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► **To cite this version:**

| Alexis Garcia. Killing vector fields for some meromorphic affine connections. 2022. hal-03680548v2

**HAL Id: hal-03680548**

**<https://hal.science/hal-03680548v2>**

Preprint submitted on 2 Jun 2022 (v2), last revised 8 Nov 2022 (v5)

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# Killing vector fields for some meromorphic affine connections.

Alexis Garcia

June 2, 2022

## Abstract

We give a geometric assumption on a meromorphic affine connection for its Killing vector fields to be univalent. For, we prove a general result on the infinitesimal automorphisms for a subcategory of meromorphic Cartan geometries, and uses the equivalence with the former geometric structures. This result is applied to the classification on complex compact manifolds with algebraic dimension zero, extending partially the main result of [3].

## 1 Introduction

Many geometric structures over complex or real manifolds are infinitesimal versions of some *homogeneous* geometric structure on a model space, i.e on which the group of global automorphisms of the structure act transitively on this space. Thus, the *infinitesimal isometries*, i.e differentiable or holomorphic local vector fields whose local flows preserve the geometric structure, are of special interest when we try to classify them. When these vector fields span the tangent space at any point, we say that the geometric structure is *locally homogeneous* (resp. *quasi-homogeneous* if an open dense subset is locally homogeneous). The classification of simply connected manifolds (complex or real) admitting a *locally homogeneous* geometric structure is a classical problem in differential geometry, and started with the riemannian case. In particular, Riemann, Killing and Hopf proved that any simply connected and locally homogeneous riemannian manifold is in fact *homogeneous* (see [18]).

The quasi-homogeneity of the holomorphic geometric structure on a compact complex manifold is sometimes obtained from complex-geometrical assumptions on the base manifold. For example, any *rigid* holomorphic geometric structures (see [8]) on a compact complex manifold whose meromorphic functions are constants is quasi-homogeneous, by a result proved by S. Dumitrescu in [11]. It was extended to *holomorphic Cartan geometries* and their *infinitesimal automorphisms* (or *Killing fields*). These are objects related with holomorphic geometric structures in the sense that any holomorphic Cartan geometry on a base complex manifold  $M$ , with model a complex Klein geometry  $(G, P)$ , *induces* a holomorphic geometric structure whose model only depends on  $(G, P)$ . For example, when  $(G, P)$  is the affine Klein geometry, the holomorphic geometric structure is a holomorphic affine connection (see Corollary 5.1).

In [3], the following classification result were proved : any simply connected complex compact manifold  $M$ , whose meromorphic functions are constants, cannot bear a holomorphic  $(G, P)$ -Cartan geometry. Two important facts are used in the proof. First, it is proved that any germ of infinitesimal automorphism of the Cartan geometry is the germ of a global infinitesimal automorphism : this is a generalization of a result by Nomizu ([17]) for analytic riemannian metrics, and follows from the fact that the former objects form a local system on  $M$ . Next, it is proved

that there is a family of global infinitesimal automorphisms, whose existence is ensured by the previous fact, which integrates into an action by automorphisms of a complex abelian Lie group  $L$  with an open dense orbit in  $M$ . The conclusion follow from detailed study of the geometry of such manifolds  $M$ , which implies that the Cartan geometry is flat.

In the meromorphic category, the two above properties may not stand : infinitesimal automorphisms could be multivaluated, and a meromorphic univaluated infinitesimal automorphism may not have a well defined flow at some point of the pole. In this paper, we give a sufficient condition on some meromorphic Cartan geometries to recover the first step, and apply it to the classification of meromorphic affine connections. The section 2 recall the dictionary between locally free modules of finite rank and vector bundles, the corresponding meromorphic sections, and recall the definition of the Atiyah's exact sequence associated with a principal bundle. The section 3 introduces meromorphic Cartan geometries and the holomorphic vector bundles naturally associated to these objects. The section 4 gives sufficient conditions for two classes of regular meromorphic *parabolic* geometries (subsection 4.1) for their infinitesimal automorphisms to be univaluated. Finally, the section 5 applies the previous results to the classification of some meromorphic affine connections on complex compact manifolds with *algebraic dimension zero*.

## 2 Preliminaries and notations

### 2.1 Locally free modules and meromorphic connections

Let  $(M, D)$  be a **pair**, i.e a complex manifold  $M$  equipped with a divisor  $D$ . Let  $\mathcal{L}$  be a coherent  $\mathcal{O}_M$ -module. The **order**  $ord_D^{\mathcal{L}}(s)$  at  $D$  of a section  $s$  of  $\mathcal{L}[*D]$  defined on an open subset  $U \subset M$  is the greatest integer  $d \in \mathbb{Z}$  such that  $s$  is also a section of  $\mathcal{L}(-dD)|_U$  on  $U$ .

**Definition 2.1.** *A meromorphic connection on  $(M, D)$  is a couple  $(\mathcal{V}, \nabla)$  where  $\mathcal{V}$  is a locally free  $\mathcal{O}_M$ -module of finite rank, and  $\nabla$  is a morphism of  $\mathbb{C}$ -sheaves from  $\mathcal{V}[*D]$  to  $\Omega_M^1 \otimes \mathcal{V}[*D]$  satisfying the Leibniz identity  $\nabla(fs) = d(f)s + f\nabla(s)$  for any  $s \in \mathcal{V}(U)$  and  $f \in \mathcal{O}_M[*D](U)$  ( $U$  is an open subset of  $M$ ).*

If  $(\mathcal{L}, \nabla)$  and  $(\mathcal{L}', \nabla')$  are two meromorphic connections related by  $\mathcal{L} = \bigoplus_{i=1}^r \mathcal{O}_M s_i$ ,  $\mathcal{L}' = \bigoplus_{i=1}^r \mathcal{O}_M t_i$  and  $t_i = \sum_{j=1}^r q_{ji} s_j$  for a meromorphic matrix  $Q$  on  $M$ , then the matrices  $A$  and  $A'$  respectively associated to the basis  $(s_i)_{i=1, \dots, r}$  and  $(t_i)_{i=1, \dots, r}$  are related by the gauge-transformation formula :

$$A' = Q^{-1}dQ + Q^{-1}AQ \quad (1)$$

where  $d$  stands for the deRham derivative.

A **meromorphic affine connection** on  $(M, D)$  is a meromorphic connection  $\nabla$  on  $TM$  with poles supported at  $D$ . The **torsion** of a meromorphic affine connection  $\nabla$  on  $(M, D)$  is the meromorphic section  $T_{\nabla}$  of  $\Omega_M^1 \otimes \text{End}(TM)$  defined by :

$$T_{\nabla}(X)(Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y]_{TM} \quad (2)$$

Let  $(M, D)$  be a pair and  $r \geq 1$  be an integer. Recall that mapping a holomorphic rank  $r$  vector bundle  $V$  over  $M$  to the locally free rank  $r$   $\mathcal{O}_M$ -module  $\mathcal{E}$  of local holomorphic sections of  $V$  is an equivalence of categories. It maps a isomorphism  $\hat{\Psi} : V_1 \rightarrow V_2$  of vector bundles

with associated sheaves of sections  $\mathcal{E}_1, \mathcal{E}_2$ , covering an isomorphism  $\varphi : M_1 \rightarrow M_2$  of complex manifolds, to the isomorphism  $\Phi$  of  $\mathcal{O}_M$ -modules between  $\mathcal{E}_1$  and  $\varphi^* \mathcal{E}_2$  defined by :

$$\begin{aligned} \Phi & : \mathcal{E}_1(U) \longrightarrow \varphi^* \mathcal{E}_2(U) \\ s & \longrightarrow \hat{\Psi} \circ s \circ \varphi^{-1} \end{aligned} \quad (3)$$

where  $U$  is an open subset of  $M$ .

**Definition 2.2.** A couple  $(\varphi, \Phi)$  as above will be called a *isomorphism of vector bundles between  $\mathcal{E}_1$  and  $\mathcal{E}_2$* . More generally, we define a *isomorphism of meromorphic bundles by replacing the sheaves of holomorphic sections by the corresponding of meromorphic sections with poles at  $D_1$  and  $D_2$* .

Let  $\nabla_1, \nabla_2$  be two meromorphic connections on  $\mathcal{E}_1, \mathcal{E}_2$  with poles supported at  $D_1, D_2$ . We say that an isomorphism of meromorphic vector bundles  $(\varphi, \Phi)$  **preserves the connections** iff :

$$(\varphi, \Phi)^* \nabla_2 = \nabla_1 \quad (4)$$

where  $(\varphi, \Phi)^* \nabla_2$  is the **pullback** of  $\nabla_2$  through  $(\varphi, \Phi)$  defined by the commutative diagram :

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{(\varphi, \phi)^* \nabla_2} & \Omega_{M_1}^1 \otimes \mathcal{E}_1[*D_1] \\ \phi \downarrow & & \uparrow (d\varphi)^* \otimes \phi^{-1} \\ \varphi^*(\mathcal{E}_2[*D_2]) & \xrightarrow{\varphi^* \nabla_2} & \varphi^* \Omega_{M_2}^1 \otimes \varphi^*(\mathcal{E}_2[*D_2]) \end{array}$$

where we denoted  $\varphi^*$  the pullback in the sheaf theoretic sense and  $d\varphi$  the differential in the sheaf theoretic sense.

A **flat** meromorphic connection  $\nabla$  on  $\mathcal{E}$  with poles at  $D$  is a meromorphic connection such that the subsheaf of **horizontal sections**  $\ker(\nabla)$  on  $M \setminus D$  defined by :

$$\forall U \subset M \setminus D, \ker(\nabla)(U) = \{s \in \mathcal{E}(U) \mid \nabla(s) = 0\} \quad (5)$$

is a **local system** (see [9]).

We recall that there is an equivalence of categories between the category of local systems of rank  $r$  on  $M \setminus D$  with arrows being the isomorphisms, and the category of representations  $\rho : \pi_1(M \setminus D, x) \rightarrow K$  (for any  $x \in M \setminus D$ , and  $K$  is a  $\mathbb{C}$ -vector space of dimension  $r$ ) with arrows being the isomorphisms of representations. Once a point  $x \in M \setminus D$  is chosen, this equivalence is obtained by associating to the local system  $\mathcal{K}$ , the **monodromy map**  $Mon^x(\mathcal{K}) : \pi_1(M \setminus D, x) \rightarrow Aut(\mathcal{K}_x)$  (see [9]).

## 2.2 Atiyah sequence of the frame bundle

The **frame bundle** of a locally free  $\mathcal{O}_M$ -module  $\mathcal{E}$  of rank  $r$  is the holomorphic  $GL_r(\mathbb{C})$ -principal bundle  $E \xrightarrow{p} M$  whose fiber at  $x \in M$  is the set of isomorphisms  $\mathbb{C}^r \simeq \mathcal{E}(x)$ . Here  $\mathcal{E}(x) = \mathcal{E}_x / \mathfrak{m}_x$  stands for the fiber of  $\mathcal{E}$  at  $x$ .

We recall that for any complex Lie group  $P$  and a holomorphic  $P$ -principal bundle  $E \xrightarrow{p} M$ , there is a notion of  **$P$ -linearization** for a  $\mathcal{O}_E$ -module  $\mathcal{V}$  : this is a family  $(\phi_b)_{b \in P}$  of isomorphisms  $\phi_b : \mathcal{V} \simeq r_b^* \mathcal{V}$  (where  $r_b$  is the right action of  $P$ ) with nice properties (see [14]). A  $\mathcal{O}_E$ -module equipped with a  $P$ -linearization is said to be  **$P$ -equivariant**. In this context, there is an

equivalence between the  $P$ -equivariant locally free  $\mathcal{O}_E$ -modules and the locally free  $\mathcal{O}_M$ -modules, and between the  $P$ -equivariant morphisms and the morphisms between the corresponding  $\mathcal{O}_M$ -modules (see []). For any representation  $\rho : P \rightarrow GL(\mathbb{V})$ , and any holomorphic  $P$ -principal bundle  $E \xrightarrow{p} M$ , we denote by  $E(\mathbb{V})$  the  $\mathcal{O}_M$ -module associated with the  $\mathcal{O}_E$ -module  $\mathcal{O}_E \otimes \mathbb{V}$ , where the  $P$ -linearization  $(\phi_b)_{b \in P}$  is given by  $\phi_b = r_b^* \otimes \rho(b^{-1})$ . We call it the **representation module** associated with  $E$  and  $\mathbb{V}$ . For any isomorphism  $\Psi : E_1 \rightarrow E_2$  of holomorphic  $P$ -principal bundles covering  $\varphi : M_1 \rightarrow M_2$ , the **representation isomorphism of associated vector bundles** corresponding to  $\Psi$  is the isomorphism

$$\Psi(\mathbb{V}) : E_1(\mathbb{V}) \rightarrow \varphi^* E_2(\mathbb{V}) \tag{6}$$

associated to the  $P$ -equivariant isomorphism  $\Psi^* \otimes Id_{\mathbb{V}}$  of trivial  $\mathcal{O}_E$ -modules.

**Definition 2.3.** *Let  $\mathbb{V}$  be a representation of a complex Lie group  $P$ . Let  $E_1 \xrightarrow{p_1} M_1$  and  $E_2 \xrightarrow{p_2} M_2$  be two holomorphic  $P$ -principal bundles, and  $D_1, D_2$  be respectively two divisors of  $M_1$  and  $M_2$ . An isomorphism  $\Psi : E_1|_{M_1 \setminus D_1} \rightarrow E_2|_{M_2 \setminus D_2}$  of holomorphic  $P$ -principal bundles is  $\mathbb{V}$ -meromorphic between  $(M_1, D_1)$  and  $(M_2, D_2)$  iff the representation isomorphism  $\Phi = \Psi(\mathbb{V})$  restricts to an isomorphism  $\Phi : E_1(\mathbb{V})[*D_1] \rightarrow \varphi^* E_2(\mathbb{V})[*D_2]$  (see 6).*

In particular, mapping holomorphic  $GL_r(\mathbb{C})$ -principal bundles  $E$  over  $M$  to the associated representation modules  $E(\mathbb{C}^r)$  gives an equivalence of categories. A pseudo-inverse is given by mapping a locally free  $\mathcal{O}_M$ -module  $\mathcal{E}$  of rank  $r$  to its frame bundle  $E$ .

