. Thus, it straightforwardly extends to the meromorphic category since such operators preserves the sheaves of meromorphic sections. □

Corollary 12 *Let (E, ω_0) be a regular holomorphic branched (G, P)-Cartan geometry on a pair (M, D). Suppose that for any irreducible component D_α of D, there exists $A \in \mathfrak{g}_- \setminus \mathfrak{g}_{-k}$ and a A-spiral Σ of (E, ω_0) with $\Sigma \cap D_\alpha = \{x_0\}$. Then (E, ω_0) satisfies the extension property for the infinitesimal automorphisms.*

Proof By Theorem 6, we can suppose that the curves Σ in the statement are holomorphic spirals, i.e. admit lifts to A-distinguished curves $\tilde{\Sigma}$ with $\tilde{\Sigma} \cap \tilde{D} = \{e_0\}$, for some $e_0 \in p^{-1}(x_0)$ and $A \in \mathfrak{g}_- \setminus \mathfrak{g}_{-k}$.

Pick an irreducible component $\tilde{D}_\alpha$, let e_0 be as above and apply the Theorem 8. Since $k > 1$, $\mathbb{C}A$ and $\mathfrak{g}_{-k}$ are independent subspaces in $\mathfrak{g}$. Thus, the projection $\pi_{-k}(ker(\nabla^{\kappa}_{\omega_0}))$ extends as a constant $\mathbb{C}$-subsheaf of $\mathcal{V}_{\mathfrak{g}_{-k}}[\star\tilde{D}]$ on a neighborhood U of e_0. The image of a constant sheaf by a morphism of $\mathbb{C}$-sheaves is a constant sheaf, so by Theorem 11, $ker(\nabla^{\kappa}_{\omega_0})$ extends as a constant $\mathbb{C}$-subsheaf of $\mathcal{V}[\star\tilde{D}]$ on U. The proof is then achieved. □

Remark 2 Theorem 12 admits a meromorphic version as in the degree one case, see Remark 1.

5 Application to the classification of meromorphic affine connections

5.1 Equivalence between meromorphic principal connections and meromorphic connections

We now prove the equivalence between meromorphic connections on a locally free $\mathcal{O}_M$-module $\mathcal{E}$ and meromorphic principal connections on its frame bundle E. It straightforwardly restricts as an equivalence between meromorphic connections preserving a holomorphic reduction $E \subset E$ to a subgroup $P \subset GL_r(\mathbb{C})$, and meromorphic P-principal connections on E. In the regular setting, this was first proved by C. Ehresmann ([4]) using the formalism of horizontal lifts for paths, and reformulated in an equivariant way by M. Atiyah ([5]). We adopt the point of view of M. Atiyah in order to extend the result to the meromorphic category.

The starting point is that for $P = GL_r(\mathbb{C})$, there is a canonical isomorphism ([5], Proposition 9):

$$E(\mathfrak{p}) = End(\mathcal{E}) \tag{36}$$

There is a bijection between the set of meromorphic connections ∇ on $\mathcal{E}$ and the one of $\mathcal{O}_M$-linear splittings $\delta : \mathcal{E}[\star D] \longrightarrow J^1(\mathcal{E})[\star D]$ of the exact sequence of $\mathbb{C}$-sheaves:

$$0 \longrightarrow \Omega^1_M[\star D] \otimes \mathcal{E} \longrightarrow J^1(\mathcal{E})[\star D] \longrightarrow \mathcal{E}[\star D] \longrightarrow 0 \tag{37}$$

where $J^1(\mathcal{E})$ is the jet-module of $\mathcal{E}$ (see [5]). Let $\sigma : U \longrightarrow E$ be a holomorphic section of the holomorphic frame bundle. This corresponds to a basis $(s_1, \ldots, s_r)$ of $\mathcal{E}|_U$, and we denote in the following lines by d the pullback of the de Rham differential through the corresponding isomorphism $\mathcal{E}|_U \simeq \mathcal{O}_U^{\oplus r}$. The former equivalence is given by $\nabla = d - \delta$. Indeed, this clearly defines a meromorphic connection, and if ∇ is a meromorphic connection on $\mathcal{E}|_U$, then $\delta_1 = d - \nabla$ is a morphism of $\mathcal{O}_U$-modules from $\mathcal{E}|_U[\star D]$ to $\Omega^1_U[\star D] \otimes \mathcal{E}|_U$, and we obtain a splitting $\delta = (Id_{\mathcal{E}|_U}, \delta_1)$ of (37).

Definition 21 A *meromorphic principal connection* on a holomorphic $GL_r(\mathbb{C})$-principal bundle $E \xrightarrow{p} M$ with poles at $\tilde{D} = p^{-1}(D)$ (shortly on $(E, \tilde{D})$) is a meromorphic one form $\tilde{\omega}$ on E with values in $\mathfrak{p}$, which is P-equivariant and such that $\tilde{\omega}$ coïncides with the Maurer-Cartan form of P when restricted to any fiber $p^{-1}(x) \subset E$.

Using the correspondance for equivariant one forms as in subsection 2.2, a meromorphic P-principal connection on $(E, \tilde{D})$ is equivalent to a morphism $\beta : At(E)[\star D] \longrightarrow E(\mathfrak{p})[\star D]$ such that $\iota \circ \beta = Id_{At(E)}$, where ι is defined in (11). Its kernel defines a splitting

$$\tau : TM[\star D] \longrightarrow At(E)[\star D] \tag{38}$$

of (11), which uniquely determines β. The following lemma straightforwardly follows from the equivalence described before between equivariant morphisms of modules over principal bundles and morphisms between the corresponding modules over the base manifolds:

Lemma 13 *Let (M, D) and (M', D') be two pairs of same dimension. Let $\tilde{\Psi} : E \longrightarrow E'$ be an isomorphism of holomorphic P-principal bundles over M and M' covering a morphism of pairs $\varphi : M \longrightarrow M'$ (i.e $\varphi(D) = D'$). Let $\tilde{\omega}_2$ be a meromorphic principal connection on $(E, \tilde{D}_1)$ where $\tilde{D}_1$ is the preimage of D (resp. $\tilde{\omega}_1 = \tilde{\Psi}^\star \tilde{\omega}_1$), and τ_1 (resp. τ_2) be the splitting as in (38). Then the diagram below is commutative:*

$$\begin{tikzcd} TM \arrow[r, "\tau_1"] \arrow[d, "d\varphi"'] & At(E)[\star D] \arrow[d, "p_{1*} d\tilde{\Psi}"] \\ \varphi^* TM' \arrow[r, "\varphi^* \tau_2"'] & \varphi^* At(E')[\star D] \end{tikzcd} \tag{39}$$

where p_1 is the footmap of E.