Consider  $\mathfrak{p} = Lie(P)$  which is the adjoint representation of  $P$ . Let  $At(E)$  be the  $\mathcal{O}_M$ -module associated with the  $P$ -equivariant locally free  $\mathcal{O}_E$ -module  $TE$  equipped with the  $P$ -linearization induced by the infinitesimal action of  $P$  on  $E$  : it is called the **Atiyah bundle** of  $E$ , and fits into the short exact sequence :

$$0 \longrightarrow E(\mathfrak{p}) \xrightarrow{\iota} At(E) \xrightarrow{q} TM \longrightarrow 0 \tag{7}$$

where  $\iota$  is the morphism associated with the  $P$ -equivariant morphism which to any  $A \in \mathcal{O}_E \otimes \mathfrak{p}$  associates the corresponding fundamental vector field on  $E$ , and  $q$  is the one associated with the  $P$ -equivariant morphism  $dp : TE \rightarrow p^*TM$ .

The previous equivalence implies that  $P$ -equivariant meromorphic one forms on  $E$ , with poles at  $\tilde{D} = p^{-1}(D)$ , and values in  $\mathbb{V}$  are in bijection with morphisms  $\beta : At(E)[*D] \rightarrow E(\mathbb{V})[*D]$ , or equivalently with sections of  $At(E) \otimes E(\mathbb{V})[*D]$ . This correspondance restricts to a bijective correspondance between :

- The set of morphisms  $\beta$  as above vanishing on the image of  $\iota$  in (7), equivalently sections of  $\Omega_M^1[*D] \otimes E(\mathbb{V})$
- The set of meromorphic one forms  $\tilde{\omega}$  on  $(E, \tilde{D})$  with values in  $\mathbb{V}$  vanishing on  $ker(dp)$

### 3 Holomorphic branched Cartan geometries and the Killing connection

In this section, we fix a pair  $(M, D)$  where  $M$  is of complex dimension  $n$ . We define meromorphic Cartan geometries, and the subcategory of branched holomorphic Cartan geometries. We describe their infinitesimal automorphisms as sections for a meromorphic connection either

on a trivial module over the principal bundle of the geometry, or on the corresponding module over the base manifold. We introduce the subcategory of *totally geodesic* meromorphic Cartan geometries : in the next section, we will see that their infinitesimal automorphisms are univaluated, in a sense that will be defined.

### 3.1 Meromorphic and holomorphic branched Cartan geometries

First, we have to define the models for Cartan geometries :

**Definition 3.1.** *A complex Klein geometry of dimension  $n \geq 1$  is a couple  $(G, P)$  where  $G$  is a complex Lie group, and  $P$  is a complex Lie subgroup with  $\dim(G) - \dim(P) = n$ .*

Let  $(G, P)$  be as in Definition 3.1 and let  $P' = \ker(\underline{ad})$  where  $\underline{ad} : P \rightarrow GL(\mathfrak{g}/\mathfrak{p})$  is the representation induced by the adjoint representation. Then any choice of a basis for  $\mathfrak{g}/\mathfrak{p}$  identifies  $Q = P/P'$  with a linear complex subgroup, and  $TG/P$  with the module  $G(\mathfrak{g}/\mathfrak{p})$  associated to the  $P$ -principal bundle  $E$  and the representation  $\mathfrak{g}/\mathfrak{p}$ . Thus, the complex manifold  $G/P$  comes equipped with a holomorphic reduction  $G \times_P Q$  of its holomorphic frame bundle  $R^1(G/P)$ , i.e a holomorphic  $Q$ -structure : namely  $G/P'$ .

This is in fact only due to the presence of a holomorphic 1- $\mathfrak{g}$ -form with special properties on the total space of the holomorphic  $P$ -principal bundle  $G \rightarrow G/P$ , namely the *Maurer-Cartan form*  $\omega_G$  of  $G$ . We can consider curved versions of these objects for which the above fact is still true replacing  $G$  by a suitable holomorphic  $P$ -principal bundle (see next subsection). Authorizing the one form to have poles on the  $P$ -principal bundle, we obtain their meromorphic analogues :

**Definition 3.2.** *Let  $(G, P)$  be a complex Klein geometry with  $\dim(G/P) = n$  and  $(M, D)$  be a pair. A meromorphic  $(G, P)$ -Cartan geometry is a couple  $(E, \omega_0)$  where  $E \xrightarrow{p} M$  is a holomorphic  $P$ -principal bundle, and  $\omega_0$  is a meromorphic 1- $\mathfrak{g}$ -form on  $E$ , with poles on  $\tilde{D} = p^{-1}(D)$ , such that :*

- (i) *For any  $x \in M \setminus D$ ,  $\iota_x^* \omega_0$  coincides with the Maurer-Cartan form  $\omega_{E,x}$  (see above).*
- (ii)  *$\omega_0$  is  $P$ -equivariant.*
- (iii) *For any  $e \in E \setminus \tilde{D}$ ,  $\omega_0(e)$  is an isomorphism between  $T_e E$  and  $\mathfrak{g}$ .*

These objects form a category :

**Definition 3.3.** *Let  $(G, P)$  be a complex Klein geometry and  $(M, D), (M', D')$  be two pairs with  $\dim(M) = \dim(M') = \dim(G/P)$ . Let  $(E, \omega_0)$  and  $(E', \omega'_0)$  be respectively two meromorphic  $(G, P)$ -Cartan geometries on  $(M, D)$  and  $(M', D')$ . An isomorphism between  $(E, \omega_0)$  and  $(E', \omega'_0)$  is an isomorphism of holomorphic  $P$ -principal fiber bundles  $\Psi : E \setminus \tilde{D} \xrightarrow{\sim} E' \setminus \tilde{D}'$  such that  $\Psi^* \omega'_0 = \omega_0$ .*

The following object is central in the study of Cartan geometries :

**Definition 3.4.** *Let  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ . Its curvature function is the meromorphic function  $k_{\omega_0}$  on  $E$  with values in  $\mathbb{W} = \mathfrak{g}^* \wedge \mathfrak{g}^* \otimes \mathfrak{g}$  and defined by :*

$$k_{\omega_0} = d\omega_0 \circ (\omega_0^{-1} \wedge \omega_0^{-1}) + [\cdot, \cdot]_{\mathfrak{g}} \quad (8)$$

where  $[\cdot, \cdot]_{\mathfrak{g}}$  is the Lie-bracket of  $\mathfrak{g}$  identified with an element of  $\mathbb{W}$ .

Fix a Klein geometry  $(G, P)$  and choose a basis  $(\mathfrak{e}_i)_{i=1, \dots, N}$  of  $\mathfrak{g}$ , with  $(\mathfrak{e}_i)_{i=1, \dots, n}$  spanning a supplementary  $\mathfrak{g}_-$  of  $\mathfrak{p}$ . Denote by  $(\mathfrak{e}_i^*)_{i=1, \dots, N}$  the dual basis of  $\mathfrak{g}^*$ .

**Definition 3.5.** Let  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ . The meromorphic functions :

$$\gamma_{i,j}^k = \mathfrak{e}_k^* \circ k_{\omega_0}(\mathfrak{e}_i, \mathfrak{e}_j) \quad (9)$$

are called the structure coefficients of  $(E, \omega_0)$ .

A natural subcategory of the meromorphic  $(G, P)$ -Cartan geometries on pairs is the following :

**Definition 3.6.** A branched holomorphic  $(G, P)$ -Cartan geometry on a pair  $(M, D)$  is a meromorphic  $(G, P)$ -Cartan geometry  $(E, \omega_0)$  on  $(M, D)$  such that  $\omega_0$  extends as a holomorphic one form on  $E$ .

An important feature of these objects for the classification is the existence of a holomorphic connection on the adjoint vector bundle. Indeed, let  $(E, \omega_0)$  be a branched holomorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ , and  $E_G = E \times_P G$  the extension of the holomorphic  $P$ -principal bundle  $E$  to the group  $G$ . By definition,  $E_G$  is the quotient of the product  $E \times G$  by the action of  $G$  given by  $(e, g) \cdot h = (e \cdot h, h^{-1}g)$ . Consider the  $G$ -equivariant holomorphic one form  $\bar{\omega}$  on  $E \times G$  with values in  $\mathfrak{g}$  given by :

$$\bar{\omega} = ad(\pi_2) \circ \pi_1^* \omega_0 + \pi_2^* \omega_G \quad (10)$$

where  $\pi_1, \pi_2$  are the projections on each factor and  $\omega_G$  is the Maurer-Cartan form of  $G$ . It is straightforward to verify that for any  $A \in \mathfrak{g}_0$ ,  $\bar{\omega}(\frac{d}{dt}|_{t=0}(e, h) \cdot exp_G(tA)) = 0$ , i.e the vectors tangent to the fibers of

$$\pi_G : E \times G \longrightarrow E_G \quad (11)$$

are in the kernel of  $\bar{\omega}$ . Thus,  $\bar{\omega}$  induces a holomorphic one form on  $E_G$ .

**Definition 3.7.** The holomorphic  $G$ -principal connection  $\tilde{\omega}$  on  $E_G$  induced by  $\bar{\omega}$  is the tractor-connection of  $(E, \omega_0)$ . We denote by  $\nabla^{\omega_0}$  the corresponding holomorphic connection on  $E_G(\mathfrak{g}) = E(\mathfrak{g})$  (see Proposition 5.0.1).

The pullback  $p^*E(\mathfrak{g})$  is the trivial module  $\mathcal{V} = \mathcal{O}_E \otimes \mathfrak{g}$ .

**Lemma 3.1.** The pullback  $p^*\nabla^{\omega_0}$  is  $d - Ad(\omega_0)$  where  $d$  is the deRham differential on the trivial module  $\mathcal{V}$ ,  $Ad(\omega_0)$  is the section of  $\Omega_E^1 \otimes End(\mathfrak{g}) = \Omega_E^1 \otimes End(\mathcal{V})$  defined by :

$$X \lrcorner Ad(\omega_0)(s) = [\omega_0(X), s]_{\mathfrak{g}}$$

for any holomorphic vector field  $X$  of  $E$  and section  $s$  of  $\mathcal{V}$ . In particular, its curvature is  $R_{p^*\nabla^{\omega_0}} = Ad(d\omega_0) + Ad(\omega_0 \wedge \omega_0)$ .

*Proof.* Since the  $\omega_0$ -constant vector fields on  $E$  span  $T_e E$  at any  $e \in E \setminus \tilde{D}$ , we can choose  $\tilde{A} = \omega_0^{-1}(A)$  for  $A \in \mathfrak{g}$  as a holomorphic vector field on  $E \setminus \tilde{D}$ . Let  $s$  be any section of  $\mathcal{V}(U)$ ,  $U \subset E \setminus \tilde{D}$  an open subset. By definition of  $\nabla^{\omega_0}$  and the remarks preceding Definition 3.7, we have :

$$\tilde{A} \lrcorner p^*\nabla^{\omega_0}(s) = (\bar{A} - \hat{A}) \lrcorner d(\tilde{s})$$

where  $\tilde{s}$  is the unique  $G$ -equivariant section of  $\mathcal{O}_{E \times G} \otimes \mathfrak{g}$  which coincides with  $s$  in restriction to  $E \subset E \times G$ ,  $\bar{A}$  is the unique  $G$ -invariant meromorphic vector field whose restriction to  $E$  coincides with  $\tilde{A}$ , and  $\hat{A}$  is the holomorphic vector field tangent to the fibers of  $E \times G \xrightarrow{\pi_1} E$  such that  $\pi_2^* \omega_G(\hat{A}) = A$ . Indeed,  $\bar{A} - \hat{A}$  is the unique vector field which belongs to  $\ker(\bar{\omega})$  and projects to  $\tilde{A}$  via  $\pi_1 : E \times G \rightarrow E$ . Now,  $\bar{A} - d(\tilde{s})$  coincides with  $\tilde{A} - d(s)$  in restriction to  $E$ , while  $\hat{A} - d(\tilde{s}) = [A, s]_{\mathfrak{g}}$  because  $\tilde{s}$  is  $G$ -equivariant. The first formula follows. For the curvature, it corresponds to the classical computation of the curvature in a trivialisation of a vector bundle.  $\square$

### 3.2 Meromorphic extension of the tangent sheaf

We now describe an object induced by any meromorphic Cartan geometry, which plays the same role as the tangent bundle of the base manifold in the regular case. It is a particular case of the following objects :

**Definition 3.8.** *Let  $(M, D)$  be a pair.*

1. *A meromorphic extension of  $(M, D)$  is a couple  $(\phi_0, \mathcal{E})$  where  $\mathcal{E}$  is a locally free  $\mathcal{O}_M$ -module and  $\phi_0 : TM[\star D] \rightarrow \mathcal{E}[\star D]$  is an isomorphism of  $\mathcal{O}_M$ -modules.*
2. *A holomorphic extension of  $(M, D)$  is a meromorphic extension  $(\phi_0, \mathcal{E})$  such that  $\phi_0(TM) \subset \mathcal{E}$ .*
3. *The category  $\mathcal{F}$  (resp.  $\mathcal{F}^0$ ) of meromorphic extensions (resp. holomorphic extensions) over pairs is defined as follow. An arrow between two meromorphic extensions  $(\phi_0, \mathcal{E})$  and  $(\phi'_0, \mathcal{E}')$  over  $(M_1, D_1)$  and  $(M_2, D_2)$  is an isomorphism  $(\varphi, \Phi)$  of meromorphic bundles (resp. of vector bundles, see Definition 2.2) between  $\mathcal{E}$  and  $\mathcal{E}'$  such that the following diagram commutes :*

$$\begin{array}{ccc}
 TM_1 & \xrightarrow{\phi_0} & \mathcal{E}[\star D_1] \\
 d\varphi \downarrow & & \downarrow \Phi \\
 \varphi^* TM_2 & \xrightarrow{\varphi^* \phi'_0} & \varphi^* \mathcal{E}'[\star D_1]
 \end{array} \tag{12}$$

4. *The category obtained by restricting to meromorphic extensions of  $(M, D)$  and to isomorphisms of meromorphic bundles of the form  $(Id_M, \Phi)$  is denoted by  $\mathcal{F}_{M,D}$  (resp.  $\mathcal{F}_{M,D}^0$ ).*

Meromorphic extensions on  $(M, D)$  are thus canonically isomorphic to submodules of maximal rank of the sheaf of tangent vector fields with poles at  $D$ . The restriction of the corresponding frame bundle to  $M \setminus D$  can thus be canonically identified with the frame bundle of  $M \setminus D$ . This gives the following alternative description :

**Definition 3.9.** *Let  $(M, D)$  be a pair.*

1. *Let  $E \xrightarrow{p} M$  be a holomorphic  $P$ -principal bundle and  $\tilde{D} = p^{-1}(D)$ . A meromorphic solderform on  $(E, \tilde{D})$  is a  $P$ -equivariant meromorphic  $1\text{-}\mathbb{C}^n$ -form  $\theta_0$  on  $E$ , with poles supported at  $\tilde{D}$ , vanishing on  $\ker(dp)$ , and such that  $\theta_0(e)$  is surjective for any  $e \in E \setminus \tilde{D}$ . A couple  $(E, \theta_0)$  is called a meromorphic solder form over  $(M, D)$*



2. An arrow between two meromorphic solderforms  $(E, \theta_0)$  and  $(E', \theta'_0)$  over  $(M_1, D_1)$  and  $(M_2, D_2)$  is an isomorphism of holomorphic  $P$ -principal bundles  $\tilde{\Psi} : E \rightarrow E'$  such that  $\theta_0 = \tilde{\Psi}^* \theta'_0$ . This defines the category  $\mathcal{D}$  of meromorphic solderforms over pairs.

**Proposition 3.0.1.** *The map which to any meromorphic solder form  $(E, \theta_0)$  over  $(M, D)$  (Definition 3.9) associates the meromorphic extension  $(\phi_0, \mathcal{E})$  where  $\phi_0 : TM[\star D] \xrightarrow{\sim} \mathcal{E}[\star D]$  is the isomorphism which corresponds to  $\theta_0$  (see remarks above), extends to an equivalence of categories  $m : \mathcal{D} \rightarrow \mathcal{E}$ .*

*Proof.* If  $\tilde{\Psi} : E \rightarrow E'$  is an arrow between two objects  $(E, \theta_0)$  and  $(E', \theta'_0)$  of the category of solderforms over  $(M, D)$ , we define  $m(\tilde{\Psi}) = \Phi$  as the image of  $\tilde{\Psi}$  through the equivalence of categories described in subsection 2.2. Consider the images  $(\phi_0, \mathcal{E})$  and  $(\phi'_0, \mathcal{E})$  of  $(E, \theta_0)$  and  $(E', \theta'_0)$ . Since  $\theta'_0 = \tilde{\Psi}^* \theta_0$ , by definition,  $\Phi \circ \phi_0 = \phi'_0$  so  $m$  is an essentially surjective functor. Since it is the restriction of the equivalence of categories described in subsection 2.2, it is an equivalence of categories.  $\square$

Now let  $(E, \omega_0)$  be any meromorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ . Then the meromorphic one form  $\pi_{\mathfrak{g}/\mathfrak{p}} \circ \omega_0$  obtained by projecting  $\omega_0$  on  $\mathfrak{g}/\mathfrak{p}$  is  $P$ -equivariant for the quotient adjoint action on  $\mathfrak{g}/\mathfrak{p}$ , and pointwise surjective on  $E \setminus p^{-1}(D)$ . Moreover, its kernel contains  $\ker(dp)$ . By the subsection 2.2, it thus corresponds to a morphism of  $\mathcal{O}_M$ -modules :

$$\phi_0 : TM[\star D] \rightarrow \mathcal{E}[\star D] \tag{13}$$

where we set  $\mathcal{E} = E(\mathfrak{g}/\mathfrak{p})$ . By construction,  $\phi_0$  is an isomorphism of meromorphic bundles and  $(\mathcal{E}, \phi_0)$  is thus a meromorphic extension on  $(M, D)$ .

**Definition 3.10.** *The meromorphic extension  $(\mathcal{E}, \phi_0)$  obtained as above is the meromorphic extension induced by  $(E, \omega_0)$ . We denote by  $f$  the map from the set of meromorphic  $(G, P)$ -Cartan geometries on pairs to the set of meromorphic extensions which maps  $(E, \omega_0)$  to its induced meromorphic extension  $(\mathcal{E}, \phi_0)$ . This extends as a functor  $f$  between the corresponding categories.*

### 3.3 Infinitesimal automorphisms as horizontal sections

Important objects in the study of meromorphic Cartan geometries are the following :

**Definition 3.11.** *Let  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ . An infinitesimal automorphism of  $(E, \omega_0)$  is a holomorphic vector field  $\bar{X}$  on an open subset  $U \subset M \setminus D$ , lifting to a vector field  $X$  on  $p^{-1}(U)$  such that  $\phi_X^* \omega_0 = \omega_0$ . We write  $\mathfrak{k}ill_{M, \omega_0}^{loc}$  for the subsheaf of  $TM \setminus D$  whose sections are the local infinitesimal automorphisms, and  $\mathfrak{k}ill_{E, \omega_0}^{loc}$  for the subsheaf of  $TE \setminus \tilde{D}$  whose sections are the lifts of sections of  $\mathfrak{k}ill_{M, \omega_0}^{loc}$ .*

In order to study the sections of  $\mathfrak{k}ill_{M, \omega_0}^{loc}$ , it is convenient to identify them with horizontal sections for a meromorphic connection on a trivial module over  $E$ . This is a classical approach for general meromorphic parallelisms (see for example [4]). Indeed, let's denote by  $T$  the torsion of the flat meromorphic connection  $\nabla^0$  whose horizontal sections are the  $\omega_0$ -constant vector fields on  $E$ . Then :

**Proposition 3.0.2.** *Let  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ .*

1. The sheaf  $\mathfrak{k}\mathrm{ill}_{E,\omega_0}^{\mathrm{loc}}$  coincides with the sheaf  $\ker(\nabla_{\omega_0}^{\mathrm{rec}})$  of horizontal sections for the reciprocal connection  $\nabla^{\mathrm{rec}}$  defined by :

$$\nabla_X^{\mathrm{rec}} = \nabla_X^0 + T(X, \cdot) \quad (14)$$

for any local vector field  $X$ .

2. The connection  $\nabla_{\omega_0}^{\mathrm{rec}}$  is invariant by the  $P$ -linearization  $(dr_b)_{b \in P}$  corresponding to the action of principal  $P$ -bundle.

*Proof.* 1. See Lemma 3.2 in [4].

2. This straightforwardly follows from the fact that the torsion of  $\nabla_0$  is  $P$ -invariant by definition. □

**Definition 3.12.** *The Killing connection of a meromorphic  $(G, P)$ -Cartan geometry  $(E, \omega_0)$  on  $(M, D)$  is the meromorphic connection  $(\mathcal{V}, \nabla^{\omega_0})$  where  $\mathcal{V} = \mathcal{O}_E \otimes \mathfrak{g}$  and*

$$\nabla^{\omega_0} = \Phi_{\omega_0}^{-1} \nabla^{\mathrm{rec}}$$

where  $\Phi_{\omega_0}$  is the isomorphism of  $\mathcal{O}_E[\star\tilde{D}]$ -modules between  $TE[\star\tilde{D}]$  and  $\mathcal{V}[\star\tilde{D}]$ .

Hence,  $\mathfrak{k}\mathrm{ill}_{E,\omega_0}$  and  $\mathfrak{k}\mathrm{ill}_{M,\omega_0}$  are respectively local systems on  $E \setminus \tilde{D}$  and  $M \setminus D$ .

As explained in the introduction, our goal is to classify quasi-homogeneous meromorphic Cartan geometries  $(E, \omega_0)$ . This hypothesis is satisfied whenever the base manifold  $M$  has only constant meromorphic functions (see [10]). In this case, there exists a point  $x_0 \in M$  and  $n$  independent germs of Killing vector fields for  $(E, \omega_0)$ . We want to see for a sufficient condition for these germs to come from global Killing vector fields, i.e for the following property to be satisfied :

**Definition 3.13.** *Let  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on a pair  $(M, D)$ . It satisfies the extension property of infinitesimal automorphisms if and only the local system  $\mathfrak{k}\mathrm{ill}_{M,\omega_0}$  on  $M \setminus D$  extends as a local system  $\mathfrak{k} \subset TM$  on  $M$ .*

### 3.4 Distinguished foliations and totally geodesic meromorphic Cartan geometries

We will restrict our attention on the following subcategory of meromorphic Cartan geometries. Let  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ , and  $A \in \mathfrak{g} \setminus \{0\}$ . Since  $\omega_0$  induces an isomorphism of meromorphic bundles between  $TE[\star\tilde{D}]$  and  $\mathcal{O}_E[\star\tilde{D}] \otimes \mathfrak{g}$ , there exists a unique distribution of rank one (thus integrable)  $\mathcal{T}_A \subset TE$  with the following property :

$$\omega_0(\mathcal{T}_A) \subset \mathcal{O}_E[\star\tilde{D}]A \quad (15)$$

We will call it the  **$A$ -distinguished foliation** of  $(E, \omega_0)$ , and a leaf  $\Sigma$  will be called a  **$A$ -distinguished curve** for  $(E, \omega_0)$ . A  **$A$ -geodesic** for  $(E, \omega_0)$  will be a complex curve  $\bar{\Sigma}$  embedded in  $M$  such that  $\bar{\Sigma} \setminus D$  lifts to a  $A$ -distinguished curve in  $E$ . When we can lift the whole  $\bar{\Sigma}$ , we say that  $\bar{\Sigma}$  is a **holomorphic geodesic**.

**Definition 3.14.** A meromorphic  $(G, P)$ -Cartan geometry  $(E, \omega_0)$  on a pair  $(M, D)$  is holomorphically totally geodesic iff for a dense subset  $W \subset \tilde{D}$ , and any  $e_0 \in W$ , there exists  $A \in \mathfrak{g} \setminus \mathfrak{p}$  and a smooth leaf  $\Sigma$  of  $\mathcal{T}_A$  (Equation 15) with  $\Sigma \cap \tilde{D} = \{e_0\}$ .

The following lemma show that it suffices to check the existence of one distinguished curve  $\Sigma$  as in Definition 3.14 for each irreducible component  $\tilde{D}_\alpha$  of  $\tilde{D}$  :

**Lemma 3.2.** Let  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on a pair  $(M, D)$ , with  $D$  a smooth irreducible divisor. Let  $e_0 \in \tilde{D} = p^{-1}(D)$ , and suppose there exists  $A \in \mathfrak{g} \setminus \mathfrak{p}$  and a leaf  $\Sigma$  of  $\mathcal{T}_A$  through  $e_0$  s.t.  $\Sigma \cap \tilde{D} = \{e_0\}$ . Then there exists a holomorphic foliation of an open neighborhood  $U$  of  $e_0$  in  $E$  such that any leaf  $\Sigma$  of  $\mathcal{T}_A|_U$  satisfies  $\Sigma \cap \tilde{D} = \{e'_0(e)\}$  for some  $e'_0(e) \in \tilde{D}$ .

*Proof.* Let's prove the first statement. Since there is a leaf  $\Sigma$  of  $\mathcal{T}_A$  through  $e_0$ , there exists an open neighborhood  $\tilde{U}$  of  $e_0$  s.t.  $\mathcal{T}_A|_{\tilde{U}}$  is a regular foliation, and defined by a non vanishing holomorphic vector field  $Z_A$  on  $\tilde{U}$ .

The condition that  $\Sigma \cap \tilde{D}$  is a discrete set of points in  $\tilde{D}$  is equivalent to  $Z_A(e_0) \notin T_{e_0}\tilde{D}$ . Denoting by  $z_1$  an equation of  $U \cap \tilde{D}$ , this means (see [5]) :

$$\dim_{\mathbb{C}} \mathcal{O}_{E, e_0} / \langle \mathcal{L}_{Z_A}(z_1), z_1 \rangle_{e_0} < \infty \quad (16)$$

Up to restriction of  $U$ , for any  $e \in U$  there is an isomorphism of vector spaces  $\rho_e$  between  $\mathcal{O}_E(U)$  and  $\mathcal{O}_{E, e}$  associating to any holomorphic function its germ at  $e$ . The morphism  $\rho_e$  restricts as an isomorphism between the subspaces  $\langle \mathcal{L}_{Z_A}(z_1), z_1 \rangle(U)$  and  $\langle \mathcal{L}_{Z_A}(z_1), z_1 \rangle_e$ . Thus, the function  $e \mapsto \dim_{\mathbb{C}} \mathcal{O}_{E, e} / \langle \mathcal{L}_{Z_A}(z_1), z_1 \rangle_e$  is constant on  $U$ . In particular, by the above remark, any leaf  $\Sigma'$  of  $\mathcal{T}_A|_U$  satisfies  $\Sigma' \cap \tilde{D} = \{e_1, \dots, e_k\}$ . Up to restriction of  $U$ , we can assume  $k = 1$ . In particular, the leaf  $\Sigma_e$  of  $\mathcal{T}_A|_U$  through  $e \in U$  intersects  $\tilde{D}$  in a unique point  $e'_0(e)$ .  $\square$

Moreover, locally, the isomorphism class of totally geodesic meromorphic Cartan geometries and the one of holomorphic totally geodesic Cartan geometries on a pair  $(M, D)$  coincide :

**Lemma 3.3.** Let  $(M, D)$  be a pair and  $(G, P)$  be a complex Klein geometry with  $\dim(G/P) = \dim(M)$ . Let  $(E, \omega_0)$  be a totally geodesic meromorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ , and denote by  $W \subset D$  the corresponding dense subset. Then for any  $x_0 \in W$ , there exists a neighborhood  $U$  of  $x_0$  in  $M$ , a holomorphically totally geodesic meromorphic  $(G, P)$ -Cartan geometry  $(E', \omega'_0)$  on  $(U, D)$  and an isomorphism  $\Psi : E|_{U \setminus D} \rightarrow E'|_{U \setminus D}$  between  $(E|_U, \omega_0)$  and  $(E', \omega'_0)$ .