Denote by $\tilde{d}$ the usual de Rham differential on $\mathcal{O}_E[\tilde{D}] \otimes \mathbb{V}$. Since the P-linearization $(\phi_b^{\mathbb{V}})_{b \in P}$ preserves the subsheaf of constant functions with values in $\mathbb{V}$ on E, the pushforward $p_* \tilde{d}$ restricts to $p_* \tilde{d} : \mathcal{E} \longrightarrow p_* \Omega_E^1 \otimes \mathcal{E}$. This defines a meromorphic connection ∇ on $\mathcal{E}$ by:

$$\nabla = \tau \lrcorner p_* \tilde{d} \tag{40}$$

where $\tilde{d}$ is defined above and $\lrcorner$ stands for the contraction by a vector field.

Proposition 14 *Mapping a meromorphic principal connection $(E, \tilde{\omega})$ over (M, D) to the meromorphic connection $(\mathcal{E}, \nabla)$ on (M, D) defined by (40) induces an equivalence of categories between:*

- *The category of principal meromorphic (resp. holomorphic) connections over pairs, where the arrows are the $\mathbb{C}^r$-meromorphic isomorphisms (see Definition 4) of principal bundles between pairs preserving the principal connections (resp. isomorphisms of holomorphic principal bundles preserving the principal connections)*
- *The category of meromorphic (resp. holomorphic) connections on (M, D) with isomorphisms of meromorphic bundle (resp. holomorphic vector bundles, see Definition 2) preserving connections (in the sense of (**??**)).*

Proof Let's first prove that this map induces a functor. Let $\tilde{\Psi} : E \longrightarrow E'$ be an isomorphism of meromorphic principal connections between $(E, \tilde{\omega}_1)$ and $(E', \tilde{\omega}_1)$ over (M, D) and $(\mathcal{E}_1, \nabla_1)$ and $(\mathcal{E}_2, \nabla_2)$ obtained as in (40). Let (φ, Φ) be the associated isomorphism of vector bundles (see (6)). Fix any open subset $U \subset M$ and a basis $(s_i)_{i=1,\ldots,r}$ of $\mathcal{E}_1|_U$ and denote by $(\varphi^* t_i)_{i=1,\ldots,r}$ its image through Φ. Denote by

$(\tilde{s}_i)_{i=1,\dots,r}$ and $(\tilde{t}_{i=1,\dots,r})$ respectively the corresponding equivariant functions on $p_1^{-1}(U)$ and $p_2^{-1}(\varphi(U))$. Thus $\tilde{t}_i = \tilde{s}_i \circ \tilde{\Psi}$ by definition of Φ. By definition of $\Phi^{-1}\varphi^\star\nabla_2$, we can compute:

$$\Phi^{-1}\varphi^\star\nabla_2(s_i) = (Id_{\Omega^1_M} \otimes \Phi^{-1})[d\varphi \lrcorner (\varphi^*\nabla_2(\varphi^* t_i))] \tag{41}$$

Using the definition of ∇_1 and ∇_2, and Theorem 13, we get:

$$\begin{aligned} \Phi^{-1}\varphi^\star\nabla_2(s_i) &= (Id_{\Omega^1_M} \otimes \Phi^{-1})[(\varphi^*\tau_2 \circ d\varphi) \lrcorner \varphi^* p_{2*}\tilde{d}_2(\tilde{t}_i))] \\ &= \tau_1 \lrcorner (p_{1*}\tilde{d}_1\tilde{t}_i \circ \tilde{\Psi}) \\ &= \nabla_1(s_i) \end{aligned} \tag{42}$$

where we denoted by $\tilde{d}_1$ and $\tilde{d}_2$ the usual de Rham differentials on $\mathcal{O}_E \otimes \mathbb{C}^r$ and $\mathcal{O}_{E'} \otimes \mathbb{C}^r$. Hence we can map $\tilde{\Psi}$ to the vector bundle isomorphism (φ, Φ) which preserves the linear meromorphic connections ∇_1 and ∇_2.

Now, we construct the pseudo-inverse. Let $(\mathcal{E}, \nabla)$ be a meromorphic connection over a pair (M, D). Denote by E its frames bundle. Let $x \in M$ and U be a neighbhoorhood equipped with a holomorphic section $\sigma : U \longrightarrow E$. Denote by $(s_1, \dots, s_r)$ the corresponding basis of $\mathcal{E}|_U$. The section σ induces a splitting $TE|_{p^{-1}(U)} = p^*TU \oplus ker(dp)$ which is P-equivariant, hence a splitting

$$At(E)|_U = TU \oplus E(\mathfrak{p})|_U \tag{43}$$

We denote by τ_0 the splitting of the exact sequence (11) restricted to U induced by (43), and by d the pullback of the de Rham differential through the trivialization associated with $(s_i)_{i=1,\dots,r}$. Let $\delta = \nabla - d$, which vanishes on the image of $E(\mathfrak{p})$ through ι (see (11)). Its kernel thus define a morphism $\Theta : TU \longrightarrow At(E)|_U[\star D]$, and we obtain a splitting

$$\tau = \tau_0 + \Theta \tag{44}$$

of (11) over U. From the remarks above, this is equivalent to a meromorphic principal connection $\tilde{\omega}_U$ on $p^{-1}(U)$ with poles at $\tilde{D} \cap p^{-1}(U)$.

Now, let U, U' be two open subset and $(s_i)_{i=1,\dots,r}$ and $(s'_i)_{i=1,\dots,r}$ be two basis of $\mathcal{E}|_U$ and $\mathcal{E}|_{U'}$ corresponding to holomorphic sections σ, σ' of E on U and U'. Let d and d' be the corresponding de Rham differentials, then:

$$d - d'(s'_i) = d(s'_i) = d(\sum_{j=1}^r b^{-1}_{ji} s_j) = \sum_{j=1}^r (bd_0(b^{-1}))_{ji} s'_j \tag{45}$$

where b is the meromorphic function on $U \cap U'$ with values in P such that $\sigma' = \sigma \cdot b$, and d_0 is the usual de Rham differential on $\mathfrak{p}$-valued functions. Denote by τ and τ' constructed as before. Thus:

$$\tau' - \tau = [(\sigma, b^\star\omega_P)] \tag{46}$$

Thus $\nabla' - \nabla = d' - d + \tau' - \tau = 0$ and the corresponding meromorphic principal connections $\tilde{\omega}$ and $\tilde{\omega}'$ coïncide over $p^{-1}(U \cap U')$. We obtain a global meromorphic principal connection $\tilde{\omega}$ on $(E, \tilde{D})$ inducing ∇ as in (40).

If (φ, Φ_0) is an isomorphism of vector bundles preserving the meromorphic connections ∇_1, ∇_2, then from subsection 2.2 it induces an isomorphism $\tilde{\Psi}$ of holomorphic principal bundles between E and E'. Since the action of P on $\mathbb{C}^r$ is free, by definition of ∇_1 and ∇_2 we get that $\varphi^*\tau_2 = \tau_1 \circ d\varphi$. By Theorem 13 we obtain $\tilde{\Psi}^\star\tilde{\omega}_2 = \tilde{\omega}_1$.