*Proof.* Pick  $x_0 \in W$ . By definition, there exists  $A \in \mathfrak{g} \setminus \{0\}$ , and a meromorphic  $A$ -distinguished curve  $\tilde{\gamma} : D(0, \epsilon) \rightarrow E$  with  $p \circ \tilde{\gamma}(0) = x_0$  and  $p \circ \tilde{\gamma}(t) \notin D$  for  $t \neq 0$ . Denote by  $\bar{\Sigma}$  the image of  $\gamma = p \circ \tilde{\gamma}$ , which is a smooth one dimensional submanifold of  $M$  by definition of  $W$ . Thus, there are local coordinates  $(\bar{u}_1, \dots, \bar{u}_n)$  on an open neighborhood  $U$  of  $x_0$  such that  $\bar{\Sigma}$  is the common zero loci of  $\bar{u}_2, \dots, \bar{u}_n$ . This enables to consider a section  $\sigma$  of  $E$  over  $U \setminus D$ , with the property  $\sigma \circ \gamma = \tilde{\gamma}$ , by assuming  $\sigma$  to be constant (with respect to any holomorphic section  $\sigma_0 : U \rightarrow E$ ) on level sets of  $(\bar{u}_2, \dots, \bar{u}_n)$ . A such section is  $\mathfrak{g}$ -meromorphic on  $(U, D)$ , in the sense that the induced trivialisation  $\psi_\sigma(\mathfrak{g}) : E(\mathfrak{g})|_{U \setminus D} \rightarrow \mathcal{O}_{U \setminus D} \otimes \mathfrak{g}$  maps the  $\mathcal{O}_U$ -submodule  $j^*E(\mathfrak{g})[*D]$  (where  $j : U \setminus D \hookrightarrow U$  is the inclusion) to the  $\mathcal{O}_U$ -submodule  $j^*\mathcal{O}_U[*D] \otimes \mathfrak{g}$ . Moreover, if  $X$  is a meromorphic vector field on  $U$  with pole at  $D \cap U$ , then the unique  $P$ -invariant holomorphic vector field  $\hat{X}$  on  $p^{-1}(U \setminus D)$  satisfying  $\hat{X} \circ \sigma = T\sigma(X)$  extends as a meromorphic vector field on

$p^{-1}(U)$ . Indeed, it suffices to check it for  $X = \frac{\partial}{\partial u_1}$ , for which it follows from  $\hat{X}|_{\Sigma} = h \circ p\omega_0^{-1}(A)$  with  $h$  a holomorphic function on  $U$ , and the equivariancy of  $\omega_0$ .

We then define the trivial holomorphic  $P$ -principal bundle  $E' = U \times P$ , with canonical holomorphic section  $\sigma'_0$  and the isomorphism of holomorphic  $P$ -principal bundles  $\Psi : E|_{U \setminus D} \rightarrow E'|_{U \setminus D}$  defined by

$$\Psi \circ \sigma = \sigma'_0$$

Then the unique holomorphic  $(G, P)$ -Cartan connection on  $E'|_{U \setminus D}$  satisfying  $\Psi^* \omega'_0 = \omega_0$  extends as a meromorphic Cartan connection on  $E'$ . Indeed,  $(\sigma'_0)^* \omega'_0 = \sigma^* \omega_0$  is a meromorphic one form on  $U$  by the above remarks. Moreover, the intersection of the smooth submanifold  $\Sigma' = \sigma'_0(\bar{\Sigma})$  with  $E'|_{U \setminus D}$  coincide with  $\Psi(\Sigma)$  by construction, thus  $\Sigma'$  is a  $A$ -distinguished curve. This means that  $\bar{\Sigma}$  is a holomorphic geodesic for  $(E', \omega'_0)$ , which ends the proof.  $\square$

## 4 Infinitesimal automorphisms of meromorphic parabolic geometries

A classical result in riemaniann geometry states that any Killing vector field  $X$  for a riemannian metric  $g$  is a Jacobi field : for any geodesic  $\gamma$ , its scalar product  $g(X(\gamma(t)), \gamma'(t))$  with the velocity of  $\gamma$  is constant. There is a natural generalization of riemannian metrics to the holomorphic category, and the corresponding objects are equivalent to torsionfree holomorphic affine connections preserving a holomorphic reduction to the orthogonal group. The holomorphic version of the previous result can be seen as a result on some torsionfree holomorphic affine Cartan geometries (see Corollary 5.1). In this section, we will see a general result for meromorphic Cartan geometries. In particular, this will imply that the local system of infinitesimal automorphisms for any totally geodesic *regular meromorphic parabolic geometry* prolongates as a local system on the whole base manifold.

### 4.1 Regular meromorphic parabolic geometries

A **complex parabolic Klein geometry** is a complex Klein geometry  $(G, P)$  where  $G$  is a complex semi-simple Lie group, and  $P$  a parabolic subgroup. A **meromorphic parabolic geometry** is a meromorphic  $(G, P)$ -Cartan geometry for some complex parabolic Klein geometry. We refer the reader to [7] for a detailed introduction.

With the subgroup  $P$  is associated a grading  $(\mathfrak{g}_i)_{i \in \mathbb{Z}}$  of the Lie algebra  $\mathfrak{g} = Lie(G)$ , meaning  $[\mathfrak{g}_{i_1}, \mathfrak{g}_{i_2}]_{\mathfrak{g}} \subset \mathfrak{g}_{i_1+i_2}$  for any indices  $i_1, i_2 \in \mathbb{Z}$ . We call it the **parabolic graduation** associated with  $P$ . It induces a grading of any representation of  $G$ , in particular  $\mathbb{W} = (\bigwedge^2 \mathfrak{g}_{-}^*) \otimes \mathfrak{g}$  is graded by **homogeneous degrees**  $\mathbb{W}_l$ , and we denote by  $\pi_l$  the corresponding projections.

The **parabolic degree** of  $(G, P)$  is the smallest positive integer  $k \geq 1$  such that  $\mathfrak{g}_i = \{0\}$  for any  $|i| > k$ . The subspaces  $\mathfrak{p} = Lie(P)$  and the subspace

$$\mathfrak{g}_{-} = \bigoplus_{i=-k}^{-1} \mathfrak{g}_i \tag{17}$$

are clearly subalgebras of  $\mathfrak{g}$ . For any  $i \in \{-k, \dots, k\}$ , we will denote  $\mathfrak{g}^i = \bigoplus_{i' \geq i} \mathfrak{g}_{i'}$ , inducing a filtration  $(\mathfrak{g}^i)_{i=-k, \dots, k}$  of  $\mathfrak{g}$ .

By a result of C. Chevalley, we can always pick a basis  $(\mathbf{e}_j^i)_{i=-k, \dots, k}$  of  $\mathfrak{g}$ , such that  $(\mathbf{e}_j^i)_{j=1, \dots, n_i}$  is a basis for  $\mathfrak{g}_i$  for any  $i \in \{-k, \dots, k\}$ , and  $[\mathbf{e}_{j_1}^{i_1}, \mathbf{e}_{j_2}^{i_2}]_{\mathfrak{g}}$  is either 0 or  $\mathbf{e}_j^{i_1+i_2}$  for some  $j \in \{1, \dots, n_{i_1+i_2}\}$ . We will refer to it as a **graded basis** of  $\mathfrak{g}$  for  $(G, P)$ .

The homogeneous space  $G/P$  associated with a complex parabolic Klein geometry  $(G, P)$  bears the following holomorphic geometric structure. Its tangent bundle is filtered by subbundles  $(T^{-i}G/P)_{i=1, \dots, k}$  where  $T^{-i}G/P$  is the projection of  $\omega_G^{-1}(\mathfrak{g}^i)$  through the tangent map  $Tp_{G/P}$  of the projection  $p_{G/P} : G \rightarrow G/P$ . The Lie bracket of holomorphic vector fields on  $G/P$  induces a Lie bracket of holomorphic vector bundle on the corresponding graded bundle  $gr(TG/P)$ . The Lie algebra bundle thus obtained is locally isomorphic to  $(U \times \mathfrak{g}_-, [, ]_{\mathfrak{g}_-})$ .

The **regular meromorphic parabolic geometries** are the infinitesimal versions of this model. More precisely, these are meromorphic  $(G, P)$ -Cartan geometries  $(E, \omega_0)$  on  $(M, D)$  for which the homogeneous component  $\pi_l(k_{\omega_0})$  of degree  $l$  of the Cartan curvature vanishes identically whenever  $l \leq 0$  (see above). This amounts to the following property. Let  $T^{-i}M[\star D]$  be the image of  $\omega_0^{-1}(\mathfrak{g}^{-i})[\star \tilde{D}]$  through  $Tp$ . This gives a filtration of  $TM[\star \tilde{D}]$ , and  $(E, \omega_0)$  is regular if and only if the Lie bracket of vector fields on  $M$  induces a structure of Lie algebras bundle on the graded  $gr(TM \setminus D)$ , locally isomorphic to  $(U \times \mathfrak{g}_-, [, ]_{\mathfrak{g}_-})$ .

## 4.2 Bott connections and infinitesimal automorphisms of Cartan geometries

Now, we come back to a general complex Klein geometry  $(G, P)$ . Let  $(M, D)$  be a complex pair of dimension  $n \geq 1$ , and  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on it. Fix  $A \in \mathfrak{g} \setminus \{0\}$  and consider the holomorphic foliation  $\mathcal{T}_A$  from Equation 15. To any such holomorphic foliation is associated a  $\mathcal{T}_A$ -partial holomorphic connection  $\nabla^{\mathcal{T}_A}$  on  $TE/\mathcal{T}_A$ , the **Bott-connection** of  $\mathcal{T}_A$ , defined as follow. Let  $X$  be a holomorphic vector field on  $U \subset E$ ,  $[X]$  its class in  $TE/\mathcal{T}_A(U)$ , and  $Z \in \mathcal{T}_A(U)$ . Then :

$$Z \cdot \nabla^{\mathcal{T}_A}([X]) = [[Z, X]_{TE}] \quad (18)$$

Let  $t \in \mathbb{C}$  and  $V \subset U$  such that the flow  $\phi = \phi_Z^t$  is well defined on  $V$ . Then clearly  $d\phi(\mathcal{T}_A) \subset \phi^*\mathcal{T}_A$ , so  $\phi$  induces a morphism  $[d\phi]$  of  $\mathcal{O}_V$ -modules defined by the commutative diagram :

$$\begin{array}{ccc} TV & \xrightarrow{d\phi} & \phi^*T\phi(V) \\ q \downarrow & & \downarrow \phi^*q \\ TV/\mathcal{T}_A & \xrightarrow{[d\phi]} & \phi^*T\phi(V)/\mathcal{T}_A \end{array} \quad (19)$$

By the formula (18), the horizontal sections for  $\nabla^{\mathcal{T}_A}$  are the  $[X]$  which are invariant by the isomorphisms of holomorphic vector bundle  $(\phi, [d\phi])$  defined as before.

It will be more convenient to work with the images of meromorphic vector fields on  $E$  through the isomorphism  $\Phi_{\omega_0}$  between  $TE[\star \tilde{D}]$  and  $\mathcal{V}[\star \tilde{D}]$ , where  $\mathcal{V} = \mathcal{O}_E \otimes \mathfrak{g}$ . We will write :

$$\mathcal{K} = \Phi_{\omega_0}(\mathfrak{fill}_{E, \omega_0}) \quad (20)$$

for the corresponding local system on  $E \setminus \tilde{D}$ . Clearly, the image of  $\mathcal{T}_A[\star \tilde{D}]$  is  $\mathcal{V}_A = \mathcal{O}_E[\star \tilde{D}]A$ . The class of a section  $s$  of  $\mathcal{V}[\star \tilde{D}](U)$  (where  $U \subset E$  is an open subset) in  $\mathcal{V}/\mathcal{V}_A[\star \tilde{D}]$  will be denoted by  $[s]_{\mathcal{V}/\mathcal{V}_A}$ . Since  $\Phi_{\omega_0}$  induces an isomorphism of  $\mathcal{O}_E$ -modules between  $TE/\mathcal{T}_A[\star \tilde{D}]$  and  $\mathcal{V}/\mathcal{V}_A[\star \tilde{D}]$ , for any  $Z \in \mathcal{T}_A(U)$ , the morphism  $[d\phi]$  defined by (19) corresponds to an isomorphism

$$\overline{d\phi} : \mathcal{V}/\mathcal{V}_A[\star \tilde{D}]|_V \rightarrow \phi^*\mathcal{V}/\mathcal{V}_A[\star \tilde{D}]|_{\phi(V)} \quad (21)$$

and thus an isomorphism  $(\phi, \overline{d\phi})$  of meromorphic bundles.

The isomorphism of meromorphic bundles  $\Phi_{\omega_0}$  (see above) maps  $\mathcal{T}_A[\star\tilde{D}]$  to  $\mathcal{V}_A[\star\tilde{D}]$ , and we denote by  $\overline{\Phi}_{\omega_0} : TE/\mathcal{T}_A[\star\tilde{D}] \rightarrow \mathcal{V}/\mathcal{V}_A[\star\tilde{D}]$  the isomorphism induced by  $\Phi_{\omega_0}$ . Then :

**Lemma 4.1.** *Let  $s$  be a section of  $\mathcal{K}$  on an open subset  $U \subset E \setminus \tilde{D}$ . Then its class  $[s]_{\mathcal{V}/\mathcal{V}_A}$  is invariant by any isomorphism of meromorphic bundles  $(\phi, \overline{d\phi})$  constructed as above.*

*Proof.* Let  $X$  be any holomorphic vector field on  $U \subset E$ , and  $[X]$  its class in  $TE/\mathcal{T}_A$ . By definition, for any  $Z_A = h\tilde{A}$  (where  $h$  is a meromorphic function on  $U$  and  $\tilde{A} = \omega_0^{-1}(A)$ ) we have :

$$\begin{aligned} 0 &= [\tilde{A}, X]_{TE} \\ &= \frac{1}{h}[Z_A, X]_{TE} \quad \text{mod} \quad \mathcal{T}_A[\star\tilde{D}](U) \end{aligned} \tag{22}$$

In other words, the classes of  $d\phi(X)$  and  $\phi^*X$  in  $TE/\mathcal{T}_A[\star\tilde{D}]$ , well defined on  $U \cap \phi(U)$ , coincides i.e  $s$  is invariant by  $(\phi, \overline{d\phi})$ .  $\square$

Now, we suppose  $M$  to be simply connected. We wish to prove the extension property for  $(E, \omega_0)$  (Definition 3.13). We will use the following general fact on meromorphic Cartan geometries :

**Proposition 4.0.1.** *Let  $(G, P)$  be a complex Klein geometry, and  $(M, D)$  be a pair with  $\dim(M) = \dim(G/P)$ . Let  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ . Let  $x_0 \in D$  belonging to a unique irreducible component  $D_\alpha$ , and suppose that there exists  $A \in \mathfrak{g} \setminus \mathfrak{p}$  and a  $A$ -geodesic  $\overline{\Sigma}$  for  $(E, \omega_0)$  such that  $\overline{\Sigma} \cap D_\alpha = \{x_0\}$ . Then there exists a neighborhood  $U$  of  $x_0$  with the following properties:*

1. *Let  $s$  be a section of  $\mathcal{K}$  on  $V \subset p^{-1}(U)$ . Then the image  $\mu \in GL(\mathcal{V}_e)$  of any loop at  $e \in V$  in  $p^{-1}(U \setminus \tilde{D})$  by the local monodromy of  $\mathcal{K}$  (at  $e$ ) satisfies :  $[\mu(s_e)]_{\mathcal{V}/\mathcal{V}_A} = [s_e]_{\mathcal{V}/\mathcal{V}_A}$ .*
2. *For any section  $s$  of  $\mathcal{K}$  on  $p^{-1}(U \setminus D)$ ,  $[s]_{\mathcal{V}/\mathcal{V}_A}$  extends as a section of  $\mathcal{V}/\mathcal{V}_A[\star\tilde{D}](p^{-1}(U))$ . In other words  $j_*\pi_{\mathcal{V}/\mathcal{V}_A}(\mathcal{K}) \subset \mathcal{V}/\mathcal{V}_A[\star\tilde{D}]$  where  $j$  is the inclusion of  $E \setminus \tilde{D}$  in  $E$  and  $\pi_{\mathcal{V}/\mathcal{V}_A}$  the morphism which map  $s$  to its class  $[s]_{\mathcal{V}/\mathcal{V}_A}$ .*
3. *Suppose moreover that  $(E, \omega_0)$  is holomorphic branched on  $(M, D)$  and let  $s$  be as in 2. Then  $\pi_{\mathcal{V}/\mathcal{V}_A}(s)$  extends as a section of  $\mathcal{V}/\mathcal{V}_A(p^{-1}(U))$ . Equivalently,  $j_*\pi_{\mathcal{V}/\mathcal{V}_A}(\mathcal{K}) \subset \mathcal{V}/\mathcal{V}_A$ .*

*Proof.* 1. It is a classical result that the image  $\mathcal{K}_A = \pi_{\mathcal{V}/\mathcal{V}_A}(\mathcal{K})$  of the local system  $\mathcal{K}$  on  $E \setminus \tilde{D}$  is a local system on  $E \setminus \tilde{D}$ . The image  $\mu_A \in GL((\mathcal{K}_A)_e)$  of any loop of  $E \setminus \tilde{D}$  at  $e$  is given by :

$$\pi_{\mathcal{V}/\mathcal{V}_A} \circ \mu = \mu_A \circ \pi_{\mathcal{V}/\mathcal{V}_A} \tag{23}$$

By the correspondance between local systems and their monodromy, it is thus sufficient to prove that  $\mathcal{K}_A$  is a constant sheaf on a open subset  $\tilde{U} \setminus \tilde{D}$  where  $\tilde{U}$  is a neighborhood of  $e_0 \in p^{-1}(x_0)$ .

By Lemma 3.3, we can suppose, without loss of generality, that  $\overline{\Sigma}$  is a holomorphic geodesic for  $(E, \omega_0)$ . By Lemma 3.2, there exists an open neighborhood  $\tilde{U}_0$  of some point  $e_0 \in p^{-1}(x_0)$  such that any leaf of  $\mathcal{T}_A|_{\tilde{U}_0}$  intersects  $\tilde{D}$  in exactly one point. Equivalently, there exists a holomorphic nonvanishing vector field  $Z$  defining  $\mathcal{T}_A|_{\tilde{U}_0}$  such that  $\phi_Z^t(e'_0) \notin \tilde{D}$  for any  $e'_0 \in \tilde{D} \cap \tilde{U}_0$  and  $t \neq 0$  with  $|t|$  small enough. Pick a point  $e = \phi_Z^{-t}(e_0)$  of the leaf  $\Sigma$

through  $e_0$ , which does't belong to  $\tilde{D}$ . Thus, there is a simply connected neighborhood  $V$  of  $e$  such that  $\mathcal{K}_A$  is a constant sheaf, and thus admits a basis  $[s_1]_{\mathcal{V}/\mathcal{V}_A}, \dots, [s_r]_{\mathcal{V}/\mathcal{V}_A}$ .

By the Lemma 4.1, the family  $\overline{d\phi}([s_1]_{\mathcal{V}/\mathcal{V}_A}), \dots, \overline{d\phi}([s_r]_{\mathcal{V}/\mathcal{V}_A})$ , where  $\overline{d\phi}$  is the morphism (21), is a basis of  $\phi^*\mathcal{K}_A(\phi(V) \setminus \tilde{D})$ . Then  $\mathcal{K}_A|_{\phi(V)}$  is a constant sheaf. The open neighborhood  $U = p(\phi(V))$  of  $x_0$  thus satisfies the required property.

2. Since (21) is an isomorphism of meromorphic bundles, we have proved in 1. that the local system  $\mathcal{K}_A = \pi_{\mathcal{V}/\mathcal{V}_A}(\mathcal{K})$  extends as a constant sheaf, included in  $\mathcal{V}/\mathcal{V}_A[\star\tilde{D}]|_{p^{-1}(U)}$  since  $(\phi, \overline{d\phi})$  is an automorphism of meromorphic bundles for  $\mathcal{V}/\mathcal{V}_A$ .
3. The meromorphic Cartan geometry  $(E, \omega_0)$  is holomorphic branched on  $(M, D)$  if and only if  $\Phi_{\omega_0}(TE) \subset \mathcal{V}$ . Suppose this is the case. Since the automorphism of meromorphic bundles  $(\phi, [d\phi])$  of  $TE/\mathcal{T}_A$  defined before (19) is an automorphism of holomorphic vector bundles. Since  $\Phi_{\omega_0}(TE)$  and  $\mathcal{V}$  coincides when restricted to  $p^{-1}(U \setminus D)$ , we obtain that the image of  $\mathcal{V}/\mathcal{V}_A|_{p^{-1}(U \setminus D)}$  through the  $\overline{d\phi}$  lies in  $\mathcal{V}/\mathcal{V}_A|_{\phi(p^{-1}(U \setminus D))}$ , where  $\phi(p^{-1}(U \setminus D))$  is a neighborhood of  $e_0$  by construction. This proves the assertion.  $\square$

### 4.3 Affine and degree one parabolic models

We now apply Proposition 4.0.1 to prove the extension property for infinitesimal automorphisms of some totally geodesic meromorphic  $(G, P)$ -Cartan geometries. More precisely, we let  $(G, P)$  be a complex parabolic Klein geometry of dimension  $n \geq 2$  and degree  $k = 1$  (see subsection 4.1), or the complex affine Klein geometry of dimension  $n \geq 2$ . For the first model, we denote by  $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  the parabolic graduation associated with  $P$  (see subsection 4.1). For the second one,  $\mathfrak{g}_{-1}$  will stand for the abelian subalgebra of infinitesimal generators for the translations in  $\mathbb{C}^n$ .

These two kind of models are of special interest because  $P$  acts transitively on  $\mathbb{P}(\mathfrak{g}/\mathfrak{p})$  through the projectivized quotient adjoint action. Indeed, if  $(G, P)$  is a degree one parabolic model, for any  $A \in \mathfrak{g}_{-1} \setminus \{0\}$ ,  $Ad(\mathfrak{g}_0)[A]$  must span  $\mathfrak{g}_{-1}$ , since  $\mathfrak{g}$  is semi-simple and  $\mathfrak{g}_{-1}$  is an abelian subalgebra. If  $(G, P)$  is the affine model, the isotropy group  $P = GL_n(\mathbb{C})$  clearly acts transitively on  $\mathfrak{g}_{-1}$ .

By the above remarks, for any  $A \in \mathfrak{g}_{-1} \setminus \{0\}$ , there exists a basis  $(\mathbf{e}_i)_{i=1, \dots, n}$  of the abelian subalgebra  $\mathfrak{g}_{-1} \subset \mathfrak{g} = Lie(G)$ , and  $n$  elements  $b_1, \dots, b_n \in P$  such that

$$ad(b_i^{-1})[A] \in \mathbb{C}\mathbf{e}_i \quad (24)$$

for  $1 \leq i \leq n$ .

**Lemma 4.2.** *Let  $(E, \omega_0)$  be a meromorphic  $(G, P)$ -Cartan geometry on  $(M, D)$ , and let  $x_0 \in D$  belonging to a unique irreducible component  $D_\alpha$ . Suppose there exists a geodesic  $\overline{\Sigma}$  for  $(E, \omega_0)$  with  $\overline{\Sigma} \cap D = \{x_0\}$ . Then, for any  $1 \leq i \leq n$ , there exists a  $\mathbf{e}_i$ -geodesic  $\overline{\Sigma}_i$  for  $(E, \omega_0)$  with  $\overline{\Sigma}_i \cap D = \{x_0\}$ .*

*Proof.* By Lemma 3.3, we can suppose, without loss of generality, that  $\overline{\Sigma}$  is a  $A$ -holomorphic geodesic for  $A \in \mathfrak{g}_{-1} \setminus \{0\}$ . Thus, there exists  $e_0 \in p^{-1}(x_0)$  and a  $A$ -distinguished smooth complex curve  $\Sigma$  with  $p(\Sigma) = \overline{\Sigma}$  and  $\Sigma \cap \tilde{D} = \{e_0\}$ . Let  $b_1, \dots, b_n \in P$  as in (24). By equivariance of  $\omega_0$ , for any  $1 \leq i \leq n$ , the  $b_i$ -translated  $\Sigma_i$  of  $\Sigma$  is a  $\mathbf{e}_i$ -distinguished smooth complex curve,

with  $\Sigma_i \cap \tilde{D} = \{e_0 \cdot b_i\}$ . By Lemma 3.2, there is an open neighborhood  $U_i$  of  $e_0 \cdot b_i$  in  $E$  such that for any  $e \in U_i$ , the leaf  $\Sigma_i$  of  $\mathcal{T}_{\mathfrak{e}_i}|_{U_i}$  through  $e$  intersects  $\tilde{D}$  in a unique point  $e'_0$ . In fact,  $U_i \cap p^{-1}(x_0)$  is a Zariski-dense subset of the fiber  $p^{-1}(x_0)$ , so that  $\tilde{V}_{x_0} = \bigcap_{i=1}^n U_i \cap p^{-1}(x_0)$  is a Zariski dense subset of  $p^{-1}(x_0)$ . Pick any  $e'_0 \in \tilde{V}$ . Then by construction, for any  $1 \leq i \leq n$  there is a  $\mathfrak{e}_i$ -distinguished smooth curve  $\Sigma'_i$  with  $\Sigma'_i \cap \tilde{D} = \{e'_0\}$ . Their projections  $\bar{\Sigma}'_i$  through  $p$  are  $\mathfrak{e}_i$ -geodesics for  $(E, \omega_0)$ , and the proof is thus achieved.  $\square$

**Corollary 4.1.** *Let  $(E, \omega_0)$  be a totally geodesic meromorphic  $(G, P)$ -Cartan connection on  $(M, D)$ , with  $M$  simply connected. Then :*

1.  $(E, \omega_0)$  satisfies the extension property for the infinitesimal automorphisms.
2. Suppose moreover that  $(E, \omega_0)$  is branched holomorphic on  $(M, D)$ . Then any section  $s$  of  $\ker(\nabla_{\omega_0}^\kappa)(U)$ , where  $U \subset E$  is an open subset, is a section of  $\mathcal{V}(U)$ .

*Proof.* 1. Since the complement of a codimension 2 subset of  $M$  has the same fundamental group as  $M$ , and in virtue of the equivalence between local systems and representations of the fundamental group, it suffices to find a codimension 1 subset  $W$  of  $D$ , pick a point  $x_0$  on an unique irreducible component  $D_\alpha$  of  $D \cap W$ , and show the existence of a neighborhood  $U$  of  $x_0$  in  $M$  such that the restriction of  $\ker(\nabla_{\omega_0}^\kappa)$  to  $p^{-1}(U \setminus D)$  extends as a local system on  $p^{-1}(U)$ , included in  $TE[\star D]|_U$ . The property on  $\mathfrak{fil}_{M, \omega_0}$  is clearly invariant under isomorphisms of meromorphic Cartan geometries, so we assume, without loss of generality, that  $(E, \omega_0)$  is holomorphically totally geodesic (Lemma 3.3).