□

5.2 Equivalence between meromorphic affine connections and meromorphic affine Cartan geometries

In this subsection, we consider the complex affine group G of dimension $n \geq 1$, and the complex linear group $P \subset G$. The restricted adjoint representation $ad : P \longrightarrow GL(\mathfrak{g})$ splits as the sum of two irreducible representations $\mathfrak{g}_-$, the subalgebra corresponding to the infinitesimal generators for the translations in $Aff(\mathbb{C}^n)$, and $\mathfrak{p} = Lie(P)$. Consequently, if $E \xrightarrow{p} M$ is a holomorphic P-principal bundle and ω_0 is a meromorphic (G, P)-Cartan connection on $(E, \tilde{D})$, then it splits as the sum:

$$\omega_0 = \theta_0 \oplus \tilde{\omega} \tag{47}$$

of a meromorphic solderform θ_0 on $(E, \tilde{D})$ (see Definition 13) and a meromorphic P-principal connection $\tilde{\omega}$ on $(E, \tilde{D})$.

Consider the category $\mathcal{F}_{conn}$ whose objects are triples $(\phi_0, \mathcal{E}, \overline{\nabla})$ formed by a meromorphic extension $(\phi_0, \mathcal{E})$ over a pair (M, D) and a meromorphic connection $(\mathcal{E}, \overline{\nabla})$ on (M, D), and the arrows are the isomorphisms of vector bunlde (see subsection 2.2) preserving the meromorphic connections (see (**??**)). Define the map f from the category $\mathcal{G}_{aff}$ of meromorphic (G, P)-Cartan geometries on (M, D) to $\mathcal{F}_{conn}$ as follows. If (E, ω_0) is an object of $\mathcal{G}_{aff}$, consider the meromorphic solderform (E, θ_0) (see Definition 13) defined by (47), and $\overline{\nabla}$ the meromorphic connection on $\mathcal{E} = E(\mathbb{C}^n)$ associated with $\tilde{\omega}$ (see Theorem 14).

Now, consider the subcategory $\mathcal{G}^0_{aff}$ of $\mathcal{G}_{aff}$ whose objects are holomorphic branched (G, P)-Cartan geometries, together with their isomorphisms. Consider a subcategory $\mathcal{F}^0_{conn}$ of $\mathcal{F}_{conn}$ obtained by intersecting with $\mathcal{F}^0$ (Definition 12).

Proposition 15 *Let (M, D) be a pair. The map f extends as an equivalence of categories between $\mathcal{G}_{aff}$ (resp. $\mathcal{G}^0_{aff}$) and $\mathcal{F}_{conn}$ (resp. $\mathcal{F}^0_{conn}$).*

Proof Let $\Psi : E \longrightarrow E'$ be an arrow between two meromorphic (G, P)-Cartan geometries (E, ω_0) and (E', ω'_0) over (M, D) and (M', D') , and $(\phi_0, \mathcal{E}, \overline{\nabla})$ and $(\phi'_0, \mathcal{E}', \overline{\nabla}')$ their images through f. So Ψ is a morphism of holomorphic principal bundles between the frame bundles and we define $f(\Psi) = (\varphi, \Phi)$ as the image of Ψ through the equivalence described in subsection 2.2. By construction, since $\Psi^\star \omega'_0 = \omega_0$, we have $\Psi^\star \theta'_0 = \theta_0$ and $\Psi^\star \tilde{\omega}' = \tilde{\omega}$. The first condition implies that (φ, Φ) is an arrow of meromorphic extensions (Definition 12), while the second one implies that it preserves the meromorphic connections ∇ and ∇' (see Theorem 13). Hence, f is a functor. Since it is the restriction of the equivalence of categories from Theorem 2, we obtain an equivalence of categories. □

Let $(\phi_0, \mathcal{E}, \overline{\nabla})$ be an object of $\mathcal{F}_{conn}$ on (M, D). Then:

$$\nabla = \phi_0^{-1} \overline{\nabla} \tag{48}$$

defines a meromorphic affine connection on (M, D), we will call it the *meromorphic affine connection induced by* $(\phi_0, \mathcal{E}, \overline{\nabla})$. Thus, there is a functor

$$\mu : \mathcal{F}_{conn} \longrightarrow \mathcal{A} \tag{49}$$

to the category $\mathcal{A}$ of meromorphic affine connections on pairs.

We denote by T_∇ the torsion of ∇ (Equation 3). There is the analogous notion of $\mathfrak{g}_-$-*torsion* for an object (E, ω_0) of $\mathcal{G}_{aff}$ on (M, D). It is the P-equivariant meromorphic function τ_{ω_0} on E with values in $\mathbb{W}_{\mathfrak{g}_-} = \bigwedge^2(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-$ defined as the projection of the Cartan curvature k_{ω_0} of (E, ω_0) (see Definition 8) on $\mathbb{W}_{\mathfrak{g}_-}$ respective to $\bigwedge^2(\mathfrak{g}_-)^* \otimes \mathfrak{p}$.

Finally, let's remark that for any object $(\mathcal{E}, \phi_0, \overline{\nabla})$ of $\mathcal{F}^0_{conn}$, the meromorphic affine connection (48) restricts as a holomorphic connection on the submodule $\mathcal{E}$. We then define:

Definition 22 The category $\mathcal{A}^0$ is the subcategory of $\mathcal{A}$ whose objects are the meromorphic affine connections on (M, D) preserving a locally free $\mathcal{O}_M$-module $\mathcal{E}$ with $TM \subset \mathcal{E} \subset TM[\star D]$, in the above sense. Its objects are called holomorphic branched affine connections.

Lemma 16 *Let ∇ be a holomorphic branched affine connection on (M, D). Then the submodule $\mathcal{E} \subset TM[\star D]$ from Definition 22 is unique.*

Proof Let E be the bundle of holomorphic frames for $\mathcal{E}$, and $\tilde{\omega}$ be the meromorphic principal connection on $R^1(M)$ corresponding to ∇ (Theorem 14). Suppose there exists another rank n locally free submodule $\mathcal{E}'$ of $TM[\star D]$ such that ∇ restricts to a holomorphic connection on $\mathcal{E}'$, and let $\tilde{\omega}'$ be the corresponding holomorphic princpal connection on its bundle of holomorphic frames E'.