Pick  $x_0 \in W \cap D_\alpha$ , where  $W$  is the dense subset in  $D$  of Definition 3.14. Thus, there exists a holomorphic geodesic  $\bar{\Sigma}$  of  $(E, \omega_0)$  with  $\bar{\Sigma} \cap D = \{x_0\}$ . We now apply Lemma 4.2 to obtain, for any  $1 \leq i \leq n$ , a holomorphic  $\mathfrak{e}_i$ -geodesic  $\bar{\Sigma}_i$  with  $\bar{\Sigma}_i \cap D = \{x_0\}$ . More precisely, the proof of the lemma implies the existence of  $e_0 \in p^{-1}(x_0)$  such that the  $\mathfrak{e}_i$ -distinguished curve  $\Sigma_i$  projecting onto  $\bar{\Sigma}_i$  satisfies  $\Sigma_i \cap \tilde{D} = \{e_0\}$ . Using the Proposition 4.0.1 for each geodesic, we obtain neighborhoods  $U_i$  of  $x_0$  such that the restriction of the local system  $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_i}}(\ker(\nabla_{\omega_0}^\kappa))$  to  $p^{-1}(U_i)$  is a constant sheaf.

Let  $U = \bigcap_{i=1}^n U_i$ . Since  $\mathfrak{e}_1, \mathfrak{e}_2$  are independant vectors of  $\mathfrak{g}$ , the morphism of  $\mathcal{O}_E$ -modules :

$$\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}} \oplus \pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}} : \mathcal{V}[\star \tilde{D}] \longrightarrow \mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}[\star \tilde{D}] \oplus \mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}[\star \tilde{D}] \quad (25)$$

is an isomorphism onto its image. Thus, it restricts to  $\ker(\nabla_{\omega_0}^\kappa)$  as an isomorphism of  $\mathbb{C}$ -sheaves onto its image, a subsheaf of the local system  $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}}(\ker(\nabla_{\omega_0}^\kappa)) \oplus \pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}}(\ker(\nabla_{\omega_0}^\kappa))$ . By the above remark, this local system is a constant sheaf when restricted to  $p^{-1}(U)$ . Thus, the same is true for  $\ker(\nabla_{\omega_0}^\kappa)$ , i.e  $(E, \omega_0)$  satisfies the extension property for the infinitesimal automorphisms.

2. Since  $(E, \omega_0)$  is a branched holomorphic Cartan geometry, we can apply the point 3. of Proposition 4.0.1 to  $A = \mathfrak{e}_1$  and  $A = \mathfrak{e}_2$ . We obtain that the image of  $\ker(\nabla_{\omega_0}^\kappa)$  through  $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}}$  and  $\pi_{\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}}$  respectively extends as subsheaves of  $\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1}$  and  $\mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}$  on  $E$ . Since the morphism (25) clearly restricts to a morphism between  $\mathcal{V}$  and  $\mathcal{V}/\mathcal{V}_{\mathfrak{e}_1} \oplus \mathcal{V}/\mathcal{V}_{\mathfrak{e}_2}$ , this proves the assertion.  $\square$



#### 4.4 Parabolic geometries of higher degree

Now, we let  $(G, P)$  be a complex parabolic Klein geometry of degree  $k > 1$ , and denote by  $\mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_k$  the parabolic graduation. We refer the reader to [7] for the definitions and a complete introduction on this subject.

The group  $P$  no longer acts transitively on  $\mathbb{P}(\mathfrak{g}/\mathfrak{p})$ . Instead, we use a result of [6] which implies the following :

**Lemma 4.3.** *Let  $(E, \omega_0)$  be a regular meromorphic  $(G, P)$ -Cartan geometry on a pair  $(M, D)$ . Then there exists a morphism of  $\mathbb{C}$ -sheaves :*

$$\mathcal{L} : \mathcal{V}_{\mathfrak{g}_{-k}}[\star\tilde{D}] \longrightarrow \mathcal{V}[\star\tilde{D}] \quad (26)$$

with the following properties :

- (i) Let  $\pi_{-k} : \mathcal{V}[\star\tilde{D}] \longrightarrow \mathcal{V}_{\mathfrak{g}_{-k}}[\star\tilde{D}]$  be the projection on  $\mathcal{V}_{\mathfrak{g}_{-k}}$  with respect to  $\mathcal{V}_{\mathfrak{g}_{-k+1}}[\star\tilde{D}]$ . Then  $\pi_{-k} \circ \mathcal{L} = \text{Id}_{\mathcal{V}_{\mathfrak{g}_{-k}}}$ .
- (ii) The restriction of  $\mathcal{L} \circ \pi_{-k}$  to  $\ker(\nabla_{\omega_0}^k)$  is the identity on  $\ker(\nabla_{\omega_0}^k)$ .

*Proof.* The Theorem 4 in [6] is exactly the regular version of this lemma, i.e when  $D$  is empty. Its proof uses only differential operators constructed with the de Rham differential on trivial modules, and morphisms of modules obtained by tensorizing linear map of complex vector spaces with the identity on holomorphic functions. Thus, it straightforwardly extends to the meromorphic category since such operators preserves the sheaves of meromorphic sections.  $\square$

**Corollary 4.2.** *Let  $(E, \omega_0)$  be a regular meromorphic  $(G, P)$ -Cartan geometry on a pair  $(M, D)$ . Suppose that for any irreducible component  $D_\alpha$  of  $D$ , there exists  $A \in \mathfrak{g}_- \setminus \mathfrak{g}_{-k}$  and a  $A$ -geodesic  $\bar{\Sigma}$  of  $(E, \omega_0)$  with  $\bar{\Sigma} \cap D_\alpha = \{x_0\}$ . Then  $(E, \omega_0)$  satisfies the extension property for the infinitesimal automorphisms.*

*Proof.* The property of regularity is invariant under isomorphisms of meromorphic Cartan geometries, so we assume as in the proof of Corollary 4.1 that  $(E, \omega_0)$  is holomorphically totally geodesic without losing any generality. We pick  $x_0 \in W \cap D_\alpha$ , where  $W$  is the dense subset of  $D$  in Definition 3.14 and  $D_\alpha$  the unique irreducible component of  $D$  containing  $x_0$ , and prove that  $\ker(\nabla_{\omega_0}^k)$  extends as constant  $\mathbb{C}$ -subsheaf of  $\mathcal{V}[\star\tilde{D}]$  when restricted to some neighborhood  $p^{-1}(U)$  of  $e_0 \in p^{-1}(x_0)$ .

By definition of  $W$ , there exists  $A \in \mathfrak{g}_- \setminus \mathfrak{g}_{-k}$  and a  $A$ -holomorphic geodesic  $\bar{\Sigma}$  of  $(E, \omega_0)$  with  $\bar{\Sigma} \cap D = \{x_0\}$ , and thus a leaf  $\Sigma$  of  $\mathcal{T}_A$  with  $\Sigma \cap \tilde{D} = \{e_0\}$  for some  $e_0 \in p^{-1}(x_0)$ . We now apply the Proposition 4.0.1 to the  $A$ -holomorphic geodesic  $\bar{\Sigma}$ . Since  $k > 1$ ,  $\mathbb{C}A$  and  $\mathfrak{g}_{-k}$  are independant subspaces in  $\mathfrak{g}$ . Thus, the projection  $\pi_{-k}(\ker(\nabla_{\omega_0}^k))$  extends as a constant  $\mathbb{C}$ -subsheaf of  $\mathcal{V}_{\mathfrak{g}_{-k}}[\star\tilde{D}]$  on a neighborhood  $U$  of  $e_0$ . The image of a constant sheaf by a morphism of  $\mathbb{C}$ -sheaves is a constant sheaf, so by Lemma 4.3,  $\ker(\nabla_{\omega_0}^k)$  extends as a constant  $\mathbb{C}$ -subsheaf of  $\mathcal{V}[\star\tilde{D}]$  on  $U$ . The proof is then achieved.  $\square$

## 5 Application to the classification of meromorphic affine connections

### 5.1 Equivalence between meromorphic principal connections and meromorphic connections

We now prove the equivalence between meromorphic connections on a locally free  $\mathcal{O}_M$ -module  $\mathcal{E}$  and meromorphic principal connections on its frame bundle  $E$ . It straightforwardly restricts as an equivalence between meromorphic connections preserving a holomorphic reduction  $E_1 \subset E$  to a subgroup  $P \subset GL_r(\mathbb{C})$ , and meromorphic  $P$ -principal connections on  $E_1$ . In the regular setting, this was first proved by C. Erhesmanh ([13]) using the formalism of horizontal lifts for paths, and reformulated in an equivariant way by M. Atiyah ([1]). We adopt the point of view of M. Atiyah in order to extend the result to the meromorphic category.

The starting point is that for  $P = GL_r(\mathbb{C})$ , there is a canonical isomorphism ([1], Proposition 9) :

$$E(\mathfrak{p}) = \text{End}(\mathcal{E}) \quad (27)$$

There is a bijection between the set of meromorphic connections  $\nabla$  on  $\mathcal{E}$  and the one of  $\mathcal{O}_M$ -linear splittings  $\delta : \mathcal{E}[\star D] \rightarrow J^1(\mathcal{E})[\star D]$  of the exact sequence of  $\mathbb{C}$ -sheaves :

$$0 \longrightarrow \Omega_M^1[\star D] \otimes \mathcal{E} \longrightarrow J^1(\mathcal{E})[\star D] \longrightarrow \mathcal{E}[\star D] \longrightarrow 0 \quad (28)$$

Let  $\sigma : U \rightarrow E$  be a holomorphic section of the holomorphic frame bundle. This corresponds to a basis  $(s_1, \dots, s_r)$  of  $\mathcal{E}|_U$ , and we denote in the following lines by  $d$  the pullback of the de Rham differential through the corresponding isomorphism  $\mathcal{E}|_U \simeq \mathcal{O}_U^{\oplus r}$ . The former equivalence is given by  $\nabla = d - \delta$ . Indeed, this clearly defines a meromorphic connection, and if  $\nabla$  is a meromorphic connection on  $\mathcal{E}|_U$ , then  $\delta_1 = d - \nabla$  is a morphism of  $\mathcal{O}_U$ -modules from  $\mathcal{E}|_U[\star D]$  to  $\Omega_U^1[\star D] \otimes \mathcal{E}|_U$ , and we obtain a splitting  $\delta = (Id_{\mathcal{E}|_U}, \delta_1)$  of (28).

**Definition 5.1.** *A meromorphic principal connection on a holomorphic  $GL_r(\mathbb{C})$ -principal bundle  $E \xrightarrow{p} M$  with poles at  $\tilde{D} = p^{-1}(D)$  (shortly on  $(E, \tilde{D})$ ) is a meromorphic one form  $\tilde{\omega}$  on  $E$  with values in  $\mathfrak{p}$ , which is  $P$ -equivariant and such that  $\tilde{\omega}$  coincides with the Maurer-Cartan form of  $P$  when restricted to any fiber  $p^{-1}(x) \subset E$ .*

Using the correspondance for equivariant one forms as in subsection 2.2, a meromorphic  $P$ -principal connection on  $(E, \tilde{D})$  is equivalent to a morphism  $\beta : \text{At}(E)[\star D] \rightarrow E(\mathfrak{p})[\star D]$  such that  $\iota \circ \beta = Id_{\text{At}(E)}$ , where  $\iota$  is defined in (7). Its kernel defines a splitting

$$\tau : TM[\star D] \rightarrow \text{At}(E)[\star D] \quad (29)$$

of (7), which uniquely determines  $\beta$ . The following lemma straightforwardly follows from the equivalence described before between equivariant morphisms of modules over principal bundles and morphisms between the corresponding modules over the base manifolds :

**Lemma 5.1.** *Let  $(M_1, D_1)$  and  $(M_2, D_2)$  be two pairs of same dimension. Let  $\tilde{\Psi} : E_1 \rightarrow E_2$  be an isomorphism of holomorphic  $P$ -principal bundles over  $M_1$  and  $M_2$  covering a morphism of pairs  $\varphi : M_1 \rightarrow M_2$  (i.e  $\varphi(D_1) = D_2$ ). Let  $\tilde{\omega}_2$  be a meromorphic principal connection on*

$(E_1, \tilde{D}_1)$  where  $\tilde{D}_1$  is the preimage of  $D_1$  (resp.  $\tilde{\omega}_1 = \tilde{\Psi}^* \tilde{\omega}_1$ ), and  $\tau_1$  (resp.  $\tau_2$ ) be the splitting as in (29). Then the diagram below is commutative :

$$\begin{array}{ccc} TM_1 & \xrightarrow{\tau_1} & At(E_1)[*D_1] \\ d\varphi \downarrow & & \downarrow p_{1*} d\tilde{\Psi} \\ \varphi^* TM_2 & \xrightarrow{\varphi^* \tau_2} & \varphi^* At(E_2)[*D_1] \end{array} \quad (30)$$

where  $p_1$  is the footmap of  $E_1$ .

Denote by  $\tilde{d}$  the usual de Rham differential on  $\mathcal{O}_E[\tilde{D}] \otimes \mathbb{V}$ . Since the  $P$ -linearization  $(\phi_b^{\mathbb{V}})_{b \in P}$  preserves the subsheaf of constant functions with values in  $\mathbb{V}$  on  $E$ , the pushforward  $p_* \tilde{d}$  restricts to  $p_* \tilde{d} : \mathcal{E} \rightarrow p_* \Omega_E^1 \otimes \mathcal{E}$ . This defines a meromorphic connection  $\nabla$  on  $\mathcal{E}$  by :

$$\nabla = \tau - p_* \tilde{d} \quad (31)$$

where  $\tilde{d}$  is defined above and  $-$  stands for the contraction by a vector field.

**Proposition 5.0.1.** *Mapping a meromorphic principal connection  $(E, \tilde{\omega})$  over  $(M, D)$  to the meromorphic connection  $(\mathcal{E}, \nabla)$  on  $(M, D)$  defined by (31) induces an equivalence of categories between :*

- *The category of principal meromorphic (resp. holomorphic) connections over pairs, where the arrows are the  $\mathbb{C}^r$ -meromorphic isomorphisms (Definition 2.3 of principal bundles between pairs preserving the principal connections (resp. isomorphisms of holomorphic principal bundles preserving the principal connections))*
- *The category of meromorphic (resp. holomorphic) connections on  $(M, D)$  with isomorphisms of meromorphic bundle (resp. holomorphic vector bundles, see Definition 2.2) preserving connections (in the sense of (4)).*

*Proof.* Let's first prove that this maps induces a functor. Let  $\tilde{\Psi} : E_1 \rightarrow E_2$  be an isomorphism of meromorphic principal connections between  $(E_1, \tilde{\omega}_1)$  and  $(E_2, \tilde{\omega}_1)$  over  $(M, D)$  and  $(\mathcal{E}_1, \nabla_1)$  and  $(\mathcal{E}_2, \nabla_2)$  obtained as in (31). Let  $(\varphi, \Phi)$  be the associated isomorphism of vector bundles (see (3)). Fix any open subset  $U \subset M_1$  and a basis  $(s_i)_{i=1, \dots, r}$  of  $\mathcal{E}_1|_U$  and denote by  $(\varphi^* t_i)_{i=1, \dots, r}$  its image through  $\Phi$ . Denote by  $(\tilde{s}_i)_{i=1, \dots, r}$  and  $(\tilde{t}_{i=1, \dots, r})$  respectively the corresponding equivariant functions on  $p_1^{-1}(U)$  and  $p_2^{-1}(\varphi(U))$ . Thus  $\tilde{t}_i = \tilde{s}_i \circ \tilde{\Psi}$  by definition of  $\Phi$ . By definition of  $\Phi^{-1} \varphi^* \nabla_2$ , we can compute :

$$\Phi^{-1} \varphi^* \nabla_2(s_i) = (Id_{\Omega_{M_1}^1} \otimes \Phi^{-1})[d\varphi - (\varphi^* \nabla_2(\varphi^* t_i))] \quad (32)$$

Using the definition of  $\nabla_1$  and  $\nabla_2$ , and Lemma 5.1, we get :

$$\begin{aligned} \Phi^{-1} \varphi^* \nabla_2(s_i) &= (Id_{\Omega_{M_1}^1} \otimes \Phi^{-1})[(\varphi^* \tau_2 \circ d\varphi) - \varphi^* p_{2*} \tilde{d}_2(\tilde{t}_i)] \\ &= \tau_1 - (p_{1*} \tilde{d}_1 \tilde{t}_i \circ \tilde{\Psi}) \\ &= \nabla_1(s_i) \end{aligned} \quad (33)$$

where we denoted by  $\tilde{d}_1$  and  $\tilde{d}_2$  the usual deRham differentials on  $\mathcal{O}_{E_1} \otimes \mathbb{C}^r$  and  $\mathcal{O}_{E_2} \otimes \mathbb{C}^r$ . Hence we can map  $\tilde{\Psi}$  to the vector bundle isomorphism  $(\varphi, \Phi)$  which preserves the linear meromorphic connections  $\nabla_1$  and  $\nabla_2$ .

Now, we construct the pseudo-inverse. Let  $(\mathcal{E}, \nabla)$  be a meromorphic connection over a pair  $(M, D)$ . Denote by  $E$  its frames bundle. Let  $x \in M$  and  $U$  be a neighborhood equipped with a holomorphic section  $\sigma : U \rightarrow E$ . Denote by  $(s_1, \dots, s_r)$  the corresponding basis of  $\mathcal{E}|_U$ . The section  $\sigma$  induces a splitting  $TE|_{p^{-1}(U)} = p^*TU \oplus \ker(dp)$  which is  $P$ -equivariant, hence a splitting

$$At(E)|_U = TU \oplus E(\mathfrak{p})|_U \quad (34)$$

We denote by  $\tau_0$  the splitting of the exact sequence (7) restricted to  $U$  induced by (34), and by  $d$  the pullback of the deRham differential through the trivialization associated with  $(s_i)_{i=1, \dots, r}$ . Let  $\delta = \nabla - d$ , which vanishes on the image of  $E(\mathfrak{p})$  through  $\iota$  (see (7)). Its kernel thus define a morphism  $\Theta : TU \rightarrow At(E)|_U[\star D]$ , and we obtain a splitting

$$\tau = \tau_0 + \Theta \quad (35)$$

of (7) over  $U$ . From the remarks above, this is equivalent to a meromorphic principal connection  $\tilde{\omega}_U$  on  $p^{-1}(U)$  with poles at  $\tilde{D} \cap p^{-1}(U)$ .

Now, let  $U, U'$  be two open subset and  $(s_i)_{i=1, \dots, r}$  and  $(s'_i)_{i=1, \dots, r}$  be two basis of  $\mathcal{E}|_U$  and  $\mathcal{E}|_{U'}$  corresponding to holomorphic sections  $\sigma, \sigma'$  of  $E$  on  $U$  and  $U'$ . Let  $d$  and  $d'$  be the corresponding de Rham differentials, then :

$$d - d'(s'_i) = d(s'_i) = d\left(\sum_{j=1}^r b_{ji}^{-1} s_j\right) = \sum_{j=1}^r (bd_0(b^{-1}))_{ji} s'_j \quad (36)$$

where  $b$  is the meromorphic function on  $U \cap U'$  with values in  $P$  such that  $\sigma' = \sigma \cdot b$ , and  $d_0$  is the usual de Rham differential on  $\mathfrak{p}$ -valued functions. Denote by  $\tau$  and  $\tau'$  constructed as before. Thus :

$$\tau' - \tau = [(\sigma, b^* \omega_P)] \quad (37)$$

Thus  $\nabla' - \nabla = d' - d + \tau' - \tau = 0$  and the corresponding meromorphic principal connections  $\tilde{\omega}$  and  $\tilde{\omega}'$  coincide over  $p^{-1}(U \cap U')$ . We obtain a global meromorphic principal connection  $\tilde{\omega}$  on  $(E, \tilde{D})$  inducing  $\nabla$  as in (31).

If  $(\varphi, \Phi_0)$  is an isomorphism of vector bundles preserving the meromorphic connections  $\nabla_1, \nabla_2$ , then from subsection 2.2 it induces an isomorphism  $\tilde{\Psi}$  of holomorphic principal bundles between  $E$  and  $E'$ . Since the action of  $P$  on  $\mathbb{C}^r$  is free, by definition of  $\nabla_1$  and  $\nabla_2$  we get that  $\varphi^* \tau_2 = \tau_1 \circ d\varphi$ . By Lemma 5.1 we obtain  $\tilde{\Psi}^* \tilde{\omega}_2 = \tilde{\omega}_1$ . □

## 5.2 Equivalence between meromorphic affine connections and meromorphic affine Cartan geometries

In this subsection, we consider the complex affine group  $G$  of dimension  $n \geq 1$ , and the complex linear group  $P \subset G$ . The restricted adjoint representation  $ad : P \rightarrow GL(\mathfrak{g})$  splits as the sum of two irreducible representations  $\mathfrak{g}_-$ , the subalgebra corresponding to the infinitesimal generators for the translations in  $Aff(\mathbb{C}^n)$ , and  $\mathfrak{p} = Lie(P)$ . Consequently, if  $E \xrightarrow{P} M$  is a holomorphic  $P$ -principal bundle and  $\omega_0$  is a meromorphic  $(G, P)$ -Cartan connection on  $(E, \tilde{D})$ , then it splits as the sum :

$$\omega_0 = \theta_0 \oplus \tilde{\omega} \quad (38)$$

of a meromorphic solderform  $\theta_0$  on  $(E, \tilde{D})$  (see Definition 3.9) and a meromorphic  $P$ -principal connection  $\tilde{\omega}$  on  $(E, \tilde{D})$ .

Consider the category  $\mathcal{F}^{conn}$  whose objects are triples  $(\phi_0, \mathcal{E}, \bar{\nabla})$  formed by a meromorphic extension  $(\phi_0, \mathcal{E})$  over a pair  $(M, D)$  and a meromorphic connection  $(\mathcal{E}, \bar{\nabla})$  on  $(M, D)$ , and the arrows are the isomorphisms of vector bundle (see subsection 2.2) preserving the meromorphic connections (see (4)). Define the map  $f$  from the category  $\mathcal{G}_{aff}$  of meromorphic  $(G, P)$ -Cartan geometries on  $(M, D)$  to  $\mathcal{F}^{conn}$  as follows. If  $(E, \omega_0)$  is an object of  $\mathcal{G}_{aff}$ , consider the meromorphic solderform  $(E, \theta_0)$  (see Definition 3.9) defined by (38), and  $\bar{\nabla}$  the meromorphic connection on  $\mathcal{E} = E(\mathbb{C}^n)$  associated with  $\tilde{\omega}$  (see Proposition 5.0.1).

Now, consider the subcategory  $\mathcal{G}_{aff}^0$  of  $\mathcal{G}_{aff}$  whose objects are holomorphic branched  $(G, P)$ -Cartan geometries, together with their isomorphisms. Consider a subcategory  $\mathcal{F}^{conn,0}$  of  $\mathcal{F}^{conn}$  obtained by intersecting with  $\mathcal{F}^0$  (Definition 3.8).

**Proposition 5.0.2.** *Let  $(M, D)$  be a pair. The map  $f$  extends as an equivalence of categories between  $\mathcal{G}_{aff}$  (resp.  $\mathcal{G}_{aff}^0$ ) and  $\mathcal{F}^{conn}$  (resp.  $\mathcal{F}^{conn,0}$ ).*

*Proof.* Let  $\Psi : E \rightarrow E'$  is an arrow between two meromorphic  $(G, P)$ -Cartan geometries  $(E, \omega_0)$  and  $(E', \omega'_0)$  over  $(M_1, D_1)$  and  $(M_2, D_2)$ , and  $(\phi_0, \mathcal{E}, \bar{\nabla})$  and  $(\phi'_0, \mathcal{E}', \bar{\nabla}')$  their images through  $f$ . So  $\Psi$  is a morphism of holomorphic principal bundles between the frame bundles and we define  $f(\Psi) = (\varphi, \Phi)$  as the image of  $\Psi$  through the equivalence described in subsection 2.2. By construction, since  $\Psi^* \omega'_0 = \omega_0$ , we have  $\Psi^* \theta'_0 = \theta_0$  and  $\Psi^* \tilde{\omega}' = \tilde{\omega}$ . The first condition implies that  $(\varphi, \Phi)$  is an arrow of meromorphic extensions (Definition 3.8), while the second one implies that it preserves the meromorphic connections  $\nabla$  and  $\nabla'$  (see Lemma 5.1). Hence,  $f$  is a functor. Since it is the restriction of the equivalence of categories from Proposition 3.0.1, we obtain an equivalence of categories.  $\square$

Let  $(\phi_0, \mathcal{E}, \bar{\nabla})$  be an object of  $\mathcal{F}_{M,D}^{conn}$ . Then :

$$\nabla = \phi_0^{-1} \bar{\nabla} \quad (39)$$

defines a meromorphic affine connection on  $(M, D)$ , we will call it the **meromorphic affine connection induced by  $(\phi_0, \mathcal{E}, \bar{\nabla})$** . Thus, there is a functor

$$\mu : \mathcal{F}^{conn} \rightarrow \mathcal{A} \quad (40)$$

to the category  $\mathcal{A}$  of meromorphic affine connections on pairs.

We denote by  $T_{\nabla}$  the torsion of  $\nabla$  (Equation 2). There is the analogous notion of  **$\mathfrak{g}_-$ -torsion** for an object  $(E, \omega_0)$  of  $\mathcal{G}_{aff}$  on  $(M, D)$ . It is the  $P$ -equivariant meromorphic function  $\tau_{\omega_0}$  on  $E$  with values in  $\mathbb{W}_{\mathfrak{g}_-} = \bigwedge^2 (\mathfrak{g}_-)^* \otimes \mathfrak{g}_-$  defined as the projection of the Cartan curvature  $k_{\omega_0}$  of  $(E, \omega_0)$  (see Definition 3.4) on  $\mathbb{W}_{\mathfrak{g}_-}$  respective to  $\bigwedge^2 (\mathfrak{g}_-)^* \otimes \mathfrak{p}$ .

Let  $(E, \omega_0)$  be an object of  $\mathcal{G}_{aff}$  on a pair  $(M, D)$  and  $(\phi_0, \mathcal{E}, \bar{\nabla})$  its image through the equivalence in Proposition 5.0.2. Let  $\nabla$  be the meromorphic affine connection on  $(M, D)$  induced by this object (see above). Identify the  $\mathfrak{g}_-$ -torsion  $\tau_{\omega_0} \circ (\omega_0 \wedge Id_{\mathfrak{g}_-})$  with a meromorphic one form on  $(E, \tilde{D})$  with values in  $\mathfrak{g}_-^* \otimes \mathfrak{g}_- = \text{End}(\mathbb{C}^n)$ . Let  $\bar{\tau}_{\omega_0}$  be the section of  $\Omega_M^1[*D] \otimes \text{End}(\mathcal{E})[*D]$  corresponding to  $\tau_{\omega_0}$  through the equivalence preceding Definition 5.1. Identify it with a section of  $\Omega_M^1 \otimes \text{End}(TM)[*D]$  through the isomorphism  $\phi_0 : TM[*D] \rightarrow \mathcal{E}[*D]$ . Then :

**Lemma 5.2.** *The two sections  $\bar{\tau}_{\omega_0}$  and  $T_{\nabla}$  defined as above coincide.*

*Proof.* It suffices to prove the statement for the restriction  $(E|_{M \setminus D}, \omega_0)$  and  $\nabla|_{M \setminus D}$  over  $M \setminus D$ , in the holomorphic categories. We refer to [16], Theorem 5.1.  $\square$

The vanishing of the torsion is clearly preserved by the isomorphisms of  $\mathcal{G}_{aff}$  and we define  $\mathcal{G}_{aff}^{norm}$  as its subcategory whose objects are meromorphic  $(G, P)$ -Cartan geometries whose torsion vanishes identically. We define  $\mathcal{A}^{norm}$  as the subcategory of  $\mathcal{A}$  whose objects are meromorphic affine connections on pairs whose torsions identically vanish.