Pick a point $x \in M$, and a neighbhoorhood U of x in M with two basis $(\overline{s}_1, \ldots, \overline{s}_n)$ of $\mathcal{E}|_U$ and $(\overline{t}_1, \ldots, \overline{t}_n)$ of $\mathcal{E}'|_U$. Denote by σ, σ' the holomorphic sections of $R^1(M \setminus D)$ on $U \setminus D$ corresponding respectively to these basis, and b be the unique holomorphic function on $U \setminus D$ with values in $GL_n(\mathbb{C})$ such that $\sigma' = \sigma \cdot b$. The classical jauge formula implies that b must be a solution of the differential equation:

$$d(b) = Ab - bA' \tag{50}$$

where A (resp. A') is the matrix of ∇ in the basis σ (resp. σ'). Since A and A' are holomorphic on U, we can use the proof of the Proposition II.2.13 in [17] to obtain that b extends on U as a holomorphic funcion. Reversing the roles of σ and σ', this is also true for b^{-1}, so that $\mathcal{E}$ and $\mathcal{E}'$ coïncide over U. We get the unicity. □

Corollary 17 *The composition of the equivalence from Theorem 15 and the map given by (49) gives an equivalence of categories between $\mathcal{G}_{aff}$ (resp. $\mathcal{G}^0_{aff}$) and $\mathcal{A}$ (resp. $\mathcal{A}^0$).*

Recall that, given any meromorphic affine connection ∇ on (M, D), a local holomorphic vector field $\overline{X}$ on an open subset $U \subset M$ is a *Killing field* for ∇ iff the pullback of ∇ by its flows is again ∇. We denote by $\mathfrak{kill}_\nabla$ the subsheaf of $TM \setminus D$ whose sections are the Killing field for ∇. By Theorem 17, we obtain:

Lemma 18 *If (E, ω_0) is an object of $\mathcal{G}_{aff}$ on a pair (M, D), and ∇ the corresponding meromorphic affine connection on (M, D), then $\mathfrak{kill}_{M,\omega_0} = \mathfrak{kill}_\nabla$.*

5.3 Geodesics of holomorphic branched affine connections and τ-affine connections

Let ∇ be a holomorphic affine connection on a complex manifold M, and (E, ω) be a holomorphic affine Cartan geometry inducing it. Recall that *geodesics* $\Sigma \subset M$ of ∇ are the projections of the A-distinguished curve of (E, ω_0) for $A \in \mathfrak{g}_-$. Equivalently, these are the images of holomorphic parametrized curves $\gamma : D(0, \epsilon) \longrightarrow M$ such that:

$$\gamma^\star \nabla(\frac{\partial}{\partial t}) = 0 \tag{51}$$

where $\frac{\partial}{\partial t}$ is the canonical vector field on the open disk $D(0, \epsilon) \subset \mathbb{C}$ and $\gamma^\star \nabla$ is the pullback (see for example [7]).

Definition 23 Let ∇ be a meromorphic affine connection on a pair (M, D). A *geodesic* of ∇ is a curve $\Sigma \subset M$ whose restriction to $M \setminus D$ is a geodesic of the holomorphic affine connection $\nabla|_{M \setminus D}$ in the above sense.

Lemma 19 *Let ∇ be a holomorphic branched affine connection on a pair (M, D). Let $\Sigma \subset M$ be a curve. Then the following assertions are equivalent:*

(i) Σ is a geodesic for ∇

(ii) For any non-constant holomorphic curve $\gamma : D(0, \epsilon) \longrightarrow \Sigma$, $\gamma^\star \nabla$ is the null morphism or there exists a holomorphic function h_γ on $D(0, \epsilon)$, which does'nt identically vanish, and such that:

$$\gamma^\star \nabla(\frac{1}{h_\gamma} \frac{\partial}{\partial t}) = 0$$

where $\frac{\partial}{\partial t}$ and $\gamma^\star \nabla$ are defined as above.

Proof Let (E, ω_0) be the unique holomorphic branched affine Cartan geometry inducing ∇ through the equivalence of Theorem 15, and such that $\mathcal{E} = E(\mathbb{C}^n)$ is the submodule of Definition 22.

We first prove that (i) implies (ii). By Theorem 6, and since $\gamma(D(0, \epsilon))$ is simply connected and not reduced to a point, there exists $A \in \mathfrak{g}_- \setminus \{0\}$ and a A-distinguished curve $\tilde{\Sigma} \subset E$ projecting onto $\gamma(D(0, \epsilon))$. Moreover, the restriction of $p : E \longrightarrow M$ to

$\tilde{\Sigma}$ is a biholomorphism to its image, and we denote by $\tilde{\gamma}$ the composition of γ with its inverse. For notational convenience, we assume from now on that $\Sigma = \gamma(D(0,\epsilon))$.

By definition of a distinguished curve, and since (E, ω_0) is holomorphic branched on (M, D), we have:

$$\tilde{\gamma}^\star \omega_0(\frac{\partial}{\partial t}) = \tilde{\gamma}^\star \theta_0(\frac{\partial}{\partial t}) = h_\gamma A \tag{52}$$

where θ_0 is the $\mathfrak{g}_-$-component of ω_0, and h_γ is a holomorphic function h_γ as in the statement. The function h_γ vanishes identically, if and only if $\gamma^\star \nabla$ is the null morphism. So from now on h_γ is supposed not to be the zero function.

The pullback $\gamma^* E$ is the frame bundle for $\gamma^* \mathcal{E}$. Moreover, $\gamma^* E = D(0,\epsilon) \times P$ and there is an isomorphism:

$$\hat{\gamma} : \gamma^* E \longrightarrow E|_\Sigma \tag{53}$$

satisfying $\hat{\gamma} \circ (t, Id) = \tilde{\gamma}(t)$. Then, by definition of the pullback, the Theorem 14 and the definition of (E, ω_0), we get:

$$\begin{aligned} \gamma^\star \nabla(\tfrac{1}{h_\gamma(t)} \tfrac{\partial}{\partial t}) &= (\gamma^* \phi_0)^{-1}([(\gamma, \tfrac{\partial}{\partial t}|_{s=t} \theta_0(T\hat{\gamma}(\tfrac{1}{h_\gamma(s)} \tfrac{\partial}{\partial t}, 0))]) \\ &= \phi_0^{-1}([(\tilde{\gamma}, \tfrac{1}{h_\gamma(t)} \tfrac{\partial}{\partial s}|_{s=t} \theta_0(\tilde{\gamma}'(s))]) \end{aligned}$$

Using (52), we obtain:

$$\gamma^\star \nabla(\frac{1}{h_\gamma} \frac{\partial}{\partial t}) = \phi_0^{-1}([(\tilde{\gamma}(t), \frac{\partial}{\partial s}|_{s=t} A)]) = 0$$

i.e (ii) is satisfied. □

Definition 24 A meromorphic affine connection ∇ on a pair (M, D) is said to be a τ-connection if no irreducible component of D is invariant by the geodesics of ∇ in the sense of Definition 23.

5.4 Holomorphic branched τ-connections in algebraic dimension zero

Now, we will give an application of the results of section 4 to the classification of affine meromorphic connections on some simply connected complex compact manifolds M. Most of them are adaptations of the arguments used in the proof of the principal theorem in [1], using the results of this article.

Theorem 20 *Let (M, D) be a pair, with M a simply connected complex compact manifold. If (M, D) bears a quasihomogeneous and totally geodesic meromorphic affine connection ∇, then it admits a meromorphic parallelism $(\overline{X}_1, \ldots, \overline{X}_n)$, such that $\overline{X}_i$ is a Killing vector field for ∇ when restricted to $M \setminus D$.*

Proof Let (E, ω_0) be the meromorphic affine Cartan geometry on M corresponding to ∇ (see Theorem 17). By the Theorem 10, it satisfies the extension property of infinitesimal automorphisms, i.e the local system $\mathfrak{kill}_\nabla$ on $M \setminus D$ extends as a local system $\mathcal{K}$ on M, with $\mathcal{K} \subset TM[\star D]$. Since M is simply connected, this is a constant sheaf on M. Since ∇ is assumed quasihomogeneous, we can pick $x \in M$ and a

$\mathcal{O}_{M,x}$-basis $\overline{X}_{1,x},\ldots,\overline{X}_{n,x}$ of $(TM)_x$ formed by germs of Killing fields for ∇. These germs are thus restrictions of global meromorphic vector fields $\overline{X}_1,\ldots,\overline{X}_n$ whose restrictions to $M\setminus D$ are elements of $\mathfrak{kill}_\nabla(M\setminus D)$. Since their germs at x are linearly independent, there exists a Zariski-dense open subset $M\setminus S$ such that the restrictions of $\overline{X}_1,\ldots,\overline{X}_n$ to any subset $U\subset M\setminus S$ are independent elements of $TM(U)$, i.e $(\overline{X}_1,\ldots,\overline{X}_n)$ is a meromorphic parallelism on M.

□

We obtain:

Theorem 21 *Let M be a compact complex manifold with finite fundamental group, and whose meromorphic functions are constants. Then M doesn't bear any totally geodesic branched holomorphic affine connection.*

Proof Suppose that ∇ is a totally geodesic branched holomorphic affine connection on (M,D), denote by $\mathcal{E}$ the submodule of $TM[\star D]$ from Theorem 16. Complete the meromorphic parallelism $(\overline{X}_i)_{i=1,\ldots,n}$ from Theorem 20 into a basis $(\overline{X}_j)_{j=1,\ldots,r}$ for the global meromorphic Killing fields of ∇. A meromorphic parallelism is a rigid geometric structure (see [26]), so by the Theorem 2 of [27], the juxtaposition of $(\overline{X}_j)_{j=1,\ldots,r}$ and ∇ is quasihomogeneous. Since ∇ satisfies the extension property for the Killing vector fields (Theorem 10) and M is simply connected, we obtain a meromorphic parallelism $(\overline{X}'_i)_{i=1,\ldots,n}$ such that the restriction of each $\overline{X}'_i$ to $M\setminus D$ is a Killing field for ∇ and commutes with each $\overline{X}_j$. In particular, each $\overline{X}'_i$ is a $\mathbb{C}$-linear combination of the $(\overline{X}_j)_{j=1,\ldots,r}$, so $(\overline{X}'_i)_{i=1,\ldots,n}$ are commuting meromorphic vector fields.

Now, let pick any Gauduchon metric on M ([28]) and let's prove that the degree $deg(\mathcal{E})$ of $\mathcal{E}$ with respect to this metric is zero. Let (E,ω_0) be the branched holomorphic affine Cartan geometry on (M,D) corresponding to ∇ (Theorem 17). Then $\mathcal{E}=E(\mathfrak{g}/\mathfrak{p})=E(\mathfrak{g})/E(\mathfrak{p})$ by definition of (E,ω_0), and since $P=GL_n(\mathbb{C})$, we have $deg(E(\mathfrak{p}))=0$ (see [12], Corollary 4.2).

We must then prove that $deg(E(\mathfrak{g}))=0$. For, it is sufficient to prove that $C_1(R_{\nabla^{\omega_0}})$ vanishes identically, where C_1 is the trace on $End(E(\mathfrak{g}))$ and ∇^{ω_0} is the tractor connection (see Definition 11). We will prove that the meromorphic one form $\eta_i=X'_i\lrcorner C_1(R_{\nabla^{\omega_0}})$ vanishes identically on M for any $1\leq i\leq n$. By Theorem 1, we have:

$$p^\star\eta_i=\tilde{X}'_i\lrcorner C_1(R_{p^\star\nabla^{\omega_0}})=\underbrace{dC_1(Ad(s_i))}_{\tilde{\eta}_i^0}+\underbrace{\tilde{X}'_i\lrcorner Ad(\omega_0\wedge\omega_0)}_{\tilde{\eta}_i^1} \tag{54}$$

where $\tilde{X}'_i$ is the lifting of X'_i to E and $s_i=\omega_0(X'_i)$.

The meromorphic one form $\tilde{\eta}_i^0$ is exact and P-equivariant. By a classical result on exact invariant forms on connected Lie groups, the restriction of $\tilde{\eta}_i^0$ to any fiber of $E\xrightarrow{p}M$ corresponds to a homomorphism $\chi:P\longrightarrow\mathbb{C}$. Because $P=GL_n(\mathbb{C})$, any such homomorphism is trivial, so that $\tilde{\eta}_i^0$ vanishes on the fibers of $E\xrightarrow{p}M$. Thus, it is the pullback of a meromorphic exact one form η_i^0 on M. Moreover, by Theorem 10, s_i is a holomorphic section of $\mathcal{V}$ on E, so that $\tilde{\eta}_i^0$ is a holomorphic one form. Thus, η_i^0 is an exact holomorphic one form on a simply connected compact complex manifold, i.e vanishes everywhere.

Now, let's prove that $\tilde{\eta}_i^1 = p^\star \eta_i$ vanishes everywhere. Consider $E_G = E \times G \xrightarrow{pi_G} E$ and the holomorphic tractor connection $\tilde{\omega}$ on it (Definition 11). Using the splitting $TE_G = ker(\tilde{\omega}) \oplus ker(d\pi_G)$, the pullback $\hat{\eta}_i^1 = \pi_G^\star \tilde{\eta}_i^1$ uniquely decomposes as a sum:

$$\hat{\eta}_i^1 = \hat{\eta}_i^H \oplus \hat{\eta}_i \tag{55}$$

with $\hat{\eta}_i^H$ a G-invariant meromorphic one form on E_G, vanishing on $ker(\tilde{\omega})$, and $\hat{\eta}_i^V$ vanishing on $ker(\tilde{\omega})$. In particular, $\hat{\eta}_i^H$ is the pullback of η_i through the composition $p_G = p \circ \pi_G$, so that $\hat{\eta}_i^V$ vanishes everywhere. Now, using Theorem 10, $\tilde{\eta}_i^1$ is a holomorphic one form, so that η_i is a holomorphic one form on M. Using the Lie-Cartan formula, we have:

$$d\eta_i(X'_j, X'_k) = \mathcal{L}_{X'_j}\eta_i(X'_k) - \mathcal{L}_{X'_k}\eta_i(X'_j) - \eta_i([X'_j, X'_k]_{TM}) \tag{56}$$

Since the only meromorphic functions on M are the constants, we obtain $\mathcal{L}_{X'_j}\eta_i(X'_k) = \mathcal{L}_{X'_k}\eta_i(X'_j) = 0$, and since the meromorphic vector fields $(X'_i)_{i=1,\dots,n}$ commute, η_i is a closed holomorphic one form. Since M is simply connected and compact, η_i vanishes everywhere. This proves that $C_1(R_{\nabla^{\omega_0}})$ vanishes everywhere, i.e $deg(E(\mathfrak{g})) = 0$.