Finally, let's remark that for any object  $(\mathcal{E}, \phi_0, \bar{\nabla})$  of  $\mathcal{F}^{conn,0}$ , the meromorphic affine connection (39) restricts as a holomorphic connection on the submodule  $\mathcal{E}$ . We then define :

**Definition 5.2.** *The category  $\mathcal{A}^0$  is the subcategory of  $\mathcal{A}$  whose objects are the meromorphic affine connections on  $(M, D)$  preserving a locally free  $\mathcal{O}_M$ -module  $\mathcal{E}$  with  $TM \subset \mathcal{E} \subset TM[\star D]$ , in the above sense. Its objects are called holomorphic branched affine connections.*

**Lemma 5.3.** *Let  $\nabla$  be a holomorphic branched affine connection on  $(M, D)$ . Then the submodule  $\mathcal{E} \subset TM[\star D]$  from Definition 5.2 is unique.*

*Proof.* Let  $E$  be the bundle of holomorphic frames for  $\mathcal{E}$ , and  $\tilde{\omega}$  be the meromorphic principal connection on  $R^1(M)$  corresponding to  $\nabla$  (Proposition 5.0.1). Suppose there exists another rank  $n$  locally free submodule  $\mathcal{E}'$  of  $TM[\star D]$  such that  $\nabla$  restricts to a holomorphic connection on  $\mathcal{E}'$ , and let  $\tilde{\omega}'$  be the corresponding holomorphic principal connection on its bundle of holomorphic frames  $E'$ .

Pick a point  $x \in M$ , and a neighborhood  $U$  of  $x$  in  $M$  with two basis  $(\bar{s}_1, \dots, \bar{s}_n)$  of  $\mathcal{E}|_U$  and  $(\bar{t}_1, \dots, \bar{t}_n)$  of  $\mathcal{E}'|_U$ . Denote by  $\sigma, \sigma'$  the holomorphic sections of  $R^1(M \setminus D)$  on  $U \setminus D$  corresponding respectively to these basis, and  $b$  be the unique holomorphic function on  $U \setminus D$  with values in  $GL_n(\mathbb{C})$  such that  $\sigma' = \sigma \cdot b$ . Thus,  $\sigma'^* \tilde{\omega} = \sigma^* \tilde{\omega} = 0$ , so that :

$$b^* \omega_{GL_n(\mathbb{C})} = \sigma'^* \tilde{\omega} - \sigma^* \tilde{\omega} = 0 \tag{41}$$

Hence,  $b$  prolongates on  $U$  as a holomorphic function. This being true for any  $x \in M$ , we obtain that the two prolongations  $E$  and  $E'$  of  $R^1(M \setminus D)$  coincide, i.e  $\mathcal{E} = \mathcal{E}'$ .  $\square$

**Corollary 5.1.** *The composition of the equivalence from Proposition 5.0.2 and the map given by (40) gives an equivalence of categories between  $\mathcal{G}_{aff}$  (resp.  $\mathcal{G}_{aff}^0$ ) and  $\mathcal{A}$  (resp.  $\mathcal{A}^0$ ). It restricts as an equivalence between  $\mathcal{G}_{aff}^{norm}$  and  $\mathcal{A}^{norm}$ .*

Recall that, given any meromorphic affine connection  $\nabla$  on  $(M, D)$ , a local holomorphic vector field  $\bar{X}$  on an open subset  $U \subset M$  is a *Killing field* for  $\nabla$  iff the pullback of  $\nabla$  by its flows is again  $\nabla$ . We denote by  $\mathfrak{kill}_{\nabla}$  the subsheaf of  $TM \setminus D$  whose sections are the Killing field for  $\nabla$ . By Corollary 5.1, we obtain :

**Lemma 5.4.** *If  $(E, \omega_0)$  is an object of  $\mathcal{G}_{aff}$  on a pair  $(M, D)$ , and  $\nabla$  the corresponding meromorphic affine connection on  $(M, D)$ , then  $\mathfrak{kill}_{M, \omega_0} = \mathfrak{kill}_{\nabla}$ .*

### 5.3 Totally geodesic branched affine connections in algebraic dimension zero

Now, we will give some application of the results of section 4 to the classification of affine meromorphic connections on some simply connected complex compact manifolds  $M$ . Most of them are adaptations of the arguments used in the proof of the principal theorem in [3], using the results of this article.

**Theorem 5.1.** *Let  $(M, D)$  be a pair, with  $M$  a simply connected complex compact manifold. If  $(M, D)$  bears a quasihomogeneous and totally geodesic meromorphic affine connection  $\nabla$ , then it admits a meromorphic parallelism  $(\bar{X}_1, \dots, \bar{X}_n)$ , such that  $\bar{X}_i$  is a Killing vector field for  $\nabla$  when restricted to  $M \setminus D$ .*

*Proof.* Let  $(E, \omega_0)$  be the meromorphic affine Cartan geometry on  $M$  corresponding to  $\nabla$  (see Corollary 5.1). By the Corollary 4.1, it satisfies the extension property of infinitesimal automorphisms, i.e the local system  $\mathfrak{kill}_\nabla$  on  $M \setminus D$  extends as a local system  $\mathcal{K}$  on  $M$ , with  $\mathcal{K} \subset TM[\star D]$ . Since  $M$  is simply connected, this is a constant sheaf on  $M$ . Since  $\nabla$  is assumed quasihomogeneous, we can pick  $x \in M$  and a  $\mathcal{O}_{M,x}$ -basis  $\bar{X}_{1,x}, \dots, \bar{X}_{n,x}$  of  $(TM)_x$  formed by germs of Killing fields for  $\nabla$ . These germs are thus restrictions of global meromorphic vector fields  $\bar{X}_1, \dots, \bar{X}_n$  whose restrictions to  $M \setminus D$  are elements of  $\mathfrak{kill}_\nabla(M \setminus D)$ . Since their germs at  $x$  are independants, there exists a Zariski-dense open subset  $M \setminus S$  such that the restrictions of  $\bar{X}_1, \dots, \bar{X}_n$  to any subset  $U \subset M \setminus S$  are independant elements of  $TM(U)$ , i.e  $(\bar{X}_1, \dots, \bar{X}_n)$  is a meromorphic parallelism on  $M$ . □

**Corollary 5.2.** *Let  $M$  be a compact complex manifold with finite fundamental group, and whose meromorphic functions are constants. Then  $M$  doesn't bear any totally geodesic branched holomorphic affine connection.*

*Proof.* Suppose that  $\nabla$  is a totally geodesic branched holomorphic affine connection on  $(M, D)$ , denote by  $\mathcal{E}$  the submodule of  $TM[\star D]$  from Lemma 5.3. Complete the meromorphic parallelism  $(\bar{X}_i)_{i=1, \dots, n}$  from Theorem 5.1 into a basis  $(\bar{X}_j)_{j=1, \dots, r}$  for the global meromorphic Killing fields of  $\nabla$ . A meromorphic parallelism is a rigid geometric structure (see [12]), so by the Theorem 2 of [11], the juxtaposition of  $(\bar{X}_j)_{j=1, \dots, r}$  and  $\nabla$  is quasihomogeneous. Since  $\nabla$  satisfies the extension property for the Killing vector fields (Corollary 4.1) and  $M$  is simply connected, we obtain a meromorphic parallelism  $(\bar{X}'_i)_{i=1, \dots, n}$  such that the restriction of each  $\bar{X}'_i$  to  $M \setminus D$  is a Killing field for  $\nabla$  and commutes with each  $\bar{X}_j$ . In particular, each  $\bar{X}'_i$  is a  $\mathbb{C}$ -linear combination of the  $(\bar{X}_j)_{j=1, \dots, r}$ , so  $(\bar{X}'_i)_{i=1, \dots, n}$  are commuting meromorphic vector fields.

Now, let pick any Gauduchon metric on  $M$  ([15]) and let's prove that the degree  $deg(\mathcal{E})$  of  $\mathcal{E}$  with respect to this metric is zero. Let  $(E, \omega_0)$  be the branched holomorphic affine Cartan geometry on  $(M, D)$  corresponding to  $\nabla$  (Corollary 5.1). Then  $\mathcal{E} = E(\mathfrak{g}/\mathfrak{p}) = E(\mathfrak{g})/E(\mathfrak{p})$  by definition of  $(E, \omega_0)$ , and since  $P = GL_n(\mathbb{C})$ , we have  $deg(E(\mathfrak{p})) = 0$  (see [2], Corollary 4.2).

We must then prove that  $deg(E(\mathfrak{g})) = 0$ . For, it is sufficient to prove that  $C_1(R_{\nabla\omega_0})$  vanishes identically, where  $C_1$  is the trace on  $End(E(\mathfrak{g}))$  and  $\nabla^{\omega_0}$  is the tractor connection (see Definition 3.7). We will prove that the meromorphic one form  $\eta_i = \tilde{X}'_i - C_1(R_{\nabla\omega_0})$  vanishes identically on  $M$  for any  $1 \leq i \leq n$ . By Lemma 3.1, we have :

$$p^* \eta_i = \tilde{X}'_i - C_1(R_{p^*\nabla\omega_0}) = \underbrace{dC_1(Ad(s_i))}_{\tilde{\eta}_i^0} + \underbrace{\tilde{X}'_i - Ad(\omega_0 \wedge \omega_0)}_{\tilde{\eta}_i^1} \quad (42)$$

where  $\tilde{X}'_i$  is the lifting of  $X'_i$  to  $E$  and  $s_i = \omega_0(X'_i)$ .

The meromorphic one form  $\tilde{\eta}_i^0$  is exact and  $P$ -equivariant. By a classical result on exact invariant forms on connected Lie groups, the restriction of  $\tilde{\eta}_i^0$  to any fiber of  $E \xrightarrow{p} M$  corresponds to a homomorphism  $\chi : P \rightarrow \mathbb{C}$ . Because  $P = GL_n(\mathbb{C})$ , any such homomorphism is trivial, so that  $\tilde{\eta}_i^0$  vanishes on the fibers of  $E \xrightarrow{p} M$ . Thus, it is the pullback of a meromorphic exact one form  $\eta_i^0$  on  $M$ . Moreover, by Corollary 4.1,  $s_i$  is a holomorphic section of  $\mathcal{V}$  on  $E$ , so that  $\tilde{\eta}_i^0$  is a holomorphic one form. Thus,  $\eta_i^0$  is an exact holomorphic one form on a simply connected compact complex manifold, i.e. vanishes everywhere.

Now, let's prove that  $\tilde{\eta}_i^1 = p^* \eta_i$  vanishes everywhere. Consider  $E_G = E \times G \xrightarrow{p_G} E$  and the holomorphic tractor connection  $\tilde{\omega}$  on it (Definition 3.7). Using the splitting  $TE_G = \ker(\tilde{\omega}) \oplus \ker(d\pi_G)$ , the pullback  $\hat{\eta}_i^1 = p_G^* \tilde{\eta}_i^1$  uniquely decomposes as a sum :

$$\hat{\eta}_i^1 = \hat{\eta}_i^H \oplus \hat{\eta}_i^V \quad (43)$$

with  $\hat{\eta}_i^H$  a  $G$ -invariant meromorphic one form on  $E_G$ , vanishing on  $\ker(\tilde{\omega})$ , and  $\hat{\eta}_i^V$  vanishing on  $\ker(\tilde{\omega})$ . In particular,  $\hat{\eta}_i^H$  is the pullback of  $\eta_i$  through the composition  $p_G = p \circ \pi_G$ , so that  $\hat{\eta}_i^V$  vanishes everywhere. Now, using Corollary 4.1,  $\tilde{\eta}_i^1$  is a holomorphic one form, so that  $\eta_i$  is a holomorphic one form on  $M$ . Using the Lie-Cartan formula, we have :

$$d\eta_i(X'_j, X'_k) = \mathcal{L}_{X'_j} \eta_i(X'_k) - \mathcal{L}_{X'_k} \eta_i(X'_j) - \eta_i([X'_j, X'_k]_{TM}) \quad (44)$$

Since the only meromorphic functions on  $M$  are the constants, we obtain  $\mathcal{L}_{X'_j} \eta_i(X'_k) = \mathcal{L}_{X'_k} \eta_i(X'_j) = 0$ , and since the meromorphic vector fields  $(X'_i)_{i=1, \dots, n}$  commute,  $\eta_i$  is a closed holomorphic one form. Since  $M$  is simply connected and compact,  $\eta_i$  vanishes everywhere. This proves that  $C_1(R_{\nabla \omega_0})$  vanishes everywhere, i.e.  $\deg(E(\mathfrak{g})) = 0$ .

Hence,  $\deg(\mathcal{E}) = 0$ . Let  $\bar{s}_1, \dots, \bar{s}_n$  be the images of  $X'_1, \dots, X'_n$  through the morphism  $\phi_0$ , where  $(\phi_0, \mathcal{E})$  is the holomorphic extension image of  $(E, \omega_0)$  as in Proposition 5.0.2. Since  $s_i = \omega_0(\tilde{X}'_i)$  is a section of  $\mathcal{V}(E)$  for any  $1 \leq i \leq n$  from Corollary 4.1, each  $\bar{s}_i$  is a section of

$\mathcal{E}(M)$ . Since they are independant, the holomorphic section  $\bigwedge_{i=1}^n \bar{s}_i$  of  $\det(\mathcal{E})$  is not identically vanishing, thus  $\det(\mathcal{E})$  is trivial and  $\bigwedge_{i=1}^n \bar{s}_i$  never vanishes. It therefore forms a basis of  $\mathcal{E}$  on  $M$ , and the dual sections  $\bar{s}_1^*, \dots, \bar{s}_n^*$  are holomorphic sections of  $\mathcal{E}^*$  on  $M$ . We obtain a branched holomorphic  $(\mathbb{C}^n, \{0\})$ -Cartan geometry  $(M, \eta)$  on  $(M, D)$ , where :

$$\eta = \bigwedge_{i=1}^n (\bar{s}_i^* \circ \phi_0) \otimes \mathbf{e}_i \quad (45)$$

where  $(\mathbf{e}_i)_{i=1, \dots, n}$  is the canonical basis of  $\mathbb{C}^n$ . Because the  $\eta$ -constant vector fields  $(X'_i)_{i=1, \dots, n}$  commute, it is a flat branched holomorphic Cartan geometry. Since  $M$  is simply connected and compact, there is a holomorphic submersion  $dev : M \rightarrow \mathbb{C}^n$ . This is impossible by the maximum principle, so  $M$  cannot bear any totally geodesic branched holomorphic affine connection.  $\square$

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