Hence, $deg(\mathcal{E}) = 0$. Let $\overline{s}_1, \dots, \overline{s}_n$ be the images of $X'_1, \dots, X'_n$ through the morphism ϕ_0, where $(\phi_0, \mathcal{E})$ is the holomorphic extension image of (E, ω_0) as in Theorem 15. Since $s_i = \omega_0(\tilde{X}'_i)$ is a section of $\mathcal{V}(E)$ for any $1 \leq i \leq n$ from Theorem 10, each $\overline{s}_i$ is a section of $\mathcal{E}(M)$. Since they are independent, the holomorphic section $\bigwedge_{i=1}^{n} \overline{s}_i$ of $det(\mathcal{E})$ is not identically vanishing, thus $det(\mathcal{E})$ is trivial and $\bigwedge_{i=1}^{n} \overline{s}_i$ never vanishes. It therefore forms a basis of $\mathcal{E}$ on M, and the dual sections $\overline{s}_1^*, \dots, \overline{s}_n^*$ are holomorphic sections of $\mathcal{E}^*$ on M. We obtain a branched holomorphic $(\mathbb{C}^n, \{0\})$-Cartan geometry (M, η) on (M, D), where:

$$\eta = \bigwedge_{i=1}^{n} (\overline{s}_i^* \circ \phi_0) \otimes \mathfrak{e}_i \tag{57}$$

where $(\mathfrak{e}_i)_{i=1,\dots,n}$ is the canonical basis of $\mathbb{C}^n$. Because the η-constant vector fields $(X'_i)_{i=1,\dots,n}$ commute, it is a flat branched holomorphic Cartan geometry. Since M is simply connected and compact, there is a holomorphic submersion $dev : M \longrightarrow \mathbb{C}^n$. This is impossible by the maximum principle, so M cannot bear any totally geodesic branched holomorphic affine connection.

□

6 Genericity of the affine τ-condition on surfaces

In this section, we fix a pair (M, D) where M is a complex surfaces and D an effective divisor of M with irreducible and reduced components $(D_\alpha)_{\alpha \in I}$. We fix a submodule $\mathcal{E} \subset TM[\star D]$ with $TM \subset \mathcal{E}$.

Definition 25 In the above setting, we denote by $\mathcal{A}^0_{\mathcal{E}}$ the set of holomorphic branched affine connections on (M, D) whose associated submodule is $\mathcal{E}$ (Definition 22). We denote by $\mathcal{A}^0_{\mathcal{E},\tau}$ the subset of $\mathcal{A}^0_{\mathcal{E}}$ consisting of τ-connection (Definition 24).

We prove a result of genericity for $\mathcal{A}^0_{\mathcal{E},\tau}$ (Theorem 24). We also give examples of flat holomorphic branched affine τ-connections on compact complex manifolds of arbitrary dimension, and one example of a non-flat holomorphic branched affine τ-connection on a complex compact threefold.

6.1 Consequence of the existence of a τ-connection on the submodule

We begin with a necessary condition on $\mathcal{E}$ for $\mathcal{A}^0_{\mathcal{E},\tau}$ not being empty.

Proposition 22 *Suppose the exitence of a τ-connection ∇ in $\mathcal{A}^0_{\mathcal{E},\tau}$. Then $\mathcal{E}$ satisfies the following property. Let D_α be an irreducible component of D, and $x \in D_\alpha$. Let (z_1, z_2) be local coordinates on an open neighborhood U of x with*

$$D_\alpha \cap U = \{z_1 = 0\}$$

The matrix of any element $\varphi \in End(\mathcal{E})(U)$, identified with an element of $End(\mathcal{E})[\star D](U)$, in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ takes the form:

$$\begin{pmatrix} * & z_1 f_1 \\ * & f_2 \end{pmatrix} \tag{58}$$

where f_1, f_2 are holomorphic functions on U.

Proof Let ∇, D_α, x and U as in the statement. Let (E, ω_0) be the unique holomorphic branched affine Cartan geometry inducing ∇ and such that E is the frame bundle of $\mathcal{E}$. The hypothesis on ∇ implies that the pullback $\tilde{D}_\alpha$ of D_α to E is invariant through the A-distinguished foliation $\mathcal{T}_A$ (see Equation 19), for any $A \in \mathfrak{g}_-$. Using the proof Theorem 5, and denoting by $(\frac{\tilde{\partial}}{\partial z_1}, \frac{\tilde{\partial}}{\partial z_2})$ two P-invariant holomorphic vector fields on $p^{-1}(U)$ projecting on $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$, we get that the ω_0-constant vector fields Y_1, Y_2 associated with $\mathfrak{e}_1, \mathfrak{e}_2$ decompose as:

$$Y_i = z_1^{n_i} g_i \frac{\tilde{\partial}}{\partial z_1} + z_1^{m_i} g_i' \frac{\tilde{\partial}}{\partial z_2} + Y_i' \tag{59}$$

where Y_i' is an element of $ker(dp)(p^{-1}(U))$, g_i, g_i' are invertible or identically vanishing holomorphic functions, and:

$$0 \geq n_i > m_i \text{ or } g_i = 0 \tag{60}$$

Now, fix any holomorphic section σ of E on U and denote by $(\overline{Y}_1, \overline{Y}_2)$ the corresponding basis of $\mathcal{E}$ on U. These are the projections of $Y_1 \circ \sigma$ and $Y_2 \circ \sigma$ through Tp, so that:

$$\overline{Y}_i = z_1^{n_i} g_i \frac{\partial}{\partial z_1} + z_1^{m_i} g_i' \frac{\partial}{\partial z_2} \tag{61}$$

The inequality (60) implies that, up to replacing σ by $\sigma \cdot b$ for some holomorphic function $b : U \longrightarrow P$, we can suppose that $g_1 = 0$, i.e the matrix Q of $(\overline{Y}_1, \overline{Y}_2)$ in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ is:

$$Q = \begin{pmatrix} 0 & z_1^{n_2} g_2 \\ z_1^{m_1} g_1' & z_1^{m_2} g_2' \end{pmatrix} \tag{62}$$

with g_2, g_1', g_2', n_2 and m_2 as before. Since $\mathcal{E}|_U$ contains TU, the inverse of this matrix must be a holomorphic matrix on U.

This implies the following identities:

$$\begin{array}{rl} n_2 + m_2 - m_1 & \geq 0 \\ n_2 & < 0 \\ m_1 & < 0 \\ m_2 & \leq 0 \end{array} \tag{63}$$

Let $\varphi \in End(\mathcal{E})(U)$ with matrix $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ in $(\overline{Y}_1, \overline{Y}_2)$. Then the matrix of the same section in the basis $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ is:

$$Q \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} Q^{-1} = \begin{pmatrix} * & g z_1^{n_2 - m_2} \\ * & g' z_1^{m_2 - m_1} \end{pmatrix} \tag{64}$$

for some holomorphic functions g, g' on U. Using (63), we get that this matrix has the desired form. □

6.2 Intermediate condition and local characterization

We continue by introducing a subset $\mathcal{A}^0_{\mathcal{E},1} \subset \mathcal{A}^0_{\mathcal{E}}$, containing the complement of $\mathcal{A}^0_{\mathcal{E},\tau}$ in $\mathcal{A}^0_{\mathcal{E}}$. This subset has the advantage that its elements ∇ can be described by their local Christoffel symbols.

Definition 26 The set $\mathcal{A}^0_{\mathcal{E},1}$ is the subset of elements $\nabla \in \mathcal{A}^0_{\mathcal{E}}$, with the following property. For any $x \in D_\alpha$, where D_α is some irreducible component of D, there exists a non-constant geodesic $\gamma : D(0, \epsilon) \longrightarrow M$ for ∇ with

$$\gamma(0) = x \text{ and } Im(\gamma) \subset D_\alpha \tag{65}$$

Lemma 23 *Let $\nabla \in \mathcal{A}^0_{\mathcal{E}}$. The following properties are equvialent:*

(i) $\nabla \in \mathcal{A}^0_{\mathcal{E},1}$

(ii) For any irreducible component D_α of D, any $x \in D_\alpha \setminus \bigcup_{\beta \neq \alpha} D_\beta$, and any open neighborhood U of x with local coordinates (z_1, z_2) as in Theorem 22, the matrix of ∇ in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ is of the form:

$$\sum_{i=1,2} dz_i \otimes \begin{pmatrix} a_i & c_i \\ b_i & d_i \end{pmatrix} \tag{66}$$

with:

$$\begin{cases} c_2 & \textit{vanishing on } D_\alpha \cap U \\ d_2 & \textit{holomorphic} \end{cases} \tag{67}$$

Proof We first prove (i) implies (ii). Suppose that ∇ is as in (i), and let γ be a non constant geodesic at $x \in D_\alpha$ with $Im(\gamma) \subset D_\alpha$. Let U and (z_1, z_2) as in (ii) and denote by $\gamma_i = z_i \circ \gamma$, so that γ_1' is identically vanishing on $D(0, \epsilon)$ by the hypothesis on γ. The first line of the generalized geodesic equation from Theorem 19 implies that $c_2 \circ \gamma$ must be identically vanishing, since γ_2' is not identically vanishing by

hypothesis. In particular, since $D_\alpha \cap U$ is irreducible, c_2 must be identically vanishing on this divisor. Moreover, the second line of the same equation implies that $y = \frac{1}{h_\gamma}\gamma_2'$ is a solution of the differential equation:

$$y' = (d_2 \circ \gamma)y \tag{68}$$

Now, let $\tilde{\gamma}$ be the lifting of γ to a A-distinguished curve of the unique holomorphic branched affine Cartan connection (E, ω_0) inducing ∇ and with $E = R(\mathcal{E})$. Then $\tilde{\gamma}^\star\omega_0$ is of constant rank. Recall that the holomorphic function h_γ in Theorem 19 is defined by $\tilde{\gamma}^\star\omega_0(\frac{\partial}{\partial t}) = h_\gamma A$. But ω_0 has constant rank on the pullback $\tilde{D}_\alpha$ of D_α, by definition of D. Thus, h_γ must be an invertible holomorphic function on $D(0,\epsilon)$, so that y is a holomorphic function. Then $d_2 \circ \gamma$ must be holomorphic, and since $\gamma'(0) \neq 0$, this implies that d_2 has no pole along $D_\alpha \cap U$. But the poles of ∇ are contained in D, so that d_2 is a holomorphic function on the whole U.

Now we prove (ii) implies (i). Let $\nabla \in \mathcal{A}^0_{\mathcal{E}}$ and suppose (ii) is satisfied. Let $x \in D_\alpha$, and $(U, (z_1, z_2))$ as above, and suppose moreover that $z_1(x) = z_2(x) = 0$. Define $\gamma_1 = 0$, and γ_2 to be a solution of (68) on $D(0,\epsilon)$ $(\epsilon > 0)$ with $\gamma_2(0) = 0$. Such a solution exists because d_2 is holomorphic. Then the unique holomorphic curve $\gamma : D(0,\epsilon)$ with $\gamma_i = z_i \circ \gamma$ is a geodesic of ∇ by Theorem 19. By construction, $Im(\gamma) \subset D_\alpha$, i.e. $\nabla \in \mathcal{A}^0_{\mathcal{E},1}$. □

6.3 Genericity result

The set $\mathcal{A}^0_{\mathcal{E}}$ has the structure of an affine space directed by $End(\mathcal{E})(M)$. Indeed, suppose there exists $\nabla \in \mathcal{A}^0_{\mathcal{E}}$. Let ∇' be any meromorphic connection on (M, D), and:

$$\Theta = \nabla' - \nabla \tag{69}$$

which is an element of $\Omega^1_M \otimes End(TM)[\star D](M) = \Omega^1_M \otimes End(\mathcal{E})[\star D]$. Then it is immediate that $\nabla' \in \mathcal{A}^0_{\mathcal{E}}$ exactly when $\Theta \in \Omega^1_M \otimes End(\mathcal{E})(M)$.

Using the above remark and Theorem 23, we get the following result:

Theorem 24 *Let (M, D) be a pair and $\mathcal{E} \subset TM[\star D]$ a submodule containing TM. Then one and only one of the following assertions is true:*

(a) $\mathcal{A}^0_{\mathcal{E},\tau} = \mathcal{A}^0_{\mathcal{E}}$

(b) $\mathcal{A}^0_{\mathcal{E},1} = \mathcal{A}^0_{\mathcal{E}}$

Proof Suppose (a) to be false and let's prove (b). By hypothesis, there exists an element $\nabla \in \mathcal{A}^0_{\mathcal{E}}$ which is not a τ-connection. Fix any element ∇' of $\mathcal{A}^0_{\mathcal{E}}$. Fix an irreducible component D_α of D, a point $x \in D_\alpha \setminus \bigcup_{\beta \neq \alpha} D_\beta$ and an open neighborhood U of x with coordinates (z_1, z_2) such that $U \cap D_\alpha = \{z_1 = 0\}$. The matrix of Θ from (69) in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ is the difference between the matrices of ∇' and ∇. It has the form (58) by Theorem 22. By Theorem 23, and since ∇ is already an element of $\mathcal{A}^0_{\mathcal{E},1}$ we get that ∇' is again an element of $\mathcal{A}^0_{\mathcal{E},1}$, that is (b) is true.

□

In particular, in order to prove that a complex manifold M admits a holomorphic branched τ-connection with poles at D, and with associated submodule $\mathcal{E}$, it is sufficient to prove that there exists a holomorphic connection ∇ on $\mathcal{E}$ which does'nt belong to $\mathcal{A}^0_{\mathcal{E},1}$, i.e such that for any irreducible component D_α of D, there exists $x \in D_\alpha$ such that no geodesic γ of ∇ at x is contained in D_α.

6.4 Example in any dimension

We now construct an example, for any $n \geq 1$, of a compact complex manifold M of dimension n, equipped with a submodule $\mathcal{E} \subset TM[\star D]$ with $TM \subset \mathcal{E}$ and an object $\nabla \in \mathcal{A}^0_{\mathcal{E},\tau}$. Namely, M are Hopf manifolds and ∇ is constructed from branched coverings between these manifolds and from the holomorphic affine structure coming from their universal covering.

Pick any $\lambda \in]0,1[$, and define Γ' and Γ to be respectively the abelian groups spanned by the linear automorphisms $\lambda^2 Id_{\mathbb{C}^n}$ and $(z_1, z_2, \ldots, z_n) \mapsto (\lambda z_1, \lambda^2 z_2, \ldots, \lambda^2 z_n)$ of $\mathbb{C}^n$. Let $\tilde{M} = \mathbb{C}^n \setminus \{0\}$, and let $M = \Gamma \backslash \tilde{M}$ and $M' = \Gamma' \backslash \tilde{M}$. M and M' are Hopf manifolds associated with Γ and Γ', so these are complex compact manifolds. Since Γ' is a subgroup of the affine group of $\mathbb{C}^n$, the canonical affine structure ∇_0 of $\mathbb{C}^n$ (restricted to $\tilde{M}$) descends as a holomorphic affine connection ∇' on M'.

The map $\tilde{f} : \tilde{M} \longrightarrow \tilde{M}$ given by $\tilde{f}(z_1, \ldots, z_n) = (z_1^2, \ldots, z_n)$ is equivariant for the actions of Γ and Γ'. Thus we obtain a map $f : M \longrightarrow M'$ defined by the commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M} \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & M' \end{array} \tag{70}$$

By construction, f is a double covering with ramification locus the divisor D of M obtained as the quotient of $\tilde{D} = \{z_1 = 0\}$ by the action of Γ. The pullback $\nabla = f^\star \nabla'$ is an object of $\mathcal{A}^0_{\mathcal{E},\tau}$ where $\mathcal{E} = \mathcal{O}_M \underset{f^{-1}\mathcal{O}_{M'}}{\otimes} f^{-1}TM'$. Indeed, by construction, ∇ pulls back to $\mathbb{C}^n$ (first to $\tilde{M}$, then to $\mathbb{C}^n$ by the Hartog's extension theorem) as the Γ-invariant meromorphic affine connection $\tilde{\nabla} = \tilde{f}^\star \nabla_0$. Now, the curve $\tilde{\Sigma} = \{z_2 = \ldots = z_n = 0\}$ projects onto itslef through $\tilde{f}$, and is a geodesic for ∇_0. Thus, its projection Σ on M is a geodesic for ∇, and by construction Σ intersects D exactly at one point, namely the class of $(0, \ldots, 0)$.

Unfortunately, the examples just constructed are flat, i.e they are obtained from a branched developping map $\tilde{f} : \tilde{M} \longrightarrow \mathbb{C}^n$ from the universal cover of the manifold M to the affine model $\mathbb{C}^n = Aff(\mathbb{C}^n)/GL_n(\mathbb{C})$. Moreover, there are no non-flat examples on the Hopf manifolds M: this can be recovered using that $End(\mathcal{E})$ is the trivial module on M, that Ω^1_M has no non-trivial global sections, and the parametrization of $\mathcal{A}^0_{\mathcal{E}}$ as in Equation 69.

However, we mention the existence of non-flat examples due to the following construction, by S. Cantat in [29],of a complex compact manifold M, equipped with a holomorphic parallelism with non-trivial structure constants, and with a multiple branched cover $f : M \longrightarrow M$ as a self-map.

Let H_3 be the complex Heisenberg group, that is the subgroup of $SL_3(\mathbb{C})$ of upper-triangular matrices with ones on the diagonal. The subgroup $\Gamma := H_3(\mathbb{Z}[i])$ of elements with coefficients in $\mathbb{Z}[i]$ is a cocompact lattice of H_3. The quotient $M = H_3/\Gamma$ is then a complex compact manifold. It is equipped with the holomorphic parallelism $(\overline{Z}_1, \overline{Z}_2, \overline{Z}_3)$ obtained from any basis of righ-invariant holomorphic vector fields on $H_3(\mathbb{C})$. We denote by ∇' the unique holomorphic affine connection on $M' = M$ such that $\nabla'(\overline{Z}_i) = 0$. By construction, the torsion of ∇' is not zero. It is proved in [29] (Exemple 5.3) that there exists a finite surjective morphism $f : M \longrightarrow M'$ with ramification locus a non-trivial divisor D of M. We can reproduce the constructions used with the Hopf manifolds to obtain an object $\nabla = f^\star\nabla'$ of $\mathcal{A}^0_{\mathcal{E}}$ where $\mathcal{E} = \mathcal{O}_M \underset{f^{-1}\mathcal{O}_{M'}}{\otimes} f^{-1}TM'$.

Then we use the:

Lemma 25 *Let $f : M \longrightarrow M'$ be a branched cover between two complex manifolds M, M', with ramification locus $D \subset M$ (we denote $D' = f_*(D)$), and $\nabla' \in \mathcal{A}^0_{\mathcal{E}',\tau}$ where $\mathcal{E}'$ is any submodule of $TM'[\star D']$ as in Definition 22. Then the pullback $\nabla = f^\star\nabla'$ is an object of $\mathcal{A}^0_{\mathcal{E},\tau}$ where $\mathcal{E} = \mathcal{O}_M \underset{f^{-1}\mathcal{O}_{M'}}{\otimes} f^{-1}\mathcal{E}'$.*

Proof The fact that ∇ is an object of $\mathcal{A}^0_{\mathcal{E}}$ is clear from the equivalence Theorem 17 and the definition of a holomorphic branched Cartan geometry. Now, let D_α be any irreducible component of D, D'_α its projection through f. Since ∇' is a τ-connection, there exists $x'_0 \in D'_\alpha$ and a geodesic Σ' for ∇' such that $\Sigma' \cap D'_\alpha = \{x'_0\}$. Let x_0 be any point of the fiber $f^{-1}(x'_0) \subset D_\alpha$, and $\Sigma = f^{-1}(\Sigma')$. Using the characterization of Theorem 19, and the definition of the pullback, we obtain that Σ is a geodesic of ∇. By construction $\Sigma \cap D_\alpha$ is the finite set of points $f^{-1}(x'_0)$, so we can consider a neighborhood U of x_0 such that $\Sigma \cap U$ is a geodesic of ∇, intersecting D_α exactly at x_0. So $\nabla \in \mathcal{A}^0_{\mathcal{E},\tau}$.

□

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